SELF-ADJOINT EXTENSIONS OF THE DIRAC HAMILTONIAN WITH A δ-SPHERE INTERACTION

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The purpose of this paper is to make an explicit construction of specific self-adjoint extensions of the Dirac Hamiltonian in the presence of a δ-sphere interaction of finite radius. The exact resolvent kernel of the free Dirac operator is given. This specifies related results that have recently appeared in the literature.

1. Introduction

Solvable Hamiltonians with singular δ-sphere potentials have been the subject of many investigations in the past since the work by Green and Moszkowsky.[1] These models belong to the small class of models that are analytically solvable. As far as we know, all these studies have used the von Neumann formalism of self-adjoint extensions of symmetric linear operators. The earlier works concern nonrelativistic quantum mechanics based on the Schrödinger Hamiltonian with δ interactions. A good summary of these nonrelativistic models is to be found in Refs. 2 and 3. Comparatively, a much smaller number of papers have dealt with the relativistic δ-Dirac Hamiltonian. To the best of our knowledge, the first rigorous treatments of singular Dirac Hamiltonians were provided by Dittrich, Exner and Šeba.[4,5] Next, in the same vein, there have been the papers by Avossevou and Hounkonnou,[6,7,8,9,10] and more recently those by Shabani and Vyabandi[11] which include quite an uncommon critique of the work in Refs. 6–10. In this contribution, we provide self-adjoint extensions of the δ-Dirac Hamiltonian with support on a sphere of finite radius, given specific boundary conditions.

The paper is organized as follows. We present the model and give the normalised solutions for the free Dirac radial operator in Secs. 2 and 3. This leads us to the construction of the resolvent of the free Dirac radial Hamiltonian in Sec. 4. In Secs. 5 and 6 we then construct the resolvent of the δ-Dirac Hamiltonian given boundary conditions of the first type.

2. Model and Boundary Conditions

Let us consider in the Hilbert space $\mathcal{H}$ the formal expression describing the interaction model

$$H = H_D + V(|\vec{x}|) \quad , \quad \vec{x} \in \mathbb{R}^3 \ , \tag{1}$$

where $V(|\vec{x}|)$ is the interaction potential to be specified hereafter. The quantity H_D is the free Dirac operator,

$$H_D \equiv -i\hbar c \underline{\alpha} \nabla + \underline{\beta} mc^2 \quad , \quad \hbar = h/2\pi \ , \tag{2}$$

where the following notations are used,

$$\underline{\alpha} = \begin{pmatrix} \mathbf{0} & \sigma \\ \sigma & \mathbf{0} \end{pmatrix} \quad , \quad \underline{\beta} = \begin{pmatrix} \mathbb{1}_2 & \mathbf{0} \\ \mathbf{0} & -\mathbb{1}_2 \end{pmatrix} \ , \tag{3}$$

$\sigma \equiv \sigma^\iota$ $(\iota = 1, 2, 3)$ being the 2×2 Pauli matrices defined by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{4}$$

so that the Hamiltonian is explicitly given by

$$H_D = \begin{pmatrix} mc^2 \mathbb{1}_2 & -i\hbar c \sigma \cdot \nabla \\ -i\hbar c \sigma \cdot \nabla & -mc^2 \mathbb{1}_2 \end{pmatrix} \quad , \quad \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{5}$$

The method adopted for the study of this Hamiltonian is based on the von Neuman theory of self-adjoint extensions of linear symmetric operators.[12,13] The first step is to consider within a Hilbert space $\mathcal{H}$ the symmetric operator $\dot{H} \equiv H_D$, $\mathcal{D}(\dot{H}) = \{\psi \in H^{1,2}(\mathbb{R}^3) \otimes \mathbb{C}^4 \ , \ \psi(S_R) = 0\}$, where S_R is the closed ball of radius R centered at the origin in $\mathbb{R}^3$, and $H^{1,2}$ the Sobolev space of indices $(1, 2)$. The closure of this operator is given by

$$\overline{\dot{H}} = \overline{H_D|_{C_0^\infty(\mathbb{R}^3 \setminus S_R)}} = H_D|_{H_0^{1,2}(\mathbb{R}^3 \setminus S_R)} \ . \tag{6}$$

The Hilbert space $\mathcal{H}$ decomposes as[4,5]

$$\mathcal{H} = \bigoplus_{j=(1/2)}^{\infty} \bigoplus_{\ell=j-(1/2)}^{j+(1/2)} U_{j\ell}^{-1} \left[L^2((0,\infty),dr)\otimes \mathbb{C}^2\right] \bigotimes \left[\Omega_{j\ell-j},\cdots,\Omega_{j\ell j}\right] \ , \tag{7}$$

where the isomorphism U_{jl} has been introduced to remove the weight factor r^2 from the measure on the original radial space. The corresponding radial self-adjoint operator writes as

$$\dot{H} = \bigoplus_{j=(1/2)}^{\infty} \bigoplus_{\ell=j-(1/2)}^{j+(1/2)} U_{j\ell}^{-1}\dot{h}_{j\ell}U_{j\ell}\otimes \mathbb{1} \ ,$$

