UCLouvain
Institut de recherche
en mathématique et physique

Aspects of internal structures in algebraic categories

Doctoral dissertation presented by

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in partial fulfillment of the requirement for the degree of
Doctor in Sciences

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June 2026

Acknowledgements

More than five years ago, in May 2021, I came to Louvain-la-Neuve with the goal of pursuing a PhD in Categorical Algebra. The encouragement, support and kindness I have experienced over the past years, the many things I have learned, the experiences I have had, and the friends I have found fill me with a profound gratitude that I can hardly put into words.

Je tiens tout d'abord à remercier mes deux promoteurs de thèse, le Professeur Marino Gran et le Dr. Pierre-Alain Jacqmin, qui m'ont permis de réaliser cette thèse à l'UCLouvain et m'ont donné l'opportunité de m'immerger dans tant de sujets mathématiques passionnants. Merci infiniment pour votre confiance en mes capacités, vos encouragements sans faille, le temps et la patience que vous m'avez accordés, ainsi que pour votre générosité à partager avec moi vos connaissances et vos idées.

I would also like to thank Prof. Tim Van der Linden, Prof. Diana Rodelo, Prof. Alan Cigoli and Prof. Pierre-Emmanuel Caprace, members of my examination committee, for their feedback on my doctoral thesis. My gratitude also extends to Prof. Enrico Vitale and Prof. Wendy Lowen, members of my advisory committee, for their suggestions during my midterm evaluation.

I would particularly like to thank Prof. George Janelidze for his time, as well as his valuable insights and advice during my two research stays at the University of Cape Town in the springs of 2023 and 2025. I also wish to express my gratitude to Prof. Jérôme Scherer and Prof. Kathryn Hess Bellwald for their exceptional hospitality during my research stay at the Ecole Polytechnique Fédérale de Lausanne in the spring of 2026. Additionally, I would like to thank Prof. Richard Blute and Prof. Paul-Eugène Parent at the University of Ottawa, as well as Prof. Emily Riehl at Johns Hopkins University for their warm openness during my research visits in the summers of 2023 and 2024.

My research trips were enriching not only on a professional level, thanks to Samantha, Nicholas and Sarah in Cape Town, and Alex, Basile, Bjørnar, Joe, Markus and Virgile in Lausanne. Thank you for welcoming me so warmly and for sharing so many experiences with me.

Un merci tout particulier va à l'équipe administrative, Cathy Brichard, Martine Furnémont, Elodie Hanoy et Carine Baras, pour leur travail essentiel.

I consider myself incredibly fortunate to have been surrounded by so many wonderful people over the past five years. Lucy, Julia and Sofia: it was an ab-

solute pleasure sharing the office with you. Maxime, Aline, Sandrine, Robynn, Feba and David: thank you for your friendship, your generosity and your willingness to help. A big thank you to the entire Category Theory team for so many interesting and enjoyable seminars and conferences. I was filled with immense pride to be part of "Team Louvain-la-Neuve". Thank you to everyone who made lunch breaks in the 'salle de convivialité' and the 4 o'clock coffee breaks so refreshing and energizing — and for all those 'after-work' moments, too.

Obwohl ihr meistens weiter weg seid, als ich es mir wünsche, kann ich doch auf unsere Freundschaft zählen. Danke, Malin, Franzi, Hanna, Juli, Kevin und Basil, für die regelmäßigen Telefonate und Besuche in Belgien und Deutschland.

Merci, François, d'être entré dans ma vie avec autant de goût pour l'aventure et d'énergie, et de toujours pouvoir compter sur ton soutien.

Zuletzt möchte ich meiner Familie danken. Ohne euren Rückhalt und eure Unterstützung wäre diese Promotion nicht möglich gewesen. Danke, Mama, dass du mir immer mit Rat und Tat zur Seite stehst. Danke, Papa, dass für dich keine Distanz zu weit ist, um mir in Not zu Hilfe zu eilen. Danke, Gerrit und Jonathan, dass ihr es nie müde werdet, mich in Belgien zu besuchen.

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Introduction

From abelian to semi-abelian via Mal'tsev categories

Category theory was introduced by S. Eilenberg and S. Mac Lane in the 1940s [EML45] to formalize the notion of natural transformation, that was used informally by the topologists of that time. In order to do so, they first had to define the notions of category and functor. Though "Initially, Eilenberg and Mac Lane had written what they thought would perhaps be the only necessary research paper on categories — for the rest, categories and functors would provide a useful language for mathematicians" [ML88, p. 354], category theory has developed immensely since then and is now used in many different fields of mathematics, theoretical computer science, physics and neuroscience.

One of the first major achievements of category theory was the introduction of the notion of abelian category in 1955 by D. Buchsbaum [Buc55]; see also the independent work [Gro57] by A. Grothendieck from 1957. Abelian categories capture typical algebraic properties of the categories of abelian groups and, more generally, of modules over a ring. In fact, sheaves of abelian groups (or modules) on a topological space also form an abelian category. Subsequently, the concept of abelian category proved particularly useful in the development of homological algebra.

In [ML88, p. 346], S. Mac Lane writes "Then in 1963 it suddenly became clear that general category theory (not just abelian categories or applications of categories) was a viable field of mathematical research. It is difficult to understand why so many instances of this development came in just this one year." He stresses the mimeograph edition of SGA IV in 1963-64 [AGV72], F.W. Lawvere's doctoral dissertation [Law63] containing the concept of what is nowadays called a Lawvere (algebraic) theory, P. Freyd's first public presentation of his adjoint functor theorem [Fre66], S. Mac Lane's article [ML63] on his coherence theorem for symmetric monoidal categories, and C. Ehresmann's article [Ehr63] on, in modern terms, internal categories, first formalized by J. Bénabou.

Abelian categories are defined by a particularly simple list of axioms. Namely, a category is abelian if and only if it is pointed, i.e. it has a zero object, it has binary products, and any morphism factors as a normal epimorphism, i.e. the cokernel of a morphism, followed by a normal monomorphism, i.e. a kernel of a morphism. In the category of abelian groups, for which we generally use the additive notation, normal epimorphisms are surjective group homomorphisms, and normal monomorphism are (up to isomorphism) subgroup inclusions. The

(normal epimorphism, normal monomorphism)-factorization of a group homomorphism $f : A \to B$ between two abelian group A, B is given by its classical image factorization $A \twoheadrightarrow f(A) \rightarrowtail B$. Tierney's equation states that a category is abelian if and only if it is exact in the sense of M. Barr [Bar71] and additive. In particular, finite products and coproducts coincide in an abelian category $\mathscr{A}$, and any hom-set $\mathrm{Hom}(A, B)$ between two objects A, B of $\mathscr{A}$ has a canonical abelian group structure. Given two abelian groups A, B, their biproduct $A \oplus B$, that serves at the same time as a binary product and coproduct, can be described as the cartesian product of the underlying sets of A, B equipped with the componentwise addition. Given two group homomorphisms $f, g : A \to B$ between two abelian groups A, B, their sum $f + g : A \to B$ is defined pointwise, i.e., $(f + g)(a) = f(a) + g(a)$ for all $a \in A$. In 1960 [Lub60], S. Lubkin showed that any small abelian category admits a faithful conservative embedding into the category of abelian groups which preserves all finite limits and finite colimits. In this sense, the category of abelian groups can be seen as a role model for all abelian categories.

The category Grp of groups, for which we generally use the multiplicative notation, although being pointed and exact, is not abelian. The normal monomorphisms in Grp are (up to isomorphism) normal subgroup inclusions, and hence not any group homomorphism factors as a normal epimorphism, i.e. a surjective group homomorphism, followed by a normal monomorphism. Moreover, the binary product $A \times B$ of two groups A, B is given by the cartesian product of their underlying sets equipped with the componentwise multiplication, whereas the binary coproduct $A + B$ is given by their so-called free product. Given two group homomorphisms $f, g : A \to B$ between groups A, B, the function $f \cdot g : A \to B$ defined by $(f \cdot g)(a) = f(a) \cdot g(a)$, where $a \in A$, is in general not a group homomorphism. In [JMT02] from 2002, G. Janelidze, L. Márki and W. Tholen write "when reading once again from [ML50, p. 507] [...] we are reminded that, right from the beginning of Category Theory, it was very much the intention to *find a list of axioms which reflect the properties of groups, rings and algebras as nicely as the abelian-category axioms do for abelian groups and modules*". However, even though there were many different proposals for such a list of axioms, related to isomorphism and decomposition theorems [Hig56, Kur59, Wyl71], radical theory [Ami54, Shu60], or non-abelian homological algebra [Frö61, Huq68, Ger70, Orz72], no generally accepted one emerged out of these investigations.

In 1990 [CLP90], A. Carboni, J. Lambek and M.C. Pedicchio introduced the notion of exact Mal'tsev category as a categorical generalization of the notion of Mal'tsev variety from universal algebra, and developed some aspects of homological lemmas in this non-abelian setting. Two years later [CPP92], A. Carboni, M.C. Pedicchio and N. Pirovano defined Mal'tsev categories in the more general finitely complete setting, as those categories in which any reflexive relation is an equivalence relation. Depending on the context, Mal'tsev categories allow for various characterizations. For instance, a regular category $\mathscr{C}$ is a Mal'tsev category if and only if the composition of equivalence relations on any fixed object in $\mathscr{C}$ is commutative. A (finitary, one-sorted) variety of universal algebras is a Mal'tsev category if and only its equational theory has a

ternary term $p(x, y, z)$ satisfying the identities $p(x, x, y) \approx y$ and $p(x, y, y) \approx x$ [Mal54]. Examples of Mal'tsev categories include the categories of groups, rings, algebras, Heyting algebras, topological groups, and the dual of any elementary topos such as the dual of the category of sets. Mal'tsev categories provide a good context in which to develop the theory of centrality of equivalence relations [Ped95, Ped96, BG02c, Bou04], Baer sums of central extensions [Bou99, Bou07], and some non-pointed homological lemmas [Bou03, GR14]. Further results on Mal'tsev categories have been proved in the study of central extensions [JK94, Eve14, DG18], and (non-abelian) embedding theorems [BJ08, Jac18, Jac19, Jac20]. We refer the reader to [BJG21] for more details on the history of Mal'tsev categories, and to [BB04, Bou17] for thorough textbook accounts on Mal'tsev categories.

That being said, the notion of exact Mal'tsev category "seems to suit, in many aspects, the needs of universal algebra perfectly, but those of homological algebra of group- and ring-like structures much less so" [JMT02, p. 369]. In [BJG21], D. Bourn, M. Gran and P.-A. Jacqmin write "The homological ambition of the initial project of Carboni, Lambek and Pedicchio [CLP90] was not totally achieved. As a matter of fact, this early work was missing the notion of pointed protomodular category [Bou91] which was necessary to define and investigate the conceptual categorical notion of short exact sequences". In 2002 [JMT02], G. Janelidze, L. Márki and W. Tholen introduced the notion of semi-abelian category, nowadays seen, to groups, as what abelian categories are to abelian groups. More precisely, a semi-abelian category is a category that is exact, pointed, protomodular and finitely cocomplete. The concept of protomodularity was introduced by D. Bourn in 1991 [Bou91] and, in the pointed setting, is equivalent to the validity of the Split Short Five Lemma. Moreover, any finitely complete protomodular category is a Mal'tsev category. A variety of universal algebras is semi-abelian if and only if its theory admits a unique constant e, binary terms $\alpha_1(x, y), \ldots, \alpha_n(x, y)$, where $n \in \mathbb{N}$ is a natural number, and an $(n + 1)$-ary term β satisfying the identities $\beta(\alpha_1(x, y), \ldots, \alpha_n(x, y), y) \approx y$ and $\alpha_i(x, x) \approx e$ for all $i \in \{1 \ldots, n\}$ [BJ03]. Examples of semi-abelian categories include the categories of groups, rings, algebras, crossed modules of groups, compact Hausdorff groups, cocommutative Hopf algebras over a field [GKV18, GSV19], and the dual of the category of pointed objects in any elementary topos such as the dual of the category of pointed sets. Many homological lemmas such as the Noether Isomorphism Theorems, the Short Five Lemma, the Nine Lemma and the Snake Lemma generalize to the semi-abelian context [BB04]. In addition, semi-abelian categories allow for good notions of internal object action [BJK05a] and of semi-direct product [BJ98].

A syntactic characterization of weakly Mal'tsev varieties

A small category consists of a set C_0 of *objects*, a set C_1 of *arrows*, a function $d : C_1 \to C_0$ mapping any arrow to its *domain*, a function $c : C_0 \to C_1$ mapping any arrow to its *codomain*, a function $e : C_0 \to C_1$ mapping any object x to its *identity* arrow whose domain and codomain is x, and a function $m : C_1 \times_{C_0} C_1 \to C_1$, where $C_1 \times_{C_0} C_1$ is the set of pairs $(f, g) \in C_1 \times C_1$ of

arrows for which the codomain of f coincides with the domain of g, mapping any *composable* pair of morphisms to its *composite* satisfying certain conditions. Note that the set $C_1 \times_{C_0} C_1$ can equally be described as the pullback

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{p_2} & C_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle d} \\ C_1 & \xrightarrow[c]{} & C_0 \end{array}$$

of the domain function $d : C_1 \to C_0$ along the codomain function $c : C_1 \to C_0$ in the category of sets. Here the functions $p_1, p_2 : C_1 \times_{C_0} C_1 \to C_1$ map a pair (f, g) to f, g, respectively. An internal category in an arbitrary category $\mathscr{C}$ with pullbacks is given by a diagram

$$C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \underset{c}{\overset{d}{\underset{\leftarrow e \rightarrow}{\rightrightarrows}}} C_0 \tag{1}$$

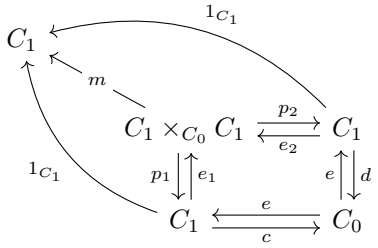
in $\mathscr{C}$, where $C_1 \times_{C_0} C_1$ is the pullback of the *domain* morphism $d : C_1 \to C_0$ along the *codomain* morphism $c : C_1 \to C_0$ in $\mathscr{C}$, such that the *identities* morphism $e : C_0 \to C_1$ is a common splitting of d, c, i.e., $de = 1_{C_0} = ce$, and the *multiplication* morphism $m : C_1 \times_{C_0} C_1 \to C_1$ satisfies the usual incidence, unitality and associativity conditions. In this way, the notions of small category and of internal category in the category of sets coincide. Similarly, one can *internalize* the notion of functor between two small categories leading to the category $\mathrm{Cat}(\mathscr{C})$ whose objects are internal categories in $\mathscr{C}$ and whose morphisms are internal functors.

Internal categories in Mal'tsev categories are well-behaved [CPP92]. For example, any reflexive graph in a Mal'tsev category $\mathscr{C}$ admits at most one internal category structure, meaning that, given objects C_0, C_1 and morphisms d, c, e as in Diagram (1) with $de = 1_{C_0} = ce$, then there exists at most one morphism $m : C_1 \times_{C_0} C_1 \to C_0$ such that (C_0, C_1, d, c, e, m) is an internal category. Furthermore, any internal category in a Mal'tsev category is automatically an *internal groupoid*. The former follows from the fact that, in a Mal'tsev category $\mathscr{C}$, *canonical pullback injections* $e_1 : A \to A \times_B C$, $e_2 : C \to A \times_B C$ are jointly extremally epimorphic, i.e., they do not factor through a common proper subobject of $A \times_B C$. Here we consider the pullback $(A \times_B C, p_1, p_2)$ of a morphism $g : C \to B$ along a morphism $f : A \to B$, that have respective splittings $s : B \to C$, $r : B \to A$, i.e., $gs = 1_B$, $fr = 1_B$, as in Diagram 2. By 'canonical pullback injections' we mean the unique morphisms $e_1 : A \to A \times_B C$, $e_2 : C \to A \times_B C$ with $p_1 e_1 = 1_A$, $p_2 e_1 = sf$, and $p_2 e_2 = 1_C$, $p_1 e_2 = rg$.

$$\begin{array}{ccc} A \times_B C & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & C \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle g} \\ A & \underset{f}{\overset{r}{\rightleftarrows}} & B \end{array} \tag{2}$$

Given an internal category (C_0, C_1, d, c, e, m), the multiplication morphism $m : C_1 \times_{C_0} C_1 \to C_1$ is thus the unique morphism with $me_1 = 1_{C_1} = me_2$ (this

is the unitality axiom of an internal category), where $e_1, e_2 : C_1 \to C_1 \times_{C_0} C_1$ are the canonical pullback injections in the pullback of $d : C_1 \to C_0$ along $c : C_1 \to C_0$ with common splitting $e : C_0 \to C_1$ as in the following diagram:



In 2008 [MF08], N. Martins-Ferreira introduced, as a generalization of the notion of Mal'tsev category, that of weakly Mal'tsev category. More precisely, a weakly Mal'tsev category is a category $\mathscr{C}$ with pullbacks of split epimorphisms along split epimorphisms such that, given an instance of Diagram (2) in $\mathscr{C}$, the canonical pullback injections $e_1 : A \to A \times_B C$, $e_2 : C \to A \times_B C$ are jointly epimorphic (a condition slightly weaker than that of being jointly extremally epimorphic). In [JMF12], it is shown that a finitely complete category is a weakly Mal'tsev category if and only if any reflexive *strong* relation is an equivalence relation. Examples of weakly Mal'tsev categories that are not Mal'tsev categories are given by the categories of distributive lattices [MF12], CancCommMon of cancellative commutative monoids, and the dual of the category of topological spaces [MF15]. While a reflexive graph in a weakly Mal'tsev category still admits at most one internal category structure, the usual preorder on the non-negative integers $\mathbb{N}_0$ (viewed as an internal category in CancCommMon) shows that there can be internal categories that are not internal groupoids in weakly Mal'tsev categories.

In Chapter 4, based on my joint work [EJMF24] with P.-A. Jacqmin and N. Martins-Ferreira, we prove a syntactic characterization of weakly Mal'tsev varieties. Namely, a variety of universal algebras is a weakly Mal'tsev category if and only if there exist non-negative integers $k, m, N \in \mathbb{N}_0$, binary terms $f_1, g_1, \ldots, f_k, g_k$, ternary terms $p_1, \ldots, p_m$, $2(k+2m+1)$-ary terms $s_1, \ldots, s_N$, $2(k+m+2)$-ary terms $\sigma_1 \ldots, \sigma_{N+1}$ and, for all $i \in \{1, \ldots, N+1\}$, $(k+m+1)$-ary terms $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ satisfying certain equations (Theorem 4.2.2). We apply this result to the variety of distributive lattices (Example 4.2.4), and observe that a slight variation of the syntactic characterization of weakly Mal'tsev varieties yields a syntactic characterization of the varieties of universal algebras in which any reflexive *regular* relation is an equivalence relation (Theorem 4.3.4).

Categorical Galois theory and generalized Hopf formulae for homology

An important modern categorical-algebraic method to analyze algebraic structures is given by categorical Galois theory, and its derived theories of (higher) central extensions and generalized Hopf formulae for homology. Categorical Galois theory was introduced and developed in the 1980s by G. Janelidze [Jan84, Jan89, Jan90b, Jan91a], and not only provides a categorical general-

ization of classical Galois theory and its various generalizations [GR71, Mag74] but it also recovers multiple notions of central extensions.

A Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ consists of an adjunction between two categories $\mathscr{C}, \mathscr{F}$ with left adjoint $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ and right adjoint $\mathsf{U} : \mathscr{F} \to \mathscr{C}$, and two classes $\mathscr{E}, \mathscr{Z}$ of morphisms in $\mathscr{C}, \mathscr{F}$, respectively, satisfying certain conditions. Relative to the Galois structure Γ, the notions of trivial, normal and central extensions in $\mathscr{C}$ are defined. If the Galois structure Γ satisfies a certain *admissibility* condition, one can think of $\mathscr{C}$ as a 'complicated' category whose class $\mathscr{E}$ of 'extensions' one wants to understand, and think of $\mathscr{F}$ as a category whose class $\mathscr{Z}$ of 'extensions' is easier to understand. The trivial extensions in $\mathscr{C}$ are those 'coming' from $\mathscr{F}$, and the Fundamental Theorem of Categorical Galois Theory provides a classification of certain central extensions in $\mathscr{C}$ in terms of actions of a certain *Galois pregroupoid*.

An example of an admissible Galois structure is given by

$$\Gamma = (\mathrm{Grp}, \mathrm{Ab}, \mathsf{ab}, \mathsf{U}, \mathrm{Surj}(\mathrm{Grp}), \mathrm{Surj}(\mathrm{Ab})),$$

where Grp is the category of groups, Ab is the category of abelian groups, $\mathsf{ab} : \mathrm{Grp} \to \mathrm{Ab}$ is the abelianization functor which maps a group G to the quotient $G/[G,G]$ by its commutator subgroup, $\mathsf{U} : \mathrm{Ab} \to \mathrm{Grp}$ is the inclusion functor, and $\mathrm{Surj}(\mathrm{Grp}), \mathrm{Surj}(\mathrm{Ab})$ are the classes of surjective homomorphisms in $\mathrm{Grp}, \mathrm{Ab}$, respectively. In this case, the notions of normal and central extension coincide, and one precisely recovers the classical notion of central extension of groups, i.e. of a surjective group homomophism $f : A \to B$ with kernel lying in the centre of A.

More generally, the 1994 article [JK94] by G. Janelidze and G.M. Kelly studies Galois structures induced by Birkhoff subcategories $\mathscr{F}$ in exact categories $\mathscr{C}$. Birkhoff subcategories are a categorical generalization of the notion of subvariety of a variety of universal algebras, in the sense that the Birkhoff subcategories of a variety of universal algebras are precisely its subvarieties [Bir35]. More precisely, a Birkhoff subcategory of a category $\mathscr{C}$ is a full reflective subcategory that is closed under isomorphisms, subobjects and regular quotients in $\mathscr{C}$. An important property of Birkhoff subcategories is that a Birkhoff subcategory of an exact category is also exact. Note that this is not the case in general of a reflective subcategory of an exact category. For example, the category $\mathrm{Ab}_{\mathrm{tf}}$ of torsion-free abelian groups is a full reflective subcategory of Ab closed under isomorphisms and subobjects but it is not closed under regular quotients in Ab and not exact.

Given a Birkhoff subcategory $\mathscr{F}$ of an exact category $\mathscr{C}$, the datum

$$\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathrm{RegEpi}(\mathscr{C}), \mathrm{RegEpi}(\mathscr{C}))$$

is a Galois structure, where $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ is the reflection functor, $\mathsf{U} : \mathscr{F} \to \mathscr{C}$ is the inclusion functor, and $\mathrm{RegEpi}(\mathscr{C}), \mathrm{RegEpi}(\mathscr{C})$ are the classes of regular epimorphisms in $\mathscr{C}, \mathscr{F}$, respectively. If $\mathscr{C}$ is moreover a Mal'tsev category, then the Galois structure Γ is automatically admissible, and the notions of normal and central extensions coincide. For $\mathscr{C}$ a variety of Ω-groups and $\mathscr{F}$ one of its subvarieties, [JK94] shows that the categorical notion of central extension recovers that of central extension in the sense of A. Fröhlich [Frö63], see also

[Lue67, FC72]. For $\mathscr{C}$ a Mal'tsev variety and $\mathscr{F}$ its subvariety of abelian algebras, [JK00a, JK00b] (see also [BG02a] for a generalization to semi-abelian categories) prove that the categorical notion of central extension recovers that coming from commutator theory in universal algebra.

In [Gra99] from 1999, M. Gran shows that the category $\mathrm{Grpd}(\mathscr{C})$ of internal groupoids in a regular Mal'tsev category $\mathscr{C}$ is also a regular Mal'tsev category. If $\mathscr{C}$ is in addition exact, then $\mathrm{Grpd}(\mathscr{C})$ is also exact. In [Gra01] published two years later, M. Gran characterizes the central extensions in $\mathrm{Grpd}(\mathscr{C})$, where $\mathscr{C}$ is an exact Mal'tsev category, relative to its Birkhoff subcategory $\mathrm{DGrpd}(\mathscr{C})$ of discrete internal groupoids. More precisely, if $\mathscr{C}$ is an exact Mal'tsev category, then $\Gamma = (\mathrm{Grpd}(\mathscr{C}), \mathscr{C}, \pi_0, \mathsf{D}, \mathrm{RegEpi}(\mathrm{Grpd}(\mathscr{C})), \mathrm{RegEpi}(\mathscr{C}))$ is an admissible Galois structure, where the left adjoint $\pi_0 : \mathrm{Grpd}(\mathscr{C}) \to \mathscr{C}$ is the so-called connected components functor which maps an internal groupoid with underlying reflexive graph (C_0, C_1, d, c, e) to the coequalizer of $d, c : C_1 \to C_0$, and the right adjoint $\mathsf{D} : \mathscr{C} \to \mathrm{Grpd}(\mathscr{C})$ is the so-called discrete functor which maps an object C to the discrete internal groupoid with underlying reflexive graph $(C, C, 1_C, 1_C, 1_C)$. Given two internal groupoids $\mathbb{C}, \mathbb{D}$ with respective underlying reflexive graphs (C_0, C_1, d, c, e), (D_0, D_1, d, c, e), an internal functor $(f_0 : C_0 \to D_0, f_1 : C_1 \to D_1)$ between $\mathbb{C}, \mathbb{D}$ (as depicted below) is a central extension relative to Γ if and only if it is a regular epimorphism, meaning that both f_0, f_1 are regular epimorphisms in $\mathscr{C}$, as well as a discrete fibration, meaning that any of the two commutative squares here below are pullbacks:

$$\begin{array}{ccc} C_1 & \xrightarrow{f_1} & D_1 \\ {\scriptstyle d}\downarrow\!\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\!\downarrow{\scriptstyle c} \\ C_0 & \xrightarrow[f_0]{} & D_0 \end{array} \tag{3}$$

Note that, if $\mathscr{C}$ is a semi-abelian category, $\mathrm{DGrpd}(\mathscr{C})$ is not only a Birkhoff subcategory but also a torsion-free subcategory of $\mathrm{Grpd}(\mathscr{C})$ whose corresponding torsion subcategory is given by the full subcategory of $\mathrm{Grpd}(\mathscr{C})$ with objects the connected internal groupoids, meaning that the induced morphism $(d, c) : C_1 \to C_0 \times C_0$ is a regular epimorphism [EG10]. Torsion theories were introduced by S.E. Dickson in 1966 [Dic66] in abelian categories but later proved also useful in more general pointed contexts, see e.g. [CDT06, BG06, JT07]. More recently, the notion of pretorsion theory in non-pointed settings was introduced and studied in [FFG21, BCGT23, BCG25]. A torsion theory in a pointed category $\mathscr{C}$ consists of two replete full subcategories $\mathscr{F}$ (the torsion-free subcategory) and $\mathscr{T}$ (the torsion subcategory) of $\mathscr{C}$ such that any morphism in $\mathscr{C}$ from an object in $\mathscr{T}$ to an object in $\mathscr{F}$ factors necessarily through the zero object, and, for any object C in $\mathscr{C}$, there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow C \longrightarrow F \longrightarrow 0,$$

where T is an object in $\mathscr{T}$ and F is an object in $\mathscr{F}$. The terminology 'torsion theory' is inspired by the fact that the subcategories of torsion and of torsion-free abelian groups form a torsion theory in Ab. Torsion subcategories turn out to be coreflective subcategories, and torsion-free subcategories are reflective

subcategories. A torsion theory $(\mathscr{T}, \mathscr{F})$ in a pointed regular category $\mathscr{C}$ induces the admissible Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathrm{RegEpi}(\mathscr{C}), \mathrm{RegEpi}(\mathscr{F}))$, where $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ is the reflection functor and $\mathsf{U} : \mathscr{F} \to \mathscr{C}$ is the inclusion functor.

In [EG10] from 2010, T. Everaert and M. Gran generalize the results of [Gra01] in the semi-abelian context, see also [EG14]. More precisely, they consider the admissible Galois structure

$$\Gamma = (\mathrm{Grpd}^n(\mathscr{C}), \mathscr{C}, \pi_0 \circ \cdots \circ \pi_0^n, \mathsf{D}^n \circ \cdots \circ \mathsf{D}, \mathrm{RegEpi}(\mathrm{Grpd}^n(\mathscr{C})), \mathrm{RegEpi}(\mathscr{C})),$$

where $\mathscr{C}$ is a semi-abelian category. In [BG02a], it is shown that, if $\mathscr{C}$ is a semi-abelian category, then $\mathrm{Grpd}(\mathscr{C})$ is also semi-abelian. This implies that the recursively defined category $\mathrm{Grpd}^n(\mathscr{C}) = \mathrm{Grpd}(\mathrm{Grpd}^{n-1}(\mathscr{C}))$ of internal n-fold groupoids in $\mathscr{C}$ is also semi-abelian. For $2 \leq i \leq n$,

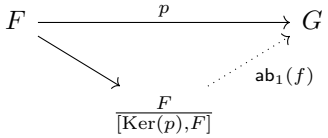
$$\pi_0^i : \mathrm{Grpd}^i(\mathscr{C}) \to \mathrm{Grpd}^{i-1}(\mathscr{C})$$

denotes the connected components functor for $\mathrm{Grpd}^i(\mathscr{C}) = \mathrm{Grpd}(\mathrm{Grpd}^{i-1}(\mathscr{C}))$, and $\mathsf{D}^i : \mathrm{Grpd}^{i-1}(\mathscr{C}) \to \mathrm{Grpd}^i(\mathscr{C})$ is the corresponding discrete functor. T. Everaert and M. Gran show that the functor $\pi_0 : \mathrm{Grpd}(\mathscr{C}) \to \mathscr{C}$ (hence also $\pi_0 \circ \cdots \circ \pi_0^n$) is protoadditive, i.e., it preserves split short exact sequences. This allows them to characterize the central extensions in $\mathrm{Grpd}^n(\mathscr{C})$ with respect to Γ. Moreover, they characterize the higher central extensions and compute generalized Hopf formulae for homology, that we now describe.

Hopf's theorem from 1942 [Hop42] states that the second integral homology group $H_2(G, \mathbb{Z})$ of a group G can be computed via a presentation of G. More precisely, given a surjective group homomorphism $p : F \to G$, where F is a free group, it holds that

$$H_2(G, \mathbb{Z}) \simeq \frac{[F, F] \cap \mathrm{Ker}(p)}{[\mathrm{Ker}(p), F]}.$$

Remember that the abelianization functor $\mathsf{ab} : \mathrm{Grp} \to \mathrm{Ab}$ maps the group F to its quotient $F/[F, F]$. Moreover, the category $\mathrm{CExt}(\mathrm{Grp})$ of central extensions of groups is a full reflective subcategory of the category $\mathrm{Ext}(\mathrm{Grp})$ of surjective group homomorphisms seen as a full subcategory of the arrow category $\mathrm{Arr}(\mathrm{Grp})$ of Grp whose objects are group homomorphisms. The reflection $\mathsf{ab}_1 : \mathrm{Ext}(\mathrm{Grp}) \to \mathrm{CExt}(\mathrm{Grp})$ maps the surjective group homomorphism $p : F \to G$ to the central extension $F/[\mathrm{Ker}(p), F] \to G$ induced by p as in the following diagram:



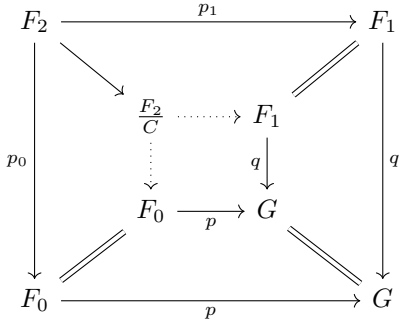
In [Jan91b] from 1991, G. Janelidze observes that the Galois structure

$$\Gamma = (\mathrm{Grp}, \mathrm{Ab}, \mathsf{ab}, \mathsf{U}, \mathrm{Surj}(\mathrm{Grp}), \mathrm{Surj}(\mathrm{Ab}))$$

induces another Galois structure

$$\Gamma_1 = (\mathrm{Ext}(\mathrm{Grp}), \mathrm{CExt}(\mathrm{Grp}), \mathsf{ab}_1, \mathsf{U}_1, \mathscr{E}^1, \mathscr{Z}^1),$$

where $\mathsf{U}_1 : \mathrm{CExt}(\mathrm{Grp}) \to \mathrm{Ext}(\mathrm{Grp})$ is the inclusion functor, and $\mathscr{E}^1, \mathscr{Z}^1$ are the classes of so-called double extensions in $\mathrm{Surj}(\mathrm{Grp}), \mathrm{Surj}(\mathrm{Ab})$, respectively. Furthermore, the category $\mathrm{CExt}^2(\mathrm{Grp})$ of double central extensions of groups, i.e. the central extensions with respect to the Galois structure Γ_1, is a full reflective subcategory of the category $\mathrm{Ext}^2(\mathrm{Grp})$ of double extensions of groups viewed as a full subcategory of $\mathrm{Arr}(\mathrm{Ext}(\mathrm{Grp}))$ with objects the morphisms in $\mathscr{E}^1$. The reflection $\mathsf{ab}_2 : \mathrm{Ext}^2(\mathrm{Grp}) \to \mathrm{CExt}^2(\mathrm{Grp})$ maps a double extension given by the outer square of the following diagram to the induced inner square, where C is the product of the two subgroups $[\mathrm{Ker}(p_0) \cap \mathrm{Ker}(p_1), F_2]$, $[\mathrm{Ker}(p_0), \mathrm{Ker}(p_1)]$ of F_2:



Note that, if the double extension as above is a *double free presentation* of G, then the third integral homology group $H_3(G, \mathbb{Z})$ of G is given by

$$H_3(G, \mathbb{Z}) \simeq \frac{[F_2, F_2] \cap \mathrm{Ker}(p_0) \cap \mathrm{Ker}(p_1)}{[\mathrm{Ker}(p_0) \cap \mathrm{Ker}(p_1), F_2][\mathrm{Ker}(p_0), \mathrm{Ker}(p_1)]}.$$

In [EGVdL08] from 2008, T. Everaert, M. Gran and T. Van der Linden show that, for a semi-abelian category $\mathscr{C}$ with a Birkhoff subcategory $\mathscr{F}$ with corresponding reflection $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ and inclusion functor $\mathsf{U} : \mathscr{F} \to \mathscr{C}$, the Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathrm{RegEpi}(\mathscr{C}), \mathrm{RegEpi}(\mathscr{F}))$ induces a sequence of Galois structures

$$\Gamma_n = (\mathrm{Ext}^n(\mathscr{C}), \mathrm{CExt}^n(\mathscr{C}), \mathsf{F}_n, \mathsf{U}_n, \mathscr{E}^n, \mathscr{Z}^n),$$

where $n \in \mathbb{N}$, similarly as we have explained above. Based on these, they define, for any $n \in \mathbb{N}_0$, the so-called $(n+1)$-st generalized Hopf formula for homology $H_{n+1}(C, \mathscr{F})$ of an object C in $\mathscr{C}$ with respect to $\mathscr{F}$ when $\mathscr{C}$ has enough regular projective objects. These coincide with the Brown-Ellis-Hopf formulae for the integral homology of groups when $\mathscr{C} = \mathrm{Grp}$ and $\mathscr{F} = \mathrm{Ab}$. More generally, the generalized Hopf formulae recover the Barr-Beck left derived functors of the reflection $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ whenever $\mathscr{C}$ is a semi-abelian category that is monadic over the category of sets. For further work on higher central extensions and generalized Hopf formulae for homology, we refer the reader to [EG10, EG14, EGVdL13, Eve14, DA17].

Galois theory and homology of quasi-abelian functor categories

When $\mathscr{A}$ is an additive category with kernels, then the category $\mathrm{Grpd}(\mathscr{A})$ of internal groupoids in $\mathscr{A}$ and the arrow category $\mathrm{Arr}(\mathscr{A})$ of $\mathscr{A}$ are equivalent.

Given an internal groupoid with underlying reflexive graph (C_0, C_1, d, c, e), the corresponding morphism is given by its so-called normalization

$$c\ker(d) : \mathrm{Ker}(d) \to C_1.$$

Given a morphism $a : A_1 \to A_0$, its corresponding internal groupoid is determined by the reflexive graph

$$A_0 \oplus A_1 \underset{[1_{A_0}, a]}{\overset{\pi_1}{\underset{\leftarrow \iota_1 \rightarrow}{\rightarrow}}} A_0,$$

where $A_0 \oplus A_1$ denotes the biproduct of A_0, A_1, and π_1, ι_1 are the corresponding product projection and coproduct injection, and the morphism

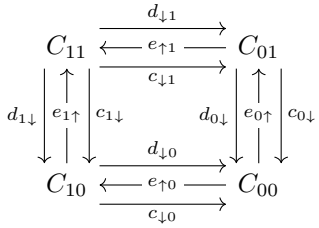
$$[1_{A_0}, a] : A_0 \oplus A_1 \to A_0$$

is induced by the universal property of the coproduct. Note that an internal functor $(f_0 : C_0 \to D_0, f_1 : C_1 \to D_1)$ between internal groupoids $\mathbb{C}, \mathbb{D}$ as in Diagram (3) in an additive category with kernels is a discrete fibration if and only if the unique morphism $\widehat{f_1} : \mathrm{Ker}(d) \to \mathrm{Ker}(d)$ with $\ker(d)\widehat{f_1} = f_1 \ker(d)$ is an isomorphism.

In Chapter 3 based on my article [Egn25], we study the functor category $\mathscr{A}^{\mathbb{T}}$, where $\mathbb{T}$ is a finite category and $\mathscr{A}$ is a quasi-abelian category. The notion of quasi-abelian category is an additive but not necessarily exact generalization of that of abelian category, see e.g. [Sch99, Rum01]. In particular, quasi-abelian categories are additive and have kernels. Note that, for suitable choices of $\mathbb{T}$, $\mathscr{A}^{\mathbb{T}}$ is given by the n-fold arrow category $\mathrm{Arr}^n(\mathscr{A})$ of $\mathscr{A}$, where $n \in \mathbb{N}$, which for additive categories with kernels is equivalent to the category $\mathrm{Grpd}^n(\mathscr{A})$ of internal n-fold groupoids in $\mathscr{A}$. We show that, for any choice of replete full subcategory $\mathbb{S}$ of $\mathbb{T}$, the full subcategory $\mathscr{F}$ of $\mathscr{A}^{\mathbb{T}}$ with objects the functors $F : \mathbb{T} \to \mathscr{A}$ with $F(T) = 0$ for all objects T that are not in $\mathbb{S}$, is torsion-free (Proposition 3.2.2). This relies on the fact that, in quasi-abelian categories, the induced morphism between the image and the coimage of a morphism is a bimorphism. Note that, for suitable choices of $\mathbb{S}$, $\mathscr{F}$ is given by the category n-$\mathrm{Arr}(\mathscr{A})$ whose objects are chains of n composable morphisms such that any composite of two consecutive ones is 0. If $\mathscr{A}$ is an additive category with kernels, then n-$\mathrm{Arr}(\mathscr{A})$ is equivalent to the category n-$\mathrm{Grpd}(\mathscr{A})$ of internal n-groupoids in $\mathscr{A}$ [Bou90]. The fact that $\mathscr{F}$ is a torsion-free subcategory of $\mathscr{A}^{\mathbb{T}}$ implies that $\Gamma = (\mathscr{A}^{\mathbb{T}}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathrm{RegEpi}(\mathscr{A}^{\mathbb{T}}), \mathrm{RegEpi}(\mathscr{F}))$ is an admissible Galois structure. We characterize the trivial extensions in $\mathscr{A}^{\mathbb{T}}$ relative to Γ when $\mathscr{A}$ is abelian (Proposition 3.3.2), and the central extensions when $\mathscr{A}$ is quasi-abelian (Proposition 3.3.4). More precisely, we prove that an extension $\alpha : F \Rightarrow G$ in $\mathscr{A}^{\mathbb{T}}$, i.e. a regular epimorphism, is central relative to Γ if and only if $\alpha_T : F(T) \to G(T)$ is an isomorphism for all objects T not in $\mathbb{S}$. Our proof relies on the fact the reflection $\mathsf{F} : \mathscr{A}^{\mathbb{T}} \to \mathscr{F}$ between the two additive categories $\mathscr{A}^{\mathbb{T}}, \mathscr{F}$ is necessarily protoadditive. In Example 3.3.5, we give, for suitable choices of $\mathscr{A}, \mathbb{T}, \mathbb{S}$, an example of a central extension in $\mathscr{A}^{\mathbb{T}}$ relative to Γ which is not trivial. In Proposition 3.4.12, we characterize the higher central extensions in $\mathscr{A}^{\mathbb{T}}$ relative to Γ, and in Examples 3.4.13 and 3.4.14, we compute some generalized Hopf formulae for homology.

N-fold groupoids and n-groupoids in regular Mal'tsev categories

In Chapter 2 partly based on my joint article [EG25] with M. Gran, we study the full subcategory n-Grpd($\mathscr{C}$) of internal n-groupoids of the category Grpd$^n(\mathscr{C})$ of internal n-fold groupoids in a regular Mal'tev category $\mathscr{C}$ with coequalizers. Recall that a small double groupoid, which is an internal double groupoid in the category of sets, consists of a set C_{00} of *objects*, a set C_{01} of *vertical morphisms*, a set C_{10} of *horizontal morphisms*, and a set C_{11} of *squares* equipped with some functions satisfying certain conditions. A small 2-groupoid is a small double groupoid in which any horizontal morphism is an identity morphism. An internal double groupoid in any category $\mathscr{C}$ with pullbacks has an underlying double reflexive graph that we depict in the following way:



This internal double groupoid is an internal 2-groupoid if the morphism

$$e_{\uparrow 0} : C_{00} \to C_{10}$$

(hence also $d_{\downarrow 0}, c_{\downarrow 0} : C_{10} \to C_{00}$) is an isomorphism. Using that, when $\mathscr{C}$ is a regular Mal'tsev category, Grpd($\mathscr{C}$) is closed under regular quotients in RG($\mathscr{C}$) [Gra99], we show in Proposition 2.2.2 that 2-Grpd($\mathscr{C}$) is a Birkhoff subcategory of Grpd$^2(\mathscr{C})$ whenever $\mathscr{C}$ is a regular Mal'tsev category with coequalizers, and construct the reflection $\mathsf{F} : \text{Grpd}^2(\mathscr{C}) \to \text{2-Grpd}(\mathscr{C})$ explicitly. This implies that, in this case, 2-Grpd($\mathscr{C}$) is itself a regular Mal'tsev category with coequalizers, and, if $\mathscr{C}$ is a semi-abelian category, then 2-Grpd($\mathscr{C}$) is also semi-abelian. Moreover, if $\mathscr{C}$ is a Mal'tsev variety of universal algebras, then 2-Grpd($\mathscr{C}$) is also (equivalent to) a Mal'tsev variety of universal algebras. If $\mathscr{C}$ is a regular *naturally* Mal'tsev category with coequalizers, then the reflection $\mathsf{F} : \text{Grpd}^2(\mathscr{C}) \to \text{2-Grpd}(\mathscr{C})$ has an additional property related to the commutator of equivalence relations (Proposition 2.2.6). A finitely complete category is a naturally Mal'tsev category [Joh89] if and only if it satisfies the so-called Lawvere condition, i.e., the forgetful functor Grpd($\mathscr{C}$) $\to$ RG($\mathscr{C}$) yields an equivalence of categories. In Theorem 2.3.4 we show more generally that n-Grpd($\mathscr{C}$) is a Birkhoff subcategory of Grpd$^n(\mathscr{C})$ for any $n \in \mathbb{N}$ and regular Mal'tsev category $\mathscr{C}$ with coequalizers. To this end, we use the recursive description of n-Grpd($\mathscr{C}$) due to D. Bourn [Bou84]. Finally, we extend some results of M. Gran and J.R.A. Gray in [GG21] from 2021 by proving that, if $\mathscr{C}$ is a semi-abelian, action representable [BJK05a], algebraically coherent [CGVdL15] category with normalizers [Gra14], then n-Grpd($\mathscr{C}$) is also semi-abelian, action representable, algebraically coherent and has normalizers (Theorem 2.4.8).

If $\mathscr{C}$ is an exact Mal'tsev category with coequalizers, then

$$\Gamma = (\text{Grpd}^n(\mathscr{C}), n\text{-Grpd}(\mathscr{C}), \mathsf{F}, \mathsf{U}, \text{RegEpi}(\text{Grpd}^n(\mathscr{C})), \text{RegEpi}(n\text{-Grpd}(\mathscr{C})))$$

is an admissible Galois structure. It would be interesting to have a characterization of the central extensions relative to Γ in this non-additive setting.

Structure of the thesis

The thesis is structured into four chapters: the first recalls important definitions and results needed for the subsequent three chapters, where we present our original contributions.

The preliminary Chapter 1 recalls the categorical contexts of Mal'tsev and semi-abelian categories, categorical Galois theory including the Galois structures induced by Birkhoff subcategories and torsion theories, and important properties of internal structures such as reflexive graphs, internal categories and groupoids, and their n-fold versions not only in Mal'tsev and semi-abelian categories but also in naturally Mal'tsev and additive categories with kernels.

In Chapter 2, we study n-fold groupoids and n-groupoids in regular Mal'tsev categories as described above. Section 2.1 can be seen as a supplement of Section 1.4 recalling the notions of internal n-category and n-groupoid.

In Chapter 3, we study Galois theory and homology of quasi-abelian functor categories as described above. For this, we recall in Section 3.1 the categorical context of quasi-abelian categories, and Section 3.4.1 can be seen as a prolongation of Section 1.3 recalling the theory of higher central extensions and generalized Hopf formulae for homology.

In Chapter 4, we present a syntactic characterization of weakly Mal'tsev varieties. Note that Section 1.1.3 is written specifically in preparation for this last chapter.

Chapter 1

Preliminaries

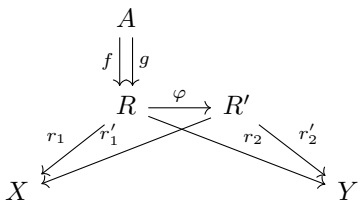
In this preliminary section, we review different categorical contexts such as those of Mal'tsev categories (Section 1.1) and of semi-abelian categories (Section 1.2). In Section 1.3, we summarize some aspects of categorical Galois theory that we will need in Chapter 3. Finally, we recall some notions of internal categorical structures, and collect various results regarding their properties in the categorical contexts revisited before (Section 1.4).

1.1 Mal'tsev categories

In this section, we want to recall the notion of Mal'tsev category and some of its fundamental properties. Before we do so, we recall the notion of internal (equivalence) relation, and its behaviour in regular and exact categories. After recalling some basic results from universal algebra, we consider more specifically Mal'tsev varieties.

1.1.1 Internal relations

Definition 1.1.1 (Internal relation)**.** Let $\mathscr{C}$ be a category. Furthermore, let $r_1 : R \to X, r_2 : R \to Y$, and $r'_1 : R' \to X, r'_2 : R' \to Y$ be two ordered pairs of jointly monomorphic morphisms in $\mathscr{C}$, meaning that, if $f, g : A \to R$ are two morphisms such that $r_1 f = r_1 g$ and $r_2 f = r_2 g$, then $f = g$, and analogously for r'_1, r'_2. The pairs (r_1, r_2) and (r'_1, r'_2) are *equivalent* if there is a (necessarily unique) isomorphism $\varphi : R \to R'$ such that $r'_1 \varphi = r_1$ and $r'_2 \varphi = r_2$.

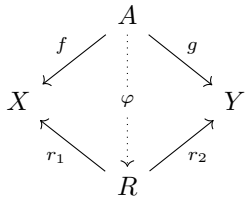


An *(internal) relation* in $\mathscr{C}$ from X to Y is an equivalence class of ordered pairs of jointly monomorphic morphisms $r_1 : R \to X, r_2 : R \to Y$.

In the following, we will refer to a relation and one of its representative ordered pairs of jointly monomorphic morphisms synonymously.

If the category $\mathscr{C}$ has binary products, a relation from X to Y is equivalently given by a subobject of the product $X \times Y$ of X, Y. Remember that a subobject of an object A in $\mathscr{C}$ is an equivalence class of monomorphisms $m : S \to A$, where two monomorphisms $m : S \to A$ and $m' : S' \to A$ are *equivalent* if there exists a (necessarily unique) isomorphism $\psi : S \to S'$ such that $m'\psi = m$. This shows that the notion of internal relation in the category Set of sets recovers the usual notion of relation given by a subset $R \subseteq X \times Y$ of the cartesian product of two sets X, Y.

Given a relation $(r_1 : R \to X, r_2 : r \to Y)$ in a category $\mathscr{C}$, one can define, for any object A in $\mathscr{C}$, a relation $R_A \subseteq \mathrm{Hom}(A, X) \times \mathrm{Hom}(A, Y)$ from the hom-set $\mathrm{Hom}(A, X)$ to the hom-set $\mathrm{Hom}(A, Y)$ in Set. Namely, a pair $(f : A \to X, g : A \to Y)$ of morphisms is in R_A if there exists a (necessarily unique) morphism $\varphi : A \to R$ such that $r_1\varphi = f$ and $r_2\varphi = g$.



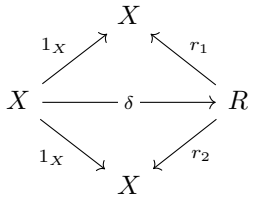
Recall that a relation $R \subseteq X \times Y$ in Set is called *difunctional* [Rig48] if, given elements $(x, y), (z, y), (z, u) \in R$, then $(x, u) \in R$.

Definition 1.1.2 (Internal difunctional relation)**.** Let $\mathscr{C}$ be a category. A relation R between two objects X, Y in $\mathscr{C}$ is *difunctional* if, for any object A in $\mathscr{C}$, the relation R_A between the hom-sets $\mathrm{Hom}(A, X)$, $\mathrm{Hom}(A, Y)$ is difunctional.

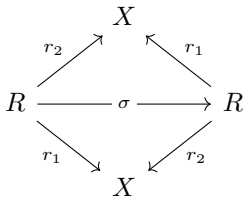
Example 1.1.3 (Internal graph)**.** Let $f : X \to Y$ be a morphism in a category $\mathscr{C}$. Then its *internal graph* $(1_X : X \to X, f : X \to Y)$ is a relation in $\mathscr{C}$ that is difunctional.

Definition 1.1.4 (Internal equivalence relation)**.** Let $\mathscr{C}$ be a category and $(r_1, r_2 : R \to X)$ be a relation on an object X in $\mathscr{C}$.

1. The relation R is *reflexive* if there exists a (necessarily unique) morphism $\delta : X \to R$ such that $r_1\delta = 1_X = r_2\delta$.



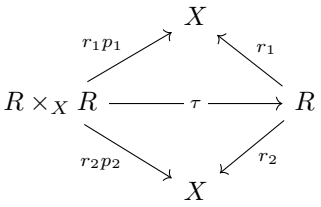
2. The relation R is *symmetric* if there exists a (necessarily unique) morphism $\sigma : R \to R$ such that $r_1\sigma = r_2$ and $r_2\sigma = r_1$.



In the following, let us assume that the category $\mathscr{C}$ has pullbacks and let us consider the pullback of r_1 along r_2:

$$\begin{array}{ccc} R \times_X R & \xrightarrow{p_2} & R \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_2]{} & X \end{array} \tag{1.1}$$

3. The relation R is *transitive* if there exists a (necessarily unique) morphism $\tau : R \times_X R \to R$ such that $r_1\tau = r_1p_1$ and $r_2\tau = r_2p_2$.



4. The relation R is an *(internal) preorder* in $\mathscr{C}$ if it is reflexive and transitive.

5. The relation R is an *(internal) equivalence relation* in $\mathscr{C}$ if it is reflexive, symmetric and transitive.

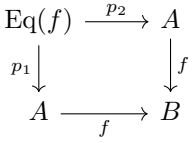
Equivalently, we could have defined the relation R in $\mathscr{C}$ to be reflexive (symmetric, transitive) if the relation $R_A \subseteq \mathrm{Hom}(A, X) \times \mathrm{Hom}(A, Y)$ in Set is reflexive (symmetric, transitive) for all objects A in $\mathscr{C}$.

Example 1.1.5 (Set-theoretic equivalence relation)**.** The notion of internal equivalence relation in Set recovers the usual notion of equivalence relation. Given an equivalence relation $R \subseteq X \times X$ on a set X, the function $\delta : X \to R$ as in Definition 1.1.4, Item 1 maps an element $x \in X$ to $(x, x) \in R$. The function $\sigma : R \to R$ as in Item 2 maps an element $(x, y) \in R$ to $(y, x) \in R$. The pullback $R \times_X R$ as in Diagram (1.1) is given by the subset

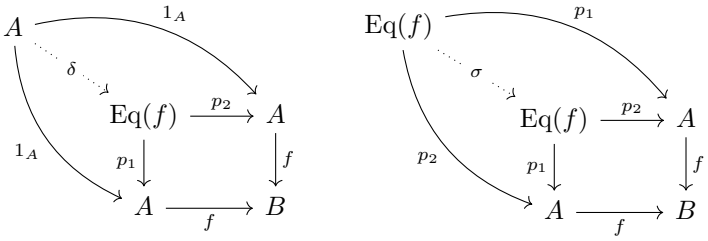
$$\{(x, y, z) \in X \times X \times X \mid (x, y), (y, z) \in R\}$$

of $X \times X \times X$, the function p_1 maps an element (x, y, z) to (x, y), and the function p_2 maps an element (x, y, z) to (y, z). The function $\tau : R \times_X R \to R$ as in Item 3 maps an element $(x, y, z) \in R \times_X R$ to $(x, z) \in R$.

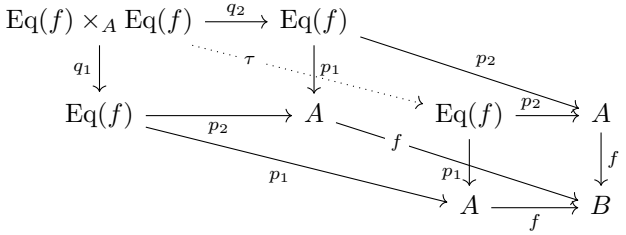
Example 1.1.6 (Kernel pair)**.** Let $f : A \to B$ be a morphism in a category $\mathscr{C}$ with pullbacks. Then the kernel pair of f, i.e. the pullback of f along itself, yields the equivalence relation $(p_1, p_2 : \mathrm{Eq}(f) \to A)$.



Namely, the morphism $\delta : A \to \mathrm{Eq}(f)$ as in Definition 1.1.4, Item 1 is the unique morphism such that $p_1\delta = 1_A = p_2\delta$ induced by the universal property of the pullback $\mathrm{Eq}(f)$. The morphism $\sigma : \mathrm{Eq}(f) \to \mathrm{Eq}(f)$ as in Item 2 is the unique morphism such that $\sigma p_1 = p_2$ and $\sigma p_1 = p_2$.



The morphism $\tau : \mathrm{Eq}(f) \times_A \mathrm{Eq}(f) \to \mathrm{Eq}(f)$ as in Item 3 is the unique morphism such that $p_1\tau = p_1q_1$ and $p_2\tau = p_2q_2$, where $q_1, q_2 : \mathrm{Eq}(f) \times_A \mathrm{Eq}(f) \to \mathrm{Eq}(f)$ are the pullback projections in the pullback of p_1 along p_2.



Note that, when $f = 1_A$, more generally, when f is a monomorphism, then $(\mathrm{Eq}(f), p_1, p_2) = (A, 1_A, 1_A)$. When $\mathscr{C}$ has a terminal object 1 and $f = \omega_A$ is the unique morphism from A to 1, then $(\mathrm{Eq}(f), p_1, p_2) = (A \times A, \pi_1, \pi_2)$, where $A \times A$ is the binary product of A with itself, and $\pi_1, \pi_2 : A \times A \to A$ are the corresponding product projections.

Regular categories provide a good categorical context for a *calculus of relations*.

Definition 1.1.7 (Regular category)**.** A category $\mathscr{C}$ is *regular* if it has finite limits, coequalizers of kernel pairs, and its regular epimorphisms are stable under pullback, meaning that, if $f : A \to B$ is a regular epimophism, i.e., the coequalizer of a parallel pair of morphisms, and $p : E \to B$ is an arbitrary

morphism, then p_1 in the pullback

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

is also a regular epimorphism.

Proposition 1.1.8. Let $\mathscr{C}$ be a regular category. Then every morphism in $\mathscr{C}$ factors as a regular epimorphism followed by a monomorphism. More precisely, given a morphism $f : A \to B$, its (regular epimorphism, monomorphism)-factorization is given as in the following diagram, where q is the coequalizer of the kernel pair $(\mathrm{Eq}(f), p_1, p_2)$ of f, and i is the unique morphism such that $iq = f$ induced by the universal property of the coequalizer $(\mathrm{Im}(f), q)$, that turns out to be a monomorphism:

$$\begin{array}{ccccc} \mathrm{Eq}(f) & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & A & \xrightarrow{\quad f \quad} & B \\ & & & {\scriptstyle q}\searrow & \nearrow{\scriptstyle i} \\ & & & & \mathrm{Im}(f) \end{array}$$

The object $\mathrm{Im}(f)$ is called the *(regular) image* of f.

In fact, a category with finite limits is a regular category if and only if it has pullback-stable (regular epimorphism, monomorphism)-factorizations.

In the following, we will also need the notion of exact category in the sense of Barr [Bar71]. Recall from Example 1.1.6 that, in any category with pullbacks, the kernel pair of any morphism can be seen as an equivalence relation.

Definition 1.1.9 (Exact category)**.** An *exact category* is a regular category in which any equivalence relation is effective, i.e., it is the kernel pair of some morphism.

Examples 1.1.10 (Regular and exact categories)**.**

1. The category Set of sets and functions is exact. The regular epimorphisms in Set are the surjective functions, and the monomorphisms in Set are the injective functions. The regular image of a function $f : A \to B$ is given by its classical image $f(A) \subseteq B$.

2. More generally, any variety $\mathbb{V}$ of universal algebras (see Section 1.1.3.1) as the categories of monoids, of groups, of rings, etc. is an exact category. The regular epimorphisms in $\mathbb{V}$ are the surjective homomorphisms, and the monomorphisms in $\mathbb{V}$ are the injective homomorphisms. The regular image of a homomorphism $f : A \to B$ is given by its classical image $f(A) \subseteq B$.

3. Any abelian category is exact and co-exact, meaning that its opposite category is also exact.

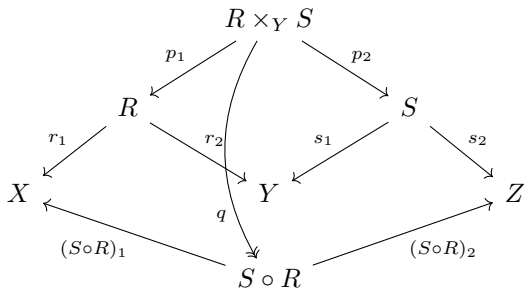
4. Any elementary topos is exact and co-exact [CKP93].

5. Quasi-varieties as the category of cancellative commutative monoids are regular but need not be exact in general.

6. The category of topological groups is regular but not exact.

Note that the category Top of topological spaces is not regular. It has all limits and colimits but there are regular epimorphism in Top that are not pullback-stable.

Definition 1.1.11 (Composition of relations)**.** Let $\mathscr{C}$ be a regular category, $(r_1, r_2) : R \to X \times Y$ be a relation from X to Y, and $(s_1, s_2) : S \to Y \times Z$ be a relation from Y to Z. The composite relation $S \circ R$ from X to Z is given by $((S \circ R)_1, (S \circ R)_2) : S \circ R \to X \times Z$, where $(R \times_Y S, p_1, p_2)$ is the pullback of r_2, s_1, and $((S \circ R)_1, (S \circ R)_2)q$ is the (regular epimorphism, monomorphism)-factorization of $(r_1 p_1, s_2 p_2) : R \times_Y S \to X \times Z$, as depicted in the following diagram:



For the category Set of sets, Definition 1.1.11 recovers the usual composition of relations. Namely, given two relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ in Set, their composite relation $S \circ R \subseteq X \times Z$ is given by

$$\{(x, z) \in X \times Z \mid \exists y \in Y \text{ such that } (x, y) \in R, (y, z) \in S\}.$$

Proposition 1.1.12. Let $\mathscr{C}$ be a regular category. Then the composition of relations as defined in Definition 1.1.11 is associative, meaning that, given relations R from X to Y, S from Y to Z, and T from Z to U, then

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

Due to Proposition 1.1.12, one can associate with any regular category $\mathscr{C}$ its category $\mathrm{Rel}(\mathscr{C})$ of relations whose objects are the objects in $\mathscr{C}$ and whose morphisms from an object X to an object Y are the relations in $\mathscr{C}$ from X to Y. The category $\mathscr{C}$ can be identified as a wide subcategory of $\mathrm{Rel}(\mathscr{C})$ by associating to any morphism $f : A \to B$ in $\mathscr{C}$ its graph $(1_A, f) : A \to A \times B$ (Example 1.1.3).

Definition 1.1.13 (Opposite relation)**.** Let $(r_1 : R \to X, r_2 : R \to Y)$ be a relation from an object X to an object Y in a category $\mathscr{C}$. Its *opposite relation* from Y to X is given by $(r_2 : R \to Y, r_1 : R \to X)$.

Proposition 1.1.14. Let $\mathscr{C}$ be a regular category.

1. A relation R between objects X and Y is difunctional if and only if

$$R \circ R^{\mathrm{op}} \circ R = R,$$

where R^{op} denotes the opposite relation of R.

2. A relation S on an object X is transitive if and only if

$$S \circ S \leq S.$$

1.1.2 Mal'tsev categories

The study of Mal'tsev categories originates with A.I. Mal'tsev's article [Mal54] from 1954 (see also [Mal63] for the English translation) in the context of varieties of universal algebras. In [CLP90] from 1990, A. Carboni, J. Lambek and M.C. Pedicchio introduce the notion of Mal'tsev category in the context of exact categories and develop some aspects of non-abelian homological algebra. Let us also mention the PhD thesis [Mei72] of J. Meisen supervised by J. Lambek from 1972. In [CPP92] from 1992, A. Carboni, M.C. Pedicchio and N. Pirovano define Mal'tsev categories in the finitely complete setting. Let us recall here some characterizations and properties of Mal'tsev categories in different contexts, see also [BB04, Bou17, BJG21].

Definition 1.1.15 (Mal'tsev category [Mei72, CPP92])**.** Let $\mathscr{C}$ be a category with finite limits. Then $\mathscr{C}$ is called a *Mal'tsev category* if any reflexive relation in $\mathscr{C}$ is an equivalence relation.

