Gap probabilities in the bulk of the Airy process

Elliot Blackstone, Christophe Charlier, Jonatan Lenells *

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Abstract

We consider the probability that no points lie on g large intervals in the bulk of the Airy point process. We make a conjecture for all the terms in the asymptotics up to and including the oscillations of order 1, and we prove this conjecture for $g = 1$.

1 Introduction and statement of results

The Airy point process is a determinantal point process on $\mathbb{R}$ which plays an important role in many seemingly unrelated topics: for example, it models the largest eigenvalues of certain large random matrices [10, 15, 29], the fluctuations at the solid-liquid boundary of certain tiling models [25], the largest parts of Young diagrams with respect to the Plancherel measure [4, 9], the fluctuations of the length of the longest increasing subsequence of a random permutation of $\{1, 2, \dots, N\}$, $N \gg 1$ [3], and the Brownian particles near the edge-curve in non-intersecting Brownian paths [1]. It has also recently been shown to appear at the liquid-gas boundary of the two-periodic Aztec diamond [5]. The correlation kernel of the Airy point process is given by

$$K^{\mathrm{Ai}}(u, v) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\mathrm{Ai}(v)}{u - v}, \qquad u, v \in \mathbb{R}, \tag{1.1}$$

where Ai denotes the Airy function. There are infinitely many random points which are almost surely all distinct, and there is almost surely a largest point. The Airy process is rigid, in the sense that given a bounded Borel set A, the restriction of any point configuration to $\mathbb{R} \setminus A$ almost surely determines the number of points in A, see [11]. We also mention that the maximum fluctuations of the points have been studied in [14, 32] and in [12, Theorem 1.4].

A *gap probability* refers to the probability that a given region is free from points. It is well-known from the general theory of determinantal point processes (see e.g. [28]) that gap probabilities are Fredholm determinants. In particular, if $\mathcal{I} \subset \mathbb{R}$ is a finite union of intervals, then the probability of finding no points in $\mathcal{I}$ for the Airy process is given by

$$F(\mathcal{I}) := \mathbb{P}(\text{no points lie in } \mathcal{I}) = \det(I - \mathcal{K}^{\mathrm{Ai}}\chi_{\mathcal{I}}), \tag{1.2}$$

where $\mathcal{K}^{\mathrm{Ai}}$ is the integral operator whose kernel is K^{Ai}, and $\chi_{\mathcal{I}}$ is the projection operator. The properties of $F(\mathcal{I})$ depend on the structure of $\mathcal{I}$: given $g \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $\tilde{x}_0, x_1, \dots, x_{2g} \in \mathbb{R}$ such that $x_{2g} < \cdots < x_1 < \tilde{x}_0$, we distinguish the cases

$$\mathcal{I}_g^{(e)} = (x_{2g}, x_{2g-1}) \cup \cdots \cup (x_2, x_1) \cup (\tilde{x}_0, +\infty),$$
$$\mathcal{I}_g^{(b)} = (x_{2g}, x_{2g-1}) \cup \cdots \cup (x_2, x_1).$$

If $x_1 < 0$, then $F(\mathcal{I}_g^{(b)})$ is the probability of finding no points on a union of g intervals in the bulk, while $F(\mathcal{I}_g^{(e)})$ is the probability of finding no points on these g intervals and a gap at the edge. Note that the

*Department of Mathematics, KTH Royal Institute of Technology, Lindstedtsvägen 25, SE-114 28 Stockholm, Sweden.
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E-mail: elliotb@kth.se, cchar@kth.se, jlenells@kth.se.

case $\mathcal{I}_0^{(b)} = \emptyset$ leads trivially to $F(\mathcal{I}_0^{(b)}) = 1$. The distribution $F(\mathcal{I}_0^{(e)}) = F((\tilde{x}_0, +\infty))$ is the well-known Tracy–Widom distribution of the largest point which is naturally expressed in terms of the solution to a Painlevé II equation [29]. More generally, $F(\mathcal{I}_g^{(e)})$ (resp. $F(\mathcal{I}_g^{(b)})$) can be expressed in terms of a solution to a system of $2g+1$ (resp. $2g$) coupled Painlevé II equations [13, 31].

Gap probabilities are transcendental functions, but in certain cases they possess convenient approximations. For example, for small values of $r > 0$, $F(r\mathcal{I})$ represents the probability of a small gap, and can be estimated from its Fredholm series representation. A classical and much harder problem is to find *large gap asymptotics*, which are asymptotics for $F(r\mathcal{I})$ as $r \to +\infty$. Large r asymptotics for $F(r\mathcal{I}_0^{(e)})$ and $F(r\mathcal{I}_1^{(e)})$ have been analysed in [2, 16, 29] and [6, 26], respectively. In this work, we focus on $F(r\mathcal{I}_g^{(b)})$ with $g \geq 1$ (the case $g = 0$ is trivial, since $F(r\mathcal{I}_0^{(b)}) = 1$ for all r).

Large gap asymptotics on g intervals in the bulk

In Conjecture 1.1 below, we formulate a conjecture for the large r asymptotics of $F(r\mathcal{I}_g^{(b)})$ for any finite number of intervals $g \geq 1$, up to and including the oscillations of order 1. This conjecture has been verified numerically for $g = 1, 2, 3$, and we prove it rigorously for $g = 1$.

To formulate the conjecture, let $g \geq 1$ be an integer and let $\vec{x} = (x_1, \ldots, x_{2g}) \in \mathbb{R}^{2g}$ be such that $x_{2g} < \cdots < x_2 < x_1$. Define the square root $\sqrt{\mathcal{R}(z)}$ by

$$\sqrt{\mathcal{R}(z)} = \prod_{j=0}^{2g} \sqrt{z - x_j},$$

where the principal branch is taken for each of the square roots, and $x_0 = x_0(\vec{x})$ is defined below. Let $q_{g+1} = -1$ and define $\{m_{ij}\}_{i,j=1}^{g}$ and $\{\tilde{m}_i\}_{i=1}^{g}$ by

$$m_{ij} = \int_{x_{2i}}^{x_{2i-1}} \frac{s^{j-1}}{\sqrt{\mathcal{R}(s)}} ds, \qquad \tilde{m}_i = -\int_{x_{2i}}^{x_{2i-1}} \frac{q_{g+1}s^{g+1} + q_g s^g}{\sqrt{\mathcal{R}(s)}} ds, \qquad \text{where} \qquad q_g = \frac{1}{2}\sum_{j=0}^{2g} x_j.$$

The conjecture involves the solution $(x_0, q_0, \ldots, q_{g-1}) = (x_0(\vec{x}), q_0(\vec{x}), \ldots, q_{g-1}(\vec{x})) \in \mathbb{R}^{g+1}$ of the following system of equations:

$$\begin{cases} \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1g} \\ m_{21} & m_{22} & \cdots & m_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ m_{g1} & m_{g2} & \cdots & m_{gg} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{g-1} \end{pmatrix} = \begin{pmatrix} \tilde{m}_1 \\ \tilde{m}_2 \\ \vdots \\ \tilde{m}_g \end{pmatrix}, \\ \sum_{j=0}^{g+1} q_j x_0^j = 0, \\ x_0 > \max\{x_1, 0\}. \end{cases} \tag{1.3}$$

Let $\mathcal{A} = \{\vec{x} \in \mathbb{R}^{2g} : x_{2g} < \cdots < x_2 < x_1 \text{ and there exists } x_0 \text{ satisfying (1.3)}\}$. For all $g \geq 1$, we expect that $\{\vec{x} \in \mathbb{R}^{2g} : x_{2g} < \cdots < x_2 < x_1 < 0\} \subset \mathcal{A}$, and we completely characterize $\mathcal{A}$ for $g = 1$ in Theorem 1.2 below (see also Proposition 3.1). The conjecture also involves the two-sheeted Riemann surface X of genus g associated to $\sqrt{\mathcal{R}(z)}$ where $\sqrt{\mathcal{R}(z)} > 0$ for $z \in (x_0, +\infty)$ on the first sheet. We define A-cycles and B-cycles on X as in Figure 1 and consider the matrix $\mathbb{A} \in \mathbb{R}^{g \times g}$ given by

$$\mathbb{A}_{jk} := \oint_{A_j} \frac{s^{k-1}}{\sqrt{\mathcal{R}(s)}} ds = 2\sum_{l=j}^{g} \int_{x_{2l-1}}^{x_{2l}} \frac{s^{k-1}}{\sqrt{\mathcal{R}(s)}} ds \in \mathbb{R}, \qquad 1 \leq j,k \leq g.$$

It follows from the general theory of Riemann surfaces [22] that $\mathbb{A}$ is invertible. Consider

$$\begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_g \end{pmatrix} = \frac{dz}{\sqrt{\mathcal{R}(z)}} \begin{pmatrix} 1 & z & \cdots & z^{g-1} \end{pmatrix} \mathbb{A}^{-1}. \tag{1.4}$$

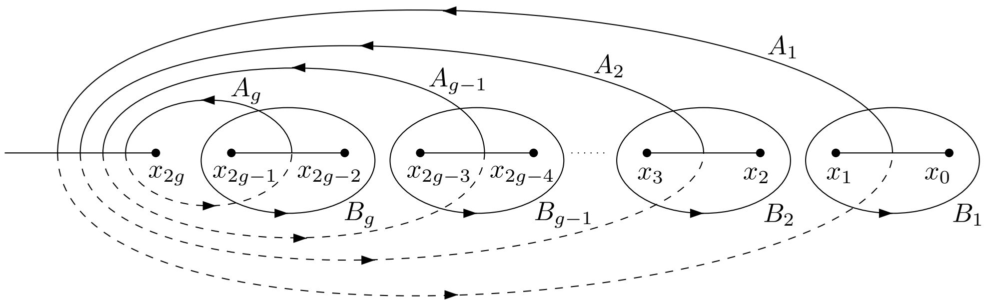

Figure 1: The canonical homology basis on the genus g Riemann surface X. The solid parts lie on the first sheet, and the dashed parts lie on the second sheet.

By definition, the holomorphic differentials ω_k, $k = 1, \ldots, g$, satisfy

$$\oint_{A_j} \omega_k = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \qquad 1 \leq j, k \leq g.$$

The period matrix τ defined by

$$\tau := \left[\oint_{B_j} \omega_k \right]_{j,k=1,\ldots,g} \in i\,\mathbb{R}^{g \times g}, \tag{1.5}$$

is symmetric with positive definite imaginary part [22, page 63]. The Riemann theta function $\theta : \mathbb{C}^g \to \mathbb{C}$ associated to τ is defined by

$$\theta(\vec{z}; \tau) = \theta(\vec{z}) := \sum_{\vec{n} \in \mathbb{Z}^g} e^{i\pi(\vec{n}^t \tau \vec{n} + 2\vec{n}^t \vec{z})}, \qquad \vec{z} \in \mathbb{C}^g, \tag{1.6}$$

where ^{t} denotes the transpose operation. This is an entire function in each component of $\vec{z}$ which satisfies

$$\theta(-\vec{z}) = \theta(\vec{z}) \qquad \text{and} \qquad \theta(\vec{z} + \vec{\lambda}' + \tau\vec{\lambda}) = e^{-i\pi(2\vec{\lambda}^t \vec{z} + \vec{\lambda}^t \tau \vec{\lambda})} \theta(\vec{z}) \tag{1.7}$$

for all $\vec{z} \in \mathbb{C}^g$ and $\vec{\lambda}, \vec{\lambda}' \in \mathbb{Z}^g$.

Conjecture 1.1. Let $g \geq 1$ be a positive integer and let $\vec{x} = (x_1, \ldots, x_{2g}) \in \mathcal{A}$. For $r > 0$, define $\vec{\nu} = \vec{\nu}(r, \vec{x}) = (\nu_0, \ldots, \nu_{g-1})$ by

$$\nu_j = -\frac{\Omega_j}{2\pi} r^{\frac{3}{2}}, \qquad \text{with} \qquad \Omega_j = 2i \int_{x_{2j+1}}^{x_{2j}} \frac{\sum_{k=0}^{g+1} q_k \xi^k}{\sqrt{\mathcal{R}(\xi)}_+} d\xi > 0, \qquad j = 0, \ldots, g-1.$$

Assume that there exist $\delta_1, \delta_2 > 0$ such that

$$\left| \sum_{j=1}^{g} m_j \Omega_{j-1} \right| \geq \delta_1 \left(\sum_{j=1}^{g} m_j^2 \right)^{-\delta_2}, \qquad \text{for all } m_1, \ldots, m_g \in \mathbb{Z} \text{ such that } \sum_{j=1}^{g} m_j \Omega_{j-1} \neq 0. \tag{1.8}$$

Also suppose that $\Omega_0, \Omega_1, \ldots, \Omega_{g-1}$ are rationally independent, that is, if $(n_1, n_2, \ldots, n_g) \in \mathbb{Z}^g$ satisfies

$$\Omega_0 n_1 + \Omega_1 n_2 + \cdots + \Omega_{g-1} n_g = 0, \tag{1.9}$$

then $n_1 = n_2 = \cdots = n_g = 0$. As $r \to +\infty$,

$$F(r\vec{x}) = \exp\left(c\, r^3 - \frac{3g}{8} \log r + \log\theta(\vec{\nu}) + C + \mathcal{O}(r^{-\frac{3}{2}}) \right), \tag{1.10}$$

where $C = C(\vec{x})$ is independent of r and $c = c(\vec{x})$ is given by

$$c = \frac{1}{12} \left[\sum_{j=0}^{2g} x_j^3 - \sum_{0 \leq j < k \leq 2g} (x_j^2 x_k + x_j x_k^2) - 2 \sum_{0 \leq j < k < l \leq 2g} x_j x_k x_l \right] - \frac{2}{3} q_{g-2} - \frac{q_{g-1}}{3} \sum_{j=0}^{2g} x_j, \tag{1.11}$$

with $q_{-1} := 0$. Furthermore, $\{\vec{x} \in \mathbb{R}^{2g} : x_{2g} < \cdots < x_2 < x_1 < 0\} \subset \mathcal{A}$.

