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Interfaces with Other Disciplines

Do stable outcomes survive in marriage problems with myopic and farsighted players?

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ABSTRACT

We consider marriage problems where myopic and farsighted players interact and analyze these problems by means of the myopic-farsighted stable set. We require that coalition members are only willing to deviate if they all strictly benefit from doing so. Our first main result establishes the equivalence of myopic-farsighted stable sets based on arbitrary coalitional deviations and those based on pairwise deviations.

We are interested in the question whether the core is still the relevant solution concept when myopic and farsighted agents interact and whether more farsighted agents are able to secure more preferred core elements. For marriage problems where all players are myopic as well as those where all players are farsighted, myopic-farsighted stable sets lead to the same prediction as the core. The same result holds for α-reducible marriage problems, without any assumptions on the set of farsighted agents. These results change when one side of the market is more farsighted than the other. For general marriage problems where all women are farsighted, only one core element can be part of a myopic-farsighted stable set, the woman-optimal stable matching. If the woman-optimal stable matching is dominated from the woman point of view by an individually rational matching, then no core element can be part of a myopic-farsighted stable set.

1. Introduction

Stability is considered to be a crucial property in marriage problems. A matching is stable if no individual player prefers to leave his or her current partner and no pair of players prefers to form a match between them. For marriage problems, this stability notion is equivalent to core stability. A matching is in the core if there is no subset of players who can all obtain a strictly preferred outcome by forming only partnerships among themselves. Gale and Shapley (1962) have shown that the set of stable matchings, and therefore the core, is non-empty. Their approach to establish non-emptiness of the core is algorithmic and has spurred an extensive literature in operations research (Gusfield & Irving, 1989). For marriage problems, the core coincides with the myopic stable set (Demuynck, Herings, Saulle, & Seel, 2019). Also farsighted solution concepts point towards the core as the set of equilibrium outcomes, but neither the myopic nor the farsighted approach is able to discriminate between different core elements.

In this paper, we allow for the interaction between myopic and farsighted players and propose to analyze this interaction by means of the pairwise myopic-farsighted stable set. We are interested in the question whether the core is still the relevant solution concept when myopic and farsighted agents interact and whether more farsighted agents are able to secure more preferred core elements.

A pairwise myopic-farsighted stable set is the set of matchings satisfying internal and external stability with respect to the notion of a pairwise myopic-farsighted improving path. That is, there is no pairwise myopic-farsighted improving path from any matching in the set to another matching in the set (internal stability) and there is a pairwise myopic-farsighted improving path from any matching outside the set to some matching in the set (external stability).

We define a pairwise myopic-farsighted improving path as a sequence of matchings that can emerge when farsighted players form or destroy matches based on the improvement the end matching offers relative to the current one while myopic players form or destroy matches based on the improvement the next matching offers relative to the current one. Each matching in the sequence differs from the previous one in that either a new match is formed or an existing one is destroyed.

In the definition of the pairwise myopic-farsighted stable set, deviations are made by pairs or single players. Standard notions of the vNM stable set allow for deviations by arbitrary coalitions and not

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only by coalitions of size one or two. We study whether allowing for deviations by arbitrary coalitions is equivalent to deviations by coalitions of size one or two in marriage problems with both myopic and farsighted players. We find that there is an equivalence between the pairwise myopic-farsighted stable set and the coalitional myopic-farsighted stable set.

We next identify conditions that result in an equivalence between the core and the myopic-farsighted stable set. We show that for α-reducible matching problems as defined in Alcalde (1995), the pairwise myopic-farsighted stable set is unique and equal to the core. This result holds independently from the distribution of myopic and farsighted players over the two sides of the marriage market.

Things become different when α-reducibility is not satisfied. Consider marriage problems where all women are farsighted and all men are myopic. The only core element that can be part of a pairwise myopic-farsighted stable set is the woman-optimal stable matching. This result shows that the farsighted side of the market is able to rule out unfavorable core elements. Moreover, if the woman-optimal stable matching is Pareto dominated from the woman point of view, then no pairwise myopic-farsighted stable set can be formed by core elements.

In this paper we identify important classes of marriage problems for which the existence of myopic-farsighted stable sets is guaranteed. Subsequent to the writing of this paper, Doğan and Ehlers (2024) extend these results and prove that a myopic-farsighted stable set exists for any one-to-one matching problem. Their Corollary 1 establishes that when all men are myopic and all women are farsighted, the efficiency-enhanced deferred acceptance (EADA) matching constitutes a singleton myopic-farsighted set. When the EADA-matching coincides with the women-optimal stable matching, there exists a myopic-farsighted stable set that is a subset of the core (Corollary 2). These results confirm our findings that the woman-optimal stable matching is the unique core element that can be a myopic-farsighted stable set. Finally, their Proposition 4 extends the result of Atay, Mauleon, and Vannetelbosch (2025) in priority-based school choice problems and establishes that, when all men are myopic and all women are farsighted, the Top Trading Cycles (TTC) matching is also a singleton myopic-farsighted stable set in one-to-one matching problems. Thus, multiple myopic-farsighted stable sets could exist.

In this paper we analyze the interaction between myopic and farsighted players by an adaptation of the vNM farsightedly stable set. Such an interaction can potentially also be analyzed by other approaches to farsightedness that have been suggested in the literature on coalition and network formation. We mention the largest consistent set of Chwe (1994), the optimistic and conservative stable standards of behavior of Xue (1998), the stable set with respect to path dominance in Page Jr. and Wooders (2009) and farsightedly stability as defined in Herings, Mauleon, and Vannetelbosch (2009). Non-cooperative approaches towards farsightedness have been used as well. Bloch (1996) studies the stationary subgame perfect equilibria of a dynamic game, whereas Herings, Mauleon, and Vannetelbosch (2004) use an approach based on rationalizability. Anesi (2010) supports the vNM stable set as a stationary subgame perfect equilibrium in legislative bargaining games. The dynamic process of coalition formation by Konishi and Ray (2003) has many features of a non-cooperative equilibrium and can be obtained as a non-cooperative equilibrium if one allows the process to be history-dependent as shown in Vartiainen (2011). Kimya (2020) proposes the equilibrium coalitional behavior and the credible equilibrium coalitional behavior and bridges the non-cooperative and the cooperative approaches to farsightedness.

The paper is organized as follows. Section 2 introduces marriage problems and standard notions of stability. Section 3 defines the pairwise and the coalitional myopic-farsighted stable set and demonstrates their equivalence. Section 4 provides an example where all women are farsighted and all men are myopic and the unique pairwise myopic-farsighted stable set contains a matching outside the core. Section 5 presents the analysis on the relation between the core and the pairwise myopic-farsighted stable set. Section 6 compares the results to those in Herings, Mauleon, and Vannetelbosch (2017), where weak improvements are sufficient for coalitions to deviate. Section 7 contains the conclusion.

2. Marriage problems

A marriage problem consists of a finite set of players N, partitioned into a set of men M and a set of women W. The set of non-empty subsets of N is denoted by $\mathcal{N}$. Players can be either farsighted or myopic. The farsighted players are collected in the set $F \subseteq N$, which is allowed to be empty or equal to N. Myopic players only care about immediate payoffs, whereas farsighted players take a long-run perspective. The implications for behavior of being either myopic or farsighted, are made precise in Section 3.

Each player $i \in N$ has a complete and transitive *preference ordering* $\succ_i$ over the players of the opposite sex and the prospect of being alone. Preferences are assumed to be strict. We write $j \sim_i j'$ if player i is indifferent between j and j', which can only be the case if $j = j'$, and $j \succeq_i j'$ if $j \succ_i j'$ or $j \sim_i j'$. Let $\succ = ((\succ_m)_{m\in M}, (\succ_w)_{w\in W})$ be a preference profile.[1]

A *marriage problem* is a tuple $(M, W, F, \succ)$. A *matching* is a function $\mu : N \to N$ satisfying that (i) for every $m \in M$, $\mu(m) \in W \cup \{m\}$, (ii) for every $w \in W$, $\mu(w) \in M \cup \{w\}$, and (iii) for every $i \in N$, $\mu(\mu(i)) = i$. The set of all matchings is denoted by $\mathcal{M}$. Given a matching $\mu \in \mathcal{M}$, player i is said to be *single* if $\mu(i) = i$. A matching μ is individually rational if each player is acceptable to his or her partner, so for every $i \in N$ it holds that $\mu(i) \succeq_i i$. Let $\mathcal{I}$ be the set of matchings that are individually rational, i.e., $\mathcal{I} = \left\{\mu \in \mathcal{M} \mid \text{for every } i \in N,\ \mu(i) \succeq_i i\right\}$. A matching μ is said to be *stable* if it is individually rational and there is no pair $(m, w) \in M \times W$ that are not matched to one another but prefer each other to their partners at μ, i.e., $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

For every $i \in N$, we extend the preference ordering $\succ_i$ over the player's potential partners to the set of matchings $\mathcal{M}$ in the following way. We say that player i prefers the matching μ' to the matching μ if $\mu'(i) \succ_i \mu(i)$ and we write $\mu' \succ_i \mu$. We write $\mu' \sim_i \mu$ if $\mu'(i) \sim_i \mu(i)$, and $\mu' \succeq_i \mu$ if $\mu' \succ_i \mu$ or $\mu' \sim_i \mu$.

For $S \in \mathcal{N}$, $\mu(S) = \{\mu(i) \in N \mid i \in S\}$ denotes the set of partners of players in S at μ. A coalition $S \in \mathcal{N}$ is said to *block* a matching $\mu \in \mathcal{M}$ if there exists a matching $\mu' \in \mathcal{M}$ such that $\mu'(S) = S$ and $\mu' \succ_S \mu$, where $\mu' \succ_S \mu$ is defined as $\mu'(i) \succ_i \mu(i)$ for every $i \in S$. The core of the marriage problem $(M, W, F, \succ)$ consists of all matchings that are not blocked by any coalition. We denote the set of matchings that belong to the core by C.

It has been shown by Gale and Shapley (1962) that the core of a marriage problem is non-empty. Also, a matching is stable if and only if it is not blocked by a coalition of size one or two if and only if it belongs to the core, see Theorem 3.3 in Roth and Sotomayor (1990). Knuth (1976) has shown that the core of a marriage problem is a distributive lattice. In particular, there is a man-optimal stable matching μ^{M} and a woman-optimal stable matching μ^{W}. For any matching μ in the core, for every $m \in M$, it holds that $\mu^{\mathrm{M}} \succeq_m \mu$. Similarly, for any matching μ in the core, for every $w \in W$, it holds that $\mu^{\mathrm{W}} \succeq_w \mu$. Roth (1985) shows that the existence of side-optimal stable matchings generalizes substantially beyond the one-to-one matching set-up.

3. Pairwise and coalitional myopic-farsighted stable sets

We propose the notions of pairwise and coalitional myopic-farsighted stable sets to study the matchings that are stable when myopic and farsighted players interact with each other. The notion of

[1] The possibility of ties is very relevant and considerably complicates the analysis, even in the absence of farsighted agents, see Irving (1994) for the notion of stability itself in the presence of indifferences, and Diebold and Bichler (2017) and Delorme et al. (2019) for a comparison of the performance of various algorithms in matching problems with indifferences.

pairwise myopic-farsighted stable set has been introduced in Herings et al. (2017) when deviating pairs are required to make weak improvements. Here we study both pairwise and coalitional deviations and require strict improvements for all deviating players.

A coalitional myopic-farsighted improving path is a sequence of matchings that can emerge when farsighted players form or destroy matches based on the improvement the end matching offers them relative to the current one while myopic players form or destroy matches based on the improvement the next matching in the sequence offers them relative to the current one.

Given a matching $\mu \in \mathcal{M}$, a coalition $S \in \mathcal{N}$ can enforce matching $\mu' \in \mathcal{M}$ if any match in μ' that does not exist in μ has to be between players in S. Moreover, if a match in μ is destroyed, then at least one of the two players involved in that match belongs to S. The next definition formalizes these ideas.

Definition 1. Given a matching $\mu \in \mathcal{M}$, a coalition $S \in \mathcal{N}$ is able to *enforce* the matching μ' if the following conditions hold: For every $i \in N$, (i) $\mu'(i) \notin \{\mu(i), i\}$ implies $\{i, \mu'(i)\} \subseteq S$ and (ii) $\mu'(i) = i \neq \mu(i)$ implies $\{i, \mu(i)\} \cap S \neq \emptyset$.

Given a matching $\mu \in \mathcal{M}$ with man $m \in M$ matched to woman $w \in W$, so $\mu(m) = w$, the matching μ' that is identical to μ, except that the match between m and w has been destroyed by either m or w, is denoted by $\mu - (m, w)$. Given a matching $\mu \in \mathcal{M}$ such that $m \in M$ and $w \in W$ are not matched to one another, the matching μ' that is identical to μ, except that the pair (m, w) has formed at μ' and their partners at μ, i.e., $\mu(w)$ and $\mu(m)$, are now single at μ', is denoted by $\mu + (m, w)$.

