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Progressivity, Inequality Reduction and Merging-Proofness in Taxation*

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Abstract

Progressivity, *inequality reduction* and *merging-proofness* are three well-known axioms in taxation. We investigate implications of each of the three axioms through characterizations of several families of taxation rules and their logical relations. We also study the preservation of these axioms under two operators on taxation rules, the so-called convexity operator and minimal-burden operator, which give intuitive procedures for determining tax schedules.

Keywords: taxation, progressivity, inequality reduction, merging-proofness, convexity operator, minimal-burden operator.

JEL Classification: C70, D63, D70, H20

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1 Introduction

In modern welfare states, income tax is a major source of state funds and is an essential policy measure for the enhancement of distributive justice. In the framework introduced by O'Neill (1982) and Young (1988), we study two principles of distributive justice, known as *progressivity* (tax rates are in the order of individual incomes) and *inequality reduction* (taxation reduces income inequality).[1] We investigate how the two principles are related to each other and to another principle that prevents any gain from strategic merging among taxpayers. This third principle, called *merging-proofness*, is studied among others by de Frutos (1999), Ju (2003) and Ju et al. (2005) in this same context, and by Moulin (2004) and Sprumont (2005) in similar contexts. We also study the robustness of the three principles under the application of two operators, known as the convexity operator and the minimal-burden operator (to be explained later).

Merging-proofness and its motivation seem to have no bearing on the two principles of distributive justice. However, we find that they are in fact related. Based on two characterization results imposing *merging-proofness* or *progressivity*, as well as some standard axioms in the literature, we show that (essentially) any *progressive* taxation rule is *merging-proof*. This gives an extra advantage of imposing *progressivity*.

In a companion paper (Ju and Moreno-Ternero, 2006), we also establish a close connection between *progressivity* and *inequality reduction* that has long been perceived by a number of authors in the literature of tax functions (which are functions from the set of real numbers to itself).[2] A formal proof in such a framework, however, is provided rather recently by Eichhorn et al. (1984).

A recent study by Thomson and Yeh (2006) gives a novel classification of rules and axioms based on *operators* that map a rule into another. We consider two types of operators that capture intuitive proposals of determining tax schedules. When two rules compete, a natural compromise is mixing the two rules by taking a convex combination of them, which is what a *convexity operator* (Thomson and Yeh, 2006) does. In this way, we are able to mix two different ideas of taxation embedded in two rules. The *minimal-burden operator* (Thomson and Yeh, 2006) gives us an intuitive procedure of identifying tax schedules. If the aggregate income of all agents except,

[1] We refer readers to Young (1994), Moulin (2002) and Thomson (2003, 2005) for extensive treatments of diverse problems (such as taxation, bankruptcy, cost sharing or surplus sharing) fitting this framework.

[2] See Le Breton et al. (1996) or Mitra and Ok (1997) for instances of this literature.

say, agent i is lower than the tax revenue to be collected, this difference can be interpreted as the minimal tax burden imposed on agent i (he is the only person who can contribute for this portion because the maximum aggregate tax payment by the remaining agents cannot cover it). Thus, the following two-step procedure, as suggested by the minimal-burden operator, seems interesting. First, let each agent pay his minimal burden. Second, the remainder of the tax revenue is collected by considering the remaining income profile.

The application of an operator may be problematic if it fails to preserve some appealing axioms; in particular, our three main axioms, *progressivity*, *inequality reduction* and *merging-proofness*.[3] We show that the two types of operators preserve *progressivity* and *inequality reduction*. Regarding *merging-proofness*, the minimal-burden operator is slightly disruptive as it requires an additional, but mild, axiom to preserve it.

The rest of the paper is organized as follows. In Section 2, we present the model and basic concepts. In Section 3, we define axioms. In Section 4, we state and prove the characterization results. In Section 5, we state and prove our results on operators. We conclude in Section 6 with some further insights. For a smooth passage, we defer some proofs and provide them in the appendix.

2 Model and basic concepts

We study taxation problems in a variable population model. The set of potential taxpayers, or *agents*, is identified by the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, with generic element N. For each $i \in N$, let $y_i \in \mathbb{R}_+$ be i's (taxable) *income* and $y \equiv (y_i)_{i\in N}$ the income profile. A (taxation) *problem* is a triple consisting of a population $N \in \mathcal{N}$, an income profile $y \in \mathbb{R}^N_+$, and a tax revenue $T \in \mathbb{R}_+$ such that $\sum_{i\in N} y_i \geq T$. Let $Y \equiv \sum_{i\in N} y_i$. To avoid unnecessary complication, we assume $Y = \sum_{i\in N} y_i > 0$. Let $\mathcal{D}^N$ be the set of taxation problems with population N and $\mathcal{D} \equiv \cup_{N\in\mathcal{N}}\mathcal{D}^N$.

Given a problem $(N, y, T) \in \mathcal{D}$, a *tax profile* is a vector $x \in \mathbb{R}^N$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq y_i$ and (ii) $\sum_{i\in N} x_i = T$. We refer to (i) as *boundedness* and (ii) as *balancedness*.[4] A (taxation) *rule* on $\mathcal{D}$, $R\colon \mathcal{D} \to \cup_{N\in\mathcal{N}}\mathbb{R}^N$, associates with each problem

[3] Preservation of an axiom means that if a rule satisfies an axiom so does the rule obtained by applying the operator.

[4] Note that *boundedness* implies that each agent with zero income pays zero tax.

$(N, y, T) \in \mathcal{D}$ a tax profile $R(N, y, T)$ for the problem. Each rule R gives the associated *post-tax income function* $S^R(\cdot)$ defined as follows: for each $(N, y, T) \in \mathcal{D}$, $S^R(N, y, T) \equiv y - R(N, y, T)$. Throughout the paper, for each $N \in \mathcal{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^N$, let $z_M \equiv (z_i)_{i \in M}$.

We now provide some examples of rules. We start with three well-known rules. The *head tax* distributes the tax burden equally, provided no agent ends up paying more than her income. The *leveling tax* equalizes post-tax income across agents, provided no agent is subsidized. The *flat tax* equalizes tax rates across agents. These three rules are examples of rules in the following family introduced by Young (1987).

