DIVIDING THE EXPECTED PAYOFF RESULTING FROM JOINT ACTIONS

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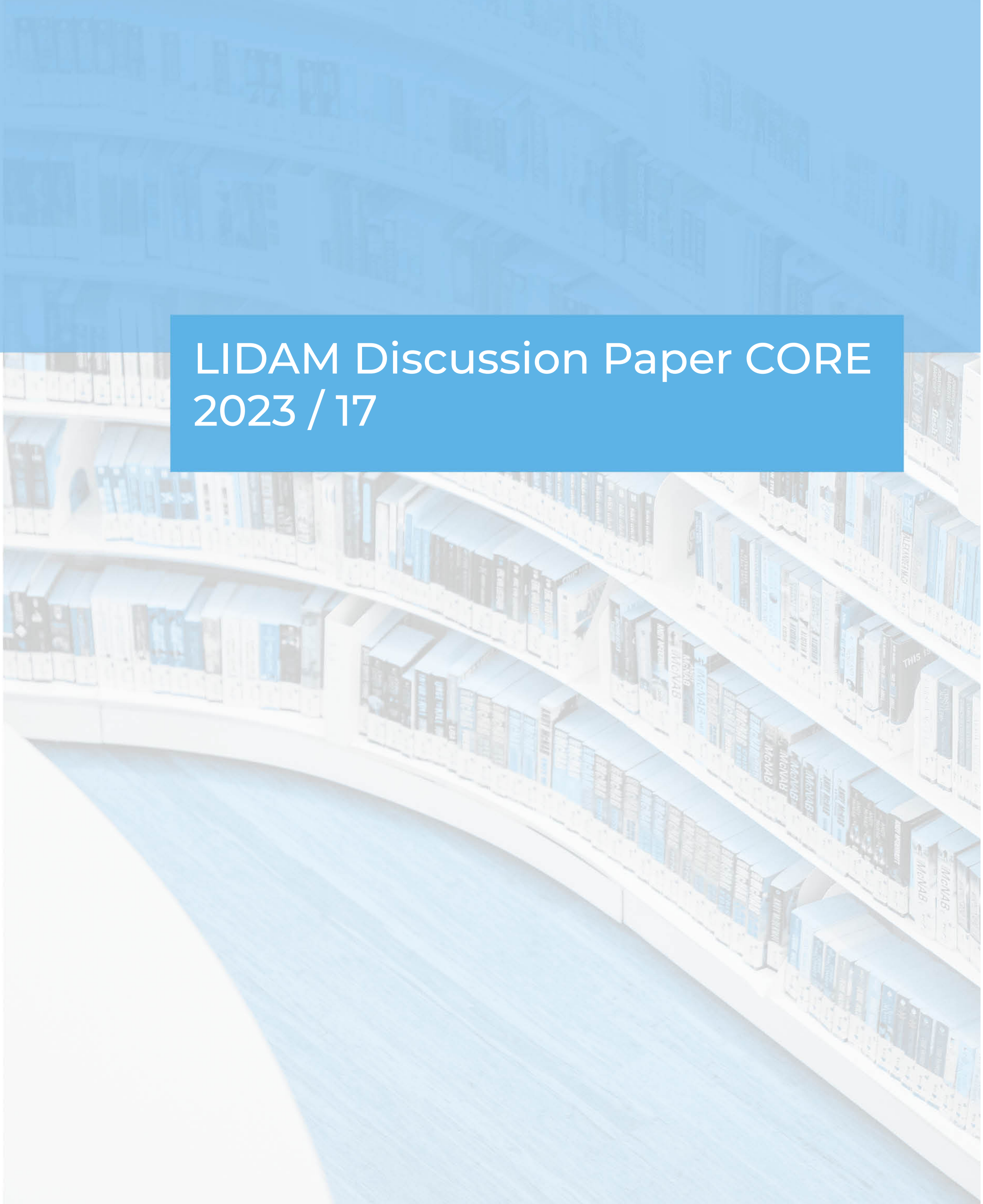

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Dividing the expected payoff resulting from joint actions

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Abstract
We consider situations where players hit targets with known probabilities and are rewarded according to given rules. The division of the expected payoff resulting from their joint actions is studied in the context of transferable utility games, using the Shapley value as the allocation rule.

