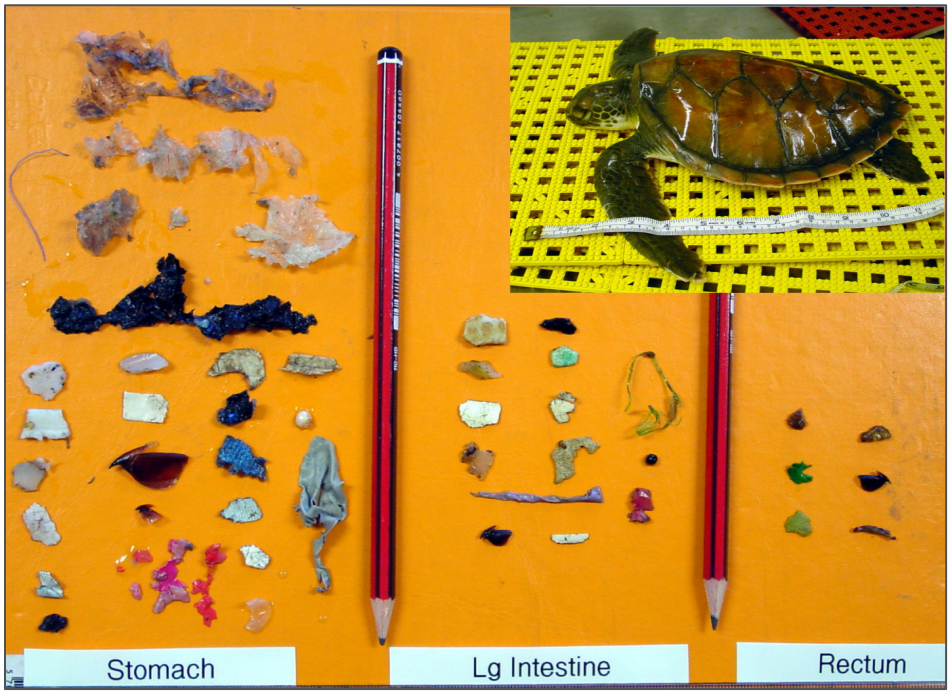
Solution of the problems pertaining to plastic debris pollution for the course

Coastal and maritime hydraulics (LGCIV2056)

4. Reactive transport II

12. Nowadays, about eight million tons of mismanaged plastic waste enter the oceans[1,2]. The size of plastic debris found in the marine environment vary over a very wide range of orders of magnitude, from the size of a soda bottle to much less than one micron. The transport and transformations of such man-made debris are still poorly known.

There is no doubt, however, that their impact on marine life and, presumably, human health is a detrimental one. For instance, a significant percentage of the mass of marine animals can consist of plastic debris, as is illustrated in the figure below, which is due to Kathy Townsend and was provided to me by Kay Critchell (personal communication, June 2017).