Definition 2. Let $(M, W, F, \succ)$ be a marriage problem. A *coalitional myopic-farsighted improving path* from a matching $\mu \in \mathcal{M}$ to a matching $\mu' \in \mathcal{M} \setminus \{\mu\}$ is a finite sequence of distinct matchings $\mu_0, \ldots, \mu_L$ with $\mu_0 = \mu$ and $\mu_L = \mu'$ such that for every $\ell \in \{0, \ldots, L-1\}$ there is a coalition $S_\ell \in \mathcal{N}$ that can enforce $\mu_{\ell+1}$ from μ_ℓ and

$$\mu_{\ell+1}(i) \succ_i \mu_\ell(i) \quad \text{if } i \in S_\ell \setminus F,$$
$$\mu_L(i) \succ_i \mu_\ell(i) \quad \text{if } i \in S_\ell \cap F.$$

A *pairwise myopic-farsighted improving path* from a matching $\mu \in \mathcal{M}$ to a matching $\mu' \in \mathcal{M} \setminus \{\mu\}$ is a coalitional myopic-farsighted improving path $\mu_0, \ldots, \mu_L$ such that for every $\ell \in \{0, \ldots, L-1\}$ either $\mu_{\ell+1} = \mu_\ell - (m, w)$ for some $(m, w) \in M \times W$ or $\mu_{\ell+1} = \mu_\ell + (m, w)$ for some $(m, w) \in M \times W$.

Without loss of generality, the moving coalition in step ℓ of a pairwise myopic-farsighted improving path is equal to $\{(m, w)\}$ when $\mu_{\ell+1} = \mu_\ell + (m, w)$ and is equal to $\{m\}$ or $\{w\}$ when $\mu_{\ell+1} = \mu_\ell - (m, w)$, i.e. is equal to a minimal coalition that can enforce the matching $\mu_{\ell+1}$ from μ_ℓ.

Let some $\mu \in \mathcal{M}$ be given. The set of matchings $\mu' \in \mathcal{M}$ such that there is a coalitional myopic-farsighted improving path from μ to μ' is denoted by $g^*(\mu)$. The set of all matchings that can be reached by a pairwise myopic-farsighted improving path is denoted by $h^*(\mu)$. It clearly holds that $h^*(\mu) \subseteq g^*(\mu)$.

The coalitional, respectively pairwise, myopic-farsighted stable set results when the conditions of internal and external stability are based on the coalitional, respectively pairwise, myopic-farsighted improving paths.

Definition 3. Let $(M, W, F, \succ)$ be a marriage problem. A set of matchings $V \subseteq \mathcal{M}$ is a *coalitional myopic-farsighted stable set* if it satisfies:

(i) Internal stability: For every $\mu, \mu' \in V$, it holds that $\mu' \notin g^*(\mu)$.

(ii) External stability: For every $\mu \in \mathcal{M} \setminus V$, it holds that $g^*(\mu) \cap V \neq \emptyset$.

A set of matchings $V \subseteq \mathcal{M}$ is a *pairwise myopic-farsighted stable set* if it satisfies (i) and (ii) with g^* replaced by h^*.

Condition (i) of Definition 3 corresponds to internal stability. For any two matchings μ and μ' in the coalitional (pairwise) myopic-farsighted stable set V there is no coalitional (pairwise) myopic-improving path connecting μ to μ'. Condition (ii) of Definition 3 expresses external stability. There always exists a coalitional (pairwise) myopic-improving path from every matching μ outside the coalitional (pairwise) myopic-farsighted stable set V to some matching in V.[2]

We first address the question whether it makes a difference whether coalitions of size larger than two may form. It turns out that such is not the case and we demonstrate in Theorem 2 that the pairwise and coalitional myopic-farsighted stable sets coincide. It then follows that all existence and uniqueness results obtained for pairwise myopic-farsighted stable sets carry over to coalitional myopic-farsighted stable sets.

Pairwise myopic-farsighted stable sets and coalitional myopic-farsighted stable sets coincide if g^* and h^* coincide. It follows from Example 2 in Herings et al. (2017) that this is not the case if all players are myopic. Example 1 extends this example to show that g^* and h^* do not coincide irrespective of whether players are myopic or farsighted.

Example 1. Consider a matching problem $(M, W, F, \succ)$, where $M = \{m_1, m_2\}$, $W = \{w_1, w_2\}$, the set F is taken arbitrary, and the preferences of the individuals are as follows.

m_1	m_2	w_1	w_2
w_1	w_2	w_1	w_2
w_2	w_1	m_1	m_2
m_1	m_2	m_2	m_1

The preferences in Example 1 are such that man m_1 prefers to be married to w_1 rather than to w_2 and m_2 prefers to be married to w_2 rather than to w_1. Also woman w_1 prefers m_1 over m_2 and woman w_2 prefers m_2 over m_1, but contrary to the men, women prefer not to be married at all. The matching μ' is defined by $\mu'(m_1) = w_2$ and $\mu'(m_2) = w_1$. The matching μ^* is defined by $\mu^*(m_1) = w_1$ and $\mu^*(m_2) = w_2$.

It is immediate that $\mu^* \in g^*(\mu')$, since the coalition N can go from μ' to μ^* in one step, leading to a strict improvement for every player.

We argue next that $\mu^* \notin h^*(\mu')$ by showing that at any $\mu \in h^*(\mu')$ it holds that at least one of the women is single. Suppose $\mu_0, \ldots, \mu_L$ is a pairwise myopic-farsighted improving path from $\mu_0 = \mu'$ to $\mu_L = \mu^*$. At μ_1, exactly one of the women is single. Irrespective of whether this woman is myopic or farsighted, she is not willing to form a match with a man again and it is not possible to reach μ^*. We have shown that $\mu^* \notin h^*(\mu')$. △

The next result demonstrates that $g^*(\mu)$ and $h^*(\mu)$ are identical for individually rational matchings μ.

Theorem 1. *Let $(M, W, F, \succ)$ be a marriage problem. For every $\mu \in \mathcal{I}$ it holds that $g^*(\mu) = h^*(\mu)$.*

Lemma 2 in the Appendix states that any element of a pairwise myopic-farsighted stable set is individually rational. Together with Theorem 1, we obtain an equivalence between the pairwise and the coalitional myopic-farsighted stable set.

Theorem 2. *Let $(M, W, F, \succ)$ be a marriage problem. A set of matchings $V \subseteq \mathcal{M}$ is a pairwise myopic-farsighted stable set if and only if $V \subseteq \mathcal{M}$ is a coalitional myopic-farsighted stable set.*

From now on, we derive results for pairwise myopic-farsighted stable sets, which extend to coalitional myopic-farsighted stable sets by virtue of Theorem 2.

[2] Doğan and Ehlers (2024) recently prove that a myopic-farsighted stable set exists for any one-to-one matching problem.

4. Coincidence of pairwise myopic-farsighted stable sets and the core

In this section, we present general conditions for which the pairwise myopic-farsighted stable sets and the core lead to the same predictions.

In the special case where all players are myopic, $F = \emptyset$, the pairwise myopic-farsighted stable set is equivalent to the pairwise CP vNM set of Herings et al. (2017) and is equal to the core.

In the special case where all players are farsighted, $F = N$, coalitional myopic-farsighted stable sets are equivalent to the vNM farsightedly stable sets of Mauleon, Vannetelbosch, and Vergote (2011), which are characterized as the singletons consisting of a core element. The pairwise myopic-farsighted stable sets are therefore characterized as singleton core elements by virtue of Theorem 2.

We now turn to the heterogeneous case. One popular restriction that guarantees uniqueness of the core is the condition of α-reducibility as introduced in Alcalde (1995) for roommate problems. This condition constitutes a special case of the top-coalition property, which was introduced in Banerjee, Konishi, and Sönmez (2001) for hedonic games.

Definition 4. Let $(M, W, F, \succ)$ be a marriage problem and $S \in \mathcal{N}$. A man $m' \in S \cap M$ and a woman $w' \in S \cap W$ are $\succ$-*reciprocal* on S if, for every $w \in S \cap W$, $w' \succeq_{m'} w$, and, for every $m \in S \cap M$, $m' \succeq_{w'} m$. The non-empty coalition $T \subseteq S$ is a *top coalition* on S if it is either a singleton consisting of a player who prefers being single to any match in S or a two-player coalition consisting of a $\succ$-reciprocal pair on S. The marriage problem $(M, W, F, \succ)$ is α-*reducible* if for every $S \in \mathcal{N}$ there is a top coalition on S.

The definition of α-reducibility requires that for any group of players there are always two players that top rank each other within the group or there is a player who prefers to remain single.

Consider an α-reducible marriage problem. For $k = 1, \ldots, \ell$, let S_k^* be a top coalition of $N \setminus \cup_{j=1}^{k-1} S_j^*$ and let $\cup_{k=1}^{\ell} S_k^* = N$. The matching μ^* such that for every $i \in N$ there is $k \in \{1, \ldots, \ell\}$ such that $S_k^* = \{i, \mu^*(i)\}$ is called the matching induced by the partition $\{S_1^*, S_2^*, \ldots, S_k^*\}$. Alcalde (1995) shows that when α -reducibility is satisfied, the core is unique and consists of the matching μ^*, which is therefore trivially equal to μ^{W}. It follows that the partition $\{S_1^*, S_2^*, \ldots, S_k^*\}$, called the *top-coalition partition*, is unique as well.

We show next that for matching problems satisfying α-reducibility, the pairwise myopic-farsighted stable set is unique and equal to the core.

Theorem 3. *Let $(M, W, F, \succ)$ be an α-reducible marriage problem. Then V is a pairwise myopic-farsighted stable set if and only if $V = C$.*

Whenever α-reducibility is satisfied, the core and the pairwise myopic-farsighted stable set make exactly the same predictions. An interesting aspect of this result is that the distribution of myopic and farsighted players over the two sides of the market does not matter.

It is relatively straightforward to construct a pairwise myopic-farsighted improving path from an arbitrary matching μ different from μ^{W} to μ^{W}. Whenever the players in S_1^* are not matched under μ, then they form a link. Next, we consider the players in S_2^*. Again, whenever they are not matched, they form a link. In this way, we proceed until the matching μ^{W} is reached. Independent of whether players are myopic or farsighted, it is in their interest to form a link whenever they are asked to do so along the path.

5. An example

In this section we present an example where the pairwise myopic-farsighted stable set is unique and does not intersect the core.

We present our results in the form of three claims. Claim 1 asserts that $\overline{\mu} \in h^*(\mu^{\mathrm{W}})$, where $\overline{\mu}$ is a particular matching that Pareto dominates μ^{W} from the perspective of the women. Claim 2 states that $\{\overline{\mu}\}$ is a pairwise myopic-farsighted stable set and Claim 3 that there is no other such set.

Consider the marriage problem $(M, W, F, \succ)$, which corresponds to Example 2.31 of Roth and Sotomayor (1990) with the roles of men and women reversed. It holds that $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Assume $F = \{w_1, w_2, w_3\}$, so all women are farsighted and all men are myopic. The preferences of the players are as follows.

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_3	w_1	m_2	m_1	m_1
w_2	w_1	w_2	m_1	m_2	m_2
w_3	w_2	w_3	m_3	m_3	m_3.

By applying the deferred acceptance algorithm of Gale and Shapley (1962), it can be easily verified that the woman-optimal stable matching is equal to $\mu^{\mathrm{W}}(m_1) = w_1$, $\mu^{\mathrm{W}}(m_2) = w_3$, and $\mu^{\mathrm{W}}(m_3) = w_2$. Moreover, as shown in Roth (1982), when the deferred acceptance algorithm is used, it is not possible for the women to obtain a more favorable matching by misrepresenting their preferences.

The matching $\overline{\mu}$ defined by $\overline{\mu}(m_1) = w_3$, $\overline{\mu}(m_2) = w_1$, and $\overline{\mu}(m_3) = w_2$, is strictly preferred by w_1 and w_3 to μ^{W} and does not make a difference for w_2. However, the pair (m_1, w_2) can block $\overline{\mu}$, so $\overline{\mu}$ does not belong to the core.

A matching $\overline{\mu} \in \mathcal{M}$ is said to W-dominate a matching $\mu \in \mathcal{M}$ if $\overline{\mu}(i) \succeq_i \mu(i)$ for all $i \in W$ and $\overline{\mu}(i) \succ_i \mu(i)$ for some $i \in W$.

Claim 1. $\overline{\mu} \in h^*(\mu^{\mathrm{W}})$

Claim 1 implies that it is possible that farsighted women leave the woman-optimal stable matching to end up in a matching that W-dominates μ^{W}. To see this, consider the pairwise myopic-farsighted improving path $\mu_0, \ldots, \mu_4$ with $\mu_0 = \mu^{\mathrm{W}}$ and $\mu_4 = \overline{\mu}$, where $\mu_1 = \mu_0 - (m_1, w_1)$, $\mu_2 = \mu_1 - (m_2, w_3)$, $\mu_3 = \mu_2 + (m_2, w_1)$, and $\mu_4 = \mu_3 + (m_1, w_3)$.