Remark 1. If condition (1.8) fails, then the error term $\mathcal{O}(r^{-\frac{3}{2}})$ in (1.10) must be replaced by $o(\log r)$ as in [19]. On the other hand, if condition (1.9) does not hold, from [19] and [7, Theorems 1.1–1.3] we expect that the log-coefficient $-\frac{3g}{8}$ should be replaced by a rather complicated hyperelliptic integral. Again from [19, 7], we expect the map $(x_1, \ldots, x_{2g}) \to (\Omega_0, \ldots, \Omega_{g-1})$ to contain open balls in $(0, +\infty)^g$. Since the set $\{(\Omega_0, \ldots, \Omega_{g-1})\} \subset (0, +\infty)^g$ for which either (1.8) or (1.9) does not hold has Lebesgue measure zero, the asymptotics (1.10) are expected to hold for generic $(x_1, \ldots, x_{2g}) \in \mathcal{A}$. Note that for $g = 1$, both (1.8) and (1.9) are satisfied for *all* $(x_1, x_2) \in \mathcal{A}$.

Remark 2. The fact that the oscillations are described in terms of the Riemann θ-function related to a genus g Riemann surface is not surprising; this also happens in the case of the sine process [19, 21, 30].

Remark 3. The above conjecture has been verified numerically for $g = 1, 2, 3$ for several choices of $\vec{x}$ by using the Bornemann Linear Algebra package [8].

Theorem 1.2. *Let $g = 1$; then Conjecture 1.1 holds and moreover $\mathcal{A}$ is given by*

$$\mathcal{A} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 < x_1 < 0\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 < -2x_1\}.$$

More precisely, taking $(x_1, x_2) \in \mathcal{A}$ and $r \to +\infty$,

$$F(r\vec{x}) = \exp\left(c\, r^3 - \frac{3}{8}\log r + \log\theta(\nu) + C + \mathcal{O}(r^{-\frac{3}{2}})\right), \tag{1.12}$$

where $C = C(\vec{x})$ is independent of r, $c = c(\vec{x})$ is given by

$$c = \frac{1}{12}\left[x_0^3 + x_1^3 + x_2^3 - (x_0 + x_1)(x_0 + x_2)(x_1 + x_2)\right] - \frac{x_0}{6}(x_0 - x_1 - x_2)(x_0 + x_1 + x_2), \tag{1.13}$$

x_0 is the implicit solution to the equation

$$\frac{\boldsymbol{E}(k)}{\boldsymbol{K}(k)} = \frac{-2(x_0 - x_1)}{x_0 + x_1 + x_2}, \qquad k := \sqrt{\frac{x_1 - x_2}{x_0 - x_2}}, \tag{1.14}$$

with $\mathbf{K}, \mathbf{E}$ denoting the complete elliptic integrals of the first, second kind, respectively, and $\nu = \nu(r, \vec{x})$ is given by

$$\nu(r, \vec{x}) = -\frac{\Omega(\vec{x})}{2\pi} r^{\frac{3}{2}}, \qquad \textit{with} \qquad \Omega(\vec{x}) = \frac{\pi\sqrt{x_0 - x_2}(x_0 + x_1 + x_2)}{-3\mathbf{K}(k)}. \tag{1.15}$$

Remark 4. The complete elliptic integrals of the first and second kind are given by (see [23, Eqs. 8.111.2 and 8.111.3])

$$\mathbf{K}(k) := \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \qquad \mathbf{E}(k) := \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 t}\, dt. \tag{1.16}$$

Outline of the paper. In Section 2, we use the method developed by Its, Izergin, Korepin and Slavnov [24], combined with a result of Claeys and Doeraene [13], to express $\partial_r \log F(r\mathcal{I}_1^{(b)}) = \partial_r \log F(r(x_2, x_1))$ in terms of the solution, denoted Ψ, of a 2×2 matrix Riemann-Hilbert (RH) problem. In Sections 3-6, we obtain the asymptotics of Ψ as $r \to +\infty$ via the Deift-Zhou steepest descent method [17, 20]. As a consequence of our RH analysis, we obtain large r asymptotics for $\partial_r \log F(r(x_2, x_1))$. The simplification of these asymptotics is the most challenging part of this work; this is done in Section 7. Finally, we prove Theorem 1.2 by integrating the asymptotics of $\partial_r \log F(r(x_2, x_1))$ with respect to r.

2 Differential identity for F

The method of Its, Izergin, Korepin and Slavnov [24] applies to kernels of the *integrable form* $K(x, y) = \frac{\vec{f}^{\,t}(x)\vec{h}(y)}{x-y}$, where $\vec{f}(x)$ and $\vec{h}(y)$ are column vectors satisfying $\vec{f}^{\,t}(x)\vec{h}(x) = 0$. Given a subset $A \subset \mathbb{R}$, we write χ_A for the characteristic function of A. It is easy to see that

$$K^{\mathrm{Ai}}\big|_{r(x_2,x_1)}(x, y) = K^{\mathrm{Ai}}(x, y)\chi_{r(x_2,x_1)}(y), \qquad x, y \in \mathbb{R},$$

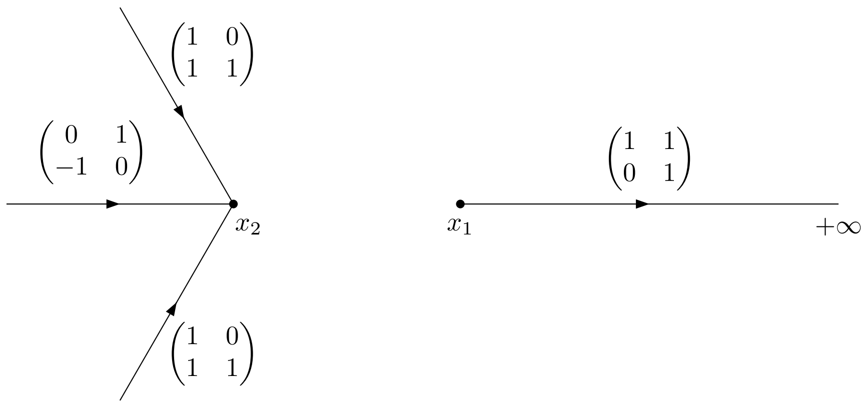

Figure 2: The jump contour Γ for the RH problem for Ψ.

is integrable with $\vec{f}$ and $\vec{h}$ given by

$$\vec{f}(x) = \begin{pmatrix} \mathrm{Ai}(x) \\ \mathrm{Ai}'(x) \end{pmatrix}, \qquad \vec{h}(y) = \begin{pmatrix} \mathrm{Ai}'(y)\chi_{r(x_2,x_1)}(y) \\ -\mathrm{Ai}(y)\chi_{r(x_2,x_1)}(y) \end{pmatrix}.$$

The associated integral operator $\mathcal{K}_r$, acting on $L^2(rx_2, +\infty)$, is given by

$$\mathcal{K}_r\phi(x) = \int_{rx_2}^{+\infty} K^{\mathrm{Ai}}(x,y)\chi_{r(x_2,x_1)}(y)\phi(y)dy, \qquad \phi \in L^2(rx_2, +\infty). \tag{2.1}$$

By definition, $F(r(x_2, x_1)) = \det(I - \mathcal{K}_r) = \mathbb{P}(\text{no points lie in } (rx_2, rx_1)) > 0$. Hence, using standard properties of trace class operators, we obtain

$$\begin{aligned} \partial_r \log\det(I - \mathcal{K}_r) &= -\mathrm{Tr}\left[(I - \mathcal{K}_r)^{-1}\partial_r\mathcal{K}_r\right] = \sum_{j=1}^{2}(-1)^j x_j \mathrm{Tr}\left[(I - \mathcal{K}_r)^{-1}\mathcal{K}_r\delta_{rx_j}\right] \\ &= \sum_{j=1}^{2}(-1)^j x_j \mathrm{Tr}\left[\mathcal{R}_r\delta_{rx_j}\right] = \sum_{j=1}^{2}(-1)^j x_j \lim_{u\to rx_j} R_r(u,u), \end{aligned} \tag{2.2}$$

where the limits $u \to rx_j$, $j = 1, 2$, are taken from the interior of (rx_2, rx_1), δ_{rx_j} is the Dirac delta operator, the integral operator $\mathcal{R}_r$ is given by

$$\mathcal{R}_r := (I - \mathcal{K}_r)^{-1}\mathcal{K}_r = (I - \mathcal{K}_r)^{-1} - I,$$

and R_r is the kernel of $\mathcal{R}_r$. Now, we invoke a result of Claeys and Doeraene [13]: for $u \in (rx_2, rx_1)$, we have

$$R_r(u,u) = \frac{1}{2\pi i}\left(\Psi_+^{-1}\Psi_+'\right)_{21}(u; rx_1, rx_2), \tag{2.3}$$

where Ψ is the solution to the following RH problem.

RH problem for $\Psi = \Psi(\cdot; x_1, x_2)$

(a) $\Psi : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$ is analytic, where

$$\Gamma = e^{\pm\frac{2\pi i}{3}}(x_2, +\infty) \cup (-\infty, x_2] \cup [x_1, +\infty) \tag{2.4}$$

is oriented as in Figure 2.

(b) $\Psi(z)$ has continuous boundary values as $\Gamma\setminus\{x_1, x_2\}$ is approached from the left (+ side) and from the right ($-$ side) and they are related by

$$\Psi_+(z) = \Psi_-(z)\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} \qquad \text{for } z \in e^{\pm\frac{2\pi i}{3}}(x_2,+\infty),$$
$$\Psi_+(z) = \Psi_-(z)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \qquad \text{for } z \in (-\infty,x_2),$$
$$\Psi_+(z) = \Psi_-(z)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \qquad \text{for } z \in (x_1,+\infty).$$

(c) As $z \to \infty$, we have

$$\Psi(z) = \left(I + \mathcal{O}(z^{-1})\right) z^{\frac{1}{4}\sigma_3} M^{-1} e^{-\frac{2}{3}z^{\frac{3}{2}}\sigma_3}, \tag{2.5}$$

where principal branches of $z^{\frac{3}{2}}$ and $z^{\frac{1}{4}}$ are taken, and

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

(d) $\Psi(z) = \mathcal{O}(\log(z-x_j))$ as $z \to x_j$, $j = 1,2$.

By combining (2.2) with (2.3), we arrive at

$$\partial_r \log F(r(x_2,x_1)) = \sum_{j=1}^{2} \frac{(-1)^j x_j}{2\pi i} \lim_{z\to x_j}\left(\Psi_+^{-1}\Psi_+'\right)_{21}(rz; rx_1, rx_2), \tag{2.6}$$

where the limits as $z \to x_j$, $j = 1,2$ are taken with $z \in (x_2,x_1)$.

3 Asymptotic analysis of Ψ: first steps

The goal of this section is to implement the first steps of the steepest descent method for the RH problem for Ψ. In Subsection 3.1, we construct the $\mathfrak{g}$-function and list its important properties. In Subsection 3.2, we use $\mathfrak{g}$ to normalize the RH problem at ∞, and then proceed with the so-called opening of the lenses. We henceforth assume that $g = 1$.

3.1 $\mathfrak{g}$-function

Our first goal is to prove that the system of equations (1.3) has a unique solution $(x_0, q_0) \in \mathbb{R}^+ \times \mathbb{R}$, where $\mathbb{R}^+ = (0,+\infty)$. According to the last equation in (1.3), the degree two polynomial $q(z) := \sum_{j=0}^{g+1} q_j z^j = q_2 z^2 + q_1 z + q_0$ satisfies $q(x_0) = 0$, and is therefore of the form

$$q(z) = -z^2 + \frac{z}{2}(x_0 + x_1 + x_2) + q_0, \qquad q_0 = \frac{x_0}{2}(x_0 - x_1 - x_2), \tag{3.1}$$

for a certain x_0 that will be determined so that the first equation in (1.3) holds, i.e., so that $m_{11}q_0 = \tilde{m}_1$. Defining the function $\mathcal{F} : [x_1,\infty) \to \mathbb{R}$ by

$$\mathcal{F}(x) := \int_{x_2}^{x_1} \frac{\sqrt{x-s}\left(s + \frac{1}{2}(x - x_1 - x_2)\right)}{\sqrt{(x_1-s)(s-x_2)}} ds, \tag{3.2}$$

the equation $m_{11}q_0 = \tilde{m}_1$ can be rewritten as $\mathcal{F}(x_0) = 0$. Hence the following proposition implies that (1.3) has a unique solution $(x_0, q_0) \in \mathbb{R}^+ \times \mathbb{R}$.