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1. Introduction

Consider a group of actors (or factors) and a target that each actor is able to reach with a given probability. In this framework, the question of the allocation among the agents of the resulting overall probability of success comes in. Framing this problem as a cooperative game with transferable utility – *a probability game* – Hou et al. (2018) have proposed to use the Shapley value. Probability games and their duals have been further studied by Dehez (2023a) together with a proper axiomatization of the Shapley value on probability games.

The present paper extends the previous setting by considering a situation where there are several targets to which payoffs are associated. The question then concerns the allocation of the resulting *expected payoff*. This is an ex-ante allocation that provides a key to the apportionment of actual payoffs. We consider different situations.

In a first situation, if a coalition forms, payoffs are allocated depending on the targets reached by its members. In a second situation, if a coalition forms and some of its members reach a target, the payoff is allocated to the coalition only once. A third situation extends the previous one by assuming that a payoff goes to a coalition only if the associated target has not been reached by the complementary coalition.

The games emerging from the first situation are additive and, as a consequence, the Shapley value allocates to each player its individual expected payoff. The other two situations lead to games that are dual to each other, thereby giving rise to the same Shapley value.

In all three situations, payoffs are being added. In contrast, we consider a situation where, if a player joins a coalition, its payoff is being multiplied by a coefficient that is specific to that player: a payoff for a player is associated to his ability to increase the payoff of a coalition. The coefficient is assumed to be a random variable with known probability distribution. The associated transferable utility game is a cooperative product game, a concept introduced by Rosales (2014) in an unpublished memo and studied in more detail in Dehez (2023b).

The paper is organized as follows. Probability games and their duals are defined in Section 2. Targets are introduced in Section 3 and three situations are considered, assuming that payoffs are added. Section 4 is devoted to a fourth situation where payoffs are multiplied. The last section covers concluding remarks.

2. Probabilistic situations

Consider a set $N = \{1,\ldots,n\}$ of players facing a target. By his actions, a player *alone* can reach the target with a given and known probability, $p_i \in [0,1]$ for player *i*. Probabilistic independence prevails: the success of a player is independent of the success of the other players. A pair (N, p) defines a *probabilistic situation*.

A transferable utility game (TU-game for short) is a pair (N, v) where *N* is the set of players and *v* is a (characteristic) function that associates a real number – its "worth" – to all subsets (coalitions) of *N*. By convention, $v(\varnothing) = 0$.

Notation: Finite sets are denoted by upper-case letters. Lower-case letters are used to denote their cardinals: $t = |T|$, $s = |S|$, … Set inclusion (non-strict) is denoted by $\subset$. For a vector *x*, $x(S)$ denotes the sum of its coordinates over the subset *S*. Set inclusion (non-strict) is denoted by $\subset$.

Braces are sometimes omitted for coalitions, for instance $v(i, j)$ replaces $v(\{i, j\})$. By convention, a sum over an empty set is zero and a product over an empty set is equal to 1.

2.1 Probability games

Given a probabilistic situation (N, p), Hou et al. (2018) define the transferable utility game (N, v) whose characteristic function *v* is given by:

$$v(S) = 1 - \prod_{i \in S} (1 - p_i) \text{ for all } S \subset N. \tag{1}$$

$v(S)$ is the probability that *at least one* member of *S* succeeds. The game (N, v) is *concave* and thereby *subadditive*. The *dual* game (N, v^d) is defined by:

$$v^d(S) = v(N) - v(N \setminus S) = \prod_{i \in N \setminus S} (1 - p_i) - \prod_{i \in N} (1 - p_i) = \left(1 - \prod_{i \in S} (1 - p_i)\right) \prod_{i \in N \setminus S} (1 - p_i). \tag{2}$$

$v^d(S)$ is the probability that at least one player in *S* succeeds, assuming that players outside *S* *all* fail. The game (N, v^d) is *convex* (and thereby *superadditive*) as dual of a concave game.[1]

By definition of the dual, the probability to be allocated among the players is the same in both cases: the question concerns the division of the collective probability of success that is given by: $v(N) = v^d(N)$.