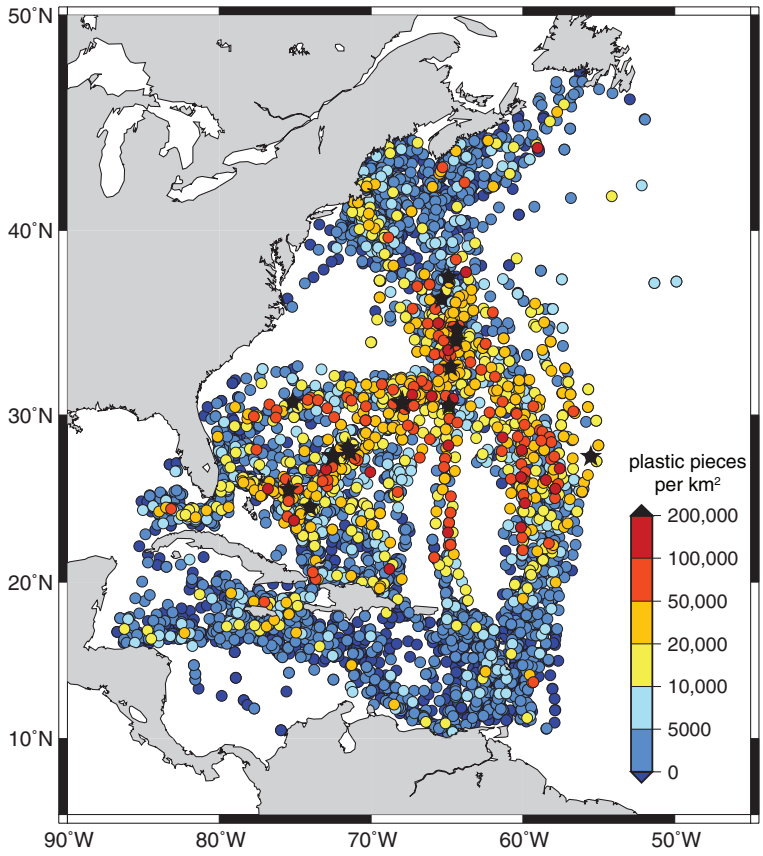
As marine plastic debris mostly originate from land and rivers, their maximum concentrations should be observed in the immediate vicinity of the coasts if their concentration obey the well-known properties of the solutions of advection-diffusion equations.

Small plastic debris typically millimetres in size located at the ocean surface or very near to it can be collected by plankton net tows. The figure below (which is Figure 1 of Law et al.

[1] Jambeck J.R., R. Geyer, W. Wilcox, T.R. Siegler, M. Perryman, A.Andrady, R. Narayan and L.L. Law, 2015, Plastic waste inputs from land into the ocean, *Science*, 347, 768-771

[2] Countless initiatives are aimed at addressing this regrettable situation, which has gone on for too long. See, for instance, *Tangaroa Blue* (www.tangaroablue.org) or *The Ocean Cleanup* (www.theoceancleanup.com).

2010[3]) shows that such debris tend to accumulate in the North Atlantic Subtropical Gyre, i.e. hundreds of kilometres away from the nearest shores.



How can one reconcile these field data with the results of the theory of classical advective and diffusive transport?

Solution

Let t and $\mathbf{x}_h = (x,y)$ represent the time and the horizontal part of the position-vector, where x and y are horizontal coordinates. The concentration $n(t,\mathbf{x}_h)$ of plastic debris located at or very near to the ocean surface (number of debris per unit area) is likely to obey the following advection-diffusion equation

$$\frac{\partial n}{\partial t} = -\lambda n - \nabla_h \bullet (n\mathbf{u} - \kappa_h \nabla_h n) \quad , \tag{S.12.1}$$

where ∇_h is the horizontal part of the del operator; $\mathbf{u}$ is the sum of the surface velocity of the water and an appropriate velocity representing the impact of the wind (e.g. Critchell et al. 2015[4]); κ_h is the relevant horizontal (turbulent) diffusivity. In the equation above, decay and sinking processes are taken into account by means of a first-order law, including non-negative

[3] Law K.L., S. Morét-Ferguson, N.A. Maximenko, G. Proskurowski, E.E. Peacock, J. Hafner and C.M. Reddy, 2010, Plastic accumulation in the North Atlantic Subtropical Gyre, *Science*, 329, 1185-1188

[4] Critchell K., A. Grech, J. Schlaefer, F.P. Andutta, J. Lambrechts, E. Wolanski and M. Hamann, 2015, Modelling the fate of marine debris along a complex shoreline: Lessons from the Great Barrier Reef, *Estuarine, Coastal and Shelf Science*, 167, 414-426

parameter λ, which is unlikely to be a constant. Clearly, (S.12.1) may be rewritten as follows

$$\frac{\partial n}{\partial t} = -\mu n - (\mathbf{u} - \nabla_h \kappa_h) \bullet \nabla_h n + \kappa_h \nabla_h^2 n \quad , \tag{S.12.2}$$

