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A subdiffusive stochastic volatility jump model

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Abstract

Subdiffusions appear as good candidates for modeling illiquidity in financial markets. Existing subdiffusive models of asset prices are indeed able to capture the motionless periods in the quotes of thinly-traded assets. However, they fail at reproducing the jumps and the time-varying volatility observed in the price of these assets. The aim of this work is hence to propose a new model of subdiffusive asset prices reproducing the main characteristics exhibited in illiquid markets. This is done by considering a stochastic volatility jump model, time-changed by an inverse subordinator. We derive the forward fractional partial differential equations (PDE) governing the probability density function of the introduced model and we prove that it leads to an arbitrage-free and incomplete market. By proposing a new procedure for estimating the model parameters and using a series expansion for solving numerically the obtained fractional PDE, we are able to price various financial derivatives on illiquid assets and to propose a corresponding hedging strategy. This way, we show that the introduced subdiffusive stochastic volatility jump model yields consistent and more reliable results in illiquid markets.

Keywords: illiquidity modeling, subdiffusion, fractional Fokker-Planck equations, stochastic volatility jump model, option pricing

1. Introduction

Small capitalization stocks tend to exhibit motionless periods in their share price which correspond to a lack of liquidity associated with these assets. This occurs when a stock stops being exchanged by the investors during several days, week or months. This feature is also common in emerging markets in which the number of participants, and thus the number of transactions, is rather low. Such periods of constant values can be observed on the upper plot of Figure 1.1, which depicts the price of Quadpack Industries (ALQP), a small capitalization listed in the Eurolist Group C. As Brownian motions and Lévy processes are perpetually moving, they are not adapted for modeling these periods of illiquidity in financial markets. A similar behavior is however observed in physical systems exhibiting subdiffusion, a well known and established phenomenon in statistical physics. The usual model of subdiffusion is developed in terms of fractional Fokker–Planck equations (FFPE) which were first derived in Metzler and Klafter (2000) from continuous-time random

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walks with heavy-tailed waiting times/trapping events. These FFPE hence provide an appropriate procedure for describing transport dynamics with trapping periods in complex systems, such as heavy particles' movements in Metzler and Klafter (2004). Subdiffusive dynamics can also be obtained from subordination, where a standard diffusion process is time-changed by the so-called inverse subordinator, as in Magdziarz (2009). Therefore, the observed motionless periods without trades in financial markets can be considered as the equivalent in physical systems of the trapping events during which the subdiffusive particles get immobilized. This explains why FFPE are heavily used in econophysics (see Bouchaud *et al.* (2003), Barrieu and El Karoui (2002) or Scalas (2006) for a survey) and this justifies their use to describe the dynamic of asset prices in illiquid markets.

From this similitude between particles transport and illiquid asset dynamic, Magdziarz (2009) first introduced a subdiffusive geometric Brownian motion as a model of asset prices in thinly-traded markets. He showed that the considered model is arbitrage-free but incomplete, and obtained the corresponding subdiffusive Black & Scholes (B&S) formula for the fair prices of European options. Gu *et al.* (2012), Guo and Yuan (2014), Wang *et al.* (2012) and Shokrollahi *et al.* (2016) then derived an extension of this subdiffusive B&S model by proposing asset dynamics based on subdiffusive fractional Brownian motions (fBm). They managed to build a pricing equation for European calls under various assumptions. However, since fBm are neither Markov processes nor semi-martingales, it has been shown in Cheridito (2003) that fBm models admit arbitrage in frictionless markets. This hence forces these authors to consider modified dynamics inspired from fBm (cfr Shokrollahi *et al.* (2016) or Guo and Yuan (2014) with mixed fBm) or to impose proportional transaction costs in a discrete setting, as in Gu *et al.* (2012). Moreover, neither of these subdiffusive models can reproduce the main characteristics observed for low-liquidity stocks in financial markets. First, these thinly-traded stocks systematically exhibit jumps in the sample path of their price process. This can be interpreted by the the arrival of news in the market which suddenly change the anticipations of investors and make them trade the stock again. These jumps in price can clearly be seen on Figure 1.1 for the share of ALQP, which exhibits similar sample paths as the majority of small capitalizations (see *e.g.* stocks listed in the Eurolist Group C). Secondly, the volatility is assumed constant in most of the models dealing with illiquidity, which is not the case in practice as suggested by the second plot of Figure 1.1, representing the annualized historical volatility of ALQP based on a monthly time window. Considering a stochastic volatility model with jumps in the price process will hence enable to take these two aspects into account while also reproducing the well-known volatility clustering effect and the observed heavy tails of the return distribution, as described in Gatheral (2011). Finally, Stanislavsky (2003) shows in the context of the classical B&S model that subordinating Brownian motions by an independent, continuous and increasing process (*i.e.* a subordinator) allows to introduce the empirically observed long-memory effect in stock returns, as investigated in Ding *et al.* (1993). The chosen subordinator in Stanislavsky (2003) is an α-stable process so that the memory effects are characterized by only one parameter (the α parameter). We therefore consider in this paper a mean-reverting stochastic volatility model with price jumps, time-changed by an inverse subordinator. We will show that this so-called *subdiffusive stochastic volatility jump* (SVJ) model is indeed arbitrage-free, that it generates more realistic sample paths incorporating periods of constant values and that it allows to better reproduce the true dynamic of asset prices in illiquid markets in comparison with the existing models developed in the literature.

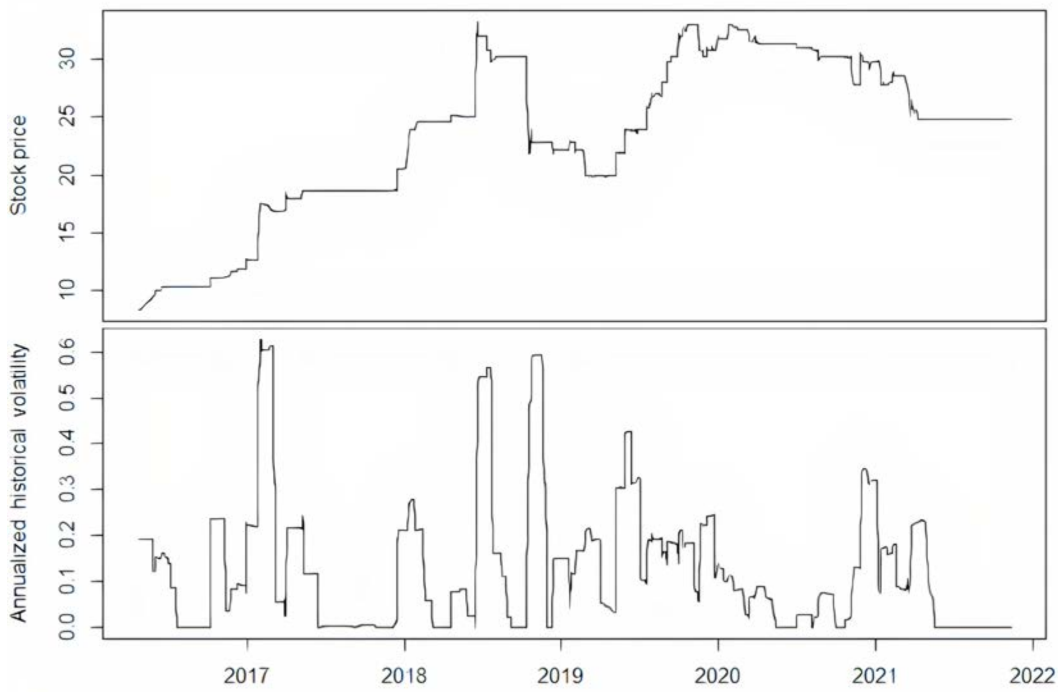


Figure 1.1: The upper plot depicts the stock price of ALQP over a 5-year period. The bottom plot represents the annualized historical volatility of ALQP computed using a monthly time window.

The contributions of this paper are multiple. We first show how to price financial derivatives under the subdiffusive SVJ model by deriving new FFPE that allow to describe the behavior of the introduced model. We generalize these forward fractional equations for a whole class of subordinators (with no drift) and we show that they are also valid if seen from a later time $s > 0$ after inception, conditionally to some information available at that time s. We next define a change of measure and from it, we show that we can obtain a risk-neutral measure Q four our model, leading to an arbitrage-free and incomplete market. Based on this risk-neutral measure and on the FFPE satisfied by the subdiffusive SVJ model, we are able to price various financial products such as European options. Since our model can be used at time s to price options after inception, we can also provide a new hedging strategy associated with the subdiffusive SVJ model. However, even if subdiffusions appear as good candidates for modeling illiquidity, options pricing in this framework remains a challenging task. We therefore show in the practical part of the paper how to solve numerically via a series expansion the FFPE driving the (Fourier-Laplace transform of) the bivariate probability density of the subdiffusive log-stock price with its inverse subordinator. Finally, we contribute with this paper to provide a full econometric estimation of the subdiffusive SVJ parameters based on the time series of observed log-returns using the Particle Monte-Carlo Markov Chain algorithm. So far, only the parameter driving the montionless periods of illiquid asset prices has been estimated (see Janczura and Wyłomańska (2009)) and we therefore propose a new procedure for evaluating simultaneously all of the model parameters.

2. A Fokker-Planck equation for jump-diffusion models

We start by deriving the Fokker-Planck equations (FPE) satisfied by the SVJ model in the non-fractional case. From these FPE, we will manage in the sequel of this paper to build their fractional counterpart so as to describe the behavior of the subdiffusive SVJ model. We hence consider in a first step the following classical jump-diffusion model under the risk-neutral measure Q :

$$\frac{dA_t}{A_t} = (r - \lambda_J m_J)dt + \sqrt{V_t}\, dW_{1,t} + \left(e^J - 1\right) dN_t\,, \tag{2.1}$$

$$dV_t = \kappa\,(\gamma - V_t)\, dt + \sigma\sqrt{V_t}\, dW_{2,t}\,. \tag{2.2}$$

Here and in the rest of the article, $(W_{j,t})_{t\geq 0}$ $(j = 1, 2)$, are two correlated Brownian motions defined on $(\Omega, \mathcal{F}, Q)$ such that $\mathrm{Cov}\,[dW_{1,t}\, dW_{2,t}] = \rho dt$. We define $(N_t)_{t\geq 0}$ as a Poisson process with intensity parameter $\lambda_J > 0$ (*i.e.* $\mathbb{E}\,[N_t] = \lambda_J t$) and which is independent from $(W_{1,t})_{t\geq 0}$ and $(W_{2,t})_{t\geq 0}$. The random variable J with probability density function (pdf) $f_J(\cdot)$ and support in $\mathbb{R}$ describes the magnitude of the jump when it occurs. We impose $m_J = \mathbb{E}[e^J - 1] = \int_{\mathbb{R}} (e^j - 1) f_J(j)\, dj$ such that the discounted price process $\left(e^{-rt} A_t\right)_{t\geq 0}$ is a martingale under Q. The risk-free rate r is given by the risk-free bond, whose value at time t is equal to

$$B_t = e^{rt}, \quad B_0 = 1\,.$$

Applying Itô's Lemma on equation (2.1), we find the following dynamic for the log-price $X_t := \ln A_t$,

$$dX_t = \left(r - \lambda_J m_J - \frac{1}{2} V_t\right) dt + \sqrt{V_t}\, dW_{1,t} + J dN_t\,. \tag{2.3}$$

Let us write $p(t, x, v\,|s, x_s, v_s)$ the conditional bivariate pdf of $(X_t, V_t \mid X_s = x_s, V_s = v_s)$. This function is such that

$$p(t, x, v\,|s, x_s, v_s) := \frac{\partial^2 P(X_t \leq x, V_t \leq v \mid X_s = x_s, V_s = v_s)}{\partial x\, \partial v}\,. \tag{2.4}$$

This pdf is solution of a forward partial differential equation (PDE) as stated in the next proposition.

