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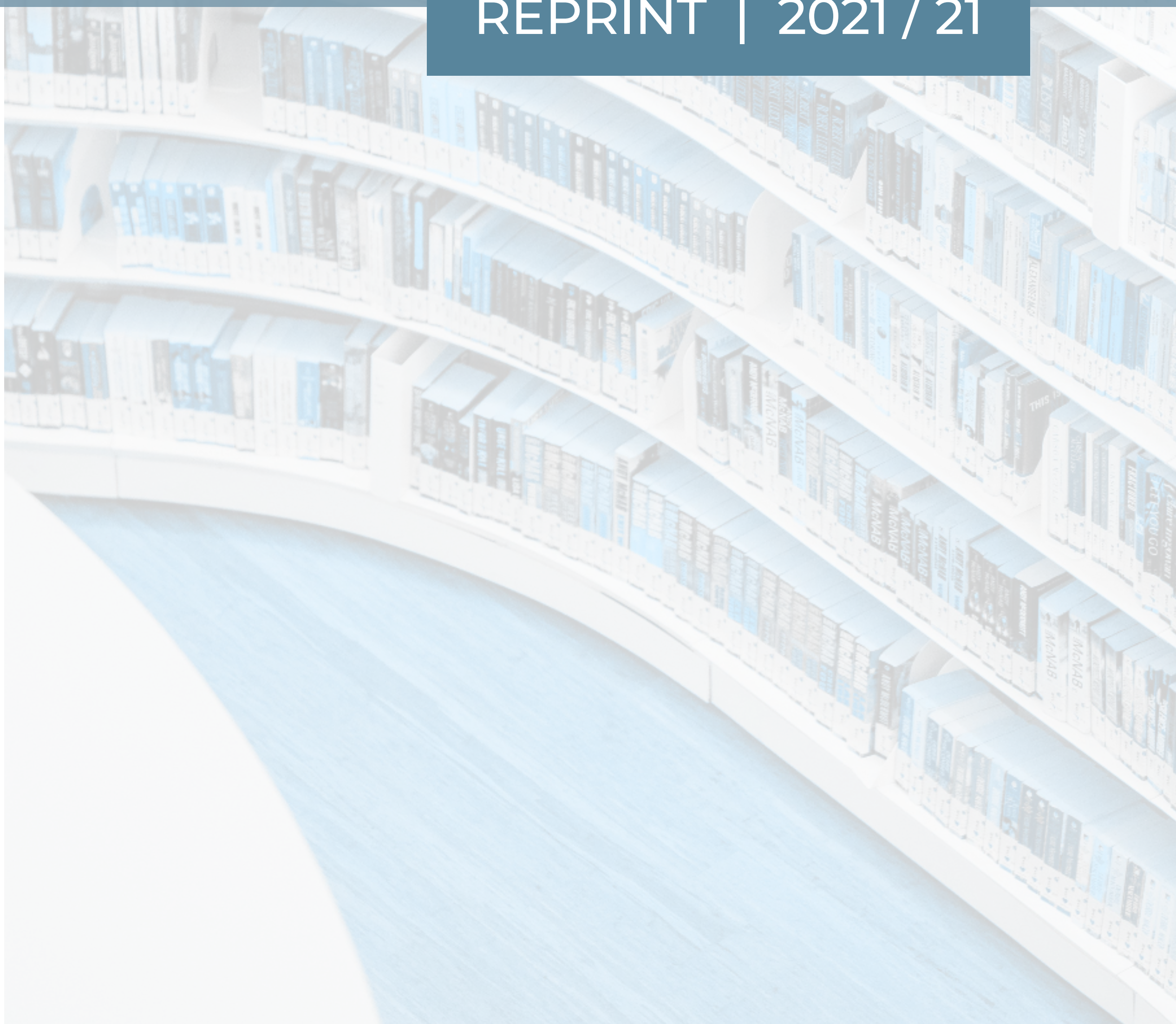

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Design of risk sharing for risk-linked annuities

Pauline NGUGNIE DIFFOUO *[1] and Pierre DEVOLDER †[2]

[1 & 2] *Institute of Statistics, Biostatistics and Actuarial Sciences*
Université Catholique de Louvain, Belgium

Abstract

In this paper we propose two financial-longevity risk-sharing methods based on an existing risk-linked annuity. Within a specific framework, we first propose the risk-sharing group self-annuitization (GSA), where different proportions of financial and longevity risks are shared between a group of annuitants and the insurer. Secondly we propose the complete risk-sharing GSA. We implicitly define a set of annuity products which depend on the risk proportion borne by a group of annuitants. The defined set of annuities goes from the classical annuity to the GSA.

Keywords : financial-longevity risk, risk sharing, annuity, group self-annuitization.

1 Introduction

Although annuities can guarantee to annuitants they won't outlive their resources, people are still reluctant to buy these products. Economists refer to this problem as the "*annuity puzzle*". In this regard, researchers have been paying attention to solve this puzzle. So far, some explanations have been given to justify the annuity puzzle such as

(i) most people prefer to invest their savings in business or account that would be accessible by their relative in case they die and buying annuity might be a bad deal for the heirs;

(ii) the fear of not living longer;

(iii) psychological reasons, in fact some retirees don't understand that annuities is a guarantee in case they live longer. Retirees seem to consider purchasing annuity as a gamble where they can die at any time and lose their premium.

*E-mail address: pauline.ngugnie@uclouvain.be
†E-mail address: pierre.devolder@uclouvain.be

Richard H. Thaler demonstrated that the Explanation (i) is not always true as a retiree can decide not to annuitize all his savings and to leave aside a proportion to his heirs. The heirs could receive that legacy either immediately or at a later date[1]. Explanations (ii) and (iii) seem to be more logical and they implicitly involve the longevity risk in the sense that they raise the question of how to predict how long a person will live?

An attempt to solve the annuity puzzle could be to design adequate annuity products both for the policymakers and policyholders so as to mitigate the longevity risk and other inherent risks such as the financial or interest rate risks. Policymakers and researchers have thus far devoted substantial efforts on the longevity risk issue throughout pricing and designing risk-linked annuities, in order to protect policyholders against outliving their resources and protect insurer against possible insolvency or high solvency capital. Even though risk-linked annuities expose the policyholder to risks, they also attempt to generate higher benefits than the fixed or classical annuities. A range of such risk-linked annuities currently exists and can further be built so as to cope with the risks in both the insurer and policyholders point of view.
Basically, risk-linked annuities are obtained based on the idea of the classical annuity with benefits contingent on financial and/or mortality risks. Further annuities can be derived from the risk-linked annuities by including death benefits and/or some fixed guarantee features. In this paper we propose a way to design a range of financial-longevity linked annuities.

Among risk contingent annuities we can mention inflation-linked annuities (see [1]), mortality-indexed annuities (MIA) proposed in 2011 by Richter & Weber [2], longevity-linked annuities (see [11]). Several annuities were developed for heterogeneous and / or homogeneous group annuitants called pooled annuities. The main idea behind the pooled annuities is to mitigate the diversifiable risk within the pool. These pooled annuity products were designed based on the common idea of tontines which consists of a group of annuitants who provide their premiums in a pool administrated by an insurance company (see [3]). For intense, in 2014 Donnelly *et al.* [4] developed a more flexible and transparent heterogeneous and homogeneous pooled annuity which provides mortality credit to both survivors and deaths in a pool. Following the pooling mechanism, the method and products proposed in this work is designed for pooled annuities. As the most famous annuities, classical annuities can be designed either for a single annuitant or a group of annuitants. The main features of classical annuities is that the whole risk is borne by the insurer as annuitants are not affected by investments or mortality experiences.

In this paper, we propose a financial-longevity linked products allowing for risk sharing between a group of annuitants and an insurer. This products are obtained by transferring a different proportions of both the financial and longevity risks to the insurer. Several authors have developed products in which the risks are either fully borne by a group of annuitants (see [3], [6], [12], [7]) or shared between these latter and the insurer. Concerning risk sharing products, Chen *et al.* [8] have developed in 2016 a product for which the whole risk is borne by the group of annuitants during the first years following the retirement and the insurer bears the whole risk afterwards. They referred to their product as tonuity which is in fact a combination of tontine and classical annuity. In 2015 Denuit *et al.* [5] revisited the concern of sharing the longevity

[1] Richard H. Thaler, The Annuity Puzzle, Economic View June 4, 2011. `https://www.nytimes.com/2011/06/05/business/economy/05view.html` accessed on November 21, 2019.

risk between a pool of annuitants and the policymakers while focusing on the deferred annuity. Denuit *et al.* [11] designed in 2011 longevity-index annuities allowing for longevity risk sharing between a group of annuitants and the insurer. Using an approach contrary to that of the group self-annuitization (GSA) (see [6]), they proposed annuities where annuitants only bear the non-diversifiable mortality risk and the insurer bears remaining risks. In 2019, Hanbali *et al.* [21] focused on the systematic longevity risk in long-term insurance businesses where they proposed a framework for risk sharing between insurer and annuitants using dynamic equivalence principle. They proposed viable risk-sharing conditions that improve the trade-off between the solvency of the insurer and a fair price for policyholders for a pure endowment contract.