with

$$\mu = \lambda + \nabla_h \bullet \mathbf{u} \quad . \tag{S.12.3}$$

Assume that concentration maximum is located at a point whose position-vector is $\mathbf{r}(t)$. Then, the value of the debris concentration at this point obeys

$$\frac{d}{dt} n[t, \mathbf{r}(t)] = \left[\frac{\partial n}{\partial t} + \frac{d\mathbf{r}}{dt} \bullet \nabla_h n \right]_{\mathbf{x}=\mathbf{r}(t)} \quad . \tag{S.12.4}$$

Then, combining (S.12.2) and (S.12.4) yields

$$\frac{d}{dt} n[t, \mathbf{r}(t)] = \left[-\mu n - \left(\mathbf{u} - \nabla_h \kappa_h - \frac{d\mathbf{r}}{dt} \right) \bullet \nabla_h n + \kappa_h \nabla_h^2 n \right]_{\mathbf{x}=\mathbf{r}(t)} \quad . \tag{S.12.5}$$

At a local maximum, $\nabla_h n = 0$, allowing one to simplify (S.12.5) as follows

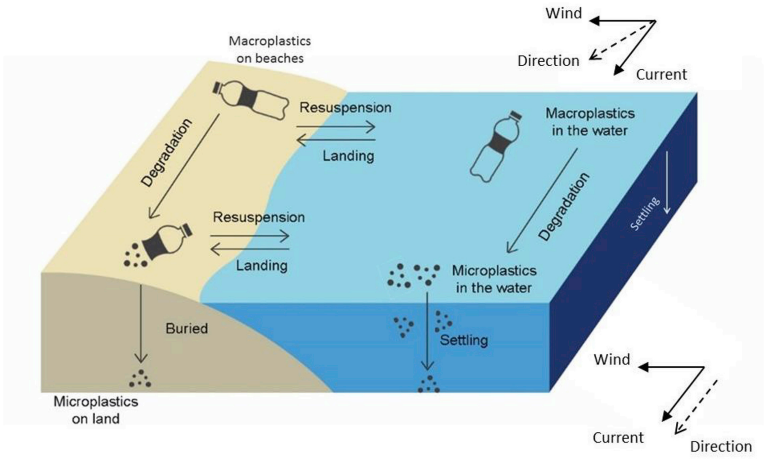
$$\frac{d}{dt} n[t, \mathbf{r}(t)] = \left[-\mu n + \kappa_h \nabla_h^2 n \right]_{\mathbf{x}=\mathbf{r}(t)} \tag{S.12.6}$$

In addition, the Laplacian $\nabla_h^2 n$ is non positive. However, if $\nabla_h \bullet \mathbf{u}$ is negative (convergent velocity) and its order of magnitude is sufficiently large, than it is possible that $dn[t, \mathbf{r}(t)] / dt$ be positive, implying that the concentration would increase. If, on the other hand, parameter μ is positive (which is the case if $\nabla_h \bullet \mathbf{u} > 0$), then the maximum value of the concentration will decrease. In the long run, the maximum value of the concentration would not occur in the domain of interest. Instead, it would be located on the boundary of the domain, i.e. at the mouth of one of the estuaries injecting plastic debris into the ocean.

The key difference between the equation governing the concentration of the plastic debris and that obeyed by a dissolved constituent (exhibiting a first order decay process) is that in the latter the velocity field is divergence-free, implying that the maximum value of the concentration cannot increase. By contrast, for plastic debris, equivalent velocity $\mathbf{u}$ is unlikely to be divergence-free, which may cause the maximum value of the concentration to be located inside the domain of interest and increase (provided $-\nabla \bullet \mathbf{u}$ is positive and sufficiently large).

7. Shallow water dynamics

12. Critchell and Lambrechts (2016[5]) developed a depth-integrated model of the transport and fate of plastic debris in the coastal zone, taking into account the processes schematically illustrated in the figure opposite (which is Figure 1 of the abovementioned article).