$$\dot{h}_{j\ell} = \tau = \begin{pmatrix} mc^2 & \hbar c\left(-\frac{d}{dr}+\frac{\kappa_{j\ell}}{r}\right) \\ \hbar c\left(\frac{d}{dr}+\frac{\kappa_{j\ell}}{r}\right) & -mc^2 \end{pmatrix} \ , \tag{8}$$

$$\mathcal{D}(\dot{h}_{j\ell}) = \left\{\psi\begin{pmatrix} f(r) \\ g(r)\end{pmatrix} \in L^2((0,\infty),dr)\otimes\mathbb{C}^2;\ \psi\in AC_{loc}((0,\infty)\setminus\{R\}); \right.$$
$$\left. \dot{h}_{j\ell}\psi \in L^2((0,\infty))\otimes\mathbb{C}^2\right\} \ , \tag{9}$$

where $f(r)$ and $g(r)$ determine the radial dependence of the upper and lower Dirac bi-spinors. The symbol $\mathbb{1}$ in Eq. (8) stands for the unit operator.

Note that the radial operator obtained in Eq. (8) is the same as the one given in Eq. (2.9a) of Ref. 4 provided a choice of units such that $\hbar = 1 = c$ is made.

The interaction potential associated to this bi-spinor decomposition writes as,[4]

$$V(|\vec{x}|) = G\delta(|\vec{x}|-R) = G\delta(r-R) \quad , \quad G = \begin{pmatrix} \alpha_o & 0 \\ 0 & \beta_o \end{pmatrix} \ , \ \alpha_o,\beta_o \in \mathbb{R} \ . \tag{10}$$

The boundary conditions we use to characterize the self-adjoint extensions are obtained as follows. Assume that the limits $\psi(R_\pm)$ exist and consider the eigenvalue equation

$$\left[\dot{h}_{j\ell} + G\delta(r-R)\right]\psi = E\psi \ . \tag{11}$$

Then, integrating over $(R-\varepsilon, R+\varepsilon)$ taking the limit $\varepsilon \to 0$, and using the relation

$$\int_{R-\varepsilon}^{R+\varepsilon} dr\delta(r-R)\psi(r) = \frac{1}{2}[\psi(R_+)+\psi(R_-)] \quad , \quad \varepsilon\to 0 \ , \tag{12}$$

we obtain

$$\left(\frac{1}{2}G + \hbar c\tau_o\right)\psi(R_+) + \left(\frac{1}{2}G - \hbar c\tau_o\right)\psi(R_-) = 0 \quad , \quad \tau_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ . \quad (13)$$

Proposition: The general boundary conditions Eq. (13) define a self-adjoint extension of $\dot{h}_{jl}$ iff $G^+ = G$.
Proof. The proof proceeds in the same manner as that of Proposition 4.1 in Ref. 4.

To solve the problem, henceforth we shall work in a given total angular momentum sector $(j\ell)$ of the Dirac spectrum, since the considered scalar potential is spherically symmetric. Hence, we shall use the standard notation as follows,

$$\ell = j \pm \frac{1}{2} \quad , \quad \kappa = \kappa_{jl} = (-1)^{j-\ell+1/2}(j+\frac{1}{2}) \quad , \quad j = 1/2, 3/2, 5/2, \cdots \ . \quad (14)$$

In what follows, we shall give expressions valid whether for $\ell = j + 1/2$ or for $\ell = j - 1/2$, the upper/lower signs corresponding to these two situations in the same order, namely $\ell = j \pm 1/2$. Thus we have

$$\begin{aligned} \ell = j + \tfrac{1}{2}: &\quad \kappa = j + \tfrac{1}{2} = \ell \ , \\ \ell = j - \tfrac{1}{2}: &\quad \kappa = -(j + \tfrac{1}{2}) = -(\ell + 1) \ . \end{aligned} \quad (15)$$

The strategy for relating appropriate choices of boundary conditions which define self-adjoint extensions of the Dirac Hamiltonian to the parameters $\mu_{ij}(z)$ in Krein's formula (see Eq. (46)) in the general setting is clear. It suffices to apply this formula to a set of functions $(F(r')\ G(r')$ and determine which boundary conditions are obeyed, as a function of the choice for the quantities $\mu_{ij}(z)$. Here, we shall not pursue the general analysis, but only restrict to the two cases which are called[14] "boundary conditions of the first type", namely when either one of the two functions $f(r)$ or $g(r)$ is continuous at $r = R$ while the other is discontinuous at $r = R$ with a discontinuity that is proportional to the value of the continuous function at that point. Namely, we shall restrict to the following two cases,

$$f(R_+) = f(R_-) = f(R) \quad , \quad g(R_+) - g(R_-) = \alpha f(R) \ , \quad (16)$$

$$f(R_+) - f(R_-) = \beta g(R) \quad , \quad g(R_+) = g(R_-) = g(R) \ , \quad (17)$$

where $\alpha = \alpha_o/(\hbar c)$ and $\beta = -\beta_o/(\hbar c)$ are arbitrary real parameters. As we shall see, since such a choice of boundary conditions may indeed be put into one-to-one correspondence with a choice for the Krein parameters $\mu_{ij}(z)$, these conditions define two separate one-parameter self-adjoint extensions of the Dirac Hamiltonian, associated to the δ-sphere scalar potential.

