

The asymptotic behaviour of the $\pi^0 \gamma^* \gamma^*$ vertex.

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To the memory of Roger Decker

Abstract

The Bjorken-Johnson-Low theorem applied to the $\gamma^* \rightarrow \gamma^* \pi^0$ process provides us with a rather remarkable asymptotic behaviour for the $\pi^0 \gamma^* \gamma^*$ vertex. We compare our result with previous QCD- inspired estimates and argue that the predicted behaviour is quite consistent with the present data on hadronic J/Ψ decays and ensures the matching of long- and short-distance radiative corrections to $\pi^+ \rightarrow e^+ \nu_e$.

Anomalies play a crucial role in elementary particle physics. Their cancellation for local currents [1] to ensure the renormalizability of the (extended) Standard Model based on the $SU(N)_C \times SU(2)_L \times U(1)$ gauge symmetries requires a relation between the electric charge and the (odd) number N of colours for the up and down quarks:

$$\begin{aligned} Q_u &= \frac{1+N}{2N} \\ Q_d &= \frac{1-N}{2N}. \end{aligned} \tag{1}$$

On the other hand, the non-vanishing triangle anomaly [2] associated with a global axial current allows the π^0 to decay into two real photons at a rate in excellent agreement with experiment if its fermionic components precisely satisfy the relations given in Eq.(1), for any N . In particular, for $N = 1$ we recover the old result [3] based on a nucleon bound state picture for the pion and for $N = 3$, the standard one [4] based on the quark model for π^0 .

It is therefore quite interesting to study extrapolations of the anomalous $\pi^0\gamma\gamma$ vertex to virtual photons. From a phenomenological point of view, the $W^+ \rightarrow \pi^+\gamma$ and $Z^0 \rightarrow \pi^0\gamma$ processes [5] are related to this vertex with one off-mass-shell photon, while the two off-mass-shell photon vertex contributes to the rare $\pi^0 \rightarrow e^+e^-$ decay amplitude [6].

Different asymptotic behaviours for the $\pi^0\gamma^*(q_1^2)\gamma^*(q_2^2)$ vertex when $(q_1^2 \rightarrow \infty, q_2^2 = 0)$ [7] or $(q_1^2 = q_2^2 \rightarrow \infty)$ [8] have been derived in the literature. On the sole basis of the Bjorken-Johnson-Low theorem [9], we will argue that the asymptotic behaviour for $(q_1^2 \rightarrow \infty, q_2^2 \text{ arbitrary})$ is in fact universal. This result is compatible with the present data on hadronic decays of the J/Ψ . Moreover it implies a smooth matching between the short-distance and long-distance electromagnetic corrections to the $\pi^+ \rightarrow e^+\nu_e$ weak process.

1. The $\gamma^* \rightarrow \gamma^*\pi^0$ transition and the Bjorken-Johnson-Low theorem.

The Bjorken-Johnson-Low (BJL) theorem [9] can be used to derive the matrix element for the $\gamma^* \rightarrow \gamma^*\pi$ process when one photon is very energetic.

If (q_1, ϵ_1) and (q_2, ϵ_2) are the momentum and polarization of the incoming and outgoing photons respectively, the amplitude for $\gamma^* \rightarrow \gamma^* \pi$ is given by

$$M = M_{\mu\nu} \epsilon_1^\mu \epsilon_2^\nu$$

where

$$\begin{aligned} M_{\mu\nu} &= ie^2 \int d^4x e^{iq_2 x} \langle \pi^0 | T \{ J_\nu(x) J_\mu(0) \} | 0 \rangle \\ &= ie^2 \int d^4x e^{iq_1 x} \langle \pi^0 | T \{ J_\mu(-x) J_\nu(0) \} | 0 \rangle. \end{aligned} \quad (2)$$

In the limit $q_1^0 \rightarrow \infty$, the well-known BJL theorem [9] applied to electromagnetic currents implies

$$M_{\mu\nu}(q_1^0 \rightarrow \infty) = -\frac{e^2}{q_1^0} \int d\vec{x} e^{-i\vec{q}_1 \vec{x}} \langle \pi^0 | [J_\mu(0, -\vec{x}), J_\nu(0, 0)] | 0 \rangle. \quad (3)$$

If we restrict ourselves to the two-flavour quark model, the electromagnetic current J_μ reads

$$J_\mu = \bar{\Psi} \gamma_\mu Q \Psi \quad (4)$$

with

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix}, \quad Q = \begin{pmatrix} Q_u & 0 \\ 0 & Q_d \end{pmatrix}.$$