Proposition 2.1. *The conditional bivariate pdf $p(t, x, v\,|s, x_s, v_s)$ at time $t \geq s$ of the pair (X_t, V_t) is solution of the following Fokker-Planck equation (FPE) :*

$$\begin{aligned}\frac{\partial}{\partial t} p(t,x, v\,|.) = &-\frac{\partial}{\partial x}\left[\left(r - \lambda_J\, m_J - \frac{1}{2} v\right) p(t, x, v\,|.)\right] - \frac{\partial}{\partial v}\left[\kappa\,(\gamma - v)\, p(t, x, v\,|.)\right] \\ &+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[v\, p(t, x, v\,|.)\right] + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\sigma^2 v\, p(t, x, v\,|.)\right] + \frac{\partial^2}{\partial x \partial v}\left[\rho\sigma v\, p(t, x, v\,|.)\right] \\ &+ \lambda_J \int_{\mathbb{R}} p(t, x - j, v\,|.)\, f_J(j)\, dj - \lambda_J\, p(t, x, v\,|.)\end{aligned} \tag{2.5}$$

with initial condition $p(s, x, v \mid s, x_s, v_s) = \delta_{\{x=x_s, v=v_s\}}$, where $\delta_{\{z\}}$ is the Dirac measure located at z.

Proof. The proof of this proposition being quite standard, it is postponed in Appendix 9.1. □

However, this FPE is hard to solve due to the initial condition involving a Dirac measure. Numerical methods for computing such forward PDE indeed fail to be reliable since the initial condition cannot be evaluated properly. We hence focus on the forward PDE of the Fourier-Laplace transform of the density, which is way simpler than equation (2.5) and which will allow to obtain more stable numerical results since its initial condition does not rely on a Dirac delta anymore.

Following Dragulescu and Yakovenko (2002), let us first denote by $\phi(t, z_1, v \mid s, x_s, v_s)$ the Fourier transform with respect to x of the conditional bivariate pdf $p(t, x, v \mid s, x_s, v_s)$ of the pair (X_t, V_t). This Fourier transform is defined as follows with $z_1 \in \mathbb{R}$:

$$\phi(t, z_1, v \mid s, x_s, v_s) := \int_{-\infty}^{\infty} e^{-iz_1 x} \, p(t, x, v \mid s, x_s, v_s) \, dx \,,$$

and we thus have that the inverse Fourier transform is given by

$$p(t, x, v \mid s, x_s, v_s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iz_1 x} \, \phi(t, z_1, v \mid s, x_s, v_s) \, dz_1 \,. \tag{2.6}$$

Inserting (2.6) into (2.5) and simplifying the information $\{X_s = x_s, V_s = v_s\}$ by $\{.\}$, we find

$$\begin{aligned} \frac{\partial \phi(t, z_1, v \,|.)}{\partial t} &= -\frac{\partial}{\partial v} \left[\kappa(\gamma - v) \, \phi(t, z_1, v \,|.) \right] + \lambda_J \left(\widehat{f}_J(z_1) - 1 \right) \phi(t, z_1, v \,|.) \\ & \quad - \left[\frac{z_1^2}{2} v + i z_1 \left(r - \lambda_J m_J - \frac{1}{2} v \right) \right) - i\rho\sigma z_1 \frac{\partial}{\partial v} v - \frac{\sigma^2}{2} \frac{\partial^2}{\partial v^2} v \right] \phi(t, z_1, v \,|.) \end{aligned} \tag{2.7}$$

where $\widehat{f}_J(z_1)$ is the Fourier transform of $f_J(.)$. The equation (2.7) is simpler than (2.5) because the number of variables has been reduced to two, v and t, whereas z_1 only plays the role of parameter. Finally, since v is positive, linear in v and quadratic in $\frac{\partial}{\partial v}$, equation (2.7) can then be simplified by taking the Laplace transform over v :

$$\phi(t, z_1, z_2 \mid s, x_s, v_s) = \int_0^{+\infty} e^{-z_2 v} \phi(t, z_1, v \mid s, x_s, v_s) \, dv \,. \tag{2.8}$$

We can then easily extend the result of Dragulescu and Yakovenko (2002) and find for our SVJ model that the Fourier-Laplace transform $\phi(t, z_1, z_2 \,|.)$ of the joint pdf $p(t, x, v \,|.)$ is solution of the following forward PDE

$$\begin{aligned} \frac{\partial \phi(t, z_1, z_2 \,|.)}{\partial t} &= -\left(\kappa\gamma z_2 + i z_1 (r - \lambda_J m_J) - \lambda_J \left(\widehat{f}_J(z_1) - 1 \right) \right) \phi(t, z_1, z_2 \,|.) \\ & \quad + \left(-z_2(\kappa + i\rho\sigma z_1) - \frac{\sigma^2 z_2^2}{2} + \frac{z_1^2 - i z_1}{2} \right) \frac{\partial \phi(t, z_1, z_2 \,|.)}{\partial z_2} \end{aligned} \tag{2.9}$$

with initial condition $\phi(s, z_1, z_2 \mid s, x_s, v_s) = e^{-iz_1 x_s - z_2 v_s}$ and boundary condition $\phi(t, 0, 0 \mid s, x_s, v_s) = 1$. This first-order partial differential equation is solved analytically in Dragulescu and Yakovenko (2002). This analytical expression is however not useful for deriving the solution in the fractional case, as we will see in Section 4.

3. Subordinators

Subordinators are $\mathbb{R}^+$-valued stochastic processes, denoted $(U_t)_{t \geq 0}$, with non-decreasing sample paths. One of their most important probabilistic applications is their use as stochastic clock for time-changed processes. Moreover, we consider Lévy subordinators for which increments are independent and homogeneously distributed. Interested readers may refer to Applebaum (2009) for an introduction to such Lévy subordinators. We have then from the Lévy-Khintchine formula that the Laplace transform of Lévy subordinators has the following form

$$\mathbb{E}\left(e^{-\lambda U_t} \right) = e^{-t f(\lambda)} \,, \tag{3.1}$$

where $f(\lambda) = b\lambda + \int_0^\infty \left(1 - e^{-\lambda z}\right)\nu(dz)$ and $b \geq 0$. The function $\nu(.)$ is a non-negative measure on $(0, \infty)$, known as the Lévy measure and satisfying the requirement

$$\int_0^\infty (z \wedge 1)\nu(dz) < \infty\,.$$

The inverse of a subordinator is a process denoted by $(S_t)_{t\geq 0}$ and defined by

$$S_t = \inf\left\{\tau \geq 0 : U_\tau > t\right\}\,.$$

This is the first passage of time of U_τ above the level t. This inverted Lévy subordinator is in general no more a Lévy process. However, $(S_t)_{t\geq 0}$ is still positive and non-decreasing and hence, it has all the required properties to be used as a stochastic clock. By construction, the inverted process $(S_t)_{t\geq 0}$ may be constant over some interval of time. Therefore, any process subordinated by S_t tend to exhibit motionless periods. The natural filtration of $(S_t)_{t\geq 0}$ is then denoted by $(\mathcal{G}_t)_{t\geq 0}$. The probability density functions of U_t and S_t are respectively denoted by $p_U(\tau, t) = \frac{d}{dt}P\left(t \leq U_\tau \leq t + dt\right)$ and $g(t, \tau) = \frac{d}{d\tau}P\left(\tau \leq S_t \leq \tau + d\tau\right)$. By definition, we have that $P(S_t > s) = P(U_s < t)$. In order to use our subdiffusive SVJ model for risk management and hedging as well as to price financial derivatives after their inception, we need to establish the features of $S_t - S_s$ conditionally to the available information up to time s. Most of the literature focuses on the properties of S_t seen from time 0, when $U_0 = S_0 = 0$. In an actuarial context with α-stable subordinators, Hainaut (2021) extends these properties when the process S_t is considered from a later time s. However, S_t is by nature non-Markovian and its natural filtration up to time $s < t$ is not sufficient to draw any conclusion about $S_t - S_s$. We hence need to adjunct additional information about the subordinator that is inverted. It is shown in this paper that most of properties available at time 0 admit an extension at later time s if the available information is the value of S_s and U_{S_s}. We can then easily extend the results of Hainaut (2021) when considering general subordinators of the form (3.1). First, we have for $0 \leq s \leq u \leq t$ that the cdf of $S_t - S_s$ is related to the cdf of U_t through the relation

$$P(S_t - S_s \leq \tau \mid S_s = w, U_{S_s} = u) = P(U_\tau \geq t - u)\,.$$

Hence, we find that the conditional pdf of $S_t - S_s$, denoted by $g(t, \tau \mid s, w, u)$ for $t \geq u \geq s \geq 0$, is the derivative of this last expression with respect to τ :

$$\begin{aligned} g(t, \tau \mid s, w, u) &= \frac{\partial}{\partial \tau}P(S_t - S_s \leq \tau \mid S_s = w, U_{S_s} = u) \\ &= -\frac{\partial}{\partial \tau}P(U_\tau \leq t - u) \\ &= -\frac{\partial}{\partial \tau}\int_0^{t-u} p_U(\tau, x)\,dx\,. \end{aligned}$$

The Laplace transform of $g(t, \tau \mid s, w, u)$ with respect to time t, denoted $\widetilde{g}(\lambda, \tau \mid s, w, u)$, is therefore equal to

$$\widetilde{g}(\lambda, \tau \mid s, w, u) = -\frac{\partial}{\partial \tau}\int_u^\infty e^{-\lambda t}\int_0^{t-u} p_U(\tau, x)dx\,dt\,. \tag{3.2}$$

The subordinator $(U_t)_{t\geq 0}$ being strictly positive, its density is null on $\mathbb{R}^-$ and hence, integrating by parts as in Hainaut (2021), we find

$$\begin{aligned} \widetilde{g}(\lambda, \tau \mid s, w, u) &= -\frac{\partial}{\partial \tau}\left(\frac{1}{\lambda}e^{-\lambda u}\,\widetilde{p_U}(\tau, \lambda)\right) \\ &= \frac{f(\lambda)}{\lambda}e^{-\lambda u - \tau f(\lambda)}\,, \end{aligned} \tag{3.3}$$

since from equation (3.1), we have $\widetilde{p_U}(\tau,\lambda) = \mathbb{E}\left(e^{-\lambda U_\tau}\right) = e^{-\tau f(\lambda)}$.

We now introduce the Dzerbayshan-Caputo (D-C) derivative. First, a Bernstein function is a function $f : (0,\infty) \to \mathbb{R}$ of class C^∞ such that $f(x) \geq 0$ for all $x > 0$ and for which

$$(-1)^k f^{(k)}(x) \leq 0\,,\ \forall x > 0\,,\ k \in \mathbb{N}\,.$$

A Bernstein function also admits a similar representation as the Laplace exponent of a Lévy subordinator

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-xz}\right)\nu(dz)\,, \tag{3.4}$$

where $a, b \geq 0$. As above, $\nu(.)$ is a non-negative Lévy measure on $(0,\infty)$. The triplet (a,b,ν) is the Lévy triplet of the Bernstein function. We then denote by $\bar{\nu}(s)$ the tail of the Lévy measure defined by

$$\bar{\nu}(s)ds = \left(a + \int_s^\infty \nu(dz)\right) ds\,. \tag{3.5}$$

Let us consider $f(.)$, a Bernstein function, and its tail Lévy measure $\bar{\nu}(s)$ that is absolutely continuous on $(0,\infty)$. We also need a function $h(t) \in AC([u,\infty])$, that is the set of absolutely continuous function on $[u,+\infty]$. The generalized D-C derivative according to the Bernstein function $f(.)$ is defined as in Toaldo (2015) by

$${}^f\mathcal{D}_t^{(u,+\infty)} h(t) = b\frac{d}{dt}h(t) + \int_0^{t-u} \frac{\partial}{\partial t} h(t-s)\bar{\nu}(s)ds\,, \quad t \in [u,\infty)\,. \tag{3.6}$$

This integral is well defined if $|h(t)| \leq Me^{\lambda_0 t}$ for some $\lambda_0, M > 0$. As we will explain in Section 4, we only consider from now on subordinators with Lévy triplet $(0,0,\nu)$ associated with Bernstein functions of the form $f(x) = \int_0^\infty \left(1 - e^{-xz}\right)\nu(dz)$ and tail Lévy measures given by $\bar{\nu}(s)ds = \int_s^\infty \nu(dz)ds$. We will indeed show in the proof of Proposition 4.1 the following Lemma.