The present paper provides more general risk sharing methods that can be applied on deferred annuities, can be used to designed a longevity-linked; financial-linked as well as the financial-longevity-linked annuities. Our methods are inspired by the group self-annuitization (GSA) introduced in 2005 by Piggot *et al.* [6] where they developed a formal analysis of the payout of the GSA which can be seen as a financial-longevity risk-pooling fund. The GSA is a scheme which allows annuitants to pool a part or their whole retirement fund with other annuitants with a view to afford benefits in retirement through a risk sharing arrangement within the pool. They proposed a detail procedure of assessing the payout while considering the mortality changes of the pool. They expressed the payouts recursively and depending on the adjustment factor which depends on the ratios of survivorship (see [6]). The main feature of the GSA is that unlike the classical annuity, the whole risk is borne by the group of annuitants whereas the insurer bears no risk. Some authors have proposed modified version of the GSA and developed detailed analysis of particular cases of GSA. For instance, Qiao & Sherris went further in 2013 [12] by using the multiple-factor stochastic mortality model to show how efficient pooling can be and quantify the limitation of pooling scheme with respect to the longevity risk. They also quantifies the volatility of benefit payment over time and found out that with their approach the benefits will decrease at old age. They hence improved the model proposed by Piggot *et al.* by combining both the changed expectation adjustment factor and the mortality adjustment factor into a single factor so as to include expected future mortality improvements. As a result, they found that considering the expected future mortalities ensure the relative stability of the benefit payment of each annuitant in the pool. In 2015 Boyle *et al.* [7] studied a special case of the GSA and they referred to it as the variable payout annuities (VPA). They made a comparative study between VPA and fixed annuity using utility approach. They hence came out with optimal investment and consumption strategy for a policyholder who chooses between VPA, fixed annuity or self-annuitizaton and shown that VPA improves the expected utility of consumption of the policyholders.

As GSA shifts the whole risk to the group of annuitants and in classical annuity to the insurer; these two products can then be seen as extreme cases. The question we address here is what happens in between these two extreme annuities. Our contribution is to design a set of annuities moving from the classical annuity to the GSA, we refer to the obtained set of annuities as the (complete) risk-sharing GSA. Our motivation of using the GSA comes from its recursive structure and from the fact that financial risk is separated from longevity risk. We compare the obtained products with both classical annuity and GSA using utility approach. Following the approach of Chen *et al.* [8] and Chen & Hieber [24], we compare the expected discounted lifetime (constant relative risk aversion) utility of classical annuity, GSA and (complete) risk-sharing GSA. We find that the more risk averse a policyholder is, the high expected discounted lifetime utility he get with complete risk-sharing GSA then classical annuity or GSA. We further find that there exist complete risk-sharing GSAs that improve the expected discounted lifetime utility of a policyholder (depending on the proportion of risk he bears) compared to the classical annuity and GSA.

The remaining parts of the paper is structured as follows. In Section 2 we give a brief recall on the GSA. The risk-sharing techniques are developed in Section 3, the obtained annuity is called the risk-sharing GSA. Section 4 presents the complete risk-sharing GSA which is in fact an alternative way to share the risk using the GSA. Numerical analysis is made in Section 5 and we conclude in Section 6.

2 Recalls on the GSA

GSA basically operates like a classical life annuity for which both the expected investment return and the policyholder's expected future mortality are captured by the benefit payout. In 2005, Piggott *et al.* [6] proposed a flat yield curve in order to capture the expected rate of return on the investment. It follows that if both investment and mortality actually follow these expectations, then the benefit payout stays constant.
Considering at time $t = 0$ a group of N_x policyholders aged x paying a total amount of F_0; we assume the cohort to be homogeneous (i.e annuitants with same initial age x and same initial premium F_0/N_x). The individual initial benefit payout is then given by

$$B_0 = \frac{F_0}{N_x \ddot{a}_x}, \tag{1}$$

with $\ddot{a}_x = \sum_{t=0}^{\infty} v^t \, \frac{N_{x+t}}{N_x}$ and $v = (1+R)^{-1}$ where R is the interest rate and N_{x+t} is the expected number of survivors at age $x + t$ for $t \geq 0$. Noting that the benefit payout of a conventional annuity, (where the whole risk is borne by the insurer) is constant and is given by Formula(1), i.e $B_t = B_0$ for all $t > 0$.

Assuming actual survivors are different from expected number of survivors, let $N^*_{x+1},\ N^*_{x+2}, ..., N^*_{x+t}, ...$ denote the yearly actual number of survivors. With a constant and fixed interest rate, they found that the benefit payout at any time t is given by

$$B_t = B_{t-1} \, \frac{p_{x+t-1}}{p^*_{x+t-1}}, \tag{2}$$

where p_{x+t} is a deterministic one year survival probability of an individual aged $x+t$, for $t \geq 0$; the variables with the superscript '*' denote the real values of the variables and $p^*_{x+t} = \frac{N^*_{x+t}}{N^*_x}$.

Similarly, considering different actual annual rate of return $R_1^*,\ R_2^*, ...$ the implied benefit at t is

$$B_t = B_{t-1}\ \frac{p_{x+t-1}}{p^*_{x+t-1}}\ \frac{1+R^*_t}{1+R}. \tag{3}$$

The two last term of the right hand side (RHS) respectively represent (from the right to left) the investment rate adjustment at t (IRA_t) and the mortality experience adjustment at t (MEA_t). Note that IRA_t represents the financial risk whereas MEA_t represents the longevity risk. From Formula(3) they derived the formulae of the benefit payout from an heterogeneous cohort (i.e different ages and different premiums) with identical annuity factor as well as different annuity factors entering the pool at different time. For this latter case the formula of benefit paid at time t to the i^{th} annuitant entering the pool at age x, k period of time ago is given by

$$ {}^{k}_{x}B_{i,t} \ = \ \frac{{}^{k}_{x}\hat{F}^*_{i,t}}{\ddot{a}_{x+k,t}} \ = \ {}^{k-1}_{\ \ x}B_{i,t-1}\ \mathrm{MEA}_t\ \mathrm{CEA}_t\ \mathrm{IRA}_t, \tag{4}$$

where ${}^{k}_{x}\hat{F}^*_{i,t}$ is the fund value at t for the i^{th} policyholder including the inheritance of those who died during the period $[t-1,t]$; IRA_t is defined as previously, MEA_t is given by

$$\mathrm{MEA}_t = \frac{F^*_t}{\displaystyle\sum_{k\geq 1}\sum_{x} p^{-1}_{x+k-1,t-1}\ \sum_{A_t} {}^{k}_{x}F^*_{i,t}} \tag{5}$$

and $\mathrm{CEA}_t = \frac{\ddot{a}_{x+k,t-1}}{\ddot{a}_{x+k,t}}$ represents the change expectation adjustment at time t. It is used to capture the new mortality information available at t. ${}^{k}_{x}F^*_{i,t}$ is the fund value at time t of the i^{th} policyholder who entered the pool at age x, k period ago; F^*_t is the total fund at time t; $p_{x,t}$ is the expected survival probability of an annuitant aged x at time t of living one more year and $p^*_{x,t}$ is the corresponding actual survival probability. A_t is the set of annuitants alive at time t. In this general case, the longevity risk is represented by the product of MEA_t and CEA_t and IRA_t represents the financial risk. Note that Formula(4) is not valid for annuitants entering the pool at time t but for those who entered before time t. The benefit of those entering the pool at time t is given by

$$ {}^{0}_{x}B_{i,t} \ = \ \frac{{}^{0}_{x}F_{i,t}}{\ddot{a}_{x,t}}. \tag{6}$$

In the GSA, the financial and the longevity risks are borne by the pool whereas the insurer borne no risk. Moreover, these two risks are separated from each other in the formula of the benefit payout. The next section proposes two ways of sharing financial and longevity risks between both insurer and policyholders using the GSA.

3 Risk sharing techniques

Recall that, in the GSA, the whole risk is shifted to the pool. Here we present two techniques which allow the policyholder to get rid of a proportion of risks (total or partial); the obtained product is called *risk-sharing GSA*. We first consider a sharing method based on the lower bound of the benefit payouts, secondly we present a more direct sharing method that can be viewed as a proportional sharing. These methods are used to define the annual benefits for a risk-sharing GSA.

3.1 Risk sharing by the mean of a lower bound threshold on benefits

Defining a lower bound threshold of the benefits, we hedge policyholders against drastic mortality or investment changes. This means they can see their benefits increase or decrease but not below a set threshold level. Let us now denote by $\varepsilon_t \in [0,1]$ the proportion at time t of the total risk (longevity and financial risks) borne by the insurer, such that the lower bound threshold of benefit is given by $B = \varepsilon_t B_0$. Hence we suggest that the benefit at a given time $t > 0$ satisfies

$$B_t \geq B = \varepsilon_t B_0. \tag{7}$$

Therefore for $\varepsilon_t = 1$, the whole risk is shifted to the insurer and for $\varepsilon_t = 0$ is the case described in the previous section, i.e where the risk is borne by the pool. One could also think of an alternative situation where for all $t > 0$ the benefits rather satisfy the condition

$$B_t \geq \varepsilon_t \; B_{t-1}. \tag{8}$$

Such case would be non-realistic as compared to the previous case because it could tend to zero in the case of continuously drastic mortality or investment changes. Both cases (Formulae (7) and 8) can be roughly illustrated for a constant proportion $\varepsilon \in (0,1)$ in the following Figure 1.