2.2 The Shapley value of a probability game

Given a set of players *N*, the set *G*(*N*) of all characteristic functions on *N* is equivalent to the vector space $\mathbb{R}^{2^n - 1}$. In proving the uniqueness of his value, Shapley (1953) shows that the collection of *unanimity games* $(N, u_T), T \subset N$, defined by

$$\begin{aligned} u_T(S) &= 1 \quad \text{if } T \subset S, \\ &= 0 \quad \text{if } T \not\subset S, \end{aligned}$$

forms a basis of *G*(*N*): for any given characteristic function $v \in G(N)$, there exists a unique $(2^n - 1)$-dimensional vector $\alpha = (\alpha_T)_{T \subset N}$ such that:

$$v(S) = \sum_{T \subset N} \alpha_T u_T(S) = \sum_{T \subset S} \alpha_T, \tag{3}$$

assuming $\alpha_\varnothing = 0$. The α_T can be defined recursively, starting with as follows:

$$\alpha_T = v(T) - \sum_{S \subsetneq T} \alpha_S \;\; \Rightarrow \;\; \alpha_T = \sum_{S \subset T} (-1)^{t-s} v(S) \text{ for all } T \subset N. \tag{4}$$

Following Harsanyi (1959), α_T is the *dividend* accruing to coalition *T* once all sub-coalitions have received their dividends. By (3), the sum of all dividends is equal to *v*(*N*). An allocation can then be obtained by distributing the dividends of each coalition among its members.[2] Harsanyi shows that the *Shapley value* gives to each player the sum of the *per capita* dividend of the coalitions of which he is a member.

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[1] A game (*N*,*v*) is concave if the game (*N*,-*v*) is convex. For convexity of TU-games, refer to Shapley (1971).

[2] See Dehez (2017) for a comprehensive analysis of the distribution of Harsanyi dividends.

Hence, we have:

$$SV_i(N,v) = \sum_{T\subset N: i\in T} \frac{1}{t}\alpha_T(N,v). \qquad (5)$$

Using (4), the Harsanyi dividends of the probability game (N,v) associated to the probability vector $p \in P_n$ are given by:

$$\alpha_T(N,v) = (-1)^{t-1}\prod_{i\in T} p_i. \qquad (6)$$

They alternate in sign according to coalition size and have no particular interpretation. The dividends of the dual game (N,v^d) are given by:

$$\alpha_T(N,v^d) = \prod_{j\in T} p_j \prod_{j\in N\setminus T}(1-p_j). \qquad (7)$$

It is the probability that players in *T* all succeed while players outside *T* all fail. We observe that dividends of dual probability games are nonnegative: dual probability games are *positive*.[3]

The Shapley value is a *self-dual* allocation rule: the value of a game coincides with the value of its dual. Introducing successively (6) in (5) and (7) in (5), we obtain two *equivalent* formulations of the Shapley value associated to a probabilistic situation (N,p):

$$SV_i(N,p) = \sum_{T\subset N: i\in T} \frac{(-1)^{t-1}}{t}\prod_{j\in T} p_j = p_i \sum_{T\subset N: i\in T} \frac{(-1)^{t-1}}{t}\prod_{j\in T\setminus i} p_j \qquad (8)$$

and

$$SV_i(N,p) = \sum_{T\subset N: i\in T} \frac{1}{t}\prod_{j\in T} p_j \prod_{j\in N\setminus T}(1-p_j) = p_i \sum_{T\subset N: i\in T} \frac{1}{t}\prod_{j\in T/i} p_j \prod_{j\in N\setminus T}(1-p_j) \qquad (9)$$

where $i = 1,\ldots,n$. Hence, whatever is the definition of the probability of success of a coalition, (1) or (2), the Shapley value defines the same allocation: (8) and (9) coincide.

The Shapley value follows a natural decomposition of the collective probability of success. In the 2-player case, the collective probability $p_1 + p_2 - p_1p_2$ is decomposed as follows:

$$x_1 = p_1 - \frac{1}{2}p_1p_2 \text{ and } x_2 = p_2 - \frac{1}{2}p_1p_2.$$

This decomposition is easily extended to any number of players. In the 3-player case, the collective probability is given by $p_1 + p_2 + p_3 - p_1p_2 - p_1p_3 - p_2p_3 + p_1p_2p_3$ and the amount allocated to player 1 is given by:

$$x_1 = p_1 - \frac{1}{2}p_1p_2 - \frac{1}{2}p_1p_3 + \frac{1}{3}p_1p_2p_3 = p_1\left(1 - \frac{1}{2}p_2 - \frac{1}{2}p_3 + \frac{1}{3}p_2p_3\right).$$