Definition 1 (Parametric Rules) *A rule R is a* parametric *rule if there is a function $f : [a, b] \times \mathbb{R}_+ \to \mathbb{R}$, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$, such that (i) f is continuous and non-decreasing in the first variable; (ii) for each $x \in \mathbb{R}_+$, $f(a, x) = 0$ and $f(b, x) = x$; (iii) for each $(N, y, T) \in \mathcal{D}$ and each $i \in N$, $R_i(N, y, T) = f(\lambda, y_i)$, where $\lambda \in [a, b]$ satisfies $\sum_{i \in N} f(\lambda, y_i) = T$.[5] We call f a parametric representation of R.*

The three rules mentioned earlier have the following parametric representations:

- Head tax: $f^H(\lambda, y) = \min\{-\frac{1}{\lambda}, y\}$, for each $\lambda \in \mathbb{R}_-$ and each $y \in \mathbb{R}_+$.
- Leveling tax: $f^L(\lambda, y) = \max\{y - \frac{1}{\lambda}, 0\}$, for each $\lambda \in \mathbb{R}_+$ and each $y \in \mathbb{R}_+$.
- Flat tax: $f^F(\lambda, y) = \lambda \cdot y$, for each $\lambda \in [0, 1]$ and each $y \in \mathbb{R}_+$.

3 Axioms

We now define our three main axioms of taxation.

Progressivity postulates that, for any pair of agents, the one with higher income should face a tax rate at least as high as the rate faced by the other.

Axiom 1 ***Progressivity.*** *For each $(N, y, T) \in \mathcal{D}$ and each $i, j \in N$, if $0 < y_i \leq y_j$,*

$$\frac{R_i(N, y, T)}{y_i} \leq \frac{R_j(N, y, T)}{y_j}.$$

[5] Existence of such λ is guaranteed by the first two conditions (i) and (ii).

Our second axiom requires that the post-tax income profile should have at least as low "income inequality" as the original income profile. This axiom is based on the following basic inequality relation over income profiles. For each population $N \equiv \{1, \ldots, n\}$ and each pair of income profiles $y, y' \in \mathbb{R}_+^N$, *y Lorenz dominates y'* if, for each $k = 1, \ldots, n-1$, the proportion of the sum of the k lowest incomes to the total income at y is greater than or equal to the same proportion at y': that is, when $y_1 \leq y_2 \leq ... \leq y_n$ and $y'_1 \leq y'_2 \leq ... \leq y'_n$, for each $k = 1, ..., n-1$,

$$\frac{\sum_{i=1}^k y_i}{\sum_{i=1}^n y_i} \geq \frac{\sum_{i=1}^k y'_i}{\sum_{i=1}^n y'_i}.$$

Axiom 2 ***Inequality reduction.*** *For each $(N, y, T) \in \mathcal{D}$, the post-tax income profile $S^R(N, y, T)$ Lorenz dominates y.*

Our third axiom prevents a rule from being manipulated by a pair of agents through merging their incomes.

Axiom 3 ***Merging-proofness.*** *For each $(N, y, T) \in \mathcal{D}$ and each pair $i, j \in N$ with $i \neq j$, if $y' \in \mathbb{R}_+^{N \backslash \{j\}}$ is such that $y'_i = y_i + y_j$ and $y'_{N \backslash \{i\}} = y_{N \backslash \{i,j\}}$,*

$$R_i(N, y, T) + R_j(N, y, T) \leq R_i(N \backslash \{j\}, y', T).$$

We will investigate logical relations between the three axioms, invoking in the process some of the following standard axioms.[6]

The next axiom requires that a rule should give the same tax profile when it is applied for any subset of agents as when it is applied for the whole population.

Axiom 4 ***Consistency.*** *For each $(N, y, T) \in \mathcal{D}$, each $M \subset N$, and each $i \in M$,*

$$R_i(M, y_M, \sum_{i \in M} x_i) = x_i,$$

where $(x_i)_{i \in N} \equiv R(N, y, T)$ and $y_M \equiv (y_i)_{i \in M}$.

The next two axioms require that tax contributions and post-tax incomes be in the order of pre-tax income.

Axiom 5 ***Tax order preservation.*** *For each $(N, y, T) \in \mathcal{D}$ and each pair $i, j \in N$, if $y_i \geq y_j$, $R_i(N, y, T) \geq R_j(N, y, T)$.*

[6] We refer readers to Thomson (2003, 2005) for a detailed discussion on these axioms.

Axiom 6 ***Income order preservation.*** *For each $(N, y, T) \in \mathcal{D}$ and each pair $i, j \in N$, if $y_i \geq y_j$, $y_i - R_i(N, y, T) \geq y_j - R_j(N, y, T)$.*

Note that *progressivity* implies *tax order preservation.*

Finally, the next axiom says that small changes in incomes or revenue do not produce a jump in tax schedules.

Axiom 7 ***Continuity.*** *For each $N \in \mathcal{N}$, each sequence $\{(N, y^n, T^n) : n \in \mathbb{N}\}$ in $\mathcal{D}^N$, and each $(N, y, T) \in \mathcal{D}^N$, if (y^n, T^n) converges to (y, T), then $R(N, y^n, T^n)$ converges to $R(N, y, T)$.*

4 Characterizations and logical relations among axioms

4.1 Progressivity and merging-proofness

Lemma 1 gives a necessary and sufficient condition for a parametric rule to satisfy *progressivity.* A parametric representation $f\colon [a, b] \times \mathbb{R}_+ \to \mathbb{R}$ is *superhomogeneous in income* if for each $\lambda \in [a, b]$, each $y_0 \in \mathbb{R}_+$ and each $\alpha \geq 1$, $f(\lambda, \alpha y_0) \geq \alpha f(\lambda, y_0)$.

Lemma 1 *A parametric rule satisfies progressivity if and only if it has a parametric representation that is superhomogeneous in income.*

Proof. Let R be a parametric rule and $f\colon [a, b] \times \mathbb{R}_+ \to \mathbb{R}$ a parametric representation of R. Assume that R is *progressive.* Let $\lambda \in [a, b]$, $y_0 > 0$ and $\alpha \geq 1$. Let $T^\lambda \equiv f(\lambda, y_0) + f(\lambda, \alpha y_0)$ and $N \equiv \{1, 2\}$. Then, $R(N, (y_0, \alpha y_0), T^\lambda) = (f(\lambda, y_0), f(\lambda, \alpha y_0))$. By *progressivity,* $f(\lambda, y_0)/y_0 \leq f(\lambda, \alpha y_0)/(\alpha y_0)$. Thus $\alpha f(\lambda, y_0) \leq f(\lambda, \alpha y_0)$, which shows that f is superhomogeneous in income.

Conversely, assume that f is superhomogeneous in income. Let $(N, y, T) \in \mathcal{D}$ and $i, j \in N$ be such that $0 < y_i \leq y_j$. Let $\lambda \in [a, b]$ be such that $R(N, y, T) = (f(\lambda, y_i))_{i \in N}$. Then, by superhomogeneity, $f(\lambda, y_j) = f(\lambda, \frac{y_j}{y_i} \cdot y_i) \geq \frac{y_j}{y_i} \cdot f(\lambda, y_i)$.Thus

$$\frac{R_j(N, y, T)}{y_j} = \frac{f(\lambda, y_j)}{y_j} \geq \frac{f(\lambda, y_i)}{y_i} = \frac{R_i(N, y, T)}{y_i},$$

which shows the *progressivity* of R. ■

It is evident that *progressivity* implies the following axiom, which says that any two agents with the same income should pay the same tax.