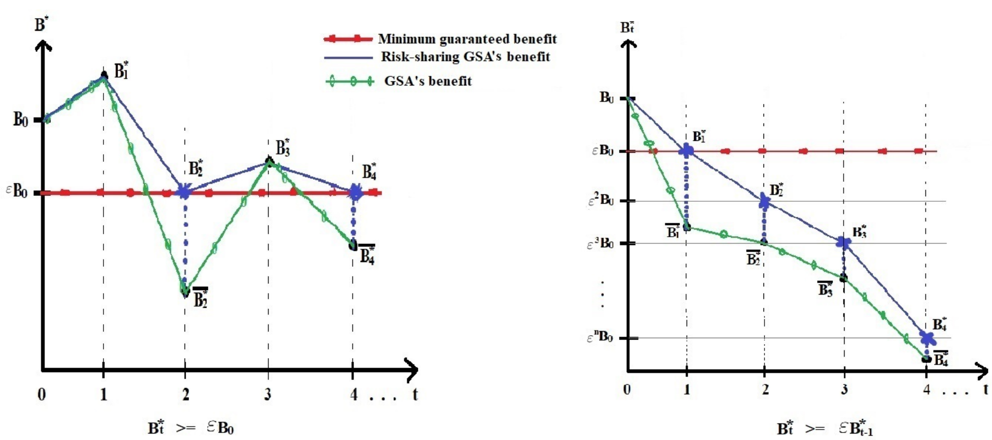

Figure 1: Example of benefit variation

In fact, in the first graph of Figure 1, we consider a volatile scenario of the GSA (green balled line) and we obtained a threshold risk-sharing GSA (blue line) which is always above the guaranteed minimum benefit (red squared line). Whereas on the second graph, we consider an extreme (strictly decreasing) scenario of the GSA (green balled line), it yields a decreasing threshold risk-sharing GSA (blue line) which goes below the guaranteed benefit (red squared line) and tends to zero.

We define the benefit payout $\overline{B_t}$ at time t of a policyholder buying a lower bound threshold risk-sharing GSA by

$$\overline{B_t} = \max\left(\overline{B_{t-1}}\ Lrisk_t\ Erisk_t, \varepsilon_t B_0\right), \tag{9}$$

where $Lrisk_t$ and $Erisk_t$ are respectively the longevity and the financial risk as defined in the GSA in Section 2. For example, considering the simple case of a close homogeneous cohort where benefit is given by Formula(3), we defined the benefit at $t > 0$ by

$$\overline{B_t} = \max\left(\overline{B_{t-1}}\ \frac{p_{x+t-1}}{p^*_{x+t-1}}\ \frac{1+R^*_t}{1+R},\ \varepsilon_t B_0\right). \tag{10}$$

In the more general case of an open heterogeneous cohort purchasing such a risk-sharing GSA (RS-GSA), we defined the lower bound threshold is

$$\overline{{}_xB_i}(t) = \varepsilon_t\ {}^0_xB_{i,t}\ =\ \varepsilon_t\ \frac{{}^0_xF_{i,t}}{\ddot{a}_{x,t}}.$$

Thus the benefit of an annuitant having survived at time t is given by

$$\overline{{}^k_xB_{i,t}}\ =\ \max\left(\overline{{}^{k-1}_{\ \ x}B_{i,t-1}}\ \mathrm{Risk}_t,\ \overline{{}_xB_i}(t)\right), \tag{11}$$

with Risk_t being the (financial and the longevity) risks generated by the GSA of Formula (4).

Note that in this case, the benefit at time t is obtained from the maximum value of the previous maximums (as compared to the threshold). An alternative way could consist of choosing between the threshold and the real benefit payout at t denoted by ${}^k_xB_i(t)$. The new annual benefit is then given by

$$\overline{{}^k_xB_{i,t}}\ =\ \max\left({}^{k-1}_{\ \ x}B_{i,t-1},\ \overline{{}_xB_i}(t)\right), \tag{12}$$

where ${}^k_xB_{i,t}$ is given by Formula (4). For instance, the simple case of Formula (10) leads to

$$\overline{B_t} = \max\left(B_t,\ \varepsilon_t B_0\right),$$

where

$$B_t = B_{t-1}\ \frac{p_{x+t-1}}{p^*_{x+t-1}}\ \frac{1+R^*_t}{1+R}. \tag{13}$$

This sharing method allows us to share the whole (financial + longevity) risk generated by the GSA. In what follows we proposed a way of sharing different proportions of each risk.

3.2 Proportional risk sharing

A more straightforward risk sharing mechanism consists of proportional or direct sharing. In other words, we consider at time t a proportion $\beta_t \in [0, 1]$ of the risk to be borne by a group of

annuitants and the remaining proportion $(1-\beta_t)$ is borne by the insurer.
Considering the same annuity benefit ${}_x^kB_{i,t}$ defined by Formula (4), we obtain the following mathematical definition of the proportional risk sharing GSA

$$ {}_x^kB_{i,t}(\beta_t) \;=\; {}_x^kB_{i,t-1}(\beta_t)\left[\beta_t \ \mathrm{Risk}_t + (1-\beta_t)\right], \tag{14}$$

where $\mathrm{Risk}_t = \mathrm{MEA}_t \times \ \mathrm{CEA}_t \times \ \mathrm{IRA}_t$.

Note that since β_t is the proportion of the risk borne by a group of policyholders at time t, it could represent either a proportion of the whole risk (i.e both the longevity and the financial risks as in Formula (14) or a proportion of a partial risk. By partial risk, we mean either the longevity risk or the financial risk. More generally, we could think of sharing at time t different proportion of longevity and financial risks. Let $\beta_t \in [0,1]$ be the proportion of the longevity risk borne by the policyholder and $\beta_t' \in [0,1]$ be the proportion of financial risk borne by the same group. The general formula of the risk sharing GSA (with different proportions) is given by

$$ {}_x^kB_{i,t}(\beta_t,\beta_t') \;=\; {}_x^kB_{i,t-1}(\beta_t,\beta_t')\left[\beta_t \ Lrisk_t + (1-\beta_t)\right]\left[\beta_t' \ Erisk_t + (1-\beta_t')\right].$$

For example, in this case Formula(14) becomes

$$ {}_x^kB_{i,t}(\beta_t,\beta_t') \;=\; {}_x^{k-1}B_{i,t-1}(\beta_t,\beta')\left[\beta_t \ \mathrm{MEA}_t \ \mathrm{CEA}_t + (1-\beta_t)\right] \ \left[\beta_t' \ \mathrm{IRA}_t + (1-\beta_t')\right]. \tag{15}$$

Note that defining the proportional RS-GSA allows us to define a set of annuities moving from the GSA (when $\beta_t = \beta_t' = 100\%$) to the classical annuity (when $\beta_t = \beta_t' = 0\%$). We observe that in both the GSA and the risk-sharing GSA, the policyholders and the insurer are not rewarded for the risk they agree to bear at the beginning of the contract. In other words, the first benefit does not depends on the risk proportions, it is equal to the benefit of a classical annuity. In the next section we propose what we called the *complete risk-sharing GSA* which defines the initial benefit depending on the risk proportions. This is done using the proportional risk-sharing method because it includes both the GSA and the classical annuity whereas the lower bound threshold method does not include the classical annuity.

4 On the complete risk-sharing GSA

In the Section 2, the individual benefit payout for a GSA is defined recursively as follows

$$B_t = B_{t-1} \times Lrisk_t \times Erisk_t, \tag{16}$$

where $Lrisk_t = \mathrm{MEA}_t \times \mathrm{CEA}_t$ and $Erisk_t = (1+R_t)/(1+R)$ are respectively the longevity and the financial risk at time t. In what follows, let's denote by $\beta, \ \beta' \in [0,1]$ respectively the proportion of the longevity and the financial risk borne by the pool and we denote the annuity factor depending of these risk proportions by $\ddot{a}_x(\beta,\beta')$. Our objective here is to introduce a discount rate that depends on the pool proportion of financial and longevity risks denoted by

$R_\gamma(\beta,\beta')$ called risk adjusted discount rate; where γ is the policyholder's risk aversion parameter as defined in Formula (20). $R(\beta,\beta')$ is defined such that the risk adjusted annuity factor given by

$$\ddot{a}_x(\beta,\beta') = \sum_{s=0}^{\infty} \left(\frac{1}{(1+R_\gamma(\beta,\beta'))} \right)^s \times \ {}_sp_x$$

could be used to annuitize individual premium. Therefore in this case the initial benefit payout of an individual aged x with a premium of F_0 will be

$$B_0(\beta,\beta') = \frac{F_0}{N_{x_0}\ \ddot{a}_x(\beta,\beta')}. \tag{17}$$

Using the proportional risk-sharing method, the individual annual benefit of a complete risk-sharing GSA is defined for $t>0$ by

$$B_t(\beta,\beta') = B_{t-1}(\beta,\beta') \times [\beta Lrisk_t + (1-\beta)] \times [\beta' Erisk_t(\beta,\beta') + (1-\beta')], \tag{18}$$

The obtained annuity is referred to as the complete risk-sharing GSA. Let's compare this annuity with the classical annuity where the individual benefits is defined for all $t \geq 0$ by

$$B_t = \frac{F_0}{N_{x_0} \sum_{s=0}^{\infty} \left(\frac{1}{1+R}\right)^s \ {}_sp_x} = \frac{F_0}{N_{x_0}\ \ddot{a}_x}. \tag{19}$$

We use the utility approach on the policyholder point of view and we consider his risk preference; namely following [8] and [24], we value the expected discounted lifetime utility of that policyholders. In line with the literature, we capture the policyholder's risk preferences using the constant relative risk aversion (CRRA) utility function given by

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{for } \gamma \neq 1 \\ \log(x) & \text{for } \gamma = 1 \end{cases} \tag{20}$$

where $\gamma \geq 0$ is the relative risk aversion parameter. The objective here is to find the values of $R_\gamma(\beta,\beta')$ such that the expected discounted lifetime utility of the complete risk-sharing GSA (CRS-GSA) is at least equal to the expected discounted lifetime utility of the classical annuity. It is important to stress that the values of $R(\beta,\beta')$ which give the same expected utility as the classical annuity can be seen as the indifference discount rate.