In the present problem, a simplified version of this model is derived and applied to a highly idealised estuary. The latter is represented as a rectilinear channel, whose width and length are constants W and L, respectively. The longitudinal coordinate reads x, with $x=0$ at the head of the estuary and $x=L$ at its mouth, i.e. the estuary-sea interface. The surface velocity of the water is constant U_w (>0). Along-channel turbulent diffusion is considered negligible.

Macrosplastic debris are floating at the water surface or very near to it. Their width-averaged concentration, $\overline{n}(t,x)$, is defined as the number of debris per unit area. It is a function of only the time t and along-channel coordinate x.

Let positive constant U_g represent the net grounding velocity[6] of the debris under study. From here on, only the width-averaged concentration will be used. Therefore, for notational simplicity, the overbar will be omitted. Demonstrate that the macroplastic debris concentration obeys the following partial differential equation

$$\frac{\partial n}{\partial t} = -U_w \frac{\partial n}{\partial x} - \frac{2U_g n}{W} \quad . \tag{12.1}$$

Explain why on each shore of the estuary the lineic concentration of the debris, $l(t,x)$, i.e. the number of debris per unit length (see, for instance, photograph below), is governed by equation

$$\frac{\partial l}{\partial t} = U_g n \quad . \tag{12.2}$$

A large number of debris (N) are abruptly discharged at time $t=0$ at the head of the estuary (x=0). It is assumed that at the initial time there are debris neither at the surface of the estuary nor on its shores. Demonstrate that the number of particles ending up in the sea is equal to

[5] Critchell K. and J. Lambrechts, 2016, Modelling accumulation of marine plastics in the coastal zone; what are the dominant physical processes?, *Estuarine, Coastal and Shelf Science*, 171, 111-122

[6] Let y represent the cross-estuary coordinate, which is defined in such a way that the shores are located at $y=\pm W/2$. Cross-channel (turbulent) diffusion is assumed to be rather efficient. The grounding fluxes are purely diffusive and are expressed by means of the relevant grounding velocity: $\left[-\kappa_h \partial n/\partial y\right]_{y=\pm W/2} = \pm U_g n$.

$$N_s \;=\; \exp\left(-\frac{2LU_g}{WU_w}\right) N \tag{12.3}$$

and that the lineic debris concentration on the shores is

$$l(t,x) \;=\; \begin{cases} 0\,, & x > U_w t \\ \exp\left(-\dfrac{2U_g}{WU_w}\,x\right)\dfrac{U_g N}{WU_w}\,, & U_w t > x \end{cases} \tag{12.4}$$

Elaborate on the physical significance of the dimensionless parameter $\mu = 2LU_g/(WU_w)$. Explain why the number of debris lying on the shores at time $t > LU_w$ of the estuary obeys the relation

$$2\int_0^L l(t,x)\,dx \;=\; (1-e^{-\mu})N \quad . \tag{12.5}$$



Source: http://www.earthlyhappenings.com/2012/06/
last viewed on 14 August 2017

Solution

The geometry of the domain is illustrated in Figure 12.1.

The (width-averaged) plastic debris concentration, $n(t,x)$, is defined as the number of debris per unit area. Hence, the physical dimension of this variable is $(\text{length})^{-2}$. On the other hand, the lineic concentration of debris on the shore, $l(t,x)$, is the number of debris per unit length, implying that its physical dimension is $(\text{length})^{-1}$.