Axiom 8 ***Equal treatment of equals.*** *For each* $(N, y, T) \in \mathcal{D}$ *and each pair* $i, j \in N$ *with* $y_i = y_j$, $R_i(N, y, T) = R_j(N, y, T)$.

Young (1987, Theorem 1) shows that the parametric rules are the only rules satisfying *consistency*, *equal treatment of equals*, and *continuity*. Therefore, using his result and Lemma 1 we obtain:

Proposition 2 *A rule satisfies progressivity, consistency, and continuity if and only if it has a parametric representation that is superhomogeneous in income.*[7]

Remark 1 *Marshall and Olkin (1979, p.453) and Bruckner and Ostrow (1962, Lemma 3) offer similar results for tax functions* $\xi\colon \mathbb{R}_+ \to \mathbb{R}$.[8] *The main difference between their model and ours is that our rules are* multivariate vector-valued *functions with the two constraints of* boundedness *or* balancedness. *Despite the differences, Proposition 2 shows that, thanks to Young's (1987) characterization of parametric rules, the earlier results can be extended to our model without much difficulty.*

Lemma 3 gives a necessary and sufficient condition for a parametric rule to satisfy *merging-proofness*. A parametric representation $f\colon [a, b] \times \mathbb{R}_+ \to \mathbb{R}$ is *superadditive in income* if for each $\lambda \in [a, b]$ and each pair $y_0, y_0' \in \mathbb{R}_+$, $f(\lambda, y_0 + y_0') \geq f(\lambda, y_0) + f(\lambda, y_0')$.[9] Ju (2003, Proposition 1) offers the following result:

Lemma 3 (Ju 2003) *A parametric rule satisfies merging-proofness if and only if it has a parametric representation that is superadditive in income.*

The next lemma, whose proof is given in the appendix, says that *consistency* and *merging-proofness* together imply *equal treatment of equals*.

Lemma 4 *Merging-proofness and consistency together imply equal treatment of equals.*[10]

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[7] It is worth noting that, although there might be different representations of a parametric rule, superhomogeneity in income is invariant; that is, either every representation is superhomogeneous in income or none of them is superhomogeneous in income.

[8] See also Proposition 2 in Thon (1987).

[9] Like superhomogeneity, superadditivity in income is also invariant with respect to the choice of the representation.

[10] Chambers and Thomson (2002, Lemma 3) show that *consistency* and *equal treatment of equals* together imply *anonymity*, which says that the chosen tax profile should not depend on the names of agents. Combining this with our lemma, *merging-proofness* and *consistency* imply *anonymity*.

Combining Lemmas 3 and 4 and Young's (1987) characterization of parametric rules, we obtain:

Proposition 5 *A rule satisfies merging-proofness, consistency, and continuity if and only if it has a parametric representation that is superadditive in income.*[11]

Now, due to Propositions 2 and 5, the logical relation between *progressivity* and *merging-proofness* can be established from the following relation between superhomogeneity and superadditivity.

Lemma 6 *Superhomogeneity in income implies superadditivity in income.*

Proof. Let y_0 and y_0' be such that $0 < y_0 \leq y_0'$. Let $\alpha \equiv (y_0+y_0')/y_0'$. Then, by *superhomogeneity*, $f(\lambda, \alpha y_0') \geq \alpha f(\lambda, y_0')$, that is, $f(\lambda, y_0 + y_0') / (y_0 + y_0') \geq f(\lambda, y_0')/y_0'$. Thus, $f(\lambda, y_0 + y_0') \geq f(\lambda, y_0') + \frac{y_0}{y_0'} f(\lambda, y_0')$. By *superhomogeneity*, $\frac{y_0}{y_0'} f(\lambda, y_0') \geq f(\lambda, y_0)$. Hence $f(\lambda, y_0 + y_0') \geq f(\lambda, y_0') + f(\lambda, y_0)$, which shows that f is *superadditive* in income. ■

It follows from Propositions 2 and 5 and Lemma 6 that:

Corollary 7 *Let R be a rule satisfying consistency and continuity. If R is progressive, then R is merging-proof. However, the converse does not hold.*[12]

Remark 2 *Without* consistency *and* continuity*, the logical relation between* progressivity *and* merging-proofness *in Corollary 7 does not hold, as shown by Example 1 in Section 5.*

Remark 3 *Since rules take only non-negative values, if a parametric representation is* superadditive in income *(or* superhomogeneous*, by Lemma 6), then it is* non-decreasing in income*. Thus the corresponding parametric rule satisfies* tax order preservation*. Therefore, among parametric rules,* merging-proofness *(or* progressivity*) implies* tax order preservation*.*

Note that any convex function that crosses the origin is superhomogeneous. This, together with Proposition 2 and Corollary 7, gives the following:

Corollary 8 *Any rule with a parametric representation that is convex in income is progressive and merging-proof.*

[11] This strengthens Theorem 2 in Ju (2003) by dropping *equal treatment of equals*.

[12] An example of a rule satisfying *merging-proofness* but violating *progressivity* can be provided upon request.

Both the leveling tax and the flat tax have parametric representations that are convex in income. Thus, they are both *progressive* and *merging-proof*. The same argument applies to show that two other classical tax rules, such as the proposals made by Cohen-Stuart and Cassel (and formulated as rules by Young, 1988), are *progressive* and *merging-proof*.

4.2 Progressivity and inequality reduction

The next proposition summarizes the logical relation between *progressivity* and *inequality reduction*. We need to consider the following additional axioms to present this proposition.

Axiom 9 ***Revenue continuity.*** *For each $N \in \mathcal{N}$, each $y \in \mathbb{R}_+^N$, each sequence $\{T^n : n \in \mathbb{N}\}$ in $\mathbb{R}_+$ and each $T \in \mathbb{R}_+$, if T^n converges to T, then $R(N, y, T^n)$ converges to $R(N, y, T)$.*

Axiom 10 ***Revenue monotonicity.*** *For each $(N, y, T) \in \mathcal{D}$ and each $T' \geq T$, $R(N, y, T') \geqq R(N, y, T)$.*

Proposition 9 (Ju and Moreno-Ternero (2006)) *The following statements hold*:
(i) *Progressivity and income order preservation together imply inequality reduction.*
(ii) *Inequality reduction and consistency together imply progressivity.*
(iii) *Inequality reduction, together with consistency and revenue continuity* (*or revenue monotonicity*), *implies income order preservation.*

The next result follows directly from Proposition 9.