Looking at (8), we observe that the Shapley value allocates to a player a share that is *proportional* to his probability of success:

$$SV_i(N,p) = p_i\ f(p_{-i})$$

[3] Positive games form an interesting class of games on which solution concepts tend to converge: the core coincides with the set of weighted Shapley values and the Harsanyi set that collects all possible distributions of dividends.

where p_{-i} denotes the vector of probabilities from which the coordinate i has been eliminated and $f:\mathbb{R}_{+}^{n-1}\to\mathbb{R}_{+}$ is the *player-independent* and *symmetric*[4] function given by

$$f(z)=1+\sum_{T\subset\{1,\dots,n-1\}}\frac{(-1)^{t}}{t+1}\prod_{j\in T}z_j. \tag{10}$$

In Dehez (2023a), it is shown that *proportionality,* together with *efficiency* and *symmetry,* characterizes the Shapley value of a probability game: there is one and only one function f verifying these three properties.[5]

3. Associating payoffs to targets

Positive payoffs are associated to targets. We first extend the previous section by considering a single target. The case of several targets is analyzed thereafter within different situations.

3.1 The case of a single target

Let r be the payoff associated to the target, $r>0.$ There are two possible situations. In the first situation, a coalition obtains the payoff if it succeeds, independently of the success or failure of the complementary coalition. If coalition S forms, its expected payoff is given by:

$$w_0(S)=\left(1-\prod_{i\in S}(1-p_i)\right)r \tag{11}$$

Referring to the probability game $(N,v),$ we have $w_0(S)=v(S)\,r.$ In the second situation, a coalition obtains the payoff if it succeeds while the complementary coalition fails. If coalition S forms, its expected payoff is given by:

$$w_0^d(S)=\left(1-\prod_{i\in S}(1-p_i)\right)\prod_{i\in N\setminus S}(1-p_i)\,r. \tag{12}$$

Both games are dual: $w_0^d(S)=v^d(S)\,r.$ In both situations, the question concerns the allocation of the collective expected payoff given by:

$$w_0(N)=w_0^d(N)=(1-\prod_{i\in N}(1-p_i))\,r.$$

By self-duality, the Shapley value can be computed indifferently from (11) or (12), and by linearity of the Shapley value, it is simply given by $SV_i(N,p,r)=SV_i(N,p)\,r$ where $SV_i(N,p)$ is given, by (8) or (9).

3.2 The case of several targets: three situations

Consider a situation involving a finite set M of targets, $m\geq 2$. Target h is associated with a real number $r_h>0$ expressed in terms of some "money". We denote by $p_{ih}\in[0,1]$ the probability that player i hits target h and by P the probability matrix $[p_{ih}].$ The case of a single target is covered by setting $m=2$ and $r_2=0.$ We assume that players hit one of the m targets with probability one:

$$\sum_{h\in M}p_{ih}=1 \text{ for all } i\in N. \tag{13}$$

[4] A function is symmetric if permuting its arguments leaves its value unchanged.

[5] Efficiency requires that the value of the game $v(N)$ is exactly distributed. Symmetry requires that an identical amount is allocated to pairs of players who contribute identically to coalitions to which they belong: their marginal contributions coincide.

We denote by X_i the (discrete) random variable associated to player *i*. It is defined by the payoffs $(r_1,...,r_m)$ and corresponding probabilities $(p_{i1},..., p_{im})$.

If $n = m = 3$, there are $3^3 = 27$ possible events. For example, the event (r_1, r_3, r_1) occurs with probability $p_{11}p_{23}p_{31}$. (13) ensures that the probabilities sum up to one. Table 1 lists the events and probabilities for the case where $n = 3$ and $m = 2$ where $p_{i2} = 1 - p_{i1}$.