Consider elemental estuarine segment $[x-\Delta x/2, x+\Delta x/2]$. In the latter, at time t, the number of plastic debris tends to $n(t,x)W\,\Delta x$ as $\Delta x \to 0$. During the time interval $[t, t+\Delta t]$ ($\Delta t \to 0$), the number of debris varies under the impact of (along-estuary) advective and (cross-estuary) grounding fluxes (Figure 12.2):

$$n(t+\Delta t,x)W\,\Delta x \;\sim\; n(t,x)W\,\Delta x - \underbrace{\left[n(t,x+\Delta x/2)W\,U_w\,\Delta t - n(t,x-\Delta x/2)W\,U_w\,\Delta t\right]}_{\text{budget of the along-estuary advective fluxes}} - \underbrace{2\,n(t,x)U_g\,\Delta t\,\Delta x}_{\text{net grounding fluxes}} \tag{S.12.1}$$

This asymptotic expression is readily transformed to

$$\frac{n(t+\Delta t,x)-n(t,x)}{\Delta t} \sim U_w \frac{n(t,x+\Delta x/2)-n(t,x-\Delta x/2)}{\Delta x} - \frac{2U_g\, n(t,x)}{W} \qquad \text{(S.12.2)}$$

and, in the limit $\Delta t, \Delta x \to 0$, one obtains

$$\frac{\partial n}{\partial t} = -U_w \frac{\partial n}{\partial x} - \frac{2U_g n}{W} \qquad \text{(S.12.3)}$$

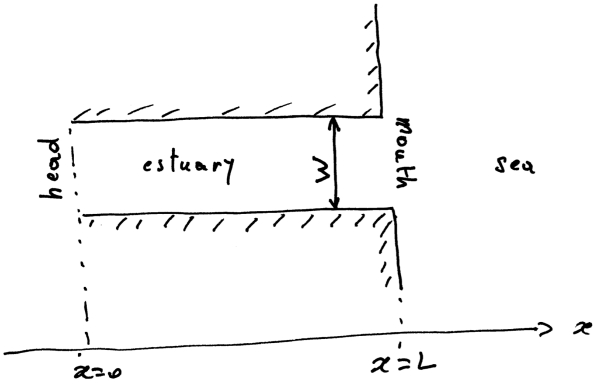


Figure 12.1. Geometry of the domain of interest.

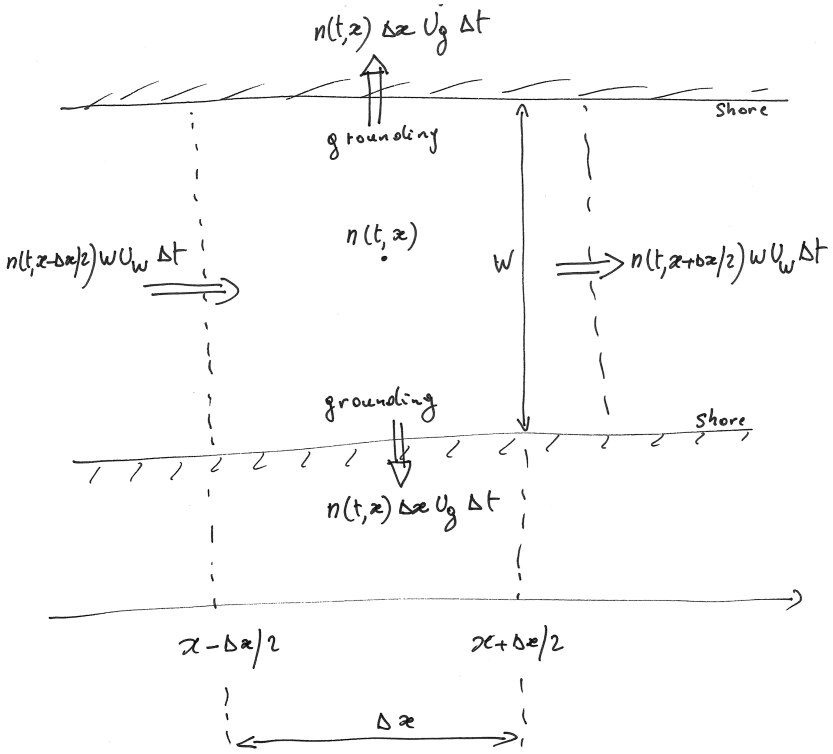


Figure 12.2. Illustration of debris transport into and out of elemental estuary segment $[x-\Delta x/2, x+\Delta x/2]$ during the time interval $[t, t+\Delta t]$, with $\Delta t, \Delta x \to 0$.