Using the canonical equal time anticommutation relations for the quark fields

$$\{ \bar{\Psi}_i^\alpha(0, -\vec{x}), \Psi_j^\beta(0, 0) \} = \gamma_0 \delta_{ij} \delta^{\alpha\beta} \delta(-\vec{x}) \quad (5)$$

with $i, j = 1 \dots N$ and $\alpha, \beta = u, d$, the colour and flavour indices respectively, we obtain the equal-time commutator

$$[J_\mu(0, -\vec{x}), J_\nu(0, 0)] = 2i \epsilon_{\mu\nu 0\sigma} \delta(-\vec{x}) J_5^\sigma(0) \quad (6)$$

in terms of the axial-vector current

$$J_5^\sigma = \bar{\Psi} \gamma^\sigma \gamma_5 Q^2 \Psi. \quad (7)$$

From the pion-to-vacuum matrix element

$$\langle 0 | \bar{u} \gamma^\sigma \gamma_5 d | \pi^-(p) \rangle \equiv i f_\pi p^\sigma \quad (8)$$

with $f_\pi \sim 132$ MeV, the $\pi \rightarrow \mu\nu$ decay constant, we infer that

$$M_{\mu\nu}(q_1^0 \rightarrow \infty) = e^2(Q_u^2 - Q_d^2) \frac{\sqrt{2}f_\pi}{q_1^0} \epsilon_{\mu\nu 0\sigma} (q_2 - q_1)^\sigma. \quad (9)$$

If we multiply and divide the right-hand side of Eq.(9) by q_1^0 and assume $|\vec{q}_1| \ll q_1^0$, we obtain

$$M_{\mu\nu}(q_1^2 \rightarrow \infty) = e^2(Q_u^2 - Q_d^2) \sqrt{2} \frac{f_\pi}{q_1^2} \epsilon_{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma. \quad (10)$$

The standard $\pi^0\gamma(q_1^2)\gamma(q_2^2)$ form factor $F(q_1^2, q_2^2)$ is defined as

$$M_{\mu\nu}(q_1^2, q_2^2) \equiv e^2 F(q_1^2, q_2^2) \epsilon_{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma. \quad (11)$$

With this normalization, the $\pi^0 \rightarrow \gamma\gamma$ triangle anomaly [2] implies

$$F(q_1^2 = 0, q_2^2 = 0) = -\frac{\sqrt{2}}{4\pi^2} \frac{1}{f_\pi}. \quad (12)$$

On the other hand, from Eq.(1) and Eq.(10) based on the BJL theorem we obtain the following asymptotic behaviour for the $\pi^0\gamma^*\gamma^*$ vertex:

$$F(q_1^2 \rightarrow \infty, q_2^2 \text{ finite}) = +\frac{\sqrt{2}}{N} \frac{f_\pi}{q_1^2}. \quad (13)$$

This result has been obtained for time-like photon momenta. One can easily show that it remains valid in the space-like region.

Since

$$f_\pi \sim \sqrt{N} \quad (14)$$

the dependence on the number of colours is the same in Eqs (12) and (13), as it should be in a quark model for the pion.

The remarkable feature of the main result displayed in Eq.(13) is its validity for any finite value of q_2^2 . Such a feature is not expected from QCD-inspired derivations.

2. A comparison with previous QCD estimates of $F(q_1^2, q_2^2)$.

The Operator Product Expansion approach (OPE) has been used [8] for the symmetric case of two highly energetic photons with equal square momenta ($q_1^2 = q_2^2 = q^2$). The resulting form-factor

$$F^{OPE}(q^2 \rightarrow \infty, q^2 \rightarrow \infty) = \frac{\sqrt{2}}{N} \frac{f_\pi}{q^2} \quad (15)$$

has very small QCD corrections and nicely extends the asymptotic behaviour given in Eq.(13) to arbitrary values of q_2^2 . This OPE approach applied to the case of one real photon ($q_2^2 = 0$) implies [10]

$$F^{OPE}(q_1^2 \rightarrow \infty, q_2^2 = 0) = \frac{2\sqrt{2}}{N} \frac{f_\pi}{q_1^2} + \text{large corrections} \quad (16)$$

with potentially large QCD corrections. For the same reason, the QCD-inspired calculation of Lepage and Brodsky (LB) [7]

$$F^{LB}(q_1^2 \rightarrow \infty, q_2^2 = 0) = \frac{3\sqrt{2}}{N} \frac{f_\pi}{q_1^2} \quad (17)$$

cannot be trusted [10].