From now on, the boundary conditions defined by Eq. (16) will be called "α-type boundary condition of the first type" and those defined by Eq. (17) will be called "β-type boundary condition of the first type". Note, as it should be, that these boundary conditions are particular cases of boundary conditions given in Eq. (13).

The self-adjoint extension of $\dot{H}$ corresponding to the specific boundary conditions in Eqs. (16) and (17) write, respectively,

$$H_\alpha = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} U_{j\ell}^{-1} h_{j\ell,\alpha} U_{j\ell} \otimes 1\!\!1 \quad , \quad \alpha = \{\alpha_{j\ell}\} \ , \tag{18}$$

with

$$\begin{aligned} h_{j\ell,\alpha_\ell} &\equiv \tau \ , \\ \mathcal{D}(h_{j\ell,\alpha_\ell}) &= \big\{ \psi \in L^2((0,\infty)) \otimes \mathbb{C}^2 \ ; \ f_{j\ell} \in AC_{loc}((0,\infty)) \ ; \\ &\quad g_{j\ell} \in AC_{loc}((0,\infty) \setminus \{R\}); \\ &\quad g_{j\ell}(R_+) - g_{j\ell}(R_-) = \alpha f_{j\ell}(R); \\ &\quad h_{j\ell,\alpha}\psi \in L^2((0,\infty)) \otimes \mathbb{C}^2 \ \big\} \ , \end{aligned} \tag{19}$$

and

$$H_\beta = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{l=j-\frac{1}{2}}^{j+\frac{1}{2}} U_{j\ell}^{-1} h_{j\ell,\beta} U_{j\ell} \otimes 1\!\!1 \quad , \quad \beta = \{\beta_{j\ell}\} \ , \tag{20}$$

with

$$\begin{aligned} h_{j\ell,\alpha_\ell} &\equiv \tau \ , \\ \mathcal{D}(h_{j\ell,\beta_{j\ell}}) &= \big\{ \psi \in L^2((0,\infty)) \otimes \mathbb{C}^2 \ ; \ g_{j\ell} \in AC_{loc}((0,\infty)) \ ; \\ &\quad f_{j\ell} \in AC_{loc}((0,\infty) \setminus \{R\}); \\ &\quad f_{j\ell}(z;R_+) - f_{j\ell}(z;R_-) = \beta_{j\ell} g_{j\ell}(z;R); \\ &\quad h_{j\ell,\beta_{j\ell}}\psi \in L^2((0,\infty)) \otimes \mathbb{C}^2 \ \big\} \ . \end{aligned} \tag{21}$$

3. Solutions to the Free Radial Dirac Hamiltonian

In the present section, we are interested in the determination of the eigenstates of the free radial operator, and in particular their absolute normalisation as required for the construction of the associated free Dirac resolvent or kernel of the corresponding Green's function. Hence, we must solve for the eigenvalue equation

$$\begin{pmatrix} mc^2 & \hbar c\left(-\frac{d}{dr}+\frac{\kappa}{r}\right) \\ \hbar c\left(\frac{d}{dr}+\frac{\kappa}{r}\right) & -mc^2 \end{pmatrix}\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = E\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} \ . \tag{22}$$

The choices of normalisation and signs for these components is standard and we required only that

$$\int_0^\infty dr\,\left(f^*(r;p) \quad g^*(r;p)\right)\begin{pmatrix} f(r;p') \\ g(r;p') \end{pmatrix} = \delta(p-p') \ . \tag{23}$$

We have to distinguish the two cases of either positive or negative energy states, since we know that the free Dirac energy spectrum is that of positive and negative energy states such that

$$E = \pm\sqrt{(pc)^2+(mc^2)^2} \quad , \quad p \geq 0 \ . \tag{24}$$

Then, introducing the following parametrisations,

$$E(p) = \pm\omega(p) \quad , \quad \omega(p) = \sqrt{(pc)^2+(mc^2)^2} \quad , \quad \rho = \frac{pc}{\hbar c} r \ , \tag{25}$$

and labelling all distinct eigenstates by the value for $p > 0$ and the sign of the energy, one finds for the normalised solutions to the free radial Dirac Hamiltonian of positive energy, for $\ell = j \pm 1/2$,

$$f_+(r;p) = \frac{pc}{\sqrt{2\hbar\omega(p)(\omega(p)-mc^2)}}\left(\frac{pr}{\hbar}\right)^{1/2} J_{\ell+1/2}\left(\frac{pr}{\hbar}\right) \ , \tag{26}$$

$$g_+(r;p) = \frac{pc}{\sqrt{2\hbar\omega(p)(\omega(p)+mc^2)}}\left(\frac{pr}{\hbar}\right)^{1/2} J_{\ell+1/2\mp1}\left(\frac{pr}{\hbar}\right) \ . \tag{27}$$