Examples 1.1.16 (Mal'tsev categories)**.**

1. The category Grp of groups is an exact Mal'tsev category. Let R be a relation on a group G, i.e. a subgroup of $G \times G$, that is reflexive. Given an element $(g, h) \in R$, we can write

$$(h, g) = (h, h)(g^{-1}, h^{-1})(g, g) = (h, h)(g, h)^{-1}(g, g).$$

This implies that $(h, g) \in R$ since $(g, g), (h, h) \in R$ because of reflexivity and R is a subgroup of $G \times G$. Hence R is symmetric. Given elements $(g, h), (h, i) \in R$, we can write

$$(g, i) = (g, h)(h^{-1}, h^{-1})(h, i) = (g, h)(h, h)^{-1}(h, i).$$

This implies that $(g, i) \in R$ since $(h, h) \in R$ because of reflexivity and R is a subgroup of $G \times G$. Hence R is transitive.

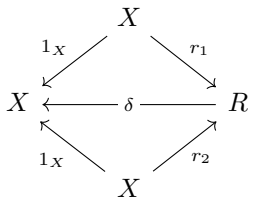
2. Using the underlying group structure, one can also show that the categories of rings, of associative algebras, of Lie algebras, etc. are exact Mal'tsev categories.

3. The categories of Heyting algebras and of Boolean algebras are also exact Mal'tsev categories [Bou96].

4. Any abelian category is an exact Mal'tsev category.

5. The category of topological groups is a regular Mal'tsev category [CKP93].

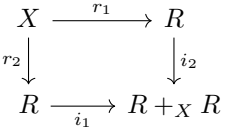
6. The opposite category of the category Set of sets is an exact Mal'tsev category. Note that a relation in $\mathrm{Set}^{\mathrm{op}}$ on a set X is given by a jointly epimorphic pair of functions $r_1, r_2 : X \to R$, that amount to a surjective function $[r_1, r_2] : X \dot{\cup} X \to R$ from the disjoint union of X with itself. The disjoint union $X \dot{\cup} X$ can be seen as the set $X \times \{1, 2\}$, and $[r_1, r_2]$ maps an element $(x, 1)$ to $r_1(x)$, and an element $(x, 2)$ to $r_2(x)$. The relation R being reflexive means that there exists a function $\delta : R \to X$ such that $\delta[r_1, r_2] = [1_X, 1_X]$.



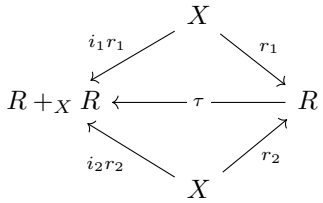
Note that the function $[1_X, 1_X] : X \dot{\cup} X \to X$ maps an element (x, i) to x. This implies that, for two elements $(x, i), (y, j) \in X \dot{\cup} X$ with $[r_1, r_2](x, i) = [r_1, r_2](y, j)$, it holds that

$$x = [1_x, 1_x](x, i) = \delta[r_1, r_2](x, i) = \delta[r_1, r_2](y, j) = y.$$

Hence the function $\sigma : R \to R$ which maps an element $[r_1, r_2](x, 1)$ to $[r_1, r_2](x, 2)$, and an element $[r_1, r_2](x, 2)$ to $[r_1, r_2](x, 1)$ is well-defined. The fact that $\sigma[r_1, r_2] = [r_2, r_1]$ shows that R is symmetric. Furthermore, let us consider the pushout



of r_2 along r_1. Then the function $\tau : R \to R +_X R$ which maps an element $[r_1, r_2](x, 1)$ to $i_1[r_1, r_2](x, 1)$, and an element $[r_1, r_2](x, 2)$ to $i_2[r_1, r_2](x, 2)$ is well-defined. The fact that $\tau[r_1, r_2] = [i_1 r_1, i_2 r_2]$ proves that R is transitive.



7. More generally, the opposite category of any elementary topos is an exact Mal'tsev category [CKP93].

Examples of categories that are not Mal'tsev categories are given by the categories of monoids, of distributive lattices, of cancellative commutative monoids and the opposite category of the category of topological spaces. We will see in Section 4.1 that these categories are examples of so-called weakly Mal'tsev categories.

Let us recall some characterizations of Mal'tsev categories in different contexts.

Proposition 1.1.17. Let $\mathscr{C}$ be a finitely complete category. Then the following conditions are equivalent:

1. $\mathscr{C}$ is a Mal'tsev category, i.e., any reflexive relation in $\mathscr{C}$ is an equivalence relation.
2. Any reflexive relation in $\mathscr{C}$ is symmetric.
3. Any reflexive relation in $\mathscr{C}$ is transitive, i.e., a preorder.
4. Any relation in $\mathscr{C}$ is difunctional.

Proposition 1.1.18. Let $\mathscr{C}$ be a regular category. Then the following conditions are equivalent:

1. $\mathscr{C}$ is a Mal'tsev category.
2. For any two equivalence relations R and S an object X in $\mathscr{C}$, the composite $S \circ R$ is also an equivalence relation.
3. The composition of equivalence relations on the same object in $\mathscr{C}$ is commutative, i.e., if R and S are two equivalence relations on an object X in $\mathscr{C}$, then it holds that $S \circ R = R \circ S$.

In the following, let $\mathscr{C}$ be a category with pullbacks. Furthermore, let $f : A \to C$ and $g : B \to C$ be two split epimorphisms in $\mathscr{C}$, meaning that there exist morphisms $r : C \to A$ and $s : C \to B$ such that $fr = 1_C$ and $gs = 1_C$. Let us consider the pullback of g along f as in the following diagram:

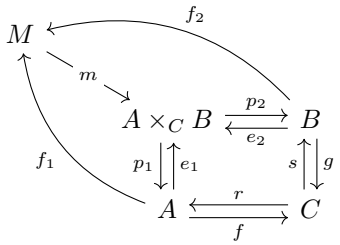
$$\begin{array}{ccc} A \times_C B & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & B \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle g} \\ A & \underset{f}{\overset{r}{\rightleftarrows}} & C \end{array} \tag{1.2}$$

Here $e_1 : A \to A \times_C B$ and $e_2 : B \to A \times_C B$ are the unique morphisms such that $p_1 e_1 = 1_A$ and $p_2 e_1 = sf$, and $p_2 e_2 = 1_B$ and $p_1 e_2 = rg$, induced by the universal property of the pullback $(A \times_C B, p_1, p_2)$. We also refer to them as the *canonical pullback injections* in this setting.

Proposition 1.1.19. [Bou96] Let $\mathscr{C}$ be a finitely complete category. Then the following conditions are equivalent:

1. $\mathscr{C}$ is a Mal'tsev category.

2. For any instance of Diagram (1.2) in $\mathscr{C}$, the morphisms $e_1 : A \to A \times_C B$ and $e_2 : B \to A \times_C B$ are jointly extremally epimorphic, i.e., if there exists an object M and morphisms $f_1 : A \to M$, $f_2 : B \to M$ and a monomorphism $m : M \to A \times_C B$ as depicted below such that $mf_1 = e_1$ and $mf_2 = e_2$, then m is an isomorphism.

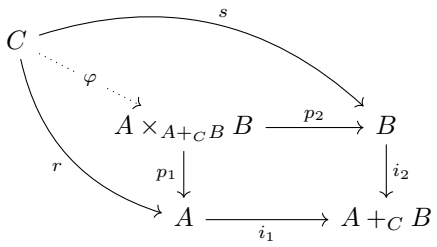


Proposition 1.1.19 can be reformulated by saying that a finitely complete category $\mathscr{C}$ is a Mal'tsev category if and only if, for any object C in $\mathscr{C}$, the category $\mathrm{Pt}_C(\mathscr{C})$ of *points* over C is *unital*. In fact, this statement remains true if one replaces unitality, by strong unitality or the Mal'tsev property itself, see [Bou96] for more details.

Proposition 1.1.20. [CKP93] Let $\mathscr{C}$ be a regular category. Then the following conditions are equivalent:

1. $\mathscr{C}$ is an exact Mal'tsev category.

2. Given two regular epimorphisms $r : C \to A$ and $s : C \to B$ with the same domain, their pushout $(A +_C B, i_1, i_2)$ as in the diagram below exists, and the unique morphism φ such that $p_1\varphi = r$ and $p_2\varphi = s$, where p_1, p_2 are the pullback projections of the pullback of i_2 along i_1, is a regular epimorphism.



We will come back to Mal'tsev categories in Section 1.4 where we will recall the behaviour of some categorical structures in Mal'tsev categories.

1.1.3 Mal'tsev varieties

Mal'tsev varieties provide a wide subclass of Mal'tsev categories. Let us start with recalling some basic notions and results from universal algebra, that we will also need in Chapter 4.

1.1.3.1 Universal algebra

Definition 1.1.21 (Algebra over a signature)**.**

1. A *signature* is a set Σ together with a function $\mathrm{ar} : \Sigma \to \mathbb{N}_0$. We call an element $\omega \in \Sigma$ *operation symbol* and $\mathrm{ar}(\omega) \in \mathbb{N}_0$ the *arity* of ω.

In the following, let (Σ, ar) be a signature.

2. A Σ-*algebra* is a set A together with, for any operation symbol $\omega \in \Sigma$, a function $\omega^A : A^{\mathrm{ar}(\omega)} \to A$. Here $A^{\mathrm{ar}(\omega)}$ denotes the cartesian product of A $\mathrm{ar}(\omega)$-times with itself.

3. Let A and B be two Σ-algebras. A *homomorphism of* Σ*-algebras* from A to B is a function $f : A \to B$ such that, for any operation symbol $\omega \in \Sigma$ and elements $a_1, \ldots, a_{\mathrm{ar}(\omega)} \in A$,

$$f(\omega^A(a_1, \ldots, a_{\mathrm{ar}(\omega)})) = \omega^B(f(a_1), \ldots, f(a_{\mathrm{ar}(\omega)})).$$

4. We denote by Σ-Alg the category whose objects are Σ-algebras and whose morphisms are homomorphisms of Σ-algebras.

For the signature with one operation symbol $\cdot$ of arity 2, Σ-Alg is given by the category of magmas. The category of groups can be seen as a subcategory of Σ-Alg for the signature which has one operation symbol $\cdot$ of arity 2, one operation symbol 1 of arity 0, and one operation symbol $(-)^{-1}$ of arity 1.

Example 1.1.22 (Term algebra)**.** Let (Σ, ar) be a signature and X be a set. We denote by $\mathsf{T}(X)$ the smallest set that satisfies the following two conditions:

1. $X \subseteq \mathsf{T}(X)$.

2. Given an operation symbol $\omega \in \Sigma$ and elements $t_1, \ldots, t_{\mathrm{ar}(\omega)} \in \mathsf{T}(X)$, then the formal expression $\omega(t_1, \ldots, t_n)$ lies also in $\mathsf{T}(X)$.

The elements of X are called *variables* in this context, and the elements of $\mathsf{T}(X)$ *terms* over X. The set $\mathsf{T}(X)$ can be turned in a Σ-algebra by defining, for any operation symbol $\omega \in \Sigma$ and terms $t_1, \ldots, t_{\mathrm{ar}(\omega)} \in \mathsf{T}(X)$,

$$\omega^{\mathsf{T}(X)}(t_1, \ldots, t_{\mathrm{ar}(\omega)}) := \omega(t_1, \ldots, t_{\mathrm{ar}(\omega)}).$$

The algebra $\mathsf{T}(X)$ is called the *term algebra* over X.

Given a term $t \in \mathsf{T}(X)$ and elements $x_1, \ldots, x_n \in X$, we write $t(x_1, \ldots, x_n)$ to indicate that the variables appearing in t are contained in the set $\{x_1, \ldots, x_n\}$.

Proposition 1.1.23. Let (Σ, ar) be a signature and $\varphi : X \to A$ be a function from a set X to a Σ-algebra A. Then there exists a unique Σ-algebra homomorphism $\widetilde{\varphi} : \mathsf{T}(X) \to A$ such that $\widetilde{\varphi}|_X = \varphi$. Constructively, $\widetilde{\varphi}$ is given by:

1. for any $x \in X$, $\widetilde{\varphi}(x) = \varphi(x)$;

2. for any $\omega \in \Sigma$ and $t_1, \ldots, t_{\mathrm{ar}(\omega)} \in \mathsf{T}(X)$,

$$\widetilde{\varphi}(\omega(t_1, \ldots, t_n)) = \omega^A(\widetilde{\varphi}(t_1), \ldots, \widetilde{\varphi}(t_{\mathrm{ar}(\omega)})).$$

This means that the forgetful functor Σ-Alg $\to$ Set has the left adjoint

$$\mathsf{T} : \mathrm{Set} \to \Sigma\text{-Alg}$$

that maps a set X to its term algebra $\mathsf{T}(X)$.

Given a term $t \in \mathsf{T}(x_1, \ldots, x_n) := \mathsf{T}(\{x_1, \ldots, x_n\})$ and a Σ-algebra A, we write $t^A(a_1, \ldots, a_n)$ for $\widetilde{\varphi}(t)$, where $\varphi : \{x_1, \ldots, x_n\} \to A$ is given by $\varphi(x_i) = a_i$ for all $i \in \{1, \ldots, n\}$. The variables $x_1, \ldots, x_n$ are replaced by the elements $a_1, \ldots, a_n \in A$, and the term is evaluated in A. Furthermore, if $f : A \to B$ is a Σ-algebra homomorphism, then $f(t^A(a_1, \ldots, a_n)) = t^B(f(a_1), \ldots, f(a_n))$.

One can show that Σ-Alg is a Barr-exact category with all limits and colimits, see e.g. [Bor94]. The forgetful functor Σ-Alg $\to$ Set creates all limits. Given two Σ-algebras A, B, their product $A \times B$ is given by the cartesian product of A, B, and the operations are defined componentwise, i.e., for any n-ary operation symbol $\omega \in \Sigma$ and elements $(a_1, b_1), \ldots, (a_n, b_n) \in A \times B$,

$$\omega^{A\times B}((a_1, b_1), \ldots, (a_n, b_n)) = (\omega^A(a_1, \ldots, a_n), \omega^B(b_1, \ldots, b_n)).$$

Given two Σ-algebra homomorphisms $f, g : A \to B$, their equalizer $\mathrm{Eq}(f, g)$ is given by the subset $\{a \in A \mid f(a) = g(a)\}$ of A and the operations are defined as in A.

Internal equivalence relations in Σ-Alg are also called *congruences*. A congruence on an algebra A is given by a subset $C \subseteq A \times A$ that is an equivalence relation on A and a subalgebra of $A \times A$. A *subalgebra* S of an algebra B is given by a subset $S \subseteq B$ such that, for any operation symbol $\omega \in \Sigma$ and elements $s_1, \ldots, s_{\mathrm{ar}(\omega)} \in S$, the element $w^A(s_1, \ldots, s_{\mathrm{ar}(\omega)})$ lies also in S.

Proposition 1.1.24. Let (Σ, ar) be a signature and A be a Σ-algebra. If $C \subseteq A \times A$ is a congruence on A, then the quotient A/C is a Σ-algebra, setting, for any n-ary operation symbol $\omega \in \Sigma$ and elements $[a_1], \ldots, [a_n] \in A/C$,

$$\omega^{A/C}([a_1], \ldots, [a_n]) = [\omega^A(a_1, \ldots, a_n)].$$

Let us now introduce identities in the sense of universal algebra.

Definition 1.1.25 (Variety of universal algebras)**.** Let (Σ, ar) be a signature.

1. Let $(t, t') \in \mathsf{T}(X)$ be a pair of terms over the set X, and A be a Σ-algebra. We say that A *satisfies the identity* $t \approx t'$, if, for any function $\varphi : X \to A$, $\widetilde{\varphi}(t) = \widetilde{\varphi}(t')$, where $\widetilde{\varphi} : \mathsf{T}(X) \to A$ is as in Proposition 1.1.23.

2. Let $E = \{(t_i, t_i') \in \mathsf{T}(X_i)\}_{i\in I}$ be a set of pairs of terms over the sets X_i, respectively, where I is an index set. We call (Σ, ar, E) an *equational theory* or, shortly, *theory*.

In the following, let (Σ, ar, E) be an equational theory.

3. A (Σ, E)-*algebra* is a Σ-algebra that satisfies the identity $t_i \approx t_i'$ for all $i \in I$.

4. We denote by (Σ, E)-Alg the full subcategory of Σ-Alg whose objects are the (Σ, E)-algebras. (Σ, E)-Alg is called a *variety of universal algebras*.

The following identities with respect to the signature with the operation symbols $\cdot$ of arity 2, 1 of arity 0, and $(-)^{-1}$ of arity 1, and the set $\{x, y, z\}$ of variables encode the variety of groups: $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$, $x \cdot 1 \approx x$, $1 \cdot x \approx x$, $x \cdot x^{-1} \approx 1$ and $x^{-1} \cdot x \approx 1$.

The following result is due to G. Birkhoff [Bir35] and is sometimes called HSP theorem. The 'H' stands for homomorphic image, the 'S' for subalgebra and the 'P' for product.

Theorem 1.1.26 (Birkhoff's HSP theorem)**.** Let (Σ, ar) be a signature. A full subcategory $\mathbb{V}$ of Σ-Alg is a variety of universal algebras, i.e., there exists a set E such that $\mathbb{V} = (\Sigma, E)$-Alg, if and only if $\mathbb{V}$ is closed in Σ-Alg under quotients, subalgebras and products.

Proposition 1.1.27. Let (Σ, ar, E) be an equational theory and A be a Σ-algebra. Then there exists a smallest congruence $C \subseteq A \times A$ on A such that A/C is a (Σ, E)-algebra. Moreover, given any Σ-algebra homomorphism $\varphi : A \to B$ to a (Σ, E)-algebra B, the function $\widehat{\varphi} : A/C \to B$ defined by $\widehat{\varphi}([a]) = \varphi(a)$ for all $a \in A$ is also a Σ-algebra homomorphism. This means that the forgetful functor (Σ, E)-Alg $\to \Sigma$-Alg has a left adjoint.

Example 1.1.28 (Free algebra)**.** Let (Σ, ar, E) be an equational theory and X be a set. Then there exists a smallest congruence C such that $\mathsf{F}(X) := \mathsf{T}(X)/C$ is a (Σ, E)-algebra. One can show that, for two terms $t, t' \in \mathsf{T}(X)$, $[t] = [t']$ in $\mathsf{F}(X)$ if and only if any (Σ, E)-algebra satisfies the identity $t \approx t'$. $\mathsf{F}(X)$ is called the *free algebra* over X.

Composing the adjunctions between Set and Σ-Alg from Proposition 1.1.23, and between Σ-Alg and (Σ, E)-Alg from Proposition 1.1.27, we obtain the following result.

Proposition 1.1.29. Let (Σ, ar, E) be an equational theory. The functor

$$\mathsf{F} : \mathrm{Set} \to (\Sigma, E)\text{-Alg}$$

which maps any set X to its free algebra $\mathsf{F}(X)$ is the left adjoint of the forgetful functor (Σ, E)-Alg $\to$ Set. Given a function $\varphi : X \to A$ from a set X to a (Σ, E)-algebra A, the unique Σ-algebra homomorphism $\bar{\varphi} : \mathsf{F}(X) \to A$ such that $\bar{\varphi}([x]) = \varphi(x)$ for all $x \in X$ is given by $\widetilde{\widehat{\varphi}}$, i.e., $\bar{\varphi}([t]) = \widetilde{\varphi}(t)$ for all $t \in \mathsf{T}(X)$, where $\widetilde{\varphi} : \mathsf{T}(X) \to A$ is as in Proposition 1.1.23.

One can show that a variety of universal algebras is a Barr-exact category with all limits and colimits. In Chapter 4, we will need an explicit description of the binary coproduct.

Proposition 1.1.30. Let $\mathbb{V}$ be a variety of universal algebras and A, B be two algebras in $\mathbb{V}$. Their coproduct $A + B$ is given by the quotient of the free algebra $\mathsf{F}(A \dot\cup B)$ over the disjoint union of A, B modulo the congruence C generated by all elements in $\mathsf{F}(A \dot\cup B) \times \mathsf{F}(A \dot\cup B)$ of the form

$$([i_1(\omega^A(a_1, \ldots, a_n))], \omega^{\mathsf{F}(A \dot\cup B)}([i_1(a_1)], \ldots, [i_1(a_n)]))$$

and

$$([i_2(\omega^B(b_1, \ldots, b_n))], \omega^{\mathsf{F}(A \dot\cup B)}([i_2(b_1)], \ldots, [i_2(b_n)])),$$

where ω is an n-ary operation symbol, $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$, and $i_1 : A \to A \dot\cup B$ and $i_2 : B \to A \dot\cup B$ are the corresponding inclusions into the disjoint union of A, B.

Note that

$$\omega^{\mathsf{F}(A \dot\cup B)}([i_1(a_1)], \ldots, [i_1(a_n)]) = [\omega^{\mathsf{T}(A \dot\cup B)}(i_1(a_1), \ldots, i_1(a_n))].$$

The congruence $C \subseteq \mathsf{F}(A \dot\cup B) \times \mathsf{F}(A \dot\cup B)$ as in Proposition 1.1.30 is the smallest congruence such that the functions $A \to \mathsf{F}(A \dot\cup B)$ and $B \to \mathsf{F}(A \dot\cup B)$ that map an element $a \in A$ to $[i_1(a)]$ and an element $b \in B$ to $[i_2(b)]$, respectively, induce Σ-algebra homomorphisms $\iota_1 : A \to A + B$ and $\iota_2 : B \to A + B$ on the quotient $\mathsf{F}(A \dot\cup B)/C = A + B$. These are the coproduct injections.

Given two homomorphisms $f : A \to D$ and $g : B \to D$ in the variety $\mathbb{V}$ of universal algebras, the unique homomorphism $[f, g] : A + B \to D$ induced by the coproduct $A + B$ is given by

$$\begin{aligned}&[f, g]([t(i_1(a_1), \ldots, i_1(a_m), i_2(b_1), \ldots, i_2(b_n))])\\ =&t^D(f(a_1), \ldots, f(a_m), g(b_1), \ldots, g(b_n)),\end{aligned}$$

where $t \in \mathsf{T}(A \dot\cup B)$.

In the case of two non-empty algebras A, B, we have an alternative description of their coproduct $A + B$. Let us fix a countable set $V = \{v_i\}_{i \in \mathbb{N}}$. By saying that t is an n-ary term, where $n \in \mathbb{N}$, we mean that $t \in \mathsf{T}(v_1, \ldots, v_n)$. A 0-ary term is a constant, i.e. a 0-ary operation symbol.

Proposition 1.1.31. Let $\mathbb{V}$ be a variety of universal algebras and A, B be two non-empty algebras in $\mathbb{V}$. Then the congruence C as in Proposition 1.1.30 is given by the transitive closure of the relation on $\mathsf{F}(A \dot\cup B)$ that contains all elements in $\mathsf{F}(A \dot\cup B) \times \mathsf{F}(A \dot\cup B)$ of the form

$$\begin{aligned}([&\tau^{\mathsf{T}(A \dot\cup B)}(i_1(a_1), \ldots, i_1(a_m), i_2(b_1), \ldots, i_2(b_n),\\ &\mu_1^{\mathsf{T}(A \dot\cup B)}(i_1(a_1), \ldots, i_1(a_m)), \mu_2^{\mathsf{T}(A \dot\cup B)}(i_2(b_1), \ldots, i_2(b_n)))],\\ [&\tau^{\mathsf{T}(A \dot\cup B)}(i_1(a_1), \ldots, i_1(a_m), i_2(b_1), \ldots, i_2(b_n),\\ &\lambda_1^{\mathsf{T}(A \dot\cup B)}(i_1(a_1), \ldots, i_1(a_m)), \lambda_2^{\mathsf{T}(A \dot\cup B)}(i_2(b_1), \ldots, i_2(b_n)))]),\end{aligned}$$

where $m, n \in \mathbb{N}_0$, τ is an $(m + n + 2)$-ary term, $a_1, \ldots, a_m \in A$, $b_1 \ldots, b_n \in B$, μ_1, λ_1 are m-ary terms and μ_2, λ_2 are n-ary terms with

$$\begin{aligned}\mu_1^A(a_1, \ldots, a_m) &= \lambda_1^A(a_1, \ldots, a_m),\\ \mu_2^B(b_1, \ldots, b_n) &= \lambda_2^B(b_1, \ldots, b_n).\end{aligned}$$

1.1.3.2 Mal'tsev's theorem

In 1954, A.I. Mal'tsev proved the following result.

Theorem 1.1.32 (Mal'tsev's theorem [Mal54, Mal63])**.** Let $\mathbb{V}$ be a variety of universal algebras. Then the following conditions are equivalent:

1. The composition of congruences on the same algebra in $\mathbb{V}$ is commutative, i.e., if R, S are two congruences on an algebra A in $\mathbb{V}$, then
$$R \circ S = S \circ R.$$
2. The equational theory of $\mathbb{V}$ has a ternary term $p(x, y, z) \in \mathsf{T}(x, y, z)$ such that $\mathbb{V}$ satisfies the identities
$$p^{\mathsf{T}(x,y)}(x, x, y) \approx y \qquad \text{and} \qquad p^{\mathsf{T}(x,y)}(x, x, y) \approx y.$$

Varieties of universal algebras that allow such a *Mal'tsev term* $p(x, y, z)$ are nowadays called *Mal'tsev varieties*, or also 2-*permutable varieties*. Proposition 1.1.18 implies that a variety of universal algebras is a Mal'tsev category if and only if it is a Mal'tsev variety. Of course the definition of a Mal'tsev category as in Definition 1.1.15 was inspired by Theorem 1.1.32 and works such as [Lam58, Fin60, Wer73, Mei72, Lam92], see also [BJG21] for more details on the history of Mal'tsev categories.

Examples 1.1.33 (Mal'tsev varieties)**.**

1. The variety of groups has the Mal'tsev term $p(x, y, z) = x \cdot y^{-1} \cdot z$.
2. Mal'tsev terms are not unique. The variety of Heyting algebras has the Mal'tsev terms $p_1(x, y, z) = ((x \Rightarrow y) \Rightarrow z) \wedge ((z \Rightarrow y) \Rightarrow x)$ and $p_2(x, y, z) = (y \Rightarrow (x \wedge z)) \wedge (x \vee z)$.

In the following, we want to give two proofs of the fact that a variety of universal algebras is a Mal'tsev category if and only if it has a Mal'tsev term. In the first proof, we will use the characterization of Mal'tsev categories as in Proposition 1.1.19. In the second, we will use the characterization of Mal'tsev categories via Proposition 1.1.17, Item 2. We will slightly adapt these proofs in Chapter 4 when we study *weakly* Mal'tsev varieties.

Note that the remainder of this section can be also found in [EJMF24, 3.2. Lemma, 3.3. Lemma, 4.3. Lemma, 4.4. Lemma].

Proof 1

Lemma 1.1.34. For a variety $\mathbb{V}$ of universal algebras, the following conditions are equivalent:

1. $\mathbb{V}$ is a Mal'tsev category.

2. For any split epimorphisms f', g' in $\mathbb{V}$ with splittings r', s', respectively, the canonical pullback injections e_1', e_2' as in the diagram

$$\begin{array}{ccc} P' & \underset{e_2'}{\overset{p_2'}{\rightleftarrows}} & B \\ {\scriptstyle p_1'}\downarrow\uparrow{\scriptstyle e_1'} & & {\scriptstyle s'}\uparrow\downarrow{\scriptstyle g'} \\ A & \underset{f'}{\overset{r'}{\rightleftarrows}} & C \end{array} \tag{1.3}$$

   are jointly extremally epimorphic.

3. The canonical pullback injections $e_1, e_2 : \mathsf{F}(x,y) \to P$ in the diagram

$$\begin{array}{ccc} P & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & \mathsf{F}(x,y) \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ \mathsf{F}(x,y) & \underset{f}{\overset{r}{\rightleftarrows}} & \mathsf{F}(x) \end{array}$$

   are jointly extremally epimorphic. Here $\mathsf{F}(x)$ and $\mathsf{F}(x,y)$ are the free algebras on variables x and x, y, respectively, $f : \mathsf{F}(x,y) \to \mathsf{F}(x)$ is the unique morphism with $f([x]) = [x] = f([y])$, $r : \mathsf{F}(x) \to \mathsf{F}(x,y)$ is the unique morphism with $r([x]) = [x]$, and $s : \mathsf{F}(x) \to \mathsf{F}(x,y)$ is the unique morphism with $s([x]) = [y]$.

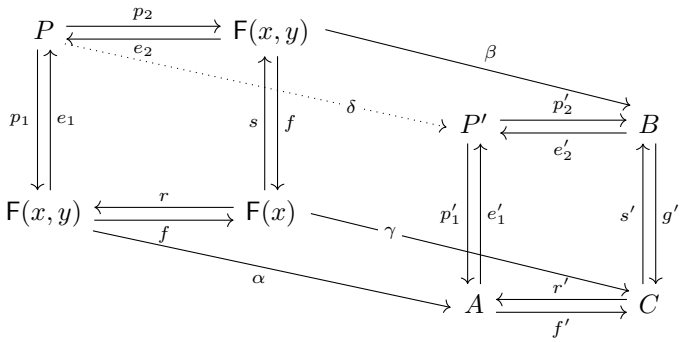
4. The induced morphism $[e_1, e_2] : \mathsf{F}(x,y) + \mathsf{F}(x,y) \to P$ is surjective.

5. The element $([y], [x])$ lies in $\mathrm{Im}([e_1, e_2])$.

Note that P is given by the subalgebra of $\mathsf{F}(x,y) \times \mathsf{F}(x,y)$ with elements $([t_1], [t_2])$, where $t_1, t_2 \in \mathsf{T}(x,y)$, such that $[t_1^{\mathsf{T}(x)}(x,x)] = [t_2^{\mathsf{T}(x)}(x,x)]$.

Proof. Item 1 is equivalent to Item 2 by Proposition 1.1.19. Trivially, Item 2 implies Item 3, and Item 4 implies Item 5. Note that, in a category with binary coproducts, two morphisms $\varphi : X \to U$ and $\psi : Y \to U$ are jointly extremally epimorphic if and only if the induced morphism $[\varphi, \psi] : X + Y \to U$ is an extremal epimorphism. In a variety of universal algebras, the extremal epimorphisms are exactly the surjective homomorphisms. We show that, if $([y], [x])$ lies in $\mathrm{Im}([e_1, e_2])$, then, for any pullback as in Diagram (1.3), the induced morphism $[e_1', e_2'] : A + B \to P'$ is surjective. For this, let (a, b) be an arbitrary element in $P' = \{(a', b') \in A \times B \mid f'(a') = g'(b')\}$. It is easy to see that the unique morphism $\alpha\colon \mathsf{F}(x,y) \to A$ such that $\alpha([x]) = r'f'(a)$ and $\alpha([y]) = a$, and the unique morphism $\beta\colon \mathsf{F}(x,y) \to B$ such that $\beta([x]) = b$ and

$\beta([y]) = s'g'(b)$ induce a unique morphism $\delta\colon P \to P'$ such that the cube



'reasonably' commutes, where $\gamma\colon \mathsf{F}(x) \to C$ is the unique morphism such that $\gamma([x]) = f'(a) = g'(b)$. Hence we get the commutative diagram

$$\begin{array}{ccc} \mathsf{F}(x,y) + \mathsf{F}(x,y) & \xrightarrow{[e_1,e_2]} & P \\ \downarrow{\scriptstyle \alpha+\beta} & & \downarrow{\scriptstyle \delta} \\ A + B & \xrightarrow[[e_1',e_2']]{} & P'. \end{array} \tag{1.4}$$

By assumption, there exists $X \in \mathsf{F}(x,y) + \mathsf{F}(x,y)$ with $[e_1, e_2](X) = ([y], [x])$. Thus,

$$(a, b) = \delta([y], [x]) = \delta[e_1, e_2](X) = [e_1', e_2'](\alpha + \beta)(X) \in \mathrm{Im}([e_1', e_2']).$$

This concludes the proof. □

Lemma 1.1.35. Let $\mathbb{V}$ be a variety of universal algebras, and the morphism $[e_1, e_2] : \mathsf{F}(x,y) + \mathsf{F}(x,y) \to P$ as in Lemma 1.1.34. Then it holds that

$$\mathrm{Im}([e_1, e_2]) = \{[p^{\mathsf{T}(x,y)}(x, x, y)], [p^{\mathsf{T}(x,y)}(x, y, y)]) \mid p \in \mathsf{T}(x, y, z)\}.$$

Proof. We note that $e_1([t]) = ([t], [t^{\mathsf{T}(y)}(y, y)])$ and $e_2([t]) = ([t^{\mathsf{T}(x)}(x, x)], [t])$, where $t \in \mathsf{T}(x, y)$.

"$\supseteq$": We have that $([x], [x]) = e_2([x])$, $([x], [y]) = e_1([x]) = e_2([y])$ and $([y], [y]) = e_1(y)$ are in $\mathrm{Im}([e_1, e_2])$. Hence, given $p \in \mathsf{T}(x, y, z)$,

$$p^P(([x], [x]), ([x], [y]), ([y], [y])) = ([p^{\mathsf{T}(x,y)}(x, x, y)], [p^{\mathsf{T}(x,y)}(x, y, y)])$$

lies also in the subalgebra $\mathrm{Im}([e_1, e_2]) \subseteq P$.

"$\subseteq$": Let us recall from Proposition 1.1.30 and Example 1.1.28 that the coproduct $\mathsf{F}(x,y) + \mathsf{F}(x,y)$ can be constructed as a quotient of $\mathsf{T}(\mathsf{F}(x,y) \dot\cup \mathsf{F}(x,y))$. Let

$$\tau(i_1([s_1]), \ldots, i_1([s_k]), i_2([t_1]), \ldots, i_2([t_\ell]))$$

be a term in $\mathsf{T}(\mathsf{F}(x,y)\dot{\cup}\mathsf{F}(x,y))$ with possible variables $i_1([s_1]),\dots,i_1([s_k])$, $i_2([t_1]),\dots,i_2([t_\ell])$, where $i_1, i_2 : \mathsf{F}(x,y) \to \mathsf{F}(x,y)\dot{\cup}\mathsf{F}(x,y)$ are the corresponding inclusions, $k,\ell \in \mathbb{N}_0$ and $s_1,\dots,s_k,t_1,\dots,t_\ell \in \mathsf{T}(x,y)$. We compute

$$
\begin{aligned}
&[e_1,e_2]([\tau(i_1([s_1]),\dots,i_1([s_k]),i_2([t_1]),\dots,i_2([t_\ell]))])\\
&= [e_1,e_2](\tau^{\mathsf{F}(x,y)+\mathsf{F}(x,y)}(\iota_1([s_1]),\dots,\iota_1([s_k]),\iota_2([t_1]),\dots,\iota_2([t_\ell])))\\
&= \tau^P([e_1,e_2]\iota_1([s_1]),\dots,[e_1,e_2]\iota_1([s_k]),[e_1,e_2]\iota_2([t_1]),\dots,[e_1,e_2]\iota_2([t_\ell]))\\
&= \tau^P(e_1([s_1]),\dots,e_1([s_k]),e_2([t_1]),\dots,e_2([t_\ell]))\\
&= \tau^P(s_1^P(e_1([x]),e_1([y])),\dots,s_k^P(e_1([x]),e_1([y])),\\
&\qquad t_1^P(e_2([x]),e_2([y])),\dots,t_\ell^P(e_2([x]),e_2([y])))\\
&= \tau^P(s_1^P(([x],[y]),([y],[y])),\dots,s_k^P(([x],[y]),([y],[y])),\\
&\qquad t_1^P(([x],[x]),([x],[y])),\dots,t_\ell^P(([x],[y]),([y],[y])))\\
&= \tau^P((s_1^{\mathsf{F}(x,y)}([x],[y]),s_1^{\mathsf{F}(x,y)}([y],[y])),\dots,(s_k^{\mathsf{F}(x,y)}([x],[y]),s_k^{\mathsf{F}(x,y)}([y],[y])),\\
&\qquad (t_1^{\mathsf{F}(x,y)}([x],[x]),t_1^{\mathsf{F}(x,y)}([x],[y])),\dots,(t_\ell^{\mathsf{F}(x,y)}([x],[x]),t_\ell^{\mathsf{F}(x,y)}([x],[y])))\\
&= (\tau^{\mathsf{F}(x,y)}([s_1^{\mathsf{T}(x,y)}(x,y)],\dots,[s_k^{\mathsf{T}(x,y)}(x,y)],[t_1^{\mathsf{T}(x,y)}(x,x)],\dots,[t_\ell^{\mathsf{T}(x,y)}(x,x)]),\\
&\quad \tau^{\mathsf{F}(x,y)}([s_1^{\mathsf{T}(x,y)}(y,y)],\dots,[s_k^{\mathsf{T}(x,y)}(y,y)],[t_1^{\mathsf{T}(x,y)}(x,y)],\dots,[t_\ell^{\mathsf{T}(x,y)}(x,y)]))\\
&= ([\tau^{\mathsf{T}(x,y)}(s_1^{\mathsf{T}(x,y)}(x,y),\dots,s_k^{\mathsf{T}(x,y)}(x,y),t_1^{\mathsf{T}(x,y)}(x,x),\dots,t_\ell^{\mathsf{T}(x,y)}(x,x))],\\
&\quad [\tau^{\mathsf{T}(x,y)}(s_1^{\mathsf{T}(x,y)}(y,y),\dots,s_k^{\mathsf{T}(x,y)}(y,y),t_1^{\mathsf{T}(x,y)}(x,y),\dots,t_\ell^{\mathsf{T}(x,y)}(x,y))])\\
&= ([p^{\mathsf{T}(x,y)}(x,x,y)],[p^{\mathsf{T}(x,y)}(x,y,y)]),
\end{aligned}
$$

where we set

$$
\begin{aligned}
p(x,y,z) := \tau^{\mathsf{T}(x,y,z)}(&s_1^{\mathsf{T}(x,y,z)}(y,z),\dots,s_k^{\mathsf{T}(x,y,z)}(y,z),\\
&t_1^{\mathsf{T}(x,y,z)}(x,y),\dots,t_\ell^{\mathsf{T}(x,y,z)}(x,y)).
\end{aligned}
$$

Hence the claim is proven. □

Together Lemmas 1.1.34 and 1.1.35 show that a variety $\mathbb{V}$ of universal algebras is a Mal'tsev category if and only if $\mathbb{V}$ has a Mal'tsev term $p(x,y,z)$.

Proof 2

Lemma 1.1.36. For a variety $\mathbb{V}$ of universal algebras, the following conditions are equivalent:

1. $\mathbb{V}$ is a Mal'tsev category.
2. Any reflexive relation in $\mathbb{V}$ is symmetric.
3. The smallest reflexive relation R on $\mathsf{F}(x,y)$ containing the element $([x],[y])$ is symmetric.
4. The element $([y],[x])$ lies in R.

Proof. Items 1 and 2 are equivalent by Proposition 1.1.17. Trivially, Item 2 implies Item 3, and Item 3 implies Item 4 since $([x],[y]) \in R$. Similarly as in the proof of Lemma 1.1.34, one can show that the condition that $([y],[x])$ lies in R implies that, for any element (a,b) in a reflexive relation S on an algebra A in $\mathbb{V}$, the element (b,a) also lies in S. □

Lemma 1.1.37. Let $\mathbb{V}$ be a variety of universal algebras. The smallest reflexive relation R on $\mathsf{F}(x,y)$ which contains $([x],[y])$ is given by

$$R := \{[p^{\mathsf{T}(x,y)}(x,x,y)], [p^{\mathsf{T}(x,y)}(x,y,y)]) \mid p \in \mathsf{T}(x,y,z)\}.$$

Proof. This is clear. □

Together Lemmas 1.1.36 and 1.1.37 prove that a variety $\mathbb{V}$ of universal algebras is a Mal'tsev category if and only $\mathbb{V}$ has a Mal'tsev term $p(x,y,z)$.

1.2 Semi-abelian categories

In this section, we recall the notion of semi-abelian category, that allows for a good notion of internal object action, leading us to the notion of action representability.

1.2.1 Semi-abelian categories

We start by recalling the notion of protomodularity that was introduced by D. Bourn in 1991 [Bou91] and that is one of the building blocks of the notion of semi-abelian category. We will need the following notation.

Let $\mathscr{C}$ be a category. We denote by $\mathrm{Pt}(\mathscr{C})$ the category of *points* of $\mathscr{C}$ whose objects are split epimorphisms $f : A \to B$ in $\mathscr{C}$ together with a chosen splitting $s : B \to A$, i.e., $fs = 1_B$, and whose morphisms between points $(f : A \to B, s : B \to A)$ and $(f' : A' \to B', s' : B' \to A')$ are given by pairs $(\alpha : A \to A', \beta : B \to B')$ of morphisms in $\mathscr{C}$ such that $\beta f = f'\alpha$ and $\alpha s = s'\beta$.

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\ B & \xrightarrow[\beta]{} & B' \end{array}$$

We denote by

$$\pi : \mathrm{Pt}(\mathscr{C}) \to \mathscr{C}$$

the functor that maps a point $(f : A \to B, s : B \to A)$ to the object B, and by $\mathrm{Pt}_B(\mathscr{C})$ the fibre of π over B, i.e., the objects of $\mathrm{Pt}_B(\mathscr{C})$ are the split epimorphisms $f : A \to B$ with codomain B with a chosen splitting $s : B \to A$, and the morphisms between two such points $(f : A \to B, s : B \to A)$ and $(f' : A' \to B, s' : B \to A')$ are given by morphisms $\alpha : A \to A'$ such that $f'\alpha = f$ and $\alpha s = s'$.

$$\begin{array}{ccccc} A & & \xrightarrow{\alpha} & & A' \\ & {\scriptstyle f}\searrow\quad\nwarrow{\scriptstyle s} & & {\scriptstyle f'}\nearrow\quad\swarrow{\scriptstyle s'} & \\ & & B & & \end{array}$$

If $\mathscr{C}$ is a category with pullbacks of split epimorphisms, then any morphism $p : B \to B'$ induces a functor $p^* : \mathrm{Pt}_{B'}(\mathscr{C}) \to \mathrm{Pt}_B(\mathscr{C})$ that maps a point $(f' : A' \to B', s' : B' \to A')$ to the point $(p_1 : B \times_{B'} A' \to B, e_1 : B \to B \times_{B'} A')$ as in the following pullback:

$$\begin{array}{ccc} B \times_{B'} A' & \xrightarrow{p_2} & A' \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\ B & \xrightarrow[p]{} & B' \end{array} \tag{1.5}$$

Here e_1 is the unique morphism such that $p_1 e_1 = 1_B$ and $p_2 e_1 = s'p$. Note that in this case $\pi : \mathrm{Pt}(\mathscr{C}) \to \mathscr{C}$ is a fibration, the so-called the *fibration of points* of $\mathscr{C}$, and that its corresponding change of base functors are given by $p^* : \mathrm{Pt}_{B'}(\mathscr{C}) \to \mathrm{Pt}_B(\mathscr{C})$, see e.g. [BB04] for more details.

Definition 1.2.1 (Protomodular category [Bou91])**.** A category is *protomodular* if it has pullbacks of split epimorphisms and the change of base functors of its fibration of points $p^* : \mathrm{Pt}_{B'}(\mathscr{C}) \to \mathrm{Pt}_B(\mathscr{C})$, where $p : B \to B'$ is any morphism in $\mathscr{C}$, are conservative, i.e., they reflect isomorphisms.

The following syntactic characterization of protomodular varieties is due to D. Bourn and G. Janelidze [BJ03].

Proposition 1.2.2. Let $\mathbb{V}$ be a variety of universal algebras. Then the following conditions are equivalent:

1. $\mathbb{V}$ is a protomodular category.
2. The equational theory of $\mathbb{V}$ admits $n \in \mathbb{N}$, constants $e_1, \ldots, e_n$, binary terms $\alpha_1, \ldots, \alpha_n$, and an $(n+1)$-ary term β such that $\mathbb{V}$ satisfies the identities

$$\alpha_i(x, x) \approx e_i \text{ for all } i \in \{1, \ldots, n\},$$
$$\beta(\alpha_1(x, y), \ldots, \alpha(x, y), y) \approx x.$$

Examples 1.2.3 (Protomodular categories)**.**

1. The variety Grp of groups has the constant 1 and the binary terms $\alpha_1(x, y) = x \cdot y^{-1}$ and $\beta(x, y) = x \cdot y$, and satisfies the identities

$$\alpha_1(x, x) \approx 1 \text{ and } \beta(\alpha_1(x, y), y) \approx x.$$

   Hence Grp is protomodular.

2. Any variety of universal algebras whose equational theory contains that of Grp is protomodular. Examples are the varieties of rings, of associative algebras and of Lie algebras over a ring.

3. The varieties of Heyting algebras and of Boolean algebras are protomodular [Bou96].

4. In [Joh04], P.T. Johnstone proves that the variety of Heyting semi-lattices is protomodular. Namely, it has the constants $1, 0$, the binary terms $\alpha_1(x, y) = (x \Rightarrow y)$ and $\alpha_2(x, y) = (((x \Rightarrow y) \Rightarrow y) \Rightarrow x)$, and the ternary term $\Theta(x, y, z) = (x \Rightarrow y) \wedge z$ satisfying the identities as in Proposition 1.2.2. Remarkably, it is impossible to find one constant e_1, one binary term α_1 and a binary term β that satisfy the conditions as in Proposition 1.2.2.

5. The category of topological models of any protomodular variety is protomodular [BB04, Example 3.1.16]. In particular, the category of topological groups is protomodular.

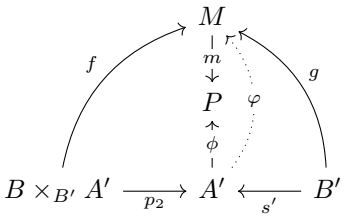
6. The dual of any elementary topos is a protomodular category [Bou96]. In particular, the dual $\mathrm{Set}^{\mathrm{op}}$ of the category of sets is protomodular.

Proposition 1.2.4. [Bou96] Any finitely complete protomodular category is a Mal'tsev category.

Proposition 1.2.5. [BB04] Let $\mathscr{C}$ be a category with pullbacks of split epimorphisms. Then the following conditions are equivalent:

1. $\mathscr{C}$ is protomodular.

2. Given a point $(f : A' \to B', s' : B' \to A')$ and a morphism $p : B \to B'$ in $\mathscr{C}$, the morphisms $s' : B' \to A'$ and $p_2 : B \times_{B'} A' \to A'$ as in Diagram (1.5) are jointly strongly epimorphic. This means that, for any any commutative diagram



   in $\mathscr{C}$ in which m is a monomorphism, there exists a unique morphism $\varphi : A' \to M$ such that $m\varphi = \phi$, and hence also $\varphi p_2 = f$ and $\varphi s' = g$.

If $\mathscr{C}$ is in addition a pointed category, i.e., it has a *zero* object, the above conditions are further equivalent to the following ones:

3. For any object B in $\mathscr{C}$, the functor $\mathrm{Ker}_B(-) : \mathrm{Pt}_B(\mathscr{C}) \to \mathscr{C}$ that maps a point $(f : A \to B, s : A \to B)$ to the kernel $\mathrm{Ker}(f)$ of f is conservative.

4. The *Split Short Five Lemma* holds in $\mathscr{C}$, i.e., given a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C & \xrightarrow{k} & A & \underset{s}{\overset{f}{\rightleftarrows}} & B \\ & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ 0 & \longrightarrow & C' & \xrightarrow[k']{} & A' & \underset{s'}{\overset{f'}{\rightleftarrows}} & B' \end{array}$$

in $\mathscr{C}$, where $fs = 1_B$, $f's' = 1_{B'}$, (C, k) is the kernel of f, (C', k') is the kernel of f', $k'c = ak$, $f'a = bf$ and $as = s'b$, if c and b are isomorphisms, then a is also an isomorphism.

5. Given a point $(f : A \to B, s : B \to A)$ in $\mathscr{C}$, the morphisms $s : B \to A$ and $\ker(f) : \mathrm{Ker}(f) \to A$ are jointly strongly epimorphic.

$$\mathrm{Ker}(f) \xrightarrow{\ker(f)} A \underset{s}{\overset{f}{\rightleftarrows}} B$$

Note that Item 3 in the above proposition means that, if $\mathscr{C}$ is a pointed category with pullbacks of split epimorphisms, then the change of base functor $p^* : \mathrm{Pt}_{B'}(\mathscr{C}) \to \mathrm{Pt}_B(\mathscr{C})$ is conservative for all morphisms $p : B \to B'$ in $\mathscr{C}$ if and only if the change of base functor $\alpha_{B'} : \mathrm{Pt}_{B'}(\mathscr{C}) \to \mathrm{Pt}_0(\mathscr{C})$ is conservative for all objects B' in $\mathscr{C}$, where $\alpha_{B'} : 0 \to B'$ denotes the unique morphism from the zero object 0 to B'. It is clear that $\mathrm{Pt}_0(\mathscr{C}) \approx \mathscr{C}$.

Proposition 1.2.6. [Bou91] Let $\mathscr{C}$ be a pointed, finitely complete, protomodular category and $f : A \to B$ be a morphism in $\mathscr{C}$. Then the following conditions are equivalent:

1. The morphism f is a monomorphism.
2. The kernel $\mathrm{Ker}(f)$ of f is 0.

Proposition 1.2.7. [BB04] Let $\mathscr{C}$ be a pointed, finitely complete, protomodular category and $f : A \to B$ be a morphism in $\mathscr{C}$. Then the following conditions are equivalent:

1. The morphism f is a regular epimorphism.
2. The morphism f is the cokernel of its kernel: $f = \mathrm{cok}(\ker(f))$.

Many homological lemmas such as the Noether Isomorphism Theorems, the Short Five Lemma, the Nine Lemma and the Snake Lemma generalize in homological categories, introduced by F. Borceux and D. Bourn in 2004 [BB04].

Definition 1.2.8 (Homological category)**.** A category is *homological* if it is pointed, regular and protomodular.

Proposition 1.2.9. [BB04, Lemma 4.2.5] Let $\mathscr{C}$ be a homological category and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{k} & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & C' & \xrightarrow[k']{} & A' & \xrightarrow[f']{} & B' & & \end{array}$$

be a diagram in $\mathscr{C}$ such that the left- and right-hand side squares commute, (C, k) is the kernel of f, (C', k') is the kernel of f', and (B, f) is the cokernel of f.Then the following conditions are equivalent:

1. The right-hand side square is a pullback.

2. The morphism $c : C \to C'$ is an isomorphism.

The notion of semi-abelian category was introduced in 2002 by G. Janelidze, L. Márki and W. Tholen in [JMT02]. In the abstract, they write "The notion of semi-abelian category as proposed in this paper is designed to capture typical algebraic properties valid for groups, rings and algebras, say, just as abelian categories allow for a generalized treatment of abelian-group and module theory."

Definition 1.2.10 (Semi-abelian category)**.** A category is *semi-abelian* if it is pointed, exact, protomodular and has binary coproducts.

Proposition 1.2.11. [JMT02] Any semi-abelian category is finitely cocomplete.

The so-called Tierney's equation (see e.g. [Bor94]) states that a category is abelian if and only if it is exact and additive. Hence the notion of semi-abelian category is an exact generalization of that of abelian category in which additivity is weakened to pointed protomodularity and the existence of finite colimits.

Proposition 1.2.12. [JMT02] For a category $\mathscr{C}$, the following conditions are equivalent:

1. The category $\mathscr{C}$ is abelian.

2. The category $\mathscr{C}$ and its opposite category $\mathscr{C}^{\mathrm{op}}$ are semi-abelian.

Since a variety of universal algebras is pointed if and only if it has a unique constant, the following result follows directly from Proposition 1.2.2.

Proposition 1.2.13. Let $\mathbb{V}$ be a variety of universal algebras. Then the following conditions are equivalent:

1. $\mathbb{V}$ is a semi-abelian category.

2. The equational theory of $\mathbb{V}$ admits a unique constant e, $n \in \mathbb{N}$, binary terms $\alpha_1, \ldots, \alpha_n$ and an $(n+1)$-ary term β such that $\mathbb{V}$ satisfies the identities $\alpha_i(x, x) = e$ for all $i \in \{1, \ldots, n\}$ and $\beta(\alpha_1(x, y), \ldots, \alpha_n(x, y), y) = y$.

Such varieties of universal algebras have been studied before by A. Ursini under the name of *BIT speciale* [Urs73] and *classically ideal determined* varieties [Urs94] in relation with commutator theory of ideals.

Examples 1.2.14 (Semi-abelian categories)**.**

1. The categories of groups, of rings, of associative algebras and of Lie algebras are semi-abelian.

2. The category of compact Hausdorff models of any semi-abelian variety is a semi-abelian category [BC05]. In particular, this is the case for the category of compact Hausdorff groups.

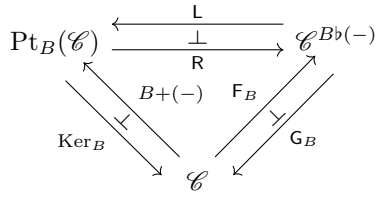
3. The category of cocommutative Hopf algebras over a field is semi-abelian [GSV19].

4. Given an elementary topos $\mathscr{C}$, the dual $\mathscr{C}_*^{\mathrm{op}}$ of the category of its pointed objects is semi-abelian [BB04]. In particular, the dual $\mathrm{Set}_*^{\mathrm{op}}$ of the category of pointed sets is semi-abelian.

1.2.2 Action representability

The fact that a category is abelian if and only if itself and its opposite category are semi-abelian (Proposition 1.2.12) motivates the terminology 'semi-abelian'. Moreover, semi-abelian categories have semi-direct products as we will see in the following. For this, we need some notation.

In the following, we assume that $\mathscr{C}$ is a pointed, finitely bicomplete category, i.e., it has all finite limits and all finite colimits. In this case, we have the following triangle of adjunctions:



(1.6)

Here B is an object in $\mathscr{C}$. Recall that the functor $\mathrm{Ker}_B : \mathrm{Pt}_B(\mathscr{C}) \to \mathscr{C}$ maps a point $(p : A \to B, s : B \to A)$ over B to $\mathrm{Ker}(p)$. It has a left-adjoint $B + (-) : \mathscr{C} \to \mathrm{Pt}_B(\mathscr{C})$ that maps an object A to the point

$$B + A \underset{\iota_1}{\overset{[1_B,0]}{\rightleftarrows}} B,$$

where the morphism $[1_B, 0] : B + A \to B$ is induced by the universal property of the coproduct, and $\iota_1 : B \to B + A$ is the corresponding coproduct injection.

The adjunction $B + (-) \dashv \mathrm{Ker}_B$ induces the monad

$$B\flat(-) = \mathrm{Ker}_B \circ (B + (-))$$

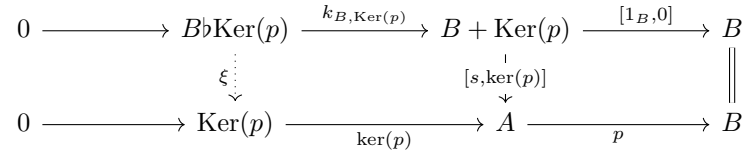
on $\mathscr{C}$ which maps an object A to the object part $B\flat A$ of the kernel $(B\flat A, k_{B,A})$ of $[1_B, 0] : B + A \to A$.

$$0 \longrightarrow B\flat A \xrightarrow{k_{B,A}} B + A \xrightarrow{[1_B,0]} B$$

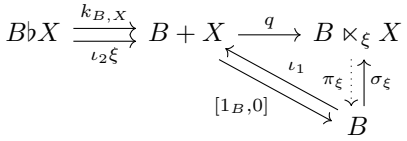
We denote by $\mathscr{C}^{B\flat(-)}$ the corresponding Eilenberg-Moore category whose objects are given by morphisms $\xi : B\flat A \to A$ satisfying the usual conditions of algebras over a monad. The corresponding free-forgetful adjunction is denoted by $\mathsf{F}_B \dashv \mathsf{G}_B$.

The comparison functor $\mathsf{R} : \mathrm{Pt}_B(\mathscr{C}) \to \mathscr{C}^{B\flat(-)}$ maps a point $(p : A \to B,$ $s : B \to A)$ over B to the $B\flat(-)$-algebra $\xi : B\flat\mathrm{Ker}(p) \to \mathrm{Ker}(p)$ induced by

the commutative right-hand side square in the following diagram:



The comparison functor has a left adjoint $\mathsf{L} : \mathscr{C}^{B\flat(-)} \to \mathrm{Pt}_B(\mathscr{C})$ which maps a $B\flat(-)$-algebra $\xi : B\flat X \to X$ to the point $(\pi_\xi : B \ltimes_\xi X \to B, \sigma_\xi : B \to B \ltimes_\xi X)$ as in the following diagram:



Here $(B \ltimes_\xi X, q)$ is the coequalizer of $k_{B,X}, \iota_2\xi : B\flat X \to B + X$, and $\sigma_\xi = q\iota_1$, and π_ξ is the unique morphism such that $\pi_\xi q = [1_B, 0]$ induced by the universal property of the coequalizer.

Example 1.2.15 (Semi-direct product of groups)**.** Let us consider the above constructions for $\mathscr{C}$ being the category Grp of groups. In this case, the coproduct $B + A$ of groups A, B is given by their free product, and the group $B\flat A$ is the normal subgroup of $B+A$ generated by the elements $bab^{-1} \in B+A$, where $a \in A$, $b \in B$. One can show that $B\flat(-)$-algebra structures on a group A correspond exactly to actions of the group B on A. Given a $B\flat(-)$-algebra $\xi : B\flat A \to A$, the corresponding action $\triangleright : B \times A \to A$ is given, for any $a \in A$, $b \in B$, by $b \triangleright a = \xi(bab^{-1})$. Given a group action $\triangleright : B \times A \to A$, the corresponding $B\flat(-)$-algebra $\xi : B\flat A \to A$ is determined by $\xi(bab^{-1}) = b \triangleright a$, where $a \in A$, $b \in B$. The comparison functor $\mathsf{R} : \mathrm{Pt}_B(\mathscr{C}) \to \mathscr{C}^{B\flat(-)}$ maps a point $(p : A \to B, s : B \to A)$ to the $B\flat(-)$-algebra $\xi : B\flat\mathrm{Ker}(p) \to \mathrm{Ker}(p)$ which corresponds to the action $\triangleright : B \times \mathrm{Ker}(p) \to \mathrm{Ker}(p)$ given by $b \triangleright k = s(b)ks(b)^{-1}$ for all $b \in B$, $k \in \mathrm{Ker}(p)$. Given a $B\flat(-)$-algebra $\xi : B\flat X \to X$, the group $B \ltimes_\xi X$ is generated by the groups B, X with the condition that conjugation of X by B coincides with the action of B on X. Hence $B \ltimes_\xi X$ is the semi-direct product of B, X with respect to the action of B on X. Note that, for groups, the adjunction $\mathsf{L} \dashv \mathsf{R}$ is actually an adjoint equivalence.

In 2005, F. Borceux, G. Janelidze and G.M. Kelly introduced the notion of internal object action [BJK05a].

Definition 1.2.16 (Internal object action)**.** Let $\mathscr{C}$ be a pointed, finitely bicomplete category and B, X be objects in $\mathscr{C}$. An *internal object action* of B on X is a $B\flat(-)$-algebra structure $\xi : B\flat X \to X$ on X.

The notion of internal object action recovers the usual notions of actions of groups, of rings, of Lie algebras, and of associative algebras.

In their article [BJ98] from 1998, D. Bourn and G. Janelidze define what is meant for a category to have semi-direct products.

Definition 1.2.17 (Semi-direct products)**.** A category with pullbacks of split epimorphisms *has semi-direct products* if, for any morphism $p : E \to B$ in $\mathscr{C}$, the functor $p^* : \mathrm{Pt}_B(\mathscr{C}) \to \mathrm{Pt}_E(\mathscr{C})$ is monadic, i.e., it has a left adjoint and the comparison functor to the Eilenberg-Moore category of the corresponding monad is an equivalence of categories.

Proposition 1.2.18. [BJ98] Let $\mathscr{C}$ be an exact category. Then the following conditions are equivalent:

1. $\mathscr{C}$ is protomodular and has pushouts of split monomorphisms.

2. $\mathscr{C}$ has semi-direct products.

Corollary 1.2.19. [JMT02] Let $\mathscr{C}$ be a pointed exact category with binary coproducts. Then the following conditions are equivalent:

1. $\mathscr{C}$ is semi-abelian.

2. $\mathscr{C}$ has semi-direct products.

In the following, we want to recall the notion of action representability due to F. Borceux, G. Janelidze and G.M. Kelly [BJK05a].

Let us assume that $\mathscr{C}$ is a pointed, finitely bicomplete category. For a fixed object X of $\mathscr{C}$, we denote by $\mathrm{Act}(-, X) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ the functor which maps an object B to the set $\mathrm{Act}(B, X)$ of internal B-actions on X. Given a morphism $\varphi : B \to B'$, the function $\mathrm{Act}(\varphi, X) : \mathrm{Act}(B', X) \to \mathrm{Act}(B, X)$ maps a B'-action $\xi : B' \flat X \to X$ to the B-action $\xi(\varphi \flat X) : B \flat X \to X$, where the morphism $\varphi \flat X : B \flat X \to B' \flat X$ is the unique one such that $k_{B',X}(\varphi \flat X) = (\varphi + X)k_{B,X}$ as in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & B \flat X & \xrightarrow{k_{B,X}} & B + X & \xrightarrow{[1_B, 0]} & B \\ & & \downarrow{\scriptstyle \varphi \flat X} & & \downarrow{\scriptstyle \varphi + X} & & \downarrow{\scriptstyle \varphi} \\ 0 & \longrightarrow & B' \flat X & \xrightarrow[k_{B',X}]{} & B' + X & \xrightarrow[[1_{B'}, 0]]{} & B' \end{array}$$

Definition 1.2.20 (Action representable category [BJK05a])**.** Let $\mathscr{C}$ be a pointed, finitely bicomplete category. Then $\mathscr{C}$ is called *action representable* if, for any object X in $\mathscr{C}$, the functor $\mathrm{Act}(-, X) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ is representable, i.e., there exists an object $[X]$, called the *actor* of X, such that the functors $\mathrm{Act}(-, X), \mathrm{Hom}(-, [X]) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ are naturally isomorphic.

Examples 1.2.21 (Action representable categories)**.**

1. The category Grp of groups is action representable. Given a group X, the functor $\mathrm{Act}(-, X) : \mathrm{Grp}^{\mathrm{op}} \to \mathrm{Set}$ is naturally isomorphic to the hom-functor $\mathrm{Hom}(-, \mathrm{Aut}(X)) : \mathrm{Grp}^{\mathrm{op}} \to \mathrm{Set}$, where $\mathrm{Aut}(X)$ is the group of automorphisms on X. An action of a group B on X is equivalently encoded by a group homomorphism from B to $\mathrm{Aut}(X)$. Hence the actor $[X]$ of a group X is its automorphism group $\mathrm{Aut}(X)$.