The move to μ_1 is initiated by w_1 who is farsighted and therefore wants to destroy her match with m_1 in the anticipation of ending up in a match with m_2. Similarly, the move from μ_1 to μ_2 is initiated by w_3 who is farsighted and is willing to divorce m_2 in the expectation of being matched with m_1. The transition to $\mu_4 = \overline{\mu}$ is completed by the subsequent marriages of the single players m_2 and w_1 and the single players m_1 and w_3. △

At the same time, it holds that $\mu^{\mathrm{W}} \notin h^*(\overline{\mu})$. Indeed, both w_1 and w_3 are strictly worse off at μ^W and w_2 is indifferent, so none of the women is willing to add or destroy a match at $\overline{\mu}$ in the anticipation of ending up at μ^{W}. Since all the men are myopic and have an individually rational match at $\overline{\mu}$, they are not willing to destroy a link, so it is not possible to find a pairwise myopic-farsighted improving path from $\overline{\mu}$ to μ^{W}. It holds that $\mu^{\mathrm{W}} \notin h^*(\overline{\mu})$.

The next claim states that $\overline{\mu}$ is a pairwise myopic-farsighted stable set. This finding is important since it shows that in heterogeneous societies, it is possible for farsighted players to end up in a matching that is better than any core element.

Claim 2. *$\{\overline{\mu}\}$ is a pairwise myopic-farsighted stable set.*

We have already shown that $\overline{\mu} \in h^*(\mu^{\mathrm{W}})$. We show now that for every $\mu \in \mathcal{M} \setminus \{\overline{\mu}\}$ it holds that $\overline{\mu} \in h^*(\mu)$. It then follows that $V = \{\overline{\mu}\}$ is a pairwise myopic-farsighted stable set.

Let some $\mu \in \mathcal{M} \setminus \{\overline{\mu}\}$ be given. We define $\mu_0 = \mu$ and distinguish four cases. These cases are illustrated in Fig. 1.

Case 1. $\mu(m_1) = w_2$ and $\mu(m_2) = w_1$.

It follows that $\mu(m_3) = m_3$ or $\mu(m_3) = w_3$. We define $\mu_1 = \mu_0 + (m_2, w_3)$. Notice that $\overline{\mu}(w_3) = m_1 \succ_{w_3} \mu_0(w_3)$ and $w_3 \succ_{m_2} \mu_0(m_2)$, so m_2 and w_3 are willing to form the match and reach μ_1. Next, we take $\mu_2 = \mu_1 + (m_1, w_1)$. The purpose of this step is to make w_2 single. Woman w_1 is strictly improving since $\overline{\mu}(w_1) = m_2 \succ_{w_1} w_1 = \mu_1(w_1)$. Man m_1 is willing to marry w_1 since $w_1 \succ_{m_1} \mu_1(m_1) = w_2$. Next, we make w_1 and w_3 single by setting $\mu_3 = \mu_2 - (m_1, w_1)$ and $\mu_4 = \mu_3 - (m_2, w_3)$. Woman w_1 is willing to destroy the match at μ_2 since $\overline{\mu}(w_1) = m_2 \succ_{w_1} m_1$ and

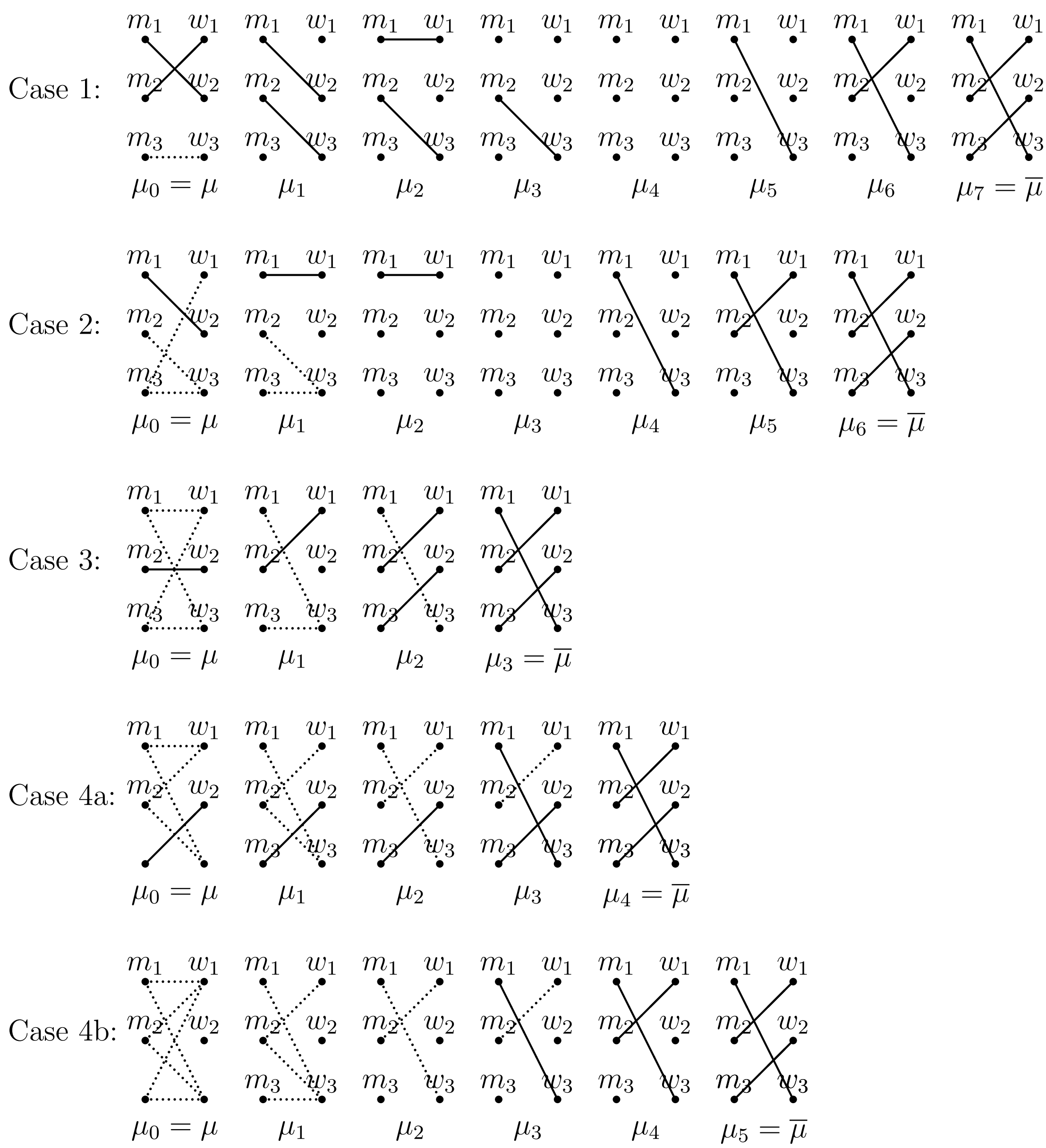

Fig. 1. Pairwise myopic-farsighted improving paths to move from μ to $\overline{\mu}$ for the four cases in Claim 2. Solid lines represent actual links and dotted lines potential links.

w_3 is willing to destroy the match at μ_3 since $\overline{\mu}(w_3) = m_1 \succ_{w_3} m_2$. Notice that at μ_4, everyone is single. We now define $\mu_5 = \mu_4 + (m_1, w_3)$, $\mu_6 = \mu_5 + (m_2, w_1)$, and $\mu_7 = \mu_6 + (m_3, w_2)$. It is easily verified that the latter three moves involve strict improvements for the marrying couple. Since $\mu_7 = \overline{\mu}$, we have shown that $\mu_0, \ldots, \mu_7$ is a pairwise myopic-farsighted improving path from μ to $\overline{\mu}$.

Case 2. $\mu(m_1) = w_2$ and $\mu(m_2) \neq w_1$.

We define $\mu_1 = \mu_0 + (m_1, w_1)$. Notice that $\overline{\mu}(w_1) = m_2 \succ_{w_1} \mu_0(w_1)$, with $\mu_0(w_1) = m_3$ or $\mu_0(w_1) = w_1$, and $w_1 \succ_{m_1} \mu_0(m_1)$, so m_1 and w_1 are willing to form the match and reach μ_1. Woman w_2 is now single. At μ_1, if w_3 is married, she is willing to destroy the match at μ_1, since $\overline{\mu}(w_3) = m_1 \succ_{w_3} m_2, m_3$, moving to $\mu_2 = \mu_1 - (\mu_1(w_3), w_3)$. Next, we also make w_1 single by setting $\mu_3 = \mu_2 - (m_1, w_1)$. Woman w_1 is willing to destroy the match at μ_2 because $\overline{\mu}(w_1) = m_2 \succ_{w_1} m_1$. Notice that at μ_3, everyone is single. We now define $\mu_4 = \mu_3 + (m_1, w_3)$, $\mu_5 = \mu_4 + (m_2, w_1)$, and $\mu_6 = \mu_5 + (m_3, w_2)$. It is easily verified that the latter three moves involve strict improvements for the marrying couple. Since $\mu_6 = \overline{\mu}$, we have shown that $\mu_0, \ldots, \mu_6$ is a pairwise myopic-farsighted improving path from μ to $\overline{\mu}$.

Case 3. $\mu(m_2) = w_2$.

We define $\mu_1 = \mu_0 + (m_2, w_1)$. The purpose of this step is to make w_2 single. Notice that $\overline{\mu}(w_1) = m_2 \succ_{w_1} \mu_0(w_1)$ and $w_1 \succ_{m_2} \mu_0(m_2)$, so m_2 and w_1 are willing to form the match and reach μ_1. Next, we take $\mu_2 = \mu_1 + (m_3, w_2)$. Woman w_2 is willing to do so since $\overline{\mu}(w_2) = m_3 \succ_{w_2} w_2 = \mu_1(w_2)$. Man m_3 is willing to match w_2 since $w_2 \succ_{m_3} \mu_1(m_3)$, with $\mu_1(m_3) = w_3$ or $\mu_1(m_3) = m_3$. Now, either $\mu_2 = \overline{\mu}$ or we form the remaining match between m_1 and w_3 moving to $\mu_3 = \mu_2 + (m_1, w_3) = \overline{\mu}$.

Case 4. $\mu(m_3) = w_2$ or $\mu(w_2) = w_2$.

At μ_0, if w_1 or w_3 are not married to their final partner at $\overline{\mu}$, then we proceed to divorce them moving to $\mu_1 = \mu_0 - (\mu_0(w_1), w_1)$ and then to $\mu_2 = \mu_1 - (\mu_1(w_3), w_3)$. Next, we form the remaining matches at $\overline{\mu}$ one by one. △

Claim 3. *$\{\overline{\mu}\}$ is the unique pairwise myopic-farsighted stable set.*

Suppose that V is a pairwise myopic-farsighted stable set that does not contain $\overline{\mu}$. By external stability of V, we know that V must contain some matching in $h^*(\overline{\mu})$.

We argue next that $\{\mu^1, \mu^2, \mu^3, \mu^4\} \subseteq h^*(\overline{\mu})$, where $\mu^1 = \overline{\mu} + (m_1, w_2)$, $\mu^2 = \mu^1 + (m_3, w_3)$, $\mu^3 = \mu^1 + (m_2, w_3)$, and $\mu^4 = \mu^3 + (m_3, w_1)$.

The move from $\overline{\mu}$ to μ^1 is done by the pair (m_1, w_2) with both m_1 and w_2 strictly preferring μ^1 to $\overline{\mu}$. Next, the move from μ^1 to μ^2 is realized by the pair (m_3, w_3) with both m_3 and w_3 strictly preferring μ^2 to μ^1. Notice that w_2 is strictly better off at μ^2 compared to $\overline{\mu}$. The move from μ^1 to μ^3 is realized by the pair (m_2, w_3) with both m_2 and w_3 strictly preferring μ^3 to μ^1. Notice that w_2 is strictly better off at μ^3 compared to $\overline{\mu}$. And, finally, μ^4 is obtained from μ^3 by forming the pair (m_3, w_1) with both m_3 and w_1 strictly preferring μ^4 to μ^3. Notice that w_2 is strictly better off at μ^4 compared to $\overline{\mu}$ and that w_3 is also strictly better off at μ^4 compared to μ^1.