Predictions [5] based on the LB form-factor given in Eq.(17) for processes like $Z^0 \rightarrow \pi^0 \gamma$ or $W^+ \rightarrow \pi^+ \gamma$ are therefore highly questionable. It is however unlikely to get any information on the form factor $F(q_1^2, 0)$ from these very rare processes in a near future. But present data on hadronic J/Ψ decays provide already a way to discriminate in favour of Eq.(13) with relatively small corrections.

3. The $J/\Psi \rightarrow \omega \pi^0$ decay and $F(q_1^2 \approx 9.6 \text{ GeV}^2, q_2^2 = 0)$.

Among the measured J/Ψ hadronic two-body decays into one vector and one pseudoscalar, the $J/\Psi \rightarrow \omega \pi^0$ is the only one to be clearly dominated by the single-photon exchange contribution ($Q_u \neq Q_d$). Indeed, the contributions induced by the isospin-violating $\eta - \pi^0$ mixing ($m_u \neq m_d$) in the $J/\Psi \rightarrow \omega \eta$ decay and $\rho - \omega$ mixing ($m_u \neq m_d$) in the $J/\Psi \rightarrow \rho \pi^0$ decay are negligible. Notice that the other isospin-violating J/Ψ decays into $\rho \eta$ and $\rho \eta'$ involve the strong anomaly. Moreover, they are contaminated by $\omega - \rho$ mixing ($m_u \neq m_d$) in the $J/\Psi \rightarrow \omega \eta, \omega \eta'$ decays since the ω width is small.

From the measured $J/\Psi \rightarrow \omega \pi^0$ decay width [11]

$$\begin{aligned} \Gamma(J/\Psi \rightarrow \omega \pi^0) &= (3.7 \pm 0.5) 10^{-8} \text{ GeV} \\ &\simeq \frac{1}{96\pi} m_\Psi^3 g_{\Psi\omega\pi^0}^2 \left[1 - \left(\frac{m_\omega}{m_\Psi}\right)^2\right]^3 \end{aligned} \quad (18)$$

we estimate the $\Psi\omega\pi^0$ vertex $g_{\Psi\omega\pi^0}$:

$$g_{\Psi\omega\pi^0} \simeq (6.8 \pm 0.5) 10^{-4} \text{ GeV}^{-1}. \quad (19)$$

The dominance of the single-photon exchange for this specific process implies then the simple relation

$$e^2 F(m_\Psi^2, 0) = \frac{f_\Psi}{f_\omega} g_{\Psi\omega\pi^0} \quad (20)$$

if the usual vector-dominance model is assumed for the $\omega - \gamma$ transition. The decay constant f_V of the vector-meson V is defined such that

$$\langle 0 | J_\mu | V \rangle = \epsilon_\mu \frac{m_V^2}{f_V}. \quad (21)$$

With this normalization, we obtain $f_\omega \simeq 17.1$ and $f_\Psi \simeq 11.5$ from the central values of the measured $\omega \rightarrow e^+e^-$ and $\Psi \rightarrow e^+e^-$ decay widths [11] given by the general expression

$$\Gamma(V \rightarrow e^+e^-) = \frac{\alpha^2}{3} \frac{4\pi}{f_V^2} m_V. \quad (22)$$

Keeping in mind the various approximations used, we get a reasonable estimate of the $\pi^0 \gamma^*(q_1^2 = m_\Psi^2) \gamma(q_2^2 = 0)$ from factor

$$e^2 F(m_\Psi^2, 0) = (4.6 \pm 0.4) \cdot 10^{-4} \text{ GeV}^{-1}. \quad (23)$$

On the other hand, assuming $q_1^2 = m_\Psi^2$ in the asymptotic expression given in Eq.(13) would imply

$$e^2 F(m_\Psi^2, 0) \rightarrow e^2 \frac{\sqrt{2}}{N} \frac{f_\pi}{m_\Psi^2} = 6 \cdot 10^{-4} \text{ GeV}^{-1} \quad (24)$$

for the standard value $N = 3$. This prediction is in good agreement with the one extracted from experiment in Eq.(23).