Concerning the utility of the classical annuity, let's consider a cohort of N_{x_0} policyholder paying a total premium of F_0 to purchase (classical) immediate lifetime annuity and denote by $n+1$ the maximum number of benefit payout. It follows that the policyholder's expected discounted lifetime utility for such a product is given by

$$Ucla = \sum_{t=0}^{n} U(B_0) v_\delta^t \ {}_tp_x^0; \tag{21}$$

where $v_\delta = \frac{1}{1+\delta}$ is the policyholder's subjective discount factor, with δ being the subjective discount rate, B_0 is given by Formula (19) and ${}_tp_x^0$ is a deterministic survival probability of the policyholder. In fact, ${}_tp_x^0$ represents the policyholder expected survival probability on the policyholder's view.

In what follows, we first consider single risk settings where we share only the financial risk (called the complete financial risk-sharing GSA), then the longevity risk (called the complete longevity risk-sharing GSA). Afterwards we combine the two cases to obtain the complete financial-longevity risk-sharing GSA.

4.1 The complete financial risk-sharing GSA (CFRS-GSA)

Let us consider a given mortality table describing the survival probability of a pool of policyholders buying an immediate lifetime annuity for a pool's premium of F_0 and let $\beta' \in [0,1]$ be the proportion of financial risk borne by the pool. We assume that there is no longevity risk occurring in this contract, in other words the proportion of the longevity risk shifted to the pool is $\beta = 0$. Therefore $R_\gamma(\beta') = R_\gamma(0,\beta')$ only reflects the investment uncertainty and the benefit payout in this case is given by Formulae (17)–(18) where $\beta = 0$. The expected discounted lifetime utility of the CFRS-GSA is given by

$$Ucfrs(\beta') = \mathbb{E}\left[\sum_{t=0}^{n} U(B_t(\beta'))\ v_\delta^t\ {}_tp_x^0\right]. \tag{22}$$

Using the Equation (18) with $\beta = 0$, one can show that for all $t > 0$

$$B_t(\beta') = B_0(\beta') \prod_{j=1}^{t} \left[\beta' Erisk_j(\beta') + (1-\beta')\right], \tag{23}$$

with

$$B_0(\beta') = \frac{F_0}{N_{x_0} \ddot{a}_x(\beta')} = \frac{F_0}{N_{x_0} \sum_{j=0}^{\infty} v_R^j(\beta')\ {}_jp_x},$$

and $v_R^j(\beta') = \left(\frac{1}{1+R_\gamma(\beta')}\right)^j$. It follows that Formula (22) becomes

$$Ucfrs(\beta') = \begin{cases} \frac{1}{1-\gamma} \sum_{t=0}^{n} \mathbb{E}\left[\left(B_0(\beta') \prod_{j=1}^{t} \left[\beta' Erisk_j(\beta') + (1-\beta')\right]\right)^{1-\gamma}\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma \neq 1 \\ \sum_{t=0}^{n} \mathbb{E}\left[\log\left(B_0(\beta') \prod_{j=1}^{t} \left[\beta' Erisk_j(\beta') + (1-\beta')\right]\right)\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma = 1. \end{cases} \tag{24}$$

For illustration, one can show that in the simplest case of a closed homogeneous pool where the benefit payout is given by Formula (3), the expected utility of a policyholder buying a CFRS-GSA is given by

$$Ucfrs(\beta') = \begin{cases} \frac{1}{1-\gamma} \sum_{t=0}^{n} \mathbb{E}\left[\left(B_0(\beta')\ \prod_{j=1}^{t} \left[\beta' \frac{1+R_j}{1+R_\gamma(\beta')} + (1-\beta')\right]\right)^{1-\gamma}\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma \neq 1 \\ \sum_{t=0}^{n} \mathbb{E}\left[\log\left(B_0(\beta')\ \prod_{j=1}^{t} \left[\beta' \frac{1+R_j}{1+R_\gamma(\beta')} + (1-\beta')\right]\right)\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma = 1. \end{cases}$$

The task is then to find the set of $R_\gamma(\beta')$ such that

$$Ucfrs(\beta') \geq Ucla, \tag{25}$$

where $Ucla$ is given by Formula (21). We cannot find an explicit formula of the risk adjusted discount rate $R_\gamma(\beta')$ satisfying the Inequality (25), but this can be solved numerically using for example the Newton-Raphson method.

4.2 The complete longevity risk-sharing GSA (CLRS-GSA)

In this case we consider an financial risk-free market made of a risk-free asset (the money market account), then the proportion of financial risk shifted to the pool is $\beta' = 0$. In this case the risk adjusted discount rate given by $R_\gamma(\beta) = R_\gamma(\beta, 0)$ captures the longevity risk. It follows that the benefit payout of the CLRS-GSA is given by Formulae (17)–(18) where $\beta' = 0$ and the expected discounted lifetime utility of an annuitant buying a CLRS-GSA is

$$Uclrs(\beta) = \mathbb{E}\left[\sum_{t=0}^{n} U(B_t(\beta))\ v_\delta^t\ {}_tp_x^0\right]. \tag{26}$$

Using the Equation (18) with $\beta' = 0$, one obtains for all $t > 0$

$$B_t(\beta) = B_0(\beta) \prod_{j=1}^{t} \left[\beta Lrisk_j + (1-\beta)\right], \tag{27}$$

with

$$B_0(\beta) = \frac{F_0}{N_{x_0}\ \ddot{a}_x(\beta)} = \frac{F_0}{N_{x_0} \sum_{j=0}^{\infty} v_R^j(\beta)\ {}_jp_x},$$

where $v_R^j(\beta) = \left(\frac{1}{1+R_\gamma(\beta)}\right)^j$. Formula (26) becomes

$$Uclrs(\beta) = \begin{cases} \frac{1}{1-\gamma} \sum_{t=0}^{n} \mathbb{E}\left[\left(B_0(\beta) \prod_{j=1}^{t} \left[\beta Lrisk_j + (1-\beta)\right]\right)^{1-\gamma}\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma \neq 1 \\ \\ \sum_{t=0}^{n} \mathbb{E}\left[\log\left(B_0(\beta) \prod_{j=1}^{t} \left[\beta Lrisk_j + (1-\beta)\right]\right)\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma = 1. \end{cases} \tag{28}$$

In the case of a closed homogeneous pool, we find

$$Uclrs(\beta) = \begin{cases} \frac{1}{1-\gamma} \sum_{t=0}^{n} \mathbb{E}\left[\left(B_0(\beta) \prod_{j=1}^{t} \left[\beta\ \frac{p_{x+j-1}}{p^*_{x+j-1}} + (1-\beta)\right]\right)^{1-\gamma}\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma \neq 1 \\ \\ \sum_{t=0}^{n} \mathbb{E}\left[\log\left(B_0(\beta) \prod_{j=1}^{t} \left[\beta\ \frac{p_{x+j-1}}{p^*_{x+j-1}} + (1-\beta)\right]\right)\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma = 1. \end{cases} \tag{29}$$

Numerical methods can be used to find the set of $R_\gamma(\beta)$ for which

$$Uclrs(\beta) \geq Ucla, \tag{30}$$

where $Ucla$ is given by Formula (21).

4.3 The complete financial-longevity risk-sharing GSA (CFLRS-GSA)

The idea here is to combine the previews two single-risk products into a double-risk one. Both the financial and the longevity risks are shared with different proportions, respectively $\beta,\ \beta' \in [0,1]$. One can show from Formula (18), that we obtain for any $t > 0$

$$B_t(\beta,\beta') = B_0(\beta,\beta')\prod_{j=1}^{t}\left[\beta Lrisk_j + (1-\beta)\right] \times \left[\beta' Erisk_j(\beta,\beta') + (1-\beta')\right], \tag{31}$$

where $B_0(\beta,\beta')$ is given by Formula (17). It follows that the expected utility of an annuitant buying a CFLRS-GSA is

$$Ucflrs(\beta,\beta') = \sum_{t=0}^{n}\mathbb{E}\left[U\left(B_0(\beta,\beta')\prod_{j=1}^{t} Lshare_j(\beta) \times Eshare_j(\beta,\beta')\right)\right] v_\delta^t\ {}_tp_x^0, \tag{32}$$

where

$$Lshare_j(\beta) = \left[\beta\ Lrisk_j + (1-\beta)\right]$$

and

$$Eshare_j(\beta,\beta') = \left[\beta'\ Erisk_j(\beta,\beta') + (1-\beta')\right],$$

with

$$Erisk_j(\beta,\beta') = \frac{1+R_j}{1+R_\gamma(\beta,\beta')}.$$

Note that the financial risk component depends on both the financial and longevity risk, as the risk adjusted discount rate depends on them. In particular, for a closed homogeneous group of annuitants, the longevity risk component is

$$Lrisk_j = \frac{p_{x+j-1}}{p^*_{x+j-1}}.$$

The problem here is to find a set of risk adjusted discount rate $R_\gamma(\beta,\beta')$ such that

$$Ucflrs(\beta,\beta') \geq Ucla, \tag{33}$$

where $Ucla$ is given by Formula (21). This leads us to a single inequality with two unknowns. Below we solve Problem (33) using numerical method, for this purpose we assume ${}_tp_x^0 = {}_tp_x$.

Note that the (complete) RS-GSA we proposed could also be designed based on the extended version of the GSA proposed by Qiao & Sherris [12] as well as on the VPA proposed by Boyle *et al.* [7].