We verify next that $h^*(\overline{\mu}) = \{\mu^1, \mu^2, \mu^3, \mu^4\}$, so there are no other matchings in $h^*(\overline{\mu})$. At $\overline{\mu}$, both w_1 and w_3 are married to their first choice man, so are not willing to make a move. The only possible change is by w_2, who can marry m_1 to reach μ^1 or divorce from m_3, after which the only possibility is to move to μ^1. It also follows that at the end matching, w_2 is either married to m_1 or m_2 as otherwise w_2 would not be willing to make a change. At μ^1, both w_1 and w_2 are married to their first choice man, so are not willing to move. The only possibility at μ^1 is therefore that w_3 marries m_3 to reach μ^2 or m_2 to reach μ^3. At μ^2, w_1 and w_2 are married to their first best choice, so are not willing to move. The only possibilities are w_3 divorcing, which brings us back to μ^1, or w_3 marring m_2, which leads us to μ^3, so does not lead to new matchings. We argue that once at μ_3, w_1 cannot be married to m_1 in the end matching. In that case w_2 would be married to m_2 in the end matching. But then there are no incentives for m_2 and w_3 to ever get divorced, and we have obtained a contradiction. By a similar argument it follows that w_1 cannot be married to m_2 in the end matching. In that case w_2 would be married to m_1 in the end matching. But then there are no incentives for m_2 and w_3 to ever get divorced, leading to a contradiction. Since w_1 never marries m_1, the match between w_2 and m_1 cannot be broken at μ_3 or later. Then it follows that the match between w_3 and m_2 cannot be broken at μ_3 or later. Therefore, at μ^3, the only possibility for w_1 is to marry m_3, which brings us to μ^4, after which further changes to the matching are not possible. Thus, we have that $h^*(\overline{\mu}) = \{\mu^1, \mu^2, \mu^3, \mu^4\} \cap V \neq \emptyset$.

We argue next that $\mu^W \in h^*(\mu^1)$, $\mu^W \in h^*(\mu^2)$, $\mu^W \in h^*(\mu^3)$, and $\mu^W \in h^*(\mu^4)$. Indeed, we have that $\mu^1+(m_2, w_3)+(m_1, w_1)+(m_3, w_2) = \mu^W$ and $\mu^2 + (m_2, w_3) + (m_1, w_1) + (m_3, w_2) = \mu^W$. The men are myopically improving in each step and the women all prefer μ^W to the matching before moving. Also, we have that $\mu^3 + (m_1, w_1) + (m_3, w_2) = \mu^W$ and $\mu^4 + (m_1, w_1) + (m_3, w_2) = \mu^W$. The men are myopically improving in each step and the women all prefer μ^W to the matching before moving. Thus, by internal stability of V, we have that $\mu^W \notin V$.

Let $\widetilde{\mu}$ be defined by $\widetilde{\mu}(m_1) = w_3$, $\widetilde{\mu}(m_2) = w_1$, and $\widetilde{\mu}(m_3) = m_3$, so $\widetilde{\mu}$ is equal to $\overline{\mu}$ without the match between m_3 and w_2. We argue next that $h^*(\mu^W) = \{\overline{\mu}, \widetilde{\mu}\}$. That $\overline{\mu} \in h^*(\mu^W)$ follows from Claim 1. To see that $\widetilde{\mu} \in h^*(\mu^W)$, consider the pairwise myopic-farsighted improving path $\mu_0, \ldots, \mu_3$ with $\mu_0 = \mu^W$ and $\mu_3 = \widetilde{\mu}$, where $\mu_1 = \mu_0 + (m_3, w_1)$, $\mu_2 = \mu_1 + (m_1, w_3)$, and $\mu_3 = \mu_2 + (m_2, w_1)$.

The move to μ_1 is initiated by w_1 who is farsighted and therefore wants to destroy her match with m_1 in the anticipation of ending up in a match with m_2. Similarly, the move from μ_1 to μ_2 is initiated by w_3 who is farsighted and is willing to divorce m_2 in the expectation of ultimately being matched with m_1. The transition to $\mu_3 = \widetilde{\mu}$ is initiated by w_1 who prefers m_2 to m_3. The men are myopically improving in each step.

We verify next that there are no other matchings in $h^*(\mu^W)$. For w_1 to break the link with m_1, w_1 must be married to m_2 at the end matching. For w_3 to break the link with m_2, w_3 must be married to m_1 at the end matching. Since w_3 is best for m_2 and w_1 is best for m_1, both w_1 and w_3 have to divorce at some point in order to move away from μ^W. So the only potential end matchings are $\widetilde{\mu}$ and $\overline{\mu}$.

In order to satisfy external stability, we therefore have that $h^*(\mu^W) \cap V = \{\widetilde{\mu}\}$. But now V violates internal stability as $\widetilde{\mu} + (m_1, w_2) = \mu^1$, $\widetilde{\mu}+(m_1, w_2)+(m_3, w_3) = \mu^2$, $\widetilde{\mu}+(m_1, w_2)+(m_2, w_3) = \mu^3$, and $\widetilde{\mu}+(m_1, w_2)+(m_2, w_3) + (m_3, w_1) = \mu^4$, where the men are myopically improving in each step and the women all prefer the end matching to the matching before moving, so we obtain a contradiction. Consequently, there is a unique pairwise myopic-farsighted stable set $V = \{\overline{\mu}\}$. △

6. Non-coincidence of pairwise myopic-farsighted stable sets and the core

In this section, we present general conditions for which the pairwise myopic-farsighted stable sets and the core lead to different predictions. In particular, we give conditions such that the former solution concept selects the optimal core matching for the more farsighted market side as well as conditions where both solution concepts are not nested.

We first turn to the case where all women are farsighted, whereas any given man can be either myopic or farsighted, i.e., $W \subseteq F$. It is convenient to define the set of women that strictly prefer a matching $\mu \in \mathcal{M}$ to μ^W by $W(\mu)$, so

$$W(\mu) = \{w \in W \mid \mu(w) \succ_w \mu^W(w)\}.$$

It follows from the lattice structure of the core that if $\mu \in C$, then $W(\mu) = \emptyset$.

A matching $\mu \in \mathcal{M}$ is said to *W-dominate* the matching μ^W if, for every $w \in W$, $\mu(w) \succeq_w \mu^W(w)$ and $W(\mu) \neq \emptyset$. The next result states that if μ^W is not W-dominated by an individually rational matching, then $\{\mu^W\}$ is a pairwise myopic-farsighted stable set.

Theorem 4. *Let $(M, W, F, \succ)$ be a marriage problem with $F \supseteq W$. Assume that μ^W is not W-dominated by a matching in $\mathcal{I}$. Then $\{\mu^W\}$ is a pairwise myopic-farsighted stable set.*

Proof. Since $\{\mu^W\}$ is a singleton, Condition (i) of Definition 3, internal stability, is satisfied. Condition (ii) of Definition 3, external stability, follows from Lemmas 3 and 4 that are proved in the appendix. □

The proof that μ^W satisfies external stability relies on Lemmas 3 and 4 in Appendix. We prove in Lemma 3 that the woman-optimal stable matching μ^W can be reached from any matching μ different from μ^W with the property that $W(\mu) = \emptyset$. This also covers the case where μ is a core element different from μ^W. The proof of Lemma 3 first identifies the set W^1 of women that strictly prefer μ^W to μ and that are not single at μ. Since all women are farsighted, they are willing to divorce their men. The resulting matching is such that all women are either married to their partner at μ^W or are single. The latter women now marry their partner at μ^W.

We prove in Lemma 4 that if μ^W is not W-dominated by a matching in $\mathcal{I}$, then the woman-optimal stable matching μ^W can be reached from any matching μ different from μ^W such that $W(\mu) \neq \emptyset$.

The proof of Lemma 4 proceeds as follows. In order to reach μ^W from μ, first the matches that are not individually rational at μ are destroyed until we arrive at an individually rational matching μ_{k^1}. If $W(\mu_{k^1}) = \emptyset$, then we can proceed as in the proof of Lemma 3. Consider the case that $W(\mu_{k^1}) \neq \emptyset$. The set of men married to $W(\mu_{k^1})$ at μ_{k^1} cannot be the same as the set of men married to $W(\mu_{k^1})$ at μ^W, since otherwise we can construct an individually rational matching that W-dominates μ^W, which violates the assumptions of Theorem 4. Since all women in $W(\mu_{k^1})$ are married, we can select a man m_0 who is married to a woman in $W(\mu_{k^1})$ at μ_{k^1}, but is not married to a woman in $W(\mu_{k^1})$ at μ^W. We show that the pair $(m_0, \mu^W(m_0))$ can profitably deviate at μ_{k^1}. More generally, we generate a sequence of pairs (m_ℓ, w_ℓ) with $m_\ell = \mu^W(w_\ell)$ that can profitably deviate at $\mu_{k^1+\ell}$. We show that after a finite number of steps k^2 equal to the cardinality of $W(\mu_{k^1})$, we arrive at a matching $\mu_{k^1+k^2}$ such that $W(\mu_{k^1+k^2}) = \emptyset$. Thus, the matching $\mu_{k^1+k^2}$ either coincides with μ^W or satisfies the assumptions of Lemma 3. In the latter case, we complete the pairwise myopic-farsighted improving path leading to μ^W as in the proof of Lemma 3.

From Lemmas 3 and 4, it follows easily that $\{\mu^W\}$ is a pairwise myopic-farsighted stable set.

We have shown that in case all women are farsighted and μ^W is not W-dominated by a matching in $\mathcal{I}$, they can achieve the woman-optimal stable matching μ^W, irrespective of whether men are myopic or farsighted.

It follows from the next result that if all women are farsighted and all men are myopic, then no subset of the core different from $\{\mu^W\}$ can be a pairwise myopic-farsighted stable set.

Theorem 5. *Let $(M, W, F, \succ)$ be a marriage problem with $F = W$. If $V \subseteq C$ is a pairwise myopic-farsighted stable set, then $V = \{\mu^W\}$.*

The proof of Theorem 5 argues that by internal stability, $V \subseteq C$ cannot contain μ^W together with another core element, so the only possibility besides $\{\mu^W\}$ for $V \subseteq C$ to be a pairwise myopic-farsighted stable set is that $V \subseteq C \setminus \{\mu^W\}$. By external stability, there should now be a myopic-farsighted improving path from μ^W to a core element. But with all women being farsighted, this is impossible.

When the women are farsighted and the men are myopic, the only possible pairwise myopic-farsighted stable set contained in the core consists of the woman-optimal stable matching. Theorem 4 presents a sufficient condition such that μ^W is indeed a pairwise myopic-farsighted stable set. Theorem 6 shows that this sufficient condition is also necessary when all women are farsighted and all men are myopic. This result generalizes the example in Section 5 by showing that if there exists an individually rational matching that W-dominates μ^W, then no subset of the core can be a pairwise myopic-farsighted stable set.

Theorem 6. *Let $(M, W, F, \succ)$ be a marriage problem with $F = W$. If μ^W is W-dominated by a matching in $\mathcal{I}$, then a pairwise myopic-farsighted stable set cannot be contained in C.*

The proof of Theorem 6 starts by observing that because of Theorem 5, $\{\mu^W\}$ is the only candidate for a pairwise myopic-farsighted stable set contained in C. But $\{\mu^W\}$ cannot satisfy external stability, since farsighted women and myopic men will not reach μ^W by a myopic-farsighted improving path when starting from an individually rational matching that W-dominates μ^W.

A matching in $\mathcal{I}$ that W-dominates μ^W need not constitute a pairwise myopic-farsighted stable set itself or be an element of such a set. To see this, consider the following variation on the example of Section 5. Add men m_4, m_5, and m_6 as well as women w_4, w_5, w_6 to the example. All the men are myopic and all the women are farsighted. Men m_4, m_5, and m_6 are at the bottom of the preference lists of w_1, w_2, and w_3, and w_4, w_5, and w_6 are at the bottom of the preference lists of m_1, m_2, and m_3. The preferences of the added men and women are similar to those of the original agents in the example. More precisely, we assume that all players of the opposite sex are acceptable and that, for $i = 4, 5, 6$, for $j, k = 1, 2, 3, 4, 5, 6$, $m_j \succeq_{w_i} m_k$ if and only if $m_{j+3} \succeq_{w_{i+3}} m_{k+3}$.[3] In this way, we obtain two isomorphic subgroups of agents, which are essentially disconnected.

It follows immediately from the analysis in Section 5 that the woman-optimal matching is equal to $\mu^W(m_1) = w_1$, $\mu^W(m_2) = w_3$, $\mu^W(m_3) = w_2$, $\mu^W(m_4) = w_4$, $\mu^W(m_5) = w_6$, and $\mu^W(m_6) = w_5$. The matching $\overline{\mu}$ defined by $\overline{\mu}(m_1) = w_3$, $\overline{\mu}(m_2) = w_1$, $\overline{\mu}(m_3) = w_2$, $\overline{\mu}(m_4) = w_6$, $\overline{\mu}(m_5) = w_4$, and $\overline{\mu}(m_6) = w_5$, W-dominates μ^W and constitutes the unique pairwise myopic-farsighted stable set in this example. However, the matching $\tilde{\mu}$ which coincides with μ^W for m_1, m_2, and m_3, and with $\overline{\mu}$ for m_4, m_5, and m_6, also W-dominates μ^W, but is not part of any pairwise myopic-farsighted stable set.