We conclude that the observed $J/\Psi \rightarrow \omega\pi^0$ hadronic decay rate alone suggests that the J/Ψ mass scale is already in the asymptotic domain where the BJL theorem applies. It also indicates the failure of the leading order QCD estimates given in Eqs (16) and (17) which predict $e^2 F(m_\Psi^2, 0) \simeq 12 \cdot 10^{-4} \text{ GeV}^{-1}$ and $18 \cdot 10^{-4} \text{ GeV}^{-1}$ respectively. A crucial test to settle the question would be the measurement of the $\Upsilon \rightarrow \omega\pi^0$ hadronic decay rate since the BJL theorem should be valid at this scale. Assuming Eq.(13), we estimate the $\Upsilon\omega\pi^0$ vertex to be

$$g_{\Upsilon\omega\pi^0} \simeq \frac{f_\omega}{f_\Upsilon} e^2 \frac{\sqrt{2}}{N} \frac{f_\pi}{m_\Upsilon^2}. \quad (25)$$

From the measured $\Upsilon \rightarrow e^+e^-$ decay width [11], we have $f_\Upsilon \approx 40$ (see Eq.(22)). For $N = 3$, the predicted $\Upsilon \rightarrow \omega\pi^0$ branching ratio is about

$$Br(\Upsilon \rightarrow \omega\pi^0) \approx 3.8 \cdot 10^{-5}. \quad (26)$$

The LB form factor in Eq.(17) would imply a branching ratio larger by a factor of nine ! Unfortunately, data on this process are not available yet [11].

We have seen that a direct test of the BJL theorem applied to the $\gamma^* \rightarrow \gamma^*\pi^0$ transition is still lacking. Now we present a theoretical argument in favour of the general asymptotic behaviour in Eq.(13).

4. Matching in the radiative corrections to $\pi^+ \rightarrow e^+\nu_e$.

The one-loop electromagnetic corrections to the zeroth-order weak $\pi^+ \rightarrow e^+\nu_e$ decay amplitude $\mathcal{M}_0$ are well-known [12]. As long as the q_2^2 square momentum of the charged W gauge boson is negligible with respect to M_W^2 , these corrections denoted $\delta\mathcal{M}_0$ can be classified according to the q_1^2 square momentum carried by the virtual photon.

For $q_1^2 > \Lambda^2$, one uses a quark model for the charged pion and considers the short-distance (SD) corrections to the generic $u\bar{d} \rightarrow e^+\nu_e$ process. Using Eq.(1), the leading dependence on the infrared euclidean cut-off Λ reads [13]

$$\delta\mathcal{M}_0^{SD} = \left[\frac{3\alpha}{8\pi}\left(1 + \frac{1}{N}\right) \ln \frac{M_W^2}{\Lambda^2}\right] \mathcal{M}_0. \quad (27)$$

For $q_1^2 < \Lambda^2$, one treats the pion as an elementary pseudoscalar and considers the long-distance (LD) corrections to $\pi^+ \rightarrow e^+\nu_e$. The leading dependence on the ultraviolet euclidean cut-off Λ reads [14]

$$\delta\mathcal{M}_0^{LD} = \left[\frac{3\alpha}{8\pi} \ln \frac{\Lambda^2}{m_\pi^2}\right] \mathcal{M}_0. \quad (28)$$

The cut-off Λ introduced by hand to separate short-distance from long-distance radiative corrections around the GeV scale is obviously unphysical. Consequently, the total $\delta\mathcal{M}_0$ corrections to the $\pi^+ \rightarrow e^+\nu_e$ weak amplitude should be Λ -independent. This smooth matching between the two complementary pictures for the pion has been successfully implemented in the case of the $\pi^+ - \pi^0$ mass difference [15].

Here, we might prematurely conclude from Eqs (27) and (28) that this matching only occurs in the large- N limit, when $Q_u = -Q_d = 1/2$. In this limit indeed, the $1/N$ term in Eq.(27) arising from the *vector* component of the weak hadronic current coupled to W^- disappears and the matching between the axial component contributions occurs.