2. The category of Lie algebras over a unital commutative ring R is action representable. The actor of a Lie algebra X is given by the Lie algebra $\mathrm{Der}(X)$ of derivations on X.

3. The category of Boolean rings is action representable. The actor of a Boolean ring X is the Boolean ring $\mathrm{End}(X)$ of endomorphisms on X [BJK05b].

4. The categories $\mathrm{XMod}(\mathrm{Grp})$ and $\mathrm{XMod}(\mathrm{Lie}_R)$ of crossed modules of groups and of Lie algebras, respectively, are action representable [BJK05a], see also [Nor90, CL98, Ram19] and compare with Proposition 2.4.4.

5. In [GSV19], it is shown that the category of cocommutative Hopf algebras over a field k is action representable, see also [GKV18] for an explicit description of the actor of a cocommutative Hopf algebra when the field k is of characteristic 0.

Note that, if $\mathscr{C}$ is an abelian category, then the adjunction $B+(-) \dashv \mathrm{Ker}_B$ as in Diagram (1.6) is an adjoint equivalence, and the kernel of $[1_B, 0] : B \to B+A$ is given by (A, ι_2). Hence $B\flat(-)$ is the identity monad on $\mathscr{C}$, and any object B has the unique action 1_X on any object X. The semi-direct product of B, X with respect to this action is given by $(B \oplus X, \pi_1, \iota_1)$. This shows that any abelian category is action representable with actor $[X] = 0$ for any object X in $\mathscr{C}$.

The categories of, for example, associative and commutative associative algebras are not action representable. D. Bourn and G. Janelidze introduced the notion of action accessibility [BJ09], and G. Janelidze introduced the notion of weak action representability [Jan22] to encompass wider classes of categories with 'reasonably good' internal object actions.

The fact that, for a semi-abelian category $\mathscr{C}$, the categories $\mathrm{Pt}_B(\mathscr{C})$ and $\mathscr{C}^{B\flat(-)}$, where B is any object in $\mathscr{C}$, are equivalent, allows for an alternative characterization of action representability.

In the following, we assume that $\mathscr{C}$ is a pointed category. Let us denote by $\mathrm{SplExt}(-, X) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$, where X is an object in $\mathscr{C}$, the functor that maps an object B to the set of isomorphism classes of split extensions of B by X. A split extension of B by X is given by a diagram

$$0 \longrightarrow X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

in $\mathscr{C}$, where $ps = 1_B$, (X, k) is the kernel of p, and (B, p) is the cokernel of k. A morphism between split extensions of B by X as in the diagram below is given by a morphism $\alpha : A \to A'$ such that $\alpha k = k'$, $p'\alpha = p$ and $\alpha s = s'$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{k} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \alpha} & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{k'} & A' & \underset{s'}{\overset{p'}{\rightleftarrows}} & B & \longrightarrow & 0 \end{array}$$

If $\mathscr{C}$ is protomodular, then α is necessarily an isomorphism (Proposition 1.2.5).

Given a morphism $\varphi : B \to B'$, the function

$$\mathrm{SplExt}(\varphi, X) : \mathrm{SplExt}(B', X) \to \mathrm{SplExt}(B, X)$$

maps the isomorphism class of the split extension of B' by X as in the diagram below to the isomorphism class of the split extension of B by X obtained by pulling back φ along p':

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\ker(p_2)} & A' \times_{B'} B & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & X & \xrightarrow{k'} & A' & \underset{s'}{\overset{p'}{\rightleftarrows}} & B' & \longrightarrow & 0 \end{array}$$

Note that the kernel of the pullback projection p_2 is indeed given by X.

Proposition 1.2.22. [BJK05a] Let $\mathscr{C}$ be a semi-abelian category. Then the following conditions are equivalent:

1. $\mathscr{C}$ is action representable.
2. For any object X in $\mathscr{C}$, the functor $\mathrm{SplExt}(-, X) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ is representable, i.e., there exists an object $[\![X]\!]$ such that the functors

   $$\mathrm{SplExt}(-, X), \mathrm{Hom}(-, [\![X]\!]) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$$

   are naturally isomorphic. In this case, $[\![X]\!]$ is called the *split extension classifier* of X.
3. Any object X in $\mathscr{C}$ has a so-called *generic split extension*

   $$0 \longrightarrow X \xrightarrow{k^*} A^* \underset{s^*}{\overset{p^*}{\rightleftarrows}} B^* \longrightarrow 0,$$

   meaning that, for any split extension

   $$0 \longrightarrow X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0,$$

   there exists a unique morphism $\varphi : B \to B^*$ such that

   $$\mathrm{SplExt}(\varphi, X)([0 \to X \to A^* \rightleftarrows B^* \to 0]) = [0 \to X \to A \rightleftarrows B \to 0].$$

In this case, the actor $[X]$ and the split extension classifier $[\![X]\!]$ of X are isomorphic, and the generic split extension of X can be chosen as

$$0 \longrightarrow X \xrightarrow{k_\xi} [X] \ltimes_\xi X \underset{\sigma_\xi}{\overset{\pi_\xi}{\rightleftarrows}} [X] \longrightarrow 0,$$

where the action $\xi : [X] \flat X \to X$ corresponds to $1_{[X]} \in \mathrm{Hom}([X], [X])$.

1.3 Categorical Galois theory

In this section, we recall some basic facts on categorical Galois theory due to G. Janelidze [Jan89, Jan90b, Jan91a] and its application to central extensions [JK94]. We will see that Birkhoff subcategories and torsion theories induce admissible Galois structures in certain contexts.

1.3.1 Galois structures and central extensions

The core concept in categorical Galois theory is that of a Galois structure.

Definition 1.3.1 (Galois structure)**.** A *Galois structure* $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ consists of an adjunction

$$\mathscr{C} \underset{\mathsf{U}}{\overset{\mathsf{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{F}, \tag{1.7}$$

and classes $\mathscr{E}, \mathscr{Z}$ of morphisms in $\mathscr{C}, \mathscr{F}$, respectively, such that the following conditions hold:

1. $\mathscr{E}$ and $\mathscr{Z}$ contain all isomorphisms in $\mathscr{C}$ and $\mathscr{F}$, respectively.
2. All pullbacks of morphisms in $\mathscr{E}$ and $\mathscr{Z}$ exist in $\mathscr{C}$ and $\mathscr{F}$, and lie in $\mathscr{E}$ and $\mathscr{Z}$, respectively.
3. Both $\mathscr{E}$ and $\mathscr{Z}$ are closed under composition.
4. $\mathsf{F}(\mathscr{E}) \subseteq \mathscr{Z}$ and $\mathsf{U}(\mathscr{Z}) \subseteq \mathscr{E}$.

The morphisms in $\mathscr{E}, \mathscr{Z}$ are often called *extensions*.

We can think of $\mathscr{C}$ as 'complicated' category whose class $\mathscr{E}$ of extensions we would like to better understand, and of $\mathscr{F}$ as category whose class $\mathscr{Z}$ of extensions is easier to understand.

A Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ induces, for any object $B \in \mathscr{C}$, the adjunction

$$\mathscr{E}(B) \underset{\mathsf{U}^B}{\overset{\mathsf{F}^B}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{Z}(\mathsf{F}(B)), \tag{1.8}$$

where $\mathscr{E}(B)$ is the full subcategory of the slice category $\mathscr{C} \downarrow B$ whose objects are given by the morphisms in $\mathscr{E}$ with codomain B, and analogously for $\mathscr{Z}(\mathsf{F}(B))$. The left adjoint F^B maps a morphism $f : A \to B$ in $\mathscr{E}$ to $\mathsf{F}(f) : \mathsf{F}(A) \to \mathsf{F}(B)$. For a morphism $\varphi : X \to \mathsf{F}(B)$ in $\mathscr{Z}$, let us consider the pullback

$$\begin{array}{ccc} B \times_{\mathsf{UF}(B)} \mathsf{U}(X) & \xrightarrow{p_2} & \mathsf{U}(X) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \mathsf{U}(\varphi)} \\ B & \xrightarrow[\eta_B]{} & \mathsf{UF}(B) \end{array} \tag{1.9}$$

of $\mathsf{U}(\varphi)$ along the B-component η_B of the unit of the adjunction (1.7). Then p_1 is the image of φ under the right adjoint U^B. By pasting the F-image of the above pullback with the corresponding naturality square of the counit ε of the

adjunction (1.7), we see that the composite $\varepsilon_X \mathsf{F}(p_2) : \mathsf{F}(B \times_{\mathsf{UF}(B)} \mathsf{U}(X)) \to X$ is a morphism from $\mathsf{F}(p_1)$ to φ in $\mathscr{Z}(\mathsf{F}(B))$:

$$\begin{array}{ccccc} \mathsf{F}(B \times_{\mathsf{UF}(B)} \mathsf{U}(X)) & \xrightarrow{\mathsf{F}(p_2)} & \mathsf{FU}(X) & \xrightarrow{\varepsilon_X} & X \\ \downarrow{\scriptstyle \mathsf{F}(p_1)} & & \downarrow{\scriptstyle \mathsf{FU}(\varphi)} & & \downarrow{\scriptstyle \varphi} \\ \mathsf{F}(B) & \xrightarrow[\mathsf{F}(\eta_B)]{} & \mathsf{FUF}(B) & \xrightarrow[\varepsilon_{\mathsf{F}(B)}]{} & \mathsf{F}(B) \end{array}$$

This is the φ-component of the counit ε^B of the adjunction (1.8).

Definition 1.3.2 (Admissibility)**.** The Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ is called *admissible* if the right adjoint U^B in the adjunction (1.8) is fully faithful for all $B \in \mathscr{C}$. Equivalently, the counit ε^B is an isomorphism for all $B \in \mathscr{C}$.

In this case, we can view the category $\mathscr{Z}(\mathsf{F}(B))$ of extensions of $\mathsf{F}(B)$ in $\mathscr{F}$ as a full subcategory of the category $\mathscr{E}(B)$ of extensions of B in $\mathscr{C}$.

The following result can be found in G. Janelidze's and G.M. Kelly's article [JK94, Proposition 3.3].

Proposition 1.3.3. Let $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ be a Galois structure in which U is fully faithful. Then Γ is admissible if and only if F preserves all pullbacks of the form (1.9), where φ lies in $\mathscr{Z}$.

Definition 1.3.4 (Central extension)**.** Let $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ be a Galois structure and $f : A \to B$ be an extension in $\mathscr{C}$.

1. f is a *trivial extension* if the naturality square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathsf{UF}(A) \\ f \downarrow & & \downarrow{\scriptstyle \mathsf{UF}(f)} \\ B & \xrightarrow[\eta_B]{} & \mathsf{UF}(B) \end{array} \tag{1.10}$$

   is a pullback. If Γ is admissible, this is equivalent to f lying in the essential image of the functor $\mathsf{U}^B : \mathscr{Z}(\mathsf{F}(B)) \to \mathscr{E}(B)$.

2. f is a *monadic extension* if it is effective $\mathscr{E}$-descent, i.e., the pullback functor $f^* : \mathscr{E}(B) \to \mathscr{E}(A)$ is monadic (see also Section 1.3.2).

3. f is a *normal extension* if it is a monadic extension and either of the pullback projections p_1, p_2 of its kernel pair

$$\begin{array}{ccc} \mathrm{Eq}(f) & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

   is a trivial extension.

4. f is a *central extension* if there exists a monadic extension $p : E \to B$ such that the pullback p_1 of f along p as in the diagram below is a trivial extension.

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

It is clear that any trivial extension is a central extension, and that any normal extension is a central extension.

Proposition 1.3.5. Let $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ be an admissible Galois structure. Then any trivial extension is a normal extension.

In this context, trivial extensions can be viewed as those arising from $\mathscr{F}$, making them particularly tractable. Before investigating arbitrary extensions in $\mathscr{C}$, we are thus led to focus on the subclasses of normal and central extensions.

Remark 1.3.6 (Fundamental theorem of categorical Galois theory)**.** We want to shortly mention the fundamental theorem of categorical Galois theory due to G. Janelidze [Jan91a]. It states that there is, for any admissible Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ and any monadic extension $p : E \to B$, an equivalence of categories $\mathrm{Split}(E, p) \approx \mathscr{F}^{\downarrow_{\mathscr{Z}} \mathrm{Gal}(E,p)}$ between the category of extensions $f : A \to B$ whose pullback along p is a trivial extension, and the category of discrete fibrations between pregroupoids in $\mathscr{F}$ with codomain the Galois pregroupoid $\mathrm{Gal}(E, p)$ and with components in $\mathscr{Z}$. The Galois pregroupoid $\mathrm{Gal}(E, p)$ is given by the F-image of the kernel pair (Eq, p_1, p_2) of p viewed as an internal groupoid in $\mathscr{C}$. When $p : E \to B$ factors through every other monadic extension over B, then we have that $\mathrm{Split}(E, p)$ is exactly the category $\mathrm{CExt}(B)$ of central extensions over B. If $\mathscr{C}$ is an exact category with enough regular projective objects (see Definition 3.4.8) and $\mathscr{E}$ is the class of regular epimorphisms, such a p always exists.

Example 1.3.7 (Central extensions of groups)**.** The categorical notion of central extension recovers the classical notion of central extension of groups. Namely, $\Gamma = (\mathrm{Grp}, \mathrm{Ab}, \mathsf{ab}, \mathsf{U}, \mathrm{Surj}(\mathrm{Grp}), \mathrm{Surj}(\mathrm{Ab}))$ is an admissible Galois structure. Here Grp is the category of groups, Ab is its full subcategory of abelian groups, $\mathsf{U} : \mathrm{Ab} \to \mathrm{Grp}$ is the inclusion functor, and $\mathsf{ab} : \mathrm{Grp} \to \mathrm{Ab}$ is the abelianization functor which maps a group G to its quotient $G/[G, G]$ by its commutator subgroup $[G, G]$. Moreover, $\mathrm{Surj}(\mathrm{Grp})$ and $\mathrm{Surj}(\mathrm{Ab})$ denote the classes of surjective homomorphisms in Grp and Ab, respectively. For any group G, Γ induces the adjunction

$$\mathrm{Surj}(\mathrm{Grp})(G) \underset{\mathsf{U}^G}{\overset{\mathsf{ab}^G}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Surj}(\mathrm{Ab})(G/[G, G]),$$

where $\mathrm{Surj}(\mathrm{Grp})(G)$ is the full subcategory of $\mathrm{Grp} \downarrow G$ with objects the surjective group homomorphisms with codomain G, and $\mathrm{Surj}(\mathrm{Ab})(G/[G, G])$ is the full subcategory of $\mathrm{Ab} \downarrow G/[G, G]$ with objects the surjective abelian group

homomorphisms with codomain $G/[G,G]$. The notions of normal and central extension coincide in this context, and an extension $f : G \to H$, i.e., a surjective group homomorphism, is central with respect to Γ if and only if the kernel $\mathrm{Ker}(f)$ of f lies in the centre $\mathcal{Z}(A)$ of A, i.e., $[\mathrm{Ker}(f), A] = \{1\}$.

The categorical notion of central extension recovers also various other notions of central extensions in algebra, see the end of Section 1.3.3 for more examples.

1.3.2 Effective descent morphisms

Given a Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$, we defined in Definition 1.3.4 an extension $f : A \to B$ in $\mathscr{C}$ to be monadic if it is effective $\mathscr{E}$-descent, i.e., the pullback functor $f^* : \mathscr{E}(B) \to \mathscr{E}(A)$ is monadic. In this section, we want to recall some basic facts on effective $\mathscr{E}$-descent morphisms.

There are different variations of the notion of epimorphism such as extremal, strong, regular and split epimorphism. In the regular context, the classes of extremal, strong and regular epimorphisms all coincide. In the exact setting, the notion of regular epimorphism can be seen as a good categorical generalization of the notion of quotient as coming from universal algebra. But already in the regular context, the notion of effective descent morphism proves often more suitable. The class $\mathscr{E}$ of effective descent morphisms in a finitely complete category $\mathscr{C}$ satisfies the following properties. It contains all isomorphisms in $\mathscr{C}$, it is closed under composition [RST93], it is stable under pullbacks [ST92], it has the strong right cancellation property [JST04] (assuming the axiom of universes [AGV72]), i.e., if $gf \in \mathscr{E}$, then $g \in \mathscr{E}$, and any morphim in it is a regular epimorphism. In [Jac23], P.-A. Jacqmin proves that, given a class $\mathscr{E}$ of morphisms in a small finitely complete category $\mathscr{C}$, there exists a fully faithful functor from $\mathscr{C}$ into a presheaf category $\mathrm{Set}^{\mathscr{D}}$, where $\mathscr{D}$ is a small category, that preserves and reflects finite limits, and sends morphisms in $\mathscr{E}$, and only those, to componentwise surjections if and only if $\mathscr{E}$ satisfies the former conditions. In this sense, the class of effective descent morphisms can be seen as a *regular surjection-like* class of morphisms. The smallest regular surjection-like class of morphisms in a finitely complete category is given by the class of split epimorphisms, and the Yoneda embedding yields a corresponding fully faithful functor. The largest regular surjection-like class of morphisms is given by the class of pullback-stable regular epimorphisms. Since, in an exact category, regular epimorphisms are pullback-stable and equivalence relations are effective, the classes of regular epimorphisms and of effective descent morphisms coincide in this setting [ST92]. In the regular context, there might be regular epimorphisms that are not effective descent morphisms, and it might be difficult to characterize effective descent morphisms.

There are various ways to define the notion of effective descent morphism. In the following, we present the monadic approach to descent theory (see e.g. [JT94]).

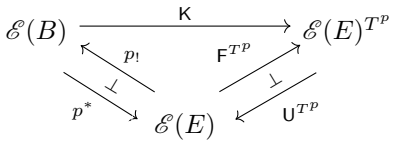
Let $\mathscr{C}$ be a category with pullbacks, and $\mathscr{E}$ be a class of morphisms in $\mathscr{C}$ that is closed under composition with isomorphisms. Furthermore, let $p : E \to B$ be a morphism in $\mathscr{C}$ such that $\mathscr{E}$ is closed under pullback along p, and under

post-composition with p. Then we have the adjunction

$$\mathscr{E}(B) \underset{p^*}{\overset{p_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathscr{E}(E), \tag{1.11}$$

where the right adjoint p^* maps a morphism $f : A \to B$ in $\mathscr{E}(B)$ to its pullback along $p : E \to B$, and the left adjoint $p_!$ maps a morphism $g : X \to E$ in $\mathscr{E}(E)$ to its composite pg with $p : E \to B$.

This adjunction induces the monad $\mathsf{T}^p := p^* p_!$ on $\mathscr{E}(E)$. Recall that we have the so-called comparison functor K from $\mathscr{E}(B)$ to the Eilenberg-Moore category $\mathscr{E}(E)^{\mathsf{T}^p}$ of T^p, which maps a morphism g in $\mathscr{E}(B)$ to the algebra $(p^*(g), p^*(\varepsilon_g) : \mathsf{T}^p(p^*(g)) \to p^*(g))$ in $\mathscr{E}(E)^{\mathsf{T}^p}$, where ε is the counit of the adjunction (1.11). The comparison functor K satisfies the two equations $\mathsf{U}^{\mathsf{T}^p}\mathsf{K} = p^*$ and $p_!\mathsf{K} = \mathsf{F}^{\mathsf{T}^p}$, where $\mathsf{U}^{\mathsf{T}^p}$ and $\mathsf{F}^{\mathsf{T}^p}$ form the free-forgetful adjunction between $\mathscr{E}(E)$ and $\mathscr{E}(E)^{\mathsf{T}^p}$, respectively.



Definition 1.3.8 (Effective descent morphism)**.** Let the category $\mathscr{C}$, the class of morphisms $\mathscr{E}$ in $\mathscr{C}$, and the morphism $p : E \to B$ in $\mathscr{C}$ be as above.

1. p is called *effective $\mathscr{E}$-descent morphism* if p^* is monadic, i.e., the comparison functor $K : \mathscr{E}(B) \to \mathscr{E}(E)^{\mathsf{T}^p}$ is an equivalence of categories.

2. p is called *effective descent morphism* if it is an effective $\mathscr{E}$-descent morphism for the class $\mathscr{E}$ of all morphisms in $\mathscr{C}$.

Proposition 1.3.9. [ST92] Let $\mathscr{C}$ be an exact category. Then the effective descent morphisms in $\mathscr{C}$ are precisely its regular epimorphisms.

Categories which are not necessarily exact but for which, however, the classes of effective descent morphisms and of regular epimorphisms coincide are called descent-exact in [DA17]. We will see in Section 3.1 that the so-called quasi-abelian categories are descent-exact.

Proposition 1.3.10. [JT94, 2.6. Proposition] Let $\mathscr{C}$ be a category with pullbacks, $\mathscr{E}_0 \subseteq \mathscr{E}_1$ be two classes of morphisms in $\mathscr{C}$ closed under composition with isomorphisms, and $p : E \to B$ be a morphism in $\mathscr{C}$ such that both classes $\mathscr{E}_0$ and $\mathscr{E}_1$ are stable under pullback along p and under post-composition with p. Furthermore, suppose that p is an effective $\mathscr{E}_1$-descent morphism. Then the following conditions are equivalent:

1. The morphism p is effective $\mathscr{E}_0$-descent.

2. For any pullback

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B, \end{array}$$

where $f \in \mathscr{E}_1$ and $\pi_1 \in \mathscr{E}_0$, then $f \in \mathscr{E}_0$.

Corollary 1.3.11. Let $\mathscr{C}$ be a regular category and p be an effective descent morphism. Then p is also an effective RegEpi($\mathscr{C}$)-descent morphism, where RegEpi($\mathscr{C}$) denotes the class of regular epimorphisms in $\mathscr{C}$.

1.3.3 Birkhoff subcategories

Birkhoff subcategories can be understood as a categorical generalization of subvarieties of universal algebras. More precisely, the Birkhoff subcategories of a variety of universal algebras are exactly its subvarieties. Moreover, Birkhoff subcategories prove to be useful in categorical Galois theory.

Definition 1.3.12 (Birkhoff subcategory)**.** Let $\mathscr{C}$ be a regular category and $\mathscr{F}$ be a replete, full, reflective subcategory of $\mathscr{C}$. Then $\mathscr{F}$ is called a *Birkhoff subcategory* of $\mathscr{C}$ when it is closed in $\mathscr{C}$ under subobjects and regular quotients, meaning that, if $f : A \to B$ is a monomorphism and B is in $\mathscr{F}$, then A is also in $\mathscr{F}$, and if $g : C \to D$ is a regular epimorphism and C is in $\mathscr{F}$, then D is also in $\mathscr{F}$.

The following two propositions provide alternative characterizations of closedness under subobjects and regular quotients in certain settings, respectively.

Proposition 1.3.13. Let $\mathscr{C}$ be a regular category and $\mathscr{F}$ be a replete, full, reflective subcategory of $\mathscr{C}$. Then the following conditions are equivalent:

1. $\mathscr{F}$ is closed in $\mathscr{C}$ under subobjects.
2. $\mathscr{F}$ is regular epi-reflective, i.e., for any object A in $\mathscr{C}$, the A-component $\eta_A : A \to \mathsf{UF}(A)$ of the unit of the adjunction is a regular epimorphism.

Proposition 1.3.14. [JK94, Proposition 3.1] Let $\mathscr{C}$ be regular category and $\mathscr{F}$ be a replete, full, regular epi-reflective subcategory of $\mathscr{C}$. Then the following conditions are equivalent:

1. $\mathscr{F}$ is closed in $\mathscr{C}$ under regular quotients.
2. For any regular epimorphism $f : A \to B$, the naturality square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathsf{UF}(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{UF}(f)} \\ B & \xrightarrow[\eta_B]{} & \mathsf{UF}(B) \end{array}$$

is a pushout.

One of the strengths of Birkhoff subcategories is that there are exact as soon as the ambient category is.

Proposition 1.3.15.

1. Let $\mathscr{C}$ be a regular category and $\mathscr{F}$ be a replete, full, regular epi-reflective subcategory of $\mathscr{C}$. Then $\mathscr{F}$ is also a regular category, and a morphism in $\mathscr{F}$ is a regular epimorphism (monomorphism) if and only if it is a regular epimorphism (monomorphism) in $\mathscr{C}$.

2. Let $\mathscr{C}$ be an exact category and $\mathscr{F}$ be a Birkhoff subcategory of $\mathscr{C}$. Then $\mathscr{F}$ is also an exact category.

Remark 1.3.16. The subcategory of torsion-free abelian groups of the category Ab of abelian groups shows that replete, full, regular epi-reflective subcategories of an exact category are not exact in general.

Proposition 1.3.17. Let $\mathscr{C}$ be a semi-abelian category and $\mathscr{F}$ be a Birkhoff subcategory of $\mathscr{C}$. Then $\mathscr{F}$ is also semi-abelian.

The following result is a consequence of Birkhoff's HSP Theorem 1.1.26.

Proposition 1.3.18. The Birkhoff subcategories of a variety of universal algebras are precisely its subvarieties.

Let us now turn our attention to the use of Birkhoff subcategories in categorical Galois theory.

Proposition 1.3.19. [JK94, Theorem 3.4, Theorem 4.8, Proposition 4.2] Let $\mathscr{C}$ be an exact Mal'tsev category and $\mathscr{F}$ be a Birkhoff subcategory of $\mathscr{C}$.

1. Then $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathrm{RegEpi}(\mathscr{C}), \mathrm{RegEpi}(\mathscr{F}))$ is an admissible Galois structure, where $\mathsf{U} : \mathscr{F} \to \mathscr{C}$ denotes the inclusion functor, $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ its left adjoint, and $\mathrm{RegEpi}(\mathscr{C})$ and $\mathrm{RegEpi}(\mathscr{F})$ the classes of regular epimorphisms in $\mathscr{C}$ and $\mathscr{F}$, respectively.

2. The classes of normal and of central extensions with respect to Γ coincide.

3. A regular epimorphism $f : A \to B$ is a trivial extension with respect to Γ if and only if $f : A \to B$ and $\eta_A : A \to \mathsf{UF}(A)$ are jointly monomorphic.

Proof. Let us prove the characterization of the trivial extensions. By definition, the regular epimorphism $f : A \to B$ is a trivial extension with respect to Γ if and only if the naturality square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathsf{UF}(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{UF}(f)} \\ B & \xrightarrow[\eta_B]{} & \mathsf{UF}(B) \end{array}$$

is a pullback. By Proposition 1.3.14, we know that it is a pushout since $\mathscr{F}$ is assumed to be a Birkhoff subcategory of $\mathscr{C}$. Hence it is also a pullback if

and only if the induced morphism φ from A to the pullback $B \times_{\mathsf{UF}(B)} \mathsf{UF}(A)$ of η_B along $\mathsf{UF}(f)$ is an isomorphism. By Proposition 1.1.20, φ is a regular epimorphism since $\mathscr{C}$ is assumed to be an exact Mal'tsev category. It is also a monomorphism if and only if f, η_A are jointly monomorphic. This shows the claim. □

We have seen in Example 1.3.7 that the categorical notion of central extension with respect to $\Gamma = (\mathrm{Grp}, \mathrm{Ab}, \mathsf{ab}, \mathsf{U}, \mathrm{Surj}(\mathrm{Grp}), \mathrm{Surj}(\mathrm{Ab}))$ recovers the classical notion of central extension of groups. Note that Ab is clearly a Birkhoff subcategory of the exact Mal'tsev category Grp. The categorical notion of central extension recovers various other notions of central extensions by considering corresponding exact Mal'tsev categories $\mathscr{C}$ and Birkhoff subcategories $\mathscr{F}$ of $\mathscr{C}$ in the admissible Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathrm{RegEpi}(\mathscr{C}), \mathrm{RegEpi}(\mathscr{F}))$. For $\mathscr{C}$ being a variety of Ω-groups and $\mathscr{F}$ being one of its subvarieties, one recovers the notion of central extension in the sense of A. Fröhlich [Frö63], see also [Lue67, FC72]. For $\mathscr{F}$ being the subvariety of abelian algebras of a Mal'tsev variety $\mathscr{C}$, one recovers the notion of central extension arising from commutator theory in universal algebra, see [JK00a]. There are also interesting notions of central extensions in exact Mal'tsev categories beyong the varietal case. One example is given by the notion of central extension of compact Hausdorff groups, which form a semi-abelian category, with respect to its Birkhoff subcategory of profinite groups, see [EG14].

1.3.4 Torsion theories

Torsion theories were introduced by S.E. Dickson [Dic66] in abelian categories to axiomatize the following two observations in the category of abelian groups. Firstly, any group homomorphism from a torsion abelian group to a torsion-free abelian group is necessarily 0. Secondly, given any abelian group A, there exist a torsion abelian group T and a torsion-free abelian group F such that T, A and F assemble into a short exact sequence:

$$0 \longrightarrow T \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} F \longrightarrow 0$$

More precisely, T is given by the subgroup of torsion elements of A, ε_A is the corresponding injection, F is given by the quotient A/T, and η_A is the corresponding projection. More recently, torsion theories have also been studied in more general pointed and non-pointed non-abelian settings [CDT06, BG06, JT07, FFG21]. Besides their own interest, torsion theories are, as Birkhoff subcategories, a source of admissible Galois structures.

Definition 1.3.20 (Torsion theory)**.** A *torsion theory* $(\mathscr{T}, \mathscr{F})$ in a pointed category $\mathscr{C}$ consists of two replete full subcategories $\mathscr{T}$ and $\mathscr{F}$ of $\mathscr{C}$ such that the following conditions hold:

1. Any morphism in $\mathscr{C}$ from an object in $\mathscr{T}$ to an object in $\mathscr{F}$ is 0.
2. For any object $C \in \mathscr{C}$, there exists a short exact sequence

$$0 \longrightarrow T \xrightarrow{\varepsilon_C} C \xrightarrow{\eta_C} F \longrightarrow 0, \tag{1.12}$$

where $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

In this case, $\mathscr{T}$ is called a *torsion subcategory* of $\mathscr{C}$, and $\mathscr{F}$ is called a *torsion-free subcategory* of $\mathscr{C}$.

It is easy to see that the short exact sequence (1.12) is unique up to unique isomorphism, i.e., given another such short exact sequence, there exist unique isomorphisms φ and ψ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \xrightarrow{\varepsilon_C} & C & \xrightarrow{\eta_C} & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \| & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & T' & \xrightarrow[\varepsilon'_C]{} & C & \xrightarrow[\eta'_C]{} & F & \longrightarrow & 0 \end{array}$$

Proposition 1.3.21. Let $(\mathscr{T}, \mathscr{F})$ be a torsion theory in a pointed category $\mathscr{C}$, and let, for any object $C \in \mathscr{C}$, a short exact sequence as in (1.12) be chosen.

1. The subcategory $\mathscr{T}$ is normal mono-coreflective in $\mathscr{C}$. More precisely, the coreflection $\mathsf{T} : \mathscr{C} \to \mathscr{T}$ maps an object C to T as in (1.12), and the C-component of the corresponding counit is given by ε_C.

2. The subcategory $\mathscr{F}$ is normal epi-reflective in $\mathscr{C}$. More precisely, the reflection $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ maps an object C to F as in (1.12), and the C-component of the corresponding unit is given by η_C.

3. The reflection $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ is semi-left exact, i.e., it preserves all pullbacks of the form

$$\begin{array}{ccc} B \times_{\mathsf{UF}(B)} \mathsf{U}(X) & \xrightarrow{p_2} & \mathsf{U}(X) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \mathsf{U}(\varphi)} \\ B & \xrightarrow[\eta_B]{} & \mathsf{UF}(B), \end{array}$$

   where X lies in $\mathscr{F}$ and φ is an arbitrary morphism.

Proposition 1.3.22. [JT07] Let $\mathscr{C}$ be a pointed category with kernels and pullback-stable normal epimorphisms, and $\mathscr{F}$ be a replete full subcategory of $\mathscr{C}$. Then the following conditions are equivalent:

1. $\mathscr{F}$ is a torsion-free subcategory of $\mathscr{C}$.

2. $\mathscr{F}$ is a normal epi-reflective subcategory of $\mathscr{C}$ and the reflection functor $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ is semi-left exact.

Proposition 1.3.23. [BG06] Let $(\mathscr{T}, \mathscr{F})$ be a torsion theory in a homological category $\mathscr{C}$.

1. The torsion subcategory $\mathscr{T}$ is closed under regular quotients in $\mathscr{C}$.

2. The torsion-free subcategory $\mathscr{F}$ is closed under subobjects in $\mathscr{C}$.

3. Both subcategories $\mathscr{T}$ and $\mathscr{F}$ are closed under extensions in $\mathscr{C}$, meaning that, if
$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$
is a short exact sequence in $\mathscr{C}$, and X and Z are in $\mathscr{T}$ (in $\mathscr{F}$), then Y is also in $\mathscr{T}$ (in $\mathscr{F}$).

Proposition 1.3.24. Let $\mathscr{C}$ be a pointed regular category and $(\mathscr{T}, \mathscr{F})$ be a torsion theory in $\mathscr{C}$. Then
$$\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathrm{RegEpi}(\mathscr{C}), \mathrm{RegEpi}(\mathscr{F}))$$
is an admissible Galois structure.

Proof. The claim follows by Propositions 1.3.3 and 1.3.21. □

1.4 Internal structures in algebraic categories

In this section, we want to recall the notions of reflexive graph, internal category and internal groupoid, and their higher-dimensional versions, and study their behaviour in algebraic categories such as Mal'tsev, naturally Mal'tsev, semi-abelian and additive categories and varieties. These recollections will highlight the strong link existing between internal structures and commutator theory.

1.4.1 Internal categorical structures

Definition 1.4.1 (Reflexive graph)**.** Let $\mathscr{C}$ be a category.

1. A *reflexive graph* in $\mathscr{C}$ is given by a diagram
$$C_1 \underset{\overset{\longleftarrow e \longrightarrow}{c}}{\overset{d}{\rightrightarrows}} C_0$$
in $\mathscr{C}$ such that $de = 1_{C_0} = ce$.

2. Given two reflexive graphs $\mathbb{C} = (C_0, C_1, d, c, e)$ and $\mathbb{D} = (D_0, D_1, d, c, e)$ in $\mathscr{C}$, a *morphism of reflexive graphs* from $\mathbb{C}$ to $\mathbb{D}$ is given by a pair $(f_0 : C_0 \to D_0, f_1 : C_1 \to D_1)$ of morphisms in $\mathscr{C}$ such that $df_1 = f_0 d$, $cf_1 = f_0 c$ and $f_1 e = e f_0$.
$$\begin{array}{ccc} C_1 & \underset{\overset{\longleftarrow e \longrightarrow}{c}}{\overset{d}{\rightrightarrows}} & C_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ D_1 & \underset{\overset{\longleftarrow e \longrightarrow}{c}}{\overset{d}{\rightrightarrows}} & D_0 \end{array}$$

3. We denote by $\mathrm{RG}(\mathscr{C})$ the category whose objects are reflexive graphs in $\mathscr{C}$ and whose morphisms are morphisms of reflexive graphs.

If $\mathscr{C}$ is the category Set of sets, the notion of reflexive graph in $\mathscr{C}$ recovers the usual notion of reflexive graph given by a set C_0 of *objects*, a set C_1 of *arrows*, a function $d : C_1 \to C_0$ which assigns to any arrow its *domain*, a function $c : C_1 \to C_0$ which assigns to any arrow its *codomain*, and a function $e : C_0 \to C_1$ which assigns to any object x an *identity* arrow whose domain and codomain is x. Given a reflexive graph (C_0, C_1, d, c, e) in an arbitrary category $\mathscr{C}$, we call C_0 its *object of objects*, C_1 its *object of arrows*, $d : C_1 \to C_0$ its *domain morphism*, $c : C_1 \to C_0$ its *codomain morphism*, and $e : C_0 \to C_1$ its *identities morphism*. A morphism of reflexive graphs in Set is given by two functions $f_0 : C_0 \to D_0$ and $f_1 : C_1 \to D_1$ that map objects to objects, and arrows to arrows, respectively, such that the domain of the f_1-image of an arrow φ is given by the f_0-image of the domain of φ, and analogously for the codomain, and such that the f_1-image of the identity arrow on an object $x \in C_0$ is the identity arrow on $f_0(x)$.

Example 1.4.2 (Reflexive relation)**.** A reflexive relation (Definition 1.1.4) on an object C_0 in a category $\mathscr{C}$ is given by a reflexive graph (C_0, C_1, d, c, e) whose domain and codomain morphisms $d, c : C_1 \to C_0$ are jointly monomorphic.

In the following, we assume that $\mathscr{C}$ is a category with pullbacks. Given a reflexive graph (C_0, C_1, d, c, e) in $\mathscr{C}$, we denote by $(C_1 \times_{C_0} C_1, p_1, p_2)$ the pullback of $d : C_1 \to C_0$ along $c : C_1 \to C_0$ as depicted below. We denote by $e_1, e_2 : C_1 \to C_{1 \times_{C_0}} C_1$ the canonical pullback injections induced by the common splitting $e : C_0 \to C_1$ of d, c. Furthermore, we denote the pullback of $d : C_1 \to C_0$ along $cp_2 : C_1 \times_{C_0} C_1 \to C_0$ by $(C_1 \times_{C_0} C_1 \times_{C_0} C_1, \pi_{1,2}, \pi_3)$, and by $(C_1 \times_{C_0} C_1 \times_{C_0} C_1, \pi_1, \pi_{2,3})$ the pullback of $dp_1 : C_1 \times_{C_0} C_1 \to C_0$ along $c : C_1 \to C_0$. Note that these two pullbacks have indeed the same object part.

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & C_1 \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle e}\uparrow\downarrow{\scriptstyle d} \\ C_1 & \underset{c}{\overset{e}{\rightleftarrows}} & C_0 \end{array} \tag{1.13}$$

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_3} & C_1 \\ {\scriptstyle \pi_{1,2}}\downarrow & & \downarrow{\scriptstyle d} \\ C_1 \times_{C_0} C_1 & \xrightarrow[cp_2]{} & C_0 \end{array} \qquad \begin{array}{ccc} C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_{2,3}} & C_1 \times_{C_0} C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle dp_1} \\ C_1 & \xrightarrow[c]{} & C_0 \end{array}$$

For $\mathscr{C} = \mathrm{Set}$, $C_1 \times_{C_0} C_1$ is given by the set of pairs $(f, g) \in C_1 \times C_1$ of arrows such that the codomain of f and the domain of g coincide. The functions p_1 and p_2 map (f, g) to f and g, respectively. The functions e_1 and e_2 map an arrow f to the pairs $(f, ec(f))$ and $(ed(f), f)$, respectively. The set $C_1 \times_{C_0} C_1 \times_{C_0} C_1$ is given by the set of triples (f, g, h) of arrows such that the codomain of f and the domain of g, and the codomain of g and the domain of h coincide. The functions $\pi_{1,2}, \pi_3, \pi_1, \pi_{2,3}$ map (f, g, h) to (f, g), h, f and (g, h), respectively.

Definition 1.4.3 (Internal category)**.** Let $\mathscr{C}$ be a category with pullbacks.

1. An *internal category* in $\mathscr{C}$ is given by a reflexive graph (C_0, C_1, d, c, e) in $\mathscr{C}$ together with a morphism $m : C_1 \times_{C_0} C_1 \to C_1$, where $C_1 \times_{C_0} C_1$

is the pullback of d along c as in Diagram (1.13), such that $dm = dp_1$, $cm = cp_2$, $me_1 = 1_{C_1} = me_2$ and $m(m \times_{1_{C_0}} 1_{C_1}) = m(1_{C_1} \times_{1_{C_0}} m)$, where $m \times_{1_{C_0}} 1_{C_1}$ is the unique morphism such that $p_1(m \times_{1_{C_0}} 1_{C_1}) = m\pi_{1,2}$ and $p_2(m \times_{1_{C_0}} 1_{C_1}) = \pi_3$, and $1_{C_1} \times_{1_{C_0}} m$ is the unique morphism such that $p_1(1_{C_1} \times_{1_{C_0}} m) = \pi_1$ and $p_2(1_{C_1} \times_{1_{C_0}} m) = m\pi_{2,3}$.

2. An *internal functor* from an internal category $\mathbb{C} = (C_0, C_1, d, c, e, m)$ to an internal category $\mathbb{D} = (D_0, D_1, d, c, e, m)$ in $\mathscr{C}$ is given by a morphism of reflexive graphs $(f_0 : C_0 \to D_0, f_1 : C_1 \to D_1)$ between the underlying reflexive graphs of $\mathbb{C}$ and $\mathbb{D}$ such that $f_1 m = m(f_1 \times_{f_0} f_1)$, where we denote by $f_1 \times_{f_0} f_1 : C_1 \times_{C_0} C_1 \to D_1 \times_{D_0} D_1$ the unique morphism such that $p_1(f_1 \times_{f_0} f_1) = f_1 p_1$ and $p_2(f_1 \times_{f_0} f_1) = f_1 p_2$.

$$\begin{array}{ccccc} C \times_{C_0} C_1 & \xrightarrow{m} & C_1 & \underset{c}{\overset{d}{\underset{\longleftarrow e \longrightarrow}{\longrightarrow}}} & C_0 \\ {\scriptstyle f_1 \times_{f_0} f_1}\downarrow & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ D_1 \times_{D_0} D_1 & \xrightarrow[m]{} & D_1 & \underset{c}{\overset{d}{\underset{\longleftarrow e \longrightarrow}{\longrightarrow}}} & D_0 \end{array}$$

3. We denote by $\mathrm{Cat}(\mathscr{C})$ the category whose objects are internal categories in $\mathscr{C}$ and whose morphisms are internal functors.

For $\mathscr{C} = \mathrm{Set}$, the notion of internal category recovers the usual notion of small category. The *multiplication* function $m : C_1 \times_{C_0} C_1$ maps a pair (f, g) of composable arrows to their composite $m(f, g)$ whose domain coincides with the domain of f, and whose codomain coincides with the codomain of g. This multiplication is unital, meaning that pre-composing an arrow f with the identity arrow on its domain, or post-composing with the identity arrow on its codomain yields f again. Furthermore, the multiplication is associative, meaning that, given a triple (f, g, h) of composable arrows, post-composing the composite $m(f, g)$ with h, or pre-composing the composite $m(g, h)$ with f yields the same result. The notion of internal functor recovers the usual notion of functor, meaning that the functions $f_0 : C_0 \to D_0$ and $f_1 : C_1 \to D_1$ are not only compatible with domain, codomain and identities, but also with composition, i.e., $m(f_1(f), f_1(g)) = f_1(m(f, g))$.

Example 1.4.4 (Preorder)**.** Given a reflexive relation $\mathbb{C} = (C_0, C_1, d, c, e)$ in a category $\mathscr{C}$ with pullbacks, there exists at most one multiplication morphism $m : C_1 \times_{C_0} C_1 \to C_1$ that turns $\mathbb{C}$ into an internal categoy. This is precisely the case when $\mathbb{C}$ is a preorder, i.e., it is not only reflexive but also transitive.

Definition 1.4.5 (Internal groupoid)**.** Let $\mathscr{C}$ be a category with pullbacks.

1. An *internal groupoid* $(C_0, C_1, d, c, e, m, i)$ in $\mathscr{C}$ consists of an internal category together with a morphism $i : C_1 \to C_1$ such that $di = c$, $ci = d$, $m(1_{C_1} \times_{1_{C_0}} i) = ed$ and $m(i \times_{1_{C_0}} 1_{C_1}) = ec$, where we denote by $1_{C_1} \times_{1_{C_0}} i : C_1 \times_{C_0} C_1 \to C_1 \times_{C_0} C_1$ the unique morphism such that $p_1(1_{C_1} \times_{1_{C_0}} i) = p_1$ and $p_2(1_{C_1} \times_{1_{C_0}} i) = ip_2$, and $i \times_{1_{C_0}} 1_{C_1} : C_1 \times_{C_0} C_1 \to C_1 \times_{C_0} C_1$ is the

unique morphism such that $p_1(i \times_{1_{C_0}} 1_{C_1}) = ip_1$ and $p_2(i \times_{1_{C_0}} 1_{C_1}) = p_2$.

$$C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \overset{i}{\circlearrowleft} \underset{c}{\overset{d}{\underset{\longleftarrow e \longrightarrow}{\rightrightarrows}}} C_0$$

2. We denote by $\mathrm{Grpd}(\mathscr{C})$ the full subcategory of $\mathrm{Cat}(\mathscr{C})$ whose objects are the internal groupoids in $\mathscr{C}$.

Remark 1.4.6. Let $\mathscr{C}$ be a category with pullbacks.

1. An internal category in $\mathscr{C}$ can be equipped with at most one internal groupoid structure, i.e., given an internal category $\mathbb{C} = (C_0, C_1, d, c, e, m)$ in $\mathscr{C}$, there exists at most one morphism $i : C_1 \to C_1$ that turns $\mathbb{C}$ into an internal groupoid.

2. An internal functor $(f_0 : C_0 \to D_0, f_1 : C_1 \to D_1)$ between two internal groupoids $(C_0, C_1, d, c, e, m, i)$ and $(D_0, D_1, d, c, m, e, i)$ is automatically compatible with the inverses morphisms, i.e, $if_1 = if_1$.

For $\mathscr{C} = \mathrm{Set}$, the notion of internal groupoid recovers the usual notion of small groupoid. Any arrow has a (necessarily unique) *inverse* arrow $i(f)$ whose domain is the codomain of f, and whose codomain is the domain of f. Post-composing f with its inverse yields the identity arrow on the domain of f, and pre-composing f with its inverse yields the identity arrow on the codomain of f.

Example 1.4.7 (Equivalence relation)**.** Internal groupoids $(C_0, C_1, d, c, e, m, i)$ in a category $\mathscr{C}$ with pullbacks, where the morphisms d, c are jointly monomorphic, yield equivalence relations on the object C_0. In particular, we can view the kernel pair $(\mathrm{Eq}(f), p_1, p_2)$ of any morphism f in $\mathscr{C}$ as an internal groupoid in $\mathscr{C}$ (recall Example 1.1.6).

Remark 1.4.8. We will recall in Definition 4.1.4 the notion of multiplicative graph in a category, which is an intermediate notion between those of reflexive graph and internal category. Note that there is also the notion of star-multiplicative graph [Jan03], which is an intermediate notion between those of reflexive graph and multiplicative graph.

Next, we want to recall the notions of higher reflexive graph, internal category and groupoid.

Definition 1.4.9 (N-fold reflexive graph)**.** Let $\mathscr{C}$ be a category. We define the category $\mathrm{RG}^n(\mathscr{C})$ of *n-fold reflexive graphs* in $\mathscr{C}$, where $n \in \mathbb{N}$, recursively as follows. We set $\mathrm{RG}^1(\mathscr{C}) := \mathrm{RG}(\mathscr{C})$ and $\mathrm{RG}^n(\mathscr{C}) := \mathrm{RG}(\mathrm{RG}^{n-1}(\mathscr{C}))$ for $2 \leq n \in \mathbb{N}$.

The category $\mathrm{RG}^2(\mathscr{C})$ is isomorphic to the category whose objects are given

by diagrams

$$\begin{array}{ccc} C_{11} & \overset{d_{\downarrow 1}}{\underset{c_{\downarrow 1}}{\overset{\longleftarrow e_{\uparrow 1} \longrightarrow}{\longrightarrow}}} & C_{01} \\ {\scriptstyle d_{1\downarrow}}\downarrow \uparrow \downarrow {\scriptstyle e_{1\uparrow}} \quad {\scriptstyle c_{1\downarrow}} & & {\scriptstyle d_{0\downarrow}}\downarrow \uparrow \downarrow {\scriptstyle e_{0\uparrow}} \quad {\scriptstyle c_{0\downarrow}} \\ C_{10} & \underset{c_{\downarrow 0}}{\overset{\longleftarrow e_{\uparrow 0} \longrightarrow}{\longrightarrow}}{\scriptstyle d_{\downarrow 0}} & C_{00} \end{array} \tag{1.14}$$

in $\mathscr{C}$ such that all four sides of the diagram yield reflexive graphs and the following equations are satisfied:

$$\begin{aligned} &d_{0\downarrow}d_{\downarrow 1} = d_{\downarrow 0}d_{1\downarrow}, && d_{0\downarrow}c_{\downarrow 1} = c_{\downarrow 0}d_{1\downarrow}, && d_{1\downarrow}e_{\uparrow 1} = e_{\uparrow 0}d_{0\downarrow}, \\ &c_{0\downarrow}d_{\downarrow 1} = d_{\downarrow 0}c_{1\downarrow}, && c_{0\downarrow}c_{\downarrow 1} = c_{\downarrow 0}c_{1\downarrow}, && c_{1\downarrow}e_{\uparrow 1} = e_{\uparrow 0}c_{0\downarrow}, \\ &d_{\downarrow 1}e_{1\uparrow} = e_{0\uparrow}d_{\downarrow 0}, && c_{\downarrow 1}e_{1\uparrow} = e_{0\uparrow}c_{\downarrow 0}, && e_{\uparrow 1}e_{0\uparrow} = e_{1\uparrow}e_{\uparrow 0}. \end{aligned}$$

One can view $C_{0\bullet} = (C_{00}, C_{01}, d_{0\downarrow}, c_{0\downarrow}, e_{0\uparrow})$ as the reflexive graph of objects of the double reflexive graph, $C_{1\bullet} = (C_{10}, C_{11}, d_{1\downarrow}, c_{1\downarrow}, e_{1\uparrow})$ as the reflexive graph of arrows, $(d_{\downarrow 0}, d_{\downarrow 1})$ as the domain morphism of reflexive graphs, $(c_{\downarrow 0}, c_{\downarrow 1})$ as the codomain morphism of reflexive graphs, and $(e_{\uparrow 0}, e_{\uparrow 1})$ as the identities morphism of reflexive graphs. In the following, we will refer to these two categories interchangeably.

More generally, the category $\mathrm{RG}^n(\mathscr{C})$ is isomorphic to the category whose objects are given by families

$$\{C_{i_1\cdots i_n}\}_{i_1,\ldots,i_n\in\{0,1\}}$$

of objects in $\mathscr{C}$ together with morphisms

$$\begin{aligned} d_{i_1\cdots i_k\downarrow i_{k+2}\cdots i_n} &: C_{i_1\cdots i_k 1 i_{k+2}\cdots i_n} \to C_{i_1\cdots i_k 0 i_{k+2}\cdots i_n}, \\ c_{i_1\cdots i_k\downarrow i_{k+2}\cdots i_n} &: C_{i_1\cdots i_k 1 i_{k+2}\cdots i_n} \to C_{i_1\cdots i_k 0 i_{k+2}\cdots i_n}, \\ e_{i_1\cdots i_k\uparrow i_{k+2}\cdots i_n} &: C_{i_1\cdots i_k 0 i_{k+2}\cdots i_n} \to C_{i_1\cdots i_k 1 i_{k+2}\cdots i_n} \end{aligned}$$

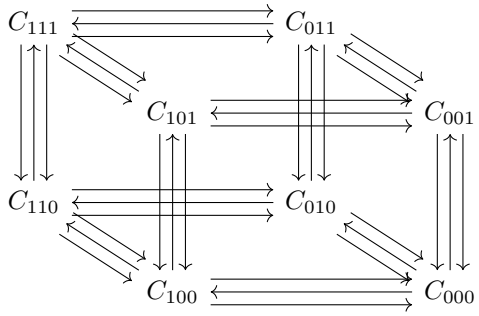
in $\mathscr{C}$, where $k \in \{0, \ldots, n-1\}$ and $i_1, \ldots, i_k, i_{k+2}, \ldots, i_n \in \{0, 1\}$, such that the diagrams

$$\begin{array}{ccc} C_{i_1\cdots i_k 1 j_1\cdots j_\ell 1 \nu_1\cdots\nu_{n-k-l-2}} & \rightrightarrows\!\!\leftrightarrows & C_{i_1\cdots i_k 0 j_1\cdots j_\ell 1 \nu_1\cdots\nu_{n-k-l-2}} \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ C_{i_1\cdots i_k 1 j_1\cdots j_\ell 0 \nu_1\cdots\nu_{n-k-l-2}} & \rightrightarrows\!\!\leftrightarrows & C_{i_1\cdots i_k 0 j_1\cdots j_\ell 0 \nu_1\cdots\nu_{n-k-l-2}} \end{array}$$

with the corresponding morphisms yield double reflexive graphs, where $k, \ell \in \mathbb{N}_0$, such that $k + \ell \leq n - 2$, and $i_1, \ldots, i_k, j_1, \ldots, j_\ell, \nu_1, \ldots, \nu_{n-k-\ell-2} \in \{0, 1\}$. In the following, we will refer to these two categories interchangeably.

Example 1.4.10 (3-fold reflexive graph)**.** We depict an object in $\mathrm{RG}^3(\mathscr{C})$ as

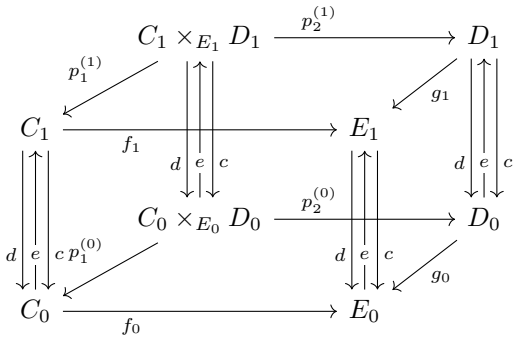
a diagram in $\mathscr{C}$ of the following form:



We view $C_{0\bullet\bullet}$ as the double reflexive graph of objects of the triple reflexive graph, and $C_{1\bullet\bullet}$ as the double reflexive graph of arrows. The double reflexive graph $C_{0\bullet\bullet}$ has itself $C_{00\bullet}$ as reflexive graph of objects, and $C_{01\bullet}$ as reflexive graph of arrows, and analogously for the double reflexive graph $C_{1\bullet\bullet}$.

Proposition 1.4.11. For $\mathscr{C}$ a category with pullbacks, the categories $\mathrm{Cat}(\mathscr{C})$ and $\mathrm{Grpd}(\mathscr{C})$ of internal categories and of internal groupoids in $\mathscr{C}$, respectively, also have pullbacks, which are computed componentwise.

Given internal functors $F = (f_0, f_1)$ and $G = (g_0, g_1)$ from internal categories $\mathbb{C}$ and $\mathbb{D}$, respectively, to an internal category $\mathbb{E}$, their pullback can be constructed as in the following diagram:



Namely, we construct the pullback of g_0 along f_0, and the pullback of g_1 along f_1. The reflexive graph structures on $\mathbb{C}, \mathbb{D}, \mathbb{E}$ induce maps d, c, e so that $(C_0 \times_{E_0} D_0, C_1 \times_{E_1} D_1, d, c, e)$ is a reflexive graph. We put $P_0 := C_0 \times_{E_0} D_0$ and $P_1 := C_1 \times_{E_1} D_1$. Then the multiplication morphisms m of $\mathbb{C}, \mathbb{D}, \mathbb{E}$ induce a multiplication morphism $m : P_1 \times_{P_0} P_1 \to C_1 \times_{E_1} D_1$. Note that $P_1 \times_{P_0} P_1$ is (isomorphic to) $(C_1 \times_{C_0} C_1) \times_{(E_1 \times_{E_0} E_1)} (D_1 \times_{D_0} D_1)$.

Definition 1.4.12 (Internal n-fold category, internal n-fold groupoid)**.** Let $\mathscr{C}$ be a category with pullbacks.

1. We define the category $\mathrm{Cat}^n(\mathscr{C})$ of *internal n-fold categories* in $\mathscr{C}$ recursively as follows. We set $\mathrm{Cat}^1(\mathscr{C}) := \mathrm{Cat}(\mathscr{C})$ and, for $2 \leq n \in \mathbb{N}$, $\mathrm{Cat}^n(\mathscr{C}) := \mathrm{Cat}(\mathrm{Cat}^{n-1}(\mathscr{C}))$.

2. We define the category $\mathrm{Grpd}^n(\mathscr{C})$ of *internal n-fold groupoids* in $\mathscr{C}$ recursively as follows. We set $\mathrm{Grpd}^1(\mathscr{C}) := \mathrm{Grpd}(\mathscr{C})$ and, for $2 \leq n \in \mathbb{N}$, $\mathrm{Grpd}^n(\mathscr{C}) := \mathrm{Grpd}(\mathrm{Grpd}^{n-1}(\mathscr{C}))$.

For $\mathscr{C} = \mathrm{Set}$, $\mathrm{Cat}^n(\mathscr{C})$ and $\mathrm{Grpd}^n(\mathscr{C})$ yield the usual definitions of the categories of small n-fold categories and of small n-fold groupoids, respectively.

As $\mathrm{Grpd}(\mathscr{C})$ is a full subcategory of $\mathrm{Cat}(\mathscr{C})$, we can also view $\mathrm{Grpd}^n(\mathscr{C})$ as a full subcategory of $\mathrm{Cat}^n(\mathscr{C})$. Furthermore, any internal n-fold category in $\mathscr{C}$ has an underlying n-fold reflexive graph in $\mathscr{C}$.

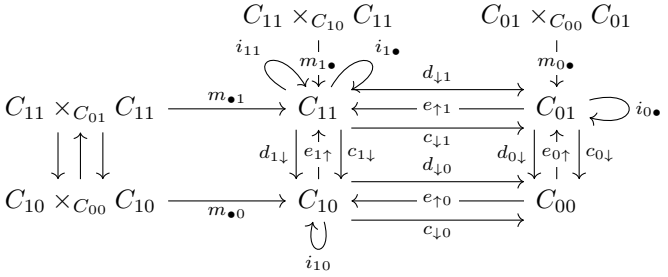
Example 1.4.13 (Internal double groupoid)**.** Let $\mathscr{C}$ be a category with pullbacks. An internal double groupoid in $\mathscr{C}$ has an internal groupoid

$$C_{0\bullet} = (C_{00}, C_{01}, d_{0\downarrow}, c_{0\downarrow}, e_{0\uparrow}, m_{0\bullet}, i_{0\bullet})$$

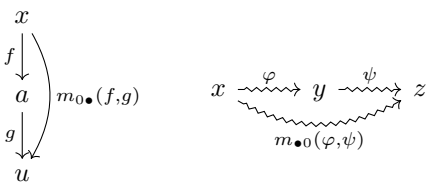
of objects, an internal groupoid

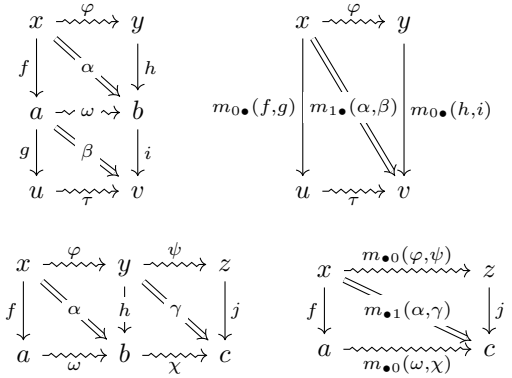
$$C_{1\bullet} = (C_{10}, C_{11}, d_{1\downarrow}, c_{1\downarrow}, e_{1\uparrow}, m_{1\bullet}, i_{1\bullet})$$

of arrows, a domain internal functor $(d_{\downarrow 0}, d_{\downarrow 1})$, a codomain internal functor $(c_{\downarrow 0}, c_{\downarrow 1})$, an identities internal functor $(e_{\uparrow 0}, e_{\uparrow 1})$, a multiplication internal functor $(m_{\bullet 0}, m_{\bullet 1})$ and an inverses internal functor (i_{10}, i_{11}) as depicted below satisfying the corresponding conditions.



An internal double groupoid in Set, i.e. a small double groupoid, has a set C_{00} of *objects*, a set C_{01} of *vertical arrows*, a set C_{10} of *horizontal arrows*, and a set C_{11} of *squares* such that vertical arrows can be composed, horizontal morphisms can be composed, and squares can be composed both 'vertically' and 'horizontally' as shown in the following diagrams, where $a, b, c, x, y, z, u, v \in C_{00}$ are objects, $f, g, h, i, j \in C_{01}$ are vertical morphims, $\varphi, \psi, \omega, \chi, \tau \in C_{10}$ are horizontal morphisms, and $\alpha, \beta, \gamma \in C_{11}$ are squares:





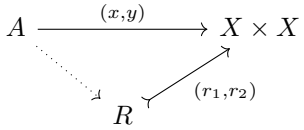
1.4.2 Centrality of equivalence relations

In this section, we want to recall the concept of centrality of equivalence relations. The notion of commutator of equivalence relations was introduced by J.D.H. Smith in 1976 in the context of Mal'tsev varieties [Smi76], and extended to congruence modular varieties by J. Hagemann and C. Herrmann in 1979 [HH79], see also [Gum83, FM87]. M.C. Pedicchio developed a categorical approach to commutator theory in the context of exact Mal'tsev categories with coequalizers using the notion of internal pregroupoid in the sense of A. Kock [Koc82] in 1995 [Ped95], see also [Ped96]. In 2001 [JP01], M.C. Pedicchio and G. Janelidze introduced the notion of internal pseudogroupoid in general categories motivated by [JP97]. D. Bourn and M. Gran introduced the notion of connector in 2002 [BG02c, BG02b] expressing centrality of equivalence relations in its simplest form. Commutators of equivalence relations in the finitely cocomplete regular Mal'tsev context are studied in [Bou04].

In the following, $\mathscr{C}$ is a finitely complete category, and $(r_1, r_2) : R \to X \times X$ and $(s_1, s_2) : S \to X \times X$ are two equivalence relations on an object X in $\mathscr{C}$. Let us consider the pullback of s_1 along r_2 as in the following diagram:

$$\begin{array}{ccc} R \times_X S & \xrightarrow{p_2} & S \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle s_1} \\ R & \xrightarrow[r_2]{} & X \end{array} \tag{1.15}$$

We have seen at the beginning of Section 1.1.1 that the internal relation R on X yields a set-theoretic relation R_A on the hom-set $\mathrm{Hom}(A, X)$ for any object A in $\mathscr{C}$. Namely, a pair (x, y) of parallel morphisms from A to X is in $R_A \subseteq \mathrm{Hom}(A, X) \times \mathrm{Hom}(A, X)$, in which case we write xR_Ay, if and only if the morphism $(x, y) : A \to X \times X$ induced by the universal property of the product factors through $(r_1, r_2) : R \to X \times X$.