Corollary 10 *For consistent and revenue-continuous* (*or revenue-monotonic*) *rules, the combination of progressivity and income order preservation is equivalent to inequality reduction.*

It follows from Proposition 9 that since the leveling tax and the flat tax satisfy both *progressivity* and *income order preservation*, they satisfy *inequality reduction*. After strengthening *revenue continuity* to (full) *continuity*, we obtain the following result.

Proposition 11 *A rule satisfies inequality reduction, consistency and continuity if and only if it has a parametric representation $f : [a, b] \times \mathbb{R}_+ \to \mathbb{R}$ such that f is superhomogeneous in income and for each $\lambda \in [a, b]$, the function $g^\lambda(x) = x - f(\lambda, x)$ is non-decreasing.*[13]

[13] This property is also invariant.

Proof. Let R be a rule satisfying *inequality reduction*, *consistency*, and *continuity*. By Proposition 9, R satisfies *progressivity* and *income order preservation*. Then, by Proposition 2, R has a parametric representation $f\colon [a,b] \times \mathbb{R}_+ \to \mathbb{R}$, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$, which is superhomogeneous in income. Let $\lambda \in [a,b]$. Let $g^\lambda : \mathbb{R}_+ \to \mathbb{R}$ be such that $g^\lambda(x) = x - f(\lambda, x)$ for all $x \in \mathbb{R}_+$. Suppose, by contradiction, that there exist $x, y \in \mathbb{R}_+$ such that $x < y$ and $g^\lambda(x) > g^\lambda(y)$. Let $T \equiv f(\lambda, x) + f(\lambda, y)$. Consider the problem $(\{1,2\}, (x,y), T)$. Then, $R(\{1,2\}, (x,y), T) = (f(\lambda,x), f(\lambda,y))$. Thus,

$$x - R_1(\{1,2\}, (x,y), T)) = g^\lambda(x) > g^\lambda(y) = y - R_2(\{1,2\}, (x,y), T)),$$

contradicting *income order preservation*.

Conversely, let R be a rule with parametric representation $f\colon [a,b] \times \mathbb{R}_+ \to \mathbb{R}$ such that f is superhomogeneous in income and for each $\lambda \in [a,b]$, $g^\lambda(x) = x - f(\lambda, x)$ is non-decreasing. By Proposition 2, R satisfies *progressivity*, *continuity* and *consistency*. Then, by Proposition 9, we only have to show that R satisfies *income order preservation*. Suppose, by contradiction, that there exist $(N, y, T) \in \mathcal{D}$ and $i, j \in N$ such that $y_i < y_j$ and $y_i - R_i(N,y,T) > y_j - R_j(N,y,T)$. Let $\lambda \in [a,b]$ be such that $R(N,y,T) = (f(\lambda, y_i))_{i \in N}$. Then

$$y_i - f(\lambda, y_i) = y_i - R_i(N,y,T) > y_j - R_j(N,y,T) = y_j - f(\lambda, y_j),$$

contradicting the non-decreasing property of $g^\lambda(\cdot)$. ■

5 Operators: what axioms are preserved?

An *operator* is a function that maps a rule into another, possibly the same, rule. An axiom is said to be *preserved* under an operator if any rule that satisfies the axiom is mapped by the operator into a rule that also satisfies the axiom. We consider two operators introduced by Thomson and Yeh (2006) and study the preservation of our three main axioms.

5.1 Convexity operators

When two rules compete, a natural compromise is to mix the two rules by a convex combination, as suggested by *convexity operators*. Formally, given a "reference rule" $\bar{R}(\cdot)$ and a weight $\alpha \in [0,1]$, the *convexity operator* associated with $\bar{R}$ and α maps each rule $R(\cdot)$ into the convex combination

$(1-\alpha)\,R(\cdot)+\alpha\bar{R}(\cdot)$.[14] The idea of mixing two rules is also useful for a smooth transition from one rule to another when such a transition is required.

Mixing two rules may lose its appeal if such an operation does not preserve some basic axioms of taxation. Fortunately, all of our three main axioms are preserved:

Proposition 12 *Consider the convexity operators associated with a reference rule $\bar{R}(\cdot)$. If $\bar{R}(\cdot)$ satisfies progressivity, then each of these convexity operators preserves progressivity. The same occurs for inequality reduction and merging-proofness.*

Proof. We skip the proofs of preservations of *progressivity* and *merging-proofness*, which are straightforward. Suppose that $R(\cdot)$ and $\bar{R}(\cdot)$ satisfy *inequality reduction*. Let $\alpha\in[0,1]$. Let $(N,y,T)\in\mathcal{D}$, $\bar{x}\equiv\bar{R}(N,y,T)$, $x\equiv R(N,y,T)$ and $x^{\alpha}\equiv R^{\alpha}(N,y,T)$. Without loss of generality, assume that $N\equiv\{1,\ldots,n\}$ and that $y_1\leq y_2\leq\cdots\leq y_n$. Let $\bar{\sigma}$, σ and $\pi\colon N\to N$ be permutations on N such that for each $i\in\{1,\ldots,n-1\}$, $y_{\bar{\sigma}(i)}-\bar{x}_{\bar{\sigma}(i)}\leq y_{\bar{\sigma}(i+1)}-\bar{x}_{\bar{\sigma}(i+1)}$, $y_{\sigma(i)}-x_{\sigma(i)}\leq y_{\sigma(i+1)}-x_{\sigma(i+1)}$, and $y_{\pi(i)}-x^{\alpha}_{\pi(i)}\leq y_{\pi(i+1)}-x^{\alpha}_{\pi(i+1)}$. Let $i\in\{1,\ldots,n-1\}$. Note that $\sum_{j=1}^{i}\left(y_{\pi(j)}-\bar{x}_{\pi(j)}\right)\geq\sum_{j=1}^{i}\left(y_{\bar{\sigma}(j)}-\bar{x}_{\bar{\sigma}(j)}\right)$ because, by definition of $\bar{\sigma}$, the right-hand side is the sum of the i lowest post-tax incomes associated with the tax profile $\bar{x}$. Similarly, $\sum_{j=1}^{i}\left(y_{\pi(j)}-x_{\pi(j)}\right)\geq\sum_{j=1}^{i}\left(y_{\sigma(j)}-x_{\sigma(j)}\right)$. Therefore,

$$
\begin{aligned}
\sum_{j=1}^{i}\left(y_{\pi(j)}-x^{\alpha}_{\pi(j)}\right) &= (1-\alpha)\sum_{j=1}^{i}\left(y_{\pi(j)}-x_{\pi(j)}\right)+\alpha\sum_{j=1}^{i}\left(y_{\pi(j)}-\bar{x}_{\pi(j)}\right)\\
&\geq (1-\alpha)\sum_{j=1}^{i}\left(y_{\sigma(j)}-x_{\sigma(j)}\right)+\alpha\sum_{j=1}^{i}\left(y_{\bar{\sigma}(j)}-\bar{x}_{\bar{\sigma}(j)}\right).
\end{aligned}
$$