The negative energy solutions are given by, with $\ell = j \pm 1/2$,

$$f_-(r;p) = \frac{pc}{\sqrt{2\hbar\omega(p)(\omega(p)+mc^2)}}\left(\frac{pr}{\hbar}\right)^{1/2} J_{\ell+1/2}\left(\frac{pr}{\hbar}\right) \ , \tag{28}$$

$$g_-(r;p) = \frac{-pc}{\sqrt{2\hbar\omega(p)(\omega(p)-mc^2)}}\left(\frac{pr}{\hbar}\right)^{1/2} J_{\ell+1/2\mp1}\left(\frac{pr}{\hbar}\right) \ . \tag{29}$$

Here, $J_\mu(\cdot)$ are Bessel functions of order μ.[15] Finally, let us note that one may readily check that the configurations of opposite energy have a vanishing inner product, given the negative sign in the function $g_-(r;p)$ for the negative energy solutions relative to that of the positive energy ones. Hence, we have constructed a complete basis of eigenstates for the free radial Dirac Hamiltonian, in terms of which it is thus possible to construct the explicit expressions for the associated resolvent.

4. The Resolvent of the Free Dirac Radial Hamiltonian

In an abstract and formal manner, let us consider a self-adjoint operator A with spectrum[a]

$$A|\psi_n >= \lambda_n|\psi_n > , \tag{30}$$

where the eigenstates are orthonormalised,

$$< \psi_n|\psi_m >= \delta_{n,m} \ . \tag{31}$$

Consequently, given a complex parameter z, the associated resolvent $[A-z]^{-1}$ is simply given by the representation

$$[A-z]^{-1} = \sum_n |\psi_n > \frac{1}{\lambda_n - z} < \psi_n| \ , \tag{32}$$

where z does not belong to the A-spectrum. We shall simply follow exactly the same construction for the (r, r') kernel of the free Dirac radial Hamiltonian, since we have constructed its spectrum in the previous section. In our case, this resolvent is also a 2×2 matrix, each of which elements are obtained in a likewise manner. Hence, we have for the resolvent kernel of the free Dirac radial Hamiltonian

$$G_0(r, r'; z) = \begin{pmatrix} G_{11}(r, r'; z) & G_{12}(r, r'; z) \\ G_{21}(r, r'; z) & G_{22}(r, r'; z) \end{pmatrix} \ . \tag{33}$$

Introducing the quantity $k = k(z)$ defined by

$$k(z)c = \sqrt{z^2 - (mc^2)^2} \ , \tag{34}$$

with the usual branch cut along the negative real axis for the square-root function, we have for the G_{11} element

$$G_{11}(r, r'; z) = \frac{i\pi}{2}\frac{z + mc^2}{(\hbar c)^2}\sqrt{rr'}H^{(1)}_{\ell+1/2}\left(\frac{k(z)c}{\hbar c}r\right)\ J_{\ell+1/2}\left(\frac{k(z)c}{\hbar c}r'\right)\ , \tag{35}$$

$$G_{11}(r, r'; z) = \frac{i\pi}{2}\frac{z + mc^2}{(\hbar c)^2}\sqrt{rr'}J_{\ell+1/2}\left(\frac{k(z)c}{\hbar c}r\right)\ H^{(1)}_{\ell+1/2}\left(\frac{k(z)c}{\hbar c}r'\right)\ , \tag{36}$$

Eqs. (35) and (36) being valid for $r > r'$ and $r < r'$, respectively. Note that as a consequence, the quantity $G_{11}(r, r'; z)$ is also well defined for $r = r'$, as it should.

[a] We assume here that the spectrum is not degenerate, since this is the situation encountered presently. The case of a degenerate spectrum is a straightforward generalisation.

The second diagonal element is given by

$$G_{22}(r,r';z) = \frac{i\pi}{2}\frac{z-mc^2}{(\hbar c)^2}\sqrt{rr'}H^{(1)}_{\ell+1/2\mp1}\left(\frac{k(z)c}{\hbar c}r\right)\ J_{\ell+1/2\mp1}\left(\frac{k(z)c}{\hbar c}r'\right)\ , \tag{37}$$

$$G_{22}(r,r';z) = \frac{i\pi}{2}\frac{z-mc^2}{(\hbar c)^2}\sqrt{rr'}J_{\ell+1/2\mp1}\left(\frac{k(z)c}{\hbar c}r\right)\ H^{(1)}_{\ell+1/2\mp1}\left(\frac{k(z)c}{\hbar c}r'\right)\ , \tag{38}$$

Eqs. (37) and (38) being valid for $r > r'$ and $r < r'$, respectively.