5 Numerical solutions

In this section we compute some numerical solutions for both the risk-sharing GSA and the complete risk-sharing GSA. First of all, we need detailed informations about the investment strategy and the mortality model.

5.1 Assumptions

Let's consider, within a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ a Black-Scholes model ([14], [15]) made of two assets : a stock (S_t) and a money market account (B_t) with a constant proportion allocation strategy with dynamic rebalancing, i.e a constant proportion $x \in [0,1]$ of the initial fund is invested on the stock and the remaining proportion invested on the money market account. We consider a constant short rate $r \in \mathbb{R}_+^*$, constant drift $\mu \in \mathbb{R}$ and constant volatility $\sigma \in \mathbb{R}_+^*$, one can show that the value at time T of an investment on such mixed asset during the period $[t,T]$ is

$$A(t,T) = A_t \ \mathrm{Asset}(t,T),$$

where

$$\mathrm{Asset}(t,T) = e^{\left(\mu x+(1-x)r-\frac{x^2\sigma^2}{2}\right)(T-t)+\sigma x(W_T-W_t)}, \tag{34}$$

with W_t being a Brownian motion under the physical measure $\mathbb{P}$, describing the stock's uncertainty. It follows that the investment return is given by $1 + R_t = \mathrm{Asset}(t-1,t)$.

We model the mortality of the policyholder using the Hull-White (HW) model (see [18]), that is we assume the force of mortality is defined by the following dynamics

$$d\mu_t^{x_0} = (\theta(t) - a\mu_t^{x_0})dt + \sigma_\mu \ dW_t^\mu, \quad \text{for all } t \geq 0,$$

where W_t^μ is a Brownian motion under $\mathbb{P}$ describing the mortality's uncertainty; x_0 is the affiliation age and $\theta(t)$ is the mean reversion level function that we assume to follow a Gompertz mortality model (see [19]) i.e

$$\theta(t) = Ae^{Bt},$$

with the baseline mortality $A > 0$ and the senescent component $B > 0$; we also assume independence between financial and longevity risk, i.e W_t is independent of W_t^μ. For more details on this mortality model, see [22]. We define the survival index at time s of an individual initially aged x_0, alive at time t and surviving $s-t$ more years by

$$I_{s-t}^{x_0+t} = e^{-\int_t^s \mu_u^{x_0} \ du}.$$

It follows that the number of survivor at time t of an initial cohort of N_{x_0} annuitants is given by

$$N_{x_0+t} = N_{x_0} \ I_t^{x_0}.$$

The contract consists of the group of $N_{x_0} = 1000$ initially aged $x_0 = 65$ and paying each a unique premium $F^0 = 1000\$$ for an immediate lifetime annuity; $F_0 = N_{x_0}F^0$. For simplicity we consider a closed homogeneous pool; an unisex mortality table of individual initially aged 65 with an ultimate age of 110 (table available on the IA|BE life table published in 2015 [25]), using the mean square error (MSE) method we obtain

μ_0^{65}	A	B	a	σ_μ	MSE
0.0105677	0.0005749505	0.1304207503	0.0014965354	0.0083530153	0.000303644

The parameters of the risky asset is calibrated using the S&P500 indexes[2] from 1871 to 2018. Using the MLE, we get $\mu = 5.8702\%$ and $\sigma = 20.4172\%$. Moreover, for sensitivity analysis we consider two additional values of the drift, i.e $\mu = 2\%$ and $\mu = 4\%$. Further sensitivity analysis are made for the risk aversion parameter by considering $\gamma = 0,\ 1$ and we assume $\delta = r = R_0 = 2\%$.

Note that an increase in the values of the risk adjusted discount rate $R_\gamma(\beta, \beta')$ implies an increase of the initial benefit given by Formula (17), which could also increase the upcoming benefits depending on both the investment and mortality experiences.

5.2 The risk-sharing GSA

Here we show how the expected utility of the FLRS-GSA changes with the annuitant's risk proportions β and β' when $\beta = \beta'$. Moreover a sensitivity analysis is made with respect to μ, γ and x. Note that in this case, $\beta = \beta' = 0\%$ represents the classical annuity and $\beta = \beta' = 100\%$ represents the GSA. The case of a FRS-GSA is shown in Figure 6 of Appendix A.

[2] http://www.multpl.com/s-p-500-historical-prices/table/by-year, March 7 2019

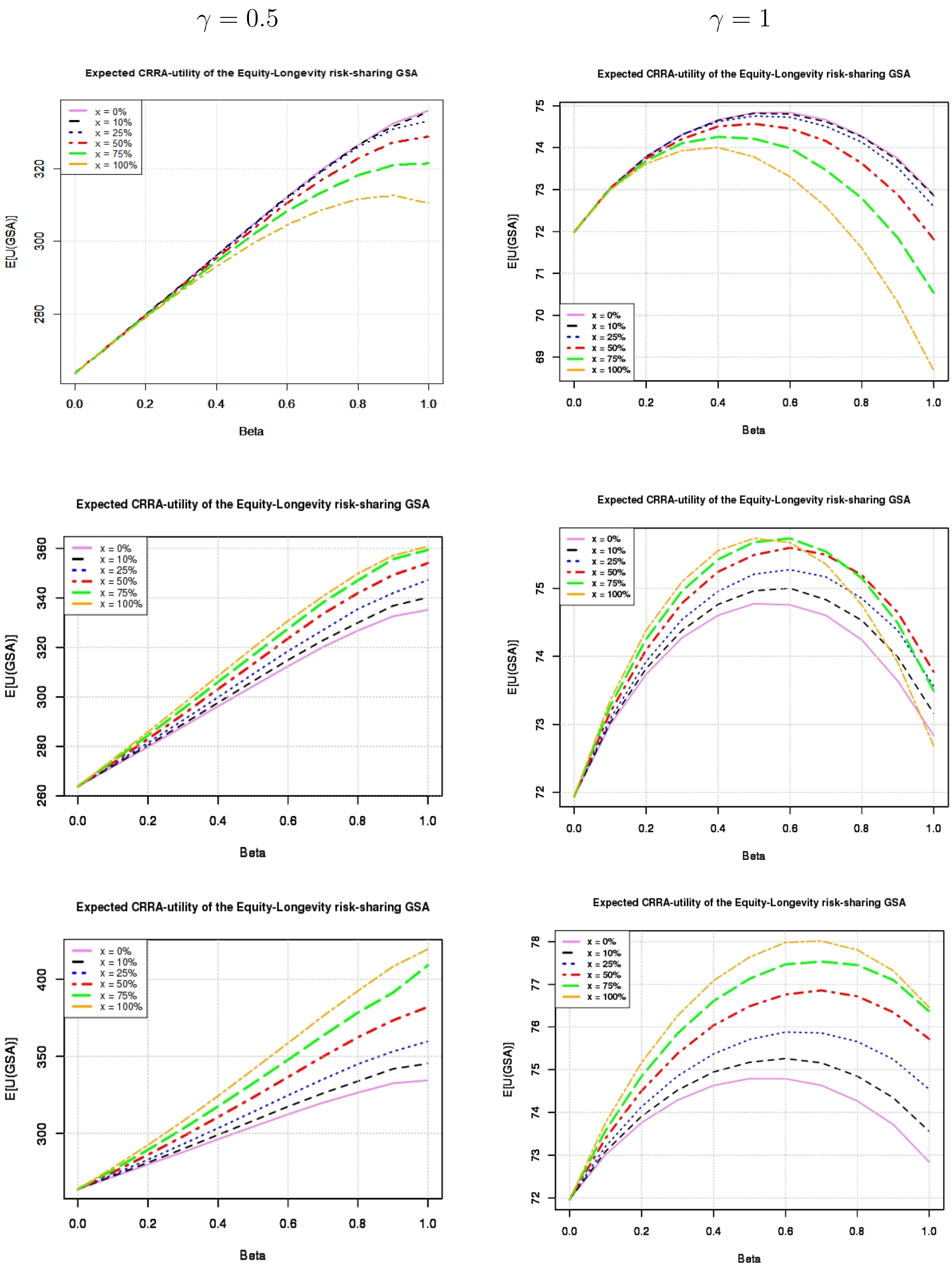

Figure 2: Expected utility of the FLRS-GSA for $\beta = \beta'$ — First row represents the case of $\mu = 2\%$; the second row represents the case of $\mu = 4\%$ and the third row is for $\mu = 5.8702\%$.

From Figure 2 we have the following observations

- The expected utility of a FLRS-GSA increases with the drift;
- When γ increases the expected utility of the classical annuity increases. In other words, for small risk aversion the GSA gives the best expected utility whereas for larger risk aversion the classical annuity gives the better expected utility compared to the GSA. Hence the concave form shown in Figure 2 for $\gamma = 1$ implies that there exist pairs (β, x) for which RS-GSA gives the best expected utility compared to both GSA and classical annuity;
- When the proportion invested on the risky asset x increases, the expected utility increases for large values of μ and decreases for small μ.

The obtained concave form (for instance when $\gamma = 1$) shows that the there exists β and x that maximise the expected utility of the policyholder. When the risk aversion increases, the product with the best expected utility moves from the GSA to RS-GSA and classical annuity. Note that in Figure 2 the pink lines represented the LRS-GSA.

5.3 The complete risk-sharing GSA

5.3.1 The CFRS-GSA

Similarly to the RS-GSA, Figure 3 illustrates changes of the expected utility of a CFRS-GSA with respect to $R_\gamma(\beta')$ for which we make a sensitivity analysis with respect to β', γ and x.