Proposition 3.1.

(a) Given $x_2 < x_1 < 0$, there is a unique $x_0 > 0$ such that $\mathcal{F}(x_0) = 0$. Moreover, $x_0 \in (0, x_1 - x_2)$.

(b) Given $x_1 \geq 0$, the equation $\mathcal{F}(x_0) = 0$ admits a solution $x_0 \in (x_1,+\infty)$ if and only if $x_2 < -2x_1$. Moreover, the solution x_0 is unique and satisfies $x_0 \in (x_1, x_1 - x_2)$.

Proof. (a) It is clear that $\mathcal{F}(x)$ is continuous and differentiable on its domain. We will show that $\mathcal{F}(0) < 0$, $\mathcal{F}(x_1 - x_2) > 0$, and that $\mathcal{F}'(x) > 0$ for all $x \in (0, \infty)$ and $\mathcal{F}'(x) \to +\infty$ as $x \to +\infty$, which implies the stated result. The inequality $\mathcal{F}(x_1 - x_2) > 0$ is easy to establish from a direct inspection of (3.2). Also, we note that

$$\mathcal{F}'(x) = \int_{x_2}^{x_1} \frac{3x - x_1 - x_2}{4\sqrt{(x-s)(x_1-s)(s-x_2)}}\, ds, \tag{3.3}$$

which is clearly positive for $x > 0$ because $x_2 < x_1 < 0$, and $\mathcal{F}'(x) \sim \frac{3\pi}{4}\sqrt{x} \to +\infty$ as $x \to +\infty$. To show that $\mathcal{F}(0) < 0$, let $x_* = \frac{1}{2}(x_1 + x_2)$, and note that $\mathcal{F}(0)$ can be written as

$$\mathcal{F}(0) = \left(\int_{x_2}^{x_*} + \int_{x_*}^{x_1} \right) \frac{\sqrt{-s}\,(s - x_*)}{\sqrt{(x_1-s)(s-x_2)}} ds < 0.$$

Since $f(s) = \frac{s - x_*}{\sqrt{(x_1-s)(s-x_2)}}$ satisfies $f(x_* + s) = -f(x_* - s)$ for all $s \in (-\frac{x_1-x_2}{2}, \frac{x_1-x_2}{2})$ and since $\sqrt{-s}$ is positive and decreases as $s \in (x_2, x_1)$ increases, this implies that $\mathcal{F}(0) < 0$.
(b) The numerator in (3.3) is positive if and only if $x > \frac{x_1+x_2}{3}$. Since $x_1 \geq 0$, we have $\frac{x_1+x_2}{3} < x_1$ and therefore $\mathcal{F}'(x) > 0$ for all $x \in (x_1, +\infty)$. By a direct computation we get

$$\mathcal{F}(x_1) = \int_{x_2}^{x_1} \frac{s - \frac{x_2}{2}}{\sqrt{s - x_2}} ds = \frac{1}{3}\sqrt{x_1 - x_2}(2x_1 + x_2).$$

This shows that $\mathcal{F}(x_1) < 0$, and therefore that there exists $x_0 \in (x_1, +\infty)$ such that $\mathcal{F}(x_0) = 0$, if and only if $x_2 < -2x_1$. Since $x_2 < 0$, the inequality $\mathcal{F}(x_1 - x_2) > 0$ still holds, which implies that $x_0 \in (x_1, x_1 - x_2)$. $\square$

Throughout the remainder of this paper, x_0 is defined as in Proposition 3.1. Since

$$q(z) = \tilde{q}(z)(z - x_0), \qquad \tilde{q}(z) := -z - \frac{1}{2}(x_0 - x_1 - x_2),$$

and $x_2 < -\frac{1}{2}(x_0 - x_1 - x_2) < x_1$, we have

$$q(x_2) < 0, \qquad q(x_1) > 0, \qquad q(x_0) = 0, \qquad q'(x_0) = \tilde{q}(x_0) < 0. \tag{3.4}$$

The square root $\sqrt{\mathcal{R}(z)} = \sqrt{z - x_0}\sqrt{z - x_1}\sqrt{z - x_2}$ is analytic on $\mathbb{C} \setminus \big((-\infty, x_2] \cup [x_1, x_0]\big)$, behaves as $\sqrt{\mathcal{R}(z)} \sim z^{\frac{3}{2}}$ as $z \to \infty$, and satisfies the jump relation

$$\sqrt{\mathcal{R}(z)}_+ + \sqrt{\mathcal{R}(z)}_- = 0, \qquad\qquad z \in (-\infty, x_2) \cup (x_1, x_0). \tag{3.5}$$

We define the $\mathfrak{g}$-function by

$$\mathfrak{g}(z) = \int_{x_0}^{z} \frac{q(s)}{\sqrt{\mathcal{R}(s)}} ds, \tag{3.6}$$

where q is given by (3.1), and the path of integration does not cross $(-\infty, x_0]$.

Lemma 3.2. *The $\mathfrak{g}$-function satisfies the following properties:*

1. *$\mathfrak{g}$ is analytic in $\mathbb{C} \setminus (-\infty, x_0]$ and satisfies $\mathfrak{g}(z) = \overline{\mathfrak{g}(\overline{z})}$ for $z \in \mathbb{C} \setminus (-\infty, x_0)$.*
2. *$\mathfrak{g}$ satisfies the jump conditions*

$$\mathfrak{g}_+(z) + \mathfrak{g}_-(z) = 0, \qquad\qquad z \in (-\infty, x_2) \cup (x_1, x_0), \tag{3.7}$$
$$\mathfrak{g}_+(z) - \mathfrak{g}_-(z) = i\Omega, \qquad\qquad z \in (x_2, x_1), \tag{3.8}$$

where $\Omega = 2i \int_{x_1}^{x_0} \mathfrak{g}'_+(s) ds = -2i\mathfrak{g}_+(x_1) > 0$.

3. *As $z \to \infty$, we have*

$$\mathfrak{g}(z) = -\frac{2}{3}z^{\frac{3}{2}} + \frac{1}{4}\left((x_1 - x_2)^2 - x_0(3x_0 - 2x_1 - 2x_2)\right) z^{-\frac{1}{2}} + \mathcal{O}(z^{-\frac{3}{2}}). \tag{3.9}$$

In particular, $\operatorname{Re}\mathfrak{g}(z) \to +\infty$ *as* $z \to \infty$ *along either of the two rays* $\arg(z - x_2) = \pm\frac{2\pi}{3}$.

4. *There exists an open neighborhood* $\mathcal{V} \subset \mathbb{C}$ *of* $(-\infty, x_0)$ *and an* $M > 0$ *such that*

$$\Big\{z \in \mathbb{C} : \arg(z - x_2) \in \Big(-\pi, -\frac{2\pi}{3}\Big] \cup \Big[\frac{2\pi}{3}, \pi\Big) \text{ and } |z| \geq M\Big\} \subset \mathcal{V} \tag{3.10}$$

and such that

$$\operatorname{Re}\mathfrak{g}(z) \geq 0 \qquad \text{for all } z \in \mathcal{V}, \tag{3.11}$$

where equality holds in (3.11) if and only if $z \in (-\infty, x_2] \cup [x_1, x_0]$.

Proof. The analyticity of $\mathfrak{g}$ follows from (3.6). The symmetry $\mathfrak{g}(z) = \overline{\mathfrak{g}(\overline{z})}$ follows from the fact that q has real coefficients. The jumps (3.5) combined with $\int_{x_2}^{x_1} \frac{q(s)}{\sqrt{\mathcal{R}(s)}}ds = -\mathcal{F}(x_0) = 0$ imply (3.7). The jump (3.8) is a consequence of (3.5) and (3.6); the fact that $\Omega > 0$ follows from (3.4) and (3.6). Expanding $\mathfrak{g}'(z)$ as $z \to \infty$, and then integrating these asymptotics gives

$$\mathfrak{g}(z) = -\frac{2}{3}z^{\frac{3}{2}} + \mathfrak{g}_0 + \frac{1}{4}\left((x_1 - x_2)^2 - x_0(3x_0 - 2x_1 - 2x_2)\right) z^{-\frac{1}{2}} + \mathcal{O}(z^{-\frac{3}{2}}) \qquad \text{as } z \to \infty,$$

for some $\mathfrak{g}_0 \in \mathbb{C}$. Equation (3.7) implies that $\mathfrak{g}_0 = 0$, which proves (3.9). Equation (3.9) implies that there exists an $M > 0$ such that $\operatorname{Re}\mathfrak{g}(z) \geq 0$ for all $z \in \mathbb{C}$ with $|z| \geq M$ and $\arg z \in (-\pi, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \pi)$. Also, from (3.7) and the symmetry $\mathfrak{g}(z) = \overline{\mathfrak{g}(\overline{z})}$, we deduce that $\operatorname{Re}\mathfrak{g}_+(z) = \operatorname{Re}\mathfrak{g}_-(z) = 0$ for $z \in (-\infty, x_2] \cup [x_1, x_0]$. Since $q(z) < 0$ for $z \in (-\infty, x_2)$ and $q(z) > 0$ for $z \in (x_1, x_0)$ by (3.4), the imaginary part of $\mathfrak{g}_+(x)$ is decreasing as $x \in (-\infty, x_2) \cup (x_1, x_0)$ increases. Hence, by the Cauchy-Riemann equations, for each $x \in (-\infty, x_2) \cup (x_1, x_0)$, there exists $\epsilon = \epsilon(x) > 0$ such that $\operatorname{Re}\mathfrak{g}(x + iu\epsilon(x)) > 0$ for all $u \in (0, 1]$. Assertion 4 will follow if we can show that ϵ can be chosen independently of x. This can be achieved by a local analysis of $\mathfrak{g}$ near each of the points x_0, x_1, x_2. As $z \to x_0$, since $q(x_0) = 0$, we have $\mathfrak{g}(z) \sim \frac{2}{3}q'(x_0)(z - x_0)^{3/2}/\sqrt{(x_0 - x_1)(x_0 - x_2)}$. Since $q'(x_0) < 0$, there exists a small neighborhood V_0 of x_0 such that $\operatorname{Re}\mathfrak{g}(z) \geq 0$ for $z \in V_0 \cap \{\operatorname{Re} z \leq x_0\}$, with equality only if $z \in [x_1, x_0] \cap V_0$. As $z \to x_1$, we have $\mathfrak{g}(z) \sim 2q(x_1)\sqrt{z - x_1}/\sqrt{(x_1 - x_2)(x_0 - x_1)}$. Recalling that $q(x_1) > 0$, $\operatorname{Re}\mathfrak{g}(z) \geq 0$ in a full open neighborhood V_1 of x_1, with equality only if $z \in [x_1, x_0] \cap V_1$. The local analysis near x_2 is similar. $\square$

Remark 5. For future reference, we note that the equation $F(x_0) = 0$ can be rewritten as

$$\frac{\mathbf{E}(k)}{\mathbf{K}(k)} = \frac{-2(x_0 - x_1)}{x_0 + x_1 + x_2}, \qquad \text{where} \qquad k := \sqrt{\frac{x_1 - x_2}{x_0 - x_2}}, \tag{3.12}$$

and $\mathbf{K}$, $\mathbf{E}$ are the complete elliptic integrals of the first and second kind defined in (1.16). Also, using [23, Eqs. 3.131.6, 3.131.11, 3.132.5, 3.141.23, 8.122] and (3.12), the constant Ω can be written as

$$\Omega = \frac{2\sqrt{x_0 - x_2}(x_0 + x_1 + x_2)}{3\mathbf{K}(k)}(\mathbf{K}(k')\mathbf{K}(k) - \mathbf{K}(k')\mathbf{E}(k) - \mathbf{E}(k')\mathbf{K}(k)) = \frac{\pi\sqrt{x_0 - x_2}(x_0 + x_1 + x_2)}{-3\mathbf{K}(k)}, \tag{3.13}$$

where $k' := \sqrt{\frac{x_0 - x_1}{x_0 - x_2}} = \sqrt{1 - k^2}$.