Now consider elemental coastline segment $[x-\Delta x/2, x+\Delta x/2]$. On the latter, at time t, the number of plastic debris tends to $l(t,x)\Delta x$ as $\Delta x \to 0$. During the time interval $[t, t+\Delta t]$ ($\Delta t \to 0$), the number of debris increases under the impact of the net grounding flux (Figure 12.2):

$$l(t+\Delta t,x)\Delta x \;\sim\; l(t,x)\Delta x \;+\; \underbrace{n(t,x)U_g\,\Delta t\,\Delta x}_{\text{net grounding flux}} \tag{S.12.4}$$

This asymptotic expression is readily transformed to

$$\frac{l(t+\Delta t,x)-l(t,x)}{\Delta t} \;\sim\; U_g\, n(t,x) \tag{S.12.5}$$

and, in the limit $\Delta t \to 0$, one obtains

$$\boxed{\frac{\partial l}{\partial t} = U_g\, n} \tag{S.12.6}$$

At the initial instant (t=0), there are debris neither in the estuary nor on its shores, leading to initial conditions

$$n(0,x)=0\,,\quad l(0,x)=0\,, \tag{S.12.7}$$

Then N debris are suddenly injected at time $t=0$ at the upstream boundary of the domain ($x=0$). Since there is no along-estuary diffusion, the associated incoming flux at x=0 is purely advective and it reads

$$WU_w\, n(t,0) \;=\; N\,\delta(t-0)\;\;, \tag{S.12.8}$$

where δ denotes the Dirac delta function. Since governing equation (S.12.3) may be transformed to $D_t n = -(2U_g/W)n$ (where $D_t = \partial/\partial t + U_w\,\partial/\partial x$ denotes the material derivative), the debris concentration will be a pulse moving at velocity U_w. Accordingly, the concentration will be zero everywhere except at $x = U_w t$, where the debris concentration will be "infinite"[7]. Such a solution is uneasy to establish rigorously by means of classical calculus. This is why it is desirable to have recourse to an appropriate integral transform, which will render the calculations necessary to obtain the solution rather straightforward.

The temporal Laplace transform of the variables under consideration are

$$\left[\tilde{n}(s,x),\tilde{l}(s,x)\right] \;=\; \int_0^\infty e^{-st}\left[n(t,x),l(t,x)\right]dt\;\;. \tag{S.12.9}$$

Taking the Laplace transform of (S.12.3), (S.12.6) and (S.12.8), and using initial conditions (S.12.7) leads to

$$s\tilde{n} \;=\; -U_w\frac{\partial \tilde{n}}{\partial x} \;-\; \frac{2U_g\tilde{n}}{W}\;\;, \tag{S.12.10}$$

$$s\tilde{l} \;=\; U_g\,\tilde{n}\;\;, \tag{S.12.11}$$

$$\tilde{n}(s,0) \;=\; \frac{N}{WU_w}\;\;. \tag{S.12.12}$$

[7] Such a debris distribution is unrealistic, since, in reality, diffusion will cause the "width" of the debris patch to be non zero — and increase as time progresses. However, in the present model, along-flow diffusion is disregarded, which is why the "width" of the debris patch must be zero.

The solution to differential problem (S.12.10)-(S.12.12) is readily derived:

$$\tilde{n}(s,x) = e^{-\mu x} e^{-sx/U_w} \frac{N}{WU_w} \quad , \tag{S.12.13}$$

$$\tilde{l}(s,x) = e^{-\mu x} \frac{e^{-sx/U_w}}{s} \frac{U_g N}{W U_w} \quad . \tag{S.12.14}$$

with $\mu = 2U_g / (WU_w)$.