The fact that one of the matchings in $\mathcal{I}$ that W-dominates μ^W constitutes a pairwise myopic-farsighted stable set is no coincidence. The matching $\overline{\mu}$ corresponds to the efficiency-enhanced deferred acceptance matching as defined in Kesten (2010). Subsequent to our work, it has been shown in Corollary 1 of Doğan and Ehlers (2024) that when the set F is equal to W, then the efficiency-enhanced deferred acceptance matching constitutes a pairwise myopic-farsighted stable set. Whenever $F = W$ and there is a matching in $\mathcal{I}$ that W-dominates μ^W, then even though the dominating matching may not be part of a pairwise myopic-farsighted stable set, another W-dominating matching, and in particular the efficiency-enhanced deferred acceptance matching, constitutes a pairwise myopic-farsighted stable set.

For the sharp results in Theorems 5 and 6, we require $F = W$. The conclusions of these theorems do not necessarily hold when some men are farsighted. For instance, in case $F = N$, Mauleon et al. (2011) show that any singleton containing a core element is a pairwise myopic-farsighted stable set.

When all the women are farsighted and all the men are myopic and the woman-optimal stable matching is W-dominated by an individually rational matching, then no pairwise myopic-farsighted stable set can be formed by core elements. When this is the case, the core and the pairwise myopic-farsighted stable set will always provide different predictions.

7. Strict versus weak improvements

In the definition of a pairwise myopic-farsighted improving path, players that match are required to have a strict preference for doing so. We will refer to this as the case with strict improvements. However, the alternative assumption that a pair of players forms a match between them if at least one player has a strict preference to do so can be reasonably entertained as well. We refer to this as the case with weak improvements.

The case with weak pairwise improvements was analyzed in Herings, Mauleon, and Vannetelbosch (2020). This section discusses how this alternative assumption affects our results.

Let $g(\mu)$ denote the matchings $\mu' \in \mathcal{M}$ that can be reached from $\mu \in \mathcal{M}$ by a coalitional myopic-farsighted improving path under weak improvements, meaning that all players in the coalition improve weakly and at least one player strictly. Replacing g^* by g in Definition 3, leads to the weak coalitional myopic-farsighted stable set.

We define $h(\mu)$ as the set of all matchings that can be reached from a matching $\mu \in \mathcal{M}$ by a pairwise myopic-farsighted improving path under weak improvements. Let us refer to the stable set that results if we replace h^* by h in Definition 3 as the weak pairwise myopic-farsighted stable set.

We first present an example to demonstrate that it is not possible to obtain the analogue of Theorem 1 for the case of weak improvements.

Example 2. Consider the marriage problem $(M, W, F, \succ)$ with $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Assume $F = \{w_1\}$. The preferences of the players are as follows, where only the individually rational part is specified.

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_2	w_3	m_1	m_1	m_2
w_2	w_3			m_2	m_3

Fig. 2 illustrates a weak coalitional myopic-farsighted improving path from a matching μ_0 to a matching μ_4.

The move from μ_0 to μ_1 is performed by the coalition $\{m_3, w_1, w_3\}$. A link is formed by the myopic players m_3 and w_3 and both improve. The link (m_1, w_1) is broken by the farsighted woman w_1, who is indifferent between μ_0 and μ_4. The move from μ_1 to μ_2 is made by coalition $\{m_1, w_2\}$. Both players are myopic and improve. The move from μ_2 to μ_3 is performed by the coalition $\{m_2, w_3\}$. Both players are myopic and improve. The move from μ_3 to μ_4 is carried out by coalition $\{m_1, w_1\}$. Both players are improving. This shows that $\mu_4 \in g(\mu_0)$.

We now argue that $\mu_4 \notin h(\mu_0)$. Suppose $\mu'_0, \ldots, \mu'_L$ is a weak pairwise myopic-farsighted improving path with $\mu'_0 = \mu_0$ and $\mu'_L = \mu_4$. At μ'_0, woman w_1 is farsighted and indifferent to the match at μ'_L. She is only willing to move if one of the myopic men would strictly improve marrying her. Since there is no such man, w_1 cannot move at μ'_0. At μ'_0, the myopic men m_1 and m_2 are married to their first choice, so not willing to move. Although the myopic woman w_2 is willing to marry m_1,

[3] We identify agent 7 with agent 1, agent 8 with agent 2, and agent 9 with agent 3.

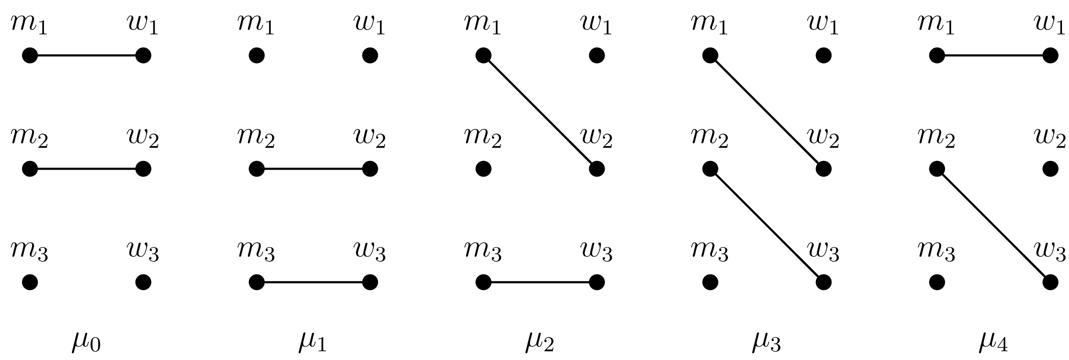

Fig. 2. Weak coalitional myopic-farsighted improving path from μ_0 to μ_4.

m_1 is not willing to marry w_2, so w_2 cannot move. The only possibility is that the match (m_3, w_3) is formed, so μ_1' is the matching where the pairs (m_1, w_1), (m_2, w_2), and (m_3, w_3) are all formed. It is easily verified that no moves away from μ_1' are possible. Since $\mu_1' \neq \mu_4$, this shows that $\mu_4 \notin h(\mu_0)$. △

Section 5 presents an example such that $\overline{\mu} \in h^*(\mu^{\mathrm{W}})$ while $\mu^{\mathrm{W}} \notin h^*(\overline{\mu})$. Replacing strict by weak improvements, we still have that $\overline{\mu} \in h(\mu^{\mathrm{W}})$, but now we also have that $\mu^{\mathrm{W}} \in h(\overline{\mu})$ despite the fact that all women are farsighted and that $\overline{\mu}$ is strictly preferred by two out of three women to μ^{W}, whereas the third woman is indifferent. Obviously, Claim 2 of Section 5 still holds and $V = \{\overline{\mu}\}$ is a weak pairwise myopic-farsighted stable set. However, on top of $\{\overline{\mu}\}$ now also $\{\mu^{\mathrm{W}}\}$ emerges as a weak pairwise myopic-farsighted stable set.

The analogue of Theorem 3 for weak improvements has not been studied so far, but can be shown along the same lines. The analogue of Theorem 4 for weak improvements holds even when μ^{W} is W-dominated by a matching in $\mathcal{I}$. The case of weak improvements offers therefore more general conditions under which the side-optimal stable matching constitutes a stable set. The analogue of Theorem 5 does not hold. There are marriage problems satisfying the conditions of Theorem 5 such that core elements different from $\{\mu^{\mathrm{W}}\}$ form a singleton weak pairwise myopic-farsighted stable set. The case of strict improvements offers therefore more general conditions under which the side-optimal stable matching is the unique stable set. The next example presents a matching problem where both $\{\mu^{\mathrm{W}}\}$ and $\{\mu^{\mathrm{M}}\}$ are weak pairwise myopic-farsighted stable sets, which implies that it is possible to reach the man-optimal stable matching μ^{M} from μ^{W}.

Example 3. Consider the marriage problem $(M, W, F, \succ)$, where $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ and the preferences of the players are as follows:

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_3	w_1	m_1	m_2	m_3
w_2	w_1	w_2	m_2	m_3	m_2
w_3	w_2	w_3.	m_3	m_1	m_1

We assume $F = W$, so all women are farsighted and all men are myopic. Using the deferred acceptance algorithm of Gale and Shapley (1962) with men proposing and with women proposing, it is immediate that the man-optimal and woman-optimal stable matchings are given by

$$\begin{aligned} \mu^{\mathrm{M}}(m_1) &= w_1, & \mu^{\mathrm{W}}(m_1) &= w_1, \\ \mu^{\mathrm{M}}(m_2) &= w_3, & \mu^{\mathrm{W}}(m_2) &= w_2, \\ \mu^{\mathrm{M}}(m_3) &= w_2. & \mu^{\mathrm{W}}(m_3) &= w_3, \end{aligned}$$

We first argue that $\mu^{\mathrm{M}} \in h(\mu^{\mathrm{W}})$. To verify this assertion, consider the weak pairwise myopic-farsighted improving path $\mu_0, \ldots, \mu_4$ with $\mu_0 = \mu^{\mathrm{W}}$ and $\mu_4 = \mu^{\mathrm{M}}$, where $\mu_1 = \mu_0 + (m_2, w_1)$, $\mu_2 = \mu_1 + (m_3, w_2)$, $\mu_3 = \mu_2 + (m_2, w_3)$, and $\mu_4 = \mu_3 + (m_1, w_1)$.

Indeed, m_2 strictly prefers $\mu_1(m_2) = w_1$ to $\mu_0(m_2) = w_2$ and the farsighted woman w_1 is indifferent between μ_4 and μ_0. Since now at μ_1 woman w_2 has become unmatched, she is willing to form a link with m_3, her partner in the end matching of the sequence, moving from μ_1 to μ_2. Since $\mu_2(m_3) = w_2 \succ_{m_3} w_3 = \mu_1(m_3)$, this is also a myopic improvement for m_3. At μ_2 woman w_3 is unmatched, so she is willing to team up with m_2, her partner in the end matching of the sequence, leading to the matching μ_3. Since $\mu_3(m_2) = w_3 \succ_{m_2} w_1 = \mu_2(m_2)$, this is also a myopic improvement for m_2. Man m_1 and woman w_1 are both single at μ_3 and are both happy to marry, which moves them to the end matching $\mu_4 = \mu^{\mathrm{M}}$.

Consider next any matching $\mu \in \mathcal{M} \setminus \{\mu^{\mathrm{M}}, \mu^{\mathrm{W}}\}$. We argue that $\mu^{\mathrm{M}} \in h(\mu)$ by constructing a weak pairwise myopic-farsighted improving path $\mu_0, \ldots, \mu_L$ with $\mu_0 = \mu$ and $\mu_L = \mu^{\mathrm{M}}$.

Assume first that $\mu(m_1) \neq w_1$. We define $\mu_1 = \mu_0 + (m_1, w_1)$. Since w_1 is the best partner for m_1, this is clearly a myopic improvement for m_1. Since m_1 is the best partner for w_1 and $\mu_L(w_1) = \mu^{\mathrm{M}}(w_1) = m_1$, this is a farsighted improvement for w_1.

If $\mu_1 = \mu^{\mathrm{M}}$, then we have shown that $\mu^{\mathrm{M}} \in h(\mu)$.

If $\mu_1 = \mu^{\mathrm{W}}$, then following the weak pairwise myopic-farsighted improving path from μ^{W} to μ^{M} constructed at the beginning of the example, we also have $\mu^{\mathrm{M}} \in h(\mu)$.

If $\mu_1 \in \mathcal{M} \setminus \{\mu^{\mathrm{M}}, \mu^{\mathrm{W}}\}$, then w_2 or w_3 is single under μ_1. If w_2 is single, then let her marry m_3 and move to $\mu_2 = \mu_1 + (m_3, w_2)$. Since at μ_1 man m_3 is not married to w_1, this is a myopic improvement for m_3. It is also clearly a farsighted improvement for w_2. If w_2 is not single, but w_3 is, then let her marry m_2 and move to $\mu_2 = \mu_1 + (m_2, w_3)$. Since w_3 is the preferred partner of m_2, this is clearly a myopic improvement for m_2. It is also clearly a farsighted improvement for w_3. Either $\mu_2 = \mu^{\mathrm{M}}$ and we are done, or μ^2 consists of two matched pairs (m_1, w_1) and (m_3, w_2), both being part of μ^{M}, and two single players, m_2 and w_3. In this case, we form the missing pair (m_2, w_3) from μ^{M} and move from μ_2 to $\mu_3 = \mu^{\mathrm{M}}$. This completes the construction of the weak pairwise myopic-farsighted improving path to μ^{M} for the case $\mu(m_1) \neq w_1$.

Assume next that $\mu(m_1) = w_1$. We can then proceed with the weak pairwise myopic-farsighted improving path starting from μ_1 as constructed in the previous paragraph.