3.2 Rescaling and opening of the lenses

The first transformation $\Psi \to T$ of the steepest descent analysis is defined by

$$T(z) = \begin{pmatrix} r^{-\frac{1}{4}} & \frac{i}{4}\left(2x_0(x_1 + x_2) - 3x_0^2 + (x_1 - x_2)^2\right) r^{\frac{7}{4}} \\ 0 & r^{\frac{1}{4}} \end{pmatrix} \Psi(rz; rx_1, rx_2) e^{-r^{\frac{3}{2}}\mathfrak{g}(z)\sigma_3}. \tag{3.14}$$

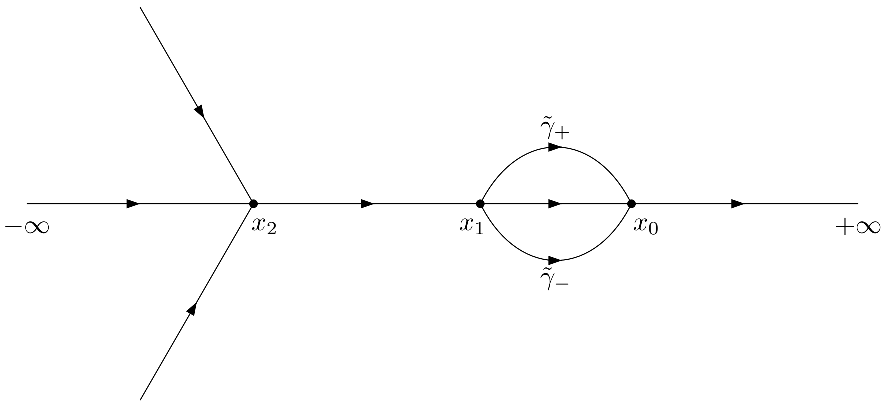

Figure 3: The jump contour Σ_S.

The asymptotics of $T(z)$ as $z \to \infty$ can be computed using the asymptotics (2.5) of Ψ and (3.9) of $\mathfrak{g}$. The first matrix on the right-hand side of (3.14) is chosen to compensate for the behavior (3.9) of $\mathfrak{g}(z)$ as $z \to \infty$. It follows that

$$T(z) = \left(I + \mathcal{O}\left(z^{-1}\right)\right) z^{\frac{\sigma_3}{4}} M^{-1} \qquad \text{as } z \to \infty, \tag{3.15}$$

where the complex powers are defined using the principal branch. Using (3.7), we note the following factorization of the jump $T_-(z)^{-1}T_+(z)$ for $z \in (x_1, x_0)$:

$$\begin{pmatrix} e^{-2r^{\frac{3}{2}}\mathfrak{g}_+(z)} & 1 \\ 0 & e^{-2r^{\frac{3}{2}}\mathfrak{g}_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-2r^{\frac{3}{2}}\mathfrak{g}_-(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-2r^{\frac{3}{2}}\mathfrak{g}_+(z)} & 1 \end{pmatrix}. \tag{3.16}$$

Let $\tilde{\gamma}_+$ and $\tilde{\gamma}_-$ be two simple curves oriented from x_1 to x_0 lying in the upper and lower half-planes, respectively, see Figure 3. The second transformation $T \mapsto S$ is defined by

$$S(z) = T(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ -e^{-2r^{\frac{3}{2}}\mathfrak{g}(z)} & 1 \end{pmatrix}, & z \text{ is below } \tilde{\gamma}_+ \text{ and } \operatorname{Im} z > 0, \\ \begin{pmatrix} 1 & 0 \\ e^{-2r^{\frac{3}{2}}\mathfrak{g}(z)} & 1 \end{pmatrix}, & z \text{ is above } \tilde{\gamma}_- \text{ and } \operatorname{Im} z < 0, \\ I, & \text{otherwise.} \end{cases} \tag{3.17}$$

S satisfies the following RH problem, whose properties can be deduced from those of Ψ, combined with Lemma 3.2, the definition (3.14) of T and the factorization (3.16).

RH problem for S

(a) $S : \mathbb{C}\backslash\Sigma_S \to \mathbb{C}^{2\times 2}$ is analytic, where $\Sigma_S := \mathbb{R} \cup \gamma_+ \cup \gamma_-$ and $\gamma_\pm := \tilde{\gamma}_\pm \cup \left(e^{\pm\frac{2\pi i}{3}}(-\infty, x_2)\right)$. The orientation of Σ_S is shown in Figure 3.

(b) The jumps for S are given by

$$S_+(z) = S_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in (-\infty, x_2) \cup (x_1, x_0), \tag{3.18}$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ e^{-2r^{\frac{3}{2}}\mathfrak{g}(z)} & 1 \end{pmatrix}, \qquad z \in \gamma_+ \cup \gamma_-, \tag{3.19}$$

$$S_+(z) = S_-(z) e^{-i\Omega r^{\frac{3}{2}}\sigma_3}, \qquad z \in (x_2, x_1), \tag{3.20}$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & e^{2r^{\frac{3}{2}}\mathfrak{g}(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in (x_0, \infty). \tag{3.21}$$

(c) As $z \to \infty$, we have $S(z) = \left(I + \mathcal{O}\left(z^{-1}\right)\right) z^{\frac{\sigma_3}{4}} M^{-1}$.

(d) As $z \to x_j$, $j = 0, 1, 2$, we have $S(z) = \mathcal{O}(\log(z - x_j))$.

Choose $\mathcal{V}$ as in Lemma 3.2. Deforming the contours γ_+ and γ_- if necessary, we may assume that they lie in $\mathcal{V}$. Since $\operatorname{Re} \mathfrak{g}(z) \geq 0$ for all $z \in \mathcal{V}$ with equality only if $z \in (-\infty, x_2] \cup [x_1, x_0]$, the jumps for S are exponentially close to I as $r \to +\infty$ on $\gamma_+ \cup \gamma_-$. This convergence is uniform except for z in small neighborhoods of x_j, $j = 0, 1, 2$.

4 Global parametrix

Ignoring the exponentially small jumps for S, and ignoring small neighborhoods of x_0, x_1, x_2, we are led to consider the following RH problem.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \backslash (-\infty, x_0] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The jumps for $P^{(\infty)}$ are given by

$$
\begin{aligned}
& P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && z \in (-\infty, x_2) \cup (x_1, x_0), \\
& P_+^{(\infty)}(z) = P_-^{(\infty)}(z) e^{-i\Omega r^{\frac{3}{2}} \sigma_3}, && z \in (x_2, x_1).
\end{aligned} \tag{4.1}
$$

(c) As $z \to \infty$, we have $P^{(\infty)}(z) = \left(I + \mathcal{O}\left(z^{-1}\right)\right) z^{\frac{\sigma_3}{4}} M^{-1}$.

(d) As $z \to x_j$, $j = 0, 1, 2$, we have $P^{(\infty)}(z) = \mathcal{O}((z - x_j)^{-\frac{1}{4}})$.

The above RH problem was solved in [6] (the points y_1, y_2 and 0 in [6] should be identified with x_0, x_1 and x_2 in the present paper, respectively). Consider the function $\varphi : \mathbb{C} \setminus (-\infty, x_0] \to \mathbb{C}$ defined by

$$
\varphi(z) = \int_{x_0}^{z} \omega, \qquad \omega = \frac{c_0 dz}{\sqrt{\mathcal{R}(z)}}, \qquad c_0 = \frac{\sqrt{x_0 - x_2}}{4\mathbf{K}(k)}, \tag{4.2}
$$

where the path of integration lies in $\mathbb{C} \setminus (-\infty, x_0]$, and k has been defined in (3.12). Note that the holomorphic differential ω_1 of (1.4) with $g = 1$ coincides with ω. By [6], $P^{(\infty)}$ is given by

$$
P^{(\infty)}(z) = \begin{pmatrix} \frac{1}{\mathcal{G}(\frac{1}{2})} & \frac{-ic_{\mathcal{G}}}{\mathcal{G}(0)} \\ 0 & \frac{1}{\mathcal{G}(0)} \end{pmatrix} \frac{\beta(z)^{\sigma_3}}{\sqrt{2}} \begin{pmatrix} \mathcal{G}(-\varphi(z)) & -i\mathcal{G}(\varphi(z)) \\ -i\mathcal{G}(-\varphi(z) - \frac{1}{2}) & \mathcal{G}(\varphi(z) - \frac{1}{2}) \end{pmatrix} \tag{4.3}
$$

with

$$
\mathcal{G}(z) = \frac{\theta(z + \nu)}{\theta(z)}, \quad \nu = -\frac{\Omega}{2\pi} r^{\frac{3}{2}}, \quad c_{\mathcal{G}} = 2c_0 (\log \mathcal{G})'(\tfrac{1}{2}), \quad \beta(z) = \frac{(z - x_0)^{1/4}(z - x_2)^{1/4}}{(z - x_1)^{1/4}}, \tag{4.4}
$$

where the principal branch is taken for the roots. The function θ is defined by (1.6) with $g = 1$, and is associated to the parameter $\tau \in i\mathbb{R}^+$ defined in (1.5).

5 Local parametrices

The goal of this section is to construct local approximations (called "parametrices") of S in small open disks $\mathbb{D}_{x_j}$ centered at x_j, $j = 0, 1, 2$. The local parametrix $P^{(x_j)}$ possesses the same jumps as S inside $\mathbb{D}_{x_j}$, $P^{(x_j)} = \mathcal{O}(\log(z - x_j))$ as $z \to x_j$, and satisfies $P^{(x_j)}(z) P^{(\infty)}(z)^{-1} = I + o(1)$ as $r \to +\infty$ uniformly for $z \in \partial\mathbb{D}_{x_j}$. In our case, the local parametrices are standard: $P^{(x_0)}$ can be built in terms of Airy functions (as in [17]), and $P^{(x_1)}$ and $P^{(x_2)}$ in terms of Bessel functions (as in [19]).

5.1 Parametrix at x_0

The function $f_0(z) := \left(-\frac{3}{2}\mathfrak{g}(z)\right)^{\frac{2}{3}}$ is a conformal map from $\mathbb{D}_{x_0}$ to a neighborhood of 0, and satisfies

$$f_0(z) = c_{x_0}(z-x_0)(1 + c_{x_0}^{(2)}(z-x_0) + c_{x_0}^{(3)}(z-x_0)^2 + \mathcal{O}((z-x_0)^3)) \qquad \text{as } z \to x_0, \tag{5.1}$$

$$c_{x_0} = \frac{(-\tilde{q}(x_0))^{\frac{2}{3}}}{(x_0-x_1)^{\frac{1}{3}}(x_0-x_2)^{\frac{1}{3}}} > 0, \qquad c_{x_0}^{(2)} = \frac{-2x_0+x_1+x_2}{5(x_0-x_1)(x_0-x_2)} - \frac{2}{5\tilde{q}(x_0)}, \tag{5.2}$$

$$175 c_{x_0}^{(3)} = \frac{43x_0^2 - 43x_0x_1 + 17x_1^2 - 43x_0x_2 + 9x_1x_2 + 17x_2^2}{(x_0-x_1)^2(x_0-x_2)^2} + \frac{18(2x_0-x_1-x_2)}{(x_0-x_1)(x_0-x_2)\tilde{q}(x_0)} - \frac{7}{\tilde{q}(x_0)^2},$$

where we recall that $-\tilde{q}(x_0) > 0$. The model RH problem of [17], which is needed for the construction of $P^{(x_0)}$, is presented in Appendix A for convenience; its solution is denoted by Φ_{Ai}. Deforming $\tilde{\gamma}_\pm$ if necessary, we may assume that $f_0(\tilde{\gamma}_\pm \cap \mathbb{D}_{x_0}) \subset e^{\pm\frac{2\pi i}{3}}\mathbb{R}^+$. It can be verified that

$$P^{(x_0)}(z) = E_{x_0}(z)\Phi_{\mathrm{Ai}}(rf_0(z))e^{-r^{\frac{3}{2}}\mathfrak{g}(z)\sigma_3}, \qquad E_{x_0}(z) = P^{(\infty)}(z)M^{-1}\Big(rf_0(z)\Big)^{\frac{\sigma_3}{4}}, \tag{5.3}$$

where $E_{x_0}(z)$ is analytic for $z \in \mathbb{D}_{x_0}$. Furthermore, due to (5.1) and (A.2),

$$P^{(x_0)}(z)P^{(\infty)}(z)^{-1} = I + \frac{P^{(\infty)}(z)\Phi_{\mathrm{Ai},1}P^{(\infty)}(z)^{-1}}{r^{\frac{3}{2}}f_0(z)^{\frac{3}{2}}} + \mathcal{O}(r^{-3}) \tag{5.4}$$

as $r \to +\infty$ uniformly for $z \in \partial\mathbb{D}_{x_0}$.