$\gamma = 0.5$ $\gamma = 1$

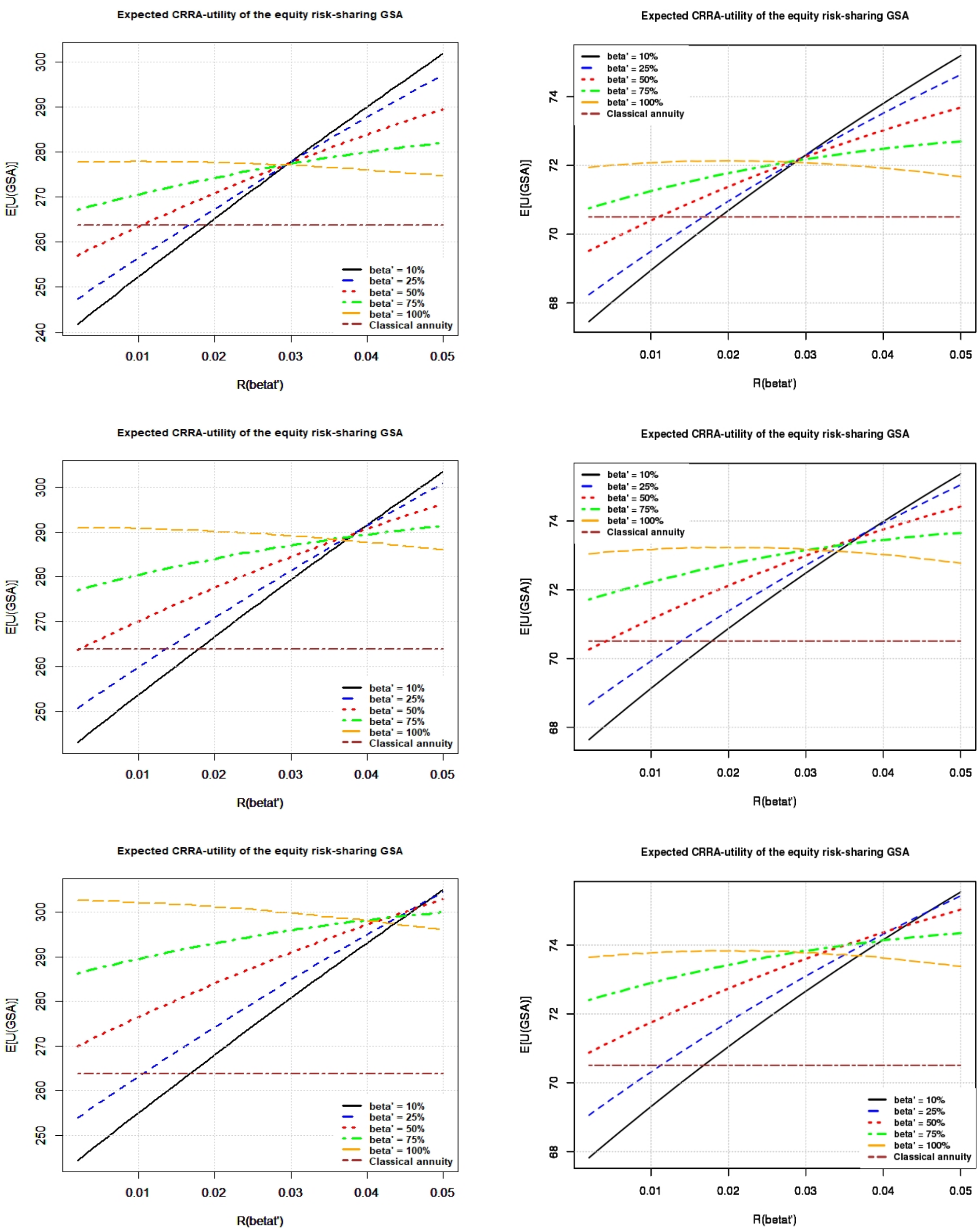

Figure 3: Expected utility of the CFRS-GSA for $\mu = 5.8702\%$ — First row represents the case of $x = 25\%$; the second row represents the case of $x = 50\%$ and the third row is for $x = 75\%$.

From this figure, we observe that there always exists pairs $(\beta', R_\gamma(\beta'))$ that improves the expected utility of a policyholder buying a CFRS-GSA compared to both GSA and classical annuity. Moreover, for all β', we can always find a $R_\gamma(\beta')$ such that GSA and CFRS-GSA give the same expected utility, unlike the classical annuity. As a result, we find that GSA and RS-GSA do not improve only the expected utility of the consumption (as concluded in [7] for the GSA), but also the expected utility of the present value of the benefits on the policyholder viewpoint. For instance, when $R_\gamma(\beta') = 2\%$, the expected utility of a CRS-GSA is greater than that of a classical annuity whereas for $R_\gamma(\beta') < 1.5\%$ the expected utility of some CRS-GSA is less than or equal to the expected utility of a classical annuity.

We further find that for $\gamma = 0$ and for all β' the expected utilities coincide at the same $R_\gamma(\beta')$; in fact we have the following proposition.

Proposition 1. *Within the financial market defined above, for a risk-neutral utility function, distinct CFRS-GSAs (with different risk proportions β's) give the same expected utilities (on the policyholder point of view) if and only if their risk adjusted discount rates are equal to the expected rate of return on the risky asset. Mathematically, for $\gamma = 0$ and for any $\beta'_1 \neq \beta'_2 \neq 0$ with $R_\gamma(\beta') := R(\beta'_1) = R(\beta'_2)$, $Ucers(\beta'_1) = Ucers(\beta'_2)$ if and only if $R_\gamma(\beta') = \log\left(\mathbb{E}[1 + R_t]\right) = \mu x + (1 - x)r$.*

We refer the reader to Appendix B for a detailed proof.

5.3.2 The CFLRS-GSA

The figure below illustrates the expected utility of a policyholder buying a CFLRS-GSA, for which me make a sensitivity study with respect to β, β' and $R(\beta, \beta')$.

$\gamma = 0.5$ $\qquad\qquad\qquad\qquad$ $\gamma = 1$

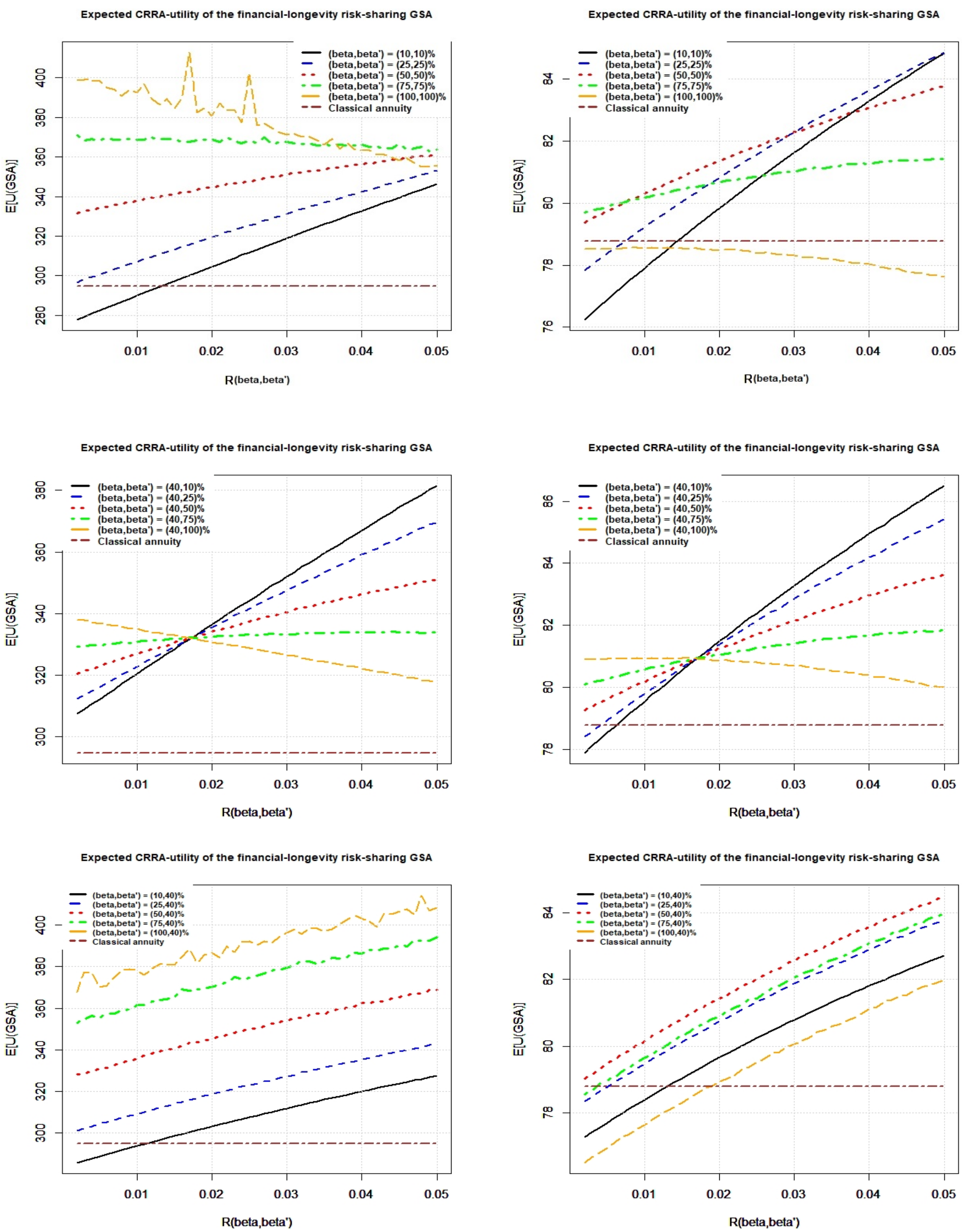

Figure 4: Expected utility of the CFLRS-GSA for $\mu = 5.8702\%$, $x = 15\%$ — First row represents the case of $\beta = \beta'$ and the second row represents the case of $\beta = 40\%$ and last row the case $\beta' = 40\%$.