Lemma 3.1. *Let* ${}^f\mathcal{D}_t^{(u,+\infty)} h(t)$, $t \geq u \geq 0$ *be defined as in equation* (3.6), *where* $f(.)$ *is a Bernstein function with Lévy triplet* $(0,0,\nu)$ *and a tail Lévy measure* $\bar{\nu}(s)ds = \int_s^\infty \nu(dz)ds$. *Then the Laplace transform* $\mathcal{L}$ *of the generalized D-C derivative according to* f *is given by*

$$\mathcal{L}\left[{}^f\mathcal{D}_t^{(u,+\infty)} h(t)\right](\lambda) = f(\lambda)\widetilde{h}(\lambda) - \frac{f(\lambda)}{\lambda} e^{-\lambda u}\, h(u)\,, \qquad \Re(\lambda) > \lambda_0. \tag{3.7}$$

We will therefore only study pure-jump subordinators with no drift in the sequel of the paper. Nevertheless, this class of subordinators encompasses most of the subordinators of interest for modeling financial illiquidity such as Poisson, α-stable and Gamma subordinators. Hainaut and Leonenko (2021) detail the form of the D-C derivative for α-stable and Poisson subordinators.

We now extend the forward PDE (2.5)-(2.9) satisfied by the probability density $p(t,x,v\,|.)$ and its Fourier-Laplace transform $\phi(t,z_1,z_2\,|.)$ in a subdiffusive and fractional context. We show in this fractional setting that they also obey a forward PDE but with respect to a D-C derivative instead of a classical derivative.

4. A Fokker-Planck equation for the subdiffusive SVJ model

In the fractional setting of the subdiffusive SVJ model, the dynamic of the subdiffusive asset price A_{S_t} (time-changed by the inverse subordinator S_t) is given under an equivalent risk-neutral measure[1] Q by

$$\frac{dA_{S_t}}{A_{S_t}} = (r - \lambda_J m_J)dS_t + \sqrt{V_{S_t}}\, dW_{1,S_t} + \left(e^J - 1\right) dN_{S_t}\,, \tag{4.1}$$

$$dV_{S_t} = \kappa\,(\gamma - V_{S_t})\, dS_t + \sigma\sqrt{V_{S_t}}\, dW_{2,S_t}\,. \tag{4.2}$$

It appears logical to subordinate both the price process and the variance process by the inverse subordinator S_t since when the price process is frozen, the variance process does not matter anymore and can be considered as being fixed (even if it is null in practice, which again has no impact on the stock price during these periods of illiquidity). As in the non-fractional case, taking the time-changed log-price $X_{S_t} := \ln A_{S_t}$ instead of A_{S_t} will allow to drastically simplify the FFPE we will have to solve. In the fractional case, we obtain via Itô's Lemma

$$dX_{S_t} = (r - \lambda_J m_J - \frac{1}{2}V_{S_t})\, dS_t + \sqrt{V_{S_t}}\, dW_{1,S_t} + J dN_{S_t}\,. \tag{4.3}$$

For $0 \le s \le u \le t$, the bivariate pdf of (X_{S_t}, V_{S_t}), conditionally to $\{S_s = w,\, U_{S_s} = u,\, X_{S_s} = x_s, V_{S_s} = v_s\}$ is denoted by $p_S(t, x, v \mid s, w, u, x_s, v_s)$. This pdf is thus defined by

$$p_S(t, x, v \mid s, w, u, x_s, v_s) := \frac{P\left(X_{S_t} \le x,\, V_{S_t} \le v \mid S_s = w, U_{S_s} = u, X_{S_s} = x_s, V_{S_s} = v_s\right)}{\partial x\, \partial v}\,,$$

and is solution of a forward fractional PDE as stated in the next proposition.

Proposition 4.1. *For $0 \le s \le u \le t$, the bivariate pdf p_S of (X_{S_t}, V_{S_t}), conditionally to $\{S_s = w,\, U_{S_s} = u,\, X_{S_s} = x_s, V_{S_s} = v_s\}$, is solution of the following fractional Fokker-Planck equation (FFPE) :*

$$\begin{aligned} {}^f\mathcal{D}_t^{(u,+\infty)} p_S(t, x, v\,|\,.\,) &= -\frac{\partial}{\partial x}\left[(r - \lambda_J m_J - \frac{1}{2}v)\, p_S(t, x, v\,|\,.\,)\right] - \frac{\partial}{\partial v}\left[\kappa\,(\gamma - v)\, p_S(t, x, v\,|\,.\,)\right] \\ &+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[v\, p_S(t, x, v\,|\,.\,)\right] + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\sigma^2 v\, p_S(t, x, v\,|\,.\,)\right] + \frac{\partial^2}{\partial x \partial v}\left[\rho\sigma v\, p_S(t, x, v\,|\,.\,)\right] \\ &+ \lambda_J \int_{\mathbb{R}} p_S(t, x - j, v\,|\,.\,) f_J(j) dj - \lambda_J\, p_S(t, x, v\,|\,.\,) \end{aligned} \tag{4.4}$$

with the initial condition $p_S(u, x, v \mid s, w, u, x_s, v_s) = \delta_{\{x = x_s,\, v = v_s\}}$. For all times t such that $0 \le s \le t < u$, then $p_S(t, x, v \mid s, w, u, x_s, v_s) = \delta_{\{x = x_s,\, v = v_s\}}$.

Proof. To lighten developments, we momentarily adopt the notations $p_S\,(t, x, v\,|.) := p_S\,(t, x, v \mid s, w, u, y_1, y_2)$ and $g(t, \tau\,|.) = g(t, \tau \mid s, w, u)$. As $g(t, \tau\,|.)$ is the pdf of $S_t - S_s \mid S_s, U_{S_s}$ and from the independence between (X_t, V_t) and S_t, we infer that

$$p_S(t, x, v\,|.) = \int_0^\infty p\,(w + \tau, x, v \mid w, x_s, v_s)\, g(t, \tau\,|.)\, d\tau\,, \tag{4.5}$$

for $0 \le s \le u \le t$. Notice that since our SVJ model (2.1)-(2.2) is Markovian

$$p(w + \tau, x, v \mid w, x_s, v_s) = p(\tau, x, v \mid 0, x_s, v_s)\,. \tag{4.6}$$

[1]We will explain in Section 5 how this equivalent measure Q is built from a Radon-Nikodym derivative and show that it is risk-neutral.

The Laplace transform of $p_S\left(t,x,v\,|.\right)$ with respect to time t is thus given by

$$\begin{aligned}\widetilde{p}_S(\lambda,x,v\,|.) &= \int_u^\infty e^{-\lambda t}\,p_S(t,x,v\,|.)\,dt \\ &= \int_u^\infty\int_0^\infty p(\tau,x,v\mid 0,x_s,v_s)\,e^{-\lambda t}g(t,\tau\,|.)\,d\tau dt \\ &= \int_0^\infty p(\tau,x,v\mid 0,x_s,v_s)\,\widetilde{g}(\lambda,\tau\,|.)\,d\tau\,,\end{aligned} \tag{4.7}$$

where $\widetilde{g}(\lambda,\tau\,|.)=\int_u^\infty e^{-\lambda t}g(t,\tau\,|.)\,dt$ is the Laplace transform of $g(t,\tau\,|.)$ with respect to time t. Since this transform is given by equation (3.3), $\widetilde{p}_S(\lambda,x,v\,|.)$ is equal to

$$\begin{aligned}\widetilde{p}_S(\lambda,x,v\,|.) &= \frac{f(\lambda)}{\lambda}e^{-\lambda u}\int_0^\infty p(\tau,x,v\mid 0,x_s,v_s)\,e^{-\tau f(\lambda)}d\tau \\ &= \frac{f(\lambda)}{\lambda}e^{-\lambda u}\,\widetilde{p}(f(\lambda),x,v\,|.)\,,\end{aligned} \tag{4.8}$$

where $\widetilde{p}(\lambda,x,v\,|.)$ denotes the Laplace transform of $p(t,x,v\mid 0,x_s,v_s)$ with respect to time t and with Laplace exponent λ. Applying the Laplace transform on both side of the PDE (2.5), we deduce that $\widetilde{p}(\lambda,x,v\,|.)$ is also solution of

$$\begin{aligned}\lambda\,\widetilde{p}(\lambda,x,v\,|.)-p(0,x,v\mid 0,x_s,v_s) &= -\frac{\partial}{\partial x}\left[(r-\lambda_J m_J-\frac{1}{2}v)\,\widetilde{p}(\lambda,x,v\,|.)\right] \\ &\quad-\frac{\partial}{\partial v}\left[\kappa\,(\gamma-v)\,\widetilde{p}(\lambda,x,v\,|.)\right]+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[v\,\widetilde{p}(\lambda,x,v\,|.)\right]+\frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\sigma^2 v\,\widetilde{p}(\lambda,x,v\,|.)\right] \\ &\quad+\frac{\partial^2}{\partial x\partial v}\left[\rho\sigma v\,\widetilde{p}(\lambda,x,v\,|.)\right]+\lambda_J\int_{\mathbb{R}}\widetilde{p}(\lambda,x-j,v\,|.)\,f_J(j)\,dj-\lambda_J\,\widetilde{p}(\lambda,x,v\,|.)\,.\end{aligned}$$

As $\widetilde{p}_S(\lambda,x,v\mid .)=\frac{f(\lambda)}{\lambda}e^{-\lambda u}\,\widetilde{p}\left(f(\lambda),x,v\right)$ from equation (4.8), replacing λ by $f(\lambda)$ leads to

$$\begin{aligned}f(\lambda)\widetilde{p}(f(\lambda),x,v\,|.)-p(0,x,v\,|0,x_s,v_s) &= -\frac{\partial}{\partial x}\left[(r-\lambda_J m_J-\frac{1}{2}v)\,\widetilde{p}(f(\lambda),x,v\,|.)\right] \\ &\quad-\frac{\partial}{\partial v}\left[\kappa\,(\gamma-v)\,\widetilde{p}(f(\lambda),x,v\,|.)\right]+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[v\,\widetilde{p}(f(\lambda),x,v\,|.)\right]+\frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\sigma^2 v\,\widetilde{p}(f(\lambda),x,v\,|.)\right] \\ &\quad+\frac{\partial^2}{\partial x\partial v}\left[\rho\sigma v\,\widetilde{p}(f(\lambda),x,v\,|.)\right]+\lambda_J\int_{\mathbb{R}}\widetilde{p}(f(\lambda),x-j,v\,|.)f_J(j)dj-\lambda_J\,\widetilde{p}(f(\lambda),x,v\,|.)\,.\end{aligned}$$