5.2 Parametrices at x_1 and x_2

We define $\tilde{f}(z) = \frac{1}{4}\left(\mathfrak{g}(z) \mp \frac{i\Omega}{2}\right)^2$, where we take the $-/+$ sign when z is above/below the real axis. As $z \to x_j$, $j = 1,2$, we have

$$\tilde{f}(z) = -c_{x_1}(z-x_1)\big(1 + c_{x_1}^{(2)}(z-x_1) + \mathcal{O}((z-x_1)^2)\big), \quad c_{x_1} = \frac{q^2(x_1)}{(x_1-x_2)(x_0-x_1)}, \quad \text{as } z \to x_1, \tag{5.5}$$

$$\tilde{f}(z) = c_{x_2}(z-x_2)\big(1 + c_{x_2}^{(2)}(z-x_2) + \mathcal{O}((z-x_2)^2)\big), \qquad c_{x_2} = \frac{q^2(x_2)}{(x_0-x_2)(x_1-x_2)}, \quad \text{as } z \to x_2. \tag{5.6}$$

Again, deforming the lenses if necessary, we may assume that $\tilde{f}(\tilde{\gamma}_\pm \cap \mathbb{D}_{x_1}) \subset e^{\mp\frac{2\pi i}{3}}\mathbb{R}^+$ and $\tilde{f}(\gamma_\pm \cap \mathbb{D}_{x_2}) \subset e^{\pm\frac{2\pi i}{3}}\mathbb{R}^+$. The local parametrices $P^{(x_1)}$ and $P^{(x_2)}$ are given by

$$P^{(x_1)}(z) = E_{x_1}(z)\sigma_3\Phi_{\mathrm{Be}}(r^3\tilde{f}(z))\sigma_3 e^{-r^{\frac{3}{2}}g(z)\sigma_3}, \quad E_{x_1}(z) = P^{(\infty)}(z)e^{\pm\frac{i\Omega}{2}r^{\frac{3}{2}}\sigma_3}M\Big(2\pi r^{\frac{3}{2}}\tilde{f}(z)^{\frac{1}{2}}\Big)^{\frac{\sigma_3}{2}}, \tag{5.7}$$

$$P^{(x_2)}(z) = E_{x_2}(z)\Phi_{\mathrm{Be}}(r^3\tilde{f}(z))e^{-r^{\frac{3}{2}}g(z)\sigma_3}, \qquad E_{x_2}(z) = P^{(\infty)}(z)e^{\pm\frac{i\Omega}{2}r^{\frac{3}{2}}\sigma_3}M^{-1}\Big(2\pi r^{\frac{3}{2}}\tilde{f}(z)^{\frac{1}{2}}\Big)^{\frac{\sigma_3}{2}}, \tag{5.8}$$

where $E_{x_j}(z)$ is analytic in $\mathbb{D}_{x_j}$, $j = 1,2$, and Φ_{Be} is the solution of the Bessel model RH problem recalled in Appendix B. As $r \to +\infty$, we have $P^{(x_j)}(z)P^{(\infty)}(z)^{-1} = I + J_R^{(1)}(z)r^{-\frac{3}{2}} + \mathcal{O}(r^{-3})$ uniformly for $z \in \partial\mathbb{D}_{x_j}$, for a certain matrix $J_R^{(1)}(z)$ that will be computed in (6.3).

6 Small norm problem

Let us define

$$R(z) = \begin{cases} S(z)P^{(x_j)}(z)^{-1}, & z \in \mathbb{D}_{x_j},\ j = 0,1,2, \\ S(z)P^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \setminus \bigcup_{j=0}^2 \overline{\mathbb{D}}_{x_j}. \end{cases} \tag{6.1}$$

From the asymptotics of $S(z)$ and $P^{(\infty)}(z)$ as $z \to \infty$, we have $R(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$. Also, by definition of $P^{(x_j)}$, $j = 0,1,2$, $R(z)$ is analytic for $z \in \cup_{j=0}^2 \mathbb{D}_{x_j}$. Therefore, the jump contour for R, denoted by Σ_R, is given by

$$\Sigma_R = \left((x_0,+\infty) \cup \gamma_+ \cup \gamma_- \cup \bigcup_{j=0}^2 \partial\mathbb{D}_{x_j}\right) \setminus \bigcup_{j=0}^2 \mathbb{D}_{x_j},$$

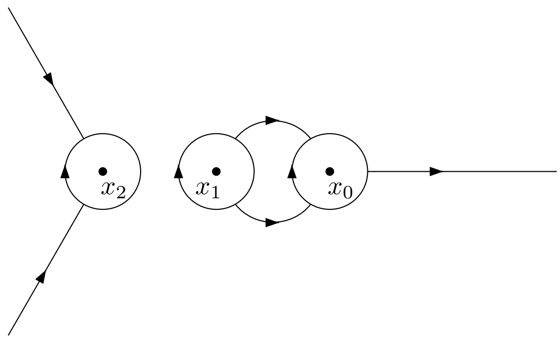

Figure 4: The jump contour Σ_R.

where we orient (for convenience) the boundaries of the disks in the clockwise direction, see Figure 4. It follows from the steepest descent analysis that

$$J_R(z) := R_-(z)^{-1}R_+(z) = \begin{cases} I + \mathcal{O}(e^{-\tilde{c}|rz|^{\frac{3}{2}}}), & \text{uniformly for } z \in \Sigma_R \cup_{j=0}^2 \partial\mathbb{D}_{x_j}, \\ I + \frac{J_R^{(1)}(z)}{r^{\frac{3}{2}}} + \mathcal{O}(r^{-3}), & \text{uniformly for } z \in \cup_{j=0}^2 \partial\mathbb{D}_{x_j}, \end{cases} \tag{6.2}$$

where $\tilde{c} > 0$ is a sufficiently small constant, and $J_R^{(1)}(z)$ is given by

$$J_R^{(1)}(z) = \begin{cases} \frac{P^{(\infty)}(z)\Phi_{\mathrm{Ai},1}P^{(\infty)}(z)^{-1}}{f_0(z)^{\frac{3}{2}}} & \text{if } z \in \partial\mathbb{D}_{x_0}, \\ \frac{P^{(\infty)}(z)e^{\pm\frac{i\Omega}{2}r^{\frac{3}{2}}\sigma_3}\sigma_3^j\Phi_{\mathrm{Be},1}\sigma_3^j e^{\mp\frac{i\Omega}{2}r^{\frac{3}{2}}}P^{(\infty)}(z)^{-1}}{\tilde{f}(z)^{1/2}} & \text{if } z \in \partial\mathbb{D}_{x_j},\ j = 1, 2. \end{cases} \tag{6.3}$$

Since $\Omega \in \mathbb{R}$, $J_R^{(1)}(z) = \mathcal{O}(1)$ as $r \to +\infty$ uniformly for $z \in \cup_{j=0}^2 \partial\mathbb{D}_{x_j}$. The jump matrix J_R is uniformly close to the identity, thus $R(z)$ exists for sufficiently large r and [17, 18, 20]

$$R(z) = I + \frac{R^{(1)}(z)}{r^{\frac{3}{2}}} + \mathcal{O}(r^{-3}), \qquad \text{where} \quad R^{(1)}(z) = \frac{1}{2\pi i}\int_{\bigcup_{j=0}^2 \partial\mathbb{D}_{x_j}} \frac{J_R^{(1)}(\xi)}{\xi - z}\, d\xi, \tag{6.4}$$

as $r \to +\infty$ uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, and these asymptotics can be differentiated with respect to z without changing the error term. In Section 7, we will need explicit expressions for

$$\Big[E_{x_j}(x_j)^{-1}R^{(1)\prime}(x_j)E_{x_j}(x_j)\Big]_{21}, \qquad j = 1, 2. \tag{6.5}$$

These quantities can be computed from (6.3) and (6.4) by residue calculations. Let us define $\mathcal{J}_{x_j}(z) := [E_{x_j}(x_j)^{-1}J_R^{(1)}(z)E_{x_j}(x_j)]_{21}$ for $j = 1, 2$. By (6.3), as $z \to x_{j'}$, $j' = 0, 1, 2$, we have

$$\mathcal{J}_{x_j}(z) = \sum_{k=-2}^{1}(\mathcal{J}_{x_j})_{x_{j'}}^{(k)}(z - x_{j'})^k + \mathcal{O}((z - x_{j'})^2), \qquad (\mathcal{J}_{x_j})_{x_1}^{(-2)} = (\mathcal{J}_{x_j})_{x_2}^{(-2)} = 0, \tag{6.6}$$

for certain matrices $(\mathcal{J}_{x_j})_{x_{j'}}^{(k)}$ that can be computed if needed. Since the circles $\partial\mathbb{D}_{x_j}$, $j = 0, 1, 2$, have clockwise orientation in (6.4), we obtain

$$\Big[E_{x_j}(x_j)^{-1}R^{(1)\prime}(x_j)E_{x_j}(x_j)\Big]_{21} = -(\mathcal{J}_{x_j})_{x_j}^{(1)} + \frac{2(\mathcal{J}_{x_j})_{x_0}^{(-2)}}{(x_0 - x_j)^3} - \sum_{\substack{j'=0 \\ j' \neq j}}^{2}\frac{(\mathcal{J}_{x_j})_{x_{j'}}^{(-1)}}{(x_j - x_{j'})^2}, \qquad j = 1, 2. \tag{6.7}$$

7 Proof of Theorem 1.2

Substituting the transformations $\Psi \mapsto T \mapsto S \mapsto R$ of Sections 3-6 into the differential identity (2.6) and using the expansion (6.4) of R, we obtain the following asymptotics:

$$\partial_r \log F(r(x_2, x_1)) = I_1(r) + I_2(r) + I_3(r) + \mathcal{O}\big(r^{-\frac{5}{2}}\big) \quad \text{as } r \to +\infty, \tag{7.1}$$

where

$$I_1(r) := r^2 \sum_{j=1}^{2} (-1)^j x_j c_{x_j}, \qquad I_2(r) := \frac{1}{2\pi i r^{\frac{5}{2}}} \sum_{j=1}^{2} (-1)^j x_j \Big[E_{x_j}(x_j)^{-1} R^{(1)\prime}(x_j) E_{x_j}(x_j) \Big]_{21},$$

$$I_3(r) := \frac{1}{2\pi i r} \sum_{j=1}^{2} (-1)^j x_j \Big[E_{x_j}(x_j)^{-1} E'_{x_j}(x_j) \Big]_{21},$$

with c_{x_j}, $j = 1, 2$, given by (5.5)-(5.6). A straightforward calculation shows that

$$I_1(r) = c \partial_r r^3, \tag{7.2}$$

where

$$c = \frac{1}{12} \left[x_0^3 + x_1^3 + x_2^3 - (x_0 + x_1)(x_0 + x_2)(x_1 + x_2) \right] - \frac{q_0}{3}(x_0 + x_1 + x_2), \tag{7.3}$$

and the constant q_0 is defined in (3.1). By a simple rearrangement of the terms, it is easy to see that c in (7.3) coincides with the constant c defined in (1.11) for $g = 1$. This establishes the leading term in (1.10) for $g = 1$; the two subleading terms are more complicated to evaluate.

7.1 Explicit expression for $I_3(r)$

From the definition of E_{x_j}, $j = 1, 2$, it is possible to obtain explicit expressions for $E_{x_j}(x_j)$ and $E'_{x_j}(x_j)$. These expressions are rather long, so we only write down $E_{x_1}(x_1)$ and $E'_{x_1}(x_1)$ (the expressions for $E_{x_2}(x_2)$ and $E'_{x_2}(x_2)$ are similar):

$$E_{x_1}(x_1) = e^{i\pi\nu} e^{-\frac{i\pi}{4}\sigma_3} \left[\frac{\beta_{x_1}^{(-\frac{1}{4})}}{\mathcal{G}(\frac{1}{2})} \begin{pmatrix} \mathcal{G}(\frac{\tau}{2}) & \varphi_{x_1}^{(\frac{1}{2})} \mathcal{G}'(\frac{\tau}{2}) \\ 0 & 0 \end{pmatrix} \right.$$
$$\left. + \frac{1}{\beta_{x_1}^{(-\frac{1}{4})} \mathcal{G}(0)} \begin{pmatrix} c_{\mathcal{G}} & c_{\mathcal{G}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\tilde{\mathcal{G}}(\frac{1+\tau}{2})}{\varphi_{x_1}^{(\frac{1}{2})}} & 0 \\ 0 & \tilde{\mathcal{G}}'(\frac{1+\tau}{2}) \end{pmatrix} \right] \left(2\pi r^{\frac{3}{2}} c_{x_1}^{\frac{1}{2}} \right)^{\frac{\sigma_3}{2}},$$