In this case, we also write xR_Ay instead of (x, y). We denote by $r_1^A, r_2^A : R_A \to \mathrm{Hom}(A, X)$ the functions mapping an element xR_Ay to x, y, respectively. Let us consider the pullback of s_1^A along r_2^A as in the following diagram:

$$\begin{array}{ccc} R_A \times_{\mathrm{Hom}(A,X)} S_A & \xrightarrow{p_2^A} & S_A \\ {\scriptstyle p_1^A}\downarrow & & \downarrow{\scriptstyle s_1^A} \\ R_A & \xrightarrow[r_2^A]{} & \mathrm{Hom}(A, X) \end{array}$$

The set $R_A \times_{\mathrm{Hom}(A,X)} S_A$ is given by triples (x, y, z) of parallel morphisms from $A \to X$ such that xR_Ay and yS_Az, which we also denote by xR_AyS_Az. Note that $R_A \times_{\mathrm{Hom}(A,X)} S_A$ is in bijection with $\mathrm{Hom}(A, R \times_X S)$ due to the universal property of the pullback $R \times_X S$. In the following, we will leave this bijection implicit. Given a morphism $p : R \times_X S \to X$, we denote by $p_A : R_A \times_{\mathrm{Hom}(A,X)} S_A \to \mathrm{Hom}(A, X)$ the function that maps an element xR_AyS_Az, viewed as a morphism from A to $R \times_X S$, to the composite $p(xR_AyS_Az)$.

Definition 1.4.14 (Connector [BG02c])**.** Let $\mathscr{C}$ be a finitely complete category, and $(r_1, r_2) : R \to X \times X$ and $(s_1, s_2) : S \to X \times X$ be two equivalence relations on an object X in $\mathscr{C}$. Let us consider the pullback of s_1 along r_2 as in Diagram (1.15). A *connector* between R and S is a morphism $p : R \times_X S \to X$ satisfying the following conditions:

1. For any object A in $\mathscr{C}$ and morphisms $x, y, z : A \to X$ such that xR_AyS_Az, it holds that $xS_Ap_A(xR_AyS_Az)$ and $p_A(xR_AyS_Az)R_Az$.

$$\begin{array}{ccc} x & \xrightarrow{S_A} & p_A(xR_AyS_Az) \\ {\scriptstyle R_A}\downarrow & & \downarrow{\scriptstyle R_A} \\ y & \xrightarrow[S_A]{} & z \end{array}$$

2. For any object A in $\mathscr{C}$ and morphisms $x, x', y, y' : A \to X$ such that xS_Ay and $x'R_Ay'$, it holds that $p_A(xR_AxS_Ay) = y$ and $p_A(x'R_Ay'S_Ay') = x'$. One also says that p is a partial Mal'tsev operation.

3. For any object A in $\mathscr{C}$ and morphisms x, y, y', z, z', w such that xR_Ay, xR_Ay', yS_Az, $y'R_Az'$, zS_Aw and $z'S_Aw$, it holds that

$$p_A(p_A(xR_AyS_Az)R_AzS_Aw) = p_A(xR_AyS_Aw),$$
$$p_A(xR_Ay'S_Ap_A(y'R_Az'S_Aw)) = p_A(xR_Az'S_Aw).$$

   One also says that p is left and right associative.

If there exists a connector between R and S, one says that R and S *centralize* each other.

Proposition 1.4.15. Let $\mathscr{C}$ be a Mal'tsev category, and $(r_1, r_2) : R \to X \times X$ and $(s_1, s_2) : S \to X \times X$ be two equivalence relations on an object X in $\mathscr{C}$. Let us consider the pullback of s_1 along r_2 as in Diagram (1.15). Then there exists at most one partial Mal'tsev operation $p : R \times_X S \to X$, that is necessarily a connector between R and S.

In the following, $\mathscr{C}$ will be an exact Mal'tsev category with coequalizers, and R, S be two equivalence relations on an object X in $\mathscr{C}$. The following diagram shows the smallest *double equivalence relation* Δ_R^S on R, S as constructed in [Ped95]:

$$\begin{array}{ccccc} & & \xrightarrow{\pi_1} & & \\ \Delta_R^S & \xleftarrow{\delta_\pi} & S & \xrightarrow{c} & S/R \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle \delta_p} & {\scriptstyle p_2}\downarrow & \xrightarrow{\pi_2} \downarrow{\scriptstyle s_1} & \uparrow{\scriptstyle \delta_S}\downarrow{\scriptstyle s_2} & {\scriptstyle \overline{s_1}}\downarrow\uparrow{\scriptstyle \overline{\delta_S}}\downarrow{\scriptstyle \overline{s_2}} \\ & & \xrightarrow{r_1} & & \\ R & \xleftarrow{\delta_R} & X & \xrightarrow[q]{} & X/R \\ & & \xrightarrow[r_2]{} & & \end{array} \tag{1.16}$$

Here $(X/R, q)$ is the coequalizer of r_1, r_2, and $(S/R, c)$ is the coequalizer of $\delta_S r_1, \delta_S r_2$. Furthermore, $\overline{\delta_S}$ is the unique morphism such that $\overline{\delta_S} q = c\delta_S$, and $\overline{s_1}, \overline{s_2}$ are the unique morphisms such that $\overline{s_1} c = q s_1$ and $\overline{s_2} c = q s_2$. The kernel pair of c yields $(\Delta_R^S, \pi_1, \pi_2, \delta_\pi)$. The morphisms between S and X, and between S/R and X/R induce the morphisms p_1, p_2, δ_p between Δ_R^S and R, that yield a reflexive relation and hence an equivalence relation on R. Furthermore, the left-hand side of Diagram (1.16) is a double reflexive graph. The following result can be found in [BG02c, 4.2. Proposition] relying on [Ped96, Proposition 1.8].

Proposition 1.4.16. Let $\mathscr{C}$ be an exact Mal'tsev category with coequalizers, and R, S be two equivalence relations on an object X in $\mathscr{C}$. Then the following conditions are equivalent:

1. The equivalence relations R and S centralize each other.
2. The double equivalence relation Δ_R^S as in Diagram (1.16) is *centralizing*, meaning that the square given by the morphisms p_1, r_1, π_1, s_1 is a pullback.

Let $\mathbb{V}$ be a variety of universal algebras, and $(r_1, r_2) : R \to A \times A$ and $(s_1, s_2) : S \to A \times A$ be two congruences on algebra A in $\mathbb{V}$. Then the pullback of s_1 along r_2 as in Diagram (1.15) is given by $R \times_X S = \{xRySz \in A \times A \times A\}$, $p_1(xRySz) = xRy$ and $p_2(xRySz) = ySz$.

Proposition 1.4.17. Let $\mathbb{V}$ be a Mal'tsev variety with a Mal'tsev term p, and R, S be two congruences on an algebra A in $\mathbb{V}$. Then the following conditions are equivalent:

1. The congruences R and S centralize each other.
2. The Mal'tsev operation $p^A : A \times A \times A \to A$ restricts to a homomorphism from $R \times_X S = \{xRySz \in A \times A \times A\}$ to A.

1.4.3 Internal structures in algebraic categories

We are now ready to recall some properties of internal categorical structures in Mal'tsev and related categories.

Proposition 1.4.18. [CPP92, Jan90a] Let $\mathscr{C}$ be a finitely complete category and $\mathbb{C} = (C_0, C_1, d, c, e)$ be a reflexive graph in $\mathscr{C}$. Then there is a bijection between the connectors between the kernel pairs $\mathrm{Eq}(d), \mathrm{Eq}(c)$ and the internal groupoid structures on $\mathbb{C}$.

Proposition 1.4.19. [CPP92] Let $\mathscr{C}$ be a Mal'tsev category.

1. Any internal category in $\mathscr{C}$ is an internal groupoid, meaning that the categories $\mathrm{Cat}(\mathscr{C})$ and $\mathrm{Grpd}(\mathscr{C})$ are isomorphic.
2. Any reflexive graph (C_0, C_1, d, c, e) in $\mathscr{C}$ admits at most one internal groupoid structure. This is exactly the case when the kernel pairs $\mathrm{Eq}(d)$ and $\mathrm{Eq}(c)$ centralize each other.
3. $\mathrm{Grpd}(\mathscr{C})$ is a full subcategory of $\mathrm{RG}(\mathscr{C})$, i.e., given a morphism of reflexive graphs between two internal groupoids in $\mathscr{C}$, it is automatically an internal functor.

Proposition 1.4.20. [Bou96] Let $\mathscr{C}$ be a finitely complete category. Then the following conditions are equivalent:

1. $\mathscr{C}$ is a Mal'tsev category.
2. $\mathrm{Grpd}(\mathscr{C})$ is closed under subobjects in $\mathrm{RG}(\mathscr{C})$, i.e., if $(f_0, f_1) : \mathbb{C} \to \mathbb{D}$ is a monomorphism of reflexive graphs, i.e., f_0, f_1 are monomorphisms in $\mathscr{C}$, and $\mathbb{D}$ is an internal groupoid, then $\mathbb{C}$ admits also an internal groupoid structure.

Proposition 1.4.21. [Gra99] Let $\mathscr{C}$ be a regular Mal'tsev category.

1. An internal functor (f_0, f_1) is a regular epimorphism in $\mathrm{Grpd}(\mathscr{C})$ if and only if it is a regular epimorphism in $\mathrm{RG}(\mathscr{C})$, i.e., if and only if f_0, f_1 are regular epimorphisms in $\mathscr{C}$.
2. $\mathrm{Grpd}(\mathscr{C})$ is closed under regular quotients in $\mathrm{RG}(\mathscr{C})$, meaning that, if $(f_0, f_1) : \mathbb{C} \to \mathbb{D}$ is a morphism of reflexive graphs, f_0, f_1 are regular epimorphisms and $\mathbb{C}$ is an internal groupoid, then $\mathbb{D}$ is also an internal groupoid.
3. $\mathrm{Grpd}(\mathscr{C})$ is a regular Mal'tsev category.
4. If $\mathscr{C}$ is in addition exact, then $\mathrm{Grpd}(\mathscr{C})$ is also exact.

Proposition 1.4.22. [Ped95, Bou04] Let $\mathscr{C}$ be a regular Mal'tsev category with finite colimits. Then $\mathrm{Grpd}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{RG}(\mathscr{C})$.

Given a reflexive graph (C_0, C_1, d, c, e) in $\mathscr{C}$, its reflection into $\mathrm{Grpd}(\mathscr{C})$ is given by

$$\begin{array}{ccc} C_1 & \underset{c}{\overset{d}{\rightleftarrows}\, e \,\rightarrow} & C_0 \\ {\scriptstyle q}\downarrow & & \| \\ C_1/[\mathrm{Eq}(d), \mathrm{Eq}(c)] & \underset{\gamma}{\overset{\delta}{\leftarrow\!\varepsilon\!\rightarrow}} & C_0, \end{array}$$

where $q : C_1 \to C_1/[\mathrm{Eq}(d), \mathrm{Eq}(c)]$ is the quotient of C_1 by the commutator equivalence relation $[\mathrm{Eq}(d), \mathrm{Eq}(c)]$, see [Bou04] for more details.

Proposition 1.4.23. [BG02a] Let $\mathscr{C}$ be a semi-abelian category. Then $\mathrm{Grpd}(\mathscr{C})$ is also semi-abelian.

Recall that Mal'tsev's Theorem 1.1.32 states that a variety $\mathbb{V}$ of universal algebras is a Mal'tsev category if and only if its equational theory has a Mal'tsev term, i.e., a ternary term $p(x,y,z) \in \mathsf{T}(x,y,z)$ satisfying the identities $p(x,x,y) \approx y$ and $p(x,y,y) \approx y$. Given an algebra A in $\mathbb{V}$, $p^A : A \times A \times A \to A$ is in general only a function and not a homomorphism. Given a category $\mathscr{C}$ with binary products and an object X in $\mathscr{C}$, a *Mal'tsev operation* on X is a morphism $p_X : X \times X \times X \to X$ such that, for any object A in $\mathscr{C}$ and morphisms $x, y : A \to X$, $p_X(x,x,y) = y$ and $p_X(x,y,y,) = x$, where $(x,x,y),(x,y,y) : A \to X \times X \times X$ are induced by the universal property of the product. The following notion is due to P.T. Johnstone [Joh89].

Definition 1.4.24 (Naturally Mal'tsev category)**.** Let $\mathscr{C}$ be a category with binary products. Then $\mathscr{C}$ is a *naturally Mal'tsev category* if it has a natural transformation p from the functor $(-)\times(-)\times(-) : \mathscr{C} \to \mathscr{C}$ that maps an object X to $X \times X \times X$ to the identity functor $1_{\mathscr{C}}$ on $\mathscr{C}$ such that $p_X : X \times X \times X \to X$ is a Mal'tsev operation for any object X in $\mathscr{C}$.

Proposition 1.4.25. [BB04, Corollary 2.4.3] Let $\mathscr{C}$ be a finitely complete, naturally Mal'tsev category. Then $\mathscr{C}$ is also a Mal'tsev category.

Proposition 1.4.26. [Joh89] Let $\mathscr{C}$ be a finitely complete category. Then the following conditions are equivalent:

1. $\mathscr{C}$ is a naturally Mal'tsev category.
2. $\mathscr{C}$ satisfies the so-called Lawvere condition, i.e., the forgetful functor $\mathrm{Grpd}(\mathscr{C}) \to \mathrm{RG}(\mathscr{C})$ yields an equivalence of categories.

Proposition 1.4.27. [BB04, Corollary 2.7.6] Let $\mathscr{C}$ be a Mal'tsev category. Then the following conditions are equivalent:

1. $\mathscr{C}$ is a naturally Mal'tsev category.
2. Any two equivalence relations on any object X in $\mathscr{C}$ centralize each other.

Proposition 1.4.28. [Joh89] Let $\mathscr{C}$ be a category with binary products. Then the following conditions are equivalent:

1. $\mathscr{C}$ is a pointed naturally Mal'tsev category.
2. $\mathscr{C}$ is an additive category.

Remark 1.4.29 (Essentially affine categories)**.** There is a slightly stronger notion than that of naturally Mal'tsev category given by that of essentially affine category due to D. Bourn [Bou91]. An *essentially affine category* is a category that has pullbacks of split epimorphisms and pushouts of split monomorphisms, and, given a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle p'}\downarrow\uparrow{\scriptstyle s'} \\ B & \xrightarrow[f']{} & B' \end{array}$$

in $\mathscr{C}$ such that $ps = 1_B$, $p's' = 1_{B'}$, $f'p = p'f$ and $fs = s'f'$, the downward directed commutative square is a pullback if and only if the upward directed commutative square is a pushout. Equivalently, the change of base functor of the fibration of points $(f')^* : \mathrm{Pt}_{B'}(\mathscr{C}) \to \mathrm{Pt}_B(\mathscr{C})$ yields an equivalence of categories for any morphism $f' : B \to B'$ in $\mathscr{C}$. The notion of essentially affineness is clearly self-dual and implies protomodularity. Any finitely complete essentially affine category is also a naturally Malt'sev category. A category with finite limits and pushouts of split monomorphisms is additive if and only if it is essentially affine and pointed, see e.g. [BB04, Proposition 7.2.7].

For certain classes of categories $\mathscr{C}$, internal groupoids allow for interesting equivalent descriptions. The following result goes back to the work [Lod82] of J.-L. Loday from 1982, where the notion of n-cat-group for $n \in \mathbb{N}$ is defined, see also [Ell84, Jan90a, CPP92].

For a category $\mathscr{C}$ with equalizers, the category $\mathrm{RG}(\mathscr{C})$ of reflexive graphs in $\mathscr{C}$ is equivalent to the category whose objects are given by objects A in $\mathscr{C}$ equipped with two endomorphisms s, t on A such that $st = t$ and $ts = s$, with expected morphisms. Given a reflexive graph (C_0, C_1, d, c, e), the triple (C_1, ed, ec) satisfies the former conditions. Given a triple (A, s, t) as before, the quintuple (A_0, A, d, c, e) is a reflexive graph in $\mathscr{C}$, where (A_0, e) is the equalizer of the parallel pair of morphisms $s, 1_A : A \to A$, $d : A \to A_0$ is the unique morphism such that $ed = s$, and $c : A \to A_0$ is the unique morphism such that $ec = t$ induced by the universal property of the equalizer.

More generally, the category $\mathrm{RG}^n(\mathscr{C})$ of n-fold reflexive graphs in $\mathscr{C}$ is equivalent to the category whose objects are given by objects A in $\mathscr{C}$ equipped with $2n$ endomorphisms s_i, t_i on A, where $i \in \{1, \dots, n\}$, such that $s_i t_i = t_i$, $t_i s_i = s_i$ and $\alpha_i \beta_j = \beta_j \alpha_i$, where $i \neq j \in \{1, \dots, n\}$ and $\alpha, \beta \in \{s, t\}$, with expected morphisms.

For example, given a double reflexive graph as in Diagram (1.14), the quintuple

$$(C_{11}, e_{\uparrow 1} d_{\downarrow 1}, e_{\uparrow 1} c_{\downarrow 1}, d_{1\downarrow} e_{1\uparrow}, c_{1\downarrow} e_{1\uparrow})$$

satisfies the former conditions.

Using Proposition 1.4.19, one can show the following.

Proposition 1.4.30. Let $\mathscr{C}$ be a Mal'tsev category and $n \in \mathbb{N}$. Then the category $\mathrm{Grpd}^n(\mathscr{C})$ of internal n-fold groupoids in $\mathscr{C}$ is equivalent to the category whose objects are given by objects A in $\mathscr{C}$ equipped with $2n$ endomorphisms s_i, t_i on A, where $i \in \{1, \dots, n\}$, such that $s_i t_i = t_i$, $t_i s_i = s_i$, the kernel pairs $\mathrm{Eq}(s_i), \mathrm{Eq}(t_i)$ centralize each other, and $\alpha_i \beta_j = \beta_j \alpha_i$, where $i \neq j \in \{1, \dots, n\}$ and $\alpha, \beta \in \{s, t\}$, with expected morphisms.

Proposition 1.4.31. Let $\mathbb{V}$ be a Mal'tsev variety and $n \in \mathbb{N}$. Then $\mathrm{Grpd}^n(\mathbb{V})$ is equivalent to a Mal'tsev variety.

Proof. We have seen in Proposition 1.4.22 that $\mathrm{Grpd}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{RG}(\mathscr{C})$ when $\mathscr{C}$ is a regular Mal'tsev category with finite colimits, in particular, when $\mathscr{C}$ is a Mal'tsev variety. The category $\mathrm{RG}(\mathbb{V})$ is equivalent to the variety $\mathbb{W}$ of universal algebras whose equational theory is given by the equational theory of $\mathbb{V}$ together with two additional unary operations

s, t, and the identities encoding that s, t yield homomorphisms in $\mathbb{V}$, and that $st = t$, $ts = s$. Since the Birkhoff subcategories of a variety of universal algebra are exactly its subvarieties (Proposition 1.3.18), this shows that $\mathrm{Grpd}(\mathbb{V})$ is (equivalent to) a Mal'tsev variety. The general case follows by induction. □

Recall that a variety of universal algebras is called *congruence modular* if, for any algebra A in $\mathbb{V}$, the lattice $\mathrm{Cong}(A)$ of congruences on A is modular, i.e., given congruences $R, S, T \in \mathrm{Cong}(A)$ such that $R \leq T$, then

$$R \wedge (S \vee T) = (R \wedge S) \vee T.$$

It is shown in [GR04, 2.4. Corollary] that a congruence modular variety is a Mal'tsev variety if and only if $\mathrm{Grpd}(\mathbb{V})$ is a subvariety of $\mathrm{RG}(\mathbb{V})$.

Example 1.4.32 (Cat-1-groups)**.** Let A be a group equipped with two group endomorphisms s, t on A such that $st = t$ and $ts = s$. The kernel pairs $\mathrm{Eq}(s), \mathrm{Eq}(t)$ centralize each other if and only if the commutator subgroup $[\mathrm{Ker}(s), \mathrm{Ker}(t)]$ is trivial, where $\mathrm{Ker}(s), \mathrm{Ker}(t)$ are the classical kernels of s, t, respectively. More abstractly, this follows from the fact that the category of groups is semi-abelian and satisfies the "Smith is Huq"-property. Note that $\mathrm{Ker}(s)$ is given by the subgroup $\{as(a)^{-1} \mid a \in A\}$ of A, and analogously for $\mathrm{Ker}(t)$. Hence the condition $[\mathrm{Ker}(s), \mathrm{Ker}(t)] = \{1\}$ is equivalent to the condition that, for all $a, b \in A$, $as(a)^{-1}bt(b)^{-1} = bt(b)^{-1}as(a)^{-1}$.

Before considering internal groupoids in the additive context, let us recall the following notations.

Definition 1.4.33 (Arrow category)**.** Let $\mathscr{C}$ be a category.

1. The *arrow category* $\mathrm{Arr}(\mathscr{C})$ of $\mathscr{C}$ is the category whose objects are given by morphisms in $\mathscr{C}$, and whose morphisms from a morphism $a : A_1 \to A_0$ to a morphism $b : B_1 \to B_0$ are given by all pairs $(f_1 : A_1 \to B_1, f_0 : A_0 \to B_0)$ of morphisms in $\mathscr{C}$ such that $bf_1 = f_0a$.

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

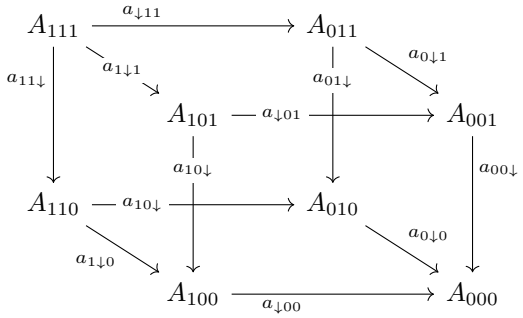
2. Let $n \in \mathbb{N}$. The *n-fold arrow category* $\mathrm{Arr}^n(\mathscr{C})$ of $\mathscr{C}$ is defined recursively by $\mathrm{Arr}^1(\mathscr{C}) := \mathrm{Arr}(\mathscr{C})$ and $\mathrm{Arr}^n(\mathscr{C}) := \mathrm{Arr}(\mathrm{Arr}^{n-1}(\mathscr{C}))$ for $2 \leq n \in \mathbb{N}$.

The double arrow category $\mathrm{Arr}^2(\mathscr{C})$ of $\mathscr{C}$ can be seen as the category with objects commutative squares

$$\begin{array}{ccc} A_{11} & \xrightarrow{a_{\downarrow 1}} & A_{01} \\ {\scriptstyle a_{1\downarrow}}\downarrow & & \downarrow{\scriptstyle a_{0\downarrow}} \\ A_{10} & \xrightarrow[a_{\downarrow 0}]{} & A_{00} \end{array} \tag{1.17}$$

in $\mathscr{C}$ and expected morphisms.

The triple arrow category $\mathrm{Arr}^3(\mathscr{C})$ can be seen as the category with objects commutative cubes



in $\mathscr{C}$ and expected morphisms.

More generally, the n-fold arrow category $\mathrm{Arr}^n(\mathscr{C})$ can be seen as the category with objects collections $\{A_{i_1\cdots i_n}\}_{i_1,\ldots,i_n\in\{0,1\}}$ of objects in $\mathscr{C}$ together with morphims $a_{i_1\cdots i_k\downarrow i_{k+1}\cdots i_{n-1}}$, where $k \in \{0,\ldots,n-1\}$ and $i_1\ldots,i_{n-1} \in \{0,1\}$ such that, for all $k,\ell \in \mathbb{N}_0$ with $k+\ell \leq n-2$, and $i_1,\ldots,i_k$, $j_1,\ldots,j_\ell$, $\nu_1,\ldots,\nu_{n-k-\ell-2} \in \{0,1\}$, the square

$$\begin{array}{ccc} A_{i_1\cdots i_k 1 j_1\cdots j_\ell 1 \nu_1\cdots\nu_{n-k-\ell-2}} & \xrightarrow{a_{i_1\cdots i_k\downarrow j_1\cdots j_\ell 1\nu_1\cdots\nu_{n-k-\ell-2}}} & A_{i_1\cdots i_k 0 j_1\cdots j_\ell 1\nu_1\cdots\nu_{n-k-\ell-2}} \\ \downarrow{\scriptstyle a_{i_1\cdots i_k 1 j_1\cdots j_\ell\downarrow\nu_1\cdots\nu_{n-k-\ell-2}}} & & \downarrow{\scriptstyle a_{i_1\cdots i_k 0 j_1\cdots j_\ell\downarrow\nu_1\cdots\nu_{n-k-\ell-2}}} \\ A_{i_1\cdots i_k 1 j_1\cdots j_\ell 0 \nu_1\cdots\nu_{n-k-\ell-2}} & \xrightarrow[a_{i_1\cdots i_k\downarrow j_1\cdots j_\ell 0\nu_1\cdots\nu_{n-k-\ell-2}}]{} & A_{i_1\cdots i_k 0 j_1\cdots j_\ell 0\nu_1\cdots\nu_{n-k-\ell-2}} \end{array}$$

commutes. Equivalently, $\mathrm{Arr}^n(\mathscr{C})$ can also be described as the functor category $\mathscr{C}^{\mathcal{P}(n)^{\mathrm{op}}}$, where $\mathcal{P}(n)$ denotes the partial order (reflexive, transitive, antisymmetric) given by the power set of n viewed as a category. Here we consider the natural numbers by their standard (von Neumann) construction, i.e., we put $0 := \emptyset$ and $n := \{0,\ldots,n-1\}$ for $n \geq 1$. We will come back to this description in Section 3.4.

Proposition 1.4.34. Let $\mathscr{A}$ be an additive category with kernels and $n \in \mathbb{N}$. Then the categories $\mathrm{Grpd}^n(\mathscr{A})$, $\mathrm{RG}^n(\mathscr{A})$ and $\mathrm{Arr}^n(\mathscr{A})$ are equivalent.

Example 1.4.35. Recall from Propositions 1.4.26 and 1.4.28 that, for an additive category $\mathscr{A}$, the forgetful functor $\mathrm{Grpd}(\mathscr{A}) \to \mathrm{RG}(\mathscr{A})$ yields an equivalence of categories. In the following, we want to illustrate the equivalence of the categories $\mathrm{RG}^n(\mathscr{A})$ and $\mathrm{Arr}^n(\mathscr{A})$, where $\mathscr{A}$ is an additive category with kernels, for $n = 1$ and $n = 2$.

1. Given a reflexive graph (C_0, C_1, d, c, e) in $\mathscr{A}$, its corresponding object in $\mathrm{Arr}(\mathscr{A})$, also called its *normalization*, is given by the morphism

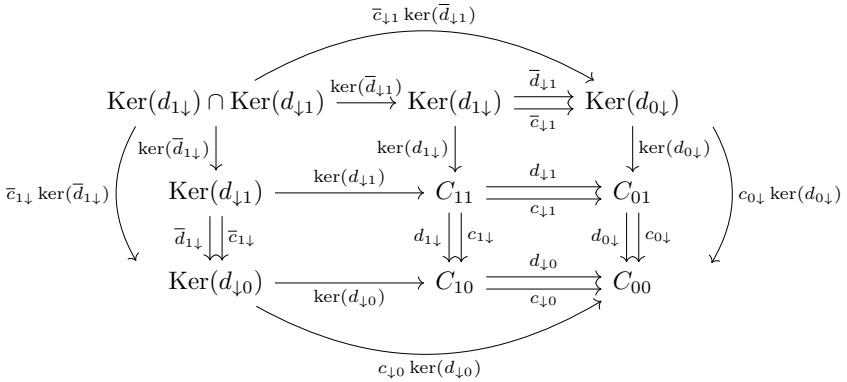
$$\mathrm{Ker}(d) \xrightarrow{c\,\ker(d)} C_0.$$

Conversely, given an arrow $a : A_1 \to A_0$ in $\mathscr{A}$, its corresponding reflexive graph, also called its *denormalization*, is given by the diagram

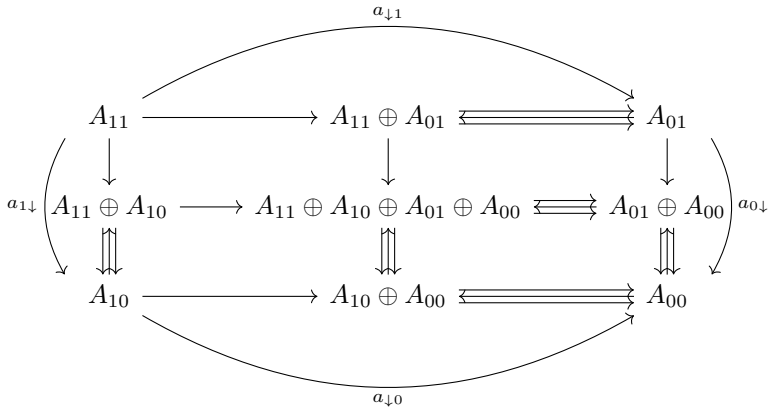
$$A_0 \oplus A_1 \underset{[1_{A_0},a]}{\overset{\pi_1}{\underset{\leftarrow \iota_1 \rightarrow}{\rightarrow}}} A_0,$$

where $A_0 \oplus A_1$ denotes the biproduct of A_0, A_1, π_1, ι_1 the corresponding product projection and coproduct injection, respectively, and $[1_{A_0}, a] : A_0 \oplus A_1 \to A_0$ is the corresponding morphism induced by the universal property of the coproduct.

2. Given a double reflexive graph in $\mathscr{A}$ as in Diagram (1.14), its corresponding commutative square is given by the outer part of the following diagram:



Note that the pullback $\mathrm{Ker}(d_{1\downarrow}) \cap \mathrm{Ker}(d_{\downarrow 1})$ of $\ker(d_{1\downarrow})$ along $\ker(d_{\downarrow 1})$ is indeed (isomorphic to) $\mathrm{Ker}(\overline{d}_{\downarrow 1})$ and $\mathrm{Ker}(\overline{d}_{1\downarrow})$. Conversely, given a commutative square in $\mathscr{A}$ as in Diagram (1.17), its corresponding double reflexive graph is indicated by the inner part of the following diagram:



Remark 1.4.36 (Crossed modules and crossed squares)**.**

1. In 1949 [Whi49], J.H.C. Whitehead introduced the notion of *crossed module of groups*, which is given by a group homomorphism $\gamma : X \to B$ together with an action $\triangleright$ of the group B on the group X satisfying equivariance and the so-called Peiffer condition, i.e., for all elements $x, x' \in X$ and $b \in B$, $\gamma(b \triangleright x) = b\gamma(x)b^{-1}$ and $\gamma(x) \triangleright x' = xx'x^{-1}$. It turns out that the category Grpd(Grp) of internal groupoids in the category Grp of groups is

equivalent to the category XMod(Grp) of crossed modules of groups, see [BS76] for the probably first published proof even though R. Brown and C.B. Spencer credit the result to J.-L. Verdier, and R. Lavendhomme might have known the result before, see [Jan03]. Given an internal groupoid in Grp with underlying reflexive graph (C_0, C_1, d, c, e), the corresponding crossed module of groups is given by the group homomorphism $c\ker(d) : \mathrm{Ker}(d) \to C_0$ together with the action of C_0 on $\mathrm{Ker}(d)$ by conjugation in C_1 via the group homomorphism e. Given a crossed module as above, the underlying reflexive graph of the corresponding internal groupoid is given by the diagram

$$B \ltimes X \underset{c}{\overset{\pi_1}{\underset{\leftarrow \iota_1 \rightarrow}{\rightleftarrows}}} B,$$

where $B \ltimes X$ is the semi-direct product of B, X relative to the action $\triangleright$ of B on X, π_1, ι_1 are the corresponding semi-direct product projection and injection, respectively, and $c(x, b) = \gamma(x)b$ for all $(x, b) \in B \rtimes X$. Similar notions of crossed modules were subsequently introduced for Lie algebras and rings [LR80], and commutative algebras [Por86], that also correspond to internal groupoids in the respective categories. In 2003 [Jan03], G. Janelidze introduced the notion of internal crossed module in any semi-abelian category $\mathscr{C}$, recovering the above notions of crossed modules, and showed that the equivalence between $\mathrm{Grpd}(\mathscr{C})$ and the category $\mathrm{XMod}(\mathscr{C})$ of internal crossed modules in $\mathscr{C}$ still holds.

2. Since $\mathrm{Grpd}(\mathscr{C})$ is a semi-abelian category whenever $\mathscr{C}$ is so, one obtains the equivalence of the category $\mathrm{Grpd}^2(\mathscr{C})$ of internal double groupoids and the category $\mathrm{XMod}^2(\mathscr{C}) := \mathrm{XMod}(\mathrm{XMod}(\mathscr{C}))$ of internal crossed squares in $\mathscr{C}$. For $\mathscr{C} = \mathrm{Grp}$, the objects of $\mathrm{XMod}^2(\mathscr{C})$ correspond to crossed squares of groups, a notion which was introduced by D. Guin-Waléry and J.-L. Loday in 1981 [GWL81] as a commutative square

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & P \end{array}$$

of group homomorphisms together with a function $h : M \times N \to L$ and group actions of P on L, M, N satisfying certain conditions. Similarly, G. Ellis defined the notion of crossed square for Lie algebras and commutative algebras in 1993 [Ell93]. In [dMVdL20], D. di Micco and T. Van der Linden use the concept of non-abelian tensor product to study an intrinsic notion of internal crossed square in a semi-abelian category recovering the notions of crossed squares of groups and of Lie algebras. Recently [Ste21], F. Sterck gave an explicit description of crossed squares in the semi-abelian category of cocommutative Hopf algebras over a field.

Chapter 2

N-fold groupoids and n-groupoids in regular Mal'tsev categories

In this chapter, we show that, when $\mathscr{C}$ is a regular Mal'tsev category with coequalizers, then the category 2-Grpd($\mathscr{C}$) of internal 2-groupoids in $\mathscr{C}$ is a Birkhoff subcategory of the category $\mathrm{Grpd}^2(\mathscr{C})$ of internal double groupoids in $\mathscr{C}$, and we provide a simple description of the left adjoint of the inclusion functor $\mathsf{U} : \text{2-Grpd}(\mathscr{C}) \to \mathrm{Grpd}^2(\mathscr{C})$. This implies in particular that, when $\mathscr{C}$ is a semi-abelian category, then 2-Grpd($\mathscr{C}$) is also semi-abelian. If $\mathscr{C}$ is an exact naturally Mal'tsev category with coequalizers, the left adjoint $\mathsf{F} : \mathrm{Grpd}^2(\mathscr{C}) \to \text{2-Grpd}(\mathscr{C})$ has an additional property related to the commutator of equivalence relations. More generally, we show that, for any $2 \leq n \in \mathbb{N}$, the category n-Grpd($\mathscr{C}$) of internal n-groupoids in $\mathscr{C}$ is a Birkhoff subcategory of the category $\mathrm{Grpd}^n(\mathscr{C})$ of internal n-fold groupoids in $\mathscr{C}$ when $\mathscr{C}$ is a regular Mal'tsev category with coequalizers. Finally, we provide sufficient conditions on $\mathscr{C}$ such that n-Grpd($\mathscr{C}$) is an action representable category. The results of this chapter for $n = 2$ are published in [EG25].

2.1 Internal n-groupoids

Let us start by recalling the definition of internal n-category following the presentation given by D. Bourn in [Bou84].

Proposition 2.1.1. Let

$$\mathscr{C}' \underset{\mathsf{G}}{\overset{\mathsf{K}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathscr{C} \tag{2.1}$$

be an adjunction, where $\mathscr{C}$ is a finitely complete category, the right adjoint G is a full embedding, and the left adjoint K preserves finite limits. Then there

is another adjunction

$$\mathscr{C} \underset{\mathsf{G}_0}{\overset{\mathsf{K}_0}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Cat}_{\mathsf{K}}(\mathscr{C})$$

such that $\mathrm{Cat}_{\mathsf{K}}(\mathscr{C})$ is a finitely complete category, the right adjoint G_0 is a full embedding, and the left adjoint K_0 preserves finite limits. Here $\mathrm{Cat}_{\mathsf{K}}(\mathscr{C})$ is the full subcategory of $\mathrm{Cat}(\mathscr{C})$ with objects the internal categories (C_0, C_1, d, c, e, m) such that $\mathsf{K}(e)$ is an isomorphism. The right adjoint $\mathsf{G}_0 : \mathscr{C} \to \mathrm{Cat}_{\mathsf{K}}(\mathscr{C})$ maps an object X to the kernel pair $\mathrm{Eq}(\eta_X)$ of the X-component η_X of the unit of the adjunction (2.1). The left adjoint $\mathsf{K}_0 : \mathrm{Cat}_{\mathsf{K}}(\mathscr{C}) \to \mathscr{C}$ maps an internal category (C_0, C_1, d, c, e, m) to C_0.

Note that the condition that $\mathsf{K}(e)$ is an isomorphism implies that also the other structural maps d, c, m get mapped to isomorphisms by K. Furthermore, the right adjoint $\mathsf{G}_0 : \mathscr{C} \to \mathrm{Cat}_K(\mathscr{C})$ is indeed well-defined since K preserves finite limits and G is fully faithful.

Recursively, we get the following tower of adjunctions:

$$\mathscr{C}' \underset{\mathsf{G}}{\overset{\mathsf{K}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathscr{C} \underset{\mathsf{G}_0}{\overset{\mathsf{K}_0}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Cat}_{\mathsf{K}}(\mathscr{C}) \underset{\mathsf{G}_1}{\overset{\mathsf{K}_1}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Cat}_{\mathsf{K}_0}(\mathrm{Cat}_{\mathsf{K}}(\mathscr{C})) \underset{\mathsf{G}_2}{\overset{\mathsf{K}_2}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Cat}_{\mathsf{K}_1}(\mathrm{Cat}_{\mathsf{K}_0}(\mathrm{Cat}_{\mathsf{K}}(\mathscr{C}))) \quad \cdots \tag{2.2}$$

In the following, we will assume that $\mathscr{C}'$ is the terminal category 1. In this case, $\mathrm{Cat}_{\mathsf{K}}(\mathscr{C})$ coincides with $\mathrm{Cat}(\mathscr{C})$.

Definition 2.1.2 (Internal n-category [Bou84])**.** Let $\mathscr{C}'$ in Diagram (2.2) be the terminal category 1, and $\mathscr{C}$ be a finitely complete category. We define the category $n\text{-Cat}(\mathscr{C})$ of internal n-categories in $\mathscr{C}$, where $2 \leq n \in \mathbb{N}$, recursively as follows. We set $2\text{-Cat}(\mathscr{C}) := \mathrm{Cat}_{\mathsf{K}_0}(\mathrm{Cat}(\mathscr{C}))$ and, for $3 \leq n \in \mathbb{N}$, $n\text{-Cat}(\mathscr{C}) := \mathrm{Cat}_{\mathsf{K}_{n-2}}((n-1)\text{-Cat}(\mathscr{C}))$.

In this way, we get the following tower of adjunctions:

$$1 \underset{\mathsf{G}}{\overset{\mathsf{K}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathscr{C} \underset{\mathsf{G}_0}{\overset{\mathsf{K}_0}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Cat}(\mathscr{C}) \underset{\mathsf{G}_1}{\overset{\mathsf{K}_1}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} 2\text{-Cat}(\mathscr{C}) \underset{\mathsf{G}_2}{\overset{\mathsf{K}_2}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} 3\text{-Cat}(\mathscr{C}) \quad \cdots$$

It is shown in [Bou84] that, for $\mathscr{C} = \mathrm{Set}$, $n\text{-Cat}(\mathscr{C})$ yields the usual definition of the category of small n-categories.

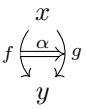
For $n = 2$, we see that an internal 2-category in a finitely complete category $\mathscr{C}$ is given by an internal double category $\mathbb{C}$ with underlying double reflexive graph

$$\begin{array}{ccc} C_{11} & \underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}} & C_{01} \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ C_{10} & \underset{\longrightarrow}{\overset{\longleftarrow e_{\uparrow 0}}{\longrightarrow}} & C_{00} \end{array}$$

such that $e_{\uparrow 0}$ is an isomorphism. Namely, K_0 maps the internal categories $C_{1\bullet}$ and $C_{0\bullet}$ to the objects C_{10} and C_{00}, respectively, and the internal functor $e_{\uparrow\bullet}$ to the morphism $e_{\uparrow 0}$. Here we ue the same notation as in Section 1.4.1.

An internal 2-category in Set, i.e. a small 2-category, has a set C_{00} of *objects*, a set C_{01} of 1-*morphisms*, and a set C_{11} of 2-*morphisms*, that are usually

depicted as follows, where x, y are objects, f, g are 1-morphisms from x to y, and α is a 2-morphism from f to g:



Definition 2.1.3 (Internal n-reflexive graph)**.** Let $\mathscr{C}$ be a category. We define the category n-RG($\mathscr{C}$) of n-reflexive graphs in $\mathscr{C}$ to be the full subcategory of the category $\mathrm{RG}^n(\mathscr{C})$ of n-fold reflexive graphs in $\mathscr{C}$ with objects $\mathbb{C}$ such that, for all $k \in \{0, \dots, n-2\}$ and $i_1, \dots, i_{n-2} \in \{0, 1\}$, the morphism

$$e_{i_1 \cdots i_k \uparrow 0 i_{k+1} \cdots i_{n-2}}$$

is an isomorphism. Here we use the same notation for internal n-fold reflexive graphs $\mathbb{C}$ as right after Definition 1.4.9.

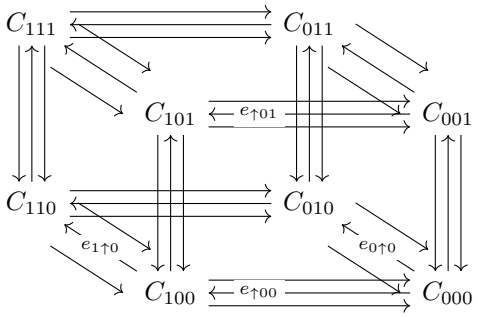
Proposition 2.1.4. Let $\mathscr{C}$ be a finitely complete category. An internal n-category in $\mathscr{C}$ is given by an internal n-fold category $\mathbb{C}$ whose underlying n-fold reflexive graph lies in n-RG($\mathscr{C}$).

Proof. We prove the statement by induction. We have just seen that the claim is true for $n = 2$. Let us now consider an internal n-category $C_{\bullet \dots \bullet}$ in $\mathscr{C}$, where $3 \leq n \in \mathbb{N}$, and suppose that we have proved the statement for all $2 \leq m \leq n-1$. Then $C_{1\bullet \dots \bullet}$ and $C_{0\bullet \dots \bullet}$ are internal $(n-1)$-categories meaning that their underlying $(n-1)$-fold reflexive graphs lie in $(n-1)$-RG($\mathscr{C}$), i.e., for all $k \in \{0 \dots, n-3\}$ and $i, i_1 \dots, i_{n-3} \in \{0, 1\}$, the morphism

$$e_{i i_1 \cdots i_k \uparrow 0 i_{k+1} \cdots i_{n-3}}$$

is an isomorphism. Furthermore, the functor $\mathsf{K}_{n-2} : (n-1)\text{-Cat}(\mathscr{C}) \to n\text{-Cat}(\mathscr{C})$ maps the internal $(n-1)$-categories $C_{1\bullet \dots \bullet}$ and $C_{0\bullet \dots \bullet}$ to the internal $(n-2)$-categories $C_{10\bullet \dots \bullet}$ and $C_{00\bullet \dots \bullet}$, respectively, and the internal $(n-1)$-functor $e_{\uparrow \bullet \dots \bullet}$ to the internal $(n-2)$-functor $e_{\uparrow 0 \bullet \dots \bullet}$, which must be an isomorphism. This means that all its components $e_{\uparrow 0 i_1 \cdots i_{k-2}}$ are isomorphisms in $\mathscr{C}$, where $i_1 \dots, i_{k-2} \in \{0, 1\}$. This concludes the proof. □

Example 2.1.5 (Internal 3-category)**.** An internal 3-category in a finitely complete $\mathscr{C}$ is given by an internal triple category $\mathbb{C}$ in $\mathscr{C}$ with underlying triple reflexive graph



such that the morphisms $e_{1\uparrow 0}$, $e_{0\uparrow 0}$, $e_{\uparrow 01}$ and $e_{\uparrow 00}$ are isomorphisms. The functor $\mathsf{K}_1 : 2\text{-Cat}(\mathscr{C}) \to \text{Cat}(\mathscr{C})$ maps the internal 2-categories $C_{1\bullet\bullet}, C_{0\bullet\bullet}$ to the internal categories $C_{10\bullet}, C_{00\bullet}$, respectively, and the internal double functor $e_{\uparrow\bullet\bullet}$ to the internal functor $e_{\uparrow 0\bullet}$.

An internal 3-category in Set, i.e. a small 3-category, has a set C_{000} of *objects*, a set C_{001} of 1-*morphisms*, a set C_{011} of 2-*morphisms*, and a set C_{111} of 3-*morphisms*.

One observes that the functor $\mathsf{G}_0 : \mathscr{C} \to \text{Cat}_{\mathsf{K}}(\mathscr{C})$ as in Proposition 2.1.1 that maps an object X to the kernel pair $\text{Eq}(\eta_X)$ factors through the inclusion functor $\mathsf{U} : \text{Grpd}_{\mathsf{K}}(\mathscr{C}) \to \text{Cat}_{\mathsf{K}}(\mathscr{C})$, where $\text{Grpd}_{\mathsf{K}}(\mathscr{C})$ is the full subcategory of $\text{Cat}_{\mathsf{K}}(\mathscr{C})$ whose objects are also internal groupoids. This allows one to define the category n-Grpd$(\mathscr{C})$ of internal n-groupoids in $\mathscr{C}$ in the same manner as n-Cat$(\mathscr{C})$, see also [Bou89].

Proposition 2.1.6. For $\mathscr{C}$ a Mal'tsev category, the categories n-Cat$(\mathscr{C})$ and n-Grpd$(\mathscr{C})$ are isomorphic.

Proof. This follows directly from the fact that Cat$(\mathscr{C})$ and Grpd$(\mathscr{C})$ are isomorphic categories when $\mathscr{C}$ is a Mal'tsev category (Proposition 1.4.19). □

Lastly, let us consider internal n-groupoids in the additive context.

Definition 2.1.7 (Internal n-arrow)**.** Let $\mathscr{C}$ be a pointed category. We define the category n-Arr$(\mathscr{C})$ of *internal n-arrows* in $\mathscr{C}$ to be the category with objects being chains of n composable arrows such that each composite of two consecutive ones is 0, and expected morphisms.

Note that, for a pointed category $\mathscr{C}$, the category n-Arr$(\mathscr{C})$ is equivalent to the full subcategory $\mathscr{X}$ of $\text{Arr}^n(\mathscr{C})$ with objects the internal n-fold arrows $\mathbb{A}$ in $\mathscr{C}$ with

$$A_{i_1 \cdots i_k 10 i_{k+1} \cdots i_{n-2}} = 0,$$

where $k \in \{0, \ldots, n-2\}$ and $i_1, \ldots, i_{n-2} \in \{0, 1\}$. Here we use the same notation for n-fold arrows $\mathbb{A}$ as right after Definition 1.4.33. For example, an internal 2-arrow 'fits' into a commutative square, and an internal 3-arrow 'fits' into a commutative cube as follows:

$$\begin{array}{ccc} A_{11} & \longrightarrow & A_{01} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A_{00} \end{array} \qquad \begin{array}{ccccc} A_{111} & \longrightarrow & A_{011} & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ 0 & \longrightarrow & 0 & \longrightarrow & A_{001} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \searrow & \\ & \searrow & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & A_{0000} \end{array}$$

A morphism $f : \mathbb{A} \to \mathbb{B}$ in $\mathscr{X}$ gets associated to the morphism

$$\begin{array}{ccccccccc} A_{1\cdots 1} & \longrightarrow & A_{01\cdots 1} & \longrightarrow & \cdots & \longrightarrow & A_{0\cdots 01} & \longrightarrow & A_{0\cdots 0} \\ \downarrow{\scriptstyle f_{1\cdots 1}} & & \downarrow{\scriptstyle f_{01\cdots 1}} & & & & \downarrow{\scriptstyle f_{0\cdots 01}} & & \downarrow{\scriptstyle f_{0\cdots 0}} \\ B_{1\cdots 1} & \longrightarrow & B_{01\cdots 1} & \longrightarrow & \cdots & \longrightarrow & B_{0\cdots 01} & \longrightarrow & B_{0\cdots 0} \end{array}$$

in n-Arr($\mathscr{C}$).

Proposition 2.1.8. [Bou90] Let $\mathscr{A}$ be an additive category with kernels. Then the categories n-Grpd($\mathscr{A}$), n-Cat($\mathscr{A}$), n-RG($\mathscr{A}$) and n-Arr($\mathscr{A}$) are equivalent.

Proof. Recall from Proposition 1.4.34 that, if $\mathscr{A}$ is an additive category with kernels, then the categories Grpd($\mathscr{A}$), Cat($\mathscr{A}$), RG($\mathscr{A}$) and Arr($\mathscr{A}$) are equivalent. This implies readily that n-Grpd($\mathscr{A}$), n-Cat($\mathscr{A}$) and n-RG($\mathscr{A}$) are equivalent. In the following, we show that n-RG($\mathscr{A}$) and n-Arr($\mathscr{A}$) are equivalent.

Let us denote by $\mathsf{F}_1 : \mathrm{RG}(\mathscr{A}) \to \mathrm{Arr}(\mathscr{A})$ the functor which maps a reflexive graph (C_0, C_1, d, c, e) to the arrow $c\ker(d) : \mathrm{Ker}(d) \to C_0$, and by $\mathsf{G}_1 : \mathrm{Arr}(\mathscr{A}) \to \mathrm{RG}(\mathscr{A})$ the functor which maps an arrow $a : A_1 \to A_0$ to the reflexive graph $(A_0, A_0\ A_1, \pi_1, [1_{A_0}, a], \iota_1)$ as in Example 1.4.35. Then $\mathsf{G}_1\mathsf{F}_1 \simeq 1_{\mathrm{RG}(\mathscr{A})}$ and $\mathsf{F}_1\mathsf{G}_1 \simeq 1_{\mathrm{Arr}(\mathscr{A})}$. This implies that also the categories $\mathrm{RG}^n(\mathscr{A})$ and $\mathrm{Arr}^n(\mathscr{A})$ are equivalent. Let $\mathsf{F}_n : \mathrm{RG}^n(\mathscr{A}) \to \mathrm{Arr}^n(\mathscr{A})$ and $\mathsf{G}_n : \mathrm{Arr}^n(\mathscr{A}) \to \mathrm{RG}^n(\mathscr{A})$, where $2 \leq n \in \mathbb{N}$, be the functors obtained in the following recursive way. We view an n-fold reflexive graph $\mathbb{C}$ as a reflexive graph $(C_{0\bullet\cdots\bullet}, C_{1\bullet\cdots\bullet}, d_{\downarrow\bullet\cdots\bullet}, c_{\downarrow\bullet\cdots\bullet}, e_{\uparrow\bullet\cdots\bullet})$ in $\mathrm{RG}^{n-1}(\mathscr{A})$. By applying F_{n-1}, we obtain the reflexive graph $(B_{0\bullet\cdots\bullet}, B_{1\bullet\cdots\bullet}, \tilde{d}_{\downarrow\bullet\cdots\bullet}, \tilde{c}_{\downarrow\bullet\cdots\bullet}, \tilde{e}_{\uparrow\bullet\cdots\bullet})$ in $\mathrm{Arr}^{n-1}(\mathscr{A})$. Then $\tilde{c}_{\downarrow\bullet\cdots\bullet}\ker(\tilde{d}_{\downarrow\bullet\cdots\bullet}) : \mathrm{Ker}(\tilde{d}_{\downarrow\bullet\cdots\bullet}) \to B_{0\bullet\cdots\bullet}$ yields an arrow in $\mathrm{Arr}^{n-1}(\mathscr{A})$, i.e. an n-fold arrow $\mathbb{A}$, which is the F_n-image of $\mathbb{C}$. Conversely, we view an n-fold arrow $\mathbb{A}$ as an arrow $a_{\downarrow\bullet\cdots\bullet} : A_{1\bullet\cdots\bullet} \to A_{0\bullet\cdots\bullet}$ in $\mathrm{Arr}^{n-1}(\mathscr{A})$. By applying G_{n-1}, we obtain the arrow $\tilde{a}_{\downarrow\bullet\cdots\bullet} : D_{1\bullet\cdots\bullet} \to D_{0\bullet\cdots\bullet}$ in $\mathrm{RG}^{n-1}(\mathscr{A})$. Then $(D_{0\bullet\cdots\bullet}, D_{0\bullet\cdots\bullet} \oplus D_{1\bullet\cdots\bullet}, \pi_{2\bullet\cdots\bullet}, [1_{D_{0\bullet\cdots\bullet}}, \tilde{a}_{\downarrow\bullet\cdots\bullet}], \iota_{2\bullet\cdots\bullet})$ yields a reflexive graph in $\mathrm{RG}^{n-1}(\mathscr{A})$, i.e. an n-fold reflexive graph $\mathbb{C}$, which is the G_n-image of $\mathbb{A}$. It is clear that $\mathsf{G}_n\mathsf{F}_n \approx 1_{\mathrm{RG}^n(\mathscr{A})}$ and $\mathsf{F}_n\mathsf{G}_n \approx 1_{\mathrm{Arr}^n(\mathscr{A})}$.

Under this equivalence, the functor $(-)_0 : \mathrm{RG}^n(\mathscr{A}) \to \mathrm{RG}^{n-1}(\mathscr{A})$ which maps an n-fold reflexive graph $\mathbb{C}$ to the $(n-1)$-fold reflexive graph $C_{0\bullet\cdots\bullet}$ corresponds to the functor $(-)_0 : \mathrm{Arr}^n(\mathscr{A}) \to \mathrm{Arr}^{n-1}(\mathscr{A})$ which maps an n-fold arrow $\mathbb{A}$ to the $(n-1)$-fold arrow $A_{0\bullet\cdots\bullet}$.

We want to show that the restriction to n-RG($\mathscr{A}$) of the functor $\mathsf{F}_n : \mathrm{RG}^n(\mathscr{A}) \to \mathrm{Arr}^n(\mathscr{A})$ factors through the inclusion of n-Arr($\mathscr{A}$) into $\mathrm{Arr}^n(\mathscr{A})$, where we view n-Arr($\mathscr{A}$) as a full subcategory $\mathscr{X}$ of $\mathrm{Arr}^n(\mathscr{A})$ as explained above. Let $\mathbb{C}$ be an n-reflexive graph. This means that $C_{1\bullet\cdots\bullet}$ and $C_{0\bullet\cdots\bullet}$ are $(n-1)$-reflexive graphs, and that the morphism $e_{\downarrow 0\bullet\cdots\bullet} : C_{10\bullet\cdots\bullet} \to C_{00\bullet\cdots\bullet}$ of $(n-2)$-reflexive graphs is an isomorphism. By applying F_{n-1}, we obtain the reflexive graph $(B_{0\bullet\cdots\bullet}, B_{1\bullet\cdots\bullet}, \tilde{d}_{\downarrow\bullet\cdots\bullet}, \tilde{c}_{\downarrow\bullet\cdots\bullet}, \tilde{e}_{\uparrow\bullet\cdots\bullet})$ in $(n-1)$-Arr($\mathscr{A}$) with $\tilde{e}_{\downarrow 0\bullet\cdots\bullet}$ being an isomorphism in $(n-2)$-Arr($\mathscr{A}$). We need to show that the n-fold arrow $\mathbb{A}$ obtained by the arrow $\tilde{c}_{\downarrow\bullet\cdots\bullet}\ker(\tilde{d}_{\downarrow\bullet\cdots\bullet}) : \mathrm{Ker}(\tilde{d}_{\downarrow\bullet\cdots\bullet}) \to B_{0\bullet\cdots\bullet}$ in $(n-1)$-Arr($\mathscr{A}$) is an n-arrow, i.e.,

$$A_{i_1\cdots i_k 10 i_{k+1}\cdots i_{n-2}} = 0$$

for all $k \in \{0, \dots, n-2\}$ and $i_1, \dots, i_{n-2} \in \{0,1\}$. For $k \geq 1$ and $i_1 = 0$, this is clear since, by induction hypothesis, $A_{0i_2\cdots i_k 10 i_{k+1}\cdots i_{n-2}} = B_{0i_2\cdots i_k 10 i_{k+1}\cdots i_{n-2}}$ and $B_{0\bullet\cdots\bullet} \in (n-1)\text{-Arr}(\mathscr{A})$. For $k \geq 1$ and $i_1 = 1$, this is true since

$$A_{1i_2\cdots i_k 10 i_{k+1}\cdots i_{n-2}} = \mathrm{Ker}(\tilde{d}_{\downarrow i_2\cdots i_k 10 i_{k+1}\cdots i_{n-2}})$$

and $B_{1\bullet\cdots\bullet}$, $B_{0\bullet\cdots\bullet} \in (n-1)\text{-Arr}(\mathscr{A})$. For $k = 0$, this is true since

$$A_{10 i_1\cdots i_{n-2}} = \mathrm{Ker}(\tilde{d}_{\downarrow 0 i_1\cdots i_{n-2}})$$

and $\tilde{d}_{\downarrow 0 i_1\cdots i_{n-2}}$ is an isomorphism.

It remains to show that the restriction to $n\text{-Arr}(\mathscr{A})$ of the functor $\mathsf{G}_n : \mathrm{Arr}^n(\mathscr{A}) \to \mathrm{RG}^n(\mathscr{A})$ factors through the inclusion of $n\text{-RG}(\mathscr{A})$ into $\mathrm{RG}^n(\mathscr{A})$. Let $\mathbb{A}$ be an n-arrow. Note that this means that all objects $A_{1i_1\cdots i_{n-1}}$ in $\mathscr{A}$, where $i_1, \dots, i_{n-1} \in \{0,1\}$, are 0 except $A_{1\cdots 1}$, and $A_{0\bullet\cdots\bullet} \in (n-1)\text{-Arr}(\mathscr{A})$. By applying G_{n-1}, we obtain the arrow $\tilde{a}_{\downarrow\bullet\cdots\bullet} : D_{1\bullet\cdots\bullet} \to D_{0\bullet\cdots\bullet}$ in $\mathrm{RG}^{n-1}(\mathscr{A})$. It is easy to show that all objects $D_{1i_1\cdots i_{n-1}}$ in $\mathscr{A}$, where $i_1, \dots, i_{n-1} \in \{0,1\}$, are 0 except $D_{1\cdots 1} = A_{1\cdots 1}$. By induction hypothesis, $D_{0\bullet\cdots\bullet} \in (n-1)\text{-RG}(\mathscr{A})$. We have to show that the n-fold reflexive graph $\mathbb{C}$ obtained by the reflexive graph $(D_{0\bullet\cdots\bullet}, D_{0\bullet\cdots\bullet} \oplus D_{1\bullet\cdots\bullet}, \pi_{2\bullet\cdots\bullet}, [1_{D_{0\bullet\cdots\bullet}}, \tilde{a}_{\downarrow\bullet\cdots\bullet}], \iota_{2\bullet\cdots\bullet})$ in $\mathrm{RG}^{n-1}(\mathscr{A})$ is an n-reflexive graph, i.e., all the structure maps $e_{i_1\cdots i_k\uparrow 0 i_{k+1}\cdots i_{n-2}}$ of $\mathbb{C}$, where $k \in \{0, \dots, n-2\}$ and $i_1, \dots, i_{n-2} \in \{0,1\}$, are isomorphisms. If $k \geq 1$ and $i_1 = 0$, this is true since $D_{0\bullet\cdots\bullet} \in (n-1)\text{-RG}(\mathscr{A})$. Otherwise, this is true since $D_{1i_1\cdots i_{n-1}} = 0$ for all $i_1, \dots, i_{n-1} \in \{0,1\}$ except for $1 = i_1 = \cdots = i_{n-1}$. □

In [Bou90], it is also shown that the categories $\mathrm{Ch}(\mathscr{A})$ of chain complexes and $\infty\text{-Grpd}(\mathscr{A})$ of *internal ∞-groupoids* in $\mathscr{A}$ are equivalent. Recall that the so-called Dold-Kan correspondence asserts that the category $\mathrm{Ch}(\mathscr{A})$ is also equivalent to the category $\mathrm{Simpl}(\mathscr{A})$ of simplicial objects in $\mathscr{A}$ [Dol58, DP61, Kan58].

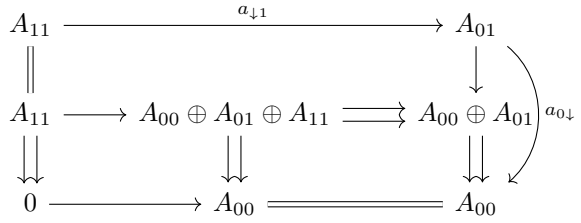
Example 2.1.9. Let us illustrate the equivalence between $n\text{-Grpd}(\mathscr{A})$ and $n\text{-Arr}(\mathscr{A})$, where $\mathscr{A}$ is an additive category with kernels, stated in Proposition 2.1.8 for $n = 2$. Given an internal 2-groupoid in $\mathscr{A}$ with underlying 2-reflexive graph depicted in the interior of the diagram

$$\begin{array}{ccccc}
\mathrm{Ker}(\tilde{d}_{\downarrow 1}) & \xrightarrow{\ker(\tilde{d}_{\downarrow 1})} & \mathrm{Ker}(d_{1\downarrow}) & \underset{\tilde{c}_{\downarrow 1}}{\overset{\tilde{d}_{\downarrow 1}}{\rightrightarrows}} & \mathrm{Ker}(d_{0\downarrow}) \\
\downarrow & & \downarrow{\scriptstyle \ker(d_{1\downarrow})} & & \downarrow{\scriptstyle \ker(d_{0\downarrow})} \\
 & & C_{11} & \underset{c_{\downarrow 1}}{\overset{d_{\downarrow 1}}{\rightrightarrows}} & C_{01} \\
 & & {\scriptstyle d_{1\downarrow}}\downarrow\uparrow{\scriptstyle c_{1\downarrow}} & & {\scriptstyle d_{0\downarrow}}\downarrow\uparrow{\scriptstyle c_{0\downarrow}} \\
0 & \longrightarrow & C_{00} & = & C_{00},
\end{array}$$

then the corresponding internal 2-arrow is given by

$$\mathrm{Ker}(\tilde{d}_{\downarrow 1}) \xrightarrow{\tilde{c}_{\downarrow 1}\ker(\tilde{d}_{\downarrow 1})} \mathrm{Ker}(d_{0\downarrow}) \xrightarrow{c_{0\downarrow}\ker(d_{0\downarrow})} C_{00}.$$

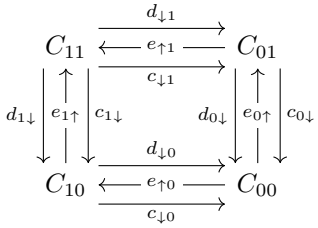
Given an internal 2-arrow $A_{11} \to A_{01} \to A_{00}$, the corresponding internal 2-groupoid is indicated in the interior of the following diagram:



2.2 Birkhoff subcategory of internal 2-groupoids

We have seen in the proof of Proposition 1.4.31 that, when $\mathbb{V}$ is a Mal'tsev variety of universal algebras, then the category $\mathrm{RG}(\mathbb{V})$ of reflexive graphs in $\mathbb{V}$ is equivalent to the Mal'tsev variety $\mathbb{W}$ whose equational theory is given by the equational theory of $\mathbb{V}$ together with two additional unary operations s, t, and the identities encoding that s, t are homomorphisms in $\mathbb{V}$, and that $st = t$, $ts = s$. Given a reflexive graph (C_0, C_1, d, c, e) in $\mathbb{V}$, its corresponding algebra in $\mathbb{W}$ is given by the algebra C_1 in $\mathbb{V}$ equipped with the two operations $s^{C_1} := ed$ and $t^{C_1} := ec$. The full subcategory $\mathrm{Grpd}(\mathbb{V})$ of $\mathrm{RG}(\mathbb{V})$ corresponds to the subvariety of $\mathbb{W}$ obtained by adding to the equational theory of $\mathbb{W}$ the identities encoding that the universal algebraic commutator $[\mathrm{Eq}(s), \mathrm{Eq}(t)]$ of the kernel pairs of s, t vanishes. More generally, when $\mathscr{C}$ is a regular Mal'tsev category with coequalizers, then so is $\mathrm{RG}(\mathscr{C})$, and $\mathrm{Grpd}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{RG}(\mathscr{C})$ [Ped95, Bou04].