Multiplying this last equation by $\frac{f(\lambda)}{\lambda}e^{-\lambda u}$ and using equation (4.8), we find

$$\begin{aligned}f(\lambda)\,\widetilde{p}_S(\lambda,x,v\,|.)-\frac{f(\lambda)}{\lambda}e^{-\lambda u}\,p(0,x,v\,|0,x_s,v_s) &= -\frac{\partial}{\partial x}\left[(r-\lambda_J m_J-\frac{1}{2}v)\,\widetilde{p}_S(\lambda,x,v\,|.)\right] \\ &\quad-\frac{\partial}{\partial v}\left[\kappa\,(\gamma-v)\,\widetilde{p}_S(\lambda,x,v\,|.)\right]+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[v\,\widetilde{p}_S(\lambda,x,v\,|.)\right]+\frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\sigma^2 v\,\widetilde{p}_S(\lambda,x,v\,|.)\right] \\ &\quad+\frac{\partial^2}{\partial x\partial v}\left[\rho\sigma v\,\widetilde{p}_S(\lambda,x,v\,|.)\right]+\lambda_J\int_{\mathbb{R}}\widetilde{p}_S(\lambda,x-j,v\,|.)\,f_J(j)\,dj-\lambda_J\,\widetilde{p}_S(\lambda,x,v\,|.)\,.\end{aligned} \tag{4.9}$$

This left-hand term is the Laplace transform $\mathcal{L}$ of the generalized D-C derivative of $p_S(t,x,v\,|.)$ associated with a Bernstein function $f(.)$ and triplet $(0,0,\nu)$, as stated in Lemma 3.1. Indeed, if we take $b=0$ in the D-C derivative (3.6) and with $t \geq U_{S_s} = u$:

$$\begin{aligned}
\mathcal{L}\left[{}^{f}\mathcal{D}_t^{(u,+\infty)} p_S(t,x,v\,|.)\right](\lambda) &= \int_u^\infty e^{-\lambda t} \int_0^{t-u} \frac{\partial}{\partial t} p_S(t-s,x,v\,|.)\bar{\nu}(s)\,dsdt \\
&= \int_u^\infty e^{-\lambda t} \int_u^{t} \frac{\partial}{\partial z} p_S(z,x,v\,|.)\bar{\nu}(t-z)\,dzdt \\
&= \int_u^\infty \frac{\partial}{\partial z} p_S(z,x,v\,|.) \int_z^\infty e^{-\lambda t}\,\bar{\nu}(t-z)\,dtdz \\
&= \int_u^\infty \frac{\partial}{\partial z} p_S(z,x,v\,|.)\,e^{-\lambda z} \int_0^\infty e^{-\lambda t'}\,\bar{\nu}(t')\,dt'dz \\
&= \int_u^\infty \frac{\partial}{\partial z} p_S(z,x,v\,|.)\,e^{-\lambda z}\,\widetilde{\bar{\nu}}(\lambda)\,dz
\end{aligned}$$

where $\widetilde{\bar{\nu}}(\lambda)$ is the Laplace transform of the tail Lévy measure (3.5). Given that $p_S(u,x,v\,|.) = p(0,x,v \mid 0,x_s,v_s)$ and from the Laplace transform (4.7), we find

$$\begin{aligned}
\mathcal{L}\left[{}^{f}\mathcal{D}_t^{(u,+\infty)} p_S(t,x,v\,|.)\right](\lambda) &= \left[p_S(z,x,v\,|.)\,\widetilde{\bar{\nu}}(\lambda)\,e^{-\lambda z}\right]_{z=u}^{\infty} + \widetilde{\bar{\nu}}(\lambda)\lambda \int_u^\infty e^{-\lambda z}\,p_S(z,x,v\,|.)\,dz \\
&= -p_S(u,x,v\,|.)\,\widetilde{\bar{\nu}}(\lambda)\,e^{-\lambda u} + \widetilde{\bar{\nu}}(\lambda)\lambda\,\widetilde{p}_S(\lambda,x,v\,|.) \\
&= \widetilde{\bar{\nu}}(\lambda)\lambda\,\widetilde{p}_S(\lambda,x,v\,|.) - p(0,x,v \mid 0,x_s,v_s)\,\widetilde{\bar{\nu}}(\lambda)\,e^{-\lambda u}\,.
\end{aligned}$$

Now, taking the Laplace transform of the tail of the Lévy measure (3.5) with $a=0$, we have

$$\widetilde{\bar{\nu}}(\lambda) = \int_0^\infty e^{-\lambda s} \int_s^\infty \nu(dz)ds = \frac{1}{\lambda}\int_0^\infty \left(1-e^{\lambda s}\right)\nu(ds) = \frac{f(\lambda)}{\lambda}\,, \tag{4.10}$$

where $f(\lambda)$ is the Bernstein function given in (3.4) with Lévy triplet $(0,0,\nu)$. Hence, we have $\widetilde{\bar{\nu}}(\lambda) = f(\lambda)/\lambda$ and we finally find

$$\mathcal{L}\left[{}^{f}\mathcal{D}_t^{(u,+\infty)} p_S(t,x,v\,|.)\right](\lambda) = f(\lambda)\,\widetilde{p}_S(\lambda,x,v\,|.) - p(0,x,v \mid 0,x_s,v_s)\,\frac{f(\lambda)}{\lambda}e^{-\lambda u}\,, \tag{4.11}$$

which is precisely the left-hand side of equation (4.9). Therefore, applying inverse Laplace transform on both sides of equation (4.9), we obtain the announced result (4.4) and hence prove Proposition 4.1. This also proves Lemma 3.1 since $p(0,x,v \mid 0,x_s,v_s) = p_S(u,x,v\,|.)$. $\square$

In the same way as for the non-fractional case in Section 2, we can obtain a forward PDE for the Fourier-Laplace transform of the fractional density $p_S(t,x,v\,|.)$ of the pair (X_{S_t},V_{S_t}), which will be a lot easier to solve and whose initial condition does not rely on a Dirac measure. However, in a first step, it is easier to analyze the Fourier-Laplace transform of the triplet $(X_{S_t},V_{S_t},S_t-S_s)$, which is defined by

$$\phi_S(t,z_1,z_2,z_3 \mid s,w,u,x_s,v_s) := \mathbb{E}\left(e^{-iz_1X_{S_t}-z_2V_{S_t}-z_3(S_t-S_s)} \mid S_s=w, U_{S_s}=u, X_{S_s}=x_s, V_{S_s}=v_s\right), \tag{4.12}$$

and which satisfies the following proposition.

Proposition 4.2. *For $0 \leq s \leq u \leq t$, the Fourier-Laplace transform $\phi_S(t,z_1,z_2,z_3 \mid s,w,u,x_s,v_s)$ of the conditional joint pdf of $(X_{S_t},V_{S_t},S_t-S_s)$ satisfies the following fractional forward PDE*

$$\begin{aligned}
{}^{f}\mathcal{D}_t^{(u,+\infty)}\phi_S(t,z_1,z_2,z_3\,|.) = &- z_3\phi_S(t,z_1,z_2,z_3\,|.) \\
&- \left(\kappa\gamma z_2 + iz_1(r-\lambda_J m_J) - \lambda_J\left(\widehat{f}_J(z_1)-1\right)\right)\phi_S(t,z_1,z_2,z_3\,|.) \\
&+ \left(-z_2(\kappa+i\rho\sigma z_1) - \frac{\sigma^2 z_2^2}{2} + \frac{z_1^2 - iz_1}{2}\right)\frac{\partial\phi_S(t,z_1,z_2,z_3\,|.)}{\partial z_2}
\end{aligned} \tag{4.13}$$

with initial condition $\phi_S(u, z_1, z_2, z_3 \mid s, w, u, x_s, v_s) = e^{-iz_1x_s - z_2v_s}$ *and boundary condition* $\phi_S(t, 0, 0, 0 \mid s, w, u, x_s, v_s) = 1$. *For all times* t *such that* $0 \le s \le t < u$, *then* $\phi_S(t, z_1, z_2, z_3 \mid s, w, u, x_s, v_s) = e^{-iz_1x_s - z_2v_s}$.

Proof. The initial and boundary conditions are straightforward given the definition (4.12) of $\phi_S(t, z_1, z_2, z_3 \,|.)$. In order to derive equation (4.13), we start by looking at the conditional joint pdf p_S of the triplet $(X_{S_t}, V_{S_t}, S_t - S_s)$ for $0 \le s \le u \le t$:

$$p_S(t, x, v, \tau \mid s, w, u, x_s, v_s) := \frac{\partial^3}{\partial x \partial v \partial \tau} P(X_{S_t} \le x, V_{S_t} \le v, S_t - S_s \le \tau \mid S_s = w, U_{S_s} = u, X_{S_s} = x_s, V_{S_s} = v_s) \tag{4.14}$$

From the definition of the conditional pdf $p_S(t, x, v, \tau \,|.)$ and from equations (4.5)-(4.6), we have

$$p_S(t, x, v, \tau \mid s, w, u, x_s, v_s) = p(\tau, x, v \mid 0, x_s, v_s)\, g(t, \tau \mid s, w, u)\,. \tag{4.15}$$

From equations (3.3) - (4.15) and simplifying the notation of the known values $\{s, w, u, x_s, v_s\}$ by $\{.\}$, it follows that

$$\begin{aligned}
\widetilde{p}_S(\lambda, x, v, \tau \,|.) &= \int_u^\infty e^{-\lambda t}\, p_S(t, x, v, \tau \,|.)\, dt \\
&= p(\tau, x, v \mid 0, x_s, v_s) \int_u^\infty e^{-\lambda t} g(t, \tau \mid s, w, u)\, dt \\
&= \frac{f(\lambda)}{\lambda} e^{-\lambda u} e^{-\tau f(\lambda)}\, p(\tau, x, v \mid 0, x_s, v_s)\,.
\end{aligned}$$

We then obtain for the Laplace transform of ϕ_S :

$$\begin{aligned}
&\widetilde{\phi}_S(\lambda, z_1, z_2, z_3 \mid .) \\
&:= \int_0^{+\infty}\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_u^{+\infty} e^{-\lambda t - iz_1x - z_2v - z_3\tau}\, p_S(t, x, v, \tau \,|.)\, dtdxdvd\tau \\
&= \frac{f(\lambda)}{\lambda} e^{-\lambda u} \int_0^{+\infty}\int_0^{+\infty}\int_{-\infty}^{+\infty} e^{-iz_1x - z_2v - z_3\tau} e^{-\tau f(\lambda)}\, p(\tau, x, v \mid 0, x_s, v_s)\, dxdvd\tau \\
&= \frac{f(\lambda)}{\lambda} e^{-\lambda u} \int_0^{+\infty} e^{-\tau(z_3 + f(\lambda))} \phi(\tau, z_1, z_2 \mid 0, x_s, v_s)\, d\tau\,,
\end{aligned} \tag{4.16}$$

where $\phi(\tau, z_1, z_2 \mid 0, x_s, v_s) = \int_0^\infty \int_{-\infty}^\infty e^{-iz_1x - z_2v}\, p(\tau, x, v \mid 0, x_s, v_s) dxdv$. Next, integrating by parts and renaming τ by t yields

$$\begin{aligned}
&\widetilde{\phi}_S\,(\lambda, z_1, z_2, z_3 \,|.) \\
&= \frac{\frac{f(\lambda)}{\lambda} e^{-\lambda u}}{z_3 + f(\lambda)} \left[\phi(0, z_1, z_2 \mid 0, x_s, v_s) + \int_0^{+\infty} e^{-t(z_3 + f(\lambda))} \frac{\partial}{\partial t} \phi(t, z_1, z_2 \mid 0, x_s, v_s) dt \right].
\end{aligned} \tag{4.17}$$