$$E'_{x_1}(x_1) = e^{i\pi\nu} e^{-\frac{i\pi}{4}\sigma_3} \left[\begin{pmatrix} \mathcal{G}(\frac{\tau}{2}) & \varphi_{x_1}^{(\frac{1}{2})} \mathcal{G}'(\frac{\tau}{2}) \\ 0 & 0 \end{pmatrix} \left(\frac{\beta_{x_1}^{(\frac{3}{4})}}{\mathcal{G}(\frac{1}{2})} I + \frac{c_{x_1}^{(2)} \beta_{x_1}^{(-\frac{1}{4})}}{4\mathcal{G}(\frac{1}{2})} \sigma_3 \right) + \right.$$
$$+ \frac{\beta_{x_1}^{(-\frac{1}{4})}}{\mathcal{G}(\frac{1}{2})} \begin{pmatrix} \frac{1}{2} (\varphi_{x_1}^{(\frac{1}{2})})^2 \mathcal{G}''(\frac{\tau}{2}) & \varphi_{x_1}^{(\frac{3}{2})} \mathcal{G}'(\frac{\tau}{2}) + \frac{1}{6} (\varphi_{x_1}^{(\frac{1}{2})})^3 \mathcal{G}'''(\frac{\tau}{2}) \\ 0 & 0 \end{pmatrix} + \frac{1}{\beta_{x_1}^{(-\frac{1}{4})} \mathcal{G}(0)} \begin{pmatrix} c_{\mathcal{G}} & c_{\mathcal{G}} \\ 1 & 1 \end{pmatrix} \times$$
$$\left. \times \begin{pmatrix} \left(\frac{c_{x_1}^{(2)}}{4} - \frac{\beta_{x_1}^{(\frac{3}{4})}}{\beta_{x_1}^{(-\frac{1}{4})}} - \frac{\varphi_{x_1}^{(\frac{3}{2})}}{\varphi_{x_1}^{(\frac{1}{2})}} \right) \frac{\tilde{\mathcal{G}}(\frac{1+\tau}{2})}{\varphi_{x_1}^{(\frac{1}{2})}} + \frac{\varphi_{x_1}^{(\frac{1}{2})}}{2} \tilde{\mathcal{G}}''(\frac{1+\tau}{2}) & 0 \\ 0 & \frac{(\varphi_{x_1}^{(\frac{1}{2})})^2}{6} \tilde{\mathcal{G}}'''(\frac{1+\tau}{2}) - \left(\frac{\beta_{x_1}^{(\frac{3}{4})}}{\beta_{x_1}^{(-\frac{1}{4})}} + \frac{c_{x_1}^{(2)}}{4} \right) \tilde{\mathcal{G}}'(\frac{1+\tau}{2}) \end{pmatrix} \right] \left(2\pi r^{\frac{3}{2}} c_{x_1}^{\frac{1}{2}} \right)^{\frac{\sigma_3}{2}},$$

where $\widetilde{\mathcal{G}}(z) = \mathcal{G}(z)(z - \frac{1+\tau}{2})$, the coefficients $\beta_{x_1}^{(-\frac{1}{4})}$, $\beta_{x_1}^{(\frac{3}{4})}$, $\varphi_{x_1}^{(\frac{1}{2})}$, $\varphi_{x_1}^{(\frac{3}{2})}$, $c_{x_1}^{(2)}$ are defined through the expansions

$$\beta(z) = \beta_{x_1}^{(-\frac{1}{4})} (z - x_1)^{-\frac{1}{4}} + \beta_{x_1}^{(\frac{3}{4})} (z - x_1)^{\frac{3}{4}} + \mathcal{O}((z - x_1)^{\frac{7}{4}}), \qquad \text{as } z \to x_1, \ \operatorname{Im} z > 0,$$
$$\varphi(z) = \varphi_{x_1}^{(\frac{1}{2})} (z - x_1)^{\frac{1}{2}} + \varphi_{x_1}^{(\frac{3}{2})} (z - x_1)^{\frac{3}{2}} + \mathcal{O}((z - x_1)^{\frac{5}{2}}), \qquad \text{as } z \to x_1, \ \operatorname{Im} z > 0,$$

and via (5.5), and they are given by

$$\beta_{x_1}^{(-\frac{1}{4})} = e^{\frac{\pi i}{4}} (x_0 - x_1)^{\frac{1}{4}} (x_0 - x_2)^{\frac{1}{4}}, \quad \beta_{x_1}^{(\frac{3}{4})} = \frac{e^{\frac{\pi i}{4}} (x_0 - 2x_1 + x_2)}{4(x_0 - x_1)^{\frac{3}{4}} (x_1 - x_2)^{\frac{3}{4}}}, \quad \varphi_{x_1}^{(\frac{1}{2})} = \frac{-2ic_0}{\sqrt{x_0 - x_1}\sqrt{x_1 - x_2}},$$
$$\varphi_{x_1}^{(\frac{3}{2})} = \frac{ic_0(x_0 - 2x_1 + x_2)}{3(x_0 - x_1)^{\frac{3}{2}} (x_1 - x_2)^{\frac{3}{2}}}, \qquad c_{x_1}^{(2)} = \frac{2q'(x_1)}{3q(x_1)} - \frac{x_0 - 2x_1 + x_2}{3(x_0 - x_1)(x_1 - x_2)}.$$

Each of the expressions for $E_{x_1}(x_1)$, $E_{x_2}(x_2)$, $E'_{x_1}(x_1)$ and $E'_{x_2}(x_2)$ contain the θ-function and its derivatives at the points 0, $\frac{1}{2}$, $\frac{\tau}{2}$ and $\frac{1+\tau}{2}$. We use θ-function identities to simplify these expressions following the strategy of [6].

The function φ can be analytically continued to the Riemann surface X defined in the introduction, and its quotient $\varphi_A(z) := \varphi(z) \mod (\mathbb{Z}+\tau\mathbb{Z})$ is a bijection from X to $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ [22]. The following proposition can be taken straight from [6] after replacing y_1, y_2 and z with x_0-x_2, x_1-x_2 and $z-x_2$, respectively, and noting that $\varphi_A(z-x_2)$ in [6] equals $\varphi_A(z)$ here.

Proposition 7.1 (Some θ-function identities). *For all $\hat{\nu}\in X$, we have*

$$\theta(\hat{\nu}+\tfrac{\tau}{2})^2 = e^{-2i\pi\hat{\nu}}\frac{D_1^2}{\hat{a}-x_2}\theta(\hat{\nu})^2, \qquad D_1 = (x_0-x_2)^{\frac14}(x_1-x_2)^{\frac14}e^{-\frac{\pi i\tau}{4}}, \tag{7.4a}$$

$$\theta(\hat{\nu}+\tfrac{1+\tau}{2})^2 = e^{-2i\pi\hat{\nu}}\frac{D_2^2(\hat{a}-x_0)}{\hat{a}-x_2}\theta(\hat{\nu})^2, \qquad D_2 = i\frac{(x_1-x_2)^{\frac14}e^{-\frac{\pi i\tau}{4}}}{(x_0-x_1)^{\frac14}}, \tag{7.4b}$$

$$\theta(\hat{\nu}+\tfrac{1}{2})^2 = \frac{D_3^2(\hat{a}-x_1)}{\hat{a}-x_2}\theta(\hat{\nu})^2, \qquad D_3 = \frac{(x_0-x_2)^{\frac14}}{(x_0-x_1)^{\frac14}}. \tag{7.4c}$$

where $\hat{a}=\varphi_A^{-1}(\hat{\nu})$. These expressions make it possible to express $\theta^{(j)}(\frac{1}{2})$, $\theta^{(j)}(\frac{\tau}{2})$, $\theta^{(j)}(\frac{1+\tau}{2})$ for $j\geq 0$ in terms of $\theta^{(j)}(0)$, $j\geq 0$. For example,

$$\theta\Big(\frac{\tau}{2}\Big) = e^{-\frac{\pi i\tau}{4}}\frac{(x_1-x_2)^{\frac14}}{(x_0-x_2)^{\frac14}}\theta(0), \qquad \theta\Big(\frac{1}{2}\Big) = \frac{(x_0-x_1)^{\frac14}}{(x_0-x_2)^{\frac14}}\theta(0), \qquad \theta'\Big(\frac{\tau}{2}\Big) = -i\pi e^{-\frac{\pi i\tau}{4}}\frac{(x_1-x_2)^{\frac14}}{(x_0-x_2)^{\frac14}}\theta(0),$$

$$\theta'\Big(\frac{1+\tau}{2}\Big) = ie^{-\frac{\pi i\tau}{4}}\frac{(x_0-x_1)^{\frac14}(x_1-x_2)^{\frac14}}{2c_0}\theta(0), \qquad \theta''\Big(\frac{1+\tau}{2}\Big) = \pi e^{-\frac{\pi i\tau}{4}}\frac{(x_0-x_1)^{\frac14}(x_1-x_2)^{\frac14}}{c_0}\theta(0),$$

$$\theta''\Big(\frac{\tau}{2}\Big) = e^{-\frac{\pi i\tau}{4}}\frac{(x_1-x_2)^{\frac14}}{(x_0-x_2)^{\frac14}}\left(\theta''(0)-\left[\pi^2+\frac{x_0-x_1}{4c_0^2}\right]\theta(0)\right),$$

$$\theta''\Big(\frac{1}{2}\Big) = \frac{(x_0-x_1)^{\frac14}}{4c_0^2(x_0-x_2)^{\frac14}}\Big((x_1-x_2)\theta(0)+4c_0^2\theta''(0)\Big).$$

As previously mentioned, we compute $E_{x_j}(x_j)$, $E'_{x_j}(x_j)$, $j=1,2$, via (5.7), (5.8) and now use Proposition 7.1 to obtain

$$\frac{-x_1}{2\pi ir}\Big[E_{x_1}(x_1)^{-1}E'_{x_1}(x_1)\Big]_{21} = \frac{-8\pi c_0x_1q(x_1)}{3\Omega(x_0-x_1)(x_1-x_2)}\frac{d}{dr}\left[\log\theta(\nu)\right],$$

$$\frac{x_2}{2\pi ir}\Big[E_{x_2}(x_2)^{-1}E'_{x_2}(x_2)\Big]_{21} = \frac{8\pi c_0x_2q(x_2)}{3\Omega(x_0-x_2)(x_1-x_2)}\frac{d}{dr}\left[\log\theta(\nu)\right],$$

where ν is defined in (4.4). It follows that $I_3(r)=c_3\,\partial_r\log\theta(\nu)$, where the constant c_3 is given by

$$c_3 = \frac{-8\pi c_0}{3\Omega(x_1-x_2)}\left(\frac{x_1q(x_1)}{x_0-x_1}-\frac{x_2q(x_2)}{x_0-x_2}\right) = -\frac{4\pi c_0}{3\Omega}(x_0+x_1+x_2).$$

Using (3.13) and (4.2), we find

$$\frac{\Omega}{c_0} = -\frac{4\pi}{3}(x_0+x_1+x_2),$$

which shows that $c_3=1$. This proves that

$$I_3(r) = \partial_r\log\theta(\nu). \tag{7.5}$$

7.2 Analysis of $I_2(r)$

By (6.7), we have

$$I_2(r) = \frac{1}{2\pi ir^{\frac52}}\left\{\sum_{j=1}^{2}(-1)^{j+1}x_j(\mathcal{J}_{x_j})^{(1)}_{x_j}+\sum_{j=1}^{2}\frac{2(-1)^jx_j(\mathcal{J}_{x_j})^{(-2)}_{x_0}}{(x_0-x_j)^3}+\sum_{j=1}^{2}\sum_{\substack{j'=0\\ j'\neq j}}^{2}\frac{(-1)^{j+1}x_j(\mathcal{J}_{x_j})^{(-1)}_{x_{j'}}}{(x_j-x_{j'})^2}\right\}. \tag{7.6}$$

To simplify the right-hand side, we need Proposition 7.1 as well as the following identities.

Proposition 7.2. *We have*

$$\frac{\theta''(0)}{\theta(0)} = -\frac{(x_1-x_2)(\gamma x_0+\delta)}{4c_0^2(\gamma x_2+\delta)} = \frac{(x_0-x_1)(3x_0+x_1-x_2)}{4c_0^2(x_0+x_1+x_2)}, \tag{7.7}$$

$$\frac{d}{dz}\left[\frac{\theta'(\varphi(z))}{\theta(\varphi(z))}\right] = \frac{\gamma z+\delta}{z-x_2}\varphi'(z), \qquad z\in X, \tag{7.8}$$

where

$$\gamma = \frac{(x_0-x_2)(3x_0-x_1+x_2)}{4c_0^2(x_0+x_1+x_2)}, \qquad \delta = -\frac{(x_0-x_2)(x_0(x_1+2x_2)+x_1(x_1-x_2))}{4c_0^2(x_0+x_1+x_2)}.$$

Proof. We will first show that there exist $\gamma,\delta\in\mathbb{C}$ such that

$$\frac{\theta''(\varphi(z))\theta(\varphi(z))-\theta'(\varphi(z))^2}{\theta(\varphi(z))^2} = \frac{\gamma z+\delta}{z-x_2}. \tag{7.9}$$

Let $h(z)$ denote the left-hand side of (7.9). Using (1.7) and the fact that $\frac{1+\tau}{2}$ is a simple zero of $\theta(u)$, we verify that $\big(\frac{\theta'(u)}{\theta(u)}\big)' = \frac{\theta''(u)\theta(u)-\theta'(u)^2}{\theta(u)^2}$ is well-defined on $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with a double pole at $\frac{1+\tau}{2}$ and no other poles. Since the Abel map $\varphi_A : X\to\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ is an isomorphism, it follows that h is a meromorphic function on X with a double pole at $\varphi_A^{-1}(\frac{1+\tau}{2}) = x_2$ and no other poles. In fact, since $\theta(u)$ is an even function of u and $\varphi(z) = -\varphi(\iota(z))$, where $\iota : X\to X$ denotes the sheet-changing involution, h descends to a meromorphic function on the Riemann sphere with a simple pole at x_2 and no other poles. This shows that (7.9), and hence also (7.8), holds for some choice of $\gamma,\delta\in\mathbb{C}$.