It follows that the category $\mathrm{RG}^2(\mathbb{V})$ of double reflexive graphs in $\mathbb{V}$ is equivalent to the Mal'tsev variety $\mathbb{U}$ whose equational theory is given by the equational theory of $\mathbb{V}$ together with four additional unary operations s, t, u, v, and the identities encoding that s, t, u, v are homomorphisms in $\mathbb{V}$, and $st = t$, $ts = s$, $uv = v$, $vu = u$, $su = us$, $sv = vs$, $tu = ut$ and $tv = vt$. Given a double reflexive graph



the corresponding algebra in $\mathbb{U}$ is given by the algebra C_{11} in $\mathbb{V}$ equipped with the operations $s^{C_{11}} := e_{\uparrow 1} d_{\downarrow 1}$, $t^{C_{11}} := e_{\uparrow 1} c_{\downarrow 1}$, $u^{C_{11}} := e_{1\uparrow} d_{1\downarrow}$, $v^{C_{11}} := e_{1\uparrow} c_{1\downarrow}$. The subcategory $\mathrm{Grpd}^2(\mathbb{V})$ of $\mathrm{RG}^2(\mathbb{V})$ corresponds to the subvariety of $\mathbb{U}$ obtained by adding the identities encoding that the commutators $[\mathrm{Eq}(s), \mathrm{Eq}(t)]$ and $[\mathrm{Eq}(u), \mathrm{Eq}(v)]$ vanish. The subcategory 2-$\mathrm{Grpd}(\mathbb{V})$ corresponds to the subvariety obtained by further adding the identities encoding that $su = tu = u$ and $sv = tv = v$. In this section, we show that 2-$\mathrm{Grpd}(\mathscr{C})$ is a Birkhoff

subcategory of $\mathrm{Grpd}^2(\mathscr{C})$ whenever $\mathscr{C}$ is a regular Mal'tsev category with coequalizers, and provide a simple description of the left adjoint of the inclusion functor $\mathsf{U} : \text{2-Grpd}(\mathscr{C}) \to \mathrm{Grpd}^2(\mathscr{C})$.

We note that, if $\mathscr{C}$ is a category with kernel pairs and coequalizers, then it has also pushouts along regular epimorphisms. Namely, let $q : X \to Y$ be a regular epimorphism and $\delta : X \to S$ be an arbitrary morphism in $\mathscr{C}$. Let us consider the kernel pair $(\mathrm{Eq}(q), q_1, q_2)$ of q, and construct the coequalizer (C, c) of $\delta q_1, \delta q_2$. This yields a unique morphism $\bar{\delta}$ such that $\bar{\delta} q = c\delta$. Then the commutative square on the right-hand side in

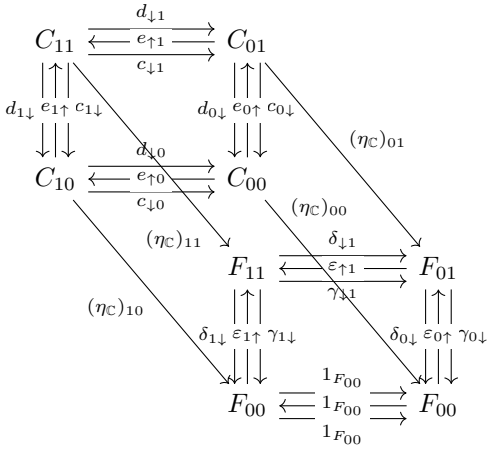
$$\begin{array}{ccc} & & S & \xrightarrow{c} & C \\ & & {\scriptstyle\delta}\uparrow & & \uparrow{\scriptstyle\bar{\delta}} \\ \mathrm{Eq}(q) & \underset{q_2}{\overset{q_1}{\rightrightarrows}} & X & \xrightarrow[q]{} & Y \end{array}$$

is easily seen to be the pushout of q and δ.

Lemma 2.2.1. Let $\mathscr{C}$ be a regular category with coequalizers. Then $\text{2-RG}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{RG}^2(\mathscr{C})$.

Proof. We first show that, for any double reflexive graph $\mathbb{C}$ in $\mathscr{C}$, there exists a regular epimorphism of double reflexive graphs $\eta_{\mathbb{C}}$ from $\mathbb{C}$ to a 2-reflexive graph $\mathsf{F}(\mathbb{C})$ such that any morphism of double reflexive graphs from $\mathbb{C}$ to any 2-reflexive graph $\mathbb{D}$ factors uniquely through $\eta_{\mathbb{C}}$. This shows that $\text{2-RG}(\mathscr{C})$ is a regular epi-reflective subcategory of $\mathrm{RG}^2(\mathscr{C})$ with $\mathsf{F} : \mathrm{RG}^2(\mathscr{C}) \to \text{2-RG}(\mathscr{C})$ the corresponding reflector and η the corresponding unit.

For a double reflexive graph $\mathbb{C}$ in $\mathscr{C}$, we set $\mathsf{F}(\mathbb{C})$ to be the 2-reflexive graph $\mathbb{F}$ in the front part of the following diagram:

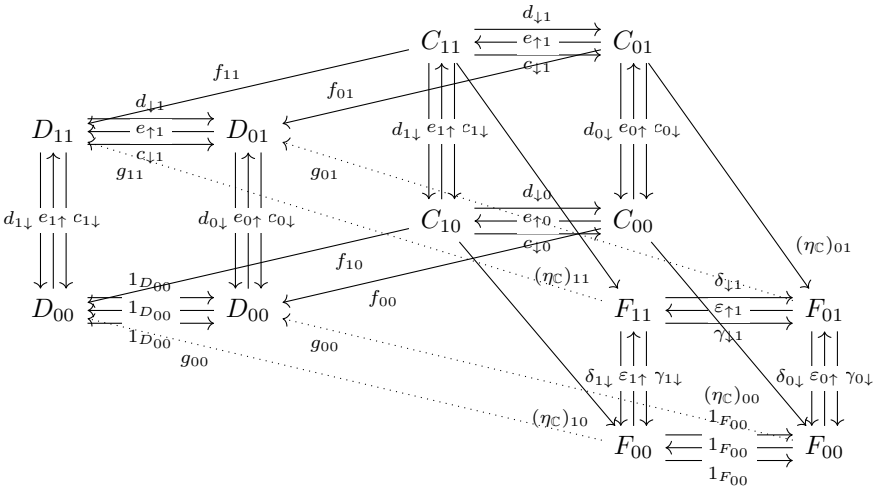


Here $(F_{00}, (\eta_{\mathbb{C}})_{00})$ is the coequalizer of $d_{\downarrow 0}, c_{\downarrow 0}$. We set $(\eta_{\mathbb{C}})_{10} := (\eta_{\mathbb{C}})_{00} d_{\downarrow 0} = (\eta_{\mathbb{C}})_{00} c_{\downarrow 0}$. $(F_{01}, (\eta_{\mathbb{C}})_{01}, \varepsilon_{0\uparrow})$ is the pushout of $(\eta_{\mathbb{C}})_{00}$ along $e_{0\uparrow}$. The morphisms $\delta_{0\downarrow}$ and $\gamma_{0\downarrow}$ are the unique ones such that $\delta_{0\downarrow}(\eta_{\mathbb{C}})_{01} = (\eta_{\mathbb{C}})_{00} d_{0\downarrow}$ and $\delta_{0\downarrow}\varepsilon_{0\uparrow} = 1_{F_{00}}$, and $\gamma_{0\downarrow}(\eta_{\mathbb{C}})_{01} = (\eta_{\mathbb{C}})_{00} c_{0\downarrow}$ and $\gamma_{0\downarrow}\varepsilon_{0\uparrow} = 1_{F_{00}}$, which are induced by the universal property of the pushout F_{01}. Similarly, $(F_{11}, (\eta_{\mathbb{C}})_{11}, \varepsilon_{1\uparrow})$ is the

pushout of $(\eta_{\mathbb{C}})_{10}$ along $e_{1\uparrow}$. The morphism $\varepsilon_{\uparrow 1}$ is the unique one such that $\varepsilon_{\uparrow 1}(\eta_{\mathbb{C}})_{01} = (\eta_{\mathbb{C}})_{11}e_{\uparrow 1}$ and $\varepsilon_{\uparrow 1}\varepsilon_{0\uparrow} = \varepsilon_{1\uparrow}$, induced by the universal property of the pushout F_{01}. The morphisms $\delta_{\downarrow 1}$ and $\gamma_{\downarrow 1}$ are the unique ones such that $\delta_{\downarrow 1}(\eta_{\mathbb{C}})_{11} = (\eta_{\mathbb{C}})_{01}d_{\downarrow 1}$ and $\delta_{\downarrow 1}\varepsilon_{1\uparrow} = \varepsilon_{0\uparrow}$, and $\gamma_{\downarrow 1}(\eta_{\mathbb{C}})_{11} = (\eta_{\mathbb{C}})_{01}c_{\downarrow 1}$ and $\gamma_{\downarrow 1}\varepsilon_{1\uparrow} = \varepsilon_{0\uparrow}$, induced by the universal property of the pushout F_{11}. We set $\delta_{1\downarrow} := \delta_{0\downarrow}\delta_{\downarrow 1}$ and $\gamma_{1\downarrow} := \gamma_{0\downarrow}\gamma_{\downarrow 1}$.

It is easily seen that the front part of the above diagram is a double reflexive graph $\mathbb{F}$ in $\mathscr{C}$. Furthermore, one checks that $\eta_{\mathbb{C}} : \mathbb{C} \to \mathbb{F}$ is a morphism of double reflexive graphs. The fact that $(\eta_{\mathbb{C}})_{ij}$ is a regular epimorphism in $\mathscr{C}$ for all $i, j \in \{0, 1\}$ implies that $\eta_{\mathbb{C}}$ is a regular epimorphism in $\mathrm{RG}^2(\mathscr{C})$.

Let now $f : \mathbb{C} \to \mathbb{D}$ be a morphism in $\mathrm{RG}^2(\mathscr{C})$, where $\mathbb{D} \in \text{2-RG}(\mathscr{C})$, as depicted in the left-hand side of the following diagram:



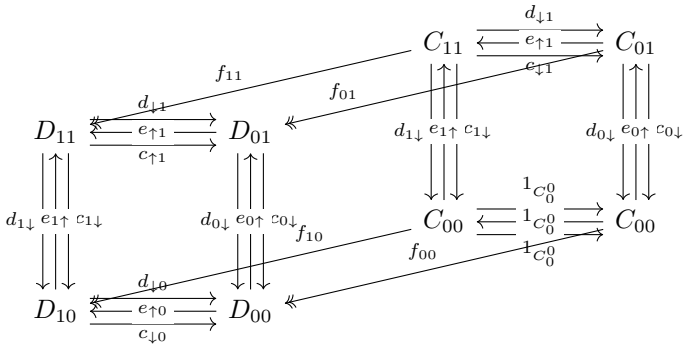
We show that there exists a unique morphism of double reflexive graphs $g : \mathbb{F} \to \mathbb{D}$ (as depicted in the above diagram) such that $g\eta_{\mathbb{C}} = f$. Since $f_{00}d_{\downarrow 0} = f_{10} = f_{00}c_{\downarrow 0}$, there exists a unique morphism g_{00} such that $g_{00}(\eta_{\mathbb{C}})_{00} = f_{00}$ by the universal property of the coequalizer F_{00}. Since $e_{0\uparrow}g_{00}(\eta_{\mathbb{C}})_{00} = e_{0\uparrow}f_{00} = f_{01}e_{0\uparrow}$, there exists a unique morphism g_{01} such that $g_{01}(\eta_{\mathbb{C}})_{01} = f_{01}$ and $g_{01}\varepsilon_{0\uparrow} = e_{0\uparrow}g_{00}$ by the universal property of the pushout F_{01}. Since

$$e_{1\uparrow}g_{00}(\eta_{\mathbb{C}})_{10} = e_{1\uparrow}g_{00}(\eta_{\mathbb{C}})_{00}d_{\downarrow 0} = e_{1\uparrow}f_{00}d_{\downarrow 0} = e_{1\uparrow}f_{10} = f_{11}e_{1\uparrow},$$

there exists a unique morphism $g_{11}\colon F_{11} \to D_{11}$ such that $g_{11}(\eta_{\mathbb{C}})_{11} = f_{11}$ and $g_{11}\varepsilon_{1\uparrow} = e_{1\uparrow}g_{00}$ by the universal property of the pushout F_{11} . It is easily seen that the constructed g is a morphism of double reflexive graphs. Its uniqueness follows from the fact that $\eta_{\mathbb{C}}$ is a regular epimorphism.

To complete the proof, it suffices to observe that 2-RG($\mathscr{C}$) is closed under regular quotients in $\mathrm{RG}^2(\mathscr{C})$. Given any regular epimorphism $f\colon \mathbb{C} \to \mathbb{D}$ in

$\mathrm{RG}^2(\mathscr{C})$



where $\mathbb{C}$ is a 2-reflexive graph, the fact that f_{10} is a regular epimorphism in $\mathscr{C}$ easily implies that $e_{\uparrow 0}$ is an isomorphism, so that $\mathbb{D} \in$ 2-RG($\mathscr{C}$), as desired. Similarly, one can see that 2-RG($\mathscr{C}$) is also closed under subobjects in $\mathrm{RG}^2(\mathscr{C})$, which also follows from the fact that 2-RG($\mathscr{C}$) is a regular epi-reflective subcategory of $\mathrm{RG}^2(\mathscr{C})$ as shown above and Proposition 1.3.13. □

Proposition 2.2.2. Let $\mathscr{C}$ be a regular Mal'tsev category with coequalizers.

1. 2-Grpd($\mathscr{C}$) is a Birkhoff subcategory of $\mathrm{Grpd}^2(\mathscr{C})$.
2. 2-Grpd($\mathscr{C}$) is a regular Mal'tsev category with coequalizers.
3. If $\mathscr{C}$ is in addition exact, then 2-Grpd($\mathscr{C}$) is also exact.
4. If $\mathscr{C}$ is a semi-abelian category, then 2-Grpd($\mathscr{C}$) is also semi-abelian.

Proof.

1. In the proof of Lemma 2.2.1, we have constructed an explicit left adjoint $\mathsf{F} : \mathrm{RG}^2(\mathscr{C}) \to$ 2-RG($\mathscr{C}$) of the inclusion $\mathsf{U} :$ 2-RG($\mathscr{C}$) $\to \mathrm{RG}^2(\mathscr{C})$ when $\mathscr{C}$ is a regular category with coequalizers. If $\mathscr{C}$ is in addition a Mal'tsev category, then F lifts to a functor $\mathsf{F} : \mathrm{Grpd}^2(\mathscr{C}) \to$ 2-Grpd($\mathscr{C}$), which is the left adjoint of the inclusion $\mathsf{U} :$ 2-Grpd($\mathscr{C}$) $\to \mathrm{Grpd}^2(\mathscr{C})$. Namely, if $\mathbb{C}$ is a double groupoid in $\mathscr{C}$, then $\mathsf{F}(\mathbb{C})$ is a 2-groupoid since Grpd($\mathscr{C}$) is closed under regular quotients in RG($\mathscr{C}$) when $\mathscr{C}$ is a regular Mal'tsev category (Proposition 1.4.21). Since the inclusion $\mathsf{U} :$ Grpd($\mathscr{C}$) $\to$ RG($\mathscr{C}$) is full when $\mathscr{C}$ is a Mal'tsev category (Proposition 1.4.19), $\eta_{\mathbb{C}} : \mathbb{C} \to \mathsf{UF}(\mathbb{C})$ is a double functor when $\mathbb{C}$ is a double groupoid. Since a morphism in Grpd($\mathscr{C}$) is a regular epimorphism if and only if it is so in RG($\mathscr{C}$) when $\mathscr{C}$ is a regular Mal'tsev category (Proposition 1.4.21), 2-Grpd($\mathscr{C}$) is a regular epi-reflective subcategory of $\mathrm{Grpd}^2(\mathscr{C})$. As in the proof of Lemma 2.2.1, it is easily seen that 2-Grpd($\mathscr{C}$) is closed under regular quotients in $\mathrm{Grpd}^2(\mathscr{C})$.
2. If $\mathscr{C}$ is a regular Mal'tsev category with coequalizers, then $\mathrm{Grpd}^2(\mathscr{C})$ satisfies the same properties (Proposition 1.4.21). The fact that 2-Grpd($\mathscr{C}$) is a regular epi-reflective subcategory of $\mathrm{Grpd}^2(\mathscr{C})$ implies that it is also a regular Mal'tsev category with coequalizers.

3. If $\mathscr{C}$ is an exact Mal'tsev category with coequalizers, then so is $\mathrm{Grpd}^2(\mathscr{C})$ (Proposition 1.4.21). The fact that 2-Grpd($\mathscr{C}$) is a Birkhoff subcategory of $\mathrm{Grpd}^2(\mathscr{C})$, implies that 2-Grpd($\mathscr{C}$) is also an exact Mal'tsev category with coequalizers.

4. If $\mathscr{C}$ is a semi-abelian category, then so is $\mathrm{Grpd}^2(\mathscr{C})$ (Proposition 1.4.23). This implies that the Birkhoff subcategory 2-Grpd($\mathscr{C}$) of $\mathrm{Grpd}^2(\mathscr{C})$ is also semi-abelian.

□

Remark 2.2.3. We have seen in Proposition 1.1.18 that a regular category $\mathscr{C}$ is a Mal'tsev category if and only if, for any two equivalence relations R, S on an object X in $\mathscr{C}$, it holds that $R \circ S = S \circ R$. In [CKP93], A. Carboni, G.M. Kelly and M.C. Pedicchio define a *Goursat category* to be a regular category $\mathscr{C}$ such that, for any two equivalence relations R, S as above, it holds that $R \circ S \circ R = S \circ R \circ S$. This shows that any regular Mal'tsev category is a Goursat category, the converse is in general not true. Whereas a finitely complete category $\mathscr{C}$ is a Mal'tsev category if and only if Grpd($\mathscr{C}$) is closed under subobjects in RG($\mathscr{C}$) (Proposition 1.4.20), it is shown in [GRTN17] that a regular category $\mathscr{C}$ is a Goursat category if and only if Grpd($\mathscr{C}$) is closed under regular quotients in RG($\mathscr{C}$). This implies that Proposition 2.2.2, 1 still holds when $\mathscr{C}$ is a Goursat category with coequalizers. However, when $\mathbb{V}$ is a Goursat variety, Grpd($\mathbb{V}$) is in general not a subvariety of RG($\mathbb{V}$) since it is not closed under subobjects. Indeed, if this was the case, then $\mathbb{V}$ would be a Mal'tsev variety since any reflexive relation on X can be seen as a sub-reflexive graph of the internal groupoid with underlying reflexive graph $(X, X \times X, \pi_2, \pi_1, \Delta_X)$. Recall that a congruence modular variety is a Mal'tsev variety of and only if Grpd($\mathbb{V}$) is a subvariety of RG($\mathbb{V}$) [GR04]. Any Goursat variety is congruence modular.

Remark 2.2.4. Let us mention another way to prove that 2-Grpd($\mathscr{C}$) is a Birkhoff subcategory of $\mathrm{Grpd}^2(\mathscr{C})$ when $\mathscr{C}$ is a regular Mal'tsev category with coequalizers. Suppose that $\mathscr{D}$ is a regular Mal'tsev category with coequalizers, and that $\mathscr{X}$ is a Birkhoff subcategory of $\mathscr{D}$ with reflection $\mathsf{F} : \mathscr{D} \to \mathscr{X}$ and unit η. Let us define $\overline{\mathscr{X}}$ to be the full subcategory of Grpd($\mathscr{D}$) with objects the internal groupoids $(D_0, D_1, d, c, e, m, i)$ such that D_0 is in $\mathscr{X}$. Then it can easily be shown that $\overline{\mathscr{X}}$ is a Birkhoff subcategory of Grpd($\mathscr{D}$). The reflection of an internal groupoid in $\mathscr{D}$ is depicted in the diagram

$$\begin{tikzcd}
D_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "q"'] & D_0 \arrow[l, "e"'] \arrow[d, "\eta_{D_0}"] \\
D_1 +_{D_0} \mathsf{F}(D_0) \arrow[r, shift left=2, "\bar{d}"] \arrow[r, shift right=2, "\bar{c}"'] & \mathsf{F}(D_0), \arrow[l, "\bar{e}"']
\end{tikzcd}$$

where $(D_1 +_{D_0} \mathsf{F}(D_0), q, \bar{e})$ is the pushout of η_{D_0} along e. If we take $\mathscr{D}$ to be Grpd($\mathscr{C}$), and $\mathscr{X}$ to be the Birkhoff subcategory DGrpd of Grpd($\mathscr{C}$) whose objects are the discrete groupoids in $\mathscr{C}$, i.e., those groupoids $(C_0, C_1, d, c, e, m, i)$,

where $d : C_1 \to C_0$ is an isomorphism, and hence also all the other structural morphisms c, m, e, i, then this implies that $\overline{\mathscr{X}} \approx 2\text{-Grpd}(\mathscr{C})$ is a Birkhoff subcategory of $\text{Grpd}(\mathscr{D}) = \text{Grpd}^2(\mathscr{C})$.

2.2.1 Naturally Mal'tsev categories

In this section we shall see that, if $\mathscr{C}$ is not only a regular Mal'tsev category with coequalizers but an exact naturally Mal'tsev category with coequalizers, then the adjunction between $2\text{-Grpd}(\mathscr{C})$ and $\text{Grpd}^2(\mathscr{C})$ has a remarkable additional property.

Let $q : X \to Y$ be a regular epimorphism and $\delta : X \to S$ be a split monomorphism with retraction $f : S \to X$ in a category $\mathscr{C}$. Let us suppose that the pushout

$$\begin{array}{ccc} S & \xrightarrow{c} & C \\ {\scriptstyle\delta}\uparrow & & \uparrow{\scriptstyle\bar{\delta}} \\ X & \xrightarrow[q]{} & Y \end{array}$$

of q and δ exists. This is for example the case when $\mathscr{C}$ has kernel pairs and coequalizers as we saw at the beginning of the last section. Then $\bar{\delta}$ is also a split monomorphism with retraction $\bar{f}$ that makes the diagram

$$\begin{array}{ccc} S & \xrightarrow{c} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle\bar{f}} \\ X & \xrightarrow[q]{} & Y \end{array} \tag{2.3}$$

commute.

In the following, we will make use of the following characterization due to D. Bourn of naturally Mal'tsev categories among exact Mal'tsev categories with coequalizers.

Proposition 2.2.5. [Bou17, Proposition 7.8.1] Let $\mathscr{C}$ be an exact Mal'tsev category with coequalizers. Then the following conditions are equivalent:

1. $\mathscr{C}$ is a naturally Mal'tsev category.
2. Given a regular epimorphism $q\colon X \to Y$ and a split monomorphism $\delta\colon X \to S$ with retraction f, the induced commutative square (2.3) is a pullback.

Proposition 2.2.6. Let $\mathscr{C}$ be an exact naturally Mal'tsev category with coequalizers, and $\mathbb{C}$ be a double groupoid in $\mathscr{C}$. Then the internal functors $(\eta_{\mathbb{C}})_{1\bullet} : C_{1\bullet} \to \mathsf{F}(\mathbb{C})_{1\bullet}$ and $(\eta_{\mathbb{C}})_{0\bullet} : C_{0\bullet} \to \mathsf{F}(\mathbb{C})_{0\bullet}$ are discrete fibrations, i.e., the commutative squares in

$$\begin{array}{ccc} C_{11} & \xrightarrow{(\eta_{\mathbb{C}})_{11}} & F_{11} \\ {\scriptstyle d_{1\downarrow}}\downarrow\downarrow{\scriptstyle c_{1\downarrow}} & & {\scriptstyle\delta_{1\downarrow}}\downarrow\downarrow{\scriptstyle\gamma_{1\downarrow}} \\ C_{10} & \xrightarrow[(\eta_{\mathbb{C}})_{10}]{} & F_{00} \end{array} \qquad \begin{array}{ccc} C_{01} & \xrightarrow{(\eta_{\mathbb{C}})_{01}} & F_{01} \\ {\scriptstyle d_{0\downarrow}}\downarrow\downarrow{\scriptstyle c_{0\downarrow}} & & {\scriptstyle\delta_{0\downarrow}}\downarrow\downarrow{\scriptstyle\gamma_{0\downarrow}} \\ C_{00} & \xrightarrow[(\eta_{\mathbb{C}})_{00}]{} & F_{00} \end{array}$$

are pullbacks. Here we use the same notation as in the proof of Lemma 2.2.1, see also Proposition 2.2.2.

Proof. The construction of the reflector $F\colon \mathrm{Grpd}^2(\mathscr{C}) \to \text{2-Grpd}(\mathscr{C})$ and Proposition 2.2.5 imply directly the claim. □

Remark 2.2.7. Recall the below-depicted construction of the smallest double equivalence relation Δ_R^S on two equivalence relations R, S on an object X in an exact Mal'tsev category $\mathscr{C}$ with coequalizers, given by M.C. Pedicchio in [Ped95], that we have already seen in Section 1.4.2:

$$\begin{array}{ccccc}
\Delta_R^S & \overset{\pi_1}{\underset{\pi_2}{\leftarrow \delta_\pi \rightarrow}} & S & \xrightarrow{c} & S/R \\
{\scriptstyle p_1}\downarrow\uparrow{\scriptstyle \delta_p}\downarrow{\scriptstyle p_2} & & {\scriptstyle s_1}\downarrow\uparrow{\scriptstyle \delta_S}\downarrow{\scriptstyle s_2} & & {\scriptstyle \overline{s_1}}\downarrow\uparrow{\scriptstyle \overline{\delta_S}}\downarrow{\scriptstyle \overline{s_2}} \\
R & \underset{r_2}{\overset{r_1}{\leftarrow \delta_R \rightarrow}} & X & \xrightarrow[q]{} & X/R
\end{array}$$

Remember that $(X/R, q)$ is the coequalizer of r_1, r_2, $(S/R, c, \overline{\delta_S})$ is the pushout of δ_S, q, and $(\Delta_R^S, \pi_1, \pi_2)$ is the kernel pair of c. Thanks to Proposition 2.2.5, when $\mathscr{C}$ is an exact naturally Mal'tsev category with coequalizers, this double equivalence relation is necessarily centralizing, which means that all the downward-directed commutative squares on the left in the diagram above are pullbacks. This implies that the categorical commutator $[R, S]$ of R, S is trivial, i.e. the smallest equivalence relation on X. This shows an interesting and unexpected connection between commutator theory and the reflector F from the category of internal double groupoids to the one of 2-groupoids considered in the previous section.

2.3 Birkhoff subcategory of internal n-groupoids

In this section, we generalize Proposition 2.2.2 by showing that n-$\mathrm{Grpd}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{Grpd}^n(\mathscr{C})$ for any $2 \leq n \in \mathbb{N}$ whenever $\mathscr{C}$ is a regular Mal'tsev category with coequalizers.

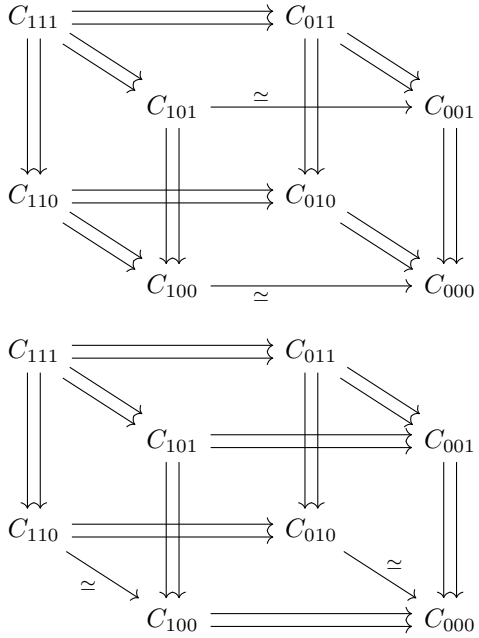
Lemma 2.3.1. Let $\mathscr{C}$ be a category. We can view the full subcategory n-$\mathrm{RG}(\mathscr{C})$ of $\mathrm{RG}^n(\mathscr{C})$ as the intersection of the full subcategories

$$\mathrm{RG}^k(\text{2-RG}(\mathrm{RG}^{n-k-2}(\mathscr{C}))),$$

where $k \in \{0, \dots, n-2\}$.

Proof. The objects of n-$\mathrm{RG}(\mathscr{C})$ are those n-fold reflexive graphs $C_{\bullet\dots\bullet}$ in $\mathscr{C}$ such that, for all $k \in \{0, \dots, n-2\}$ and $i_1, \dots, i_{n-2} \in \{0, 1\}$, the morphism $e_{i_1 \dots i_k \uparrow 0 i_{k+1} \dots i_{n-2}}$ is an isomorphism. For fixed $k \in \{0, \dots, n-2\}$, we can describe the objects of $\mathrm{RG}^k(\text{2-RG}(\mathrm{RG}^{n-k-2}(\mathscr{C})))$ as those n-fold reflexive graphs $C_{\bullet\dots\bullet}$ such that, for all $i_1, \dots, i_{n-k-2} \in \{0, 1\}$, the morphism $e_{i_1 \cdots i_k \uparrow 0 i_{k+1} \cdots i_{n-k-2}}$ is an isomorphism. This shows the claim. □

Example 2.3.2 (3-reflexive graph)**.** We can view 3-RG($\mathscr{C}$) as the intersection of the categories 2-RG(RG($\mathscr{C}$)) and RG(2-RG($\mathscr{C}$)). Below, we depict firstly an object in 2-RG(RG($\mathscr{C}$)), and secondly an object in RG(2-RG($\mathscr{C}$)):



For the object in 2-RG(RG($\mathscr{C}$)), the objects C_{101} and C_{001}, and C_{100} and C_{000} in $\mathscr{C}$ are isomorphic. For the object in RG(2-RG($\mathscr{C}$)), the objects C_{110} and C_{100}, and C_{010} and C_{000} are isomorphic.

Lemma 2.3.3. Let $\mathscr{C}$ be a regular category with coequalizers. Then n-RG($\mathscr{C}$) is a Birkhoff subcategory of RG$^n(\mathscr{C})$.

Proof. By Lemma 2.2.1, the inclusion functor $\mathsf{U} : 2\text{-RG}(\mathscr{C}) \to \text{RG}^2(\mathscr{C})$ has the left adjoint $\mathsf{F} : \text{RG}^2(\mathscr{C}) \to 2\text{-RG}(\mathscr{C})$. For $k, \ell \in \mathbb{N}_0$, we denote by $\text{RG}^k(\mathsf{F}_{\text{RG}^\ell(\mathscr{C})})$ the induced functor from $\text{RG}^{k+\ell+2}(\mathscr{C})$ to $\text{RG}^k(2\text{-RG}(\text{RG}^\ell(\mathscr{C})))$. We claim that the inclusion functor $\mathsf{U} : n\text{-RG}(\mathscr{C}) \to \text{RG}^n(\mathscr{C})$ has a left adjoint which is given by the composite

$$\tilde{\mathsf{F}} := \text{RG}^{n-2}(\mathsf{F})\text{RG}^{n-3}(\mathsf{F}_{\text{RG}(\mathscr{C})}) \cdots \text{RG}(\mathsf{F}_{\text{RG}^{n-3}(\mathscr{C})})\mathsf{F}_{\text{RG}^{n-2}(\mathscr{C})}.$$

Since 2-RG($\mathscr{C}$) is a regular epi-reflective subcategory of RG$^2(\mathscr{C})$, the image of the restriction of

$$\text{RG}(F_{\text{RG}^{n-3}(\mathscr{C})}) : \text{RG}^n(\mathscr{C}) \to \text{RG}(2\text{-RG}(\text{RG}^{n-3}(\mathscr{C})))$$

to 2-RG(RG$^{n-2}(\mathscr{C})$), which contains the image of the functor

$$\mathsf{F}_{\text{RG}^{n-2}(\mathscr{C})} : \text{RG}^n(\mathscr{C}) \to 2\text{-RG}(\text{RG}^{n-2}(\mathscr{C})),$$

lies in RG(2-RG(RG$^{n-3}(\mathscr{C})$)) and 2-RG(RG$^{n-2}(\mathscr{C})$), i.e., in 3-RG(RG$^{n-3}(\mathscr{C})$) (compare with Lemma 2.3.1). Inductively, one can show that $\tilde{\mathsf{F}}$ is a functor

from $\mathrm{RG}^n(\mathscr{C})$ to n-$\mathrm{RG}(\mathscr{C})$. Then it follows easily that n-$\mathrm{RG}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{RG}^n(\mathscr{C})$. □

Theorem 2.3.4. Let $\mathscr{C}$ be a regular Mal'tsev category with coequalizers.

1. n-$\mathrm{Grpd}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{Grpd}^n(\mathscr{C})$.
2. n-$\mathrm{Grpd}(\mathscr{C})$ is a regular Mal'tsev category with coequalizers.
3. If $\mathscr{C}$ is in addition exact, then n-$\mathrm{Grpd}(\mathscr{C})$ is also exact.
4. If $\mathscr{C}$ is a semi-abelian category, then n-$\mathrm{Grpd}(\mathscr{C})$ is also semi-abelian.

Proof. Similarly as in the proof of Proposition 2.2.2, the facts that n-$\mathrm{RG}(\mathscr{C})$ is a regular epi-reflective subcategory of $\mathrm{RG}^n(\mathscr{C})$ with left adjoint $\tilde{\mathsf{F}}$ as in the proof of Lemma 2.3.3, that $\mathrm{Grpd}(\mathscr{C})$ is closed under regular quotients in $\mathrm{RG}(\mathscr{C})$ when $\mathscr{C}$ is a regular Mal'tsev category, and that $\mathrm{Grpd}(\mathscr{C})$ is a full subcategory of $\mathrm{RG}(\mathscr{C})$ when $\mathscr{C}$ is a Mal'tsev category, imply that n-$\mathrm{Grpd}(\mathscr{C})$ is a regular epi-reflective subcategory of $\mathrm{Grpd}^n(\mathscr{C})$. It is easy to see that n-$\mathrm{Grpd}(\mathscr{C})$ is closed under regular quotients in $\mathrm{Grpd}^n(\mathscr{C})$. The remaining statements follow analogously as in the proof of Proposition 2.2.2. □

2.4 Action representability of the category of internal n-groupoids

In Examples 1.2.21, we have seen that the categories of crossed modules of groups and of Lie algebras are action representable. In [GG21], M. Gran and J.R.A. Gray prove that, under the additional assumptions that the category $\mathscr{C}$ is algebraically coherent and has normalizers, $\mathrm{Grpd}(\mathscr{C})$ is action representable whenever $\mathscr{C}$ is a semi-abelian, action representable category.

The notion of algebraically coherent category was introduced by A. Cigoli, J.R.A. Gray and T. Van der Linden in 2015 [CGVdL15].

Definition 2.4.1 (Algebraically coherent category)**.** A finitely complete category $\mathscr{C}$ is *algebraically coherent* if the change of base functors of its fibration of points $f^* : \mathrm{Pt}_Y(\mathscr{C}) \to \mathrm{Pt}_X(\mathscr{C})$, where $f : X \to Y$ is any morphism in $\mathscr{C}$, preserve finite limits and jointly strongly epimorphic pairs of morphisms.

Examples 2.4.2 (Algebraically coherent categories [CGVdL15])**.**

1. The varieties of groups, of rings, of associative and of Lie algebras are algebraically coherent.
2. The category of cocommutative Hopf algebras over a field is algebraically coherent.
3. Any finitely complete naturally Mal'tsev category is algebraically coherent. In particular, any abelian category is algebraically coherent.

The variety of Heyting semi-lattices is not algebraically coherent.

Normalizers and their relation to action representability were studied by J.R.A. Gray in 2014 [Gra14].

Definition 2.4.3 (Normalizers)**.** Let $\mathscr{C}$ be a pointed category.

1. The *normalizer* of a morphism f in $\mathscr{C}$ is the terminal object, if it exists, in the category of factorizations of f by a normal monomorphism followed by a monomorphism.
2. One says that $\mathscr{C}$ *has normalizers* if any monomorphism in $\mathscr{C}$ has a normalizer.

The categories of groups, of rings, of associative algebras, of Lie algebras, of crossed modules of groups, and of cocommutative Hopf algebras over a field have normalizers.

Proposition 2.4.4. [GG21] For a category $\mathscr{C}$, the following conditions are equivalent:

1. $\mathscr{C}$ is a semi-abelian, action representable, algebraically coherent category with normalizers.
2. $\mathrm{Grpd}(\mathscr{C})$ is a semi-abelian, action representable, algebraically coherent category with normalizers.

This implies that, if $\mathscr{C}$ is a semi-abelian, action representable, algebraically coherent category with normalizers, then so is $\mathrm{Grpd}^n(\mathscr{C})$ for any $2 \leq n \in \mathbb{N}$. We will show that the same holds true for n-$\mathrm{Grpd}(\mathscr{C})$. Examples of semi-abelian, action representable, algebraically coherent categories with normalizers are given by the category of groups, of Lie algebras over a ring, and of cocommutative Hopf algebras over a field, see [GG21].

Let us give a sketch of the proof that 1 implies 2 in Proposition 2.4.4. We refer to [GG21] for more details.

Proof. Let $\mathscr{C}$ be a semi-abelian, action representable, algebraically coherent category with normalizers. The category $\mathrm{RG}(\mathscr{C})$ is equivalent to the category with objects $\mathbf{A} = (A, s, t)$, where A is an object in $\mathscr{C}$ and $s, t : A \to A$ are morphisms in $\mathscr{C}$ such that $st = t$ and $ts = s$, and expected morphisms. The category $\mathrm{Grpd}(\mathscr{C})$ is equivalent to the full subcategory of $\mathrm{RG}(\mathscr{C})$ with objects (A, s, t) such that the *Huq* commutator $[\mathrm{Ker}(s), \mathrm{Ker}(t)]$ vanishes. In [Gra14], it is shown that, when $\mathscr{C}$ is a semi-abelian category, $\mathscr{C}$ is action representable and has normalizers if and only if, any functor category $\mathscr{C}^{\mathbb{D}}$, where $\mathbb{D}$ is a finite category, is action representable. This implies that $\mathrm{RG}(\mathscr{C})$ is action representable. Let now (X, s_X, t_X) be an internal groupoid in $\mathscr{C}$, and let

$$0 \longrightarrow \mathbf{X} \xrightarrow{\mathbf{x}} \mathbf{A} \underset{\mathbf{s}}{\overset{\mathbf{f}}{\rightleftarrows}} \mathbf{B} \longrightarrow 0$$

be the generic split extension of $\mathbf{X}$ in $\mathrm{RG}(\mathscr{C})$. Then one can show that

$$\tilde{B} := ((Z_A(\mathrm{Ker}(s_X)) \wedge \mathrm{Ker}(t_B)) \vee B_0) \vee ((Z_A(\mathrm{Ker}(t_X)) \wedge \mathrm{Ker}(s_B)) \vee B_0)$$

yields the largest sub-reflexive graph of $\mathbf{B}$ such that the two commutators

$$[\mathrm{Ker}(s_{\tilde{B}}), \mathrm{Ker}(t_X)], [\mathrm{Ker}(t_{\tilde{B}}), \mathrm{Ker}(s_X)]$$

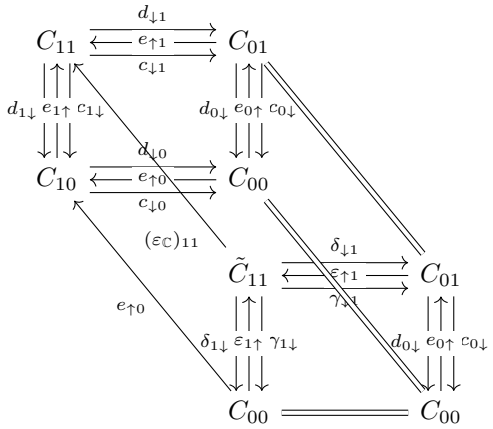
vanish. Here we denote by $Z_V(U, f)$ the centralizer of a morphism $f : U \to V$, i.e. the terminal object of the category of morphisms that commute with f in the sense of Huq, and we write $Z_V(U)$ if f is clear from the context. Note that, in our setting, centralizers of normal subobjects exist and are normal. The generic split extension of $\mathbf{X}$ in $\mathrm{Grpd}(\mathscr{C})$ can then be obtained by pulling the split extension of $\mathbf{X}$ in $\mathrm{RG}(\mathscr{C})$ back along the inclusion $\tilde{\mathbf{B}} \rightarrowtail \mathbf{B}$. □

The following is a well-known result.

Lemma 2.4.5. Let $\mathscr{C}$ be a category with finite limits. Then the inclusion $\mathsf{U} : n\text{-Grpd}(\mathscr{C}) \to \mathrm{Grpd}^n(\mathscr{C})$ has a right adjoint $\mathsf{G} : \mathrm{Grpd}^n(\mathscr{C}) \to n\text{-Grpd}(\mathscr{C})$.

Examples 2.4.6. Let $\mathscr{C}$ be a category with finite limits.

1. Let $\mathbb{C}$ be a double groupoid in $\mathscr{C}$. The coreflection $\varepsilon_{\mathbb{C}} : \mathsf{UG}(\mathbb{C}) \to \mathbb{C}$ of $\mathbb{C}$ into 2-$\mathrm{Grpd}(\mathscr{C})$ is depicted in the following diagram:



(2.4)

Here $\tilde{C}_{11}$, $(\varepsilon_{\mathbb{C}})_{11}$, $\delta_{1\downarrow}$ and $\gamma_{1\downarrow}$ are given by the pullback

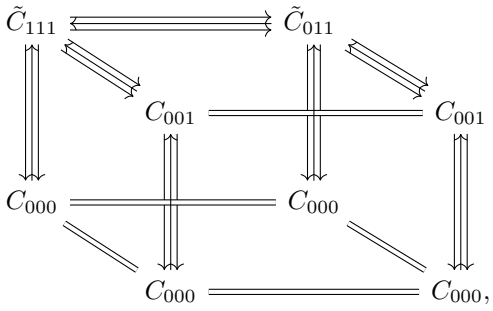
$$\begin{array}{ccc} \tilde{C}_{11} & \xrightarrow{(\varepsilon_{\mathbb{C}})_{11}} & C_{11} \\ {\scriptstyle (\delta_{1\downarrow},\gamma_{1\downarrow})}\downarrow & & \downarrow{\scriptstyle (d_{1\downarrow},c_{1\downarrow})} \\ C_{00} \times C_{00} & \xrightarrow[e_{\uparrow 0} \times e_{\uparrow 0}]{} & C_{10} \times C_{10}. \end{array}$$

The morphism $\varepsilon_{\uparrow 1}$ is the unique one such that $(\delta_{1\downarrow}, \gamma_{1\downarrow})\varepsilon_{\uparrow 1} = (d_{0\downarrow}, c_{0\downarrow})$ and $(\varepsilon_{\mathbb{C}})_{11}\varepsilon_{\uparrow 1} = e_{\uparrow 1}$.

It is easily seen that the front part of Diagram (2.4) is a double reflexive graph in $\mathscr{C}$, which is underlying a double groupoid structure. Furthermore, $\varepsilon_{\mathbb{C}}$ is a double functor, and it is not difficult to prove that it has the universal property of the counit of the adjunction.

When $\mathscr{C}$ is the category of sets, then the coreflection of a small double groupoid $\mathbb{C}$ into 2-Grpd is given by the small 2-groupoid which has the same objects as $\mathbb{C}$, as 1-morphisms all vertical morphisms of $\mathbb{C}$, and as 2-morphisms the squares of $\mathbb{C}$ whose horizontal components are identities.

2. Let $\mathbb{C}$ be a triple groupoid in $\mathscr{C}$. Then the 3-groupoid associated to it by $\mathsf{G} : \mathrm{Grpd}^3(\mathscr{C}) \to \text{3-Grpd}(\mathscr{C})$ as in Lemma 2.4.5 is given by



where $\tilde{C}_{011}$ and $\tilde{C}_{111}$ are given by the following two pullbacks:

$$\begin{array}{ccc} \tilde{C}_{011} & \xrightarrow{\varepsilon_{001}} & C_{011} \\ \downarrow & & \downarrow{\scriptstyle (d_{01\downarrow},c_{01\downarrow})} \\ C_{000}\times C_{000} & \xrightarrow[e_{0\uparrow 0}\times e_{0\uparrow 0}]{} & C_{010}\times C_{010} \end{array}$$

$$\begin{array}{ccc} \tilde{C}_{111} & \longrightarrow & C_{111} \\ \downarrow & & \downarrow{\scriptstyle (d_{11\downarrow},c_{11\downarrow},d_{1\downarrow 1},c_{1\downarrow 1},d_{\downarrow 11},c_{\downarrow 11})} \\ C^2_{000}\times C^2_{001}\times \tilde{C}^2_{011} & \xrightarrow[(e_{1\uparrow 0}e_{\uparrow 00})^2\times(e_{\uparrow 01})^2\times(\varepsilon_{001})^2]{} & C^2_{110}\times C^2_{101}\times C^2_{011} \end{array}$$

Lemma 2.4.7. [GG21] Suppose that an adjunction between two semi-abelian categories $\mathscr{C}, \mathscr{F}$ with fully faithful, protadditive left adjoint $\mathsf{U} : \mathscr{F} \to \mathscr{C}$ and right adjoint $\mathsf{G} : \mathscr{C} \to \mathscr{F}$ is given. Then, if $\mathscr{C}$ is an action representable category, then $\mathscr{F}$ is also action representable.

Proof. Let X be an object in $\mathscr{F}$ and suppose that

$$\mathsf{U}(X) \xrightarrow{\kappa} [\mathsf{U}(X)] \ltimes \mathsf{U}(X) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} [\mathsf{U}(X)]$$

is the generic split extension of $\mathsf{U}(X)$ in $\mathscr{C}$. Let η denote the unit of the adjunction $\mathsf{U} \dashv \mathsf{G}$. Then, since G is a right adjoint and U is fully faithful,

$$X \xrightarrow{\mathsf{G}(\kappa)\eta_X} \mathsf{G}([\mathsf{U}(X)] \ltimes \mathsf{U}(X)) \underset{\mathsf{G}(\sigma)}{\overset{\mathsf{G}(\pi)}{\rightleftarrows}} G([\mathsf{U}(X)])$$

is a split short exact sequence in $\mathscr{F}$. Furthermore, since U is protoadditive, the U-image of any split short exact sequence in $\mathscr{F}$ is a split short exact sequence

in $\mathscr{C}$. The claim follows by Proposition 1.2.22 and the observation that the adjunction $\mathsf{U} \dashv \mathsf{G}$ yields a bijection between morphisms of split extensions of the form

$$\begin{array}{ccccc} \mathsf{U}(X) & \xrightarrow{\mathsf{U}(k)} & \mathsf{U}(A) & \underset{\mathsf{U}(s)}{\overset{\mathsf{U}(p)}{\rightleftarrows}} & \mathsf{U}(B) \\ \| & & \vdots & & \vdots \\ \mathsf{U}(X) & \xrightarrow{\kappa} & [\mathsf{U}(X)] \ltimes \mathsf{U}(X) & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & [\mathsf{U}(X)] \end{array}$$

in $\mathscr{F}$ and of split extensions of the form

$$\begin{array}{ccccc} X & \xrightarrow{k} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ \| & & \vdots & & \vdots \\ X & \xrightarrow{\mathsf{G}\eta_X} & \mathsf{G}([\mathsf{U}(X)] \ltimes \mathsf{U}(X)) & \underset{\mathsf{G}(\sigma)}{\overset{\mathsf{G}(\pi)}{\rightleftarrows}} & G([\mathsf{U}(X)]) \end{array}$$

in $\mathscr{C}$. □

Theorem 2.4.8. Let $\mathscr{C}$ be a semi-abelian, action representable, algebraically coherent category with normalizers, and $2 \leq n \in \mathbb{N}$. Then n-Grpd($\mathscr{C}$) is also a semi-abelian, action representable, algebraically coherent category with normalizers.

Proof. Proposition 2.4.4 implies that $\mathrm{Grpd}^n(\mathscr{C})$ is a semi-abelian, action representable, algebraically coherent category with normalizers. The fact that n-Grpd($\mathscr{C}$) is a Birkhoff subcategory of $\mathrm{Grpd}^n(\mathscr{C})$ implies that it is also a semi-abelian algebraically coherent category with normalizers, see Proposition 1.3.17 and [CGVdL15, 3.7. Proposition]. Furthermore, Lemma 2.4.5 shows that the inclusion functor $\mathsf{U} : n\text{-Grpd}(\mathscr{C}) \to \mathrm{Grpd}^n(\mathscr{C})$ is a fully faithful left adjoint that is protoadditive. Lemma 2.4.7 concludes the proof. □

Chapter 3

Galois theory and homology of quasi-abelian functor categories

In this chapter, we consider the functor category $\mathscr{A}^{\mathbb{T}}$, where $\mathbb{T}$ is a finite category and $\mathscr{A}$ is a quasi-abelian category. We show that, for any replete full subcategory $\mathbb{S}$ of $\mathbb{T}$, there is a torsion theory $(\mathscr{T}, \mathscr{F})$ in $\mathscr{A}^{\mathbb{T}}$, that yields an admissible Galois structure Γ on $\mathscr{A}^{\mathbb{T}}$. We characterize the (higher) central extensions in $\mathscr{A}^{\mathbb{T}}$ with respect to Γ, and compute generalized Hopf formulae for homology. The results of this chapter are published in [Egn25].

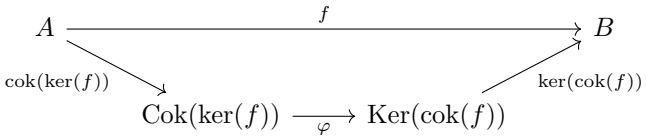
3.1 Quasi-abelian categories

Let us start by recalling the notion of quasi-abelian category and some of its properties. By the so-called Tierney's equation, a category is abelian if and only if it is (Barr-)exact and additive. The notion of quasi-abelian category is a regular additive but not necessarily exact generalization of the notion of abelian category. The history of the notion of quasi-abelian category is somewhat involved and we refer the reader to [Rum08, 2. Historical remark] for details. There W. Rump writes in particular "It seems that the term 'quasi-abelian' goes back to Yoneda's [...] paper [Yon60] on Ext^n."

Definition 3.1.1 (Quasi-abelian category)**.** A category is called *quasi-abelian* if it is additive, has kernels and cokernels, and pullback-stable normal epimorphisms and pushout stable normal monomorphisms.

Proposition 3.1.2. Let $\mathscr{A}$ be a quasi-abelian category and $f : A \to B$ be a morphism in $\mathscr{A}$. Then the unique morphism $\varphi : \mathrm{Cok}(\ker(f)) \to \mathrm{Ker}(\mathrm{cok}(f))$ from the cokernel of the kernel of f to the kernel of the cokernel of f making

the diagram



commute is a bimorphism, i.e. an epimorphism and a monomorphism but not necessarily an isomorphism.

Proof. In [Rum01, Proposition 1], W. Rump shows that, if $\mathscr{A}$ is an additive category with kernels and cokernels, for any f, φ is a monomorphism if and only if the pullback of any normal epimorphism is an epimorphism. Analogously, for any f, φ is an epimorphism if and only if the pushout of any normal monomorphism is a monomorphism. □

Proposition 3.1.3. [GNN13] Any quasi-abelian category $\mathscr{A}$ is descent-exact, i.e., the effective descent morphisms in $\mathscr{A}$ are exactly its regular epimorphisms.

More generally, M. Gran and O. Ngaha Ngaha show in [GNN13] that any normal category in the sense of Z. Janelidze [Jan10], i.e. a pointed regular category such that any regular epimorphism is a normal epimorphism, with (epimorphism, normal monomonorphism)-factorizations is descent-exact.

Proposition 3.1.4. For a category $\mathscr{A}$, the following conditions are equivalent:

1. $\mathscr{A}$ is quasi-abelian.
2. The opposite category $\mathscr{A}^{\mathrm{op}}$ of $\mathscr{A}$ is quasi-abelian.
3. $\mathscr{A}$ is homological and co-homological, i.e., its opposite category is also homological.
4. $\mathscr{A}$ is the torsion subcategory of an abelian category.
5. $\mathscr{A}$ is the torsion-free subcategory of an abelian category.

Proof. As reported in [RT07, 5.3 Remarks], G. Janelidze observed the equivalence of the statements 1 and 3. The equivalence of the assertions 1, 4 and 5 can be found in [Rum01, §4. Corollary]. □

Many examples of quasi-abelian categories arise in topological algebra and in functional analysis. There they provide a good framework for non-abelian homological algebra, K-theory and cohomological theory of sheaves [Sch99].

Examples 3.1.5 (Quasi-abelian categories)**.**

1. Any abelian category is quasi-abelian.
2. It follows from Proposition 3.1.4 that the category of torsion-free abelian groups is quasi-abelian. Note that the category of torsion abelian groups is abelian.

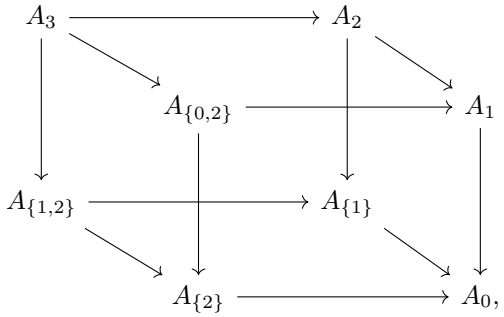
3. The categories of topological abelian groups and of locally compact abelian groups are quasi-abelian. Note that the category of compact Hausdorff abelian groups is abelian. The categories of topological vector spaces, of normed spaces, of Banach spaces, of locally convex spaces and of Fréchet spaces are also quasi-abelian.

Remark 3.1.6 (Semi-abelian categories)**.** On the one hand, the term 'semi-abelian category' is used in the sense of G. Janelidze, L. Márki and W. Tholen [JMT02] as an exact but not necessarily additive generalization of the notion of abelian category. On the other hand, it is used in the literature [Gru66, Rum01] for a slightly weaker concept than that of quasi-abelian category, namely for additive categories with kernels and cokernels such that the morphism φ as in Proposition 3.1.2 is a bimorphism.

3.2 Torsion theory $(\mathscr{T}, \mathscr{F})$ in $\mathscr{A}^{\mathbb{T}}$

In this section, we consider the functor category $\mathscr{A}^{\mathbb{T}}$, where $\mathbb{T}$ is a finite category and $\mathscr{A}$ is a quasi-abelian category. We show that, for any replete full subcategory $\mathbb{S}$ of $\mathbb{T}$, there is a torsion theory $(\mathscr{T}, \mathscr{F})$ in $\mathscr{A}^{\mathbb{T}}$. The full subcategory $\mathscr{F}$ of $\mathscr{A}^{\mathbb{T}}$ has as objects the functors $\mathsf{F} : \mathbb{T} \to \mathscr{A}$ such that $\mathsf{F}(T) = 0$ for all $T \notin \mathbb{S}$. The generality of $\mathscr{A}^{\mathbb{T}}$ and $\mathscr{F}$ allows us to treat many examples simultaneously.

Example 3.2.1. We can choose $\mathbb{T}$ such that $\mathscr{A}^{\mathbb{T}}$ is equivalent to the n-fold arrow category $\mathrm{Arr}^n(\mathscr{A})$ of $\mathscr{A}$, and $\mathbb{S}$ such that $\mathscr{F}$ is equivalent to the category $n\text{-}\mathrm{Arr}(\mathscr{A})$ of internal n-arrows in $\mathscr{A}$ as defined in Definition 2.1.7. Namely, we can set $\mathbb{T}$ to be $\mathcal{P}(n)^{\mathrm{op}}$, where $\mathcal{P}(n)$ denotes the partial order given by the power set of n viewed as a category. Here we consider the natural numbers by their standard (von Neumann) construction, i.e., we put $0 := \emptyset$ and $n := \{0, \dots, n-1\}$ for $n \geq 1$. As $\mathbb{S}$, we may choose the full subcategory of $\mathcal{P}(n)^{\mathrm{op}}$ whose objects are given by $0, \dots, n$. For $n = 3$, an object in $\mathrm{Arr}^3(\mathscr{A})$ can be depicted as



and an object in $3\text{-}\mathrm{Arr}(\mathscr{A})$ as

$$A_3 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0.$$

We recall from Propositions 1.4.34 and 2.1.8 that $\mathrm{Arr}^n(\mathscr{A})$ and $n\text{-}\mathrm{Arr}(\mathscr{A})$ are equivalent to the category $\mathrm{Grpd}^n(\mathscr{A})$ of internal n-fold groupoids and to the

category n-Grpd($\mathscr{A}$) of internal n-groupoids in $\mathscr{A}$, respectively, whenever $\mathscr{A}$ is additive and has kernels.

Given an object $T \in \mathbb{T}$, we denote by $[T]$ the finite set of morphisms $t : T' \to T$ with $T' \notin \mathbb{S}$, and set $d(t) := T'$ for such a morphism. For a functor $F \in \mathscr{A}^{\mathbb{T}}$, we denote by

$$[F(t)]_{t\in[T]} : \bigoplus_{t\in[T]} F(d(t)) \to F(T)$$

the unique morphism such that

$$[F(t)]_{t\in[T]}\iota_{t'} = F(t'),$$

where $\iota_{t'} : F(d(t')) \to \bigoplus_{t\in[T]} F(d(t))$ is the corresponding coproduct injection for $t' \in [T]$. We now show that $\mathscr{F}$ is the torsion-free subcategory of the torsion theory $(\mathscr{T}, \mathscr{F})$ in $\mathscr{A}^{\mathbb{T}}$, where $\mathscr{T}$ is the full subcategory of $\mathscr{A}^{\mathbb{T}}$ whose objects are given by the functors $F \in \mathscr{A}^{\mathbb{T}}$ with $[F(t)]_{t\in[T]}$ being an epimorphism for all $T \in \mathbb{T}$.

Proposition 3.2.2. $(\mathscr{T}, \mathscr{F})$ is a torsion theory in $\mathscr{A}^{\mathbb{T}}$.

Proof. Firstly, let $\alpha : F \to G$ be a morphism in $\mathscr{A}^{\mathbb{T}}$ from an object $F \in \mathscr{T}$ to an object $G \in \mathscr{F}$. We show that $\alpha = 0$. Let T be an object in $\mathbb{T}$. We consider the commutative square

$$\begin{array}{ccc} \bigoplus_{t\in[T]} F(d(t)) & \xrightarrow{[F(t)]_{t\in[T]}} & F(T) \\ {\scriptstyle \bigoplus_{t\in[T]} \alpha_{d(t)}}\downarrow & & \downarrow{\scriptstyle \alpha_T} \\ \bigoplus_{t\in[T]} G(d(t)) & \xrightarrow[[G(t)]_{t\in[T]}]{} & G(T). \end{array}$$

By assumption, $[F(t)]_{t\in[T]}$ is an epimorphism and $G(d(t)) = 0$ for all $t \in [T]$. Thus, $\alpha_T = 0$.

Secondly, let $F \in \mathscr{A}^{\mathbb{T}}$ be arbitrary. We define a short exact sequence

$$0 \longrightarrow \mathsf{T}(F) \xrightarrow{\varepsilon_F} F \xrightarrow{\eta_F} \mathsf{F}(F) \longrightarrow 0 \qquad (*)$$

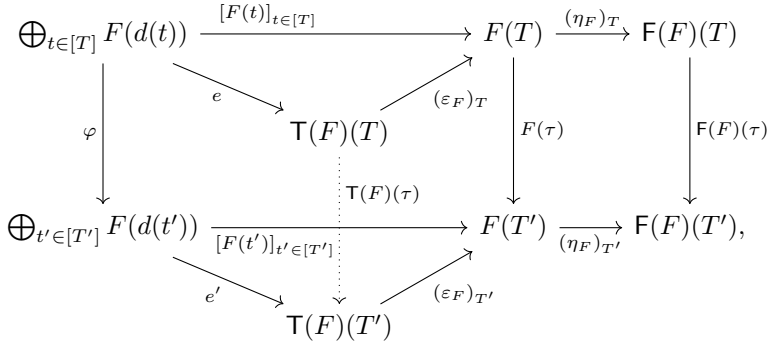
in $\mathscr{A}^{\mathbb{T}}$, where $\mathsf{T}(F) \in \mathscr{T}$ and $\mathsf{F}(F) \in \mathscr{F}$. For $T \in \mathbb{T}$, we set $\mathsf{F}(F)(T) := \mathrm{Cok}([F(t)]_{t\in[T]})$ and $(\eta_F)_T := \mathrm{cok}([F(t)]_{t\in[T]})$. For a morphism $\tau : T \to T'$ in $\mathbb{T}$, we define $\mathsf{F}(F)(\tau)$ to be the unique morphism such that the right-hand side of the diagram

$$\begin{array}{ccccc} \bigoplus_{t\in[T]} F(d(t)) & \xrightarrow{[F(t)]_{t\in[T]}} & F(T) & \xrightarrow{(\eta_F)_T} & \mathsf{F}(F)(T) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle F(\tau)} & & \vdots{\scriptstyle \mathsf{F}(F)(\tau)} \\ \bigoplus_{t'\in[T']} F(d(t')) & \xrightarrow[[F(t')]_{t'\in[T']}]{} & F(T') & \xrightarrow[(\eta_F)_{T'}]{} & \mathsf{F}(F)(T') \end{array}$$

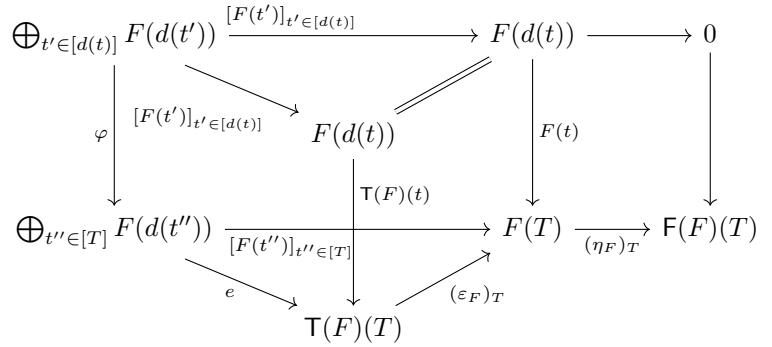
commutes, where φ is the unique morphism such that $\varphi\iota_t = \iota_{\tau t}$ for any $t \in [T]$. It is immediate to see that $\mathsf{F}(F)$ is a functor from $\mathbb{T}$ to $\mathscr{A}$ and that η_F is a

natural transformation from F to $\mathsf{F}(F)$. Furthermore, for $T \notin \mathbb{S}$, $\mathsf{F}(F)(T) = 0$ since in this case $1_T \in [T]$. Thus, $\mathsf{F}(F) \in \mathscr{F}$.