Proposition (2.9) gives the expression of $\frac{\partial}{\partial t}\phi(t, z_1, z_2 \mid 0, x_s, v_s)$ so as to obtain

$$\begin{aligned}
&\int_0^{+\infty} e^{-t(z_3 + f(\lambda))} \frac{\partial}{\partial t} \phi(t, z_1, z_2 \mid 0, x_s, v_s) dt = \\
&\quad \int_0^{+\infty} e^{-t(z_3 + f(\lambda))} \Big[- \Big(\kappa\gamma z_2 + iz_1(r - \lambda_J m_J) - \lambda_J \left(\widehat{f}_J(z_1) - 1 \right) \Big) \phi(t, z_1, z_2 \,|.) \\
&\quad + \left(-z_2(\kappa + i\rho\sigma z_1) - \frac{\sigma^2 z_2^2}{2} + \frac{z_1^2 - iz_1}{2} \right) \frac{\partial \phi(t, z_1, z_2 \,|.)}{\partial z_2} \Big] dt\,.
\end{aligned} \tag{4.18}$$

From this last expression and equation (4.16), we find for equation (4.17) :

$$\begin{aligned}(z_3 + f(\lambda))\, \widetilde{\phi}_S(\lambda, z_1, z_2, z_3\,|.) = \frac{f(\lambda)}{\lambda} e^{-\lambda u} \phi(0, z_1, z_2 \mid 0, x_s, v_s) \\ - \left(\kappa\gamma z_2 + i z_1 (r - \lambda_J m_J) - \lambda_J \left(\widehat{f}_J(z_1) - 1\right)\right) \widetilde{\phi}_S(\lambda, z_1, z_2, z_3\,|.) \\ + \left(-z_2(\kappa + i\rho\sigma z_1) - \frac{\sigma^2 z_2^2}{2} + \frac{z_1^2 - i z_1}{2}\right) \frac{\partial \widetilde{\phi}_S(\lambda, z_1, z_2, z_3\,|.)}{\partial z_2}\,.\end{aligned}$$

Rearranging the previous equation, we have

$$\begin{aligned}f(\lambda)\, \widetilde{\phi}_S(\lambda, z_1, z_2, z_3\,|.) - \frac{f(\lambda)}{\lambda} e^{-\lambda u}\, \phi(0, z_1, z_2 \mid 0, x_s, v_s\,) = -z_3 \widetilde{\phi}_S(\lambda, z_1, z_2, z_3\,|.) \\ - \left(\kappa\gamma z_2 + i z_1 (r - \lambda_J m_J) - \lambda_J \left(\widehat{f}_J(z_1) - 1\right)\right) \widetilde{\phi}_S(\lambda, z_1, z_2, z_3\,|.) \\ + \left(-z_2(\kappa + i\rho\sigma z_1) - \frac{\sigma^2 z_2^2}{2} + \frac{z_1^2 - i z_1}{2}\right) \frac{\partial \widetilde{\phi}_S(\lambda, z_1, z_2, z_3\,|.)}{\partial z_2}\,. \qquad (4.19)\end{aligned}$$

The left-hand side of equation (4.19) is the Laplace transform of the generalized D-C derivative of $\phi_S(t, z_1, z_2, z_3\,|.)$, as given in Lemma 3.1. We have indeed that $\phi_S(u, z_1, z_2, z_3 \mid s, w, u, x_s, v_s) = e^{-iz_1 x_s - z_2 v_s}$, which is equal to $\phi(0, z_1, z_2 \mid 0, x_s, v_s) = e^{-iz_1 x_s - z_2 v_s}$. Applying the inverse Laplace transform on both sides leads to the announced result of Proposition 4.2. $\square$

From the previous Proposition 4.2, we can also directly infer the Fourier-Laplace transform of the pair (X_{S_t}, V_{S_t}), which is defined by

$$\phi_S(t, z_1, z_2 \mid s, w, u, x_s, v_s) := \mathbb{E}\left[e^{-iz_1 X_{S_t} - z_2 V_{S_t}} \mid S_s = w, U_{S_s} = u, X_{S_s} = x_s, V_{S_s} = v_s\right]. \qquad (4.20)$$

This transform also obeys a fractional forward PDE as stated in the following corollary.

Corollary 4.3. *For $0 \le s \le u \le t$, the Fourier-Laplace transform $\phi_S(t, z_1, z_2 \mid s, w, u, x_s, v_s)$ of the conditional pdf of (X_{S_t}, V_{S_t}) satisfies the following fractional forward PDE :*

$$\begin{aligned}{}^{f}\mathcal{D}_t^{(u,+\infty)} \phi_S(t, z_1, z_2\,|.) = - \left(\kappa\gamma z_2 + i z_1 (r - \lambda_J m_J) - \lambda_J \left(\widehat{f}_J(z_1) - 1\right)\right) \phi_S(t, z_1, z_2\,|.) \\ + \left(-z_2(\kappa + i\rho\sigma z_1) - \frac{\sigma^2 z_2^2}{2} + \frac{z_1^2 - i z_1}{2}\right) \frac{\partial \phi_S(t, z_1, z_2\,|.)}{\partial z_2} \qquad (4.21)\end{aligned}$$

with initial condition $\phi_S(u, z_1, z_2 \mid s, w, u, x_s, v_s) = e^{-iz_1 x_s - z_2 v_s}$ and boundary condition $\phi_S(t, 0, 0|s, w, u, x_s, v_s) = 1$. For all times t such that $0 \le s \le t < u$, then $\phi_S(t, z_1, z_2 \mid s, w, u, x_s, v_s) = e^{-iz_1 x_s - z_2 v_s}$.

Proof. Apply Proposition 4.2 with $z_3 = 0$. $\square$

5. Change of measure

We have defined our subdiffusive SVJ model (4.1)-(4.2) directly under a risk neutral measure Q. Since we will provide an econometric estimation of this model based on the observed log-returns on the market, we now propose a change of measure so as to evaluate our model under the real measure P. We also show in this section that this subdiffusive SVJ model is arbitrage-free and leads to an incomplete market.

Recall first that $(S_t)_{t\ge 0}$ is an inverse Lévy subordinator defined on $(\Omega, \mathcal{G}, P)$ which is independent of the natural filtration $\mathcal{F}_t$ of the asset price $(A_t)_{t\geq 0}$. We then denote in this section by $\mathcal{H}_t$ the augmented filtration $\mathcal{G}_t \vee \mathcal{F}_{S_t}$. We thus work here in a probability space $(\Omega, \mathcal{H}, P)$ on which is defined the subdiffusive asset $(A_{S_t})_{t\geq 0}$. Under P, we assume in a first step that the stock and variance processes are solutions of the following time-changed stochastic differential equations

$$\frac{dA_{S_t}}{A_{S_t}} = \mu dS_t + \sqrt{V_{S_t}}\, dW^P_{1,S_t} + \overbrace{(e^{J^P} - 1)}^{Y^P}\, dN^P_{S_t} \tag{5.1}$$

$$dV_{S_t} = \widetilde{\kappa}\,(\widetilde{\gamma} - V_{S_t})\, dS_t + \sigma\sqrt{V_{S_t}}\, dW^P_{2,S_t} \tag{5.2}$$

where $\left(W^P_{j,S_t}\right)_{t\geq 0}$ $(j = 1, 2)$ are two correlated Brownian motions under $(\Omega, \mathcal{H}, P)$ such that $\mathrm{Cov}[dW^P_{1,t}, dW^P_{2,t}] = \rho dt$. We can therefore rewrite $W^P_{1,S_t} = \rho\, W^P_{2,S_t} + \sqrt{1-\rho^2}\, W^P_{\perp,S_t}$ where $W^P_{\perp,S_t}$ is also a Brownian motion under P, independent of $W^P_{2,t}$. Moreover, we denote $Y^P := (e^{J^P} - 1)$ and $f^P_Y(\cdot)$ is the pdf of this random variable under the physical measure. As in Section 2, we write $m^P_j = \mathbb{E}[e^{J^P} - 1] = \mathbb{E}[Y^P]$. Finally, $\left(dN^P_{S_t}\right)_{t\geq 0}$ is a P-Poisson process with intensity $\lambda_J dS_t$. Therefore, applying Itô's Lemma, we find the following expression for the real dynamic of the time-changed stock price

$$A_{S_t} = A_0 \exp\left(\int_0^{S_t} (\mu - \frac{1}{2}V_s)ds + \rho\int_0^{S_t} \sqrt{V_s}\, dW^P_{2,s} + \sqrt{1-\rho^2}\int_0^{S_t} \sqrt{V_s}\, dW^P_{\perp,s}\right)\prod_{k=1}^{N^P_{S_t}} (1 + Y^P_k),$$

where Y^P_k are i.i.d random variables with pdf $f^P_Y(.)$.

We now show how to build a change of measure such that the subdiffusive SVJ model (4.1)-(4.2) under Q that we defined in Section 4 is equivalent to the real-world dynamic (5.1)-(5.2) of this model under P, as introduced above. Let us first define $\psi(.\,, \eta) = \ln\left(\eta\frac{f^Q_Y(.)}{f^P_Y(.)}\right)$ where $\eta \in \mathbb{R}^+$. We consider two $\mathcal{H}_t$-adapted and square integrable processes, denoted $(\theta_{2,S_t})_{t\geq 0}$ and $(\theta_{\perp,S_t})_{t\geq 0}$. We define next the time-changed process $(Z_{S_t})_{t\geq 0}$ as follows

$$\begin{aligned} Z_{S_t} = \exp\Bigg(-\frac{1}{2}\int_0^{S_t} \left(\theta^2_{2,s} + \theta^2_{\perp,s}\right) ds - \int_0^{S_t} \theta_{2,s}\, dW^P_{2,s} - \int_0^{S_t} \theta_{\perp,s}\, dW^P_{\perp,s} \\ + \sum_{k=1}^{N^P_{S_t}} \psi\left(Y^P_k, \eta\right) - (\eta - 1)\lambda_J S_t\Bigg), \end{aligned}$$

which admits an equivalent infinitesimal representation by Itô's Lemma

$$dZ_{S_t} = -Z_{S_t}\left(\theta_{2,S_t}\, dW^P_{2,S_t} + \theta_{\perp,S_t}\, dW^P_{\perp,S_t}\right) - Z_{S_t}(\eta - 1)\lambda_J dS_t + Z_{S_t}\left(e^{\psi(Y^P,\eta)} - 1\right) dN^P_{S_t}. \tag{5.3}$$

For all $t \leq T$, we can check that $\mathbb{E}\left(dZ_{S_t} \mid \mathcal{H}_t \vee \mathcal{G}_T\right) = 0$ and therefore $\mathbb{E}\left(Z_{S_T} \mid \mathcal{H}_t \vee \mathcal{G}_T\right) = Z_{S_t} + \int_{S_t}^{S_T} \mathbb{E}\left(dZ_{S_t} \mid \mathcal{H}_t \vee \mathcal{G}_T\right) = Z_{S_t}$. Using nested expectations, the process Z_{S_t} is then a martingale with $Z_{S_0} = 1$. Therefore, since Z_{S_t} is $\mathcal{H}_t$-adapted and since $Z_{S_t} = \mathbb{E}\left[Z_{S_T} \,|\, \mathcal{H}_t\right] = \mathbb{E}\left[\frac{dQ}{dP} \,\middle|\, \mathcal{H}_t\right]$, we have that Z_{S_t} is well a Radon-Nikodym derivative defining an equivalent measure Q. The next proposition states the dynamics of the Brownian motions and the jump process under Q.

Proposition 5.1. *Under the equivalent measure Q defined by the Radon-Nikodym derivative* (5.3)*,*

1. $dW_{2,S_t} = dW^P_{2,S_t} + \theta_{2,S_t} dS_t$ *and* $dW_{\perp,S_t} = dW^P_{\perp,S_t} + \theta_{\perp,S_t} dS_t$ *are time-changed Brownian motions.*

2. *the process* $Y^Q dN^Q_{S_t}$ *is a point process with an intensity equal to* $\eta\lambda_J dS_T$ *and the pdf of jump is* $f^Q_Y(.)$ *such that* $\mathbb{E}[Y^Q] = m^Q_J$.