Taking $z\to\infty$ in (7.9) and using Proposition 7.1, we see that

$$\gamma = \frac{\theta''(\frac{1}{2})}{\theta(\frac{1}{2})} = \frac{x_1-x_2}{4c_0^2} + \frac{\theta''(0)}{\theta(0)}. \tag{7.10}$$

Multiplying (7.9) by $\varphi'(z)$, integrating from x_0 to $x>x_0$, and using (4.2) and [23, Eqs. 3.131.7, 3.133.17], we obtain

$$\frac{\theta'(\varphi(x))}{\theta(\varphi(x))} = \frac{\gamma x_1+\delta}{x_1-x_2}\varphi(x) + \frac{2(\gamma x_2+\delta)(x-x_0)}{x_0-x_2}\varphi'(x) - \frac{\gamma x_2+\delta}{2(x_1-x_2)}\frac{\mathbf{E}\left(\sin^{-1}\sqrt{\frac{x-x_0}{x-x_1}},k\right)}{\mathbf{K}(k)}, \tag{7.11}$$

where $k=\sqrt{\frac{x_1-x_2}{x_0-x_2}}$, and $\mathbf{E}(\varphi,k) := \int_0^\varphi \sqrt{1-k^2\sin^2 t}\, dt$ is the elliptic integral of the second kind. Sending $x\to\infty$ in (7.11) and using (3.12), we obtain δ in terms of γ. Then, taking $z\to x_0$ in (7.9), we find (7.7). Substituting (7.7) into (7.10), we find the asserted expression for γ and as an immediate consequence we find the asserted expression for δ. □

The quantity $a := \varphi_A^{-1}(\nu)\in X$ appears in the large r asymptotics of (7.6). From (4.2), we note that $\varphi(x)$ is monotone for $x\in(x_0,+\infty)$, and satisfies $\varphi_A(x_0)=0$ and $\varphi_A(\infty)=\frac{1}{2}$. Hence, as r increases, a oscillates between x_0 and $+\infty$ (and changes sheet). More precisely, a belongs to the upper sheet and satisfies $\sqrt{\mathcal{R}(a)}>0$ if $\nu \bmod 1\in(0,\frac{1}{2})$, while a belongs to the lower sheet and satisfies $\sqrt{\mathcal{R}(a)}<0$ if $\nu \bmod 1\in(\frac{1}{2},1)$.

Proposition 7.3. *We have*

$$\begin{aligned} I_2(r) = \frac{d}{dr}\Bigg\{\tilde{c}_0\log(r) + \int^r \Bigg[&\frac{\tilde{c}_{-2}}{\tilde{r}(\tilde{a}-x_2)^2} + \frac{\tilde{c}_{-1}}{\tilde{r}(\tilde{a}-x_2)} + \frac{\tilde{c}_1(\tilde{a})}{\tilde{r}}\frac{\theta'(\tilde{\nu})}{\theta(\tilde{\nu})} \\ &+ \frac{\tilde{c}_2(\tilde{a})}{\tilde{r}^{\frac{3}{2}}}\frac{d}{d\tilde{r}}\left(\frac{\theta'(\tilde{\nu})}{\theta(\tilde{\nu})}\right) + \frac{\tilde{c}_3}{\tilde{r}^{\frac{3}{2}}}\frac{d}{d\tilde{r}}\left(\frac{\theta'(\tilde{\nu})^3}{\theta(\tilde{\nu})^3}\right)\Bigg]d\tilde{r}\Bigg\} \end{aligned} \tag{7.12}$$

where $\tilde{\nu} = -\frac{\Omega}{2\pi}\tilde{r}^{\frac{3}{2}}$, $\tilde{a} = \varphi_A^{-1}(\tilde{\nu})$, *and*

$$\tilde{c}_{-2} = \frac{17(x_0-x_2)(x_1-x_2)\,(x_1\tilde{q}(x_1)-x_2\tilde{q}(x_2))}{48\tilde{q}(x_0)(x_0-x_1)},$$

$$\begin{aligned}\tilde{c}_{-1} = \frac{1}{48}\Bigg\{&\frac{3x_2q(x_2)}{q(x_1)} - \frac{3x_1\tilde{q}(x_1)}{\tilde{q}(x_2)} + \frac{x_1\tilde{q}(x_1)(-29x_0^2-20x_0x_1+21x_0x_2-11x_1^2+9x_1x_2+30x_2^2)}{\tilde{q}(x_0)(x_0-x_1)(x_0+x_1+x_2)} \\ &+ \frac{x_2\tilde{q}(x_2)\left(29x_0^2+13x_0x_1-14x_0x_2+4x_1^2-9x_1x_2-23x_2^2\right)}{\tilde{q}(x_0)(x_0-x_1)(x_0+x_1+x_2)} + \frac{3(x_0-x_2)(x_1\tilde{q}(x_1)-x_2\tilde{q}(x_2))}{\tilde{q}(x_0)^2}\Bigg\},\end{aligned}$$

$$
\begin{aligned}
\tilde{c}_0 = {} & \frac{1}{48}\Bigg\{ \frac{2x_2\tilde{q}(x_2)\left(-11x_0^2+2x_0(x_1+x_2)+3x_1^2+x_1x_2+3x_2^2\right)+7x_1\tilde{q}(x_1)(x_0-x_2)(x_0+x_1+x_2)}{\tilde{q}(x_0)(x_0-x_1)(x_0-x_2)(x_0+x_1+x_2)} \\
& + \frac{3x_2\tilde{q}(x_2)}{\tilde{q}(x_0)^2} + \frac{3x_1\tilde{q}(x_1)}{\tilde{q}(x_2)(x_1-x_2)} - \frac{3x_2 q(x_2)}{q(x_1)(x_1-x_2)} + 3x_0\left(\frac{6}{x_0+x_1+x_2}+\frac{1}{x_0-x_1}+\frac{1}{x_0-x_2}\right) \\
& - \frac{3x_1 q'(x_1)}{q(x_1)} - \frac{3x_2 q'(x_2)}{q(x_2)}\Bigg\}, \\
\tilde{c}_1(\tilde{a}) = {} & \frac{c_0\left(x_1\tilde{q}(x_1)-x_2\tilde{q}(x_2)\right)}{\tilde{q}(x_0)(x_0-x_1)}\frac{\sqrt{\mathcal{R}(\tilde{a})}}{(\tilde{a}-x_2)^2}, \\
\tilde{c}_2(\tilde{a}) = {} & \frac{-\pi c_0^2}{9\Omega\tilde{q}(x_0)(x_0-x_1)}\left(\frac{5(x_1\tilde{q}(x_1)-x_2\tilde{q}(x_2))}{\tilde{a}-x_2}+\frac{5x_2\tilde{q}(x_2)-9x_0\tilde{q}(x_0)}{x_0-x_2}\right), \\
\tilde{c}_3 = {} & \frac{-8\pi c_0^4(x_0+x_1+x_2)}{9\Omega\tilde{q}(x_0)(x_0-x_1)(x_0-x_2)}.
\end{aligned}
$$

Proof. First, using (6.3), we compute the coefficients in the expansion of $J_R^{(1)}(z)$ as $z\to x_j$, $j=1,2$. By combining these coefficients with the expressions for $E_{x_j}(x_j)$ (already needed in Subsection 7.1), this gives us expressions for the quantities $(\mathcal{J}_{x_j})_{x_j}^{(1)}$, $(\mathcal{J}_{x_j})_{x_0}^{(-2)}$ and $(\mathcal{J}_{x_j})_{x_{j'}}^{(-1)}$, for $j=1,2$ and $j'=0,1,2$, $j\neq j'$. These expressions contain the θ-function and its derivatives evaluated at the eight points 0, $\frac{1}{2}$, $\frac{\tau}{2}$, $\frac{1+\tau}{2}$, ν, $\nu+\frac{1}{2}$, $\nu+\frac{\tau}{2}$ and $\nu+\frac{1+\tau}{2}$. Second, we use Proposition 7.1 to express the quantities $\theta^{(j)}(\frac{1}{2})$, $\theta^{(j)}(\frac{\tau}{2})$, $\theta^{(j)}(\frac{1+\tau}{2})$, $\theta^{(j)}(\nu+\frac{1}{2})$, $\theta^{(j)}(\nu+\frac{\tau}{2})$ and $\theta^{(j)}(\nu+\frac{1+\tau}{2})$ in terms of only a, $\theta^{(j)}(0)$ and $\theta^{(j)}(\nu)$, $j\geq 0$. This allows us to obtain the following simplified expressions:

$$
(\mathcal{J}_{x_1})_{x_1}^{(1)} = \frac{i\pi r^{\frac{3}{2}}}{-8}\left[\frac{q'(x_1)}{q(x_1)} + \frac{4c_0^2\left(\frac{\theta'(\nu)}{\theta(\nu)}\right)^2+12c_0^2\left(\frac{\theta''(\nu)}{\theta(\nu)}-\frac{\theta''(0)}{\theta(0)}\right)+x_0-2x_1+x_2}{(x_0-x_1)(x_1-x_2)}\right],
$$

$$
(\mathcal{J}_{x_2})_{x_2}^{(1)} = \frac{i\pi r^{\frac{3}{2}}}{8}\left[\frac{q'(x_2)}{q(x_2)} - \frac{4c_0^2\left(\frac{\theta'(\nu)}{\theta(\nu)}\right)^2+12c_0^2\left(\frac{\theta''(\nu)}{\theta(\nu)}-\frac{\theta''(0)}{\theta(0)}\right)+x_0-2x_1+x_2}{(x_0-x_1)(x_1-x_2)}\right],
$$

$$
(\mathcal{J}_{x_1})_{x_0}^{(-2)} = \frac{5\pi i r^{\frac{3}{2}}(x_0-x_1)(x_0-x_2)q(x_1)}{-24(a-x_2)\tilde{q}(x_0)}, \qquad (\mathcal{J}_{x_2})_{x_0}^{(-2)} = \frac{5\pi i r^{\frac{3}{2}}(a-x_0)(x_0-x_2)q(x_2)}{24(a-x_2)\tilde{q}(x_0)},
$$

$$
(\mathcal{J}_{x_2})_{x_1}^{(-1)} = \frac{i\pi r^{\frac{3}{2}}(x_1-x_2)q(x_2)}{8q(x_1)}\left(1-\frac{x_1-x_2}{a-x_2}\right), \quad (\mathcal{J}_{x_1})_{x_2}^{(-1)} = \frac{i\pi r^{\frac{3}{2}}(x_1-x_2)\tilde{q}(x_1)}{8\tilde{q}(x_2)}\left(1-\frac{x_1-x_2}{a-x_2}\right).
$$

The expressions for $(\mathcal{J}_{x_j})_{x_0}^{(-1)}$, $j=1,2$, are significantly larger and also contain the quantities $\left(\frac{\theta'(\nu)}{\theta(\nu)}\right)^2$, $\frac{\theta''(\nu)}{\theta(\nu)}$, $\frac{\theta''(0)}{\theta(0)}$ and a. In addition, they contain terms involving $\sqrt{\mathcal{R}(a)}\frac{\theta'(\nu)}{\theta(\nu)}$. Third, we use (7.7) to express $\frac{\theta''(0)}{\theta(0)}$ in terms of the x_j, and we also use the relation $\frac{\theta''(\nu)}{\theta(\nu)} = \left(\frac{\theta'(\nu)}{\theta(\nu)}\right)' + \frac{\theta'(\nu)^2}{\theta(\nu)^2}$. Collecting all of the terms involving $\frac{\theta'(\nu)^2}{\theta(\nu)^2}$, we obtain

$$
\frac{3\tilde{c}_3}{r^{\frac{3}{2}}}\frac{\theta'(\nu)^2}{\theta(\nu)^2}\frac{\gamma a+\delta}{a-x_2}\partial_r\nu,
$$

which can be rewritten as $\frac{\tilde{c}_3}{r^{\frac{3}{2}}}\frac{d}{dr}\left[\frac{\theta'(\nu)^3}{\theta(\nu)^3}\right]$ by using (7.8). □

The next proposition generalizes [6, Proposition 8.6].

Proposition 7.4. *Let $M>0$ and let $\mathcal{H}\in L^1([0,1])$ be a periodic function of period 1 satisfying $\mathcal{H}(-u)=\mathcal{H}(u)$ for all $u\in[0,1]$. As $r\to+\infty$,*

$$
\int_M^r \mathcal{H}(\tilde{\nu})\frac{d\tilde{r}}{\tilde{r}} = \log(r)\int_0^1\mathcal{H}(s)ds + \tilde{C} + \mathcal{O}(r^{-\frac{3}{2}}),
$$

where $\tilde{\nu} = -\frac{\Omega}{2\pi}\tilde{r}^{\frac{3}{2}}$ and $\tilde{C}$ is a constant independent of r. In particular, there are no oscillations of order 1.