We set $\mathsf{T}(F)(T) := \mathrm{Ker}((\eta_F)_T)$ for any $T \in \mathbb{T}$ and $(\varepsilon_F)_T := \ker((\eta_F)_T)$. For $\tau : T \to T'$ in $\mathbb{T}$, we define $\mathsf{T}(F)(\tau)$ to be the unique morphism such that the front right-hand side of the diagram



commutes, i.e., $(\varepsilon_F)_{T'}\mathsf{T}(F)(\tau) = F(\tau)(\varepsilon_F)_T$, where e and e' are the unique morphisms that make the upper and lower triangles commute, respectively. It is clear that $\mathsf{T}(F)$ is a functor from $\mathbb{T}$ to $\mathscr{A}$ and that ε_F is a natural transformation from $\mathsf{T}(F)$ to F. Let $T \in \mathbb{T}$ and $t \in [T]$. For $\tau = t$, the above diagram looks as follows:



We compute

$$(\varepsilon_F)_T e\iota_t = [F(t'')]_{t''\in[T]}\iota_t = F(t).$$

Hence $\mathsf{T}(F)(t) = e\iota_t$ and $[\mathsf{T}(F)(t)]_{t\in[T]} = e$. Thus, $\mathsf{T}(F) \in \mathscr{T}$ since e is an epimorphism due to Proposition 3.1.2. The exactness of the short sequence $(*)$ is then clear. □

We note that the properties of a quasi-abelian category that we used in the proof of Proposition 3.2.2 were only the existence of finite coproducts, kernels and cokernels, and the fact that the induced morphism from the domain of a morphism to the kernel of its cokernel is an epimorphism.

Corollary 3.2.3. $\mathscr{F}$ is a normal epi-reflective subcategory of $\mathscr{A}^{\mathbb{T}}$ with semi-left exact reflector $\mathsf{F} : \mathscr{A}^{\mathbb{T}} \to \mathscr{F}$. For $F \in \mathscr{A}^{\mathbb{T}}$ and $T \in \mathbb{T}$, we have that

$\mathsf{F}(F)(T) = \mathrm{Cok}([F(t)]_{t\in[T]})$, and the corresponding unit η is given by $(\eta_F)_T = \mathrm{cok}([F(t)]_{t\in[T]})$.

Proof. This follows directly from Propositions 1.3.21 and 3.2.2. □

Remark 3.2.4. The full subcategory $\mathscr{F}$ is a normal epi-reflective subcategory of $\mathscr{A}^{\mathbb{T}}$ as soon as $\mathbb{T}$ is a finite category and $\mathscr{A}$ is a pointed category with finite coproducts and cokernels. Moreover, $\mathsf{F}(F)$ is given by a pointwise left Kan extension as we explain now. Let us denote by $\mathbb{T}^*$ the category with set of objects $\mathrm{Ob}(\mathbb{T}^*) := \mathrm{Ob}(\mathbb{T})\dot\cup\{0\}$ and sets of morphisms

$$\mathrm{Hom}_{\mathbb{T}^*}(T,T') := \begin{cases} \mathrm{Hom}(T,T')\dot\cup\{0\} & \text{if } T,T'\in\mathbb{T}, \\ \{0\} & \text{else,} \end{cases}$$

and analogously for $\mathbb{S}^*$. Let us denote by $\mathbb{W}$ the category with set of objects $\mathrm{Ob}(\mathbb{W}) := \mathrm{Ob}(\mathbb{S}^*) = \mathrm{Ob}(\mathbb{S})\dot\cup\{0\}$ and sets of morphisms

$$\mathrm{Hom}_{\mathbb{W}}(S,S') := \mathrm{Hom}_{\mathbb{S}^*}(S,S')/_\sim,$$

where we identify a morphism $s : S \to S'$ with 0 if it factors in $\mathbb{T}$ through an object which is not in $\mathbb{S}$. This yields a functor $\pi : \mathbb{T}^* \to \mathbb{W}$ with

$$\pi(T) := \begin{Bmatrix} T & \text{if } T\in\mathbb{S}, \\ 0 & \text{else} \end{Bmatrix}, \quad \pi(t) := \begin{Bmatrix} [t] & \text{if } t\in\mathbb{S}, \\ 0 & \text{else} \end{Bmatrix}.$$

Let us fix a zero object $0_{\mathscr{A}}$ in $\mathscr{A}$. Then we get the diagram

$$\begin{array}{ccc} \mathscr{A}^{\mathbb{T}} & \xleftarrow{\quad\mathsf{U}\quad} & \mathscr{F} \\ {\scriptstyle\mathsf{L}}\downarrow\uparrow{\scriptstyle\mathsf{R}} & & {\scriptstyle\overline{\mathsf{L}}}\downarrow\uparrow{\scriptstyle\overline{\mathsf{R}}} \\ \mathrm{Pointed}[\mathbb{T}^*,\mathscr{A}] & \xleftarrow[\pi^*]{} & \mathrm{Pointed}[\mathbb{W},\mathscr{A}], \end{array}$$

where $\mathrm{Pointed}[\mathbb{T}^*,\mathscr{A}]$ denotes the full subcategory of $\mathscr{A}^{\mathbb{T}^*}$ whose objects are the pointed functors, and analogously for $\mathrm{Pointed}[\mathbb{W},\mathscr{A}]$. Furthermore, π^* is the functor induced by precomposition with π, and $\mathsf{L},\mathsf{R},\overline{\mathsf{L}},\overline{\mathsf{R}}$ are the functors such that

$$\mathsf{L}(F)(T) := \begin{Bmatrix} F(T) & \text{if } T\in\mathbb{T}, \\ 0_{\mathscr{A}} & \text{if } T=0 \end{Bmatrix}, \quad \mathsf{R}(F) := F|_{\mathbb{T}},$$

$$\overline{\mathsf{L}}(F)(S) := \begin{Bmatrix} F(S) & \text{if } S\in\mathbb{S}, \\ 0_{\mathscr{A}} & \text{if } S=0 \end{Bmatrix}, \quad \overline{\mathsf{R}}(F)(T) := \begin{Bmatrix} F(T) & \text{if } T\in\mathbb{S}, \\ 0_{\mathscr{A}} & \text{if } T\notin\mathbb{S} \end{Bmatrix},$$

and $\mathsf{U}\overline{\mathsf{R}} \simeq \mathsf{R}\pi^*$ and $\mathsf{L}\mathsf{U} \simeq \pi^*\overline{\mathsf{L}}$. It is well-known [ML71] that, if $\mathscr{A}$ is finitely cocomplete, π^* has a left adjoint $\mathrm{Lan}_\pi : \mathrm{Pointed}[\mathbb{T}^*,\mathscr{A}] \to \mathrm{Pointed}[\mathbb{W},\mathscr{A}]$ for which

$$(\mathrm{Lan}_\pi F)(S) = \operatorname*{colim}_{(T,f)\in\pi\downarrow S}(F\circ\varphi_S)$$

for any $F \in \mathrm{Pointed}[\mathbb{T}^*,\mathscr{A}]$ and $S \in \mathbb{W}$, where $\pi\downarrow S$ is the category whose objects are the morphisms $f : \pi(T) \to S$ in $\mathbb{W}$, where $T \in \mathbb{T}^*$, and morphisms

from (T, f) to (T', f') are the morphisms $t : T \to T'$ in $\mathbb{T}^*$ such that $\pi(t)f' = f$, and $\varphi_S : \pi \downarrow S \to \mathbb{T}^*$ is the expected forgetful functor. Since $\mathscr{A}^{\mathbb{T}}$ and Pointed$[\mathbb{T}^*, \mathscr{A}]$, and $\mathscr{F}$ and Pointed$[\mathbb{W}, \mathscr{A}]$ are equivalent, respectively, U has $\mathsf{R} \circ \mathrm{Lan}_\pi \circ \bar{\mathsf{L}}$ as left adjoint. It is easily shown that, for any $S \in \mathbb{S}$,

$$\operatorname*{colim}_{(T,f)\in\pi\downarrow S}(F \circ \varphi_S) = \mathrm{Cok}([F(t)]_{t\in[S]}).$$

Corollary 3.2.5. The torsion theory $(\mathscr{T}, \mathscr{F})$ in $\mathscr{A}^{\mathbb{T}}$ is $\mathscr{M}$-hereditary, where $\mathscr{M}$ is the class of protosplit monomorphisms in $\mathscr{A}^{\mathbb{T}}$. This means that, if $\alpha : F \to G$ is the kernel of a split epimorphism in $\mathscr{A}^{\mathbb{T}}$ and $G \in \mathscr{T}$, then F is also in $\mathscr{T}$.

Proof. In [EG14, Theorem 2.6], T. Everaert and M. Gran show that, if $(\mathscr{T}, \mathscr{F})$ is a torsion theory in a homological category $\mathscr{C}$, then the reflector $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ is protoadditive if and only if $\mathscr{T}$ is closed under protosplit subobjects. In our case, $\mathsf{F} : \mathscr{A}^{\mathbb{T}} \to \mathscr{F}$ is protoadditive being a left adjoint between additive categories. More precisely, in the additive context, any split short exact sequence is isomorphic to a split short exact sequence of the following form:

$$A \xrightarrow{\iota_1} A \oplus B \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} B$$

Here $A \oplus B$ denotes the biproduct of A, B, and ι_1, ι_2 and π_2 the corresponding biproduct injections and projections, respectively. As a left adjoint, F preserves coproducts, which coincide with biproducts in the additive context. Hence F preserves split short exact sequences. □

We will see in Remark 3.3.7 that $\mathscr{T}$ is in general not closed under regular subobjects.

3.3 Central extensions in $\mathscr{A}^{\mathbb{T}}$ with respect to $\mathscr{F}$

The fact that $(\mathscr{T}, \mathscr{F})$ is a torsion theory in $\mathscr{A}^{\mathbb{T}}$ when $\mathscr{A}$ is quasi-abelian implies that

$$\Gamma = (\mathscr{A}^{\mathbb{T}}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathrm{RegEpi}(\mathscr{A}^{\mathbb{T}}), \mathrm{RegEpi}(\mathscr{F}))$$

is an admissible Galois structure, see Proposition 1.3.24. Recall that here $\mathsf{U} : \mathscr{F} \to \mathscr{A}^{\mathbb{T}}$ is the inclusion functor, $\mathsf{F} : \mathscr{A}^{\mathbb{T}} \to \mathscr{F}$ its left adjoint, and $\mathrm{RegEpi}(\mathscr{A}^{\mathbb{T}})$ the class of regular epimorphisms in $\mathscr{A}^{\mathbb{T}}$, and analogously for $\mathrm{RegEpi}(\mathscr{F})$. In this section, we characterize the trivial and central extensions in $\mathscr{A}^{\mathbb{T}}$ with respect to Γ (we also say with respect to $\mathscr{F}$) when $\mathscr{A}$ is abelian and quasi-abelian, respectively.

The subcategory $\mathscr{F}$ is not only torsion-free in $\mathscr{A}^{\mathbb{T}}$ but also a Birkhoff subcategory. This allows us to characterize the trivial exentensions in $\mathscr{A}^{\mathbb{T}}$ with respect to $\mathscr{F}$ when $\mathscr{A}$ is abelian.

Proposition 3.3.1. $\mathscr{F}$ is a Birkhoff subcategory of $\mathscr{A}^{\mathbb{T}}$.

Proof. We have already seen that $\mathscr{F}$ is a normal epi-reflective subcategory of the quasi-abelian category $\mathscr{A}^{\mathbb{T}}$ which implies that $\mathscr{F}$ is closed under subobjects

in $\mathscr{A}^{\mathbb{T}}$ (see also Proposition 1.3.13). Hence it remains to show that $\mathscr{F}$ is closed under regular quotients in $\mathscr{A}^{\mathbb{T}}$. For this, let $\alpha : F \to G$ be a regular epimorphism in $\mathscr{A}^{\mathbb{T}}$, and $F \in \mathscr{F}$. Thus, $\alpha_T : 0 \to G(T)$ is a regular epimorphism for all $T \notin \mathbb{S}$. Hence $G(T) = 0$ for all $T \notin \mathbb{S}$. This proves the claim. □

Proposition 3.3.2. Let $\mathscr{A}$ be abelian and $\alpha : F \to G$ be an extension in $\mathscr{A}^{\mathbb{T}}$. Then the following conditions are equivalent:

1. α is a trivial extension with respect to $\mathscr{F}$.
2. α_T and $(\eta_F)_T$ are jointly monomorphic for all $T \in \mathbb{T}$. If $T \notin \mathbb{S}$, this amounts to α_T being an isomorphism.

Proof. We have shown in Proposition 3.3.1 that $\mathscr{F}$ is a Birkhoff subcategory of $\mathscr{A}^{\mathbb{T}}$. Hence Proposition 1.3.19, Item 3 implies that α is a trivial extension if and only if α and η_F are jointly monomorphic. This is equivalent to α_T and $(\eta_F)_T$ being jointly monomorphic for all $T \in \mathbb{T}$. If $T \notin \mathbb{S}$, then $(\eta_F)_T = 0$ and α_T is a monomorphism. Since α_T is a regular epimorphism by assumption, this implies that α_T is an isomorphism. □

In the quasi-abelian setting, we are able to characterize the central extensions in $\mathscr{A}^{\mathbb{T}}$ with respect to $\mathscr{F}$ thanks to the following result due to T. Everaert and M. Gran [EG10].

Proposition 3.3.3. Let $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ be a Galois structure, where $\mathscr{C}$ is a pointed, finitely complete, protomodular category, and $\mathscr{F}$ is a replete, full, reflective subcategory of $\mathscr{C}$ with protoadditive reflection F and inclusion U. Furthermore, we assume that any morphism in $\mathscr{E}$ is a monadic extension. For an extension $f : A \to B$, the following conditions are equivalent:

1. f is a normal extension.
2. f is a central extension.
3. $\mathrm{Ker}(f)$ lies in $\mathscr{F}$.

Proof. We repeat the argument used in [EG10, 2.7. Proposition]. By definition, (1) implies (2). Let us assume that f is a central extension, i.e., there exists a monadic extension $p : E \to B$ such that the pullback p_1 of f along p is a trivial extension. This means that the naturality square on the left-hand side in the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ker}(\mathsf{UF}(p_1)) & \longleftarrow & \mathrm{Ker}(p_1) & \longrightarrow & \mathrm{Ker}(f) \\
{\scriptstyle \ker(\mathsf{UF}(p_1))}\downarrow & & {\scriptstyle \ker(p_1)}\downarrow & & \downarrow{\scriptstyle \ker(f)} \\
\mathsf{UF}(E \times_B A) & \xleftarrow{\eta_{E\times_B A}} & E \times_B A & \xrightarrow{p_2} & A \\
{\scriptstyle \mathsf{UF}(p_1)}\downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\
\mathsf{UF}(E) & \xleftarrow[\eta_E]{} & E & \xrightarrow[p]{} & B
\end{array}$$

is a pullback. Hence we have that the induced morphisms from $\mathrm{Ker}(p_1)$ to $\mathrm{Ker}(f)$, and from $\mathrm{Ker}(p_1)$ to $\mathrm{Ker}(\mathsf{UF}(p_1))$ are isomorphisms. Thus, we have

that $\mathrm{Ker}(f) \in \mathscr{F}$ since $\mathrm{Ker}(\mathsf{UF}(p_1)) = \mathsf{U}(\mathrm{Ker}(\mathsf{F}(p_1)))$. It remains to show that (3) implies (1). Let us assume that the kernel of f lies in $\mathscr{F}$. By assumption, f is a monadic extension. Hence it suffices to show that any of the pullback projections of the kernel pair of f is a trivial extension. We consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ker}(\mathsf{UF}(p_1)) & \longleftarrow & \mathrm{Ker}(p_1) & \longrightarrow & \mathrm{Ker}(f) \\
{\scriptstyle \ker(\mathsf{UF}(p_1))}\downarrow & & {\scriptstyle \ker(p_1)}\downarrow & & \downarrow{\scriptstyle \ker(f)} \\
\mathsf{UF}(A \times_B A) & \xleftarrow{\eta_{A\times_B A}} & A \times_B A & \xrightarrow{p_2} & A \\
{\scriptstyle \mathsf{UF}(p_1)}\downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\
\mathsf{UF}(A) & \xleftarrow[\eta_A]{} & A & \xrightarrow[f]{} & B.
\end{array}$$

Recall from Proposition 1.2.7 that, in a pointed, finitely complete, protomodular category, a regular epimorphism is the cokernel of its kernel. Hence, since p_1 is a split epimorphism and F is a protoadditive functor, $\ker(\mathsf{UF}(p_1))$ equals $\mathsf{UF}(\ker(p_1))$ and the induced morphism from $\mathrm{Ker}(p_1)$ to $\mathrm{Ker}(\mathsf{UF}(p_1)) = \mathsf{UF}(\mathrm{Ker}(p_1))$ is given by $\eta_{\mathrm{Ker}(p_1)}$. The assumption that $\mathrm{Ker}(f)$ lies in $\mathscr{F}$ implies that $\mathrm{Ker}(p_1)$ is also in $\mathscr{F}$ since, as before, the induced morphism from $\mathrm{Ker}(p_1)$ to $\mathrm{Ker}(f)$ is an isomorphism. Hence $\eta_{\mathrm{Ker}(p_1)}$ is an isomorphism. Since $\mathscr{C}$ is pointed, finitely complete, protomodular, this implies that the naturality square on the left-hand side is a pullback (Proposition 1.2.9). □

Proposition 3.3.4. Let $\mathscr{A}$ be a quasi-abelian category and $\alpha : F \to G$ be an extension in $\mathscr{A}^{\mathbb{T}}$. Then the following conditions are equivalent:

1. α is a normal extension with respect to $\mathscr{F}$.
2. α is a central extension with respect to $\mathscr{F}$.
3. $\mathrm{Ker}(\alpha)$ lies in $\mathscr{F}$.
4. α_T is an isomorphism for all $T \notin \mathbb{S}$.

Proof. The equivalence of 1, 2 and 3 is clear by Proposition 3.3.3. Namely, the left adjoint $\mathsf{F} : \mathscr{A}^{\mathbb{T}} \to \mathscr{F}$ is protoadditive as it is a colimit-preserving functor between two additive categories (see also the proof of Corollary 3.2.5 for more details). Combining Proposition 3.1.3 and Corollary 1.3.11, we obtain that any extension is monadic. Moreover, we have that $\mathrm{Ker}(\alpha)$ lies in $\mathscr{F}$ if and only if $\mathrm{Ker}(\alpha_T) = 0$ for all $T \notin \mathbb{S}$. Since $\mathscr{A}^{\mathbb{T}}$ is pointed, finitely complete, protomodular, this is equivalent to α_T being a monomorphism (Proposition 1.2.6). By assumption, α_T is a regular epimorphism. Thus, this is equivalent to α_T being an isomorphism for all $T \notin \mathbb{S}$. □

Alternatively, one could have used the results of M. Gran and G. Janelidze in [GJ09] to prove the above proposition. This result is also included in Proposition 3.4.12.

We illustrate Propositions 3.3.2 and 3.3.4 in two concrete situations. In particular, we give an example of a central extension which is not trivial.

Example 3.3.5. Let $\mathbb{T}$ be the 'walking arrow' category with exactly two objects and one non-trivial morphism:

$$1 \longrightarrow 0$$

Let 0 be the only object of $\mathbb{S}$. Note that 1 (0) does not mean here that 1 (0) is a terminal (initial) object. Hence $\mathscr{A}^{\mathbb{T}}$ corresponds to the arrow category $\mathrm{Arr}(\mathscr{A})$ of $\mathscr{A}$, and $\mathscr{F}$ corresponds to $\mathscr{A}$. U maps an object A of $\mathscr{A}$ to the morphism $0 \to A$, and F maps a morphism $a : A_1 \to A_0$ to $\mathrm{Cok}(a)$. If $\mathscr{A}$ is abelian, an extension (f_0, f_1) in $\mathrm{Arr}(A)$ from a to b as depicted in the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

is trivial if and only if f_1 is an isomorphism, and f_0 and $\mathrm{cok}(a)$ are jointly monomorphic. If $\mathscr{A}$ is quasi-abelian, it is central if and only if f_1 is an isomorphism. The diagram

$$\begin{array}{ccc} \mathbb{Z} & = & \mathbb{Z} \\ \| & & \downarrow \\ \mathbb{Z} & \longrightarrow & 0 \end{array}$$

in the category Ab of abelian groups depicts an extension which is central but not trivial.

Remark 3.3.6. In [Gra01], M. Gran considers the admissible Galois structure

$$\Gamma = (\mathrm{Grpd}(\mathscr{C}), \mathscr{C}, \pi_0, \mathsf{D}, \mathrm{RegEpi}(\mathrm{Grpd}(\mathscr{C})), \mathrm{RegEpi}(\mathscr{C})),$$

where $\mathscr{C}$ is an exact Mal'tsev category. Recall from Proposition 1.4.21 that this implies that $\mathrm{Grpd}(\mathscr{C})$ is also an exact Mal'tsev category. The so-called connected components functor $\pi_0 : \mathrm{Grpd}(\mathscr{C}) \to \mathscr{C}$ maps an internal groupoid with underlying reflexive graph (C_0, C_1, d, c, e) to the object part of the coequalizer of d, c. The so-called discrete functor $\mathsf{D} : \mathscr{C} \to \mathrm{Grpd}(\mathscr{C})$ maps an object C to the discrete internal groupoid with underlying reflexive graph $(C_0, C_0, 1_{C_0}, 1_{C_0}, 1_{C_0})$. By means of D, $\mathscr{C}$ can be seen as a torsion-free Birkhoff subcategory of $\mathrm{Grpd}(\mathscr{C})$, equivalent to the full subcategory $\mathrm{DGrpd}(\mathscr{C})$ of discrete internal groupoids of $\mathrm{Grpd}(\mathscr{C})$. The corresponding torsion subcategory is given by the full subcategory of $\mathrm{Grpd}(\mathscr{C})$ whose objects are the so-called connected internal groupoids, i.e., the induced morphism $(d, c) : C_1 \to C_0 \times C_0$ is a regular epimorphism. M. Gran shows, inter alia, that an extension (f_0, f_1) between two internal groupoids $\mathbb{C}, \mathbb{D}$ is central with respect to Γ if and only if it is an internal discrete fibration, meaning that either of the two commutative diagrams in

$$\begin{array}{ccc} C_1 & \xrightarrow{f_1} & D_1 \\ {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ C_0 & \xrightarrow[f_0]{} & D_0, \end{array}$$

is a pullback. If $\mathscr{C}$ is a pointed, finitely complete, protomodular category, this is the case if and only if the induced morphism between the kernels of the domain morphisms of the internal groupoids is an isomorphism.

As mentioned before, if $\mathscr{C}$ is an additive category with kernels, then the categories $\mathrm{Arr}(\mathscr{C})$ and $\mathrm{Grpd}(\mathscr{C})$ are equivalent. This implies that our framework recovers the above result in the quasi-abelian context.

In [EG10] (see also [EG14]), T. Everaert and M. Gran study the admissible Galois structure

$$\Gamma = (\mathrm{Grpd}^n(\mathscr{C}), \mathscr{C}, \pi_0 \circ \cdots \circ \pi_0^n, \mathsf{D}^n \circ \cdots \circ \mathsf{D}, \mathrm{RegEpi}(\mathrm{Grpd}^n(\mathscr{C})), \mathrm{RegEpi}(\mathscr{C})),$$

where $\mathscr{C}$ is a semi-abelian category. Recall that this implies that $\mathrm{Grpd}^n(\mathscr{C})$ is also semi-abelian. For $2 \leq i \leq n$, $\pi_0^i : \mathrm{Grpd}^i(\mathscr{C}) \to \mathrm{Grpd}^{i-1}(\mathscr{C})$ denotes the connected components functor for $\mathrm{Grpd}^i(\mathscr{C}) = \mathrm{Grpd}(\mathrm{Grpd}^{i-1}(\mathscr{C}))$, and $\mathsf{D}^i : \mathrm{Grpd}^{i-1}(\mathscr{C}) \to \mathrm{Grpd}^i(\mathscr{C})$ the discrete functor for $\mathrm{Grpd}^i(\mathscr{C})$. The authors show that π_0, and hence also $\pi_0 \circ \cdots \circ \pi_0^n$, is a protoadditive functor. This allows them to characterize the central extensions in $\mathrm{Grpd}^n(\mathscr{C})$ with respect to Γ. Moreover, they characterize the higher central extensions and compute the generalized Hopf formulae for homology (see Section 3.4.1). Recall that, if $\mathscr{C}$ is an additive category with kernels, then the categories $\mathrm{Grpd}^n(\mathscr{C})$ and $\mathrm{Arr}^n(\mathscr{C})$ are equivalent. This means that our framework recovers their results in the quasi-abelian context, see also Section 3.4.2.

Remark 3.3.7. In [GR07, Proposition 4.10], M. Gran and V. Rossi consider a *quasi-hereditary* torsion theory $(\mathscr{T}, \mathscr{F})$ in a semi-abelian category $\mathscr{C}$, meaning that $\mathscr{T}$ is closed under regular subobjects, which induces the admissible Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{U}, \mathsf{F}, \mathrm{RegEpi}(\mathscr{C}), \mathrm{RegEpi}(\mathscr{F}))$. They show that, if $\mathscr{F}$ is closed under regular quotients in $\mathscr{C}$, i.e., if $\mathscr{F}$ is a Birkhoff subcategory of $\mathscr{C}$, then any central extension in $\mathscr{C}$ with respect to $\mathscr{F}$ is trivial. The fact that we were able, in Example 3.3.5, to give an example of a central extension in $\mathscr{A}^{\mathbb{T}}$ which is not trivial shows that in our case $\mathscr{T}$ is in general not closed under regular subobjects. Recall that we showed in Corollary 3.2.5 that $\mathscr{T}$ is closed under protosplit subobjects.

Example 3.3.8. Let $\mathbb{T}$ be the category as in the diagram

$$\begin{array}{ccc} (1\,1) & \longrightarrow & (0\,1) \\ \downarrow & \searrow & \downarrow \\ (1\,0) & \longrightarrow & (0\,0), \end{array}$$

where we depicted all arrows except the identities, and let $(1\,0)$ be the only object not in $\mathbb{S}$. Then $\mathscr{A}^{\mathbb{T}}$ corresponds to the double arrow category $\mathrm{Arr}^2(\mathscr{A})$ of $\mathscr{A}$, $\mathscr{F}$ corresponds to the category 2-$\mathrm{Arr}(\mathscr{A})$ whose objects are given by pairs of composable morphisms $a_2 : A_2 \to A_1$ and $a_1 : A_1 \to A_0$ whose composite is 0, U maps a corresponding pair (a_2, a_1) to the commutative square

$$\begin{array}{ccc} A_2 & \xrightarrow{a_2} & A_1 \\ \downarrow & & \downarrow{\scriptstyle a_1} \\ 0 & \longrightarrow & A_0, \end{array}$$

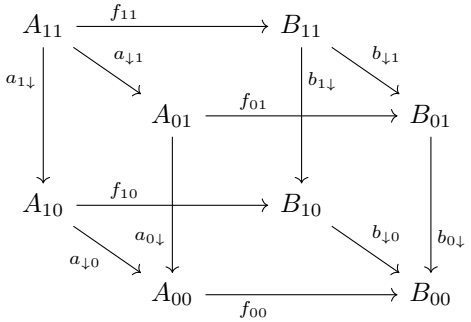
and F maps a commutative square

$$\begin{array}{ccc} A_{11} & \xrightarrow{a_{\downarrow 1}} & A_{01} \\ {\scriptstyle a_{1\downarrow}}\downarrow & & \downarrow{\scriptstyle a_{0\downarrow}} \\ A_{10} & \xrightarrow[a_{\downarrow 0}]{} & A_{00} \end{array}$$

to

$$A_{11} \xrightarrow{a_{\downarrow 1}} A_{01} \xrightarrow{\mathrm{cok}(a_{\downarrow 0})a_{0\downarrow}} \mathrm{Cok}(a_{\downarrow 0}).$$

If $\mathscr{A}$ is abelian, an extension $(f_{00}, f_{10}, f_{01}, f_{11})$ as depicted in the diagram



is trivial if and only if f_{10} is an isomorphism, and f_{00} and $\mathrm{cok}(a_{\downarrow 0})$ are jointly monomorphic. If $\mathscr{A}$ is quasi-abelian, it is central if and only if f_{10} is an isomorphism. Recall from Propositions 1.4.34 and 2.1.8 that, if $\mathscr{A}$ is additive and has kernels, $\mathrm{Arr}^2(\mathscr{A})$ is equivalent to the category $\mathrm{Grpd}^2(\mathscr{A})$ of internal double groupoids and 2-$\mathrm{Arr}(\mathscr{A})$ is equivalent to the category 2-$\mathrm{Grpd}(\mathscr{A})$ of internal 2-groupoids.

3.4 Higher central extensions and Hopf formulae in $\mathscr{A}^{\mathbb{T}}$

In this section, we characterize the higher central extensions in $\mathscr{A}^{\mathbb{T}}$ with respect to $\mathscr{F}$ and compute some generalized Hopf formulae for homology. Let us start with motivating the concepts of higher central extensions and generalized Hopf formulae for homology, and recalling the necessary theoretical background.

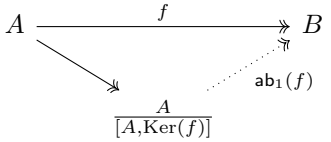
3.4.1 Higher central extensions and generalized Hopf formulae for homology

There is a remarkable link between central extensions and their higher-dimensional versions, that we will define in the following, and homological algebra.

Hopf's theorem [Hop42] states that, given a surjective group homomorphism $p : F \to G$ from a free group F to a group G, the second integral homology group of G is given by

$$H_2(G, \mathbb{Z}) \simeq \frac{[F, F] \cap \mathrm{Ker}(p)}{[\mathrm{Ker}(p), F]}.$$

Note that the abelianization functor $\mathsf{ab} : \mathrm{Grp} \to \mathrm{Ab}$ maps a group A to the quotient $A/[A, A]$ by its commutator subgroup $[A, A]$. Moreover, the category $\mathrm{CExt}(\mathrm{Grp})$ of central extensions of groups is a reflective full subcategory of the category $\mathrm{Ext}(\mathrm{Grp})$ of surjective group homomorphisms, viewed as a full subcategory of the arrow category $\mathrm{Arr}(\mathrm{Grp})$. The left adjoint $\mathsf{ab}_1 : \mathrm{Ext}(\mathrm{Grp}) \to \mathrm{CExt}(\mathrm{Grp})$ to the inclusion functor $\mathsf{U}_1 : \mathrm{CExt}(\mathrm{Grp}) \to \mathrm{Ext}(\mathrm{Grp})$ maps a surjective group homomorphism $f : A \to B$ to the central extension from $A/[A, \mathrm{Ker}(f)]$ to B induced by f:



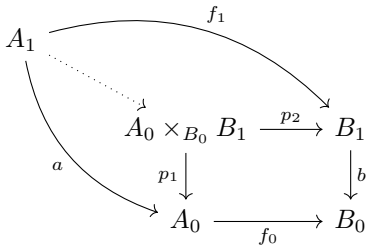
In [Jan91b], G. Janelidze observes that, starting from the Galois structure

$$\Gamma = (\mathrm{Grp}, \mathrm{Ab}, \mathsf{ab}, \mathsf{U}, \mathrm{Surj}(\mathrm{Grp}), \mathrm{Surj}(\mathrm{Ab})),$$

that we already saw in Example 1.3.7, one can construct another Galois structure

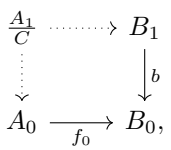
$$\Gamma_1 = (\mathrm{Ext}(\mathrm{Grp}), \mathrm{CExt}(\mathrm{Grp}), \mathsf{ab}_1, \mathsf{U}_1, \mathscr{E}^1, \mathscr{Z}^1),$$

where $\mathscr{E}^1$ is the class of so-called double extensions of groups and $\mathscr{Z}^1$ is the class of double extensions of groups in $\mathrm{CExt}(\mathrm{Grp})$. More precisely, a double extension of groups can be seen as a morphism $(f_0, f_1) : a \to b$ in $\mathrm{Ext}(\mathrm{Grp})$ as in the diagram below such that f_0, f_1 and the induced morphism from A_1 to the pullback $A_0 \times_{B_0} B_1$ of b along f_0 are all surjective.



(3.1)

A double extension $(f_0, f_1) : a \to b$ of groups is central with respect to Γ_1 if and only if both the commutator subgroups $[\mathrm{Ker}(a) \cap \mathrm{Ker}(f_1), A_1]$ and $[\mathrm{Ker}(a), \mathrm{Ker}(f_1)]$ are trivial. Moreover, the category $\mathrm{CExt}^2(\mathrm{Grp})$ of double central extensions of groups, i.e. the double extensions of groups that are central with respect to Γ_1, is a reflective full subcategory of the category $\mathrm{Ext}^2(\mathrm{Grp})$ of double extensions of groups, viewed as a full subcategory of $\mathrm{Arr}(\mathrm{Ext}(\mathrm{Grp}))$. The left adjoint $\mathsf{ab}_2 : \mathrm{Ext}^2(\mathrm{Grp}) \to \mathrm{CExt}^2(\mathrm{Grp})$ to the inclusion functor $\mathsf{U}_2 : \mathrm{CExt}^2(\mathrm{Grp}) \to \mathrm{Ext}^2(\mathrm{Grp})$ maps a double extension $(f_0, f_1) : a \to b$ of groups to the double central extension



(3.2)

induced by a and f_1, where C is the product of the groups $[\mathrm{Ker}(a)\cap\mathrm{Ker}(f_1), A_1]$ and $[\mathrm{Ker}(a), \mathrm{Ker}(f_1)]$.

Given a double free presentation of a group G, i.e. a double extension of groups

$$\begin{array}{ccc} F_2 & \xrightarrow{p_1} & F_1 \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle q} \\ F_0 & \xrightarrow[p]{} & G, \end{array}$$

where F_0, F_1 and F_2 are all free groups, the third integral homology group of G is given by

$$H_3(G,\mathbb{Z}) \simeq \frac{[F_2,F_2]\cap\mathrm{Ker}(p_0)\cap\mathrm{Ker}(p_1)}{[\mathrm{Ker}(p_0)\cap\mathrm{Ker}(p_1),F_2][\mathrm{Ker}(p_0),\mathrm{Ker}(p_1)]}.$$

In [EGVdL08], T. Everaert, M. Gran and T. Van der Linden consider the Galois structure

$$\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathrm{RegEpi}(\mathscr{C}), \mathrm{RegEpi}(\mathscr{F})),$$

where $\mathscr{C}$ is a semi-abelian category, $\mathscr{F}$ is a Birkhoff subcategory of $\mathscr{C}$ with inclusion $\mathsf{U}:\mathscr{C}\to\mathscr{F}$ and reflection $\mathscr{F}:\mathscr{C}\to\mathscr{F}$, and $\mathrm{RegEpi}(\mathscr{C})$ and $\mathrm{RegEpi}(\mathscr{F})$ are the classes of regular epimorphisms in $\mathscr{C}$ and $\mathscr{F}$, respectively. They show that such a Galois structure induces a sequence of Galois structures

$$\Gamma_n = (\mathrm{Ext}^n(\mathscr{C}), \mathrm{CExt}^n(\mathscr{C}), \mathsf{F}_n, \mathsf{U}_n, \mathscr{E}^n, \mathscr{Z}^n),$$

where $n\in\mathbb{N}$, $\mathrm{CExt}^n(\mathscr{C})$ is the reflective full subcategory of $\mathrm{Ext}^n(\mathscr{C})$ with objects the central extensions with respect to the Galois structure Γ_{n-1}, $\mathsf{U}_n : \mathrm{CExt}^n(\mathscr{C})\to\mathrm{Ext}^n(\mathscr{C})$ is the inclusion functor, and $\mathsf{F}_n:\mathrm{Ext}^n(\mathscr{C})\to\mathrm{CExt}^n(\mathscr{C})$ is its left adjoint. The central extensions with respect to Γ_{n-1} are also called n-fold central extensions with respect to Γ. Based on these, the authors define, for any $n\in\mathbb{N}_0$, the so-called $(n+1)$-st generalized Hopf formula for homology $H_{n+1}(C,\mathscr{F})$ of an object C in $\mathscr{C}$ with respect to $\mathscr{F}$ when $\mathscr{C}$ has enough regular projective objects. These coincide with the Brown-Ellis-Hopf formulae for the integral homology of groups when $\mathscr{C}=\mathrm{Grp}$ and $\mathscr{F}=\mathrm{Ab}$. More generally, the generalized Hopf formulae recover the Barr-Beck left derived functors of the left adjoint $\mathsf{F}:\mathscr{C}\to\mathscr{F}$ whenever $\mathscr{C}$ is a semi-abelian category that is monadic over the category Set of sets.

Since then, higher central extensions and generalized Hopf formulae for homology have been further studied. In particular, the contexts in which they can be defined have been extended to a more axiomatic framework. Imposing suitable conditions on a Galois structure $\Gamma=(\mathscr{C},\mathscr{F},\mathsf{F},\mathsf{U},\mathscr{E},\mathscr{Z})$, one can construct a Galois structure

$$\Gamma_1 = (\mathrm{Ext}(\mathscr{C}), \mathrm{NExt}(\mathscr{C}), \mathsf{F}_1, \mathsf{U}_1, \mathscr{E}^1, \mathscr{Z}^1)$$

that satisfies the same conditions. Here $\mathrm{Ext}(\mathscr{C})$ is the full subcategory of $\mathrm{Arr}(\mathscr{C})$ with objects the morphisms in $\mathscr{E}$, $\mathrm{NExt}(\mathscr{C})$ is the full subcategory of $\mathrm{Ext}(\mathscr{C})$ with objects the normal extensions with respect to Γ with inclusion

functor $\mathsf{U}_1 : \mathrm{NExt}(\mathscr{C}) \to \mathrm{Ext}(\mathscr{C})$ and its left adjoint $\mathsf{F}_1 : \mathrm{Ext}(\mathscr{C}) \to \mathrm{NExt}(\mathscr{C})$, $\mathscr{E}^1$ is a suitable class of morphisms in $\mathrm{Ext}(\mathscr{C})$, called double extensions, and $\mathscr{Z}^1$ is the class of double extensions in $\mathrm{NExt}(\mathscr{C})$. This allows one to define the higher Galois structures $\Gamma_n = (\mathrm{Ext}^n(\mathscr{C}), \mathrm{NExt}^n(\mathscr{C}), \mathsf{F}_n, \mathsf{U}_n, \mathscr{E}^n, \mathscr{Z}^n)$, where $n \in \mathbb{N}$, recursively. Note that, for a semi-abelian category $\mathscr{C}$ and $\mathscr{E} = \mathrm{RegEpi}(\mathscr{C})$, the category $\mathrm{Ext}(\mathscr{C})$ is not necessarily semi-abelian (it is in general not exact) and $\mathscr{E}^1$ can in general be not chosen as $\mathrm{RegEpi}(\mathrm{Ext}(\mathscr{C}))$.

The following sections are mostly based on M. Duckert-Antoine's article [DA17] since its presentation and results serve our purposes in Section 3.4.2 best, see also [EGVdL08, Eve10, EG10, EG14, Eve14].

3.4.1.1 Higher extensions

Let us first see how to define the classes $\mathscr{E}^n$ of higher extensions.

In the following, $\mathscr{E}$ will be a class of morphisms in a category $\mathscr{C}$ satisfying some, not always all of the following conditions:

(E1) $\mathscr{E}$ contains all isomorphisms in $\mathscr{C}$.

(E2) Pullbacks of morphisms in $\mathscr{E}$ exist in $\mathscr{C}$ and lie in $\mathscr{E}$.

(E3) $\mathscr{E}$ is closed under composition.

(E4) If the composite gf lies in $\mathscr{E}$, then g lies in $\mathscr{E}$.

(E5) For a commutative diagram

$$\begin{array}{ccccc} \mathrm{Ker}(a) & \xrightarrow{\ker(a)} & A_1 & \xrightarrow{a} & A_0 \\ {\scriptstyle k}\downarrow & & {\scriptstyle f}\downarrow & & \| \\ \mathrm{Ker}(b) & \xrightarrow[\ker(b)]{} & B_1 & \xrightarrow[b]{} & A_0, \end{array}$$

the condition that k and a lie in $\mathscr{E}$ implies that f lies in $\mathscr{E}$.

(M) Every morphism in $\mathscr{E}$ is a monadic extension.

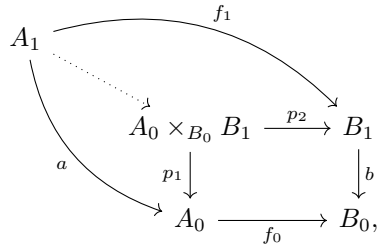
Note that the classes $\mathscr{E}, \mathscr{Z}$ in a Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ have to satisfy the conditions (E1), (E2) and (E3).

Example 3.4.1. If $\mathscr{C}$ is a descent-exact homological category, then the class $\mathrm{RegEpi}(\mathscr{C})$ of regular epimorphisms in $\mathscr{C}$ satisfies all of the above axioms.

Given a class $\mathscr{E}$ of morphisms in a category $\mathscr{C}$, we denote by $\mathrm{Ext}_{\mathscr{E}}(\mathscr{C})$ (or just $\mathrm{Ext}(\mathscr{C})$) the full subcategory of $\mathrm{Arr}(\mathscr{C})$ with objects the morphisms in $\mathscr{E}$.

Definition 3.4.2 (Double extension)**.** Let $\mathscr{C}$ be an arbitrary category and $\mathscr{E}$ be a class of morphisms in $\mathscr{C}$ satisfying (E2), i.e., all pullbacks of $\mathscr{E}$ exist in $\mathscr{C}$

and lie in $\mathscr{E}$. A *double $\mathscr{E}$-extension* (or just *double extension*) is a morphism $(f_0, f_1) : a \to b$ in $\mathrm{Ext}(\mathscr{C})$ such that all morphisms in the commutative diagram



lie in $\mathscr{E}$, i.e., not only f_0 and f_1 lie in $\mathscr{E}$ but also the induced morphism from A_1 to the pullback $A_0 \times_{B_0} B_1$ of b along f_0. The class of double extensions is denoted by $\mathscr{E}^1$.

Proposition 3.4.3. [DA17, Theorem 2.2] If $\mathscr{C}$ is a pointed protomodular category and $\mathscr{E}$ satisfies the conditions (E1) to (E5) and (M), then $(\mathrm{Ext}(\mathscr{C}), \mathscr{E}^1)$ satisfies the same conditions.

Recursively, one gets, for every $n \geq 1$, a class $\mathscr{E}^n := (\mathscr{E}^{n-1})^1$ of so-called $(n+1)$-*fold extensions*, satisfying the properties (E1) to (E5) and (M) in the pointed protomodular category $\mathrm{Ext}^n(\mathscr{C}) := \mathrm{Ext}_{\mathscr{E}^{n-1}}(\mathrm{Ext}^{n-1}(\mathscr{C}))$, where we set $\mathscr{C}^0 := \mathscr{C}$ and $\mathscr{E}^0 := \mathscr{E}$.

3.4.1.2 Higher Galois structures

We are now ready to give the definition of the higher Galois structures Γ_n.

Definition 3.4.4 (Closed Galois structure)**.** A Galois structure

$$\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$$

is called *closed* if U is fully faithful and each component η_C of the unit of the adjunction $\mathsf{F} \dashv \mathsf{U}$ lies in $\mathscr{E}$.

Note that the right adjoint U is fully faithful if and only if the counit ε of the adjunction $\mathsf{F} \dashv \mathsf{U}$ is an isomorphism. Without loss of generality, we will assume in the following that, given a closed Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$, $\mathscr{F}$ is a full subcategory of $\mathscr{C}$, U is the inclusion functor, and the counit of the adjunction is the identity natural transformation.

Proposition 3.4.5. [DA17, Theorem 2.6] Let $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ be a closed Galois structure in which $\mathscr{C}$ is pointed protomodular, $\mathscr{E}$ satisfies the conditions (E4), (E5) and (M), and the reflection $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ preserves pullbacks of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[h]{} & C, \end{array} \tag{3.3}$$

where $g \in \mathscr{E}$ and $h \in \mathrm{Split}(\mathscr{E})$. Here $\mathrm{Split}(\mathscr{E})$ denotes the class of morphisms which are split epimorphisms and lie in $\mathscr{E}$.

Then there exists a closed Galois structure

$$\Gamma_1 = (\mathrm{Ext}(\mathscr{C}), \mathrm{NExt}(\mathscr{C}), \mathsf{F}_1, \mathsf{U}_1, \mathscr{E}^1, \mathscr{Z}^1),$$

where $\mathrm{NExt}(\mathscr{C})$ is the reflective full subcategory of $\mathrm{Ext}(\mathscr{C})$ with objects the normal extensions with respect to Γ, $\mathsf{U}_1 : \mathrm{NExt}(\mathscr{C}) \to \mathrm{Ext}(\mathscr{C})$ is its inclusion functor and $\mathsf{F}_1 : \mathrm{Ext}(\mathscr{C}) \to \mathrm{NExt}(\mathscr{C})$ is its left adjoint, such that $\mathrm{Ext}(\mathscr{C})$ is pointed protomodular, $\mathscr{E}^1$ satisfies (E4), (E5) and (M), and F_1 preserves pullbacks of the form (3.3), where g lies in $\mathscr{E}^1$ and h lies in $\mathrm{Split}(\mathscr{E}^1)$.

Recursively, one gets, for every $n \geq 1$, a closed Galois structure

$$\Gamma_n = (\mathrm{Ext}^n(\mathscr{C}), \mathrm{NExt}^n(\mathscr{C}), \mathsf{F}_n, \mathsf{U}_n, \mathscr{E}^n, \mathscr{Z}^n),$$

where $\mathrm{NExt}^n(\mathscr{C})$ is the full subcategory of $\mathrm{Ext}^n(\mathscr{C})$ whose objects are given by the n-fold extensions which are normal with respect to Γ_{n-1}, where we set $\Gamma_0 := \Gamma$. We denote by η^n the unit of the adjunction in Γ_n.

Example 3.4.6. In [EGVdL08, Lemma 4.4], it is in particular shown that any semi-abelian category $\mathscr{C}$ with Birkhoff subcategory $\mathscr{F}$ yields a Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathrm{RegEpi}(\mathscr{C}), \mathrm{RegEpi}(\mathscr{F}))$ that satisfies all the conditions required above. Moreover, [EGVdL08, Proposition 4.5] shows that

$$\mathrm{NExt}^n(\mathscr{C}) = \mathrm{CExt}^n(\mathscr{C}),$$

where $\mathrm{CExt}^n(\mathscr{C})$ denotes the full subcategory of $\mathrm{Ext}^n(\mathscr{C})$ whose objects are given by the n-fold extensions which are central with respect to Γ_{n-1}, see also [Eve14].

3.4.1.3 Generalized Hopf formulae for homology

Let us finally see the definition of the generalized Hopf formulae for homology.

Let the Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ be as in Proposition 3.4.5. Given an object $A \in \mathscr{C}$, we denote by $[A]$ the kernel of $\eta_A : A \to \mathsf{UF}(A)$. For $n \geq 1$ and $A \in \mathrm{Ext}^n(\mathscr{C})$, we denote by $[A]^n$ the kernel of $\eta^n_A : A \to \mathsf{U}^n\mathsf{F}^n(A)$. This defines a functor $[-]^n : \mathrm{Ext}^n(\mathscr{C}) \to \mathrm{Ext}^n(\mathscr{C})$. Let us consider the natural numbers by their standard (von Neumann) construction, i.e., we put $0 := \emptyset$ and $n := \{0, \dots, n-1\}$ for $n \geq 1$. Let $n \geq 0$. We denote by $\mathcal{P}(n)$ the partial order given by the power set of n and view it is a category. Let us consider the functor category $\mathscr{C}^{\mathcal{P}(n)^{\mathrm{op}}}$. For an object A in $\mathscr{C}^{\mathcal{P}(n)^{\mathrm{op}}}$, we set, for any $S \subseteq T \subseteq n$, $A_S := A(S)$ and $a^T_S := A(S \subseteq T)$, which is an arrow in $\mathscr{C}$ from A_T to A_S. We define $a_i := a^n_{n\setminus\{i\}}$ for any $0 \leq i \leq n-1$. We write $(A_S)_{S\subseteq n}$ synonymously for A. For $n \geq 1$, the functor $\delta_{n-1} : \mathscr{C}^{\mathcal{P}(n)^{\mathrm{op}}} \to \mathrm{Arr}(\mathscr{C}^{\mathcal{P}(n-1)^{\mathrm{op}}})$ which maps a morphism $f : A \to B$ in $\mathscr{C}^{\mathcal{P}(n)^{\mathrm{op}}}$ to the commutative square

$$\begin{array}{ccc} (A_{S\cup\{n-1\}})_{S\subseteq n-1} & \xrightarrow{(f_{S\cup\{n-1\}})_{S\subseteq n-1}} & (B_{S\cup\{n-1\}})_{S\subseteq n-1} \\ {\scriptstyle (a_S^{S\cup\{n-1\}})_{S\subseteq n-1}}\downarrow & & \downarrow{\scriptstyle (b_S^{S\cup\{n-1\}})_{S\subseteq n-1}} \\ (A_S)_{S\subseteq n-1} & \xrightarrow{(f_S)_{S\subseteq n-1}} & (B_S)_{S\subseteq n-1} \end{array}$$

defines an isomorphism of categories. Hence the composite

$$\phi := \mathrm{Arr}^{n-1}(\delta_0) \cdots \mathrm{Arr}(\delta_{n-2})\delta_{n-1}$$

yields an isomorphism between $\mathscr{C}^{\mathcal{P}(n)^{\mathrm{op}}}$ and $\mathrm{Arr}^n(\mathscr{C})$. We set $\underline{\mathrm{Ext}}^n(\mathscr{C}) := \phi^{-1}(\mathrm{Ext}^n(\mathscr{C}))$. Even though ϕ is just one possible choice of isomorphism between $\mathscr{C}^{\mathcal{P}(n)^{\mathrm{op}}}$ and $\mathrm{Arr}^n(\mathscr{C})$, T. Everaert, J. Goedecke and T. Van der Linden show in [EGVdL12, Proposition 1.16] that $\underline{\mathrm{Ext}}^n(\mathscr{C})$ is the full subcategory of $\mathscr{C}^{\mathcal{P}(n)^{\mathrm{op}}}$ whose objects are given by the functors $\mathcal{P}(n)^{\mathrm{op}} \to \mathscr{C}$ such that the limit $\lim_{J \subsetneq I} A_J$ exists for all $0 \neq I \subseteq n$ and the induced morphism $A_I \to \lim_{J \subsetneq I} A_J$ lies in $\mathscr{E}$. In the following, we will not distinguish between $\mathrm{Ext}^n(\mathscr{C})$ and $\underline{\mathrm{Ext}}^n(\mathscr{C})$, and assume the application of ϕ implicitly. We define $\iota^n : \mathscr{C} \to \mathrm{Ext}^n(\mathscr{C})$ to be the functor which maps an object A in $\mathscr{C}$ to the functor which maps n to A, and S to 0 for any $S \subsetneq n$.

Proposition 3.4.7. [DA17, Theorem 2.15] The functor

$$[-]^n : \mathrm{Ext}^n(\mathscr{C}) \to \mathrm{Ext}^n(\mathscr{C})$$

factors through ι^n, i.e., there exists a functor $[-]_n : \mathrm{Ext}^n(\mathscr{C}) \to \mathscr{C}$ such that $[A]^n = \iota^n([A]_n)$ for all $A \in \mathrm{Ext}^n(\mathscr{C})$.

For the Galois structure $\Gamma = (\mathrm{Grp}, \mathrm{Ab}, \mathsf{ab}, \mathsf{U}, \mathrm{Surj}(\mathrm{Grp}), \mathrm{Surj}(\mathrm{Ab}))$, we have seen at the beginning of Section 3.4.1 that the functor $[-]_2 : \mathrm{Ext}^2(\mathrm{Grp}) \to \mathrm{Grp}$ maps a double extension of groups as in Diagram (3.1) to the group $C := [\mathrm{Ker}(a) \cap \mathrm{Ker}(f_1), A] \cdot [\mathrm{Ker}(a), \mathrm{Ker}(f_1)]$, see Diagram (3.2).

Definition 3.4.8 (Projective object, projective presentation)**.**

1. An object P in $\mathscr{C}$ is called *$\mathscr{E}$-projective* if in any diagram

$$\begin{array}{ccc} & & P \\ & {\scriptstyle p'} \swarrow & \downarrow {\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$$

   in $\mathscr{C}$, where f is an extension and p is an arbitrary morphism, there exists a morphism p' such that $fp' = p$. If $\mathscr{E}$ is the class of regular epimorphisms, one speaks of *regular projective objects*.

2. An *$\mathscr{E}$-projective presentation* of an object A in $\mathscr{C}$ is given by an extension $p : P \to A$, where P is an $\mathscr{E}$-projective object. One says that $\mathscr{C}$ *has enough $\mathscr{E}$-projective objects* if any object in $\mathscr{C}$ admits an $\mathscr{E}$-projective presentation.

3. An *n-fold $\mathscr{E}$-projective presentation* of an object A in $\mathscr{C}$ is given by an object P in $\mathrm{Ext}^n(\mathscr{C})$ with $P_0 = A$ and P_S an $\mathscr{E}$-projective object for all $\emptyset \neq S \subseteq n$.

Proposition 3.4.9. [DA17, Lemma 2.13] If $\mathscr{E}$ satisfies the axioms (E1), (E2) and (E3), and $\mathscr{C}$ has enough $\mathscr{E}$-projective objects, then any A in $\mathscr{C}$ admits at least one n-fold $\mathscr{E}$-projective presentation.

Proposition 3.4.10. [DA17, Theorem 3.13] The so-called *generalized Hopf formula* for the $(n+1)$-st homology of A with respect to $\mathscr{F}$

$$H_{n+1}(A,\mathscr{F}) := \frac{[P_n] \cap \bigcap_{0\leq i\leq n-1} \mathrm{Ker}(p_i)}{[P]_n},$$

where P is an n-fold $\mathscr{E}$-projective presentation of an object A in $\mathscr{C}$, does not depend on the chosen presentation. Furthermore, $H_{n+1}(A,\mathscr{F})$ lies in $\mathrm{Ab}(\mathscr{F})$, i.e. the full subcategory of $\mathscr{F}$ with objects the internal abelian groups in $\mathscr{F}$.

The following result follows from [DA17, Theorem 3.16], see also [DA17, 4.4. Torsion theories] and [EG14].

Proposition 3.4.11. Let $\Gamma = (\mathscr{C}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ be a closed Galois structure, where $\mathscr{C}$ is a pointed, finitely complete, protomodular category and $\mathscr{E}$ satisfies the conditions (E4), (E5) and (M). Furthermore, we assume that F is a protoadditive functor. Let $n \geq 1$, $A \in \mathrm{Ext}^n(\mathscr{C})$, $B \in \mathscr{C}$ and $P \in \mathrm{Ext}^n(\mathscr{C})$ be an n-fold $\mathscr{E}$-projective presentation of B.

1. The following conditions are equivalent:
   (a) $A \in \mathrm{NExt}^n(\mathscr{C})$.
   (b) $A \in \mathrm{CExt}^n(\mathscr{C})$.
   (c) $\bigcap_{0\leq i\leq n-1} \mathrm{Ker}(a_i) \in \mathscr{F}$.
2. We have that
$$[A]_n = [\bigcap_{0\leq i\leq n-1} \mathrm{Ker}(a_i)].$$
3. The Hopf formula for the $(n+1)$-st homology of B with respect to $\mathscr{F}$ is given by
$$H_{n+1}(B,\mathscr{F}) := \frac{[P_n] \cap \bigcap_{0\leq i\leq n-1} \mathrm{Ker}(p_i)}{[\bigcap_{0\leq i\leq n-1} \mathrm{Ker}(p_i)]}.$$

3.4.2 Higher central extensions and Hopf formulae in $\mathscr{A}^{\mathbb{T}}$

Having studied the central extensions in $\mathscr{A}^{\mathbb{T}}$ with respect to $\mathscr{F}$ in Section 3.3, we apply the theory recalled in the previous section to obtain a characterization of the higher central extensions in $\mathscr{A}^{\mathbb{T}}$. Indeed, $\Gamma = (\mathscr{A}^{\mathbb{T}}, \mathscr{F}, \mathsf{F}, \mathsf{U}, \mathscr{E}, \mathscr{Z})$ is a closed Galois structure, $\mathscr{A}^{\mathbb{T}}$ is pointed, finitely complete, protomodular, $\mathscr{E}$ satisfies the conditions (E4), (E5) and (M), and the reflection F is protoadditive.

Let $n \geq 1$ and $A \in \mathrm{Ext}^n(\mathscr{A}^{\mathbb{T}})$. As explained in Section 3.4.1.3, we denote, for all $0 \leq i \leq n-1$, by a_i the image of the morphism $n \setminus \{i\} \subseteq n$ seeing A as an object in $(\mathscr{A}^{\mathbb{T}})^{\mathcal{P}(n)^{\mathrm{op}}}$. In addition to Proposition 3.3.4, Proposition 3.4.11 implies the following:

Proposition 3.4.12. Let $n \geq 1$ and $\mathscr{A}$ be a quasi-abelian category. The following conditions are equivalent:

1. $A \in \mathrm{NExt}^n(\mathscr{A}^{\mathbb{T}})$.

2. $A \in \mathrm{CExt}^n(\mathscr{A}^{\mathbb{T}})$.
3. $\bigcap_{0\leq i\leq n-1} \mathrm{Ker}(a_i) \in \mathscr{F}$.
4. $\{(a_i)_T\}_{0\leq i\leq n-1}$ are jointly monomorphic for all $T \notin \mathbb{S}$.

Proof. We note that $\bigcap_{0\leq i\leq n-1} \mathrm{Ker}(a_i) \in \mathscr{F}$ is equivalent to

$$\bigcap_{0\leq i\leq n-1} \mathrm{Ker}((a_i)_T) = 0$$

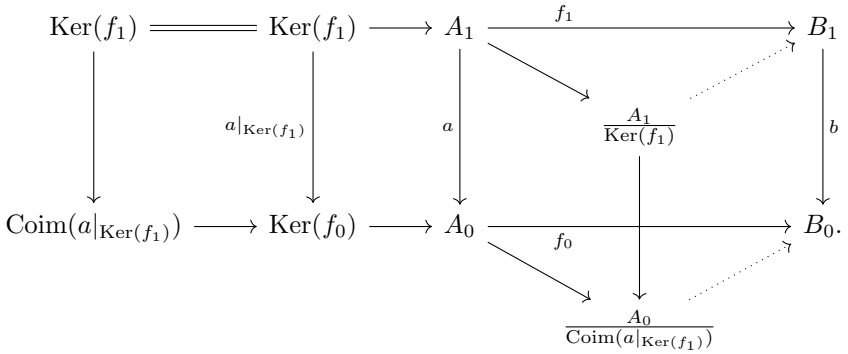
for all $T \notin \mathbb{S}$. This means that $\{(a_i)_T\}_{0\leq i\leq n-1}$ are jointly monomorphic for all $T \notin \mathbb{S}$. □

We continue the study of the Examples 3.3.5 and 3.3.8.