Proof. The proof of Hainaut and Leonenko (2021) can be easily extended when considering two independent Brownian motions W_{2,S_t} and $W_{\perp,S_t}$. $\square$

By the Fundamental Theorem of asset pricing, the market model defined by $(\Omega, \mathcal{H}, P)$ and asset prices $(A_{S_t})_{t\geq 0}$ is arbitrage-free if and only if there exists a probability measure Q equivalent to P such that the asset $(A_{S_t})_{t\geq 0}$ is martingale with respect to Q. This measure Q is then called a risk-neutral measure. This is true if and only if discounted prices are martingale under Q. We therefore need to show that the equivalent measure Q defined by the Radon-Nikodym derivative (5.3) is a risk-neutral one. The next corollary states the conditions that θ_{2,S_t} and $\theta_{\perp,S_t}$ must fulfill for defining such risk-neutral measure Q and hence to show that our subdiffusive SVJ model does not admit arbitrage.

Corollary 5.2. *If* $(\theta_{2,S_t})_{t\geq 0}$ *and* $(\theta_{\perp,S_t})_{t\geq 0}$ *are* $\mathcal{H}_t$*-adapted and square integrable processes satisfying*

$$\frac{\mu - r + \eta\lambda_J m^Q_J}{\sqrt{V_{S_t}}} = \rho\,\theta_{2,S_t} + \sqrt{1-\rho^2}\,\theta_{\perp,S_t}\,, \tag{5.4}$$

then the process $(Z_{S_t})_{t\geq 0}$ *in equation* (5.3) *defines a risk-neutral measure* Q *such that the subdiffusive SVJ model* (5.1)-(5.2) *is arbitrage-free.*

Proof. According to Proposition 5.1, the dynamic of the stock price (5.1) can be rewritten under Q as follows

$$\frac{dA_{S_t}}{A_{S_t}} = \mu\, dS_t + \sqrt{V_{S_t}}\left(\rho\,(dW_{2,S_t} - \theta_{2,S_t} dS_t) + \sqrt{1-\rho^2}\,(dW_{\perp,S_t} - \theta_{\perp,S_t} dS_t)\right) + Y^Q dN^Q_{S_t}\,.$$

The discounted stock price is a martingale under Q if and only if the drift in the last equation is equal to the risk-free rate r. Hence, we impose that

$$\mathbb{E}^Q\left[\frac{d\left(e^{-rS_t} A_{S_t}\right)}{A_{S_t}}\right] = 0\,,$$

which gives

$$\left(\mu - r - \sqrt{V_{S_t}}\left(\rho\,\theta_{2,S_t} + \sqrt{1-\rho^2}\,\theta_{\perp,S_t}\right) + \eta\lambda_J m^Q_J\right) dS_t = 0\,.$$

This last condition leads to the announced result. $\square$

Condition (5.4) emphasizes that the market in the subdiffusive SVJ model is incomplete. The risk-neutral measure Q is indeed not unique since there exist an infinity of solutions to (5.4). For a pair of processes $(\theta_{2,S_t}, \theta_{\perp,S_t})$ satisfying relation (5.4), the dynamics of A_{S_t} and V_{S_t} under Q obey respectively the equations

$$\frac{dA_{S_t}}{A_{S_t}} = (r - \eta\,\lambda_J\, m^Q_J)\, dS_t + \sqrt{V_{S_t}}\left(\rho\, dW_{2,S_t} + \sqrt{1-\rho^2}\, dW_{\perp,S_t}\right) + Y^Q dN^Q_{S_t}\,, \tag{5.5}$$

and

$$dV_{S_t} = \widetilde{\kappa}\left(\widetilde{\gamma} - \frac{\sigma\,\theta_{2,S_t}\sqrt{V_{S_t}}}{\widetilde{\kappa}} - V_{S_t}\right) dS_t + \sigma\sqrt{V_{S_t}}\, dW_{2,S_t}\,. \tag{5.6}$$

Equation (5.6) reveals that if $\theta_{2,S_t} \neq 0$, the variance reverts to a mean level, denoted $\widetilde{\gamma}_{2,S_t} = \widetilde{\gamma} - \frac{\sigma\, \theta_{2,S_t}\sqrt{V_{S_t}}}{\widetilde{\kappa}}$, which is itself an $\mathcal{H}_t$-adapted process under the risk neutral measure. For a general form of $\theta_{2,S_t} \neq 0$, the variance process is no more a CIR process and the joint pdf $p(t, x, v \mid s, x_s, v_s)$ is unknown (as well as its Fourier-Laplace transform). Since this prevents us to price options, we consider exclusively processes θ_{2,S_t} of the form

$$\theta_{2,S_t} = \theta_2 \sqrt{V_{S_t}}\ , \tag{5.7}$$

where $\theta_2 \in \mathbb{R}$ is a constant called the market price of volatility is risk. With this assumption, we have that $\theta_{\perp,S_t} = \frac{\mu - r + \eta\lambda_J m_J^Q - \rho\,\theta_2 V_{S_t}}{\sqrt{1-\rho^2}\sqrt{V_{S_t}}}$ and the subdiffusive variance dynamic is now form-invariant under Q (it is in fact again a CIR process) :

$$dV_{S_t} = \kappa^* \left(\gamma^* - V_{S_t}\right) dS_t + \sigma\sqrt{V_{S_t}}\, dW_{2,S_t}\ , \tag{5.8}$$

where κ^* and γ^* are the risk-neutral speed and level of mean reversion given by

$$\kappa^* = \widetilde{\kappa} + \theta_2\, \sigma\ , \qquad \gamma^* = \frac{\widetilde{\kappa}\widetilde{\gamma}}{\widetilde{\kappa} + \theta_2\, \sigma}\ . \tag{5.9}$$

Notice that $\theta_2 > -\widetilde{\kappa}/\sigma$, otherwise the mean reversion level γ^* is negative under Q.

To lighten notations, we can assume without loss of generality that the frequency and size of jumps are identical under the real and risk neutral measures, *i.e.* $\eta = 1$ and $Y := Y^Q \overset{d}{=} Y^P$. We therefore have $\forall t \geq 0$ that $dN_{S_t} := dN_{S_t}^Q \overset{d}{=} dN_{S_t}^P$ and $m_J := m_J^Q = m_J^P$. If we then identify in equation (2.2) $\kappa := \kappa^*$ and $\gamma := \gamma^*$, we obtain from equation (5.5)-(5.8) that the subdiffusive SVJ model is given under the risk-neutral measure Q by

$$\frac{dA_{S_t}}{A_{S_t}} = (r - \lambda_J m_J)\, dS_t + \sqrt{V_{S_t}}\, dW_{1,S_t} + Y dN_{S_t} \tag{5.10}$$

$$dV_{S_t} = \kappa \left(\gamma - V_{S_t}\right) dS_t + \sigma\sqrt{V_{S_t}}\, dW_{2,S_t} \tag{5.11}$$

where $W_{1,S_t} = \rho\, W_{2,S_t} + \sqrt{1-\rho^2}\, W_{\perp,S_t}$. This is exactly the same risk-neutral subdiffusive SVJ model as defined in Section 4 with equations (4.1)-(4.2) and with $Y = e^J - 1$. Therefore, the change of measure defined by the Radon-Nikodym derivative (5.3) allows to work both with the real-world and risk-neutral dynamics of the subdiffusive SVJ model provided that we apply the change of parameters (5.9).

6. An estimation procedure

In order to estimate the parameters of our model, we consider daily time series of log-returns from April 2016 to November 2021 for the ALQP stock price, making a total of 1428 data points. Indeed, having no option quotes available in illiquid markets, we are forced to use these log-return data. To the best of our knowledge, such econometric estimation of the parameters of subdiffusive models under the real probability measure P is new. We here consider the Particle Markov Chain Monte Carlo (PMCMC) procedure of Andrieu *et al.* (2010) which is built from a sequential Monte-Carlo filter that we now describe. We therefore work with the real-world dynamics (5.1)-(5.2) of our subdiffusive SVJ model. Moreover, we need to specify the exact form of the subordinator $(U_t)_{t\geq 0}$ and to define the jump size distribution J^P as well as the counting process $(N_t^P)_{t\geq 0}$ (denoted from now on J and $(N_t)_{t\geq 0}$, respectively). We take here for $(U_t)_{t\geq 0}$ an α-stable process which is well a pure-jump subordinator with no drift satisfying Lemma (3.1) with $f(\lambda) = \lambda^\alpha$. This particular type of subordinator has indeed already been successfully implemented in finance (cfr Tankov (2003) or Cartea and Howison (2009)). As in the classical Bates model, we consider that $J \sim N(\mu_J, \sigma_J^2)$ and that $N_t \sim Poi(\lambda_J t)$. The following estimation procedure can of course be extended easily with other kinds of subordinators (*e.g.* Poisson subordinator) and with other jump processes.

6.1. Filtering

This section introduces the filtering technique that is used to determine the most likely evolution of the hidden variance process $(V_t)_{t\geq 0}$, the inverse subordinator $(S_t)_{t\geq 0}$ and the counting process $(N_t)_{t\geq 0}$, which are indeed not observed on the market. There is a considerable literature on the development of simulation-based methods to perform filtering of nonlinear Gaussian models. Leading contributions are reviewed by Doucet *et al.* (2001). We consider in particular for this paper the *boostrap filter*, which is a Sequential Importance Resampling (SIR) filter in which the transition prior probability distribution is taken as importance function (see Gordon *et al.* (1993) for more details). This boostrap filter will be combined in the sequel with a Monte-Carlo Markov Chain to estimate the parameters of the subdiffusive SVJ model but for the moment, we assume that its parameters are known. We consider a sample of discrete observations of log-returns $X_t = \ln \frac{A_{t+\Delta t}}{A_t}$ where Δt is the time step between two observations. The sample is denoted by $x = \{x_1, x_2, \ldots, x_n\}$ where $n = T/\Delta t$. We denote s_j, v_j and x_j the realizations of $S_j = S_{t_j}$, $V_j = V_{S_j}$, and $X_j = X_{S_j}$ for $j = 1, \ldots, n$ with $t_n = T$. We also denote $\Delta S_j = S_j - S_{j-1}$ and $\Delta N_j \sim Poi(\lambda_J \Delta S_j)$ with their realizations Δs_j and Δn_j. The dynamic of log-returns is approached by applying Itô's Lemma on equation (5.1) and then discretizing the obtained equation, which gives

$$X_j = \left(\mu - \frac{V_{j-1}}{2}\right) \Delta S_j + \sqrt{V_{j-1}}\, \Delta W_{1,j}^P + J \Delta N_j \,.$$

The increment $\Delta W_{1,j}^P$ can be rewritten as $\Delta W_{1,j}^P = \rho\, \Delta W_{2,j}^P + \sqrt{1-\rho^2}\, \Delta W_{\perp,j}^P$ where $\Delta W_{2,j}^P \sim N(0, \sqrt{\Delta S_j})$ and where $\Delta W_{\perp,j}^P \sim N(0, \sqrt{\Delta S_j})$. Therefore, we have

$$X_j = \left(\mu - \frac{V_{j-1}}{2}\right) \Delta S_j + \rho \sqrt{V_{j-1}}\, \Delta W_{2,j}^P + \sqrt{1-\rho^2}\, \sqrt{V_{j-1}}\, \Delta W_{\perp,j}^P + J \Delta N_j \,. \tag{6.1}$$

The variance process (5.2) is discretized in the same manner by

$$V_j = V_{j-1} + \widetilde{\kappa}\,(\widetilde{\gamma} - V_{j-1})\, \Delta S_j + \sigma \sqrt{V_{j-1}}\, \Delta W_{2,j}^P \,. \tag{6.2}$$

The dynamic of S_j is tuned by the parameter α and following Magdziarz (2009), we can simulate efficiently sample paths of the discretized version of the inverse α-stable subordinator S_j with

$$S_j = (\min\{k \in \mathbb{N} \mid U_{k\Delta t} > j\Delta t\} - 1)\Delta t \,. \tag{6.3}$$

The α-stable subordinator $U_j = U_{t_j}$, $j = 1, \ldots, n$ is simulated by the Euler method and then summing up the increments

$$\begin{cases} U_0 = 0 \\ U_{k\Delta t} = U_{(k-1)\Delta t} + (\Delta t)^{\frac{1}{\alpha}} \xi_k \quad k \geq 1 \end{cases}$$

where ξ_k are i.i.d random variables. The ξ_k are defined by

$$\xi_k = \frac{\sin\left(\alpha\left(V+\frac{\pi}{2}\right)\right)}{(\cos(V))^{\frac{1}{\alpha}}}\left(\frac{\cos\left(V-\alpha\left(V+\frac{\pi}{2}\right)\right)}{W}\right)^{\frac{1-\alpha}{\alpha}},$$

where V is an uniform random variable on $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ and W is an exponential random variable with a mean equal to one.