event	probabilities	
r_1, r_1, r_1	$p_{11}p_{21}p_{31}$	0.126
r_1, r_1, r_2	$p_{11}p_{21}p_{32}$	0.054
r_1, r_2, r_1	$p_{11}p_{22}p_{31}$	0.294
r_1, r_2, r_2	$p_{11}p_{22}p_{32}$	0.126
r_2, r_1, r_1	$p_{12}p_{21}p_{31}$	0.084
r_2, r_1, r_2	$p_{12}p_{21}p_{32}$	0.036
r_2, r_2, r_1	$p_{12}p_{22}p_{31}$	0.196
r_2, r_2, r_2	$p_{12}p_{22}p_{32}$	0.084

Table 1: Events and probabilities

Several situations are possible. The first situation is the simplest one: once a player hits a target, he obtains the corresponding payoff. This is illustrated in Table 2 for the case where $n = 3$ and $m = 2$.

event	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
r_1, r_1, r_1	r_1	r_1	r_1	$2r_1$	$2r_1$	$2r_1$	$3r_1$
r_1, r_1, r_2	r_1	r_1	r_2	$2r_1$	$r_1 + r_2$	$r_1 + r_2$	$2r_1 + r_2$
r_1, r_2, r_1	r_1	r_2	r_1	$r_1 + r_2$	$2r_1$	$r_1 + r_2$	$2r_1 + r_2$
r_1, r_2, r_2	r_1	r_2	r_2	$r_1 + r_2$	$r_1 + r_2$	$2r_2$	$r_1 + 2r_2$
r_2, r_1, r_1	r_2	r_1	r_1	$r_1 + r_2$	$r_1 + r_2$	$2r_1$	$2r_1 + r_2$
r_2, r_1, r_2	r_2	r_1	r_2	$r_1 + r_2$	$2r_2$	$r_1 + r_2$	$r_1 + 2r_2$
r_2, r_2, r_1	r_2	r_2	r_1	$2r_2$	$r_1 + r_2$	$r_1 + r_2$	$r_1 + 2r_2$
r_2, r_2, r_2	r_2	r_2	r_2	$2r_2$	$2r_2$	$2r_2$	$3r_2$

Table 2: The additive case.

The expected payoffs are given by:

$$w_1(1) = p_{11}r_1 + p_{12}r_2 = E[X_1],\ldots$$
$$w_1(1,2) = (p_{11} + p_{21})r_1 + (p_{12} + p_{22})r_2 = E[X_1] + E[X_2],\ldots$$
$$w_1(1,2,3) = (p_{11} + p_{21} + p_{31})r_1 + (p_{12} + p_{22} + p_{32})r_2 = E[X_1] + E[X_2] + E[X_3].$$

In general, the expected payoff of coalition S is given by:

$$w_1(S) = \sum_{i\in S} E[X_i] = \sum_{h\in M} r_h \sum_{i\in S} p_{ih}.$$

The game (N, w_1) is *additive*. The Shapley value being an additive allocation rule, it simply allocates to a player his individual expected payoff:

$$SV_i(N, w_1) = E[X_i] = \sum_{h\in M} p_{ih}r_h \quad (i = 1,\ldots,n).$$

Example 1 Consider a 3-player case and two targets where $(p_{11}, p_{12}, p_{13}) = (0.6, 0.3, 0.7)$. The resulting probabilities are given in Table 1. If the payoffs are given by $(r_1, r_2) = (5, 7)$, the associated game (N, w_1) is defined by $w_1 = (5.8, 6.4, 5.6 \mid 12.2, 11.4, 12 \mid 17.8)$ and its Shapley value is given by $(5.8, 6.4, 5.6)$. The distribution key is then given by (0.33, 0.38, 0.31).

In a second situation, a coalition obtains a payoff once the corresponding target has been reached by one of its members, independently of the number of members that have succeeded. This is illustrated in Table 3.

event	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
r_1, r_1, r_1	r_1	r_1	r_1	r_1	r_1	r_1	r_1
r_1, r_1, r_2	r_1	r_1	r_2	r_1	$r_1 + r_2$	$r_1 + r_2$	$r_1 + r_2$
r_1, r_2, r_1	r_1	r_2	r_1	$r_1 + r_2$	r_1	$r_1 + r_2$	$r_1 + r_2$
r_1, r_2, r_2	r_1	r_2	r_2	$r_1 + r_2$	$r_1 + r_2$	r_2	$r_1 + r_2$
r_2, r_1, r_1	r_2	r_1	r_1	$r_1 + r_2$	$r_1 + r_2$	r_1	$r_1 + r_2$
r_2, r_1, r_2	r_2	r_1	r_2	$r_1 + r_2$	r_2	$r_1 + r_2$	$r_1 + r_2$
r_2, r_2, r_1	r_2	r_2	r_1	r_2	$r_1 + r_2$	$r_1 + r_2$	$r_1 + r_2$
r_2, r_2, r_2	r_2	r_2	r_2	r_2	r_2	r_2	r_2

Table 3: A same payoff is not repeated within a coalition.