For the off-diagonal element $G_{12}(r,r';z)$, we obtain the following two expressions valid for $r > r'$ and $r < r'$, respectively,

$$G_{12}(r,r';z) = \frac{i\pi}{2}\frac{k(z)c}{(\hbar c)^2}\sqrt{rr'}H^{(1)}_{\ell+1/2}\left(\frac{k(z)c}{\hbar c}r\right)\ J_{\ell+1/2\mp1}\left(\frac{k(z)c}{\hbar c}r'\right)\ , \tag{39}$$

$$G_{12}(r,r';z) = \frac{i\pi}{2}\frac{k(z)c}{(\hbar c)^2}\sqrt{rr'}J_{\ell+1/2}\left(\frac{k(z)c}{\hbar c}r\right)\ H^{(1)}_{\ell+1/2\mp1}\left(\frac{k(z)c}{\hbar c}r'\right)\ . \tag{40}$$

Finally for $G_{21}(r,r';z)$ element, we obtain

$$G_{21}(r,r';z) = \frac{i\pi}{2}\frac{k(z)c}{(\hbar c)^2}\sqrt{rr'}H^{(1)}_{\ell+1/2\mp1}\left(\frac{k(z)c}{\hbar c}r\right)\ J_{\ell+1/2}\left(\frac{k(z)c}{\hbar c}r'\right)\ , \tag{41}$$

$$G_{21}(r,r';z) = \frac{i\pi}{2}\frac{k(z)c}{(\hbar c)^2}\sqrt{rr'}J_{\ell+1/2\mp1}\left(\frac{k(z)c}{\hbar c}r\right)\ H^{(1)}_{\ell+1/2}\left(\frac{k(z)c}{\hbar c}r'\right)\ , \tag{42}$$

Eqs. (41) and (42) being valid for $r > r'$ and $r < r'$, respectively.

5. The Deficient Index Subspaces

In order to use Krein's formula to express the Dirac radial resolvent for a given choice of boundary conditions which define a self-adjoint extension of the Dirac Hamiltonian related to a δ-sphere scalar potential, it is first necessary to establish a basis for the deficiency subspaces. This amounts to identifying the eigensolutions to the previous eigenvalue problem in which the energy eigenvalue E is now replaced by a complex parameter z which does not belong to the spectrum of the free and interacting Hamiltonians. The only restriction is that the obtained solutions must be normalisable for the norm $\int_0^\infty dr$.

Consequently, for a given complex value of z, let us introduce again the quantity

$$k(z)c = \sqrt{z^2-(mc^2)^2}\ , \tag{43}$$

where the square-root function is defined as previously. It is then straightforward to show that there exist two unique linearly independent normalisable solutions, for $\mathrm{Im}\,k(z)>0$. The first solution is given by,

$$\begin{array}{ll} \text{. if } r<R: & f_1(r;z)=\frac{k(z)c}{\sqrt{2\hbar z(z-mc^2)}}\left(\frac{k(z)c}{\hbar c}r\right)^{1/2}J_{\ell+1/2}\left(\frac{k(z)c}{\hbar c}r\right)\ , \\ & g_1(r;z)=\frac{k(z)c}{\sqrt{2\hbar z(z+mc^2)}}\left(\frac{k(z)c}{\hbar c}r\right)^{1/2}J_{\ell+1/2\mp 1}\left(\frac{k(z)c}{\hbar c}r\right)\ ; \\ \text{. if } r>R: & f_1(r;z)=0\ , \\ & g_1(r;z)=0\ . \end{array} \tag{44}$$

The second solution is given by

$$\begin{array}{ll} \text{. if } r<R: & f_2(r;z)=0\ , \\ & g_2(r;z)=0\ ; \\ \text{. if } r>R: & f_2(r;z)=\frac{k(z)c}{\sqrt{2\hbar z(z-mc^2)}}\left(\frac{k(z)c}{\hbar c}r\right)^{1/2}H^{(1)}_{\ell+1/2}\left(\frac{k(z)c}{\hbar c}r\right)\ , \\ & g_2(r;z)=\frac{k(z)c}{\sqrt{2\hbar z(z+mc^2)}}\left(\frac{k(z)c}{\hbar c}r\right)^{1/2}H^{(1)}_{\ell+1/2\mp 1}\left(\frac{k(z)c}{\hbar c}r\right)\ . \end{array} \tag{45}$$

If $\mathrm{Im}\,k(z)<0$, the corresponding deficiency subspace is spanned by the complex conjugates of the above expressions, except for the fact that $k(z)$ is then still given by the above definition in terms of z rather than in terms of $z^*=\bar{z}$. In other words, for $\mathrm{Im}\,k(z)<0$, the deficiency subspace is spanned by the same solutions $(f_1(r;z)\ g_1(r;z))$ as above and by functions $(f'_2(r;z)\ g'_2(r;z))$ which are given by the same expressions as those for $(f_2(r;z)\ g_2(r;z))$ with the function $H^{(1)}_{\ell+1/2\mp 1}$ replaced by $H^{(2)}_{\ell+1/2\mp 1}$.

Since Krein's formula reads[13]

$$G(r,r';z)=G_0(r,r';z)+\sum_{i,j=1}^{2}\mu_{ij}(z)\begin{pmatrix} f_i(r;z) \\ g_i(r;z)\end{pmatrix}\begin{pmatrix} f^*_j(r';\bar{z}) & g^*_j(r';\bar{z})\end{pmatrix}\ , \tag{46}$$

where $G(r,r';z)$ stands for the resolvent kernel of the interacting Hamiltonian, it is clear that the specific values for the coefficients $\mu_{ij}(z)$ are defined up to a normalisation which is correlated to the chosen normalisation for the solutions $(f_i\ g_i)$ constructed above. As mentioned already, the latter normalisation is simply chosen by analogy with that of the free energy eigenstates, and it might thus turn out that there could be a choice of normalisation rendering the expressions for the coefficients $\mu_{ij}(z)$ simpler. This is indeed the case, but in a manner which is dependent on the specific choice of boundary conditions which is associated to a specific self-adjoint extension of the Dirac Hamiltonian. It is for this reason that we have kept the above normalisation, leaving it as an exercise to change the normalisation to make it simpler once a specific choice of self-adjoint boundary conditions is considered. This is rather straightforward on the basis of the results to be presented in the next section.