We denote the set of parameters $\vartheta = (\alpha, \mu, \widetilde{\kappa}, \widetilde{\gamma}, \sigma, \rho, \mu_J, \sigma_J, \lambda_J)$ and as mentioned above, it is assumed to be known for the moment. The particle filter estimates the most likely sample paths of the inverse subordinator, the variance process and the jump process based on the observed log-returns. First, we write $\Delta w_{2,j}$ for the realization of $\Delta W^P_{2,j}$. The information $\Delta S_j = \Delta s_j$, $V_j = v_j$, $\Delta N_j = \Delta n_j$ and $\Delta W^P_{2,j} = \Delta w_{2,j}$ is then stored in a *particle*, *i.e.* a vector denoted $u_j = (\Delta s_j, v_j, \Delta n_j, \Delta w_{2,j})$. The visible information (log-returns) up to time t_j is also stored in a vector $x_{1:j} = \{x_1, \ldots, x_j\}$ and the density of the observation $f(x_j|u_j)$, conditionally to the information in the particle, is then a Gaussian probability density function. More precisely, we have from equation (6.1) that

$$f(x_j \,|\, u_j) = N(\mu_{r,j}\,, \sigma^2_{r,j}) \tag{6.4}$$

where

$$\mu_{r,j} = \left(\mu - \frac{v_{j-1}}{2}\right)\Delta s_j + \rho\sqrt{v_{j-1}}\,\Delta w_{2,j} + \Delta n_j\,\mu_J \tag{6.5}$$

and

$$\sigma^2_{r,j} = (1-\rho)^2\, v_{j-1}\,\Delta s_j + \Delta n_j\,\sigma^2_J \tag{6.6}$$

On the other hand, the transition density $f(u_{j+1}|u_j)$ is simulated based on equation (6.2), (6.3) and on the fact that $\Delta n_j \sim Poi(\lambda_J \Delta s_j)$. The initial pdf of u_0 is denoted $f(u_0)$. Then, using Bayesian arguments, one can show that the posterior distribution of u_j conditionally to the available information at t_j is equal to

$$f(u_j \,|\, x_{1:j}) = \frac{f(x_j \,|\, u_j)}{\int f(x_j \,|\, u_j) f(u_j \,|\, x_{1:j-1}) du_j} f(u_j \,|\, x_{1:j-1})\,, \tag{6.7}$$

where we predict

$$f(u_j \,|\, x_{1:j-1}) = \int f(u_j \,|\, u_{j-1}) f(u_{j-1} \,|\, x_{1:j-1})\, du_{j-1}\,. \tag{6.8}$$

Based on the boostrap filter of Gordon *et al.* (1993), we have that the calculation of $f(u_j \,|\, x_{1:j})$ can be performed in three steps according to Algorithm 1. First, in the prediction step, we approach $f(u_j \,|\, x_{1:j-1})$ by simulations, based on relation (6.8). In practice, the integral in (6.8) is replaced by a Monte-Carlo simulation of N particles, denoted $u^{(i)}_j$ for $i = 1, \ldots, N$. In the correction step, the density $f(u_j \,|\, x_{1:j})$ is computed via equation (6.7) and is replaced in simulations by the discrete probability of occurrence denoted $p^{(i)}_j$ for the particle $i = 1, \ldots, N$. We hence have that $p^{(i)}_j = P(u^{(i)}_j \,|\, x_{1:j})$ and is given by equation (6.9). Also note in equation (6.10) that if $\Delta s^{(i)}_j = 0$, the pdf $f(x_j \mid p^{(i)}_j)$ is a Dirac measure located at x_j. In practice, we replace this measure by a normal pdf with very small variance.

In the third step, we perform a resampling with replacement and with probabilities $p^{(i)}_j$, $i = 1, \ldots, N$, in order to keep track of the most likely sample paths of the different hidden processes. For

Algorithm 1 Particle filtering of the subdiffusive SVJ model.

Initial step : draw N values of $v_0^{(i)}$ for $i = 1, \cdots, N$ from the initial distribution $f(v_0)$, which is assumed to be uniform $\mathcal{U}_{[0,0.36]}$.

Main procedure :

for $j = 1$ to n **do**

Prediction step :

Simulate N times $s_j^{(i)}$ based on equation (6.3) and compute $\Delta s_j^{(i)} = s_j^{(i)} - s_{j-1}^{(i)}$.

Draw a sample $\Delta w_{2,j}^{(i)}$ from a $N(0, \sqrt{\Delta s_j^{(i)}})$, $i = 1, \cdots N$.

Update $v_j^{(i)}$ using the relation :

$$v_j^{(i)} = v_{j-1}^{(i)} + \widetilde{\kappa}\left(\widetilde{\gamma} - v_{j-1}^{(i)}\right)\Delta s_j^{(i)} + \sigma\sqrt{v_j^{(i)}}\,\Delta w_{2,j}^{(i)}$$

Draw N times $\Delta n_j^{(i)}$ from a $Poi(\lambda_J\,\Delta s_j^{(i)})$.

Correction step :

The particle $u_j^{(i)} = \left(\Delta s_j^{(i)}, v_j^{(i)}, \Delta n_j^{(i)}, \Delta w_{2,j}^{(i)}\right)$ has a probability of occurrence

$$p_j^{(i)} = \frac{f(x_j \mid u_j^{(i)})}{\sum_{i=1:N} f(x_j \mid u_j^{(i)})}\,, \tag{6.9}$$

where

$$f(x_j \mid u_j^{(i)}) = N(\mu_{r,j}\,, \sigma_{r,j}^2)\,, \tag{6.10}$$

with

$$\mu_{r,j} = \left(\mu - \frac{v_{j-1}^{(i)}}{2}\right)\Delta s_j^{(i)} + \rho\sqrt{v_{j-1}^{(i)}}\,\Delta w_{2,j}^{(i)} + \Delta n_j^{(i)}\,\mu_J\,,$$

and

$$\sigma_{r,j}^2 = (1-\rho)^2\, v_{j-1}^{(i)}\,\Delta s_j^{(i)} + \Delta n_j^{(i)}\,\sigma_J^2\,.$$

Calculate the effective sample size, $N_{eff}(j)$:

$$N_{eff}(j) = 1 \Big/ \sum_{i=1}^{N}\left(p_j^{(i)}\right)^2\,. \tag{6.11}$$

Resampling step :

If $N_{eff}(j) < N_{eff,\,up} = 0.5$:

Resample with replacement N particles of the the inverse subordinator $s_j^{(i)}$, of the the variance process $v_j^{(i)}$ and of the counting process $\Delta n_j^{(i)}$ with importance weights $p_j^{(i)}$. The new importance weights are then set to $p_j^{(i,2)} = \frac{1}{N}$.

If $N_{eff}(j) > N_{eff,\,up} = 0.5$:

No resampling. The new importance weights $p_j^{(i,2)}$ are kept as $p_j^{(i)}$.

The filtered time scale and variance processes for the period j are finally computed as the sum of particles weighted by their probability of occurrence :

$$\widehat{S_j} = \mathbb{E}(S_j \mid x_{i:j}) = \sum_{i=1}^{N} s_j^{(i)} p_j^{(i,2)}$$

$$\widehat{V_j} = \mathbb{E}(V_j \mid x_{i:j}) = \sum_{i=1}^{N} v_j^{(i)} p_j^{(i,2)}$$

end for

this resampling step, we first need to calculate an estimate of the effective sample size $N_{eff}(j)$ with equation (6.11), as initially introduced in Doucet *et al.* (2001). N_{eff} is defined as the equivalent number of independent samples generated directly from the target distribution. The effective sample size may be equal (or close) to N when the observed process is frozen during a long period of time. In this case, the resampling selects exclusively particles with $\Delta s_j^{(i)} = 0$ which causes degeneracy of the algorithm and which may lead to an over-representation of the same particles after a few iterations. To avoid this, we skip the resampling step if N_{eff} is above $N_{eff,up}$ (which is typically chosen close to $0.5N$). When a resampling is done, the importance sampling weights are then set to $1/N$. The filtered time scale for the period j is finally computed as the sum of particles weighted by their probability of occurrence

$$\widehat{S_j} = \mathbb{E}(S_j \mid x_{i:j}) = \sum_{i=1}^{N} s_j^{(i)} p_j^{(i,2)} \tag{6.12}$$

where $p_j^{(i,2)} = 1/N$ in case of resampling and $p_j^{(i,2)} = p_j^{(i)}$ if no resampling is performed. Similarly, the filtered variance process for the period j is computed as

$$\widehat{V_j} = \mathbb{E}(V_j \mid x_{i:j}) = \sum_{i=1}^{N} v_j^{(i)} p_j^{(i,2)} \,,$$

where $p_j^{(i,2)}$ is defined as in (6.12). Following Doucet *et al.* (2001), the log-likelihood of the whole sample path is approached before resampling by the sum

$$\begin{aligned} \ln f(x \mid \vartheta) = \sum_{j=1}^{n} \ln f(x_j | x_{j-1}) &= \sum_{j=1}^{n} \ln \left(\int f(x_j | u_j) f(u_j | x_{j-1}) du_j \right) \\ &= \sum_{j=1}^{n} \ln \left(\sum_{i=1}^{N} f(x_j \mid u_j^{(i)})\, p_{j-1}^{(i,2)} \right). \end{aligned} \tag{6.13}$$

So as to illustrate this section on filtering, we have simulated the sample path of A_{S_t}. Under the assumption that

$$\begin{aligned} \alpha = 0.8,\ \mu = 0.05,\ \widetilde{\kappa} = 0.8,\ \widetilde{\gamma} = 0.15,\ \sigma = 0.9, \\ \rho = -0.6,\ \mu_J = -0.2,\ \sigma_J = 0.3,\ \lambda_J = 0.9\,, \end{aligned} \tag{6.14}$$

we consider 1260 observations of A_{S_t} over five years of trading (time step $\Delta t = 1/252$). The particle filter runs with 10 000 particles. Figure 6.1 compares the sample path of S_t and V_{S_t} to their filtered estimate obtained via Algorithm 1. This confirms the ability of the filter to estimate the sample paths of the stochastic clock and the subordinated variance process based on the observed log-returns in illiquid markets.