We have the following observations from Figure 4,

- there exists some values of $R(\beta, \beta')$ that give better expected utility than classical annuity and GSA. In other words, one can always find triples $(\beta, \beta', R(\beta, \beta'))$ that yield better expected utility than both GSA and classical annuity. These are represented by the values of β, β' and $R(\beta, \beta')$ for which the expected utility is above that of the classical annuity and GSA.

- We further observe that the expected utility of the CFLRS-GSA increases with the risk adjusted rate whereas the utility of the GSA decreases depending on the risk aversion level and the proportions of risks borne by annuitants.

- When $\gamma = 0.5$, we get curves with an increasing volatility with respect to the proportion of longevity risk borne by policyholders. This come from the fact that for this value of γ, we deal with the expectation of a product of dependent survival probabilities, hence the randomness (or vertexes) reflects high changes on mortality experience. Whereas for $\gamma = 1$ we deal with a sum of expected survival probabilities which gives a linear function as we can see in the first column of Figure 4.

It is important to stress that for any risk adjusted rate obtained with pair (β, β'), the corresponding CRS-GSA behaves like the RS-GSA with interest rate given by the risk adjusted rate; i.e with $R = R(\beta, \beta')$. In this case we have almost similar observations and conclusions as the RS-GSA for both expected utilities and benefits. Thus the only difference will be the level of the corresponding expected utility and benefits payouts which increase with the interest rate R. Moreover, form Figures 2 and 4, we see that for larger risk proportions $(\beta,\ \beta')$ and proportion x invested on the stock, RS-GSA yields better expected utility than CRS-GSA whereas we have other way round for smaller proportions of risks and stock's proportion.

5.3.3 Benefit payouts for a FLRS-GSA

In Figure 5 below we show the changes of the benefit per unit of the initial benefit $B_t(\beta, \beta')/B_0$ with respect to time t for a 15-years term proportional FLRS-GSA and for different values of β and β'. We also show how sensitive $B_t(\beta, \beta')/B_0$ is with respect to the drift μ.

$\mu = 2\%$ $\mu = 4\%$ $\mu = 5.8702\%$

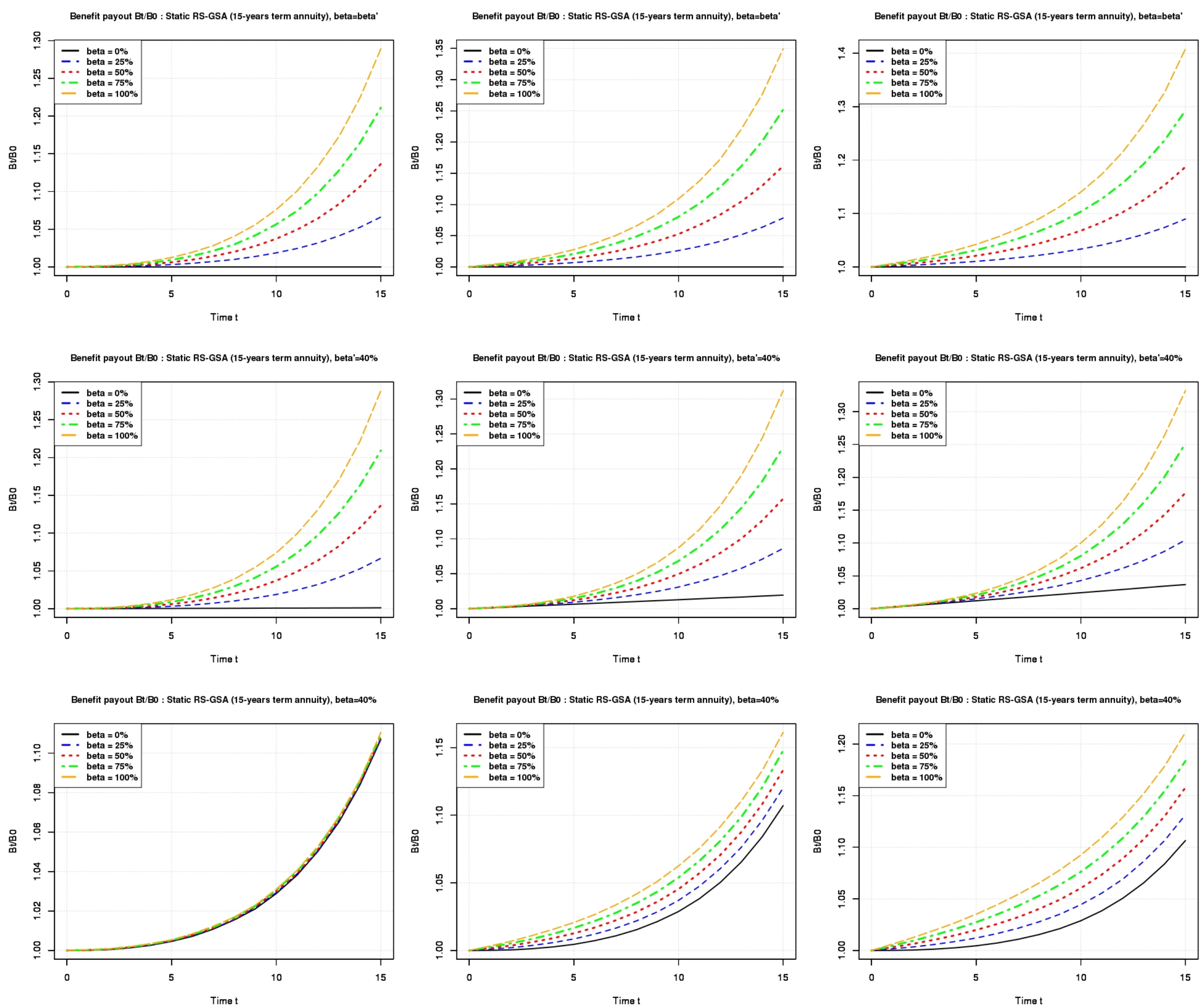

Figure 5: Representation of benefit payout $B_t(\beta, \beta')/B_0$ of a 15-years term proportional FLRS-GSA with respect to time t. The first row represents the case $\beta = \beta'$, the second is for fixed $\beta' = 40\%$ and the third is for fixed $\beta = 40\%$.

We observe that for any value of the risk proportions (i.e β, β'), the benefit payout of a proportional risk-sharing GSA increases with t, β and β'. When $\beta = \beta$ (first row), letting $\beta = \beta' = 0\%$ gives a classical annuity and letting $\beta = \beta' = 100\%$ gives a GSA. For $\beta' = 40\%$ (second row), when $\beta = 0\%$ the benefit is linear and increase with time with a tilt angle depending on μ. Whereas for fixed $\beta = 40\%$, the curves obtained with $\beta' = 0\%$ have a convex form. Furthermore, $B_t(\beta, \beta')/B_0$ increases with the drift μ.

It is important to stress that the pool or size effect is taken into account in both RS-GSA and the CRS-GSA, because it is taken into account in the definition of the GSA as shown in the respective articles (see [6], [12], [7]). Furthermore, the proportional risk sharing GSA gives larger (respectively lower) values of both the utility and the benefit payout on the policyholder point of view as compare to the classical annuity (respectively the GSA).

6 Conclusion

Aiming to design a risk-sharing annuity, we proposed in this paper different risk-sharing methods and different products based on the GSA as defined by Piggott *et al.* [6]. Our motivation to use the GSA comes from its structure in which the financial and the longevity risks are separated from each other; this allowed us to share different proportions of each of these risks between the insurer and the annuitants.

We captured the financial risk with a Black-Scholes stock and the longevity risk with a force of mortality following a Hull-White model. Our approach consisted of defining a modified GSA called the risk-sharing GSA aiming to share different proportions of financial and / or longevity (partial or whole) risk between the insurer and the annuitants. The proposed two sharing methods are the lower bound threshold risk sharing and the proportional risk sharing. The first consists of preventing the benefit payouts to go below a defined threshold; in this case only the whole risk sharing is possible. For the second, we proposed the proportional risk sharing method, which is more straightforward and intuitive as it consists of splitting the (partial or whole) risk in two parts with one part in charge of the insurer and the second in charge of the policyholders. This allowed us to define not only the risk-sharing GSA but also a range of annuity products moving from the classical annuity (where the whole risk is borne by the insurer) to the GSA (where the whole risk is borne by the annuitants) depending of the proportions of risk shifted to the annuitants. From the proportional risk-sharing GSA, we defined what we called the complete risk-sharing GSA (CRS-GSA). The CRS-GSA is obtained by defining an annuity discount rate in terms of the proportion of the financial risk borne by the annuitants called the risk adjusted discount rate and a survival probability in terms of the proportion of longevity risk borne by the annuitants called the risk adjusted survival probability. We used the CRRA utility approach (on the policyholder viewpoint) in order to find sets of the risk adjusted discount rate and probability such that the CRRA expected utility of a CRS-GSA is at least equal to that of the classical annuity, for different values of the risk aversion parameter and different values of the drift.