By *inequality reduction* of $R(\cdot)$ and $\bar{R}(\cdot)$,

$$
\frac{\sum_{j=1}^{i}\left(y_{\sigma(j)}-x_{\sigma(j)}\right)}{Y-T}\geq\frac{\sum_{j=1}^{i}y_j}{Y}\text{ and }\frac{\sum_{j=1}^{i}\left(y_{\bar{\sigma}(j)}-\bar{x}_{\bar{\sigma}(j)}\right)}{Y-T}\geq\frac{\sum_{j=1}^{i}y_j}{Y}.
$$

Therefore,

$$
\frac{\sum_{j=1}^{i}\left(y_{\pi(j)}-x^{\alpha}_{\pi(j)}\right)}{Y-T}\geq\frac{\sum_{j=1}^{i}y_j}{Y}.
$$

[14] This definition is slightly different from the definition in Thomson and Yeh (2006). The convexity operator in Thomson and Yeh (2006) maps an ordered list of a finite number of rules into the weighted average rule. Our results can easily be adapted to establish the same results for their convexity operator.

showing that R satisfies *inequality reduction*. ■

5.2 Minimal-burden operator

At each problem (N, y, T), if $T - \sum_{j \in N \setminus \{i\}} y_j > 0$ for an agent $i \in N$, this part of the revenue cannot be covered even if everyone other than i pays his full income. Thus this part can be viewed as the minimal burden imposed on agent i. For each $i \in N$, let $m_i(N, y, T) \equiv \min\{0, T - \sum_{j \neq i} y_j\}$ be i's *minimal burden*. Let $m(N, y, T) \equiv (m_i(N, y, T))_{i \in N}$ and $M(N, y, T) \equiv \sum_N m_i(N, y, T)$. The *minimal-burden operator* associates with each rule R the rule R^m defined by the following two-step payment procedure. For each problem, first each agent pays his minimal burden; second, each agent pays his tax according to R at the revised problem obtained by reducing agents' incomes by the amounts of their minimal burdens and the tax revenue by the total minimal burdens. That is, for each $(N, y, T) \in \mathcal{D}$,

$$R^m(N, y, T) \equiv m(N, y, T) + R(N, y - m(N, y, T), T - M(N, y, T)).$$

The next proposition shows what axioms are preserved under the minimal-burden operator.

Proposition 13 *The minimal burden operator preserves progressivity and inequality reduction. However, it does not preserve merging-proofness.*

The proof of the statement regarding progressivity and inequality reduction is provided in the appendix. Example 1 below shows that the minimal-burden operator does not preserve *merging-proofness*.

Example 1 *For each* $(N, y, T) \in \mathcal{D}$, *let*

$$R(N, y, T) \equiv \begin{cases} R^L(N, y, T) & \text{if } T \geq 10 \\ R^F(N, y, T) & \text{if } T < 10 \end{cases},$$

where R^L *denotes the leveling tax and* R^F *the flat tax. Since both* R^L *and* R^F *are* merging-proof, R *is* merging-proof. *However,* R^m *is not* merging-proof. *To show this, consider the problem* $(N, y, T) = (\{1, 2, 3\}, (5, 55, 100), 70)$. *Then,*

$$R^m(N, y, T) = (0, 0, 10) + R^L(\{1, 2, 3\}, (5, 55, 90), 60) = \left(0, \frac{25}{2}, \frac{115}{2}\right).$$

Consider now the resulting problem in which agents 2 *and* 3 *merge their incomes and are represented by agent* 3*, i.e.,* $(N\backslash\{2\}, y', T) = (\{1,3\}, (5,155), 70)$*. Then,*

$$R^m\left(N\backslash\{2\}, y', T\right) = (0,65) + R^F\left(\{1,3\}, (5,90), 5\right) = \left(\frac{5}{19}, \frac{1325}{19}\right).$$

Consequently,

$$R_3^m(N\backslash\{2\}, y', T) < R_2^m\left(N, y, T\right) + R_3^m\left(N, y, T\right),$$

which shows that R^m *is not* merging-proof*. Note that* R *is* progressive*. By Proposition 13, so is* R^m*. Therefore,* R^m *is an example showing that* progressivity *does not imply* merging-proofness*, as claimed in Remark 2.*

For rules satisfying the following mild axiom, however, we show that the minimal-burden operator preserves *merging-proofness*.

Suppose that an agent donates part of his income and that the donation is used to finance tax revenue. Then both the donor's income and the tax revenue go down by the amount of the donation. The next axiom says that the donor's total payment (tax plus donation) should not be lower than his total payment without donation.

Axiom 11 ***No Donation Paradox.*** *For all* $(N, y, T) \in \mathcal{D}$*, all* $i \in N$ *and all* $t \in [0, \min\{T, y_i\}]$,

$$R_i\left(N, y, T\right) \leq t + R_i\left(N, \left(y_i - t, y_{-i}\right), T - t\right).$$ [15]

Ju and Moreno-Ternero (2005) characterize a large family of rules satisfying *no donation paradox* and some other axioms. The family includes most of the well-known parametric rules, which shows that *no donation paradox* is indeed a mild condition.

Note that the rule in Example 1 violates *no donation paradox*. To show this, consider the problem $(N, y, T) = (\{1,2\}, (3,15), 11)$. Then, $R\left(N, y, T\right) = (0, 11)$ and $R\left(N, (3,13), 9\right) = (27/16, 117/16)$. Thus, $R_2\left(N, y, T\right) = 11 > 2 + 117/16 = 2 + R_2\left(N, (3,13), 9\right)$.

Proposition 14 *On the family of rules satisfying no donation paradox, the minimal-burden operator preserves merging-proofness.*

The proof is provided in the appendix.

[15] In bankruptcy problems, this axiom is introduced by Thomson and Yeh (2006). It is the dual of "claims monotonicity" (see p.100 and p.161 in Thomson 2005).

6 Concluding remarks

There are two other operators, also introduced by Thomson and Yeh (2006), that have been used in the literature.