Example 3.4.13. As in Example 3.3.5, let us consider $\mathbb{T}$ to be the walking arrow category $1 \to 0$, and $\mathbb{S}$ to be the full subcategory of $\mathbb{T}$ which contains only 0 as an object. We saw that an extension (f_0, f_1) in $\mathrm{Arr}(\mathscr{A})$ from a to b as depicted in the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

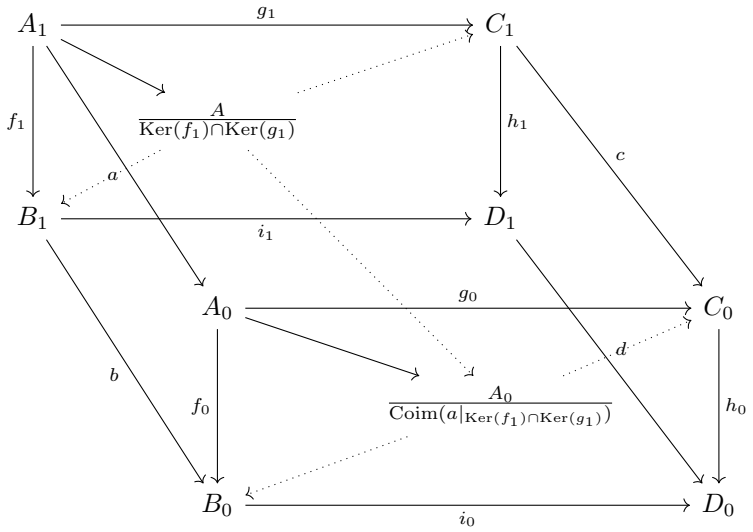
is central if and only if f_1 is an isomorphism. We note that, for an object $a : A_1 \to A_0$ of $\mathrm{Arr}(\mathscr{A})$, $[a]$ is given by the epimorphism part $\varphi \circ \mathrm{cok}(\mathrm{ker}(a)) : A_1 \to \mathrm{Ker}(\mathrm{cok}(a))$ of the (epimorphism, normal monomorphism)-factorization of a (see Proposition 3.1.2). We set $\mathrm{Coim}(a) := \mathrm{Ker}(\mathrm{cok}(a))$. The centralization of the extension (f_0, f_1) is given by the induced morphism in the commutative diagram



If (f_0, f_1) yields a regular projective presentation for (B_0, B_1), then the Hopf formula for the second homology is given by

$$H_2(b, \mathscr{A}) = \frac{\mathrm{Coim}(a) \cap \mathrm{Ker}(f_0)}{\mathrm{Coim}(a|_{\mathrm{Ker}(f_1)})}.$$

A double extension as the outer part of the diagram

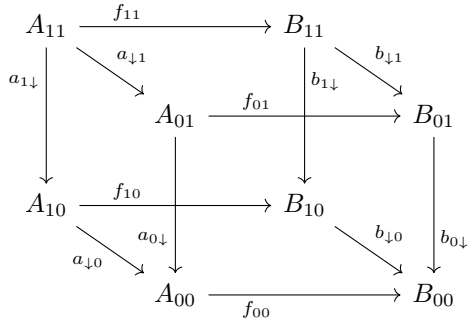


is central if and only if f_1 and g_1 are jointly monomorphic. Its centralization is given via the induced morphisms depicted in the above diagram. If the double extension yields a double regular projective presentation for $d : D_1 \to D_0$, then the Hopf formula for the third homology is given by

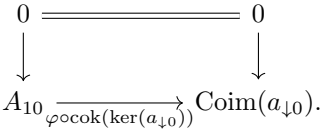
$$H_3(d, \mathscr{A}) = \frac{\mathrm{Coim}(a) \cap \mathrm{Ker}(f_0) \cap \mathrm{Ker}(g_0)}{\mathrm{Coim}(a|_{\mathrm{Ker}(f_1)\cap\mathrm{Ker}(g_1)})}.$$

It is clear how this formula generalizes to arbitrary $n \geq 2$.

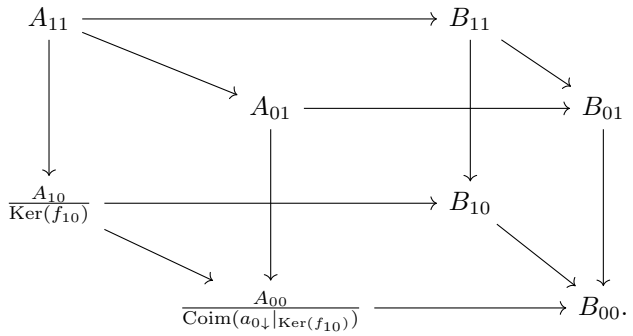
Example 3.4.14. Let $\mathbb{T}$ and $\mathbb{S}$ be as in Example 3.3.8. We saw that an extension $(f_{00}, f_{10}, f_{01}, f_{11})$ in $\mathrm{Arr}^2(\mathscr{A})$ from $\mathbb{A}$ to $\mathbb{B}$ as depicted in the diagram



is central if and only if f_{10} is an isomorphism. We observe that $[\mathbb{A}]$ is given by



Hence the centralization of $(f_{00}, f_{10}, f_{01}, f_{11})$ is given by the following induced extension:

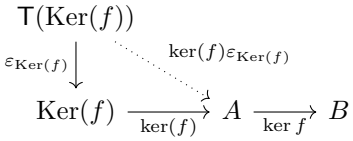


Moreover, if $(f_{00}, f_{10}, f_{01}, f_{11})$ is a regular projective presentation of $\mathbb{B}$, the Hopf formula for the second homology is given by

$$H_2(\mathbb{B}, 2\text{-Arr}(\mathscr{A})) = \left(0 \;=\!=\; 0 \longrightarrow \tfrac{\mathrm{Coim}(a_{\downarrow 0}) \cap \mathrm{Ker}(f_{00})}{\mathrm{Coim}(a_{\downarrow 0}|_{\mathrm{Ker}(f_{10})})}\right).$$

It is easy to see how this generalizes for higher central extensions.

In [EG14, Theorem 4.11], T. Everaert and M. Gran show that a torsion theory $(\mathscr{T}, \mathscr{F})$ in a homological category $\mathscr{C}$, where the reflection $\mathsf{F} : \mathscr{C} \to \mathscr{F}$ is protoadditive and the composite $\ker(f)\varepsilon_{\mathrm{Ker}(f)} : \mathsf{T}(\mathrm{Ker}(f)) \to A$ as in the diagram below is a normal monomorphism for any morphism $f : A \to B$ in $\mathscr{C}$, induces, for any $n \geq 1$, a torsion theory $(\mathscr{T}_n, \mathscr{F}_n)$ in the category $\mathrm{Ext}^n(\mathscr{C})$ of n-fold extensions with respect to the class of extensions in $\mathscr{C}$ consisting of all normal epimorphisms in $\mathscr{C}$. It would be interesting to study the consequences of this result in our setting.



Chapter 4

A syntactic characterization of weakly Mal'tsev varieties

In this chapter, we give a syntactic characterization of weakly Mal'tsev varieties, and apply it to the variety of distributive lattices. Furthermore, we give a syntactic characterization of the varieties of universal algebras in which any reflexive regular relation is an equivalence relation. The results of this chapter are published in [EJMF24].

4.1 Weakly Mal'tsev categories

Let us start with recalling the notion of weakly Malt'sev category in the sense of N. Martins-Ferreira [MF08] and some of its basic properties.

Definition 4.1.1 (Weakly Mal'tsev category)**.** A *weakly Mal'tsev category* is a category $\mathscr{C}$ in which pullbacks of split epimorphisms along split epimorphisms exist, and the canonical pullback injections e'_1, e'_2 in any pullback

$$\begin{array}{ccc} P' & \underset{e'_2}{\overset{p'_2}{\rightleftarrows}} & B \\ {\scriptstyle p'_1}\downarrow\uparrow{\scriptstyle e'_1} & & {\scriptstyle s'}\uparrow\downarrow{\scriptstyle g'} \\ A & \underset{f'}{\overset{r'}{\rightleftarrows}} & C \end{array}$$

of split epimorphisms f', g' with respective splittings r', s' are jointly epimorphic.

Examples 4.1.2.

1. By Proposition 1.1.19, any Mal'tsev category is a weakly Mal'tsev category. Note that N. Martins-Ferreira and T. Van der Linden construct in [MFVdL14] a category which is a weakly Mal'tsev and Goursat but not a Mal'tsev category.

2. In [MF12], N. Martins-Ferreira shows that a variety of lattices is a weakly Mal'tsev category if and only if it is a variety of distributive lattices.

3. In [MF15], N. Martins-Ferreira proves that a quasi-variety with ternary terms p_i, where $i \in I$ and I is an index set, that satisfy the conditions

   (a) $p_i(x, y, y) \approx p_i(y, y, x)$ for all $i \in I$,

   (b) $(\exists a, \forall i \in I, p_i(x, a, a) \approx p_i(y, a, a)) \Rightarrow x \approx y$,

   is a weakly Mal'tsev category. In particular, the quasi-variety of cancellative commutative monoids is a weakly Mal'tsev category, for which one can set $I := \{1\}$ and $p_1(x, y, z) := x \cdot z$. Note that $I = \{1, 2\}$, and $p_1(x, y, z) = x \vee z$ and $p_2(x, y, z) = x \wedge z$ yield that the variety of distributive lattices is a weakly Mal'tsev category.

4. In [MF15], one can also find many examples of co-weakly Mal'tsev categories, i.e. categories whose dual is a weakly Mal'tsev category, as it is the case for the category of topological spaces.

Recall from Proposition 1.1.17 that Mal'tsev categories allow various characterizations by means of relations. Weakly Mal'tsev categories can be characterized in terms of strong relations as Z. Janelidze and N. Martins-Ferreira show in [JMF12]. A *strong relation* is a relation whose representative monomorphism is a strong monomorphism.

Proposition 4.1.3. Let $\mathscr{C}$ be a finitely complete category. Then the following conditions are equivalent:

1. The category $\mathscr{C}$ is a weakly Mal'tsev category.

2. Any reflexive strong relation in $\mathscr{C}$ is an equivalence relation.

3. Any reflexive strong relation in $\mathscr{C}$ is symmetric.

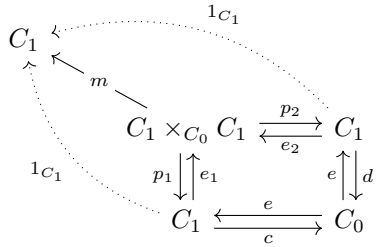
4. Any reflexive strong relation in $\mathscr{C}$ is transitive.

5. Any strong relation in $\mathscr{C}$ is difunctional.

As in the Mal'tsev context (Proposition 1.4.19), a reflexive graph in a weakly Mal'tsev category allows for at most one internal category structure. However, there are internal categories that are not internal groupoids.

Definition 4.1.4 (Multiplicative graph)**.** Let $\mathscr{C}$ be a category with pullbacks of split epimorphisms along split epimorphisms. A *multiplicative graph* (C_0, C_1, d, c, e, m) in $\mathscr{C}$ is given by a reflexive graph (C_0, C_1, d, c, e) in $\mathscr{C}$ together with a morphism $m : C_1 \times_{C_0} C_1 \to C_1$ such that $me_1 = 1_{C_1} = me_2$. Here $C_1 \times_{C_0} C_1$ is the pullback of d along c, and $e_1, e_2 : C_1 \to C_1 \times_{C_0} C_1$ are

the canonical pullback injections induced by the common splitting e of d, c .



(4.1)

Proposition 4.1.5. [MF08] Let $\mathscr{C}$ be a weakly Mal'tsev category. Then any reflexive graph (C_0, C_1, d, c, e) in $\mathscr{C}$ admits at most one multiplicative graph structure.

Proof. This follows immediately from the definition of a weakly Mal'tsev category. In this context, the canonical pullback injections e_1, e_2 in Diagram (4.1) are jointly epimorphic. □

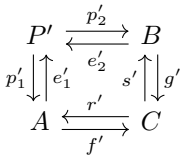
Proposition 4.1.6. [MF08] Let $\mathscr{C}$ be a weakly Mal'tsev category. Then any multiplicative graph (C_0, C_1, d, c, e, m) in $\mathscr{C}$ is an internal category.

Example 4.1.7. Let us consider the weakly Mal'tsev category CancCommMon of cancellative commutative monoids. Then the usual order on $\mathbb{N}_0$ yields a preorder in CancCommMon which is not an equivalence relation. This shows that, in a weakly Mal'tsev category, internal categories are not necessarily internal groupoids.

Remark 4.1.8. In [MFVdL14], N. Martins-Ferreira and T. Van der Linden prove that, given a weakly Mal'tsev category $\mathscr{C}$, any internal category in $\mathscr{C}$ is an internal groupoid if and only if any preorder in $\mathscr{C}$ is an equivalence relation. In [MFRVdL14], N. Martins-Ferreira, D. Rodelo and T. Van der Linden show that the same holds true in a given regular category. Furthermore, in a variety $\mathbb{V}$ of universal algebras, these two statements are further equivalent to $\mathbb{V}$ being n-permutable for some $2 \leq n \in \mathbb{N}$.

4.2 Weakly Mal'tsev varieties

In Section 1.1.3.2 (Proof 1), we saw a proof of Mal'tsev's theorem relying on the characterization of Mal'tsev categories as those finitely complete categories in which the canonical pullback injections e_1', e_2' in any pullback



of split epimorphisms f', g' with respective splittings r', s' are jointly extremally epimorphic. Here we adapt this proof to obtain a syntactic characterization of weakly Mal'tsev varieties.

Lemma 4.2.1. For a variety $\mathbb{V}$ of universal algebras, the following conditions are equivalent:

1. $\mathbb{V}$ is a weakly Mal'tsev category.
2. The canonical pullback injections e_1', e_2' in any pullback

$$\begin{array}{ccc} P' & \underset{e_2'}{\overset{p_2'}{\rightleftarrows}} & B \\ {\scriptstyle p_1'}\downarrow\uparrow{\scriptstyle e_1'} & & {\scriptstyle s'}\uparrow\downarrow{\scriptstyle g'} \\ A & \underset{f'}{\overset{r'}{\rightleftarrows}} & C \end{array}$$

   of split epimorphisms f', g' with respective splittings r', s' are jointly epimorphic.

3. The canonical pullback injections e_1, e_2 in the pullback

$$\begin{array}{ccc} P & \underset{e_2}{\overset{p_2}{\rightleftarrows}} & \mathsf{F}(x,y) \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ \mathsf{F}(x,y) & \underset{f}{\overset{r}{\rightleftarrows}} & \mathsf{F}(x) \end{array}$$

   are jointly epimorphic. Here $\mathsf{F}(x)$ and $\mathsf{F}(x,y)$ are the free algebras on variables x and x, y, respectively, $f : \mathsf{F}(x,y) \to \mathsf{F}(x)$ is the unique morphism with $f([x]) = [x] = f([y])$, $r : \mathsf{F}(x) \to \mathsf{F}(x,y)$ is the unique morphism with $r([x]) = [x]$, and $s : \mathsf{F}(x) \to \mathsf{F}(x,y)$ is the unique morphism with $s([x]) = [y]$.

4. The cokernel pair injections $q_1, q_2 : P \to Q$ of the induced morphism $[e_1, e_2] : \mathsf{F}(x,y) + \mathsf{F}(x,y) \to P$ as in the diagram below are equal.

$$\begin{array}{ccc} \mathsf{F}(x,y) + \mathsf{F}(x,y) & \xrightarrow{[e_1,e_2]} & P \\ {\scriptstyle [e_1,e_2]}\downarrow & & \downarrow{\scriptstyle q_2} \\ P & \xrightarrow[q_1]{} & Q \end{array}$$

5. It holds that $q_1([y],[x]) = q_2([y],[x])$. Note that P is given by the subalgebra of $\mathsf{F}(x,y) \times \mathsf{F}(x,y)$ with elements $([t_1],[t_2])$, where $t_1, t_2 \in \mathsf{T}(x,y)$, such that $[t_1^{\mathsf{T}(x)}(x,x)] = [t_2^{\mathsf{T}(x)}(x,x)]$.

Proof. Items 1 and 2 are equivalent by definition. Trivially, Item 2 implies Item 3, and Item 4 implies Item 5. Note that, in a category $\mathscr{C}$ with binary coproducts, two morphisms $\varphi : X \to U$ and $\psi : Y \to U$ are jointly epimorphic if and only if the induced morphism $[f, g] : X + Y \to U$ is an epimorphism. Furthermore, this is equivalent to the cokernel pair injections of $[f, g]$ being equal. The fact that Item 5 implies Item 2 can be shown similarly to how we proved that Item 5 implies Item 2 in Lemma 1.1.34. □

Recall that, in Section 1.1.3.1, we fixed a countable set $V = \{v_i\}_{i\in\mathbb{N}}$ and said that t is an n-ary term, where $n \in \mathbb{N}$, if $t \in \mathsf{T}(v_1, \ldots, v_n)$. A 0-ary term is a constant. In the following, we will omit the superscripts in expressions as $t^{\mathsf{T}(x,y)}(x, t^{\mathsf{T}(y)}(y, y))$, where t is a binary term, when they are clear from the context. We will also write t instead of $[t]$ when it is clear that we are only interested in the equivalence class of a term $t \in \mathsf{T}(X)$ in a quotient of the term algebra $\mathsf{T}(X)$.

Theorem 4.2.2. A variety $\mathbb{V}$ of universal algebras is a weakly Mal'tsev category if and only if there exist $k, m, N \in \mathbb{N}_0$, binary terms $f_1, g_1, \ldots, f_k, g_k$, ternary terms $p_1, \ldots, p_m$, $(2(k+2m+1))$-ary terms $s_1, \ldots, s_N$, $(2(k+m+2))$-ary terms $\sigma_1, \ldots, \sigma_{N+1}$ and, for all $i \in \{1, \ldots, N+1\}$, $(k+m+1)$-ary terms $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ in the theory of $\mathbb{V}$ such that the following identities (on variables x, y, u, u', $v_1, \ldots, v_k$, $v'_1, \ldots, v'_k$, $w_1, \ldots, w_m$ and $w'_1, \ldots, w'_m$) hold, where we write $\vec{v}$ for $v_1, \ldots, v_k$ and $\vec{w}$ for $w_1, \ldots, w_m$, and analogously for $\vec{v}'$ and $\vec{w}'$:

$$f_i(x, x) \approx g_i(x, x) \tag{4.2}$$

for all $i \in \{1, \ldots, k\}$;

$$\begin{aligned} &\eta_\alpha^{(i)}(y, f_1(x,y), \ldots, f_k(x,y), p_1(x,x,y), \ldots, p_m(x,x,y)) \\ \approx &\epsilon_\alpha^{(i)}(y, f_1(x,y), \ldots, f_k(x,y), p_1(x,x,y), \ldots, p_m(x,x,y)), \end{aligned} \tag{4.3a}$$

$$\begin{aligned} &\eta_\alpha^{(i)}(x, g_1(x,y), \ldots, g_k(x,y), p_1(x,y,y), \ldots, p_m(x,y,y)) \\ \approx &\epsilon_\alpha^{(i)}(x, g_1(x,y), \ldots, g_k(x,y), p_1(x,y,y), \ldots, p_m(x,y,y)), \end{aligned} \tag{4.3b}$$

for all $i \in \{1, \ldots, N+1\}$ and $\alpha \in \{1, 2\}$;

$$\begin{aligned} &\sigma_i(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(i)}(u, \vec{v}, \vec{w}), \epsilon_2^{(i)}(u', \vec{v}', \vec{w}')) \\ \approx &s_i(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}'), \end{aligned} \tag{4.4a}$$

$$\begin{aligned} &s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}) \\ \approx &\sigma_{i+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(i+1)}(u, \vec{v}, \vec{w}), \eta_2^{(i+1)}(u', \vec{v}', \vec{w}')), \end{aligned} \tag{4.4b}$$

for all $i \in \{1, \ldots, N\}$ and

$$u \approx \sigma_1(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(1)}(u, \vec{v}, \vec{w}), \eta_2^{(1)}(u', \vec{v}', \vec{w}')), \tag{4.5a}$$

$$u' \approx \sigma_{N+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(N+1)}(u, \vec{v}, \vec{w}), \epsilon_2^{(N+1)}(u', \vec{v}', \vec{w}')). \tag{4.5b}$$

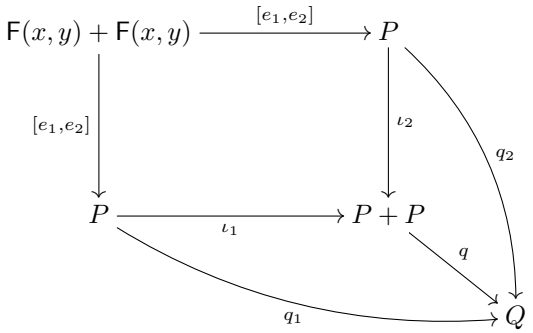
Proof. We start by showing that if $\mathbb{V}$ is a weakly Mal'tsev category, then the terms mentioned in the statement exist. We first treat the case where $\mathbb{V}$ is a Mal'tsev category separately. If a Mal'tsev term $p \in \mathsf{T}(x, y, z)$, which fulfills $p(x, x, y) \approx y$ and $p(x, y, y) \approx x$, exists, we can easily construct terms that satisfy the requirements of the theorem. We can choose for instance k to be 0

and m, N to be 1, and we can set

$$\begin{aligned} p_1(x,y,z) &:= p(x,y,z), \\ s_1(u,w,\tilde{w},u',w',\tilde{w}') &:= \tilde{w}, \\ \sigma_1(u,w,u',w',a,b) &:= a, \\ \sigma_2(u,w,u',w',a,b) &:= b, \\ \eta_1^{(1)}(u,w) := \epsilon_2^{(2)}(u,w) &:= u, \\ \epsilon_1^{(1)}(u,w) := \eta_2^{(2)}(u,w) &:= w. \end{aligned}$$

Then the binary terms $\eta_2^{(1)}, \epsilon_2^{(1)}, \eta_1^{(2)}, \epsilon_1^{(2)}$ have only to satisfy the Equations (4.3), which can easily be attained.

Let us now suppose that $\mathbb{V}$ is a weakly Mal'tsev category but not a Mal'tsev category. By Lemma 4.2.1, we know that $\mathbb{V}$ is a weakly Mal'tsev category if and only if $q_1(y,x) = q_2(y,x)$. We construct the cokernel pair $(q_1, q_2 \colon P \to Q)$ of the map $[e_1, e_2] \colon \mathsf{F}(x,y) + \mathsf{F}(x,y) \to P$ by means of the coequalizer q of the maps $\iota_1[e_1,e_2], \iota_2[e_1,e_2]$, where ι_1, ι_2 are the coproduct injections into $P + P$, as in the following diagram:



The algebra Q is given by the quotient of $P + P$ with respect to, due to Lemma 1.1.35, the smallest congruence $C \subseteq (P+P)^2$ containing the subset

$$\begin{aligned} S := \Big\{ \Big(\iota_1\big(p(x,x,y), p(x,y,y)\big), \iota_2\big(p(x,x,y), p(x,y,y)\big) \Big) \in (P+P)^2 \\ \mid p \in \mathsf{T}(x,y,z) \Big\}. \end{aligned}$$

More precisely, we first define the subset

$$\begin{aligned} S_{\mathrm{S}} := & \Big\{ \Big(\iota_1\big(p(x,x,y), p(x,y,y)\big), \iota_2\big(p(x,x,y), p(x,y,y)\big) \Big) \in (P+P)^2 \\ & \mid p \in \mathsf{T}(x,y,z) \Big\} \\ \cup & \Big\{ \Big(\iota_2\big(p(x,x,y), p(x,y,y)\big), \iota_1\big(p(x,x,y), p(x,y,y)\big) \Big) \in (P+P)^2 \\ & \mid p \in \mathsf{F}(x,y,z) \Big\}, \end{aligned}$$

which yields the smallest symmetric relation on $P+P$ containing S. Secondly, we define

$$S_{\mathrm{SR}} := S_{\mathrm{S}} \cup \{(X,X) \in (P+P)^2 \mid X \in P+P\},$$

which yields the smallest reflexive symmetric relation on $P+P$ containing S. Thirdly, we define

$$\begin{aligned} S_{\mathrm{SRO}} := \{t((X_1,Y_1),\dots,(X_n,Y_n)) \in (P+P)^2 \mid{}& n \geqslant 0 \text{ is an integer,} \\ & (X_1,Y_1),\dots,(X_n,Y_n) \in S_{\mathrm{SR}} \\ & \text{and } t \text{ is an } n\text{-ary term}\}, \end{aligned}$$

yielding the smallest symmetric reflexive relation on $P+P$ containing S and being closed under operations. Fourthly, we define

$$\begin{aligned} S_{\mathrm{SROT}} := \{(X,Y) \in (P+P)^2 & \\ \mid{}& \text{there exists an integer } n \geqslant 1 \text{ and } X_1,\dots,X_{n+1} \in P+P \\ & \text{such that } (X_1,X_2),(X_2,X_3),\dots,(X_n,X_{n+1}) \in S_{\mathrm{SRO}} \\ & \text{and } X = X_1 \text{ and } X_{n+1} = Y\} \end{aligned}$$

to be the transitive closure of S_{SRO}. It is well-known and easy to prove that C coincides with S_{SROT}. Thus, the condition $q_1(y,x) = q_2(y,x)$ is equivalent to $\big(\iota_1(y,x),\iota_2(y,x)\big) \in S_{\mathrm{SROT}}$. This means that there exist an integer $n \geqslant 1$ and elements $X_1,\dots,X_{n+1} \in P+P$ such that

$$(X_1,X_2),(X_2,X_3),\dots,(X_{n-1},X_n),(X_n,X_{n+1}) \in S_{\mathrm{SRO}},$$

$\iota_1(y,x) = X_1$ and $X_{n+1} = \iota_2(y,x)$. The condition $(X_i,X_{i+1}) \in S_{\mathrm{SRO}}$, where $i \in \{1,\dots,n\}$, means that there exist an integer $a_i \geqslant 0$, elements

$$(X_1^{(i)},Y_1^{(i)}),\dots,(X_{a_i}^{(i)},Y_{a_i}^{(i)}) \in S_{\mathrm{SR}}$$

and an a_i-ary term t_i such that

$$(X_i,X_{i+1}) = (t_i(X_1^{(i)},\dots,X_{a_i}^{(i)}),t_i(Y_1^{(i)},\dots,Y_{a_i}^{(i)})).$$

By definition of S_{SR}, each pair $(X_j^{(i)},Y_j^{(i)})$ is either of the form

$$\Big(\iota_1\big(p_j^{(i)}(x,x,y),p_j^{(i)}(x,y,y)\big),\iota_2\big(p_j^{(i)}(x,x,y),p_j^{(i)}(x,y,y)\big)\Big),$$

where $p_j^{(i)} \in \mathsf{T}(x,y,z)$, of the form

$$\Big(\iota_2\big(q_j^{(i)}(x,x,y),q_j^{(i)}(x,y,y)\big),\iota_1\big(q_j^{(i)}(x,x,y),q_j^{(i)}(x,y,y)\big)\Big),$$

where $q_j^{(i)} \in \mathsf{T}(x,y,z)$, or of the form

$$(Z_j^{(i)},Z_j^{(i)}),$$

where $Z_j^{(i)} = X_j^{(i)} = Y_j^{(i)} \in P+P$. Hence we see that, by possibly permuting the variables in the term t_i, the condition $(X_i,X_{i+1}) \in S_{\mathrm{SRO}}$ is equivalent

to the existence of integers $m_i, m_i', m_i'' \geqslant 0$, ternary terms $p_1^{(i)}, \dots, p_{m_i}^{(i)}$ and $q_1^{(i)}, \dots, q_{m_i'}^{(i)}$, elements $Z_1^{(i)}, \dots, Z_{m_i''}^{(i)} \in P + P$, and an $(m_i + m_i' + m_i'')$-ary term t_i such that the identities

$$\begin{aligned} X_i = t_i\Big(&Z_1^{(i)}, \dots, Z_{m_i''}^{(i)}, \\ &\iota_1\big(p_1^{(i)}(x,x,y), p_1^{(i)}(x,y,y)\big), \dots, \iota_1\big(p_{m_i}^{(i)}(x,x,y), p_{m_i}^{(i)}(x,y,y)\big), \\ &\iota_2\big(q_1^{(i)}(x,x,y), q_1^{(i)}(x,y,y)\big), \dots, \iota_2\big(q_{m_i'}^{(i)}(x,x,y), q_{m_i'}^{(i)}(x,y,y)\big)\Big) \end{aligned}$$

and

$$\begin{aligned} X_{i+1} = t_i\Big(&Z_1^{(i)}, \dots, Z_{m_i''}^{(i)}, \\ &\iota_2\big(p_1^{(i)}(x,x,y), p_1^{(i)}(x,y,y)\big), \dots, \iota_2\big(p_{m_i}^{(i)}(x,x,y), p_{m_i}^{(i)}(x,y,y)\big), \\ &\iota_1\big(q_1^{(i)}(x,x,y), q_1^{(i)}(x,y,y)\big), \dots, \iota_1\big(q_{m_i'}^{(i)}(x,x,y), q_{m_i'}^{(i)}(x,y,y)\big)\Big) \end{aligned}$$

hold in $P+P$. Given a term $t \in \mathsf{T}(x_1, \dots, x_v)$, where $v \geqslant 0$ is an integer, we can add a variable x_{v+1} to t in the sense that we define a term $t' \in \mathsf{T}(x_1, \dots, x_{v+1})$ by setting $t'(x_1, \dots, x_{v+1}) := t(x_1, \dots, x_v)$. In this spirit, we can consider the unions

$$\bigcup_{i \in \{1,\dots,n\}} \{p_j^{(i)} \mid j \in \{1, \dots, m_i\}\},$$

$$\bigcup_{i \in \{1,\dots,n\}} \{q_j^{(i)} \mid j \in \{1, \dots, m_i'\}\}$$

and

$$\bigcup_{i \in \{1,\dots,n\}} \{Z_j^{(i)} \mid j \in \{1, \dots, m_i''\}\},$$

and assume, without loss of generality, that $m_1 = \dots = m_n =: m$, $m_1' = \dots = m_n' =: m'$ and $m_1'' = \dots = m_n'' =: m''$. Hence we can omit the upper index from the elements $p_j^{(i)}, q_j^{(i)}, Z_j^{(i)}$. Furthermore, by looking at the union

$$\{p_j \mid j \in \{1, \dots, m\}\} \cup \{q_j \mid j \in \{1, \dots, m'\}\},$$

we can assume that $m = m'$ and $p_j = q_j$ for all $j \in \{1, \dots, m\}$ without loss of generality. Thus, the condition that $(X_i, X_{i+1}) \in S_{\mathrm{SRO}}$ for all $i \in \{1, \dots, n\}$ is equivalent to the existence of integers $m, m'' \geqslant 0$, terms $p_1, \dots, p_m \in \mathsf{T}(x, y, z)$, elements $Z_1, \dots, Z_{m''} \in P + P$ and $(2m + m'')$-ary terms $t_1, \dots, t_n$ such that the identities

$$\begin{aligned} X_i = t_i\Big(&Z_1, \dots, Z_{m''}, \\ &\iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big) \\ &\iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big)\Big) \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} X_{i+1} = t_i\Big(& Z_1, \dots, Z_{m''}, \\ & \iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big) \\ & \iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big)\Big) \end{aligned} \tag{4.7}$$

hold in $P + P$ for all $i \in \{1, \dots, n\}$. For all $i \in \{1, \dots, m''\}$, there exist integers $\hat{k}_i, \tilde{k}_i \geqslant 0$, elements $(\hat{f}_1^{(i)}, \hat{g}_1^{(i)}), \dots, (\hat{f}_{\hat{k}_i}^{(i)}, \hat{g}_{\hat{k}_i}^{(i)})$ and $(\tilde{f}_1^{(i)}, \tilde{g}_1^{(i)}), \dots, (\tilde{f}_{\tilde{k}_i}^{(i)}, \tilde{g}_{\tilde{k}_i}^{(i)})$ in P, and a $(\hat{k}_i + \tilde{k}_i)$-ary term s_i such that

$$Z_i = s_i\big(\iota_1(\hat{f}_1^{(i)}, \hat{g}_1^{(i)}), \dots, \iota_1(\hat{f}_{\hat{k}_i}^{(i)}, \hat{g}_{\hat{k}_i}^{(i)}), \iota_2(\tilde{f}_1^{(i)}, \tilde{g}_1^{(i)}), \dots, \iota_2(\tilde{f}_{\tilde{k}_i}^{(i)}, \tilde{g}_{\tilde{k}_i}^{(i)})\big).$$

Proceeding analogously as we did for the $p_1^{(i)}, \dots, p_{m_i}^{(i)}$ and $q_1^{(i)}, \dots, q_{m_i'}^{(i)}$ from above, we can first assume, without loss of generality, that $\hat{k}_1 = \dots = \hat{k}_{m''} =: \hat{k}$ and $\tilde{k}_1 = \dots = \tilde{k}_{m''} =: \tilde{k}$ and omit the upper index from the elements $\hat{f}_j^{(i)}, \hat{g}_j^{(i)}, \tilde{f}_j^{(i)}, \tilde{g}_j^{(i)}$, and then that $\hat{k} = \tilde{k} =: k$, and $\hat{f}_j = \tilde{f}_j$ and $\hat{g}_j = \tilde{g}_j$ for all $j \in \{1, \dots, k\}$. In a next step, we see that we can assume, without loss of generality, that we have an integer $k \geqslant 0$, binary terms $f_1, g_1, \dots, f_k, g_k$ such that $(f_j, g_j) \in P$, i.e., $f_j(x,x) = g_j(x,x)$, for all $j \in \{1, \dots, k\}$, and $(2(k+m+1))$-ary terms $s_1, \dots, s_{m''}$ such that the $k + m + 1$ elements

$$(y,x), (f_1, g_1) \dots, (f_k, g_k), (p_1(x,x,y), p_1(x,y,y)) \dots, (p_m(x,x,y), p_m(x,y,y))$$

of P are pairwise distinct and

$$\begin{aligned} Z_i = s_i(& \iota_1(y,x), \iota_1(f_1,g_1), \dots, \iota_1(f_k,g_k), \\ & \iota_1(p_1(x,x,y), p_1(x,y,y)), \dots, \iota_1(p_m(x,x,y), p_m(x,y,y)), \\ & \iota_2(y,x), \iota_2(f_1,g_1), \dots, \iota_2(f_k,g_k), \\ & \iota_2(p_1(x,x,y), p_1(x,y,y)), \dots, \iota_2(p_m(x,x,y), p_m(x,y,y))) \end{aligned}$$

for all $i \in \{1, \dots, m''\}$. Notice that, since we supposed that $\mathbb{V}$ is not a Mal'tsev variety, (y,x) cannot be of the form $(p_j(x,x,y), p_j(x,y,y))$. By setting, for all $i \in \{1, \dots, n\}$, the $(2(k+2m+1))$-ary term

$$\begin{aligned} & t_i'(u, v_1, \dots, v_k, w_1, \dots, w_m, \widetilde{w}_1, \dots, \widetilde{w}_m, \\ & \quad u', v_1', \dots, v_k', w_1', \dots, w_m', \widetilde{w}_1', \dots, \widetilde{w}_m') \\ := & \, t_i(s_1(u, v_1, \dots, v_k, w_1, \dots, w_m, u', v_1', \dots, v_k', w_1', \dots, w_m'), \dots, \\ & \quad s_{m''}(u, v_1, \dots, v_k, w_1, \dots, w_m, u', v_1', \dots, v_k', w_1', \dots, w_m'), \\ & \quad \widetilde{w}_1, \dots, \widetilde{w}_m, \widetilde{w}_1', \dots, \widetilde{w}_m'), \end{aligned}$$

Equations (4.6) and (4.7) translate into

$$\begin{aligned} X_i = t'_i\Big(&\iota_1(y,x), \iota_1(f_1,g_1), \dots, \iota_1(f_k,g_k),\\ &\iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big),\\ &\iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big),\\ &\iota_2(y,x), \iota_2(f_1,g_1), \dots, \iota_2(f_k,g_k),\\ &\iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big),\\ &\iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big)\Big)\\ =:\ & t'_i\{1,2\} \end{aligned}$$

and

$$\begin{aligned} X_{i+1} = t'_i\Big(&\iota_1(y,x), \iota_1(f_1,g_1), \dots, \iota_1(f_k,g_k),\\ &\iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big),\\ &\iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big),\\ &\iota_2(y,x), \iota_2(f_1,g_1), \dots, \iota_2(f_k,g_k),\\ &\iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big),\\ &\iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big)\Big)\\ =:\ & t'_i\{2,1\} \end{aligned}$$

respectively. In the notation $t'_i\{1,2\}$ and $t'_i\{2,1\}$, the first (respectively the second) entry inside the brackets corresponds to which of ι_1 or ι_2 is used in the third (respectively the sixth) row in the above expressions. Hence we see that the condition $q_1(y,x) = q_2(y,x)$ implies the existence of integers $k, m \geqslant 0$ and $n \geqslant 1$, binary terms $f_1, g_1, \dots, f_k, g_k$ with, for all $i \in \{1, \dots, k\}$, $f_i(x,x) = g_i(x,x)$, ternary terms $p_1, \dots, p_m$, and $(2(k+2m+1))$-ary terms $t'_1, \dots, t'_n$ such that the equations

$$\begin{aligned} &\iota_1(y,x)\\ =\ & t'_1\Big(\iota_1(y,x), \iota_1(f_1,g_1), \dots, \iota_1(f_k,g_k), \qquad \text{(a)}\\ &\quad \iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big),\\ &\quad \iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big),\\ &\quad \iota_2(y,x), \iota_2(f_1,g_1), \dots, \iota_2(f_k,g_k),\\ &\quad \iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big),\\ &\quad \iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big)\Big) \end{aligned}$$

and, for all $i \in \{1, \dots, n-1\}$,

$$\begin{aligned}
&t'_i\Big(\iota_1(y,x), \iota_1(f_1,g_1), \dots, \iota_1(f_k,g_k),\\
&\quad \iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big),\\
&\quad \iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big),\\
&\quad \iota_2(y,x), \iota_2(f_1,g_1), \dots, \iota_2(f_k,g_k),\\
&\quad \iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big),\\
&\quad \iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big)\Big)\\
&= t'_{i+1}\Big(\iota_1(y,x), \iota_1(f_1,g_1), \dots, \iota_1(f_k,g_k), \qquad (\mathrm{b}(i))\\
&\quad \iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big),\\
&\quad \iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big),\\
&\quad \iota_2(y,x), \iota_2(f_1,g_1), \dots, \iota_2(f_k,g_k),\\
&\quad \iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big),\\
&\quad \iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&t'_n\Big(\iota_1(y,x), \iota_1(f_1,g_1), \dots, \iota_1(f_k,g_k),\\
&\quad \iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big),\\
&\quad \iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big),\\
&\quad \iota_2(y,x), \iota_2(f_1,g_1), \dots, \iota_2(f_k,g_k),\\
&\quad \iota_2\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_2\big(p_m(x,x,y), p_m(x,y,y)\big),\\
&\quad \iota_1\big(p_1(x,x,y), p_1(x,y,y)\big), \dots, \iota_1\big(p_m(x,x,y), p_m(x,y,y)\big)\Big)\\
&= \iota_2(y,x) \qquad (\mathrm{c})
\end{aligned}$$

hold in $P + P$. We display the Equations (a), (b(i)) for all $i \in \{1, \dots, n-1\}$ and (c) in the diagram

$$\begin{array}{ccccccc}
\iota_1(y,x) & \overset{(\mathrm{a})}{=\!=} & t'_1\{1,2\} & \cdots\cdots & t'_1\{2,1\} & \overset{(\mathrm{b}(1))}{=\!=} & t'_2\{1,2\} \\
 & & & & & & \vdots \\
\iota_2(y,x). & \overset{(\mathrm{c})}{=\!=} & t'_n\{2,1\} & \cdots\cdots & t'_n\{1,2\} & \overset{(\mathrm{b}(n-1))}{=\!=} & t'_{n-1}\{2,1\}
\end{array} \qquad (4.8)$$

By Proposition 1.1.31, Equation (a) means that there exist an integer $\ell_0 \geqslant 0$

and elements $c_{(0,1)},\dots,c_{(0,\ell_0)} \in \mathsf{F}(P\dot\cup P)$ such that

$$\begin{aligned}
&(y,x)^1 \sim c_{(0,1)},\\
&c_{(0,1)} \sim c_{(0,2)},\dots,c_{(0,\ell_0-1)} \sim c_{(0,\ell_0)},\\
&c_{(0,\ell_0)} \sim t_1'\big((y,x)^1,(f_1,g_1)^1,\dots,(f_k,g_k)^1,\\
&\qquad (p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\
&\qquad (p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\
&\qquad (y,x)^2,(f_1,g_1)^2,\dots,(f_k,g_k)^2,\\
&\qquad (p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&\qquad (p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2\big)
\end{aligned}$$

where we write, for an element $X \in P$, X^1 and X^2 to indicate that we view them as variables in $\mathsf{F}(P\dot\cup P)$ coming from the first and second copy of P in $P\dot\cup P$ respectively, and $\sim$ is the relation on $\mathsf{F}(P\dot\cup P)$ from Proposition 1.1.31. More explicitly, this means that there exist, for all $j \in \{0,\dots,\ell_0\}$, integers $a_{(0,j)}, b_{(0,j)} \geqslant 0$, elements $D_1^{(0,j)},\dots,D_{a_{(0,j)}}^{(0,j)},E_1^{(0,j)},\dots,E_{b_{(0,j)}}^{(0,j)} \in P$, $a_{(0,j)}$-ary terms $\mu_1^{(0,j)},\lambda_1^{(0,j)}$, $b_{(0,j)}$-ary terms $\mu_2^{(0,j)},\lambda_2^{(0,j)}$ and $(a_{(0,j)}+b_{(0,j)}+2)$-ary terms $\tau_{(0,j)}$ such that the equations

$$\begin{aligned}
&(y,x)^1\\
=&\tau_{(0,0)}\Big((D_1^{(0,0)})^1,\dots,(D_{a_{(0,0)}}^{(0,0)})^1,(E_1^{(0,0)})^2,\dots,(E_{b_{(0,0)}}^{(0,0)})^2,\\
&\quad \mu_1^{(0,0)}\big((D_1^{(0,0)})^1,\dots,(D_{a_{(0,0)}}^{(0,0)})^1\big),\mu_2^{(0,0)}\big((E_1^{(0,0)})^2,\dots,(E_{b_{(0,0)}}^{(0,0)})^2\big)\Big)
\end{aligned} \tag{B1}$$

and

$$\begin{aligned}
&\tau_{(0,0)}\Big((D_1^{(0,0)})^1,\dots,(D_{a_{(0,0)}}^{(0,0)})^1,(E_1^{(0,0)})^2,\dots,(E_{b_{(0,0)}}^{(0,0)})^2,\\
&\quad \lambda_1^{(0,0)}\big((D_1^{(0,0)})^1,\dots,(D_{a_{(0,0)}}^{(0,0)})^1\big),\lambda_2^{(0,0)}\big((E_1^{(0,0)})^2,\dots,(E_{b_{(0,0)}}^{(0,0)})^2\big)\Big)\\
=&c_{(0,1)}
\end{aligned} \tag{C(0,1)}$$

hold in $\mathsf{F}(P\dot\cup P)$ and the equations

$$\mu_1^{(0,0)}\big(D_1^{(0,0)},\dots,D_{a_{(0,0)}}^{(0,0)}\big) = \lambda_1^{(0,0)}\big(D_1^{(0,0)},\dots,D_{a_{(0,0)}}^{(0,0)}\big), \tag{M(0,0)}$$

$$\mu_2^{(0,0)}\big(E_1^{(0,0)},\dots,E_{b_{(0,0)}}^{(0,0)}\big) = \lambda_2^{(0,0)}\big(E_1^{(0,0)},\dots,E_{b_{(0,0)}}^{(0,0)}\big) \tag{N(0,0)}$$

hold in P; the equations

$$\begin{aligned}
&c_{(0,j)}\\
=&\tau_{(0,j)}\Big((D_1^{(0,j)})^1,\dots,(D_{a_{(0,j)}}^{(0,j)})^1,(E_1^{(0,j)})^2,\dots,(E_{b_{(0,j)}}^{(0,j)})^2,\\
&\quad \mu_1^{(0,j)}\big((D_1^{(0,j)})^1,\dots,(D_{a_{(0,j)}}^{(0,j)})^1\big),\mu_2^{(0,j)}\big((E_1^{(0,j)})^2,\dots,(E_{b_{(0,j)}}^{(0,j)})^2\big)\Big),
\end{aligned} \tag{D(0,j)}$$

$$\begin{aligned}&\tau_{(0,j)}\Big((D_1^{(0,j)})^1,\dots,(D_{a_{(0,j)}}^{(0,j)})^1,(E_1^{(0,j)})^2,\dots,(E_{b_{(0,j)}}^{(0,j)})^2,\\&\qquad\lambda_1^{(0,j)}\big((D_1^{(0,j)})^1,\dots,(D_{a_{(0,j)}}^{(0,j)})^1\big),\lambda_2^{(0,j)}\big((E_1^{(0,j)})^2,\dots,(E_{b_{(0,j)}}^{(0,j)})^2\big)\Big)\\=&c_{(0,j+1)}\end{aligned}\tag{C$(0,j+1)$}$$

hold in $\mathsf{F}(P\dot\cup P)$ and the equations

$$\mu_1^{(0,j)}\big(D_1^{(0,j)},\dots,D_{a_{(0,j)}}^{(0,j)}\big)=\lambda_1^{(0,j)}\big(D_1^{(0,j)},\dots,D_{a_{(0,j)}}^{(0,j)}\big),\tag{M$(0,j)$}$$

$$\mu_2^{(0,j)}\big(E_1^{(0,j)},\dots,E_{b_{(0,j)}}^{(0,j)}\big)=\lambda_2^{(0,j)}\big(E_1^{(0,j)},\dots,E_{b_{(0,j)}}^{(0,j)}\big)\tag{N$(0,j)$}$$

are satisfied in P for all $j\in\{1,\dots,\ell_0-1\}$; and moreover the equations

$$\begin{aligned}&c_{(0,\ell_0)}\\=&\tau_{(0,\ell_0)}\Big((D_1^{(0,\ell_0)})^1,\dots,(D_{a_{(0,\ell_0)}}^{(0,\ell_0)})^1,(E_1^{(0,\ell_0)})^2,\dots,(E_{b_{(0,\ell_0)}}^{(0,\ell_0)})^2,\\&\qquad\mu_1^{(0,\ell_0)}\big((D_1^{(0,\ell_0)})^1,\dots,(D_{a_{(0,\ell_0)}}^{(0,\ell_0)})^1\big),\mu_2^{(0,\ell_0)}\big((E_1^{(0,\ell_0)})^2,\dots,(E_{b_{(0,\ell_0)}}^{(0,\ell_0)})^2\big)\Big),\end{aligned}\tag{D$(0,\ell_0)$}$$

$$\begin{aligned}&\tau_{(0,\ell_0)}\Big((D_1^{(0,\ell_0)})^1,\dots,(D_{a_{(0,\ell_0)}}^{(0,\ell_0)})^1,(E_1^{(0,\ell_0)})^2,\dots,(E_{b_{(0,\ell_0)}}^{(0,\ell_0)})^2,\\&\qquad\lambda_1^{(0,\ell_0)}\big((D_1^{(0,\ell_0)})^1,\dots,(D_{a_{(0,\ell_0)}}^{(0,\ell_0)})^1\big),\lambda_2^{(0,\ell_0)}\big((E_1^{(0,\ell_0)})^2,\dots,(E_{b_{(0,\ell_0)}}^{(0,\ell_0)})^2\big)\Big)\\=&t_1'\Big((y,x)^1,(f_1,g_1)^1,\dots,(f_k,g_k)^1,\\&(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\&(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\&(y,x)^2,(f_1,g_1)^2,\dots,(f_k,g_k)^2,\\&(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2,\\&(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2\Big)\end{aligned}\tag{T(1)}$$

hold in $\mathsf{F}(P\dot\cup P)$ and the equations

$$\mu_1^{(0,\ell_0)}\big(D_1^{(0,\ell_0)},\dots,D_{a_{(0,\ell_0)}}^{(0,\ell_0)}\big)=\lambda_1^{(0,\ell_0)}\big(D_1^{(0,\ell_0)},\dots,D_{a_{(0,\ell_0)}}^{(0,\ell_0)}\big),\tag{M$(0,\ell_0)$}$$

$$\mu_2^{(0,\ell_0)}\big(E_1^{(0,\ell_0)},\dots,E_{b_{(0,\ell_0)}}^{(0,\ell_0)}\big)=\lambda_2^{(0,\ell_0)}\big(E_1^{(0,\ell_0)},\dots,E_{b_{(0,\ell_0)}}^{(0,\ell_0)}\big)\tag{N$(0,\ell_0)$}$$

hold in P. For $i\in\{1,\dots,n-1\}$, Equation (b(i)) means that there exist an

integer $\ell_i \geqslant 0$ and elements $c_{(i,1)}, \dots, c_{(i,\ell_i)} \in \mathsf{F}(P \dot\cup P)$ such that

$$
\begin{aligned}
&t'_i\big((y,x)^1, (f_1,g_1)^1, \dots, (f_k,g_k)^1,\\
&\quad (p_1(x,x,y), p_1(x,y,y))^1, \dots, (p_m(x,x,y), p_m(x,y,y))^1,\\
&\quad (p_1(x,x,y), p_1(x,y,y))^2, \dots, (p_m(x,x,y), p_m(x,y,y))^2,\\
&\quad (y,x)^2, (f_1,g_1)^2, \dots, (f_k,g_k)^2,\\
&\quad (p_1(x,x,y), p_1(x,y,y))^2, \dots, (p_m(x,x,y), p_m(x,y,y))^2,\\
&\quad (p_1(x,x,y), p_1(x,y,y))^1, \dots, (p_m(x,x,y), p_m(x,y,y))^1\big) \sim c_{(i,1)},\\
&c_{(i,1)} \sim c_{(i,2)}, \dots, c_{(i,\ell_i-1)} \sim c_{(i,\ell_i)},\\
&c_{(i,\ell_i)} \sim t'_{i+1}\big((y,x)^1, (f_1,g_1)^1, \dots, (f_k,g_k)^1,\\
&\qquad\qquad (p_1(x,x,y), p_1(x,y,y))^1, \dots, (p_m(x,x,y), p_m(x,y,y))^1,\\
&\qquad\qquad (p_1(x,x,y), p_1(x,y,y))^1, \dots, (p_m(x,x,y), p_m(x,y,y))^1,\\
&\qquad\qquad (y,x)^2, (f_1,g_1)^2, \dots, (f_k,g_k)^2,\\
&\qquad\qquad (p_1(x,x,y), p_1(x,y,y))^2, \dots, (p_m(x,x,y), p_m(x,y,y))^2,\\
&\qquad\qquad (p_1(x,x,y), p_1(x,y,y))^2, \dots, (p_m(x,x,y), p_m(x,y,y))^2\big).
\end{aligned}
$$

More precisely, there exist, for all $j \in \{0, \dots, \ell_i\}$, integers $a_{(i,j)}, b_{(i,j)} \geqslant 0$, elements $D_1^{(i,j)}, \dots, D_{a_{(i,j)}}^{(i,j)}$, $E_1^{(i,j)}, \dots, E_{b_{(i,j)}}^{(i,j)} \in P$, $a_{(i,j)}$-ary terms $\mu_1^{(i,j)}, \lambda_1^{(i,j)}$, $b_{(i,j)}$-ary terms $\mu_2^{(i,j)}$, $\lambda_2^{(i,j)}$ and $(a_{(i,j)} + b_{(i,j)} + 2)$-ary terms $\tau_{(i,j)}$ such that the equations

$$
\begin{aligned}
&t'_i\Big((y,x)^1, (f_1,g_1)^1, \dots, (f_k,g_k)^1,\\
&\quad (p_1(x,x,y), p_1(x,y,y))^1, \dots, (p_m(x,x,y), p_m(x,y,y))^1,\\
&\quad (p_1(x,x,y), p_1(x,y,y))^2, \dots, (p_m(x,x,y), p_m(x,y,y))^2,\\
&\quad (y,x)^2, (f_1,g_1)^2, \dots, (f_k,g_k)^2,\\
&\quad (p_1(x,x,y), p_1(x,y,y))^2, \dots, (p_m(x,x,y), p_m(x,y,y))^2,\\
&\quad (p_1(x,x,y), p_1(x,y,y))^1, \dots, (p_m(x,x,y), p_m(x,y,y))^1\Big)\\
=&\tau_{(i,0)}\Big((D_1^{(i,0)})^1, \dots, (D_{a_{(i,0)}}^{(i,0)})^1, (E_1^{(i,0)})^2, \dots, (E_{b_{(i,0)}}^{(i,0)})^2, \qquad \text{(S}(i)\text{)}\\
&\qquad \mu_1^{(i,0)}\big((D_1^{(i,0)})^1, \dots, (D_{a_{(i,0)}}^{(i,0)})^1\big), \mu_2^{(i,0)}\big((E_1^{(i,0)})^2, \dots, (E_{b_{(i,0)}}^{(i,0)})^2\big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\tau_{(i,0)}\Big((D_1^{(i,0)})^1, \dots, (D_{a_{(i,0)}}^{(i,0)})^1, (E_1^{(i,0)})^2, \dots, (E_{b_{(i,0)}}^{(i,0)})^2,\\
&\qquad \lambda_1^{(i,0)}\big((D_1^{(i,0)})^1, \dots, (D_{a_{(i,0)}}^{(i,0)})^1\big), \lambda_2^{(i,0)}\big((E_1^{(i,0)})^2, \dots, (E_{b_{(i,0)}}^{(i,0)})^2\big)\Big)\\
=&c_{(i,1)} \qquad \text{(C}(i,1)\text{)}
\end{aligned}
$$

hold in $\mathsf{F}(P \dot\cup P)$ and the equations

$$\mu_1^{(i,0)}\big(D_1^{(i,0)},\dots,D_{a_{(i,0)}}^{(i,0)}\big) = \lambda_1^{(i,0)}\big(D_1^{(i,0)},\dots,D_{a_{(i,0)}}^{(i,0)}\big), \tag{M$(i,0)$}$$
$$\mu_2^{(i,0)}\big(E_1^{(i,0)},\dots,E_{b_{(i,0)}}^{(i,0)}\big) = \lambda_2^{(i,0)}\big(E_1^{(i,0)},\dots,E_{b_{(i,0)}}^{(i,0)}\big) \tag{N$(i,0)$}$$

hold in P; the equations

$$\begin{aligned} &c_{(i,j)}\\ =&\tau_{(i,j)}\Big((D_1^{(i,j)})^1,\dots,(D_{a_{(i,j)}}^{(i,j)})^1,(E_1^{(i,j)})^2,\dots,(E_{b_{(i,j)}}^{(i,j)})^2,\\ &\quad\mu_1^{(i,j)}\big((D_1^{(i,j)})^1,\dots,(D_{a_{(i,j)}}^{(i,j)})^1\big),\mu_2^{(i,j)}\big((E_1^{(i,j)})^2,\dots,(E_{b_{(i,j)}}^{(i,j)})^2\big)\Big), \end{aligned} \tag{D(i,j)}$$

$$\begin{aligned} &\tau_{(i,j)}\Big((D_1^{(i,j)})^1,\dots,(D_{a_{(i,j)}}^{(i,j)})^1,(E_1^{(i,j)})^2,\dots,(E_{b_{(i,j)}}^{(i,j)})^2,\\ &\quad\lambda_1^{(i,j)}\big((D_1^{(i,j)})^1,\dots,(D_{a_{(i,j)}}^{(i,j)})^1\big),\lambda_2^{(i,j)}\big((E_1^{(i,j)})^2,\dots,(E_{b_{(i,j)}}^{(i,j)})^2\big)\Big)\\ =&c_{(i,j+1)} \end{aligned} \tag{C$(i,j+1)$}$$

hold in $\mathsf{F}(P \dot\cup P)$ and the equations

$$\mu_1^{(i,j)}\big(D_1^{(i,j)},\dots,D_{a_{(i,j)}}^{(i,j)}\big) = \lambda_1^{(i,j)}\big(D_1^{(i,j)},\dots,D_{a_{(i,j)}}^{(i,j)}\big), \tag{M(i,j)}$$
$$\mu_2^{(i,j)}\big(E_1^{(i,j)},\dots,E_{b_{(i,j)}}^{(i,j)}\big) = \lambda_2^{(i,j)}\big(E_1^{(i,j)},\dots,E_{b_{(i,j)}}^{(i,j)}\big) \tag{N(i,j)}$$

are satisfied in P for all $j \in \{1,\dots,\ell_i - 1\}$; and moreover the equations

$$\begin{aligned} &c_{(i,\ell_i)}\\ =&\tau_{(i,\ell_i)}\Big((D_1^{(i,\ell_i)})^1,\dots,(D_{a_{(i,\ell_i)}}^{(i,\ell_i)})^1,(E_1^{(i,\ell_i)})^2,\dots,(E_{b_{(i,\ell_i)}}^{(i,\ell_i)})^2,\\ &\quad\mu_1^{(i,\ell_i)}\big((D_1^{(i,\ell_i)})^1,\dots,(D_{a_{(i,\ell_i)}}^{(i,\ell_i)})^1\big),\mu_2^{(i,\ell_i)}\big((E_1^{(i,\ell_i)})^2,\dots,(E_{b_{(i,\ell_i)}}^{(i,\ell_i)})^2\big)\Big), \end{aligned} \tag{D(i,ℓ_i)}$$

$$\begin{aligned} &\tau_{(i,\ell_i)}\Big((D_1^{(i,\ell_i)})^1,\dots,(D_{a_{(i,\ell_i)}}^{(i,\ell_i)})^1,(E_1^{(i,\ell_i)})^2,\dots,(E_{b_{(i,\ell_i)}}^{(i,\ell_i)})^2,\\ &\quad\lambda_1^{(i,\ell_i)}\big((D_1^{(i,\ell_i)})^1,\dots,(D_{a_{(i,\ell_i)}}^{(i,\ell_i)})^1\big),\lambda_2^{(i,\ell_i)}\big((E_1^{(i,\ell_i)})^2,\dots,(E_{b_{(i,\ell_i)}}^{(i,\ell_i)})^2\big)\Big)\\ =&t'_{i+1}\Big((y,x)^1,(f_1,g_1)^1,\dots,(f_k,g_k)^1,\\ &\quad(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\ &\quad(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\ &\quad(y,x)^2,(f_1,g_1)^2,\dots,(f_k,g_k)^2,\\ &\quad(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2,\\ &\quad(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2\Big) \end{aligned} \tag{T$(i+1)$}$$

hold in $\mathsf{F}(P\dot\cup P)$ and the equations

$$\mu_1^{(i,\ell_i)}\big(D_1^{(i,\ell_i)},\dots,D_{a_{(i,\ell_i)}}^{(i,\ell_i)}\big)=\lambda_1^{(i,\ell_i)}\big(D_1^{(i,\ell_i)},\dots,D_{a_{(i,\ell_i)}}^{(i,\ell_i)}\big),\tag{M(i,ℓ_i)}$$
$$\mu_2^{(i,\ell_i)}\big(E_1^{(i,\ell_i)},\dots,E_{b_{(i,\ell_i)}}^{(i,\ell_i)}\big)=\lambda_2^{(i,\ell_i)}\big(E_1^{(i,\ell_i)},\dots,E_{b_{(i,\ell_i)}}^{(i,\ell_i)}\big)\tag{N(i,ℓ_i)}$$

hold in P. Equation (c) means that there exist an integer $\ell_n\geqslant 0$ and elements $c_{(n,1)},\dots,c_{(n,\ell_n)}$ in $\mathsf{F}(P\dot\cup P)$ such that

$$\begin{aligned}
&t'_n\big((y,x)^1,(f_1,g_1)^1,\dots,(f_k,g_k)^1,\\
&\quad(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\
&\quad(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&\quad(y,x)^2,(f_1,g_1)^2,\dots,(f_k,g_k)^2,\\
&\quad(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&\quad(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1\big)\sim c_{(n,1)},\\
&c_{(n,1)}\sim c_{(n,2)},\dots,c_{(n,\ell_n-1)}\sim c_{(n,\ell_n)},\\
&c_{(n,\ell_n)}\sim (y,x)^2.
\end{aligned}$$

More explicitly, this means that there exist, for all $j\in\{0,\dots,\ell_n\}$, integers $a_{(n,j)},b_{(n,j)}\geqslant 0$, elements $D_1^{(n,j)},\dots,D_{a_{(n,j)}}^{(n,j)},E_1^{(n,j)},\dots,E_{b_{(n,j)}}^{(n,j)}\in P$, $a_{(n,j)}$-ary terms $\mu_1^{(n,j)},\lambda_1^{(n,j)}$, $b_{(n,j)}$-ary terms $\mu_2^{(n,j)},\lambda_2^{(n,j)}$ and $(a_{(n,j)}+b_{(n,j)}+2)$-ary terms $\tau_{(n,j)}$ such that the equations

$$\begin{aligned}
&t'_n\Big((y,x)^1,(f_1,g_1)^1,\dots,(f_k,g_k)^1,\\
&\quad(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\
&\quad(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&\quad(y,x)^2,(f_1,g_1)^2,\dots,(f_k,g_k)^2,\\
&\quad(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&\quad(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1\Big)\\
=&\tau_{(n,0)}\Big((D_1^{(n,0)})^1,\dots,(D_{a_{(n,0)}}^{(n,0)})^1,(E_1^{(n,0)})^2,\dots,(E_{b_{(n,0)}}^{(n,0)})^2,\\
&\qquad\mu_1^{(n,0)}\big((D_1^{(n,0)})^1,\dots,(D_{a_{(n,0)}}^{(n,0)})^1\big),\mu_2^{(n,0)}\big((E_1^{(n,0)})^2,\dots,(E_{b_{(n,0)}}^{(n,0)})^2\big)\Big)
\end{aligned}\tag{S(n)}$$

and

$$\begin{aligned}
&\tau_{(n,0)}\Big((D_1^{(n,0)})^1,\dots,(D_{a_{(n,0)}}^{(n,0)})^1,(E_1^{(n,0)})^2,\dots,(E_{b_{(n,0)}}^{(n,0)})^2,\\
&\qquad\lambda_1^{(n,0)}\big((D_1^{(n,0)})^1,\dots,(D_{a_{(n,0)}}^{(n,0)})^1\big),\lambda_2^{(n,0)}\big((E_1^{(n,0)})^2,\dots,(E_{b_{(n,0)}}^{(n,0)})^2\big)\Big)\\
=&c_{(n,1)}
\end{aligned}\tag{C$(n,1)$}$$

hold in $\mathsf{F}(P \dot\cup P)$ and the equations

$$\mu_1^{(n,0)}\big(D_1^{(n,0)},\dots,D_{a_{(n,0)}}^{(n,0)}\big) = \lambda_1^{(n,0)}\big(D_1^{(n,0)},\dots,D_{a_{(n,0)}}^{(n,0)}\big), \tag{M$(n,0)$}$$

$$\mu_2^{(n,0)}\big(E_1^{(n,0)},\dots,E_{b_{(n,0)}}^{(n,0)}\big) = \lambda_2^{(n,0)}\big(E_1^{(n,0)},\dots,E_{b_{(n,0)}}^{(n,0)}\big) \tag{N$(n,0)$}$$

hold in P; the equations

$$\begin{aligned} &c_{(n,j)}\\ =&\tau_{(n,j)}\Big((D_1^{(n,j)})^1,\dots,(D_{a_{(n,j)}}^{(n,j)})^1,(E_1^{(n,j)})^2,\dots,(E_{b_{(n,j)}}^{(n,j)})^2,\\ &\quad\mu_1^{(n,j)}\big((D_1^{(n,j)})^1,\dots,(D_{a_{(n,j)}}^{(n,j)})^1\big),\mu_2^{(n,j)}\big((E_1^{(n,j)})^2,\dots,(E_{b_{(n,j)}}^{(n,j)})^2\big)\Big), \end{aligned} \tag{D(n,j)}$$

$$\begin{aligned} &\tau_{(n,j)}\Big((D_1^{(n,j)})^1,\dots,(D_{a_{(n,j)}}^{(n,j)})^1,(E_1^{(n,j)})^2,\dots,(E_{b_{(n,j)}}^{(n,j)})^2,\\ &\quad\lambda_1^{(n,j)}\big((D_1^{(n,j)})^1,\dots,(D_{a_{(n,j)}}^{(n,j)})^1\big),\lambda_2^{(n,j)}\big((E_1^{(n,j)})^2,\dots,(E_{b_{(n,j)}}^{(n,j)})^2\big)\Big)\\ =&c_{(n,j+1)} \end{aligned} \tag{C$(n,j+1)$}$$

hold in $\mathsf{F}(P \dot\cup P)$ and the equations

$$\mu_1^{(n,j)}\big(D_1^{(n,j)},\dots,D_{a_{(n,j)}}^{(n,j)}\big) = \lambda_1^{(n,j)}\big(D_1^{(n,j)},\dots,D_{a_{(n,j)}}^{(n,j)}\big), \tag{M(n,j)}$$

$$\mu_2^{(n,j)}\big(E_1^{(n,j)},\dots,E_{b_{(n,j)}}^{(n,j)}\big) = \lambda_2^{(n,j)}\big(E_1^{(n,j)},\dots,E_{b_{(n,j)}}^{(n,j)}\big) \tag{N(n,j)}$$

are satisfied in P for all $j \in \{1,\dots,\ell_n - 1\}$; and moreover the equations

$$\begin{aligned} &c_{(n,\ell_n)}\\ =&\tau_{(n,\ell_n)}\Big((D_1^{(n,\ell_n)})^1,\dots,(D_{a_{(n,\ell_n)}}^{(n,\ell_n)})^1,(E_1^{(n,\ell_n)})^2,\dots,(E_{b_{(n,\ell_n)}}^{(n,\ell_n)})^2,\\ &\quad\mu_1^{(n,\ell_n)}\big((D_1^{(n,\ell_n)})^1,\dots,(D_{a_{(n,\ell_n)}}^{(n,\ell_n)})^1\big),\\ &\quad\mu_2^{(n,\ell_n)}\big((E_1^{(n,\ell_n)})^2,\dots,(E_{b_{(n,\ell_n)}}^{(n,\ell_n)})^2\big)\Big), \end{aligned} \tag{D(n,ℓ_n)}$$

$$\begin{aligned} &\tau_{(n,\ell_n)}\Big((D_1^{(n,\ell_n)})^1,\dots,(D_{a_{(n,\ell_n)}}^{(n,\ell_n)})^1,(E_1^{(n,\ell_n)})^2,\dots,(E_{b_{(n,\ell_n)}}^{(n,\ell_n)})^2,\\ &\quad\lambda_1^{(n,\ell_n)}\big((D_1^{(n,\ell_n)})^1,\dots,(D_{a_{(n,\ell_n)}}^{(n,\ell_n)})^1\big),\\ &\quad\lambda_2^{(n,\ell_n)}\big((E_1^{(n,\ell_n)})^2,\dots,(E_{b_{(n,\ell_n)}}^{(n,\ell_n)})^2\big)\Big)\\ =&(y,x)^2 \end{aligned} \tag{B2}$$