Note also that since the deficiency indices of the Hamiltonian are $(2, 2)$, the set of its self-adjoint extensions is characterized by four independent real parameters, which are thus in correspondence with the four parameters $\mu_{ij}(z)$. In other words, any choice of boundary conditions for the functions $f(r)$ and $g(r)$ obeying the interacting Hamiltonian energy eigenvalue problem which may be put into unique correspondence with the parameters $\mu_{ij}(z)$ in Krein's formula determines a self-adjoint extension of the Dirac Hamiltonian. The next section displays two such choices, each characterized by a single real parameter, leaving it aside how to construct the general case and how to identify its relationship to δ-sphere scalar and vector interactions.

6. The Resolvent Equation

6.1. *The α-Type Boundary Condition of the First Type*

Requiring $f(r)$ to be continuous at $r = R$ implies that we must have

$$\mu_{11}(z) = \mu_1(z) H^{(1)}_{\ell+1/2}(\frac{k(z)R}{\hbar}) \quad , \quad \mu_{21}(z) = \mu_1(z) J_{\ell+1/2}(\frac{k(z)R}{\hbar}) \ , \qquad (47)$$

$$\mu_{12}(z) = \mu_2(z) H^{(1)}_{\ell+1/2}(\frac{k(z)R}{\hbar}) \quad , \quad \mu_{22}(z) = \mu_2(z) J_{\ell+1/2}(\frac{k(z)R}{\hbar}) \ , \qquad (48)$$

where $\mu_1(z)$ and $\mu_2(z)$ are real parameters still to be determined. After some algebra, one finds that the boundary condition involving the difference $g(R_+) - g(R_-)$ implies the following relations for the quantities $\mu_1(z)$ and $\mu_2(z)$ in terms of α, for $\ell = j \pm 1/2$,

$$\begin{aligned} \mu_1(z) = \quad & \frac{i\pi}{2} \frac{R(z+mc^2)}{\hbar c} \frac{i\pi}{2} \frac{2\hbar z}{\hbar c} \frac{1}{k(z)c} \times \\ & \times \frac{\alpha H^{(1)}_{\ell+1/2}(\frac{k(z)R}{\hbar})}{\mp 1 - \alpha \frac{i\pi}{2} \frac{R(z+mc^2)}{\hbar c} H^{(1)}_{\ell+1/2}(\frac{k(z)R}{\hbar}) J_{\ell+1/2}(\frac{k(z)R}{\hbar})} \ , \qquad (49) \end{aligned}$$

$$\mu_2(z) = \frac{i\pi}{2}\frac{R(z+mc^2)}{\hbar c}\frac{i\pi}{2}\frac{2\hbar z}{\hbar c}\frac{1}{k(z)c} \times \\ \times \frac{\alpha J_{\ell+1/2}(\frac{k(z)R}{\hbar})}{\mp 1 - \alpha\frac{i\pi}{2}\frac{R(z+mc^2)}{\hbar c}H^{(1)}_{\ell+1/2}(\frac{k(z)R}{\hbar})J_{\ell+1/2}(\frac{k(z)R}{\hbar})} . \qquad (50)$$

Note that the quantity that multiplies α in the denominator of these two expressions is simply the value for $G_{11}(R, R; z)$ for the free Dirac resolvent, in exactly the same manner as for the nonrelativistic Schrödinger problem. Furthermore, it should be clear that most of the overall normalisation factors multiplying the last fraction could be absorbed directly through a renormalisation of the functions $f_i(r; z)$ and $g_i(r; z)$ spanning the deficiency subspace. We shall leave this point open, since this is rather trivial, even though it simplifies somewhat the expression for the coefficients $\mu_1(z)$ and $\mu_2(z)$. Finally, notice that the final expressions for all four coefficients $\mu_{ij}(z)$ generalise similar expressions relevant to the nonrelativistic Schrödinger problem. In the latter case, all the factors involving the Bessel functions evaluated at $k(z)R/\hbar$ may be absorbed into the normalisation of the vectors spanning the deficiency subspace, leaving over a rather simple expression for the coefficient $\mu(z)$ in that case. For the Dirac Hamiltonian, this is no longer the case in a simple way, but still the general structure is maintained for the solutions of the coefficients $\mu_{ij}(z)$.