Numerical analysis shows that the classical annuity never improve the expected utility of the annuitant and depending on the level of risk aversion, we can find proportions of risk such that the PFLRS-GSA gives higher expected utility as compare to the GSA. Moreover, we found that a policyholder with low risk aversion will prefer the GSA whereas a policyholder with high risk aversion will go for a RS-GSA. This comes from the obtained concave behaviours of the RS-GSA's expected utility which implies the existence of optimal risk proportions; i.e the risk proportions of the annuitants in $(0, 1)$ that maximise their CRRA expected utility. In line with the results obtained by Boyle *et al.* (in term of the utility of consumption), we found that the RS-GSA have a CRRA expected utility higher than that of a classical annuity and that for a highly risk aversion policyholder, the RS-GSA improves his expected utility. Additionally, we observed that there exists some values of the risk aversion, the drift and the annuitant's risk proportions such that the classical annuity gives a better expected utility than the GSA. Numerical analysis also showed that the benefit payout of a proportional RS-GSA increases with drift, time and annuitant's risk proportions. Concerning the CRS-GSA, we found that there always exists some pairs $(\beta', R_\gamma(\beta'))$ (for the CFRS-GSA) ; $(\beta, R_\gamma(\beta))$ (for the CLRS-GSA) and $(\beta', \beta, R_\gamma(\beta, \beta'))$ (for the CFLRS-GSA) that improve the expected utility of a policyholder as compare to both the classical annuity and the GSA.

Our contribution in the literature is found in the definition of a large range of annuity products moving from the classical annuity to the GSA which in some cases improve the expected utility of a policyholder. Hence, one can think of the cost of such products depending on the risk proportions. In other words, future research could consist of analysing the trade-offs involved in the sharing process. Furthermore, it could be interesting to analyse the cost of the RS-GSA on the insurer point of view. This could be achieved on the one hand by computing the risk premium, i.e the price of a given proportions of risk shifted to the insurer. On the second hand, one could value the insurer solvency capital, so as to identify the proportions of risk that minimise the solvency capital of the insurer.

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Appendix

A) Expected utility of a FRS-GSA

$\gamma = 0.5$ $\gamma = 1$

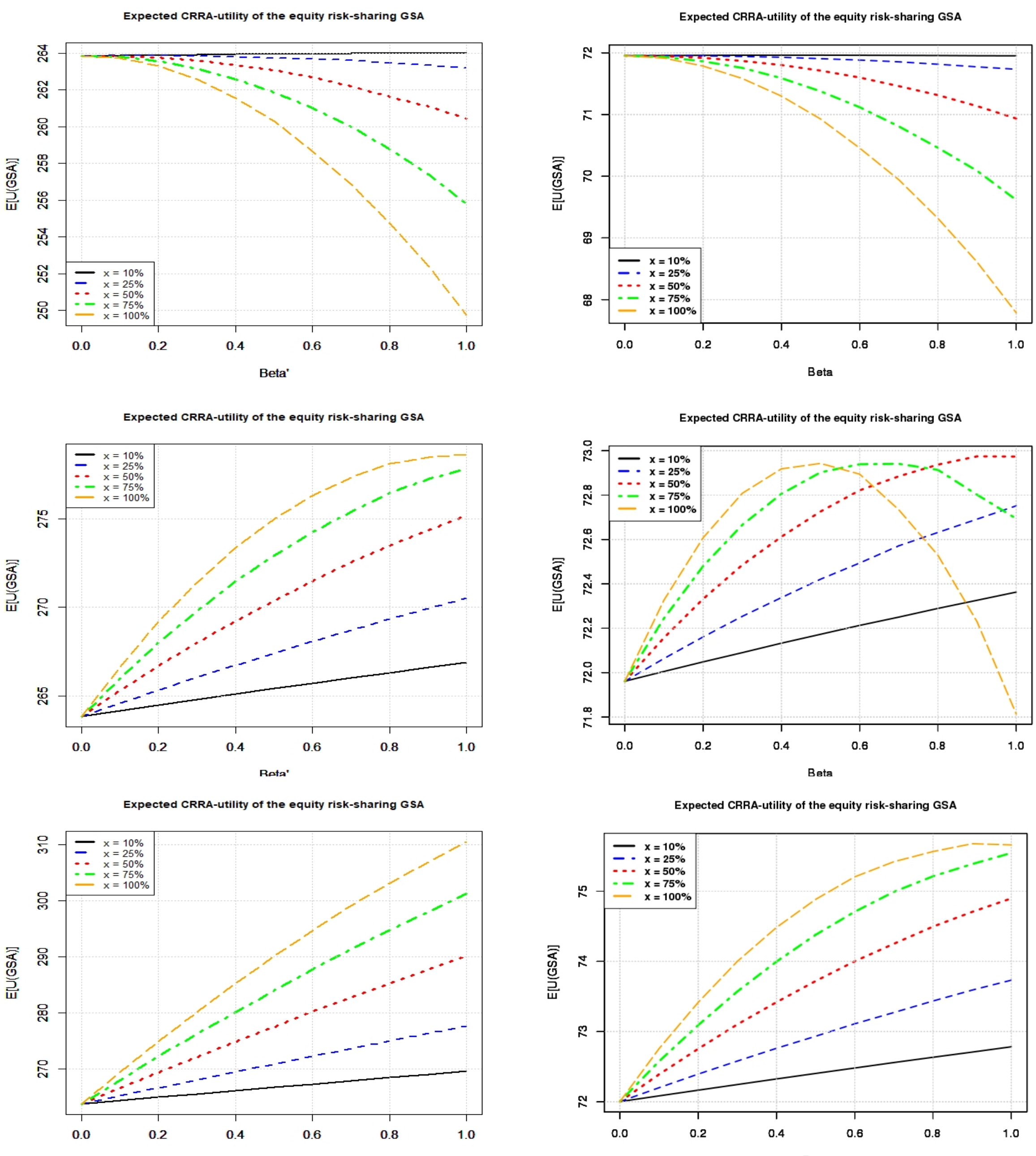

Figure 6: Expected utility of the FRS-GSA for $\beta = \beta'$ — First row represents the case of $\mu = 2\%$; the second row represents the case of $\mu = 4\%$ and the third row is for $\mu = 5.8702\%$.

B) Proof of Proposition 1

Proof. For $\gamma = 0$, we have a risk-neutral utility $U(x) = x$. Let $\beta_1' \neq \beta_2'$ and $R_\gamma(\beta') := R(\beta_1') = R(\beta_2')$; we want to show that if $Ucers(\beta_1') = Ucers(\beta_2')$ then $R_\gamma(\beta') = \log\left(\mathbb{E}[1+R_t]\right) = \mu x + (1-x)r$.

$$\begin{aligned}
Ucers(\beta_1') &= \sum_{t=0}^{n} \mathbb{E}\left[B_0(\beta_1')\prod_{j=1}^{t}\left[\beta_1'\frac{1+R_j}{1+R_\gamma(\beta')} + (1-\beta_1')\right]\right] v_\delta^t \ {}_tp_x^0 \\
&= B_0(\beta_1') + B_0(\beta_1')\sum_{t=1}^{n}\mathbb{E}\left[\prod_{j=1}^{t}\left[\beta_1'\frac{1+R_j}{1+R_\gamma(\beta')} + (1-\beta_1')\right]\right] v_\delta^t \ {}_tp_x^0 \\
&= B_0(\beta_1') + B_0(\beta_1')\sum_{t=1}^{n}\prod_{j=1}^{t}\left[\beta_1'\frac{\mathbb{E}\left[1+R_j\right]}{1+R_\gamma(\beta')} + (1-\beta_1')\right] v_\delta^t \ {}_tp_x^0 \\
&= B_0(\beta_1') + B_0(\beta_1')\sum_{t=1}^{n}\left[\beta_1'\frac{e^{\mu x+(1-x)r}}{1+R_\gamma(\beta')} + (1-\beta_1')\right]^t v_\delta^t \ {}_tp_x^0.
\end{aligned}$$

Using the first order Taylor expansion of $g(x) = x^t$, we obtain

$$\begin{aligned}
Ucers(\beta_1') &= B_0(\beta_1') + B_0(\beta_1')\sum_{t=1}^{n} v_\delta^t \ {}_tp_x^0\left(1 + t\log\left[\beta_1'\frac{e^{\mu x+(1-x)r}}{1+R_\gamma(\beta')} + (1-\beta_1')\right]\right) \\
&= B_0(\beta_1') + B_0(\beta_1')\sum_{t=1}^{n} v_\delta^t \ {}_tp_x^0 + B_0(\beta_1')\log\left[\beta_1'\frac{e^{\mu x+(1-x)r}}{1+R_\gamma(\beta')} + (1-\beta_1')\right]\sum_{t=1}^{n} t\ v_\delta^t \ {}_tp_x^0.
\end{aligned}$$

Hence for any distinct proportions β_1' and β_2', if $R(\beta_1') = R(\beta_2')$ then $B_0(\beta_1') = B_0(\beta_2')$ and $Ucers(\beta_1') = Ucers(\beta_2')$ implies that

$$\log\left[\beta_1'\frac{e^{\mu x+(1-x)r}}{1+R_\gamma(\beta')} + (1-\beta_1')\right] = \log\left[\beta_2'\frac{e^{\mu x+(1-x)r}}{1+R_\gamma(\beta')} + (1-\beta_2')\right].$$

From the first order Taylor expansion of $h(x) = e^x$, it follows that

$$R_\gamma(\beta') = e^{\mu x+(1-x)r} - 1 \simeq \mu x + (1-x)r.$$

If we assume $R_\gamma(\beta') = \log\left(\mathbb{E}[1+R_t]\right) = \mu x + (1-x)r$ then it is easy to show that $Ucers(\beta_1') = Ucers(\beta_2')$. □