Proof. Recall that $\Omega > 0$. Hence,

$$\int_M^r \mathcal{H}\left(-\frac{\Omega}{2\pi}\tilde{r}^{\frac{3}{2}}\right)\frac{d\tilde{r}}{\tilde{r}} = \frac{2}{3}\int_{M'}^{\nu}\mathcal{H}(\tilde{\nu})\frac{d\tilde{\nu}}{\tilde{\nu}} = \frac{2}{3}\left(\int_{M'}^{n_0} + \int_{n_r}^{\nu}\right)\mathcal{H}(\tilde{\nu})\frac{d\tilde{\nu}}{\tilde{\nu}} + \frac{2}{3}\sum_{j=n_0}^{n_r-1}\int_0^{-1}\frac{\mathcal{H}(s)}{s+j}ds,$$

where $M' = -\frac{\Omega}{2\pi}M^{\frac{3}{2}} < 0$, $n_0 = \lfloor M' \rfloor$ and $n_r = \lceil \nu \rceil$. Because $\mathcal{H} \in L^1([0,1])$ and $j \le n_0 < 0$, we have

$$\frac{2}{3}\left(\int_{M'}^{n_0} + \int_{n_r}^{\nu}\right)\mathcal{H}(\tilde{\nu})\frac{d\tilde{\nu}}{\tilde{\nu}} = \tilde{C}_1 + \mathcal{O}(r^{-\frac{3}{2}}),$$
$$\sum_{j=n_0}^{n_r-1}\int_0^{-1}\frac{\mathcal{H}(s)}{s+j}ds = \sum_{j=n_0}^{n_r-1}\frac{1}{j}\int_0^{-1}\mathcal{H}(s)ds + \tilde{C}_2 + \mathcal{O}(r^{-3}),$$

as $r \to +\infty$, for certain constants $\tilde{C}_1$, $\tilde{C}_2$. Finally, since $\mathcal{H}(-s) = \mathcal{H}(s)$ and $n_r < 0$, we deduce that

$$\int_0^{-1}\mathcal{H}(s)ds\sum_{j=n_0}^{n_r-1}\frac{1}{j} = \frac{3}{2}\log r\int_0^1\mathcal{H}(s)ds + \tilde{C}_3 + \mathcal{O}(r^{-\frac{3}{2}}), \qquad \text{as } r \to +\infty,$$

for some constant $\tilde{C}_3$. □

We next extract the leading order behavior from Proposition 7.3.

Proposition 7.5. *Fix $M > 0$. As $r \to +\infty$,*

$$\int_M^r I_2(\tilde{r})d\tilde{r} = c_2\log(r) + \tilde{C}_4 + \mathcal{O}(r^{-\frac{3}{2}}), \tag{7.13}$$

where $\tilde{C}_4$ is independent of r and

$$\begin{aligned} c_2 = \tilde{c}_0 &+ \frac{\tilde{c}_{-1}(3x_0 - x_1 + x_2)}{(x_1 - x_2)(x_0 + x_1 + x_2)} + \tilde{c}_{-2}\frac{2x_0^2 + x_0(x_1 - 3x_2) - (x_1 - x_2)^2}{(x_0 - x_2)(x_1 - x_2)^2(x_0 + x_1 + x_2)} \\ &- \frac{7}{6}\int_{x_0}^{\infty}\tilde{c}_1(x)\frac{\theta'(\varphi(x))}{\theta(\varphi(x))}\varphi'(x)dx, \end{aligned} \tag{7.14}$$

with the integration contour from x_0 to ∞ taken on the upper sheet of X.

Proof. The first term on the right-hand side of (7.14) follows trivially from (7.12). Since $u \mapsto \varphi_A^{-1}(u)$ is even and periodic of period 1, we can apply Proposition 7.4 with $\mathcal{H}(\tilde{\nu}) = \frac{\tilde{c}_{-2}}{(\varphi_A^{-1}(\tilde{\nu})-x_2)^2} + \frac{\tilde{c}_1}{(\varphi_A^{-1}(\tilde{\nu})-x_2)}$. Moreover, with the help of [23, Eqs. 3.133.18, 3.134.18], (3.12), and (4.2), we compute

$$\int_0^1\frac{ds}{\varphi_A^{-1}(s) - x_2} = 2\int_{x_0}^{\infty}\frac{\varphi'(\xi)d\xi}{\xi - x_2} = \frac{3x_0 - x_1 + x_2}{(x_1 - x_2)(x_0 + x_1 + x_2)}, \tag{7.15a}$$

$$\int_0^1\frac{ds}{(\varphi_A^{-1}(s) - x_2)^2} = 2\int_{x_0}^{\infty}\frac{\varphi'(\xi)d\xi}{(\xi - x_2)^2} = \frac{2x_0^2 + x_0(x_1 - 3x_2) - (x_1 - x_2)^2}{(x_0 - x_2)(x_1 - x_2)^2(x_0 + x_1 + x_2)}, \tag{7.15b}$$

where the integration contours in the integrals with respect to $d\xi$ lie on the upper sheet. Using these identities to evaluate $\int_0^1\mathcal{H}(s)ds$ explicitly, we obtain the second and third terms on the right-hand side of (7.14). Next, we use Proposition 7.4 with $\mathcal{H}(\tilde{\nu}) = \tilde{c}_1(\tilde{a})\frac{\theta'(\tilde{\nu})}{\theta(\tilde{\nu})}$ to obtain

$$\int_M^r\frac{\tilde{c}_1(\tilde{a})}{\tilde{r}}\frac{\theta'(\tilde{\nu})}{\theta(\tilde{\nu})}d\tilde{r} = -2\log(r)\int_{x_0}^{\infty}\tilde{c}_1(x)\frac{\theta'(\varphi(x))}{\theta(\varphi(x))}\varphi'(x)dx + \tilde{C}_5 + \mathcal{O}(r^{-\frac{3}{2}}), \tag{7.16}$$

where the contour in the x-integral lies on the upper sheet. Furthermore, integrating by parts and then using Proposition 7.4 again, we find

$$\begin{aligned} \int_M^r\frac{\tilde{c}_2(a)}{\tilde{r}^{\frac{3}{2}}}\frac{d}{d\tilde{r}}\left[\frac{\theta'(\tilde{\nu})}{\theta(\tilde{\nu})}\right]d\tilde{r} &= -\frac{5}{12}\int_M^r\frac{\tilde{c}_1(a)}{\tilde{r}}\frac{\theta'(\tilde{\nu})}{\theta(\tilde{\nu})}d\tilde{r} + \frac{3}{2}\int_M^r\frac{\tilde{c}_2(a)}{\tilde{r}^{\frac{5}{2}}}\frac{\theta'(\tilde{\nu})}{\theta(\tilde{\nu})}d\tilde{r} \\ &= \frac{5}{6}\log(r)\int_{x_0}^{\infty}\tilde{c}_1(x)\frac{\theta'(\varphi(x))}{\theta(\varphi(x))}\varphi'(x)dx + \tilde{C}_6 + \mathcal{O}(r^{-\frac{3}{2}}), \end{aligned} \tag{7.17}$$

where again the contour in the x-integral lies on the upper sheet. We obtain the last term on the right-hand side of (7.14) by adding (7.16) and (7.17). Lastly, an integration by parts shows that

$$\int_M^r \frac{\tilde{c}_3}{\tilde{r}^{\frac{3}{2}}} \frac{d}{d\tilde{r}} \left[\frac{\theta'(\tilde{\nu})^3}{\theta(\tilde{\nu})^3} \right] d\tilde{r} = \tilde{C}_7 + \mathcal{O}(r^{-\frac{3}{2}}),$$

which completes the proof. □

We finally simplify the constant c_2.

Proposition 7.6. *We have* $c_2 = -\frac{3}{8}$.

Proof. We start by evaluating the integral in (7.14), which can be written as

$$\int_{x_0}^{\infty} \tilde{c}_1(x) \frac{\theta'(\varphi(x))}{\theta(\varphi(x))} \varphi'(x) dx = \frac{c_0^2 (x_1 \tilde{q}(x_1) - x_2 \tilde{q}(x_2))}{\tilde{q}(x_0)(x_0 - x_1)} \int_{x_0}^{\infty} \frac{\theta'(\varphi(x))}{\theta(\varphi(x))} \frac{dx}{(x - x_2)^2}.$$

Using Proposition 7.2, an integration by parts gives

$$\int_{x_0}^{\infty} \frac{\theta'(\varphi(x))}{\theta(\varphi(x))} \frac{dx}{(x - x_2)^2} = \int_{x_0}^{\infty} \frac{\gamma x + \delta}{\sqrt{\mathcal{R}(x)}} \frac{dx}{(x - x_2)^2}.$$

Computing the integral on the right-hand side as in (7.15), we infer that

$$\int_{x_0}^{\infty} \tilde{c}_1(x) \frac{\theta'(\varphi(x))}{\theta(\varphi(x))} \varphi'(x) dx = \frac{x_0(7x_0 - 2x_1 - 2x_2) - (x_1 - x_2)^2}{8(x_0 + x_1 + x_2)(3x_0 - x_1 - x_2)}$$

and substituting this into (7.14), we obtain the result. □

Theorem 1.2 follows by combining (7.1), (7.2), (7.3), (7.5), (7.13), and Proposition 7.6.

A Airy model RH problem

The Airy model RH problem consists of finding a function Φ_{Ai} satisfying the following properties.

(a) $\Phi_{\mathrm{Ai}} : \mathbb{C} \setminus \Sigma_{\mathrm{Ai}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_{Ai} is shown in Figure 5.

(b) Φ_{Ai} has the jump relations

$$\begin{aligned} \Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } \mathbb{R}^-, \\ \Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, && \text{on } \mathbb{R}^+, \\ \Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } e^{\frac{2\pi i}{3}}\mathbb{R}^+, \\ \Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } e^{-\frac{2\pi i}{3}}\mathbb{R}^+. \end{aligned} \tag{A.1}$$

(c) As $z \to \infty$, $z \notin \Sigma_{\mathrm{Ai}}$, we have

$$\Phi_{\mathrm{Ai}}(z) = z^{-\frac{\sigma_3}{4}} M \left(I + \sum_{k=1}^{\infty} \frac{\Phi_{\mathrm{Ai,k}}}{z^{3k/2}} \right) e^{-\frac{2}{3} z^{\frac{3}{2}} \sigma_3}, \tag{A.2}$$

where $M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ and $\Phi_{\mathrm{Ai},1} = \frac{1}{8} \begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix}$.

(d) As $z \to 0$, we have $\Phi_{\mathrm{Ai}}(z) = \mathcal{O}(1)$.

The unique solution Φ_{Ai} can be written in terms of Airy functions [17], but its exact expression is unimportant for us. The quantity we need is the explicit expression for $\Phi_{\mathrm{Ai},1}$.

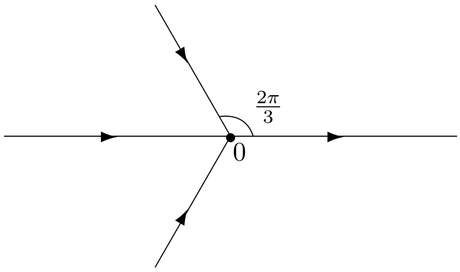

Figure 5: The jump contour Σ_{Ai}.

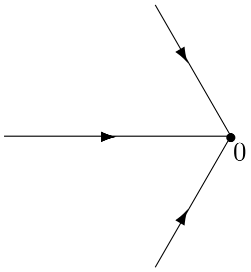

Figure 6: The jump contour Σ_{Be}.

B Bessel model RH problem

The Bessel model RH problem consists of finding a function Φ_{Be} satisfying the following properties.

(a) $\Phi_{\mathrm{Be}} : \mathbb{C} \setminus \Sigma_{\mathrm{Be}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_{Be} is shown in Figure 6.

(b) Φ_{Be} satisfies the jump conditions

$$\begin{aligned} \Phi_{\mathrm{Be},+}(z) &= \Phi_{\mathrm{Be},-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && z \in \mathbb{R}^-, \\ \Phi_{\mathrm{Be},+}(z) &= \Phi_{\mathrm{Be},-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && z \in e^{\frac{2\pi i}{3}}\mathbb{R}^+, \\ \Phi_{\mathrm{Be},+}(z) &= \Phi_{\mathrm{Be},-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && z \in e^{-\frac{2\pi i}{3}}\mathbb{R}^+. \end{aligned} \tag{B.1}$$

(c) As $z \to \infty$, $z \notin \Sigma_{\mathrm{Be}}$, we have

$$\Phi_{\mathrm{Be}}(z) = (2\pi z^{\frac{1}{2}})^{-\frac{\sigma_3}{2}} M \left(I + \frac{\Phi_{\mathrm{Be},1}}{z^{\frac{1}{2}}} + \mathcal{O}(z^{-1}) \right) e^{2z^{\frac{1}{2}}\sigma_3}, \tag{B.2}$$

where $\Phi_{\mathrm{Be},1} = \frac{1}{16}\begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix}$ and $M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$.

(d) As $z \to 0$, we have

$$\Phi_{\mathrm{Be}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(|\log z|) \\ \mathcal{O}(z) & \mathcal{O}(|\log z|) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathcal{O}(|\log z|) & \mathcal{O}(|\log z|) \\ \mathcal{O}(|\log z|) & \mathcal{O}(|\log z|) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi. \end{cases} \tag{B.3}$$

The unique solution Φ_{Be} is given in terms of Bessel functions [19], but its exact expression is unimportant for our purposes. Again, the quantity we need is the explicit expression for $\Phi_{\mathrm{Be},1}$.

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