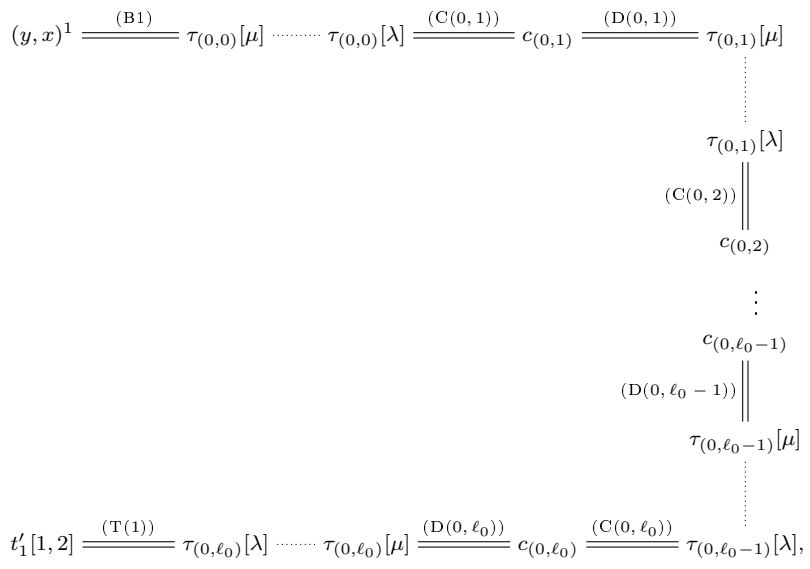
and

$$\mu_1^{(n,\ell_n)}\big(D_1^{(n,\ell_n)},\dots,D_{a_{(n,\ell_n)}}^{(n,\ell_n)}\big) = \lambda_1^{(n,\ell_n)}\big(D_1^{(n,\ell_n)},\dots,D_{a_{(n,\ell_n)}}^{(n,\ell_n)}\big), \tag{M(n,ℓ_n)}$$

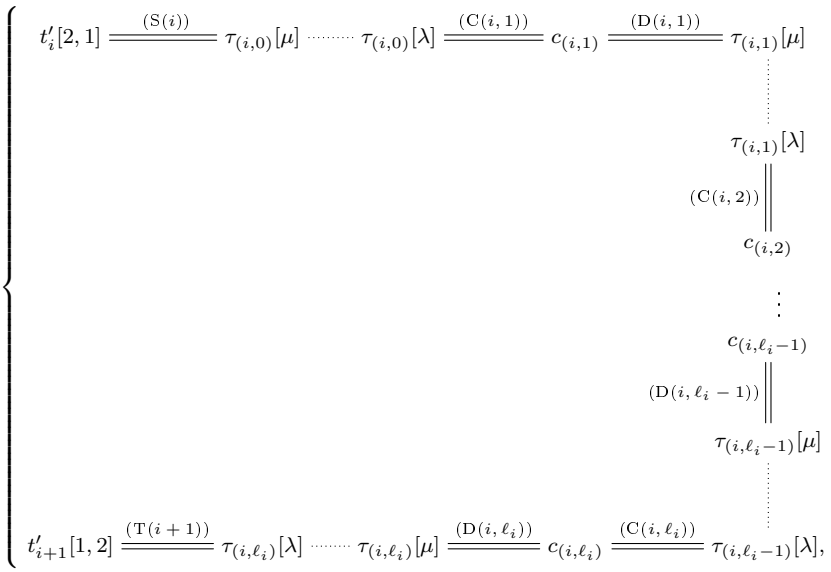
$$\mu_2^{(n,\ell_n)}\big(E_1^{(n,\ell_n)},\dots,E_{b_{(n,\ell_n)}}^{(n,\ell_n)}\big) = \lambda_2^{(n,\ell_n)}\big(E_1^{(n,\ell_n)},\dots,E_{b_{(n,\ell_n)}}^{(n,\ell_n)}\big) \tag{N(n,ℓ_n)}$$

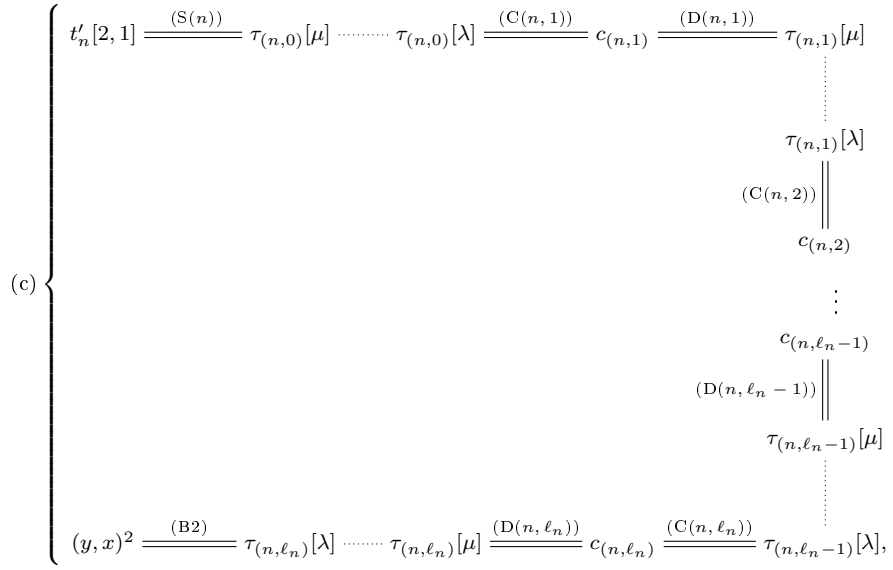
hold in $\mathsf{F}(P \dot{\cup} P)$. Similar to Diagram (4.8), we display the Equations (B1), (B2), (T(i)), (S(i)), (C(i,j)) and (D(i,j)) in the diagrams

(a)



(b(i))



(c)


where

$$
\begin{aligned}
t'_i[1,2] := t'_i\Big(&(y,x)^1,(f_1,g_1)^1,\dots,(f_k,g_k)^1,\\
&(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\
&(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\
&(y,x)^2,(f_1,g_1)^2,\dots,(f_k,g_k)^2,\\
&(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2\Big),\\
t'_i[2,1] := t'_i\Big(&(y,x)^1,(f_1,g_1)^1,\dots,(f_k,g_k)^1,\\
&(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1,\\
&(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&(y,x)^2,(f_1,g_1)^2,\dots,(f_k,g_k)^2,\\
&(p_1(x,x,y),p_1(x,y,y))^2,\dots,(p_m(x,x,y),p_m(x,y,y))^2,\\
&(p_1(x,x,y),p_1(x,y,y))^1,\dots,(p_m(x,x,y),p_m(x,y,y))^1\Big),\\
&\tau_{(i,j)}[\mu]\\
:=&\tau_{(i,j)}\Big((D_1^{(i,j)})^1,\dots,(D_{a_{(i,j)}}^{(i,j)})^1,(E_1^{(i,j)})^2,\dots,(E_{b_{(i,j)}}^{(i,j)})^2,\\
&\quad\mu_1^{(i,j)}\big((D_1^{(i,j)})^1,\dots,(D_{a_{(i,j)}}^{(i,j)})^1\big),\mu_2^{(i,j)}\big((E_1^{(i,j)})^2,\dots,(E_{b_{(i,j)}}^{(i,j)})^2\big)\Big),\\
&\tau_{(i,j)}[\lambda]\\
:=&\tau_{(i,j)}\Big((D_1^{(i,j)})^1,\dots,(D_{a_{(i,j)}}^{(i,j)})^1,(E_1^{(i,j)})^2,\dots,(E_{b_{(i,j)}}^{(i,j)})^2,\\
&\quad\lambda_1^{(i,j)}\big((D_1^{(i,j)})^1,\dots,(D_{a_{(i,j)}}^{(i,j)})^1\big),\lambda_2^{(i,j)}\big((E_1^{(i,j)})^2,\dots,(E_{b_{(i,j)}}^{(i,j)})^2\big)\Big).
\end{aligned}
$$

Since the relation $\sim$ as in Proposition 1.1.31 is reflexive, we can assume, without loss of generality, that $\ell_0 = \cdots = \ell_n =: \ell$. As before, by possibly artificially increasing the number of variables the terms of the form $\tau_{(i,j)}$, $\mu_1^{(i,j)}$, $\mu_2^{(i,j)}$, $\lambda_1^{(i,j)}$ or $\lambda_2^{(i,j)}$ depend on, we can also assume, without loss of generality, that we have integers $a, b \geqslant 0$ such that $a_{(i,j)} = a$ and $b_{(i,j)} = b$ for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, \ell\}$, and omit the upper index from the elements $D_\alpha^{(i,j)}, E_\alpha^{(i,j)}$. Moreover, we can, without loss of generality, assume that $a = b$ and $D_\alpha = E_\alpha$ for all $\alpha \in \{1, \ldots, a\}$. Furthermore, up to adding some D_α's to the list of (f_j, g_j)'s, and up to adding (y, x), some (f_j, g_j)'s and some $(p_j(x, x, y), p_j(x, y, y))$'s to the list of D_α's, we can assume that $a = k + m + 1$ and $D_1 = (y, x)$, $D_2 = (f_1, g_1), \ldots$, $D_{k+1} = (f_k, g_k)$ and $D_{k+2} = (p_1(x, x, y), p_1(x, y, y)), \ldots$, $D_{k+m+1} = (p_m(x, x, y), p_m(x, y, y))$, keeping the elements

$$(y, x), (f_1, g_1) \ldots, (f_k, g_k), (p_1(x, x, y), p_1(x, y, y)) \ldots, (p_m(x, x, y), p_m(x, y, y))$$

of P pairwise distinct. Thus, Equations (a), (b(i)) for all $i \in \{1, \ldots, n-1\}$ and (c) imply the existence of an integer $\ell \geqslant 0$, terms $c_{(i,1)}, \ldots, c_{(i,\ell)}$, $(k + m + 1)$-ary terms $\mu_1^{(i,j)}, \mu_2^{(i,j)}, \lambda_1^{(i,j)}, \lambda_2^{(i,j)}$ and $(2(k + m + 2))$-ary terms $\tau_{(i,j)}$ for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, \ell\}$ such that the following equalities hold. Using variables $u, v_1, \ldots, v_k, w_1, \ldots, w_m$ and $u', v_1', \ldots, v_k', w_1', \ldots, w_m'$, Equations (B1) and (B2) read

$$u = \tau_{(0,0)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \mu_1^{(0,0)}(u, \vec{v}, \vec{w}), \mu_2^{(0,0)}(u', \vec{v}', \vec{w}')), \tag{B1'}$$

$$u' = \tau_{(n,\ell)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \lambda_1^{(n,\ell)}(u, \vec{v}, \vec{w}), \lambda_2^{(n,\ell)}(u', \vec{v}', \vec{w}')), \tag{B2'}$$

where we write $\vec{v}$ for $v_1 \ldots, v_k$ and $\vec{w}$ for $w_1, \ldots, w_m$, and analogously for $\vec{v}'$ and $\vec{w}'$. Equations (T(1)) and (T($i+1$)), and Equations (S(i)) and (S(n)) read

$$\begin{aligned} &t_i'(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}') \\ =&\tau_{(i-1,\ell)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \lambda_1^{(i-1,\ell)}(u, \vec{v}, \vec{w}), \lambda_2^{(i-1,\ell)}(u', \vec{v}', \vec{w}')), \qquad \text{(T(i)')} \\ &t_i'(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}) \\ =&\tau_{(i,0)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \mu_1^{(i,0)}(u, \vec{v}, \vec{w}), \mu_2^{(i,0)}(u', \vec{v}', \vec{w}')), \qquad \text{(S(i)')} \end{aligned}$$

for all $i \in \{1, \ldots, n\}$. Equations (C(0, 1)), (C(0, $j+1$)), (C(i, 1)), (C(i, $j+1$)), (C(n, 1)), (C(n, $j+1$)), and Equations (D(0, j)), (D(0, ℓ_0)), (D(i, j)), (D(i, ℓ_i)), (D(n, j)), (D(n, ℓ_n)) read

$$c_{(i,j)} = \tau_{(i,j-1)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \lambda_1^{(i,j-1)}(u, \vec{v}, \vec{w}), \lambda_2^{(i,j-1)}(u', \vec{v}', \vec{w}')), \quad \text{(C(i, j)')}$$

$$c_{(i,j)} = \tau_{(i,j)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \mu_1^{(i,j)}(u, \vec{v}, \vec{w}), \mu_2^{(i,j)}(u', \vec{v}', \vec{w}')), \tag{D(i, j)'}$$

for all $i \in \{0, \ldots, n\}$ and $j \in \{1, \ldots, \ell\}$ where the variables on which the $c_{(i,j)}$'s depend are left inexplicit. Equations (M(0, 0)), (M(0, j)), (M(0, ℓ_0)), (M(i, 0)), (M(i, j)), (M(i, ℓ_i)), (M(n, 0)), (M(n, j)), (M(n, ℓ_n)), and Equations (N(0, 0)), (N(0, j)), (N(0, ℓ_0)), (N(i, 0)), (N(i, j)), (N(i, ℓ_i)), (N(n, 0)), (N(n, j)), (N(n, ℓ_n))

read

$$\begin{aligned}
&\mu_\alpha^{(i,j)}\Big(\big(y,x\big),\big(f_1,g_1\big),\dots,\big(f_k,g_k\big),\\
&\qquad\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,\big(p_m(x,x,y),p_m(x,y,y)\big)\Big)\\
=&\lambda_\alpha^{(i,j)}\Big(\big(y,x\big),\big(f_1,g_1\big),\dots,\big(f_k,g_k\big),\\
&\qquad\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,\big(p_m(x,x,y),p_m(x,y,y)\big)\Big)
\end{aligned}\tag{Z(α,i,j)}$$

for all $i \in \{0,\dots,n\}$, $j \in \{0,\dots,\ell\}$ and $\alpha \in \{1,2\}$. Equations (C(i,j)') and (D(i,j)') imply the equation

$$\begin{aligned}
&\tau_{(i,j)}(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\lambda_1^{(i,j)}(u,\vec{v},\vec{w}),\lambda_2^{(i,j)}(u',\vec{v}',\vec{w}'))\\
=\,&\tau_{(i,j+1)}(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\mu_1^{(i,j+1)}(u,\vec{v},\vec{w}),\mu_2^{(i,j+1)}(u',\vec{v}',\vec{w}'))
\end{aligned}\tag{W(i,j)}$$

for all $i \in \{0,\dots,n\}$ and $j \in \{0,\dots,\ell-1\}$. Equation (Z(α,i,j)) means more explicitly that

$$\begin{aligned}
&\mu_\alpha^{(i,j)}(y,f_1,\dots,f_k,p_1(x,x,y),\dots,p_m(x,x,y))\\
=\,&\lambda_\alpha^{(i,j)}(y,f_1,\dots,f_k,p_1(x,x,y),\dots,p_m(x,x,y))
\end{aligned}\tag{Z(α,i,j)a}$$

and

$$\begin{aligned}
&\mu_\alpha^{(i,j)}(x,g_1,\dots,g_k,p_1(x,y,y),\dots,p_m(x,y,y))\\
=\,&\lambda_\alpha^{(i,j)}(x,g_1,\dots,g_k,p_1(x,y,y),\dots,p_m(x,y,y))
\end{aligned}\tag{Z(α,i,j)b}$$

for all $i \in \{0,\dots,n\}$, $j \in \{0,\dots,\ell\}$ and $\alpha \in \{1,2\}$. We display the Equa-

tions (B1'), (B2'), (T(i)'), (S(i)') and (W(i,j)) in the diagram

$$
\begin{array}{ccccc}
u & \overset{(\mathrm{B1'})}{=\!=} & \tau_{(0,0)}[[\mu]] & \cdots & \tau_{(0,0)}[[\lambda]] \overset{(\mathrm{W}(0,0))}{=\!=} \tau_{(0,1)}[[\mu]] \\
 & & & & \vdots \\
 & & & & \tau_{(0,1)}[[\lambda]] \\
 & & & & \vdots \\
 & & & & \tau_{(0,\ell-1)}[[\mu]] \\
 & & & & \vdots \\
t_1'[[1,2]] & \overset{(\mathrm{T}(1)')}{=\!=} & \tau_{(0,\ell)}[[\lambda]] & \cdots & \tau_{(0,\ell)}[[\mu]] \overset{(\mathrm{W}(0,\ell-1))}{=\!=} \tau_{(0,\ell-1)}[[\lambda]] \\
\vdots & & & & \\
t_1'[[2,1]] & \overset{(\mathrm{S}(1)')}{=\!=} & \tau_{(1,0)}[[\mu]] & \cdots & \tau_{(1,0)}[[\lambda]] \overset{(\mathrm{W}(1,0))}{=\!=} \tau_{(1,1)}[[\mu]] \\
 & & & & \vdots \\
 & & & & \tau_{(1,1)}[[\lambda]] \\
 & & & & \vdots \\
 & & & & \vdots \\
 & & & & \tau_{(n-1,\ell-1)}[[\mu]] \\
 & & & & \vdots \\
t_n'[[1,2]] & \overset{(\mathrm{T}(n)')}{=\!=} & \tau_{(n-1,\ell)}[[\lambda]] & \cdots & \tau_{(n-1,\ell)}[[\mu]] \overset{(\mathrm{W}(n-1,\ell-1))}{=\!=} \tau_{(n-1,\ell-1)}[[\lambda]] \\
\vdots & & & & \\
t_n'[[2,1]] & \overset{(\mathrm{S}(n)')}{=\!=} & \tau_{(n,0)}[[\mu]] & \cdots & \tau_{(n,0)}[[\lambda]] \overset{(\mathrm{W}(n,0))}{=\!=} \tau_{(n,1)}[[\mu]] \\
 & & & & \vdots \\
 & & & & \tau_{(n,1)}[[\lambda]] \\
 & & & & \vdots \\
 & & & & \tau_{(n,\ell-1)}[[\mu]] \\
 & & & & \vdots \\
u' & \overset{(\mathrm{B2'})}{=\!=} & \tau_{(n,\ell)}[[\lambda]] & \cdots & \tau_{(n,\ell)}[[\mu]] \overset{(\mathrm{W}(n,\ell-1))}{=\!=} \tau_{(n,\ell-1)}[[\lambda]],
\end{array}
\tag{4.9}
$$

where

$$\tau_{(i,j)}[[\mu]] := \tau_{(i,j)}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \mu_1^{(i,j)}(u, \vec{v}, \vec{w}), \mu_2^{(i,j)}(u', \vec{v}', \vec{w}')),$$

$$\begin{aligned}\tau_{(i,j)}[[\lambda]] &:= \tau_{(i,j)}(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\lambda_1^{(i,j)}(u,\vec{v},\vec{w}),\lambda_2^{(i,j)}(u',\vec{v}',\vec{w}')),\\ t_i'[[1,2]] &:= t_i'(u,\vec{v},\vec{w},\vec{w},u',\vec{v}',\vec{w}',\vec{w}'),\\ t_i'[[2,1]] &:= t_i'(u,\vec{v},\vec{w},\vec{w}',u',\vec{v}',\vec{w}',\vec{w}).\end{aligned}$$

In comparison to $t_i'[[1,2]]$, the second occurrences of $\vec{w}$ and $\vec{w}'$, respectively, are swapped in $t_i'[[2,1]]$. Furthermore, for each $\tau_{(i,j)}$, the terms $\mu_1^{(i,j)}, \mu_2^{(i,j)}$, $\lambda_1^{(i,j)}, \lambda_2^{(i,j)}$ satisfy the Equations (Z(α, i, j)a) and (Z(α, i, j)b). We call an equation of form (W) if it is of the form

$$\begin{aligned}&\omega(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\gamma_1(u,\vec{v},\vec{w}),\gamma_2(u',\vec{v}',\vec{w}'))\\ =&\chi(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\delta_1(u,\vec{v},\vec{w}),\delta_2(u',\vec{v}',\vec{w}')),\end{aligned}$$

of form (T) if it is of the form

$$\begin{aligned}&\xi(u,\vec{v},\vec{w},\vec{w},u',\vec{v}',\vec{w}',\vec{w}')\\ =&\omega(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\gamma_1(u,\vec{v},\vec{w}),\gamma_2(u',\vec{v}',\vec{w}')),\end{aligned}$$

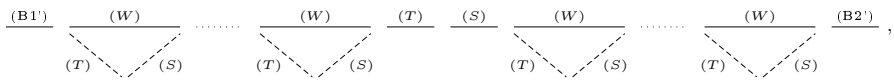
and of form (S) if it is of the form

$$\begin{aligned}&\xi(u,\vec{v},\vec{w},\vec{w}',u',\vec{v}',\vec{w}',\vec{w})\\ =&\chi(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\delta_1(u,\vec{v},\vec{w}),\delta_2(u',\vec{v}',\vec{w}'))\end{aligned}$$

where ω, χ, ξ and $\gamma_1, \gamma_2, \delta_1, \delta_2$ are arbitrary terms with appropriate arity. In comparison to an equation of form (T), the second occurrences of the variables $\vec{w}$ and $\vec{w}'$ in the first line, respectively, are swapped in an equation of form (S). Then all Equations (W(i,j)), $i \in \{0,\dots,n\}$ and $j \in \{0,\dots,\ell-1\}$, are of form (W), all Equations (T(i)'), $i \in \{1,\dots,n\}$, are of form (T), and all Equations (S(i)'), $i \in \{1,\dots,n\}$, are of form (S), and we transform Diagram (4.9) for illustrative reasons into the following diagram:



We note that here the first line (B1') encodes the equality of u and $\tau_{(0,0)}[[\mu]]$, the first of the lines (W) encodes the equality of $\tau_{(0,0)}[[\lambda]]$ and $\tau_{(0,1)}[[\mu]]$, and so on. We show that we can replace each equation of form (W) by an equation of form (T) followed by an equation of form (S) in the following sense. Then we get the diagram



where, following the dashed lines, equations of form (T) and of form (S) alternate. More precisely, we show that, for all $i \in \{0,\dots,n\}$ and $j \in \{0,\dots,\ell-1\}$, Equation (W(i,j)) implies the existence of a $(2(k+2m+1))$-ary term $\pi_{(i,j)}$ such that

$$\begin{aligned}&\tau_{(i,j)}(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\lambda_1^{(i,j)}(u,\vec{v},\vec{w}),\lambda_2^{(i,j)}(u',\vec{v}',\vec{w}'))\\ =\ &\pi_{(i,j)}(u,\vec{v},\vec{w},\vec{w},u',\vec{v}',\vec{w}',\vec{w}'),\end{aligned} \qquad \text{(T}(i,j)\text{')}$$

$$\begin{aligned}&\pi_{(i,j)}(u,\vec{v},\vec{w},\vec{w}',u',\vec{v}',\vec{w}',\vec{w})\\ =\,&\tau_{(i,j+1)}(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\mu_1^{(i,j+1)}(u,\vec{v},\vec{w}),\mu_2^{(i,j+1)}(u',\vec{v}',\vec{w}')).\end{aligned}\tag{S(i,j)'}$$

Equation (T(i,j)') is of form (T) and Equation (S(i,j)') is of form (S). One sees immediately that the choice

$$\begin{aligned}&\pi_{(i,j)}(u,\vec{v},\vec{w},\vec{\tilde{w}},u',\vec{v}',\vec{w}',\vec{\tilde{w}}')\\ :=&\tau_{(i,j)}(u,\vec{v},\vec{w},u',\vec{v}',\vec{w}',\lambda_1^{(i,j)}(u,\vec{v},\vec{w}),\lambda_2^{(i,j)}(u',\vec{v}',\vec{w}'))\end{aligned}$$

fulfills both equations. By setting $N := n(\ell+1)+\ell$,

$$\begin{aligned}\sigma_{i(\ell+1)+j+1} &:= \tau_{(i,j)} \text{ for all } i\in\{0,\dots,n\},\, j\in\{0,\dots,\ell\},\\ s_{i(\ell+1)+j+1} &:= \pi_{(i,j)} \text{ for all } i\in\{0,\dots,n\},\, j\in\{0,\dots,\ell-1\},\\ s_{i(\ell+1)} &:= t_i' \text{ for all } i\in\{1,\dots,n\},\\ \eta_\alpha^{(i(\ell+1)+j+1)} &:= \mu_\alpha^{(i,j)} \text{ for all } i\in\{0,\dots,n\},\, j\in\{0,\dots,\ell\},\alpha\in\{1,2\},\\ \epsilon_\alpha^{(i(\ell+1)+j+1)} &:= \lambda_\alpha^{(i,j)} \text{ for all } i\in\{0,\dots,n\},\, j\in\{0,\dots,\ell\},\alpha\in\{1,2\},\end{aligned}$$

we get an integer $N\geqslant 1$, $(2(k+m+2))$-ary terms $\sigma_1,\dots,\sigma_{N+1}$, $(2(k+2m+1))$-ary terms $s_1,\dots,s_N$ and, for all $i\in\{1,\dots,N+1\}$, $(k+m+1)$-ary terms $\eta_1^{(i)},\eta_2^{(i)},\epsilon_1^{(i)},\epsilon_2^{(i)}$ satisfying, by construction, all the desired equations.

Now we show that the existence of the terms as in the statement implies that $\mathbb{V}$ is a weakly Mal'tsev category. By Equation (4.2), $(f_i(x,y),g_i(x,y))\in P$ for all $i\in\{1,\dots,k\}$. Hence Equation (4.5a) implies that

$$\begin{aligned}&q_1(y,x)\\ =\,&\sigma_1\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),\\ &\quad q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big),\\ &\quad q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),\\ &\quad q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big),\\ &\eta_1^{(1)}\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),\\ &\quad q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big)\Big),\\ &\eta_2^{(1)}\Big(q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),\\ &\quad q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big)\Big)\Big),\end{aligned}$$

where we write (f_i,g_i) as shorter notation for $(f_i(x,y),g_i(x,y))$. Furthermore, we have, for all $i\in\{1,\dots,N+1\}$ and $\alpha\in\{1,2\}$, that

$$\begin{aligned}&\eta_\alpha^{(i)}\Big(q_\alpha\big(y,x\big),q_\alpha\big(f_1,g_1\big),\dots,q_\alpha\big(f_k,g_k\big),\\ &\quad q_\alpha\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_\alpha\big(p_m(x,x,y),p_m(x,y,y)\big)\Big)\\ =\,&q_\alpha\Big(\eta_\alpha^{(i)}\Big(\big(y,x\big),\big(f_1,g_1\big),\dots,\big(f_k,g_k\big),\end{aligned}$$

$$\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,\big(p_m(x,x,y),p_m(x,y,y)\big)\big)\Big)$$

$$=q_\alpha\Big(\epsilon_\alpha^{(i)}\Big(\big(y,x\big),\big(f_1,g_1\big),\dots,\big(f_k,g_k\big),$$

$$\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,\big(p_m(x,x,y),p_m(x,y,y)\big)\big)\Big)$$

$$=\epsilon_\alpha^{(i)}\Big(q_\alpha\big(y,x\big),q_\alpha\big(f_1,g_1\big),\dots,q_\alpha\big(f_k,g_k\big),$$

$$q_\alpha\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_\alpha\big(p_m(x,x,y),p_m(x,y,y)\big)\big)\Big),$$

where we used Equations (4.3) and the definition of the algebra structure on $P\subseteq \mathsf{F}(x,y)\times\mathsf{F}(x,y)$. Thus, we have that

$$\sigma_i\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),$$

$$q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big),$$

$$q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),$$

$$q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big),$$

$$\eta_1^{(i)}\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),$$

$$q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big)\Big),$$

$$\eta_2^{(i)}\Big(q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),$$

$$q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big)\big)\Big)$$

$$=\sigma_i\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),$$

$$q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big),$$

$$q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),$$

$$q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big),$$

$$\epsilon_1^{(i)}\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,$$

$$q_1\big(f_k,g_k\big),q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big)\Big),$$

$$\epsilon_2^{(i)}\Big(q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,$$

$$q_2\big(f_k,g_k\big),q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big)\big)\Big)$$

for all $i\in\{1,\dots,N+1\}$. Next we have that

$$\sigma_i\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),$$

$$q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big),$$

$$q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),$$

$$
\begin{aligned}
&q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big),\\
&\epsilon_1^{(i)}\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),\\
&\qquad q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big)\Big),\\
&\epsilon_2^{(i)}\Big(q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),\\
&\qquad q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big)\Big)\Big)\\
&= s_i\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),\\
&\qquad q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big),\\
&\qquad q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big),\\
&\qquad q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),\\
&\qquad q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big),\\
&\qquad q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big)\Big)\\
&= s_i\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),\\
&\qquad q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big),\\
&\qquad q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big),\\
&\qquad q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),\\
&\qquad q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big),\\
&\qquad q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big)\Big)\\
&= \sigma_{i+1}\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),\\
&\qquad q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big),\\
&\qquad q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),\\
&\qquad q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big),\\
&\eta_1^{(i+1)}\Big(q_1\big(y,x\big),q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),\\
&\qquad q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big)\Big),\\
&\eta_2^{(i+1)}\Big(q_2\big(y,x\big),q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),\\
&\qquad q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big)\Big)\Big)
\end{aligned}
$$

for all $i \in \{1,\dots,N\}$, where we applied Equation (4.4a) in the first step. In the second step we used that

$$q_1(p_i(x,x,y),p_i(x,y,y)) = q_2(p_i(x,x,y),p_i(x,y,y)) \tag{4.10}$$

for all $i \in \{1,\dots,m\}$, which is true by Lemma 1.1.35 and the definition of

(q_1, q_2) as the cokernel pair of $[e_1, e_2]$. In the third step we utilized Equation (4.4b). Equation (4.5b) implies that

$$\begin{aligned}
\sigma_{N+1}\Big(&q_1\big(y,x\big), q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),\\
&q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big),\\
&q_2\big(y,x\big), q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),\\
&q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big),\\
\epsilon_1^{(N+1)}\Big(&q_1\big(y,x\big), q_1\big(f_1,g_1\big),\dots,q_1\big(f_k,g_k\big),\\
&q_1\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_1\big(p_m(x,x,y),p_m(x,y,y)\big)\Big),\\
\epsilon_2^{(N+1)}\Big(&q_2\big(y,x\big), q_2\big(f_1,g_1\big),\dots,q_2\big(f_k,g_k\big),\\
&q_2\big(p_1(x,x,y),p_1(x,y,y)\big),\dots,q_2\big(p_m(x,x,y),p_m(x,y,y)\big)\Big)\Big)\\
=&q_2(y,x).
\end{aligned}$$

Combining the above identities, we get that $q_1(y,x) = q_2(y,x)$ which means that $\mathbb{V}$ is a weakly Mal'tsev category by Lemma 4.2.1. □

Example 4.2.3. We consider the special case $N = 0$ in Theorem 4.2.2, i.e., we suppose that the variety $\mathbb{V}$ admits binary terms $f_1, g_1, \dots, f_k, g_k$, ternary terms $p_1, \dots, p_m$, where $k, m \geqslant 0$ are integers, a $(2(k+m+2))$-ary term σ_1 and $(k+m+1)$-ary terms $\eta_1^{(1)}, \eta_2^{(1)}, \epsilon_1^{(1)}, \epsilon_2^{(1)}$ that fulfill all the required conditions. Using Equations (4.5), Equation (4.3a) for $\eta_1^{(1)}$ and $\epsilon_1^{(1)}$, and Equation (4.3b) for $\eta_2^{(1)}$ and $\epsilon_2^{(1)}$, we get that the identity

$$\begin{aligned}
y =&\sigma_1(y, f_1, \dots, f_k, p_1(x,x,y), \dots, p_m(x,x,y),\\
&\quad x, g_1, \dots, g_k, p_1(x,y,y), \dots, p_m(x,y,y),\\
&\quad \eta_1^{(1)}(y, f_1, \dots, f_k, p_1(x,x,y), \dots, p_m(x,x,y)),\\
&\quad \eta_2^{(1)}(x, g_1, \dots, g_k, p_1(x,y,y), \dots, p_m(x,y,y)))\\
=&\sigma_1(y, f_1, \dots, f_k, p_1(x,x,y), \dots, p_m(x,x,y),\\
&\quad x, g_1, \dots, g_k, p_1(x,y,y), \dots, p_m(x,y,y),\\
&\quad \epsilon_1^{(1)}(y, f_1, \dots, f_k, p_1(x,x,y), \dots, p_m(x,x,y)),\\
&\quad \epsilon_2^{(1)}(x, g_1, \dots, g_k, p_1(x,y,y), \dots, p_m(x,y,y)))\\
=&x
\end{aligned}$$

holds in $\mathbb{V}$. In particular, $\mathbb{V}$ is a Mal'tsev variety with Mal'tsev term $p(x,y,z) := x$.

Example 4.2.4 (Distributive lattices)**.** Recall from Examples 4.1.2, Item 2 that a variety $\mathbb{L}$ of lattices is a weakly Mal'tsev category if and only if it is a variety of distributive lattices. For such a variety $\mathbb{L}$, we want to find integers $k, m, N \geqslant 0$ and terms $f_1, g_1, \dots, f_k, g_k,\ p_1, \dots, p_m,\ s_1, \dots, s_N,\ \sigma_1, \dots, \sigma_{N+1}$

and $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ for all $i \in \{1, \ldots, N+1\}$ that fulfill the conditions of Theorem 4.2.2. We observe that we can perform the following calculation in $\mathbb{L}$ to show that $q_1(y,x) = q_2(y,x)$:

$$
\begin{aligned}
& q_1(y,x) \\
=& q_1(y,x) \wedge (q_1(y,x) \vee q_1(x,y)) && \text{(absorption law)} \\
=& q_1(y,x) \wedge (q_1(y,y) \vee q_1(x,x)) && (q_1 \text{ morphism and commutativity of } \vee) \\
=& q_1(y,x) \wedge (q_2(y,y) \vee q_2(x,x)) && (q_1(y,y) = q_2(y,y) \text{ and } q_1(x,x) = q_2(x,x)) \\
=& q_1(y,x) \wedge (q_2(y,x) \vee q_2(x,y)) && (q_2 \text{ morphism and commutativity of } \vee) \\
=& q_1(y,x) \wedge (q_2(y,x) \vee q_1(x,y)) && (q_2(x,y) = q_1(x,y)) \\
=& (q_1(y,x) \wedge q_2(y,x)) \vee (q_1(y,x) \wedge q_1(x,y)) && (\text{distributivity of } \wedge \text{ over } \vee) \\
=& (q_1(y,x) \wedge q_2(y,x)) \vee (q_1(y,y) \wedge q_1(x,x)) \\
& && (q_1 \text{ morphism and commutativity of } \wedge) \\
=& (q_2(y,x) \wedge q_1(y,x)) \vee (q_1(y,y) \wedge q_1(x,x)) && (\text{commutativity of } \wedge) \\
=& (q_2(y,x) \wedge q_1(y,x)) \vee (q_2(y,y) \wedge q_2(x,x)) \\
& && (q_1(y,y) = q_2(y,y) \text{ and } q_1(x,x) = q_2(x,x)) \\
=& (q_2(y,x) \wedge q_1(y,x)) \vee (q_2(y,x) \wedge q_2(x,y)) \\
& && (q_2 \text{ morphism and commutativity of } \wedge) \\
=& q_2(y,x) \wedge (q_1(y,x) \vee q_2(x,y)) && (\text{distributivity of } \wedge \text{ over } \vee) \\
=& q_2(y,x) \wedge (q_1(y,x) \vee q_1(x,y)) && (q_2(x,y) = q_1(x,y)) \\
=& q_2(y,x) \wedge (q_1(y,y) \vee q_1(x,x)) && (q_1 \text{ morphism and commutativity of } \vee) \\
=& q_2(y,x) \wedge (q_2(y,y) \vee q_2(x,x)) && (q_1(y,y) = q_2(y,y) \text{ and } q_1(x,x) = q_2(x,x)) \\
=& q_2(y,x) \wedge (q_2(y,x) \vee q_2(x,y)) && (q_2 \text{ morphism and commutativity of } \vee) \\
=& q_2(y,x). && \text{(absorption law)}
\end{aligned}
$$

In the proof of Theorem 4.2.2, we showed, given terms satisfying the conditions as in the statement, that $q_1(y,x) = q_2(y,x)$ in the following way:

$$
\begin{aligned}
& q_1(y,x) \\
=& \sigma_1(\eta) && \text{(Equation (4.5a))} \\
=& \sigma_1(\epsilon) && (\text{(M) and Equations (4.3) for } i = 1) \\
=& s_1(1,2) && (\text{Equation (4.4a) for } i = 1) \\
=& s_1(2,1) && \text{(Equation (4.10))} \\
=& \sigma_2(\eta) && (\text{Equation (4.4b) for } i = 1) \\
=& \sigma_2(\epsilon) && (\text{(M) and Equations (4.3) for } i = 2) \\
=& s_2(1,2) && (\text{Equation (4.4a) for } i = 2) \\
=& s_2(2,1) && \text{(Equation (4.10))} \\
=& \sigma_3(\eta) && (\text{Equation (4.4b) for } i = 2) \\
& \vdots \\
=& \sigma_{N+1}(\epsilon) && (\text{(M) and Equations (4.3) for } i = N+1) \\
=& q_2(y,x), && \text{(Equation (4.5b))}
\end{aligned}
$$

where we set

$$\begin{aligned}
\sigma_i(\eta) := \sigma_i\Big(&q_1(y,x), q_1(f_1,g_1),\dots,q_1(p_1(x,x,y),p_1(x,y,y)),\dots,\\
&q_2(y,x), q_2(f_1,g_1),\dots,q_2(p_1(x,x,y),p_1(x,y,y)),\dots,\\
&\eta_1^{(i)}\big(q_1(y,x), q_1(f_1,g_1),\dots,q_1(p_1(x,x,y),p_1(x,y,y)),\dots\big),\\
&\eta_2^{(i)}\big(q_2(y,x), q_2(f_1,g_1),\dots,q_2(p_1(x,x,y),p_1(x,y,y)),\dots\big)\Big),\\
\sigma_i(\epsilon) := \sigma_i\Big(&q_1(y,x), q_1(f_1,g_1),\dots,q_1(p_1(x,x,y),p_1(x,y,y)),\dots,\\
&q_2(y,x), q_2(f_1,g_1),\dots,q_2(p_1(x,x,y),p_1(x,y,y)),\dots,\\
&\epsilon_1^{(i)}\big(q_1(y,x), q_1(f_1,g_1),\dots,q_1(p_1(x,x,y),p_1(x,y,y)),\dots\big),\\
&\epsilon_2^{(i)}\big(q_2(y,x), q_2(f_1,g_1),\dots,q_2(p_1(x,x,y),p_1(x,y,y)),\dots\big)\Big),\\
s_j(1,2) := s_i\Big(&q_1(y,x), q_1(f_1,g_1),\dots,\\
&q_1(p_1(x,x,y),p_1(x,y,y)),\dots,\\
&q_1(p_1(x,x,y),p_1(x,y,y)),\dots,\\
&q_2(y,x), q_2(f_1,g_1),\dots,\\
&q_2(p_1(x,x,y),p_1(x,y,y)),\dots,\\
&q_2(p_1(x,x,y),p_1(x,y,y)),\dots\Big),\\
s_j(2,1) := s_i\Big(&q_1(y,x), q_1(f_1,g_1),\dots,\\
&q_1(p_1(x,x,y),p_1(x,y,y)),\dots,\\
&q_2(p_1(x,x,y),p_1(x,y,y)),\dots,\\
&q_2(y,x), q_2(f_1,g_1),\dots,\\
&q_2(p_1(x,x,y),p_1(x,y,y)),\dots,\\
&q_1(p_1(x,x,y),p_1(x,y,y)),\dots\Big)
\end{aligned}$$

for all $i \in \{1,\dots,N+1\}$ and all $j \in \{1,\dots,N\}$ and (M) means that we used that q_1 and q_2 are morphisms. Hence our ansatz is that the first step, where we use the absorption law, should correspond to the step where we use Equation (4.5a). And the last step, where we use again the absorption law, should correspond to the step where we use Equation (4.5b). A step in our concrete calculation for distributive lattices where we use an equation should either correspond to a step from $\sigma_i(\epsilon)$ to $s_i(1,2)$ or from $s_i(2,1)$ to $\sigma_{i+1}(\eta)$ for some i. A step where we use that q_1 or q_2 is a morphism and an equation should correspond to a step from $\sigma_i(\eta)$ to $\sigma_i(\epsilon)$. A step where we use that $q_1(y,y) = q_2(y,y)$, $q_1(x,x) = q_2(x,x)$ or $q_1(x,y) = q_2(x,y)$ should correspond to a step from $s_i(1,2)$ to $s_i(2,1)$. For $N = 5$ we found integers $k, m \geq 0$ and terms $f_1, g_1, \dots, f_k, g_k$, $p_1, \dots, p_m$, $s_1, \dots, s_5$, $\sigma_1, \dots, \sigma_6$ and $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ for all $i \in \{1,\dots,6\}$ such that

$$\begin{aligned}
&q_1(y,x) \wedge (q_1(y,x) \vee q_1(x,y)) = \sigma_1(\eta),\\
&q_1(y,x) \wedge (q_1(y,y) \vee q_1(x,x)) = \sigma_1(\epsilon) = s_1(1,2),
\end{aligned}$$

$$
\begin{aligned}
q_1(y,x) \wedge (q_2(y,y) \vee q_2(x,x)) &= s_1(2,1) = \sigma_2(\eta),\\
q_1(y,x) \wedge (q_2(y,x) \vee q_2(x,y)) &= \sigma_2(\epsilon) = s_2(1,2),\\
q_1(y,x) \wedge (q_2(y,x) \vee q_1(x,y)) &= s_2(2,1),\\
(q_1(y,x) \wedge q_2(y,x)) \vee (q_1(y,x) \wedge q_1(x,y)) &= \sigma_3(\eta),\\
(q_1(y,x) \wedge q_2(y,x)) \vee (q_1(y,y) \wedge q_1(x,x)) &= \sigma_3(\epsilon),\\
(q_2(y,x) \wedge q_1(y,x)) \vee (q_1(y,y) \wedge q_1(x,x)) &= s_3(1,2),\\
(q_2(y,x) \wedge q_1(y,x)) \vee (q_2(y,y) \wedge q_2(x,x)) &= s_3(2,1) = \sigma_4(\eta),\\
(q_2(y,x) \wedge q_1(y,x)) \vee (q_2(y,x) \wedge q_2(x,y)) &= \sigma_4(\epsilon),\\
q_2(y,x) \wedge (q_1(y,x) \vee q_2(x,y)) &= s_4(1,2),\\
q_2(y,x) \wedge (q_1(y,x) \vee q_1(x,y)) &= s_4(2,1) = \sigma_5(\eta),\\
q_2(y,x) \wedge (q_1(y,y) \vee q_1(x,x)) &= \sigma_5(\epsilon) = s_5(1,2),\\
q_2(y,x) \wedge (q_2(y,y) \vee q_2(x,x)) &= s_5(2,1) = \sigma_6(\eta),\\
q_2(y,x) \wedge (q_2(y,x) \vee q_2(x,y)) &= \sigma_6(\epsilon),
\end{aligned}
$$

and the conditions of Theorem 4.2.2 are satisfied. One can check that $k := 0$, $m := 3$ and the following terms fulfill the requirements:

$$p_1(x,y,z) := x, \qquad p_2(x,y,z) := y, \qquad p_3(x,y,z) := z,$$

$$
\begin{aligned}
&s_1(u,w_1,w_2,w_3,\tilde{w}_1,\tilde{w}_2,\tilde{w}_3,u',w_1',w_2',w_3',\tilde{w}_1',\tilde{w}_2',\tilde{w}_3') := u \wedge (\tilde{w}_3 \vee \tilde{w}_1),\\
&s_2(u,w_1,w_2,w_3,\tilde{w}_1,\tilde{w}_2,\tilde{w}_3,u',w_1',w_2',w_3',\tilde{w}_1',\tilde{w}_2',\tilde{w}_3') := u \wedge (u' \vee \tilde{w}_2'),\\
&s_3(u,w_1,w_2,w_3,\tilde{w}_1,\tilde{w}_2,\tilde{w}_3,u',w_1',w_2',w_3',\tilde{w}_1',\tilde{w}_2',\tilde{w}_3') := (u' \wedge u) \vee (\tilde{w}_3 \wedge \tilde{w}_1),\\
&s_4(u,w_1,w_2,w_3,\tilde{w}_1,\tilde{w}_2,\tilde{w}_3,u',w_1',w_2',w_3',\tilde{w}_1',\tilde{w}_2',\tilde{w}_3') := u' \wedge (u \vee \tilde{w}_2'),\\
&s_5(u,w_1,w_2,w_3,\tilde{w}_1,\tilde{w}_2,\tilde{w}_3,u',w_1',w_2',w_3',\tilde{w}_1',\tilde{w}_2',\tilde{w}_3') := u' \wedge (\tilde{w}_3 \vee \tilde{w}_1),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1(u,w_1,w_2,w_3,u',w_1',w_2',w_3',a,b) &:= u \wedge a,\\
\sigma_2(u,w_1,w_2,w_3,u',w_1',w_2',w_3',a,b) &:= u \wedge b,\\
\sigma_3(u,w_1,w_2,w_3,u',w_1',w_2',w_3',a,b) &:= (u \wedge u') \vee a,\\
\sigma_4(u,w_1,w_2,w_3,u',w_1',w_2',w_3',a,b) &:= (u' \wedge u) \vee b,\\
\sigma_5(u,w_1,w_2,w_3,u',w_1',w_2',w_3',a,b) &:= u' \wedge a,\\
\sigma_6(u,w_1,w_2,w_3,u',w_1',w_2',w_3',a,b) &:= u' \wedge b,
\end{aligned}
$$

$$
\begin{aligned}
\eta_1^{(1)}(u,w_1,w_2,w_3) &:= u \vee w_2, & \epsilon_1^{(1)}(u,w_1,w_2,w_3) &:= w_3 \vee w_1,\\
\eta_2^{(1)}(u,w_1,w_2,w_3) &:= u, & \epsilon_2^{(1)}(u,w_1,w_2,w_3) &:= u,\\
\eta_1^{(2)}(u,w_1,w_2,w_3) &:= u, & \epsilon_1^{(2)}(u,w_1,w_2,w_3) &:= u,\\
\eta_2^{(2)}(u,w_1,w_2,w_3) &:= w_3 \vee w_1, & \epsilon_2^{(2)}(u,w_1,w_2,w_3) &:= u \vee w_2,\\
\eta_1^{(3)}(u,w_1,w_2,w_3) &:= u \wedge w_2, & \epsilon_1^{(3)}(u,w_1,w_2,w_3) &:= w_3 \wedge w_1,\\
\eta_2^{(3)}(u,w_1,w_2,w_3) &:= u, & \epsilon_2^{(3)}(u,w_1,w_2,w_3) &:= u,
\end{aligned}
$$

$$\begin{aligned}
\eta_1^{(4)}(u,w_1,w_2,w_3) &:= u, & \epsilon_1^{(4)}(u,w_1,w_2,w_3) &:= u,\\
\eta_2^{(4)}(u,w_1,w_2,w_3) &:= w_3 \wedge w_1, & \epsilon_2^{(4)}(u,w_1,w_2,w_3) &:= u \wedge w_2,\\
\eta_1^{(5)}(u,w_1,w_2,w_3) &:= u \vee w_2, & \epsilon_1^{(5)}(u,w_1,w_2,w_3) &:= w_3 \vee w_1,\\
\eta_2^{(5)}(u,w_1,w_2,w_3) &:= u, & \epsilon_2^{(5)}(u,w_1,w_2,w_3) &:= u,\\
\eta_1^{(6)}(u,w_1,w_2,w_3) &:= u, & \epsilon_1^{(6)}(u,w_1,w_2,w_3) &:= u,\\
\eta_2^{(6)}(u,w_1,w_2,w_3) &:= w_3 \vee w_1, & \epsilon_2^{(6)}(u,w_1,w_2,w_3) &:= u \vee w_2.
\end{aligned}$$

4.3 Another Mal'tsev-like property

In this section, we discuss the observation that the condition which is obtained from the syntactic characterization of weakly Mal'tsev varieties from Theorem 4.2.2 by dropping Equation (4.2) characterizes the varieties in which any reflexive regular relation is an equivalence relation.

We call a class $\mathcal{M}$ of binary relations in a finitely complete category $\mathscr{C}$ pullback-stable if, for any relation $r\colon R \rightarrowtail X \times Y$ in $\mathcal{M}$ and any morphisms $f\colon A \to X$ and $g\colon B \to Y$, the relation $\rho\colon P \rightarrowtail A \times B$ given by the pullback

$$\begin{array}{ccc}
P & \xrightarrow{\rho} & A \times B \\
{\scriptstyle p_R}\downarrow & \lrcorner & \downarrow{\scriptstyle f\times g} \\
R & \xrightarrow[r]{} & X \times Y,
\end{array}$$

lies also in $\mathcal{M}$. The classes $\mathcal{M}_{\mathrm{all}}$ of all binary relations in $\mathscr{C}$, $\mathcal{M}_{\mathrm{strong}}$ of strong binary relations and $\mathcal{M}_{\mathrm{reg}}$ of regular binary relations are pullback-stable.

Definition 4.3.1 ($\mathcal{M}$-Mal'tsev category)**.** Let $\mathscr{C}$ be a finitely complete category and $\mathcal{M}$ be a pullback-stable class of binary relations in $\mathscr{C}$. We call $\mathscr{C}$ an *$\mathcal{M}$-Mal'tsev category* if any reflexive relation $r\colon R \rightarrowtail X \times X$ in $\mathcal{M}$ is an equivalence relation.

Let $\mathscr{C}$ be a finitely complete category. By Definition 1.1.15, it is an $\mathcal{M}_{\mathrm{all}}$-Mal'tsev category if and only if it is a Mal'tsev category. Furthermore, by Proposition 4.1.3, it is an $\mathcal{M}_{\mathrm{strong}}$-Mal'tsev category if and only if it is a weakly Mal'tsev category. In this section, we are interested in $\mathcal{M}_{\mathrm{reg}}$-Mal'tsev categories.

The following proposition recovers Proposition 1.1.17 and parts of Proposition 4.1.3. In [JMF12], Z. Janelidze and N. Martins-Ferreira show similarly the following equivalences for $\mathcal{M}$ being a proper class of binary relations in a category with pullbacks and equalizers (see also [MF20, Theorem 1]).

Proposition 4.3.2. Let $\mathscr{C}$ be a finitely complete category and $\mathcal{M}$ be a pullback-stable class of binary relations in $\mathscr{C}$. Then the following conditions are equivalent:

1. The category $\mathscr{C}$ is an $\mathcal{M}$-Mal'tsev category.
2. Any reflexive relation in $\mathcal{M}$ is symmetric.

3. Any reflexive relation in $\mathcal{M}$ is transitive.

4. Any relation in $\mathcal{M}$ is difunctional.

Proof. It is clear that Item 1. implies Items 2. and 3.. Item 4. implies Item 1. since, as it is well-known, any reflexive relation is an equivalence relation if and only if it is difunctional. For Item 2. implies Item 4., we consider a relation $r = (r_1, r_2) \colon R \rightarrowtail X \times Y$ in $\mathcal{M}$. Since $\mathcal{M}$ is assumed to be pullback-stable, the relation $\rho \colon P \rightarrowtail R \times R$ in the pullback

$$\begin{array}{ccc} P & \xrightarrow{\rho} & R \times R \\ {\scriptstyle p_R}\downarrow & \lrcorner & \downarrow{\scriptstyle r_1 \times r_2} \\ R & \xrightarrow[r]{} & X \times Y, \end{array}$$

lies also in $\mathcal{M}$. Furthermore, it is clear that it is reflexive. Hence, by assumption, ρ is symmetric. It is a classical result that ρ being symmetric is equivalent to r being difunctional. For Item 3. implies Item 4., we note that r is difunctional if and only if ρ is transitive. □

In Section 1.1.3.2 (Proof 2), we saw a proof of Mal'tsev's theorem relying on the characterization of Mal'tsev categories as those finitely complete categories in which any reflexive relation is symmetric. Here we adapt this proof to obtain a syntactic characterization of those varieties of universal algebras that are $\mathscr{M}_{\mathrm{reg}}$-Mal'tsev categories.

Lemma 4.3.3. For a variety $\mathbb{V}$ of universal algebras, the following conditions are equivalent:

1. $\mathbb{V}$ is an $\mathscr{M}_{\mathrm{reg}}$-Mal'tsev category.

2. Any reflexive regular relation in $\mathbb{V}$ is symmetric.

3. The smallest reflexive regular relation E on $\mathsf{F}(x, y)$ containing the element $([x], [y])$ is symmetric.

4. The element $([y], [x])$ lies in E.

Proof. The equivalence of Items 1 and 2 is clear by Proposition 4.3.2. Trivially, Item 2 implies Item 3, and Item 3 implies Item 4. Similarly as in the proof of Lemma 1.1.34, one can show that Item 4 implies Item 2. □

Theorem 4.3.4. A variety $\mathbb{V}$ of universal algebras is an $\mathcal{M}_{\mathrm{reg}}$-Mal'tsev category if and only if there exist $k, m, N \in \mathbb{N}_0$, binary terms $f_1, g_1, \ldots, f_k, g_k$, ternary terms $p_1, \ldots, p_m$, $(2(k+2m+1))$-ary terms $s_1, \ldots, s_N$, $(2(k+m+2))$-ary terms $\sigma_1, \ldots, \sigma_{N+1}$ and, for all $i \in \{1, \ldots, N+1\}$, $(k+m+1)$-ary terms $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ such that the following identities (on variables x, y, u, u', $v_1, \ldots, v_k$, $v'_1, \ldots, v'_k$, $w_1, \ldots, w_m$ and $w'_1, \ldots, w'_m$) hold, where we write $\vec{v}$ for $v_1, \ldots, v_k$ and $\vec{w}$ for $w_1, \ldots, w_m$, and analogously for $\vec{v}'$ and $\vec{w}'$:

$$\begin{aligned} &\eta_\alpha^{(i)}(y, f_1(x,y), \ldots, f_k(x,y), p_1(x,x,y), \ldots, p_m(x,x,y)) \\ \approx &\epsilon_\alpha^{(i)}(y, f_1(x,y), \ldots, f_k(x,y), p_1(x,x,y), \ldots, p_m(x,x,y)), \end{aligned} \tag{4.11a}$$

$$\begin{aligned}&\eta_\alpha^{(i)}(x, g_1(x,y), \dots, g_k(x,y), p_1(x,y,y), \dots, p_m(x,y,y))\\ \approx&\epsilon_\alpha^{(i)}(x, g_1(x,y), \dots, g_k(x,y), p_1(x,y,y), \dots, p_m(x,y,y)),\end{aligned} \tag{4.11b}$$

for all $i \in \{1, \dots, N+1\}$ and $\alpha \in \{1, 2\}$;

$$\begin{aligned}&\sigma_i(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(i)}(u, \vec{v}, \vec{w}), \epsilon_2^{(i)}(u', \vec{v}', \vec{w}'))\\ \approx&s_i(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}'),\end{aligned} \tag{4.12a}$$

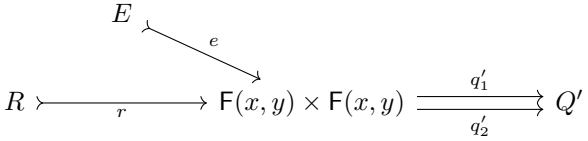
$$\begin{aligned}&s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w})\\ \approx&\sigma_{i+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(i+1)}(u, \vec{v}, \vec{w}), \eta_2^{(i+1)}(u', \vec{v}', \vec{w}')),\end{aligned} \tag{4.12b}$$

for all $i \in \{1, \dots, N\}$ and

$$u \approx \sigma_1(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(1)}(u, \vec{v}, \vec{w}), \eta_2^{(1)}(u', \vec{v}', \vec{w}')), \tag{4.13a}$$

$$u' \approx \sigma_{N+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(N+1)}(u, \vec{v}, \vec{w}), \epsilon_2^{(N+1)}(u', \vec{v}', \vec{w}')). \tag{4.13b}$$

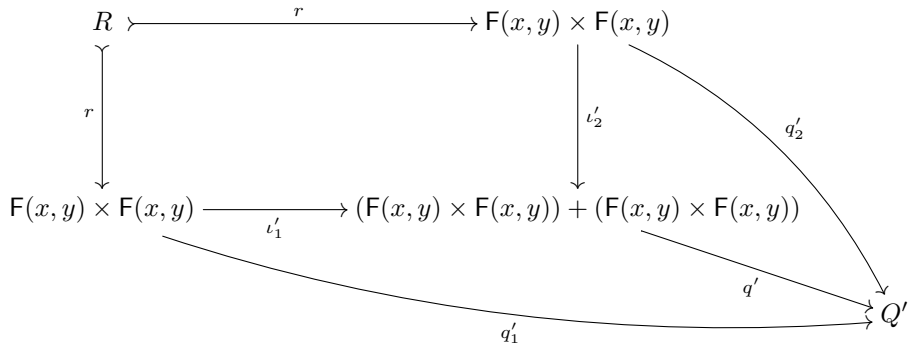
Proof. By Lemma 4.3.3, we know that $\mathbb{V}$ is an $\mathcal{M}_{\mathrm{reg}}$-Mal'tsev category if and only if the element (y, x) lies in the smallest reflexive regular relation E on $\mathsf{F}(x, y)$ containing the element (x, y). It is clear that the relation E is given by the equalizer e of the cokernel pair injections q_1', q_2' of the smallest reflexive relation R on $\mathsf{F}(x, y)$ containing the element (x, y):



By Lemma 1.1.37, R is given by

$$\{(p(x,x,y), p(x,y,y) \in \mathsf{F}(x,y) \times \mathsf{F}(x,y)) \mid p \in \mathsf{F}(x,y,z)\}.$$

To construct E, we first build the cokernel pair (Q', q_1', q_2') by means of the coequalizer q' of the maps $\iota_1' r, \iota_2' r$, where ι_1', ι_2' are the coproduct injections into $\mathsf{F}(x,y)^2 + \mathsf{F}(x,y)^2$, as in the following diagram:



The algebra Q' is given by the quotient of $\mathsf{F}(x,y)^2 + \mathsf{F}(x,y)^2$ with respect to the smallest congruence $C' \subseteq (\mathsf{F}(x,y)^2 + \mathsf{F}(x,y)^2)^2$ containing all pairs of the form

$$\Big(\iota_1'\big(p(x,x,y),p(x,y,y)\big),\iota_2'\big(p(x,x,y),p(x,y,y)\big)\Big),$$

where $p \in \mathsf{F}(x,y,z)$. The algebra E consists of all pairs $(s,t) \in \mathsf{F}(x,y) \times \mathsf{F}(x,y)$ such that $q'(\iota_1'(s,t)) = q'(\iota_2'(s,t))$. Thus, the condition $(y,x) \in E$ is equivalent to $(\iota_1'(y,x),\iota_2'(y,x)) \in C'$. Proceeding from here analogously as in the proof of Theorem 4.2.2 yields the claim. We note that, in comparison to the proof of Theorem 4.2.2, here

$$P = \{(t_1(x,y),t_2(x,y)) \in \mathsf{F}(x,y) \times \mathsf{F}(x,y) \mid t_1(x,x) = t_2(x,x)\}$$

is replaced by $\mathsf{F}(x,y) \times \mathsf{F}(x,y)$, which explains why the condition $f_i(x,x) = g_i(x,x)$ for all $i \in \{1,\dots,k\}$ does not appear here. □

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