The singleton set $V = \{\mu^{\mathrm{M}}\}$ trivially satisfies internal stability. Since we have shown that $\mu^{\mathrm{M}} \in h(\mu)$ for every $\mu \neq \mu^{\mathrm{M}}$, it also satisfies external stability. It follows that $V = \{\mu^{\mathrm{M}}\}$ is a weak pairwise myopic-farsighted stable set. △

The analogue for weak improvements of Theorem 6 does not hold. It follows from Theorem 7 in Herings et al. (2020) that the set $\{\mu^{\mathrm{W}}\}$ is a weak pairwise myopic-farsighted stable set even when it is W-dominated by an individually rational matching $\overline{\mu} \in \mathcal{I}$. To reach μ^{W} from $\overline{\mu}$ crucially depends on the willingness of women that are indifferent between $\overline{\mu}$ and μ^{W} to move away from $\overline{\mu}$, since the women that strictly prefer $\overline{\mu}$ to μ^{W} will not do so. That it is always possible to find a blocking pair involving such a woman follows from the blocking lemma, which is due to J.S. Hwang and presented as Lemma 3.5 in

Roth and Sotomayor (1990). Interestingly, the discussion in Section 4.2 of Newton and Sawa (2015) about stochastic stability in marriage problems under different specifications of the process by which players makes mistakes, also depends on the willingness of players who ultimately are indifferent to act as catalysts in the reordering of the remainder of society.

8. Conclusion

We use the pairwise myopic-farsighted stable set to study marriage problems where myopic and farsighted players interact. We assume that players only deviate if they improve strictly. For homogeneous cases, where either all players are myopic or all players are farsighted, every element of a myopic-farsighted stable set is a core element and every core element belongs to some myopic-farsighted stable set. We cannot discriminate between different core elements and matchings outside of the core cannot result. The same result holds for heterogeneous cases when the marriage problem is α-reducible.

But in general, the interaction between myopic and farsighted players leads to selection of core elements and to matchings that do not belong to the core. When all women are farsighted and all men are myopic, then the only subset of the core that can be a pairwise myopic-farsighted stable set is the singleton consisting of the woman-optimal stable matching. In case the woman-optimal stable matching is dominated from the woman point of view by an individually rational matching, then the woman-optimal stable matching cannot be a pairwise myopic-farsighted stable set and the core and the pairwise myopic-farsighted stable set provide different predictions. These results are robust to the case where coalitional deviations are allowed, since the pairwise myopic-farsighted stable set is shown to be equivalent to the coalitional pairwise myopic-farsighted stable set.

CRediT authorship contribution statement

P. Jean-Jacques Herings: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Ana Mauleon:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Vincent Vannetelbosch:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

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Appendix

Lemma 1. *Let $(M, W, F, \succ)$ be a marriage problem. Let $\mu_0, \ldots, \mu_L$ be a coalitional myopic-farsighted improving path from a matching $\mu \in \mathcal{M}$ to a matching $\mu' \in \mathcal{M}$. If $i \in N \setminus F$ and $\mu(i) \succeq_i i$, then it holds that, for every $\ell \in \{0, \ldots, L\}$, $\mu_\ell(i) \succeq_i i$. If $i \in F$ and $\mu(i) \succeq_i i$, then it holds that $\mu'(i) \succeq_i i$. If there is $\ell \in \{1, \ldots, L\}$ such that $i \in S_\ell \cap F$ and $\mu(i) \succeq_i i$, then it holds that $\mu'(i) \succ_i i$.*

Proof. For every $\ell \in \{0, \ldots, L-1\}$, let $S_\ell \in \mathcal{N}$ be the coalition that can enforce $\mu_{\ell+1}$ from μ_ℓ and

$$\begin{aligned} &\mu_{\ell+1}(i) \succ_i \mu_\ell(i) && \text{if } i \in S_\ell \setminus F, \\ &\mu_L(i) \succ_i \mu_\ell(i) && \text{if } i \in S_\ell \cap F. \end{aligned}$$

We distinguish two cases.

Case 1. $i \in N \setminus F$.

We prove by induction that, for every $\ell = 0, \ldots, L$, $\mu_\ell(i) \succeq_i i$. By assumption, $\mu_0(i) \succeq_i i$. Assume, for some $\ell = 0, \ldots, L-1$, it holds that $\mu_\ell(i) \succeq_i i$. We show that $\mu_{\ell+1}(i) \succeq_i i$. We distinguish between three cases when moving from μ_ℓ to $\mu_{\ell+1}$:

Case (i). i creates a match with a player different from $\mu_\ell(i)$.

Case (ii). i becomes single.

Case (iii). i is not affected.

In Case (i) it holds that $\mu_{\ell+1}(i) \succ_i \mu_\ell(i) \succeq_i i$, in Case (ii) that $\mu_{\ell+1}(i) = i \succeq_i i$, and in Case (iii) that $\mu_{\ell+1}(i) = \mu_\ell(i) \succeq_i i$. We have shown that $\mu_{\ell+1}(i) \succeq_i i$. It follows that $\mu'(i) = \mu_L(i) \succeq_i i$.

Case 2. $i \in F$.

Suppose $i \succ_i \mu'(i)$. Since $\mu_0(i) \succeq_i i$, it holds that $i \notin S_0$, so $\mu_1(i) = i$ (in case $\mu_0(i) \in S_0$) or $\mu_1(i) = \mu_0(i) \succeq_i i$. In both cases, it holds that $\mu_1(i) \succeq_i i$. Repeating this argument, it follows that $\mu'(i) = \mu_L(i) \succeq_i i$, leading to a contradiction. Consequently, it holds that $\mu'(i) = \mu_L(i) \succeq_i i$.

Assume $\mu_L(i) = i$. Since $\mu_0(i) \succeq_i i$, it follows that $i \notin S_0$ and $\mu_1(i) = \mu_0(i)$ or $\mu_1(i) = i$, so $\mu_1(i) \succeq_i i$. By repeating this argument, we find that, for every $\ell \in \{0, \ldots, L-1\}$, $\mu_\ell(i) \succeq_i i$. Since moves by coalitions involve strict improvement, we have that, for every $\ell \in \{0, \ldots, L-1\}$, $i \in N \setminus S_\ell$. It follows that if there is $\ell \in \{0, \ldots, L-1\}$ such that $i \in S_\ell$, then $\mu_L(i) \succ_i i$. □

Proof of Theorem 1. Let some $\mu \in \mathcal{I}$ be given.

Obviously, it holds that $h^*(\mu) \subseteq g^*(\mu)$.

We show now that $g^*(\mu) \subseteq h^*(\mu)$. Let $\mu' \in g^*(\mu)$ be given. There is a finite sequence of distinct matchings $\mu_0, \ldots, \mu_L$ with $\mu_0 = \mu$ and $\mu_L = \mu'$ such that, for every $\ell \in \{0, \ldots, L-1\}$, there is a coalition $S_\ell \in \mathcal{N}$ that can enforce $\mu_{\ell+1}$ from μ_ℓ and

$$\begin{aligned} &\mu_{\ell+1}(i) \succ_i \mu_\ell(i), && \text{if } i \in S_\ell \setminus F, \\ &\mu_L(i) \succ_i \mu_\ell(i), && \text{if } i \in S_\ell \cap F. \end{aligned}$$

Let some $\ell \in \{0, \ldots, L-1\}$ be given. We define

$$\begin{aligned} D_\ell^0 &= \{m \in S_\ell \mid \mu_\ell(m) \in W,\ \mu_{\ell+1}(m) = m, \text{ and } \mu_{\ell+1}(\mu_\ell(m)) = \mu_\ell(m)\}, \\ D_\ell^1 &= \{w \in S_\ell \mid \mu_\ell(w) \in M \setminus D_\ell^0,\ \mu_{\ell+1}(w) = w, \text{ and } \mu_{\ell+1}(\mu_\ell(w)) = \mu_\ell(w)\}, \end{aligned}$$

as the men in S_ℓ who at μ_ℓ destroy their match with a woman and the women in S_ℓ who at μ_ℓ destroy their match with a man outside D_ℓ^0. The reason to exclude men in D_ℓ^0 is to avoid double counting. Let $D_\ell = D_\ell^0 \cup D_\ell^1$. The cardinality of D_ℓ is denoted by d_ℓ. We will have the players in D_ℓ successively destroying their matches when constructing a pairwise myopic-farsighted improving path from μ_ℓ to $\mu_{\ell+1}$. By Lemma 1 it holds for every $i \in N \setminus F$ that $\mu_\ell(i) \succeq_i i$, so myopic players in S_ℓ are never destroying matches when going from μ_ℓ to $\mu_{\ell+1}$, and therefore $D_\ell \subseteq F$. We define the set $A_\ell = \{m \in S_\ell \mid \mu_\ell(m) \neq \mu_{\ell+1}(m) \in W\}$ as the set of men that form a match when going from μ_ℓ to $\mu_{\ell+1}$. Note that enforceability of $\mu_{\ell+1}$ implies that $\mu_{\ell+1}(m) \in S_\ell$. The requirement $\mu_\ell(m) \neq \mu_{\ell+1}(m)$ is added to ensure that m does not simply keep his partner at μ_ℓ. The cardinality of A_ℓ is denoted by a_ℓ.

Take an arbitrary order of the individuals in D_ℓ, say $i_0, \ldots, i_{d_\ell - 1}$, and an arbitrary order of the men in A_ℓ, say $m_0, \ldots, m_{a_\ell - 1}$. We define $\nu_{\ell,0} = \mu_\ell$. For $k \in \{0, \ldots, d_\ell - 1\}$, we define the matching $\nu_{\ell,k+1} = \nu_{\ell,k} - (i_k, \mu_\ell(i_k))$. For $k \in \{0, \ldots, a_\ell - 1\}$, we define the matching $\nu_{\ell,d_\ell+k+1} = \nu_{\ell,d_\ell+k} + (m_k, \mu_{\ell+1}(m_k))$.

Consider the sequence of distinct matchings

$$\nu_{0,0}, \ldots, \nu_{0,d_0+a_0-1}, \nu_{1,0}, \ldots, \nu_{L-1,0}, \ldots, \nu_{L-1,d_{L-1}+a_{L-1}}.$$

Notice that $\nu_{L-1,d_{L-1}+a_{L-1}} = \mu_L$. We argue that this sequence is a pairwise myopic-farsighted improving path from $\nu_{0,0} = \mu$ to $\nu_{L-1,d_{L-1}+a_{L-1}} = \mu'$.

Let some $\ell \in \{0,\ldots,L-1\}$ and some $k \in \{0,\ldots,d_\ell - 1\}$ be given. It holds that $\nu_{\ell,k+1} = \nu_{\ell,k} - (i_k, \mu_\ell(i_k))$. Since $i_k \in D_\ell$, we have that $i_k \in F$. It holds that $\nu_{\ell,k}(i_k) = \mu_\ell(i_k) \prec_{i_k} \mu_L(i_k)$, where the last step uses that $i_k \in S_\ell$.

Let some $\ell \in \{0,\ldots,L-1\}$ and some $k \in \{0,\ldots,a_\ell - 1\}$ be given. For $w_k = \mu_{\ell+1}(m_k)$, it holds that $\nu_{\ell,d_\ell+k+1} = \nu_{\ell,d_\ell+k} + (m_k, w_k)$. First, consider the case where $m_k \in F$. We have $\mu_L(m_k) \succ_{m_k} \mu_\ell(m_k)$ and, by Lemma 1, $\mu_L(m_k) \succ_{m_k} m_k$. It also holds that $\nu_{\ell,d_\ell+k}(m_k) = \mu_\ell(m_k)$ or $\nu_{\ell,d_\ell+k}(m_k) = m_k$. In both cases, we find that $\mu_L(m_k) \succ_{m_k} \nu_{\ell,d_\ell+k}(m_k)$. The same argument can be used to show that in case $w_k \in F$ it holds that $\mu_L(w_k) \succ_{w_k} \nu_{\ell,d_\ell+k}(w_k)$. Second, consider the case where $m_k \in M \setminus F$. Using Lemma 1, we have $\mu_{\ell+1}(m_k) \succ_{m_k} \mu_\ell(m_k) \succeq_{m_k} m_k$. It also holds that $\nu_{\ell,d_\ell+k+1}(m_k) = \mu_{\ell+1}(m_k)$ and $[\nu_{\ell,d_\ell+k}(m_k) = \mu_\ell(m_k)$ or $\nu_{\ell,d_\ell+k}(m_k) = m_k]$. In both cases we find that $\nu_{\ell,d_{\ell+k+1}}(m_k) \succ_{m_k} \nu_{\ell,d_\ell+k}(m_k)$. The same argument can be used to show that in case $w_k \in W \setminus F$ it holds that $\nu_{\ell,d_{\ell+k+1}}(w_k) \succ_{w_k} \nu_{\ell,d_\ell+k}(w_k)$.