The expected payoffs are then computed on the basis of the probability games (N, v_h):

$$w_2(S) = \sum_{h\in M} v_h(S)\, r_h$$

where $v_h(S)$ is the probability that at least one member of S hits target h, as given by (1):

$$v_h(S) = 1 - \prod_{i\in S}(1 - p_{ih}).$$

Being a positive linear combination of concave games, the game (N, w_2) is itself concave.

In the preceding situation, if a coalition *S* and the complementary coalition $N \setminus S$ hit the same target, both obtain the corresponding payoff. This is excluded in the third situation illustrated in Table 4 where a payoff can be obtained only once: if a coalition forms, it obtains the payoff associated to a target only if no player outside the coalition has hit the same target.

event	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
r_1, r_1, r_1	0	0	0	0	0	0	r_1
r_1, r_1, r_2	0	0	r_2	r_1	r_2	r_2	$r_1 + r_2$
r_1, r_2, r_1	0	r_2	0	r_2	r_1	r_2	$r_1 + r_2$
r_1, r_2, r_2	r_1	0	0	r_1	r_1	r_2	$r_1 + r_2$
r_2, r_1, r_1	r_2	0	0	r_2	r_2	r_1	$r_1 + r_2$
r_2, r_1, r_2	0	r_1	0	r_1	r_2	r_1	$r_1 + r_2$
r_2, r_2, r_1	0	0	r_1	r_2	r_1	r_1	$r_1 + r_2$
r_2, r_2, r_2	0	0	0	0	0	0	r_2

Table 4: A same payoff does not go to a coalition and its complement.

The expected payoffs are then computed on the basis of the dual probability games (N, v_h^d):

$$w_3(S) = \sum_{h \in M} v_h^d(S)\, r_h$$

where $v_h^d(S)$ is the probability that at least one member of *S* hit target *h* while those outside *S* all fail, as given by (2):

$$v_h^d(S) = \left(1 - \prod_{i \in S}(1 - p_{ih})\right) \prod_{i \in N \setminus S}(1 - p_{ih}) = \left(1 - \prod_{i \in S}(1 - p_{ih})\right) \prod_{i \in N \setminus S}(1 - p_{ih}).$$

Being a positive linear combination of convex games, the game (N, w_3) is itself convex. The Shapley value is self-dual and linear. Hence, these two situations lead to the same allocation. Using (8), the Shapley can be written as:

$$SV_i(N, P, r) = \sum_{h \in M} SV_i(N, p_h)\, r_h = \sum_{h \in M} p_{ih}\, r_h \sum_{T \subset N : i \in T} \frac{(-1)^{t-1}}{t} \prod_{j \in T \setminus i} p_{jh} = \sum_{h \in M} p_{ih}\, r_h\, f(p_{-ih})$$

where the function *f* is given in (10).

Example 1 (continued) The games (N, w_2) and (N, w_3) associated with the probabilities $(p_{11}, p_{12}, p_{13}) = (0.6, 0.3, 0.7)$ and payoffs $(r_1, r_2) = (5, 7)$, are defined respectively by:

$$w_2 = (5.8,\ 6.4,\ 5.6 \,|\, 9.34,\ 8.46,\ 9.48 \,|\, 10.698) \text{ and}$$

$$w_3 = w_2^d = (1.218,\ 2.238,\ 1.358 \,|\, 5.098,\ 4.298,\ 4.898 \,|\, 10.698).$$

The (common) Shapley value and the associated distribution key are respectively given by $(3.306, 4.116, 3.276)$ and $(0.31,\ 0.38,\ 0.31)$.

4. Multiplicative effects

In the preceding section, payoffs were added. Consider now a situation where adding a player to a coalition multiplies its payoff by a coefficient that is specific to that player: a payoff for a player is associated to his ability to increase the payoff of a coalition.