The *Truncation Operator* maps each rule $R(\cdot)$ into $R^t(\cdot)$, defined as follows: for each $(N, y, T) \in \mathcal{D}$ and each $i \in N$,

$$R_i^t(N, y, T) \equiv R_i(N, (\min\{y_j, T\})_{j \in N}, T).$$

The *Duality Operator* maps each rule $R(\cdot)$ into $R^d(\cdot)$, defined as follows: for each $(N, y, T) \in \mathcal{D}$ and each $i \in N$,

$$R_i^d(N, y, T) \equiv y_i - R_i(N, y, \sum_{j \in N} y_j - T).$$

Both of these operators lack appealingness in our context of taxation, as they do not provide intuitive procedures to create new rules from a given rule. This view is reinforced when one realizes that they both fail to preserve the three main axioms we have considered throughout the paper.[16]

The truncation operator, however, is more intuitive in the dual framework of taxation: *bankruptcy*. The dual axioms of *progressivity* and *merging-proofness* are *regressivity* and *splitting-proofness* respectively.[17] Theorem 6 in Thomson and Yeh (2006) says that an axiom is preserved under the truncation operator if and only if the dual axiom is preserved under the minimal-burden operator. Therefore, from Proposition 13, we obtain that the truncation operator preserves *regressivity* and the dual axiom of *inequality reduction*, but not *splitting-proofness*. Also, from Proposition 14, we obtain that, on the family of rules satisfying *income monotonicity* (which is the dual axiom of *no donation paradox* that says that if an agent increases his income, ceteris paribus, then his tax burden cannot decrease), the truncation operator preserves *splitting-proofness*. Finally, it is straightforward to show that the duality operator also fails to preserve the axioms of *regressivity*, *splitting-proofness* and the dual axiom of *inequality reduction*.

[16] Consider the leveling tax (L) which is, as we know, a progressive, inequality-reducing and merging-proof rule. It is straightforward to show that the head tax (the image of L under the duality operator) and the so-called *concede-and-divide* (the image of L under the truncation operator, in the two-agent case) violate the three properties.

[17] Thus, it is not difficult to provide parallel results to those in Section 4.1 upon replacing *progressivity* (resp. *merging-proofness*) by *regressivity* (resp. *splitting-proofness*) and *superhomogeneity* (resp. *superadditivity*) by *subhomogeneity* (resp. *subadditivity*).

1	2	$3,\ldots,n$	$n+1$	$n+2$
a	a	$y_{-\{1,2\}}$		
a	a	$y_{-\{1,2\}}$	0	0
	a	$y_{-\{1,2\}}$	a	0
	a	$y_{-\{1,2\}}$	a	
0	a	$y_{-\{1,2\}}$	a	0
a	a	$y_{-\{1,2\}}$		0
a	a	$y_{-\{1,2\}}$		

(a) Income profiles

1	2	$3,\ldots,n$	$n+1$	$n+2$
x_1	x_2	$x_{-1,2}$		
x_1	x_2	$x_{-\{1,2\}}$	0	0
	x'_2	$x'_{-\{1,2\}}$	x'_{n+1}	0
	x'_2	$x'_{-\{1,2\}}$	x'_{n+1}	
0	x'_2	$x'_{-\{1,2\}}$	x'_{n+1}	0
x_1	x_2	$x_{-\{1,2\}}$		0
x_1	x_2	$x_{-\{1,2\}}$		

(b) Tax profiles

Table 1: Proof of Lemma 4.

A Proofs

Proof. [**Proof of Lemma 4**] Let $(N, y, T) \in \mathcal{D}$ and $i, j \in N$ be such that $i \neq j$ and $y_i = y_j$. For simplicity, let $i = 1$ and $j = 2$ and $N \equiv \{1, \ldots, n\}$ (this problem is illustrated in the second row of Table 1-(a)). Let $x \equiv R(N, y, T)$ and $a \equiv y_1 = y_2$ (x is illustrated in the second row of Table 1-(b)). Let $N' \equiv N \cup \{n+1, n+2\}$. Consider the problem $(N', (y, 0, 0), T)(= (N', (a, a, y_{-\{1,2\}}, 0, 0), T))$ where $n+1$ and $n+2$ have zero income and all agents in N have the same incomes as in (N, y, T) (see the third row of Table 1-(a)). By *boundedness*, $R_{\{n+1,n+2\}}(N', (y, 0, 0), T) = (0, 0)$. By *balancedness* and *consistency*, $R_N(N', (y, 0, 0), T) = R(N, y, T)$ (see the third row of Table 1-(b)). Now consider the problem $(N' \backslash \{1\}, (a, y_{-\{1,2\}}, a, 0), T)$ obtained by merging the incomes of agents 1 and $n+1$ at $(N', (y, 0, 0), T)$ into the income of agent $n+1$ (see the fourth row of Table 1-(a)). Let $x' \equiv R(N' \backslash \{1\}, (a, y_{-\{1,2\}}, a, 0), T)$ (see the fourth row of Table 1-(b)). Then $x'_{n+2} = 0$ and by *merging-proofness*, $x'_{n+1} \geq x_1$. By *consistency*, $(x'_2, x'_{-\{1,2\}}, x'_{n+1}) = R(\{2, \ldots, n+1\}, (a, y_{-\{1,2\}}, a), T)$.

Consider the problem $(N', (0, a, y_{-\{1,2\}}, a, 0), T)$ where 1 and $n+2$ have zero income and all others in N' have the same incomes as in $(\{2, \ldots, n+1\}, (a, y_{-\{1,2\}}, a), T)$ (see the sixth row of Table 1-(a)). Then, by *boundedness* and *consistency*, $R(N', (0, a, y_{-\{1,2\}}, a, 0), T) = (0, x'_2, x'_{-\{1,2\}}, x'_{n+1}, 0)$. Now making the reverse argument but merging the incomes of 1 and $n+1$ at $(N', (0, a, y_{-\{1,2\}}, a, 0), T)$ into 1's income and applying *merging-proofness*, we can show $x_1 \geq x'_{n+1}$, as $x_1 = R_1(N' \backslash \{n+1\}, (a, a, y_{-\{1,2\}}, 0), T)$.

Therefore, $x_1 = x'_{n+1}$. By *balancedness*, $x_2 + \cdots + x_n = x'_2 + \cdots + x'_n$. Thus, the two reduced problems of (N, y, T) and $(\{2, \ldots, n+1\}, (a, y_{-\{1,2\}}, a), T)$

for the coalition $\{2,\ldots,n\}$ are identical. By *consistency*, $(x_2,\ldots,x_n) = (x'_2,\ldots,x'_n)$.

To summarize, by replacing agent 1's income at $(N,(a,a,y_{-\{1,2\}}),T)$ with agent $(n+1)$'s income, we transformed the problem into $(\{2,\ldots,n,n+1\},(a,y_{-\{1,2\}},a),T)$ and showed that 1's tax at the original problem is equal to $(n+1)$'s tax in the new problem and the taxes of all others do not change.

Now, transforming $(\{2,\ldots,n,n+1\},(a,y_{-\{1,2\}},a),T)$ into $(\{3,\ldots,n,n+1,n+2\},(y_{-\{1,2\}},a,a),T)$ and letting $\bar{x} \equiv R(\{3,\ldots,n,n+1,n+2\},(y_{-\{1,2\}},a,a),T)$, we can show that $\bar{x}_{\{3,\ldots,n+1\}} = x'_{\{3,\ldots,n+1\}} = (x_{\{3,\ldots,n\}},x'_{n+1}) = (x_{\{3,\ldots,n\}},x_1)$ and $x_2 = \bar{x}_{n+2}$. Therefore, $x_1 = \bar{x}_{n+1}$ and $x_2 = \bar{x}_{n+2}$.