Finally the resolvent of the extended Dirac Hamiltonian for the α-type boundary conditions is given by

$$(h_{l,\alpha_l} - z)^{-1} = (h_{l,0} - z)^{-1} + A(z)\mathbb{1}_2\psi(r; z)\left(\psi^*(r; \overline{z}), \cdot\right) , \qquad (51)$$

with the following notations,

$$A(z) = \frac{i\pi}{2}\frac{R(z+mc^2)}{\hbar c}\frac{i\pi}{2}\frac{2\hbar z}{\hbar c}\frac{1}{k(z)c} \times \\ \times \frac{\alpha}{\mp 1 - \alpha\frac{i\pi}{2}\frac{R(z+mc^2)}{\hbar c}H^{(1)}_{\ell+1/2}(\frac{k(z)R}{\hbar})J_{\ell+1/2}(\frac{k(z)R}{\hbar})} , \qquad (52)$$

$$\psi(r; z) = \begin{pmatrix} \psi_1(r; z) \\ \psi_2(r; z) \end{pmatrix} , \qquad (53)$$

$$\psi_1(r; z) = a(z) \times \left(\frac{k(z)}{\hbar}r\right)^{1/2}\left[J_{\ell+1/2}(\frac{k(z)r}{\hbar})H^{(1)}_{\ell+1/2}(\frac{k(z)R}{\hbar}) + \right. \\ \left. + J_{\ell+1/2}(\frac{k(z)R}{\hbar})H^{(1)}_{\ell+1/2}(\frac{k(z)r}{\hbar})\right] , \qquad (54)$$

$$\psi_2(r;z) = b(z) \times \left(\frac{k(z)}{\hbar}r\right)^{1/2} \Big[J_{\ell+1/2\mp 1}(\frac{k(z)r}{\hbar}) H^{(1)}_{\ell+1/2}(\frac{k(z)R}{\hbar}) + \\ + J_{\ell+1/2}(\frac{k(z)R}{\hbar}) H^{(1)}_{\ell+1/2\mp 1}(\frac{k(z)r}{\hbar}) \Big] \ , \quad (55)$$

$$a(z) = \frac{k(z)c}{\sqrt{2\hbar z(z-mc^2)}} \quad , \quad b(z) = \frac{k(z)c}{\sqrt{2\hbar z(z+mc^2)}} \ . \quad (56)$$

6.2. *The β-Type Boundary Condition of the First Type*

A similar analysis is possible when the continuity requirement is imposed for $g(r)$,

$$g(R_+) = g(R_-) = g(R) \ . \quad (57)$$

It then turns out that the β-type boundary conditions of the first type, Eq. (17), are then possible, and provide thus a one-parameter self-adjoint extension of the Dirac Hamiltonian. In this case, we must have

$$\mu_{11}(z) = \mu_1(z) H^{(1)}_{\ell+1/2\mp 1}(\frac{k(z)R}{\hbar}) \ , \ \mu_{21}(z) = \mu_1(z) J_{\ell+1/2\mp 1}(\frac{k(z)R}{\hbar}) \ , \quad (58)$$

$$\mu_{12}(z) = \mu_2(z) H^{(1)}_{\ell+1/2\mp 1}(\frac{k(z)R}{\hbar}) \ , \ \mu_{22}(z) = \mu_2(z) J_{\ell+1/2\mp 1}(\frac{k(z)R}{\hbar}) \ , \quad (59)$$

while the coefficients $\mu_1(z)$ and $\mu_2(z)$ are given as follows in terms of the real parameter β defining the boundary conditions Eq. (17), for $\ell = j \pm 1/2$,

$$\mu_1(z) = \frac{i\pi}{2}\frac{R(z-mc^2)}{\hbar c}\frac{i\pi}{2}\frac{2\hbar z}{\hbar c}\frac{1}{k(z)c} \times \\ \times \frac{\beta H^{(1)}_{\ell+1/2\mp 1}(\frac{k(z)R}{\hbar})}{\pm 1 - \beta \frac{i\pi}{2}\frac{R(z-mc^2)}{\hbar c} H^{(1)}_{\ell+1/2\mp 1}(\frac{k(z)R}{\hbar}) J_{\ell+1/2\mp 1}(\frac{k(z)R}{\hbar})} \ , \quad (60)$$

$$\mu_2(z) = \frac{i\pi}{2}\frac{R(z-mc^2)}{\hbar c}\frac{i\pi}{2}\frac{2\hbar z}{\hbar c}\frac{1}{k(z)c} \times \\ \times \frac{\beta J_{\ell+1/2\mp 1}(\frac{k(z)R}{\hbar})}{\pm 1 - \beta \frac{i\pi}{2}\frac{R(z-mc^2)}{\hbar c} H^{(1)}_{\ell+1/2\mp 1}(\frac{k(z)R}{\hbar}) J_{\ell+1/2\mp 1}(\frac{k(z)R}{\hbar})} \ . \quad (61)$$