The coefficients are assumed to be independent random variables with known probability distribution function and finite mean. More precisely, the coefficient specific to player *i* is a random variable X_i defined on the interval $[1,+\infty)$. Hence, $\mu_i \geq 1$ and the case where $\mu_i = 1$ occurs only if $X_i = 1$ with probability one. Given the assumption of independence, the associated transferable utility game is defined by:

$$w(S) = E\left[\prod_{i\in S} X_i\right] - 1 = \prod_{i\in S}\mu_i - 1. \tag{14}$$

Hence, a player contributes to a coalition once his expected coefficient is larger than 1. Indeed, $w(i) = \mu_i - 1$ and the marginal contributions of player *i* are given by:

$$w(S) - w(S\setminus i) = (\mu_i - 1)\prod_{j\in S\setminus i}\mu_j \quad \text{for all } S \ni i.$$

The game (N,w) belongs to the class of *cooperative product games* introduced by Rosales (2014) and revisited in Dehez (2023b).[6] Obviously, marginal contributions are increasing. As a consequence, product games are convex (and thereby superadditive). They are *strictly* convex if $\mu_i > 1$ for all $i \in N$.

Using (4), the Harsanyi dividends are given by:

$$\alpha_T(N,w) = \prod_{i\in T}(\mu_i - 1).$$

Consequently, using (5), the Shapley value is given by:

$$SV_i(N,\mu_1,...,\mu_n) = (\mu_i - 1)\sum_{T\subset N: i\in T}\frac{1}{t}\prod_{j\in T\setminus i}(\mu_j - 1).$$

We observe that proportionality applies to product games as well. The Shapley value can indeed be written as:

$$SV_i(N,\mu_1,...,\mu_n) = (\mu_i - 1)\, g(\mu_{-i})$$

where the function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ is the player independent and symmetric function given by:

$$g(z) = 1 + \sum_{T\subset\{1,...,n-1\}}\frac{1}{t+1}\prod_{j\in T}(z_j - 1).$$

Dehez (2023b) shows that efficiency, symmetry and proportionality characterize the Shapley value on the class of product games.

Example 2 The 3-player game associated to the means $(\mu_1,\mu_2,\mu_3) = (1.3,\ 1.8,\ 2.1)$ is given by $(0.3,\ 0.8,\ 1.1 \,|\, 1.34,\ 1.73,\ 2.78 \,|\, 3.914)$. The Shapley value and the associated distribution key are respectively given by $(0.673,\ 1.448,\ 1.793)$ and $(0.172, 0.370, 0.458)$.

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[6] Rosales defines *v*(*S*) as the product of the coefficients. This would be consistent if he had assumed that $v(\varnothing) = 1$.

5. Concluding remarks

The assumption of probabilistic independence is clearly a limitation. However, in the multiplicative case, the independence assumption can be dispensed with by considering the *geometric mean* instead of the linear mean. For an arbitrary random variable *X*, the geometric mean is defined by:

$$E_G[X] = \mathrm{e}^{E[\log(X)]}.$$

It is an alternative measure of the central tendencies of random variables defined on $]0,+\infty[$. It is used in different contexts.[7] The geometric expectation of a discrete random variable *Y* defined by $\{(x_1,q_1),\ldots,(x_m,q_m)\}$, where $x_h > 0$ for all *h* and $\sum q_h = 1$, is then simply given by:

$$E_G[X] = \prod_{h=1}^{m} x_h^{q_h}$$

hence the name "geometric" mean. The geometric mean is linear and bounded above by the linear mean: $E_G[\alpha Y] = \alpha E_G[Y]$ for all $\alpha > 0$ and $E_G[Y] \leq E[Y]$. The first property is immediate and the second property is a direct consequence of Jensen inequality. Furthermore, the geometric mean of a product of random variables defined on $]0,+\infty[$ is the product of their geometric means: $E_G[\prod X_i] = \prod E_G[X_i]$. This property applies without requiring independence. Hence, we could define the game (N,w) in (14) by setting $\mu_i = E_G\left[X_i\right]$.

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[7] There is a large literature on the geometric mean and its applications, in particular in finance and insurance. See for instance Jasiulewicz and Kordecki (2016).