We have shown that $\mu' \in h^*(\mu)$. □

Lemma 2. *Let $(M, W, F, \succ)$ be a marriage problem. Let V be a pairwise or a coalitional myopic-farsighted stable set. It holds that $V \subseteq \mathcal{I}$.*

Proof. Let V be a pairwise myopic-farsighted stable set. Suppose $\mu \in V$ is not individually rational. Then there is $m \in M$ and $w \in W$ such that $\mu(m) = w$ and $[m \succ_m w$ or $w \succ_w m.]$ Without loss of generality, assume $m \succ_m w$. It follows from Lemma 1 that $\mu \notin h^*(\mu - (m,w))$.

On the other hand, $\mu - (m,w) \in h^*(\mu)$, so by $\mu \in V$ and internal stability of V it follows that $\mu - (m,w) \notin V$. By external stability of V it holds that there is $\mu' \in h^*(\mu - (m,w))$ such that $\mu' \in V$. Since $\mu \notin h^*(\mu - (m,w))$, we have $\mu' \neq \mu$.

Let $\mu_1,\ldots,\mu_L$ with $\mu_1 = \mu-(m,w)$ and $\mu_L = \mu'$ be a pairwise myopic-farsighted improving path from $\mu-(m,w)$ to μ'. We distinguish between two cases.

Case 1. $m \in M \setminus F$.

It holds that $\mu - (m,w) \succ_m \mu$. Therefore, it follows that $\mu_0, \mu_1, \ldots, \mu_L$ with $\mu_0 = \mu$ is a pairwise myopic-farsighted improving path from μ to μ' and therefore $\mu' \in h^*(\mu)$. This contradicts the fact that V is internally stable.

Case 2. $m \in F$.

Since $\mu' \in h^*(\mu - (m,w))$ and $m \in F$, it follows from Lemma 1 that $\mu'(m) \succeq_m m$. Therefore it holds that $\mu_0, \mu_1, \ldots, \mu_L$ with $\mu_0 = \mu$ is a pairwise myopic-farsighted improving path from μ to μ' and therefore $\mu' \in h^*(\mu)$. This contradicts the fact that V is internally stable.

Let V be a coalitional myopic-farsighted stable set. We can repeat every step of the proof for the case where V is a pairwise myopic-farsighted stable set with h^* replaced by g^*. □

Proof of Theorem 2. Let V be a pairwise myopic-farsighted stable set.

For every $\mu, \mu' \in V$, it holds that $\mu' \notin h^*(\mu)$. By Lemma 2, μ is individually rational, so by Theorem 1 we have that $\mu' \notin g^*(\mu)$. This shows that V satisfies internal stability with respect to g^* as defined in Definition 3.

For every $\mu \in \mathcal{M} \setminus V$, it holds that $h^*(\mu) \cap V \neq \emptyset$. Since $h^*(\mu)red \subseteq g^*(\mu)$, it follows that $g^*(\mu) \cap V \neq \emptyset$. This shows that V satisfies external stability with respect to g^* as defined in Definition 3.

We conclude that V is a coalitional myopic-farsighted stable set.

Let V be a coalitional myopic-farsighted stable set.

For every $\mu, \mu' \in V$, it holds that $\mu' \notin g^*(\mu)$. Since $h^*(\mu)red \subseteq g^*(\mu)$, it follows that $\mu' \notin h^*(\mu)$. This shows that V satisfies internal stability with respect to h^* as defined in Definition 3.

By Lemma 2, every matching in V is individually rational Let some $\mu \in \mathcal{M} \setminus V$ be given. We distinguish between two cases.

Case 1. $\mu \in \mathcal{I}$.

Since V is a coalitional myopic-farsighted stable set, it holds that $g^*(\mu) \cap V \neq \emptyset$. By Theorem 1 it holds that $h^*(\mu) = g^*(\mu)$, so $h^*(\mu) \cap V \neq \emptyset$.

Case 2. $\mu \in \mathcal{M} \setminus \mathcal{I}$.

We define

$$D^0 = \{m \in M \mid m \succ_m \mu(m)\},$$
$$D^1 = \{w \in W \mid w \succ_w \mu(w) \text{ and } \mu(w) \notin D^0\},$$

as the set of men that prefer to be single over their match at μ and the set of women that are not married to a man in D^0 and prefer to be single over their match at μ. Let $D = D^0 \cup D^1$ and denote the cardinality of D by $d \geq 1$. Take an arbitrary order of the individuals in D, say $i_0, \ldots, i_d - 1$. Define $\mu_0 = \mu$. For $\ell \in \{0,\ldots,d-1\}$, we define the matching $\mu_{\ell+1} = \mu_\ell - (i_\ell, \mu(i_\ell))$. We distinguish two subcases.

Subcase 2.1. $\mu_d \in V$.

We show that $\mu_0, \ldots, \mu_d$ is a pairwise myopic-farsighted improving path from $\mu_0 = \mu$ to $\mu_d \in V$. Let some $\ell \in \{0,\ldots,d-1\}$ be given. If $i_\ell \in N \setminus F$, then $\mu_{\ell+1}(i_\ell) = i_\ell \succ_{i_\ell} \mu(i_\ell) = \mu_\ell(i_\ell)$. If $i_\ell \in F$, then $\mu_d(i_\ell) = i_\ell \succ_{i_\ell} \mu(i_\ell) = \mu_\ell(i_\ell)$.

Subcase 2.2 $\mu_d \notin V$.

If $\mu_d \notin V$, then let $\mu' \in V$ be such that $\mu' \in g^*(\mu_d)$. The existence of μ' follows since V satisfies external stability with respect to g^* as defined in Definition 3. Since μ_d is individually rational by construction, we can use Theorem 1 to conclude that $h^*(\mu_d) = g^*(\mu_d)$, so $\mu' \in h^*(\mu_d)$. Let $\mu_d, \mu_{d+1}, \ldots, \mu_{d+L}$ be a pairwise myopic-farsighted improving path from μ_d to $\mu_{d+L} = \mu'$. We show next that $\mu_0, \mu_1, \ldots, \mu_{d+L}$ is a pairwise myopic-farsighted improving path from μ_0 to μ_{d+L}.

Let some $\ell \in \{0,\ldots,d-1\}$ be given. If $i_\ell \in N \setminus F$, then $\mu_{\ell+1}(i_\ell) = i_\ell \succ_{i_\ell} \mu(i_\ell) = \mu_\ell(i_\ell)$. If $i_\ell \in F$, then $\mu_{d+L}(i_\ell) \succeq_{i_\ell} i_\ell \succ_{i_\ell} \mu(i_\ell) = \mu_\ell(i_\ell)$, where the first step uses the fact that μ' belongs to V, so is individually rational. For $\ell \in \{d,\ldots,d+L-1\}$, the moving players make a strict improvement in the appropriate sense since $\mu_d, \mu_{d+1}, \ldots, \mu_{d+L}$ is a pairwise myopic-farsighted improving path from μ_d to μ_{d+L}.

We have shown that $h^*(\mu) \cap V \neq \emptyset$ and therefore that V satisfies external stability as defined in Definition 3.

We conclude that V is a pairwise myopic-farsighted stable set. □

Proof of Theorem 3. Let $\{S_1^*, S_2^*, \ldots, S_k^*\}$ be the top-coalition partition and μ^* the induced matching, so $C = \{\mu^*\}$.

First, we show that $V = \{\mu^*\}$ is a pairwise myopic-farsighted stable set by showing that for every $\mu \in \mathcal{M} \setminus \{\mu^*\}$ we have that $\mu^* \in h^*(\mu)$.

Let $\mu \in \mathcal{M} \setminus \{\mu^*\}$ be given. We construct a pairwise myopic-farsighted improving path $\mu_0, \ldots, \mu_L$ from $\mu = \mu_0$ to $\mu^* = \mu_L$ such that the coalition S_ℓ that moves in step ℓ belongs to the top-coalition partition.

If the players in S_1^* are not matched under μ, then let S_1^* move from μ_0 to μ_1, with $\mu_1(S_1^*) = S_1^*$. Obviously, it holds that $\mu_1(i) \succ_i \mu_0(i)$ if $i \in S_1^* \setminus F$, and $\mu_L(i) \succ_i \mu_0(i)$ if $i \in S_1^* \cap F$. Next, if the players in S_2^* are not matched under μ_1, then let S_2^* move from μ_1 to μ_2, with $\mu_2(S_2^*) = S_2^*$. Since S_2^* is a top coalition of $N \setminus S_1^*$ and $\mu_1(S_1^*) = S_1^*$, it holds that $\mu_2(i) \succ_i \mu_1(i)$ if $i \in S_2^* \setminus F$, and $\mu_L(i) \succ_i \mu_1(i)$ if $i \in S_2^* \cap F$. By repeating this argument, we obtain the matching μ^* after a maximum of k steps. We have shown that $\mu^* \in h^*(\mu)$.

Second, we show that $V = \{\mu^*\}$ is the unique pairwise myopic-farsighted stable set. We start by showing that $h^*(\mu^*) = \emptyset$.

Suppose there exists a matching $\mu \in \mathcal{M}$ such that $\mu \in h^*(\mu^*)$. Then there is a finite sequence of distinct matchings $\mu_0, \ldots, \mu_L$ with $\mu_0 = \mu^*$ and $\mu_L = \mu$ such that, for every $\ell \in \{0,\ldots,L-1\}$, there is a coalition $S_\ell \in \mathcal{N}$ with cardinality at most two that can enforce $\mu_{\ell+1}$ from μ_ℓ and

$$\begin{aligned}
&\mu_{\ell+1}(i) \succ_i \mu_\ell(i), && \text{if } i \in S_\ell \setminus F,\\
&\mu_L(i) \succ_i \mu_\ell(i), && \text{if } i \in S_\ell \cap F.
\end{aligned}$$

We use induction to show that, for every $j \in \{1,\ldots,k\}$, $S_j^* \cap (S_0 \cup \cdots \cup S_{L-1}) = \emptyset$, which leads to the desired contradiction. Since S_1^* is a top coalition of N and $\mu^*(S_1^*) = S_1^*$, it clearly holds that $S_1^* \cap (S_0 \cup \cdots \cup S_{L-1}) = \emptyset$.

Assume that, for some $j \in \{1,\ldots,k-1\}$, $(S_1^* \cup \cdots \cup S_j^*) \cap (S_0 \cup \cdots \cup S_{L-1}) = \emptyset$. It follows that, for every $\ell \in \{0,\ldots,L\}$, for every $i \in (S_1^* \cup \cdots \cup S_j^*)$, $\mu_\ell(i) = \mu^*(i)$. Since S_{j+1}^* is a top coalition of $N \setminus (S_1^* \cup \cdots \cup S_j^*)$, it follows that, for every $i \in S_{j+1}^*$, $\mu_1(i) \preceq_i \mu_0(i)$ and $\mu_L(i) \preceq_i \mu_0(i)$, so $i \notin S_0$ and $\mu_1(i) = \mu^*(i)$. Repeating this argument, it follows that $S_{j+1}^* \cap (S_0 \cup \cdots \cup S_{L-1}) = \emptyset$.

We have shown that $h^*(\mu^*) = \emptyset$.

By external stability as in Definition 3 it follows that $\mu^* \in V$. Since $\mu^* \in h^*(\mu)$ for all $\mu \in \mathcal{M} \setminus \{\mu^*\}$, internal stability as in Definition 3 implies that $V = \{\mu^*\}$ is the unique pairwise myopic-farsighted stable set. □

Lemma 3. *Let $(M, W, F, \succ)$ be a marriage problem with $F \supseteq W$. For every $\mu \in \mathcal{M} \setminus \{\mu^{\mathrm{W}}\}$ such that $W(\mu) = \emptyset$ it holds that $\mu^{\mathrm{W}} \in h^*(\mu)$.*

Proof. Let $\mu \in \mathcal{M} \setminus \{\mu^{\mathrm{W}}\}$ be a matching such that $W(\mu) = \emptyset$. We construct a pairwise myopic-farsighted improving path $\mu_0,\ldots,\mu_L$ from $\mu_0 = \mu$ to $\mu_L = \mu^{\mathrm{W}}$. Let

$$W^1 = \{w \in W \mid \mu^{\mathrm{W}}(w) \succ_w \mu(w) \text{ and } \mu(w) \in M\}$$

be the, possibly empty, set of women who strictly prefer $\mu^{\mathrm{W}}(w)$ to $\mu(w)$ and who are married at μ. Let k^1 be the cardinality of W^1. Take an arbitrary order of the women in W^1, say $w_0,\ldots,w_{k^1-1}$.

For $\ell \in \{0,\ldots,k^1-1\}$, we define the matching $\mu_{\ell+1} = \mu_\ell - (\mu_\ell(w_\ell), w_\ell)$, so the k^1 women in W^1 sequentially destroy their matches. We show that the sequence of matchings $\mu_0,\ldots,\mu_{k^1}$ is the first part of a pairwise myopic-farsighted improving path from μ to μ^{W} by showing that for every $\ell \in \{0,\ldots,k^1-1\}$ we have $\mu_L(w_\ell) = \mu^{\mathrm{W}}(w_\ell) \succ_{w_\ell} \mu_\ell(w_\ell)$. This follows since $\mu_\ell(w_\ell) = \mu(w_\ell)$ and by the definition of W^1.