The same remarks as those made for the α-type boundary conditions are of application in this case, of course, with in particular the value for $G_{22}(R,R;z)$ explicitly multiplying the parameter β in the denominator of the last fraction factor. Finally, the resolvent in a compact form, for the β-type boundary conditions, is given by

$$(h_{l,\beta_l} - z)^{-1} = (h_{l,0} - z)^{-1} + \widetilde{A}(z)\mathbb{1}_2\widetilde{\psi}(r;z)\left(\widetilde{\psi}^*(r;\overline{z}),\cdot\right) , \qquad (62)$$

with the notations

$$\widetilde{\psi}(r;z) = \begin{pmatrix} \widetilde{\psi}_1(r;z) \\ \widetilde{\psi}_2(r;z) \end{pmatrix} , \qquad (63)$$

$$\begin{aligned}\widetilde{\psi}_1(r;z) = a(z) \times \left(\frac{k(z)}{\hbar}r\right)^{1/2} \Big[& J_{\ell+1/2}(\frac{k(z)r}{\hbar})H^{(1)}_{\ell+1/2\mp1}(\frac{k(z)R}{\hbar}) + \\ & + J_{\ell+1/2\mp1}(\frac{k(z)R}{\hbar})H^{(1)}_{\ell+1/2}(\frac{k(z)r}{\hbar})\Big] ,\end{aligned} \qquad (64)$$

$$\begin{aligned}\widetilde{\psi}_2(r;z) = b(z) \times \left(\frac{k(z)}{\hbar}r\right)^{1/2} \Big[& J_{\ell+1/2\mp1}(\frac{k(z)r}{\hbar})H^{(1)}_{\ell+1/2\mp1}(\frac{k(z)R}{\hbar}) + \\ & + J_{\ell+1/2\mp1}(\frac{k(z)R}{\hbar})H^{(1)}_{\ell+1/2\mp1}(\frac{k(z)r}{\hbar})\Big] ,\end{aligned} \qquad (65)$$

$$\begin{aligned}\widetilde{A}(z) = \; & \frac{i\pi}{2}\frac{R(z-mc^2)}{\hbar c}\frac{i\pi}{2}\frac{2\hbar z}{\hbar c}\frac{1}{k(z)c} \times \\ & \times \frac{\beta}{\pm 1 - \beta\frac{i\pi}{2}\frac{R(z-mc^2)}{\hbar c}H^{(1)}_{\ell+1/2\mp1}(\frac{k(z)R}{\hbar})J_{\ell+1/2\mp1}(\frac{k(z)R}{\hbar})} .\end{aligned} \qquad (66)$$

7. Conclusion

We have thus constructed the resolvent kernel $G(r, r'; z)$ for the one-parameter self-adjoint extensions of the Dirac Hamiltonian associated to the δ-sphere scalar potential and characterized by the α- and β-type boundary conditions of the first type defined in Eqs. (16) and (17). More general cases could be considered on the basis of the general expressions obtained above for the values of $f(R_\pm)$ and $g(R_\pm)$, in order to identify all possible self-adjoint extensions parametrized by four independent real parameters.

Associated to the boundary conditions Eqs. (16) and (17), these results should now enable the analysis of the scattering, spectral and resonance properties of the constructed self-adjoint extensions of the Dirac operator.[14]

Finally, let us say that it is also possible to construct a subclass of self-adjoint extensions characterized by two parameters. Shabani and Vyabandi attempted such a construction in their work[11] but it is not clear to us how the resolvent kernel of the free Dirac operator which they obtained and on which the remainder of their analysis rests in a crucial way, compares to its explicit contruction as detailed in the present contribution.

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References

1. I.M. Green and S.A. Moszkowski, *Phys. Rev.* **139**, B790 (1965).
2. J.-P. Antoine, F. Gesztesy and J. Shabani, *J. Phys. A* **20**, 3687 (1987).
3. M.N. Hounkonnou, M. Hounkpe and J. Shabani, *J. Math. Phys.* **38**, 2832 (1997).
4. J. Dittrich, P. Exner and P. Šeba, *J. Math. Phys.* **30**, 2875 (1989).
5. J. Dittrich, P. Exner and P. Šeba, *J. Math. Phys.* **33**, 2207 (1992).
6. G.Y.H. Avossevou and M.N. Hounkonnou, *J. Math. Phys.* **41**, 24 (2000).
7. M.N. Hounkonnou and G.Y.H. Avossevou, *J. Math. Phys.* **41**, 1718 (2000).
8. M.N. Hounkonnou and G.Y.H. Avossevou, *J. Math. Phys.* **41**, 1735 (2000).
9. G.Y.H. Avossevou and M.N. Hounkonnou, *Analytically solvable models of relativistic δ'-sphere interactions with first type boundary conditions*, in the Proceedings of the 1^{st} International Workshop on Contemporary Problems in Mathematical Physics, J. Govaerts, M.N. Hounkonnou and W. A. Lester, Jr., eds. (World Scientific, Singapore, 2000), pp. 347–365.
10. M.N. Hounkonnou and G.Y.H. Avossevou, *J. Math. Phys.* **42**, 30 (2001).
11. J. Shabani and A. Vyabandi, *J. Math. Phys.* **43**, 6064 (2002); *ibid.*, 6380 (2002).
12. S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, Texts and Monographs in Physics (Springer Verlag, Berlin, 1988).
13. N.I. Akhiezer and I.M. Glazman, *Theory of Linear Operators in Hilbert Space*, Vol. 2 (Pitman, Boston, 1981).
14. G.Y.H. Avossevou, Ph.D. Thesis (Institut de Mathématiques et de Sciences Physiques, Republic of Benin, April 2002), unpublished.
15. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).