Applying the symmetric argument (the whole argument above) switching the role of $n+1$ and the role of $n+2$, we can show that $x_2 = \bar{x}_{n+1}$ and $x_1 = \bar{x}_{n+2}$. Therefore, $x_1 = x_2$. ■

Proof. [**Proof of Proposition 13**] *Progressivity*: Let R be a rule satisfying *progressivity*. Let $(N,y,T) \in \mathcal{D}$ and $x^m \equiv R^m(N,y,T)$. Assume, without loss of generality, that $N = \{1,2,...,n\}$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. Let $k \in N$ be the first agent whose minimal burden is strictly positive, i.e., $y_{k-1} \leq Y-T < y_k$. Then, $m_1(N,y,T) = \cdots = m_{k-1}(N,y,T) = 0 < m_k(N,y,T) \leq m_{k+1}(N,y,T) \leq \cdots \leq m_n(N,y,T)$. For each $i \geq k$, $m_i(N,y,T) = y_i - Y + T$. Let $y' \equiv y - m(N,y,T) = (y_1,\ldots,y_{k-1},Y-T,\ldots,Y-T)$ and $T' \equiv T - \sum_{i=1}^n m_i(N,y,T) = T - \sum_{i=k}^n (y_i - Y + T)$. Let $x' \equiv R(N,y',T')$. Then

$$x_i^m = \begin{cases} x'_i \text{ if } i \leq k-1; \\ y_i - Y + T + x'_i \text{ if } i \geq k. \end{cases} \tag{1}$$

Let $i,j \in N$ be such that $y_i \leq y_j$. There are three cases.

Case 1: $y_i \leq y_j < y_k$. By *progressivity* of R at (N,y',T'), $x_i^m/y_i = x'_i/y'_i \leq x'_j/y'_j = x_j^m/y_j$.

Case 2: $y_k \leq y_i \leq y_j$. By *equal treatment of equals* of R at (N,y',T') (implied by the *progressivity* of R), $x'_i = x'_j = a$. By *boundedness,* $x'_i = x'_j = a \leq Y-T$ and so $Y-T-x'_i = Y-T-x'_j = Y-T-a \geq 0$. Therefore, since $y_i \leq y_j$,

$$\frac{x_i^m}{y_i} = \frac{y_i - Y + T + x'_i}{y_i} = 1 - \frac{Y-T-a}{y_i} \leq 1 - \frac{Y-T-a}{y_j} = \frac{y_j - Y + T + x'_j}{y_j} = \frac{x_j^m}{y_j}.$$

Case 3: $y_i < y_k \leq y_j$. By *progressivity* of R at (N,y',T'),

$$\frac{x'_i}{y_i} \leq \frac{x'_j}{Y-T}. \tag{2}$$

Now, since $Y - T < y_j$ and, by *boundedness*, $x'_i \le y_i$, then $x'_i y_j \le (Y-T)(x'_i - y_i) + y_i y_j$. Hence,

$$x'_i \le \frac{y_i(y_j - Y + T) + x'_i(Y-T)}{y_j}. \tag{3}$$

Therefore, combining (2) and (3),

$$\frac{x_i^m}{y_i} = \frac{x'_i}{y_i} \le \frac{y_i(y_j - Y + T) + x'_i(Y-T)}{y_i y_j} \le \frac{y_j - Y + T + x'_j}{y_j} = \frac{x_j^m}{y_j}.$$

Inequality Reduction: Let R be a rule satisfying *inequality reduction*. Let $(N, y, T) \in \mathcal{D}$, (N, y', T'), x^m and x' be given as in the above proof. Note that $y - x^m = y' - x'$. By the *inequality reduction* of R at (N, y', T'), $y' - x'$ Lorenz dominates y'. Thus, we only have to show that y' Lorenz dominates y.[18] It is clear that for each $l \le k-1$, $\sum_{i=1}^{l} y'_i / Y' \ge \sum_{i=1}^{l} y_i / Y$. Assume $l \ge k$. Then, it can be shown that $\sum_{i=1}^{l} y'_i / Y' \ge \sum_{i=1}^{l} y_i / Y$ holds, if and only if,

$$\sum_{i=1}^{k} y_i \left(\sum_{i=l+1}^{n} (y_i - Y + T) \right) \ge (Y-T) \left((n-l) \sum_{i=k+1}^{l} y_i - (l-k) \sum_{i=l+1}^{n} y_i \right),$$

which is true because the left-hand side is non-negative and the right-hand side is non-positive.[19] ■

To prove Proposition 14, we need the following additional axiom and lemma.

No donation paradox and *merging-proofness* together imply the following useful property, as shown in the next lemma. Suppose that two agents i and j merge their income into j's income and agent j donates i's income. The property says that the total payment by the two agents should not be lowered by such a donation.

Axiom 12 ***Donation-Proofness.*** *For all $(N, y, T) \in \mathcal{D}$ and all $i, j \in N$, such that $T \ge y_i$*

$$R_i(N, y, T) + R_j(N, y, T) \le y_i + R_j\left(N \backslash \{i\}, y_{N \backslash \{i\}}, T - y_i\right).$$

[18] Note that if y is increasingly ordered, so is y'.

[19] Note that $(n-l) \sum_{i=k+1}^{l} y_i \le (n-l)(l-k) y_l$ and $(l-k) \sum_{i=l+1}^{n} y_i \ge (l-k)(n-l) y_{l+1}$. Thus, $(n-l) \sum_{i=k+1}^{l} y_i - (l-k) \sum_{i=l+1}^{n} y_i \le (n-l)(l-k)(y_l - y_{l+1}) \le 0$.

Lemma 15 *Merging-proofness and no donation paradox together imply donation-proofness.*

Proof. Let R be a rule satisfying *merging-proofness* and *no donation paradox*. Let $(N, y, T) \in \mathcal{D}$ and $i, j \in N$ such that $T \geq y_i$. By *merging-proofness*,

$$R_i(N, y, T) + R_j(N, y, T) \leq R_j(N \backslash \{i\}, \left(y_i + y_j, y_{N \backslash \{i,j\}}\right), T).$$

By *no donation paradox*, applied to agent j with donation y_i at $\left(N \backslash \{i\}, \left(y_i + y_j, y_{N \backslash \{i,j\}}\right), T\right)$,

$$R_j(N \backslash \{i\}, \left(y_i + y_j, y_{N \backslash \{i,j\}}\right), T) \leq y_i + R_j(N \backslash \{i\}, \left(y_j, y_{N \backslash \{i,j\}}\right), T - y_i).$$

Combining the two inequalities, we obtain

$$R_i(N, y, T) + R_j(N, y, T) \leq y_i + R_j\left(N \backslash \{i\}, y_{N \backslash \{i\}}, T - y_i\right),$$

which shows *donation-proofness*. ■

Now we are ready to prove Proposition 14.