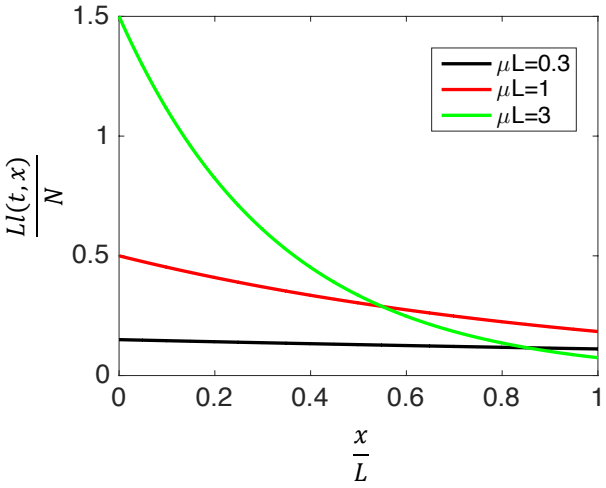


Figure 12.3. Graphical representation for various values of dimensionless parameter μL of the normalised lineic debris concentration on one of the shores of the estuary, $Ll(t,x)/N$, as a function of the dimensionless distance to the head of the estuary x/L. The curves correspond to the steady-state situation prevailing for $t > L/U_w$, i.e. after the debris pulse has left the estuary.

The originals of the above Laplace transforms are (Figure 12.3)

$$n(t,x) = e^{-\mu x} \delta(t - x/U_w) \frac{N}{WU_w} \quad , \tag{S.12.15}$$

$$l(t,x) = \begin{cases} 0 \, , & t < x/U_w \\ \dfrac{\mu N}{2} e^{-\mu x} \, , & x/U_w < t \end{cases} \tag{S.12.16}$$

The number of debris reaching the sea, N_s, is the time integral of the advective flux of debris taken at the downstream boundary (x=L), i.e. (Figure 12.4)

$$N_s = \int_0^\infty W U_w \, n(t,L) \, dt = e^{-\mu L} N \tag{S.12.17}$$

The floating debris enter the sea at $t = L/U_w$. After this time, i.e. for $t > L/U_w$, the distribution of plastic debris on the shores of the estuary will remain unchanged. The number

of debris lying on the shores may be obtained from the lineic concentration,

$$2\int_0^L l(t,x)\,dx \;=\; (1-e^{-\mu})N \quad , \tag{S.12.18}$$

or, equivalently, from the overall budget of the debris in the domain of interest. Accordingly, the number of debris lying on the shores is the difference between the number of debris injected at the initial time and the number of debris that entered the sea, i.e. (Figure 12.4)

$$N \;-\; N_s \;=\; N \;-\; e^{-\mu L}N \;=\; (1-e^{-\mu L})N \quad . \tag{S.12.19}$$

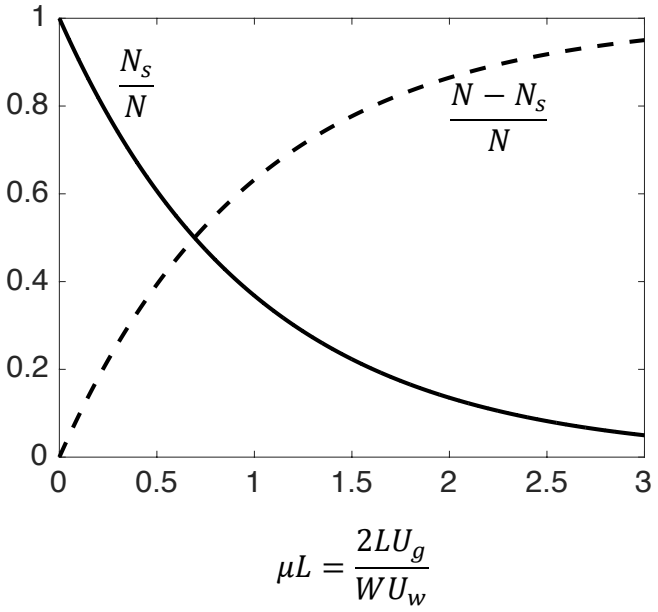


Figure 12.4. Graphical representation of the fraction of the debris reaching the sea, N_s / N, and the fraction of the debris grounded on the shores of the estuary, $(N - N_s)/N$, as a function of dimensionless parameter μL.