However, we have seen that, for arbitrary N , the structure of the pion coupled to two vector currents implies a $1/N$ -suppressed form-factor whose asymptotic behaviour is given in Eq.(13). This also applies for the isospin related $\pi^+\gamma^*(q_1^2)W^-(q_2^2)$ vertex appearing in the long-distance radiative corrections. Integrating over large photon momenta, we obtain:

$$\delta\mathcal{M}_0^{structure} = \left[\frac{3\alpha}{8\pi} \int^{\Lambda^2} dq_1^2 \quad F(q_1^2, q_2^2 < M_W^2)\right] \frac{\mathcal{M}_0}{\sqrt{2}f_\pi}. \quad (29)$$

We emphasize that the general asymptotic behaviour advocated in Eq.(13) is valid for finite q_2^2 and ensures therefore a complete cancellation of the leading cut-off dependence in $\delta\mathcal{M}_0 = \delta\mathcal{M}_0^{SD} + \delta\mathcal{M}_0^{LD} + \delta\mathcal{M}_0^{structure}$, for any value of N .

As expected from PCAC applied on π^0 , we reach the same conclusion in the case of the electromagnetic corrections to the $\pi^+ \rightarrow \pi^0 e^+ \nu_e$ process. For this pion β -decay, CVC automatically ensures the matching between the short-distance and long-distance contributions induced by the vector component of the weak hadronic current. The short-distance logarithmic dependence on the cut-off Λ , arising now from the *axial* component of the weak hadronic current, is then cancelled by the $\pi^+\pi^0\gamma^*W^-$ vertex. The asymptotic behaviour of this vertex is again simply deduced from the BJL theorem and was already known about thirty years ago [16] !

In conclusion, we have shown how the Bjorken-Johnson-Low theorem provides us with a unique asymptotic behaviour for the $\pi^0\gamma^*(q_1^2)\gamma^*(q_2^2)$ vertex when $|q_1^2|$ tends to infinity. This surprising result was not expected from previous QCD-inspired approaches. We then argued that present data on J/Ψ hadronic decays support this result and emphasized the relevance of measuring the $\Upsilon \rightarrow \omega\pi^0$ branching ratio. Finally, we explained how the matching between short-distance and long-distance electromagnetic corrections to $\pi^+ \rightarrow e^+ \nu_e$ is naturally implemented in this framework.

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References

- [1] C. Bouchiat, J. Iliopoulos and Ph. Meyer, Phys. Lett. **38B** (1972) 519;
D. Gross and R. Jackiw, Phys. Rev. **D6** (1972) 477.
- [2] S. Adler, Phys. Rev. **177** (1969) 2426;
J.S. Bell and R. Jackiw, Nuovo Cimento **60** (1969) 37.
- [3] J. Steinberger, Phys. Rev. **76** (1949) 1180.
- [4] See e.g. W.A. Bardeen, H. Fritzsch and M. Gell-Mann, in “Scale and Conformal Symmetry in Hadron Physics” (ed. by R. Gatto), New York, 1973.
- [5] L. Arnellos, W.J. Marciano and Z. Parsa, Nucl. Phys. **B196** (1982) 378.
- [6] S.D. Drell, Nuovo Cimento **11** (1959) 693.
- [7] G.P. Lepage and S.J. Brodsky, Phys. Lett. **87B** (1979) 359; Phys. Rev. **D22** (1980) 2157.
- [8] V.A. Novikov, M.A. Shifman, A.I. Vainshtein, M.B. Voloshin and V.I. Zakharov, Nucl. Phys. **B237** (1984) 525.
- [9] J.D. Bjorken, Phys. Rev. **148** (1966) 1467;
K. Johnson and F.E. Low, Suppl. Progr. Theor. Phys. **37, 38** (1966) 74;
See also S.B. Treiman, R. Jackiw and D.J. Gross, in Lectures on Current Algebra and its Applications (Princeton, New Jersey 1972).
- [10] A.V. Manohar, Phys. Lett. **244B** (1990) 101.
- [11] Particle Data Group: Review of Particle Properties, Phys. Rev. **D50** (1994).
- [12] W.J. Marciano and A. Sirlin, Phys. Rev. Lett. **71** (1993) 3629;
R. Decker and M. Finkemeier, Nucl. Phys. **B438** (1995) 17.

- [13] A. Sirlin, Nucl. Phys. **B196** (1982) 83.
- [14] T. Kinoshita, Phys. Rev. Lett. **2** (1959) 477.
- [15] W.A. Bardeen, J. Bijnens and J.-M. Gérard, Phys. Rev. Lett. **62** (1989) 1343.
- [16] J.D. Bjorken, Fourth Coral Gables Conf. on Symmetry Principles at High Energies (1967), (ed. by A. Perlmutter and B. Kursunuglu);
N. Cabibbo, L. Maiani and G. Preparata, Phys. Lett. **25B** (1967) 132.