Proof. [**Proof of Proposition 14**] Let R be a rule satisfying *no donation paradox* and *merging-proofness*. By Lemma 15, R satisfies *donation-proofness*. Let $(N, y, T) \in \mathcal{D}$. Assume, without loss of generality, that $N = \{1, 2, ..., n\}$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. Let $k \in N$ be the first agent whose minimal burden is strictly positive, i.e., k is such that $y_{k-1} \leq Y - T < y_k$. Let $i, j \in N$ and $\hat{y} \in \mathbb{R}_+^{N \backslash \{i\}}$ be such that $\hat{y}_j = y_i + y_j$ and $\hat{y}_{N \backslash \{i,j\}} = y_{N \backslash \{i,j\}}$. Let $x \equiv R(N, y, T)$ and $\hat{x} \equiv R(N \backslash \{i\}, \hat{y}, T)$. Let $x^m \equiv R^m(N, y, T)$ and $\hat{x}^m \equiv R^m(N \backslash \{i\}, \hat{y}, T)$. We show $x_i^m + x_j^m \leq \hat{x}_j^m$ below.

Let $M \equiv M(N, y, T)$ and $\hat{M} \equiv M(N \backslash \{i\}, \hat{y}, T)$. Let $y' \equiv (y_1, \ldots, y_{k-1}, Y - T, \ldots, Y - T)$ and $x' \equiv R(N, y', T - M)$.

Case 1: $y_i + y_j \leq Y - T$. Then $y_i, y_j \leq Y - T$ and so $x_i^m = x_i'$ and $x_j^m = x_j'$. Note that $M = \hat{M}$. Then, $R_j^m(N \backslash \{i\}, \hat{y}, T)$ equals j's award under $R(\cdot)$ at the problem obtained from y' after merging i and j's incomes. Therefore, *merging-proofness* of R at $(N, y', T - M)$ implies $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 2: $y_i, y_j > Y - T$. Without loss of generality, suppose $y_i \leq y_j$. In this case,

$$x_i^m + x_j^m = \left(\begin{array}{c} y_i - (Y - T) + R_i(N, y_1, \ldots, y_{k-1}, \underbrace{Y - T, \ldots, Y - T}_{n-k+1}, T - M) \\ + y_j - (Y - T) + R_j(N, y_1, \ldots, y_{k-1}, \underbrace{Y - T, \ldots, Y - T}_{n-k+1}, T - M) \end{array}\right),$$

$$\hat{x}_j^m = y_i + y_j - (Y - T) + R_j(N \backslash \{i\}, y_1, \ldots, y_{k-1}, \underbrace{Y - T, \ldots, Y - T}_{n-k}, T - \hat{M}).$$

Since $\hat{M} = M + Y - T$, then by *donation-proofness*,

$$R_i(N, y_1, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k+1}, T-M) + R_j(N, y_1, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k+1}, T-M)$$
$$\leq (Y-T) + R_j(N\backslash\{i\}, y_1, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k}, T-\hat{M}).$$

Therefore, $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 3: $y_i \leq Y-T < y_j$. Note that $\hat{M} = M + y_i$. We have

$$x_i^m + x_j^m = \left(\begin{array}{c} R_i(N, y_1, \dots, y_{i-1}, y_i, y_{i+1}, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k+1}, T-M) + y_j - (Y-T) \\ +R_j(N, y_1, \dots, y_{i-1}, y_i, y_{i+1}, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k+1}, T-M) \end{array} \right),$$
$$\hat{x}_j^m = y_i + y_j - (Y-T) + R_j(N\backslash\{i\}, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k+1}, T-\hat{M}).$$

Since $\hat{M} = M + y_i$, then by *donation-proofness*,

$$\left(\begin{array}{c} R_i(N, y_1, \dots, y_{i-1}, y_i, y_{i+1}, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k+1}, T-M) \\ +R_j(N, y_1, \dots, y_{i-1}, y_i, y_{i+1}, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k+1}, T-M) \end{array} \right)$$
$$\leq y_i + R_j(N\backslash\{i\}, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k+1}, T-\hat{M}).$$

Therefore, $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 4: $y_j \leq Y-T < y_i$. Note that $\hat{M} = M + y_j$. We have

$$x_i^m + x_j^m = \left(\begin{array}{c} y_i - (Y-T) + R_i(N, y_1, \dots, y_j, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k+1}, T-M) \\ +R_j(N, y_1, \dots, y_j, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k+1}, T-M) \end{array} \right),$$
$$\hat{x}_j^m = y_i + y_j - (Y-T) + R_j(N\backslash\{i\}, y_1, \dots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{Y-T}, \dots, y_{k-1}, \underbrace{Y-T, \dots, Y-T}_{n-k}, T-\hat{M}).$$

By *merging-proofness*,

$$\left(\begin{array}{c} R_i(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ +R_j(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \end{array}\right)$$
$$\leq R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{y_j + Y - T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-M).$$

By *no donation paradox* applied to agent j with donation y_j,

$$\begin{aligned} & R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{y_j + Y - T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-M) \\ & \leq y_j + R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{Y-T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-\hat{M}). \end{aligned}$$

Combining the two inequalities, we obtain

$$\begin{aligned} & R_i(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ & +R_j(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ & \leq y_j + R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{Y-T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-\hat{M}), \end{aligned}$$

which implies $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 5: $y_i, y_j \leq Y-T$ and $y_i+y_j > Y-T$. Then $\hat{M} = M+T-(Y-(y_i+y_j))$.
We have

$$x_i^m + x_j^m = \left(\begin{array}{c} R_i(N, y_1, \ldots, y_i, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ +R_j(N, y_1, \ldots, y_i, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M), \end{array}\right)$$
$$\hat{x}_j^m = T-(Y-(y_i+y_j)) + R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{Y-T}, \cdots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-\hat{M}).$$

By *merging-proofness*,

$$x_i^m + x_j^m \leq R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{y_i + y_j}, \cdots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M).$$

Since $T - \hat{M} = T - M - (T - (Y - (y_i + y_j)))$, then applying *no donation paradox for* j with donation $T - (Y - (y_i + y_j))$,

$$R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{y_i + y_j}, \cdots, y_{k-1}, \underbrace{Y - T, \ldots, Y - T}_{n-k+1}, T - M)$$
$$\leq T - (Y - (y_i + y_j)) + R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{Y - T}, \cdots, y_{k-1}, \underbrace{Y - T, \ldots, Y - T}_{n-k+1}, T - \hat{M}).$$

Therefore, $x_i^m + x_j^m \leq \hat{x}_j^m$. ■

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