The analytical expressions derived above point to the importance of parameter μ, whose physical dimension is $(\text{length})^{-1}$. The larger this parameter, the smaller the fraction of debris that will eventually reach the sea. This is because μ is an increasing function of the grounding velocity and the length of the estuary. In addition, increasing the water velocity or the width of the estuary (thereby decreasing the debris concentration and, hence, the grounding flux) causes μ to decrease. All this in agreement with elementary physical intuition.

One might object that dimensionless parameter μL should be more relevant, for it is this parameter that determines the fraction of debris exported to the sea, N_s / N, and the fraction lying on the shores of the estuary, $(N - N_s)/N$. This argument is not without merit, which is why it may be appropriate to re-write (S.12.16) using this dimensionless parameter. This leads to (Figure 12.3)

$$\frac{L\,l(t,x)}{N} = \begin{cases} 0\,, & t < x/U_w \\ \dfrac{\mu L}{2} e^{-\mu L (x/L)}\,, & x/U_w < t \end{cases} \tag{S.12.20}$$

A physical interpretation of dimensionless parameter μL may be arrived at by having recourse to relevant timescales. The time needed to travel at velocity U_w from the head of the estuary to its mouth is $T_a = L/U_w$, which may be regarded as the timescale characterising longitudinal advection in the domain of interest. On the other hand, the mean time needed for plastic debris to reach one of the shores of the estuary, i.e. the grounding timescale, is readily seen to be $T_g = W/(2U_g)$. Then, parameter μL may be transformed as follows:

$$\mu L = \frac{2LU_g}{WU_w} = \frac{L/U_w}{W/(2U_g)} = \frac{T_a}{T_g} = \frac{\text{advection timescale}}{\text{grounding timescale}} \tag{S.12.21}$$

If $T_a >> T_g$, then grounding proceeds at a much faster pace than along-estuary advection, implying that only a small fraction of the plastic debris will reach the sea ($N_s/N << 1$). By contrast, if $T_g >> T_a$, there is not enough time for grounding process to significantly impact the fate of the plastic debris. In this case, most of the debris will reach the sea so that ratio N_s/N will be close to unity.

13. Revisit the previous problem. Assume that the number of debris entering the estuary per unit time at $x = 0$ is $\Phi(t)$. Show that the flux of debris exported to the sea is

$$\Phi_s(t) = e^{-\mu L}\,\Phi(t - L/U_w)\ . \tag{13.1}$$

For the sake of simplicity, further assume that the incoming debris flux is constant. Then, demonstrate that the amount of debris that accumulated on the shores of the estuary during the time interval $[t_1, t_2]$ is $(1 - e^{-\mu L})(t_2 - t_1)\Phi$.

Solution

The debris that entered the estuary (through its upstream boundary) at time $t - L/U_w$ leave it at time t or are deposited on its shores during time interval $[t - L/U_w, t]$. The fraction of those debris that eventually reach the sea is $e^{-\mu L}$. Therefore, if $\Phi(t - L/U_w)$ denotes the number of debris crossing the head of the estuary per unit time, then the associated flux of debris entering the sea is $\Phi_s(t) = e^{-\mu L}\,\Phi(t - L/U_w)$.

If the incoming debris flux is constant, then the outgoing debris flux reads $e^{-\mu L}\Phi$, implying that the grounding flux is $(1 - e^{-\mu L})\Phi$. Thus, the number of debris grounded during time interval $[t_1, t_2]$ obviously is $(1 - e^{-\mu L})(t_2 - t_1)\Phi$.

QED.