At μ_{k^1} every woman is either single or married to her partner at μ^{W}. It then follows that at μ_{k^1} every man is either single or married to his partner at μ^{W}. Let

$$W^2 = \{w \in W \mid \mu^{\mathrm{W}}(w) \succ_w \mu_{k^1}(w)\}$$

be the set of women that strictly prefer the match at μ^{W} to being single. Let k^2 be the cardinality of the set W^2. Take an arbitrary order of the women in W^2, say $w_0,\ldots,w_{k^2-1}$. For $\ell = 0,\ldots,k^2-1$, we define $m_\ell = \mu^{\mathrm{W}}(w_\ell)$.

For $\ell = 0,\ldots,k^2-1$, we define the matching $\mu_{k^1+\ell+1} = \mu_{k^1+\ell} + (m_\ell, w_\ell)$, so the k^2 women in W^2 sequentially marry their partners at μ^{W}. It holds that $\mu_L = \mu_{k^1+k^2} = \mu^{\mathrm{W}}$. Observe that the sequence of matchings $\mu_{k^1},\ldots,\mu_{k^1+k^2}$ is the final part of a pairwise myopic-farsighted improving path from μ to μ^{W} since it holds that, for every $\ell \in \{0,\ldots,k^2-1\}$,

$$\begin{aligned} \mu_{k^1+\ell+1}(m_\ell) &= \mu^{\mathrm{W}}(m_\ell) \succ_{m_\ell} m_\ell = \mu_{k^1+\ell}(m_\ell), && \text{if } m_\ell \in M \setminus F, \\ \mu_L(m_\ell) &= \mu^{\mathrm{W}}(m_\ell) \succ_{m_\ell} m_\ell = \mu_{k^1+\ell}(m_\ell), && \text{if } m_\ell \in F, \\ \mu_L(w_\ell) &= \mu^{\mathrm{W}}(w_\ell) \succ_{w_\ell} w_\ell = \mu_{k^1+\ell}(w_\ell). \end{aligned}$$ □

Lemma 4. *Let $(M, W, F, \succ)$ be a marriage problem with $F \supseteq W$. Assume that μ^{W} is not W-dominated by a matching in $\mathcal{I}$. For every $\mu \in \mathcal{M}$ such that $W(\mu) \neq \emptyset$ it holds that $\mu^{\mathrm{W}} \in h^*(\mu)$.*

Proof. Let $\mu \in \mathcal{M}$ be a matching such that $W(\mu) \neq \emptyset$. We construct a pairwise myopic-farsighted improving path $\mu_0,\ldots,\mu_L$ from $\mu_0 = \mu$ to $\mu_L = \mu^{\mathrm{W}}$. Let

$$W^1 = \{w \in \mu(M) \mid \mu(w) \succ_{\mu(w)} w \text{ or } w \succ_w \mu(w)\}$$

be the, possibly empty, set of women that are involved in a match that is not individually rational for at least one of the partners and denote the cardinality of W^1 by k^1. Take an arbitrary order of the women in W^1, say $w_0,\ldots,w_{k^1-1}$. For $\ell \in \{0,\ldots,k^1-1\}$, we define $m_\ell = \mu(w_\ell)$ to be the man married to w_ℓ at μ and we define the matching $\mu_{\ell+1} = \mu_\ell - (m_\ell, w_\ell)$, so the player who is involved in a match under μ that is not individually rational destroys his or her match. We argue that the sequence of matchings $\mu_0,\ldots,\mu_{k^1}$ is the first part of a pairwise myopic-farsighted improving path from μ to μ^{W}. Notice that $\mu_{k^1} \in \mathcal{I}$.

Let some $\ell \in \{0,\ldots,k^1-1\}$ be given. It holds that $m_\ell \succ_{m_\ell} w_\ell$ or $w_\ell \succ_{w_\ell} m_\ell$. In the former case, we have that

$$\begin{aligned} \mu_{\ell+1}(m_\ell) &= m_\ell \succ_{m_\ell} \mu_\ell(m_\ell), && \text{if } m_\ell \in M \setminus F, \\ \mu_L(m_\ell) &= \mu^{\mathrm{W}}(m_\ell) \succeq_{m_\ell} m_\ell \succ_{m_\ell} \mu_\ell(m_\ell), && \text{if } m_\ell \in F, \end{aligned}$$

whereas in the latter case it holds that

$$\mu_L(w_\ell) = \mu^{\mathrm{W}}(w_\ell) \succeq_{w_\ell} w_\ell \succ_{w_\ell} \mu_\ell(w_\ell),$$

so the conditions of Definition 2 are satisfied.

Let

$$W^2 = W(\mu_{k^1})$$

be the set of women that strictly prefer $\mu_{k^1}(w)$ to $\mu^{\mathrm{W}}(w)$. If $W^2 = \emptyset$, then the matching μ_{k^1} is equal to μ^{W} or satisfies the assumptions of Lemma 3. In the former case we are done, in the latter case we proceed as in the proof of Lemma 3 to complete the construction of the pairwise myopic-farsighted improving path leading to μ^{W}. It remains to consider the situation where $W^2 \neq \emptyset$.

We consider two cases.

Case 1. $\mu^{\mathrm{W}}(W^2) = \mu_{k^1}(W^2)$.

In this case, starting from μ_{k^1}, we sequentially marry all women outside W^2 to their partner at μ^{W}, resulting in a matching $\overline{\mu}$. The matching $\overline{\mu}$ belongs to $\mathcal{I}$ and W-dominates the matching μ^{W}, leading to a contradiction with the assumptions of the lemma. This case can therefore not occur.

Case 2. $\mu^{\mathrm{W}}(W^2) \neq \mu_{k^1}(W^2)$.

We proceed by induction in ℓ. For $\ell \in \{0,1,\ldots\}$, whenever $\mu^{\mathrm{W}}(W^{2+\ell}) \neq \mu_{k^1+\ell}(W^{2+\ell})$, select some $m_\ell \in \mu_{k^1+\ell}(W^{2+\ell}) \cap M \setminus \mu^{\mathrm{W}}(W^{2+\ell})$ and let $w_\ell = \mu^{\mathrm{W}}(m_\ell)$. This selection of m_ℓ is possible since all women in $W^{2+\ell}$ are married at $\mu_{k^1+\ell}$ as they strictly prefer $\mu_{k^1+\ell}$ to μ^{W}. The cardinality of $\mu_{k^1+\ell}(W^{2+\ell}) \cap M$ is therefore greater than or equal to the cardinality of $\mu^{\mathrm{W}}(W^{2+\ell}) \cap M$, whereas these two sets are distinct.

We argue first that (m_ℓ, w_ℓ) blocks $\mu_{k^1+\ell}$. Let $w' = \mu_{k^1+\ell}(m_\ell)$. By definition of $W^{2+\ell}$, w' prefers m_ℓ to $\mu^{\mathrm{W}}(w')$, so m_ℓ prefers $\mu^{\mathrm{W}}(m_\ell) = w_\ell$ to w', because otherwise μ^{W} would not be a core element. Since $m_\ell \notin \mu^{\mathrm{W}}(W^{2+\ell})$, we have that $w_\ell \notin W^{2+\ell}$, so w_ℓ prefers $m_\ell = \mu^{\mathrm{W}}(w_\ell)$ to $\mu_{k^1+\ell}(w_\ell)$. We have shown that (m_ℓ, w_ℓ) blocks $\mu_{k^1+\ell}$.

We define the matching $\mu_{k^1+\ell+1} = \mu_{k^1+\ell} + (m_\ell, w_\ell)$ and $W^{2+\ell+1} = W(\mu_{k^1+\ell+1})$. Let k^2 denote the cardinality of $W(\mu_{k^1})$. For every $\ell \geq 0$, at $\mu_{k^1+\ell}$ man m_ℓ is married to a woman in $W^{2+\ell}$ and is married to a woman not in $W^{2+\ell}$ at $\mu_{k^1+\ell+1}$, it follows that the cardinality of the set $W^{2+\ell}$ is strictly decreasing in ℓ. By the argument of Case 1, as long as $W^{2+\ell} \neq \emptyset$, it cannot occur that $\mu^{\mathrm{W}}(W^{2+\ell}) = \mu_{k^1+\ell}(W^{2+\ell})$. Therefore, we have that $W^{2+k^2} = \emptyset$.

We argue that the sequence of matchings $\mu_{k^1},\ldots,\mu_{k^1+k^2}$ is the second part of a pairwise myopic-farsighted improving path from μ to μ^{W} since it holds that, for every $k \in \{0,\ldots,k^2-1\}$,

$$\begin{aligned} \mu_{k^1+\ell+1}(m_\ell) &= \mu^{\mathrm{W}}(m_\ell) \succ_{m_\ell} \mu_{k^1+\ell}(m_\ell), && \text{if } m_\ell \in M \setminus F, \\ \mu_L(m_\ell) &= \mu^{\mathrm{W}}(m_\ell) \succ_{m_\ell} \mu_{k^1+\ell}(m_\ell), && \text{if } m_\ell \in F, \\ \mu_L(w_\ell) &= \mu^{\mathrm{W}}(w_\ell) \succ_{w_\ell} \mu_{k^1+\ell}(w_\ell). \end{aligned}$$

The conditions of Definition 2 are therefore satisfied.

The matching $\mu_{k^1+k^2}$ is equal to μ^{W} or satisfies the assumptions of Lemma 3. In the former case we are done, in the latter case we proceed as in the proof of Lemma 3 to complete the construction of the pairwise myopic-farsighted improving path leading to μ^{W}. □

Proof of Theorem 5. Suppose that $V \subseteq C$ is a pairwise myopic-farsighted stable set different from $\{\mu^{\mathrm{W}}\}$. It holds that $V \subseteq C \setminus \{\mu^{\mathrm{W}}\}$, since otherwise Lemma 3 implies that internal stability would be violated. Since $\mu^{\mathrm{W}} \notin V$, by external stability of V, there should be a pairwise myopic-farsighted improving path $\mu_0,\ldots,\mu_L$ from $\mu_0 = \mu^{\mathrm{W}}$ to $\mu_L \in V$.

It holds that $\mu^{\mathrm{W}} \succeq_w \mu$ for all $\mu \in C$. Thus, since all women are farsighted, no woman $w \in W$ will form a match with a man $m \in M$ to move from $\mu_0 = \mu^{\mathrm{W}}$ to $\mu_1 = \mu_0 + (m, w)$ in order to finish at $\mu \in C$ because she cannot strictly improve. For the same reason, no woman $w \in W$ will destroy a match with a man $m \in M$ to move from $\mu_0 = \mu^{\mathrm{W}}$ to $\mu_1 = \mu_0 - (m, w)$ in order to finish at $\mu \in C$. The only remaining possibility is that a man destroys a link at $\mu_0 = \mu^{\mathrm{W}}$. But since μ^{W} is individually rational, this is not profitable. Hence, we conclude that there is no pairwise myopic-farsighted improving path $\mu_0, \ldots, \mu_L$ from $\mu_0 = \mu^{\mathrm{W}}$ to $\mu_L \in V$, and therefore $V \subseteq C \setminus \{\mu^{\mathrm{W}}\}$ cannot be a pairwise myopic-farsighted stable set. □

Proof of Theorem 6. Let $V \subseteq C$ be a pairwise myopic-farsighted stable set. By Theorem 5, the only possibility is $V = \{\mu^{\mathrm{W}}\}$. In order to satisfy external stability in Definition 3, it should hold that $\mu^{\mathrm{W}} \in h^*(\mu)$ for every $\mu \in \mathcal{M} \setminus \{\mu^{\mathrm{W}}\}$. However, notice that $\mu^{\mathrm{W}} \notin h^*(\overline{\mu})$ if $\overline{\mu} \in \mathcal{I}$ W-dominates μ^{W}. Since all women are farsighted, no woman $w \in W$ will form a match with a man $m \in M$ to move from $\mu_0 = \overline{\mu}$ to $\mu_1 = \mu_0 + (m, w)$ in order to end at μ^{W} because she cannot strictly improve. For the same reason, no woman $w \in W$ will destroy a match with a man $m \in M$ to move from $\mu_0 = \overline{\mu}$ to $\mu_1 = \mu_0 - (m, w)$ in order to terminate at μ^{W}. Also, since $\overline{\mu} \in \mathcal{I}$, no man can profitably deviate from $\mu_0 = \overline{\mu}$ to $\mu_1 = \mu_0 - (m, w)$. Hence, we conclude that there is no pairwise myopic-farsighted improving path $\mu_0, \ldots, \mu_L$ from $\mu_0 = \overline{\mu}$ to $\mu_L = \mu^{\mathrm{W}}$, and therefore $\{\mu^{\mathrm{W}}\}$ cannot be a pairwise myopic-farsighted stable set. □

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