This filtering technique based on a boostrap SIR filter has already been successfully implemented in finance, cfr Johannes and Polson (2010) or Dupret and Hainaut (2021). Other filters exist for filtering SVJ models, such as the Auxiliary Particle Filtering (APF) which is implemented in Johannes *et al.* (2009). However, for the Bayesian estimation of SVJ models, Jacobs and Liu (2018) find that the APF yields more variable likelihood evaluations and is therefore less suitable than the bootstrap filter.

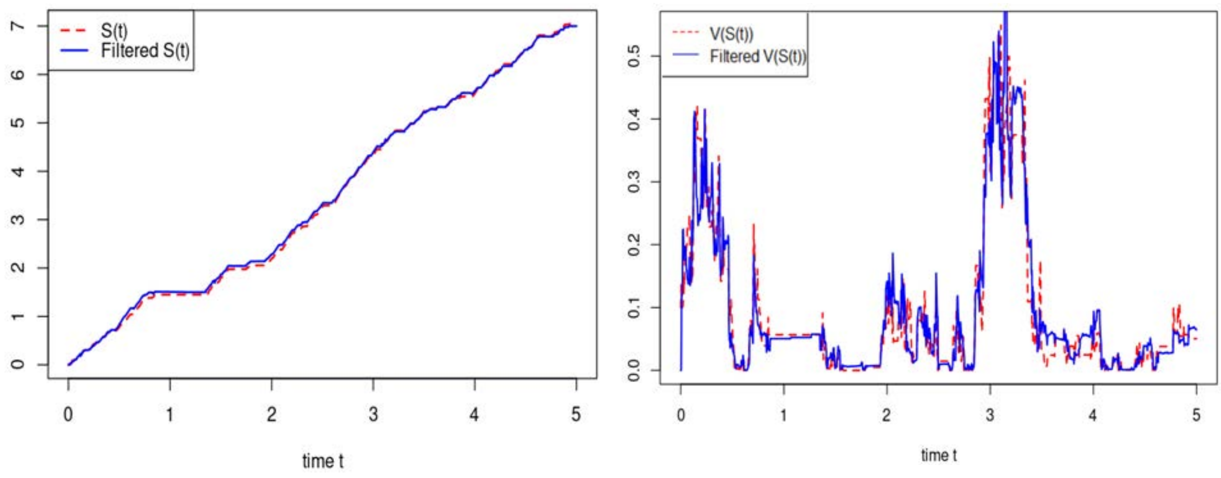


Figure 6.1: Filtered inverse subordinator S_t (left plot) and filtered variance process V_{S_t} (right plot).

6.2. *Econometric estimation*

We have so far considered that the set of parameters ϑ is known, which is not the case in practice. An inherent problem of particle filters is that the estimate of the likelihood is not a smooth function of parameters. Practically, the calibration of a model by maximizing the log-likelihood (*e.g.* with a gradient descent) is then uncertain. We here prefer using the PMCMC procedure which has been shown by Johannes and Polson (2010) to be highly efficient for such financial estimation of stochastic volatility models. We now denote the set of unknown parameters by $\Theta = (\alpha, \mu, \widetilde{\kappa}, \widetilde{\gamma}, \sigma, \rho, \mu_J, \sigma_J, \lambda_J)$. We here again adopt a Bayesian approach and Θ is a multivariate random variable with realizations ϑ and with state space χ. The parameters posterior distribution, based on the observed sample $x = \{x_1, \cdots, x_n\}$ is

$$f(\vartheta \mid x) = \frac{f(\vartheta) f(x \mid \vartheta)}{\int_\chi f(\vartheta') f(x \mid \vartheta')\, d\vartheta'}\,, \tag{6.15}$$

where $f(\vartheta)$ and $f(x \mid \vartheta)$ denotes respectively the parameter prior distribution and the likelihood of the data. The density $f(\vartheta \mid x)$ is built by the PMCMC method described in Appendix 9.2.

We first use this PMCMC algorithm on the simulated sample path A_{S_t} above to see if we can obtain a consistent estimation of the true parameters (6.14) used to generated this sample path. The standard deviation σ_ϑ of the transition distribution is set to

$$\sigma_\vartheta = (0.1\%, 0.5\%, 0.6\%, 0.1\%, 0.6\%, 0.5\%, 0.2\%, 0.1\%, 0.2\%)$$

Even if the procedure converges theoretically to parameter estimates, it is advisable to provide realistic initial parameters to improve the speed of convergence. Therefore, we impose $\vartheta_0 = (0.7, 0.01, 0.2, 0.1, 0.1, -0.2, 0, 0.1, 0.7)$. The filter runs with $N = 10\,000$ particles and we perform $K = 1000$ iterations of the PMCMC procedure. We hence obtain Table 6.1 with a burn-in period of 500 iterations. The estimated parameters are quite close to the true ones (6.14), except for κ which is typically known to have a higher standard deviation in such SVJ model and hence to be harder to estimate from the PMCMC procedure (cfr Table 6.1 and Dupret and Hainaut (2021)). Moreover, the obtained log-likelihood (6.13) is here equal to 15 141.73, which is very close to the one derived with the true parameters equal to 15 142.12. This confirms the overall good quality of

Table 6.1: Parameter estimates of the subdiffusive SVJ model based on the simulated sample path of A_{S_t} from the parameters (6.14). The second line of the table gives the standard error of these estimates.

	$\widehat{\alpha}$	$\widehat{\mu}$	$\widehat{\widehat{\kappa}}$	$\widehat{\widehat{\gamma}}$	$\widehat{\sigma}$	$\widehat{\rho}$	$\widehat{\mu}_J$	$\widehat{\sigma}_J$	$\widehat{\lambda}_J$
Estimators PMCMC	0.782	0.051	0.495	0.194	0.862	−0.531	−0.283	0.282	1.087
PMCMC std. err.	0.041	0.017	0.173	0.040	0.077	0.115	0.029	0.018	0.044

the fit when using this PMCMC procedure with boostrap filter.

We now use the PMCMC algorithm to calibrate the set of parameters Θ of the subdiffusive SVJ model (4.1)-(4.2), based on the observed time series of log-returns for the ALQP stock price between April 2016 and November 2021. The same settings $(\sigma_\vartheta, \theta_0, N, K)$ as in the simulated example above are used in the algorithm and we find in Table 6.2 the estimated parameters with their associated standard error, leading to a log-likelihood of 25 059.34

Table 6.2: Parameter estimates of the subdiffusive SVJ model for the ALQP stock price and standard error of these estimates.

	$\widehat{\alpha}$	$\widehat{\mu}$	$\widehat{\widehat{\kappa}}$	$\widehat{\widehat{\gamma}}$	$\widehat{\sigma}$	$\widehat{\rho}$	$\widehat{\mu}_J$	$\widehat{\sigma}_J$	$\widehat{\lambda}_J$
Estimators PMCMC	0.581	−0.021	0.237	0.050	0.309	−0.406	−0.056	0.037	0.947
PMCMC std. err.	0.0128	0.0191	0.1272	0.0239	0.0183	0.0659	0.021	0.031	0.076

7. Risk management, option pricing and hedging

Recall that the estimation procedure was performed using an α-stable subordinator $(U_t)_{t\geq 0}$. The Lévy triplet of U_t is in this case $(0,0,\nu)$ where $\nu(dz) = \frac{\alpha}{\Gamma(1-\alpha)}\frac{dz}{z^{1+\alpha}}$ and hence satisfies Lemma 3.1. The tail of the Lévy measure is given by

$$\bar{\nu}(s)ds = ds\int_s^\infty \frac{\alpha z^{-\alpha-1}}{\Gamma(1-\alpha)}dz = \frac{s^{-\alpha}}{\Gamma(1-\alpha)}ds\,.$$

The D-C derivative becomes in this case the Caputo fractional derivative, denoted ${}_uD^\alpha h(t)$. Indeed, if we perform the change of variable $s' = t-s$, we infer via equation (3.6) that

$$\int_0^{t-u} \frac{\partial}{\partial t}h(t-s)\bar{\nu}(s)ds = \int_u^t \frac{\partial}{\partial s'}h(s')\,\bar{\nu}\left(t-s'\right)ds'\,,$$

and thus, we obtain the following Caputo fractional derivative as defined in Ishteva *et al.* (2005) :

$$^f\mathcal{D}_t^{(u,+\infty)}h(t) = {}_uD^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)}\int_u^t \frac{h'(s)}{(t-s)^\alpha}ds\,. \tag{7.1}$$

In the case of α-stable subordinators, the general fractional forward PDE (4.13) of the Fourier-Laplace transform of the pdf $p_S(t,x,v,\tau\,|,)$ associated with the triplet $(X_{S_t}, V_{S_t}, S_t - S_s)$ hence becomes the following fractional Caputo PDE for $t \geq u \geq s \geq 0$:

$$\begin{aligned} {}_uD^\alpha\phi_S(t,z_1,z_2,z_3\,|.) = &-\left(z_3 + \kappa\gamma z_2 + iz_1(r-\lambda_J m_J) - \lambda_J\left(\widehat{f}_J(z_1)-1\right)\right)\phi_S(t,z_1,z_2,z_3\,|.) \\ &+\left(-z_2(\kappa+i\rho\sigma z_1) - \frac{\sigma^2 z_2^2}{2} + \frac{z_1^2 - iz_1}{2}\right)\frac{\partial\phi_S(t,z_1,z_2,z_3\,|.)}{\partial z_2}\,, \end{aligned} \tag{7.2}$$

with the same initial condition $\phi_S(u, z_1, z_2, z_3\,|.) = e^{-iz_1 x_s - z_2 v_s}$ and boundary condition $\phi_S(t, 0, 0, 0\,|\,.) = 1$. Since we consider a Gaussian density for modeling jump size in our SVJ model, we obtain the following Fourier transform of the normal density

$$\widehat{f}_J(z_1) = \int_{-\infty}^{+\infty} e^{-iz_1 j} f_J(j)dj = e^{-i\mu_J z_1 - \frac{1}{2}z_1^2\sigma_J^2}\,, \text{ where } f_J(j) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{j-\mu_J}{\sigma_J}\right)^2}.$$

Building from the results of Hainaut (2020), we now propose a numerical method for solving the forward Caputo PDE (7.2). Since for risk management and pricing[2] we are only interested in approximating the Fourier-Laplace transform $\phi_S(t, z_1, z_2, z_3\,|.)$ at $z_2 = 0$, our numerical method is based on a series expansion of $\phi_S(t, z_1, z_2, z_3\,|.)$ in the neighborhood of $z_2 = 0$. This approximation is standard in physics for the non-fractional case, see *e.g.* Bildik *et al.* (2006). Moreover, we know from Usero (2008) that any continuous differentiable function $l(t, z)$ with respect to time t and variable z may be expressed as an infinite sum

$$l(t, z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} \left[\left(\frac{\partial}{\partial t^\alpha}\right)^k l(t, z)\right]_{t=t_0} (t - t_0)^{k\alpha}\,. \tag{7.3}$$

We understand $\left(\frac{\partial}{\partial t^\alpha}\right)^k$ as the k^{th} order Caputo derivative. Moreover, if we assume that $l(t, z)$ is the product of two $\mathcal{C}^\infty$ functions $f(t)$ and $g(z)$, a Taylor's development of $g(z)$ around z_0 leads to the following expansion for $l(t, z)$:

$$\begin{aligned} l(t, z) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} \frac{1}{h!} \left[\left(\frac{\partial}{\partial t^\alpha}\right)^k f(t)\right]_{t=t_0} \left[\frac{\partial^h g(z)}{\partial z^h}\right]_{z=z_0} (t - t_0)^{k\alpha} (z - z_0)^h \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} w_\alpha(k, h)\,(t - t_0)^{k\alpha} (z - z_0)^h\,, \end{aligned} \tag{7.4}$$

where $(w_\alpha(k, h))_{k,h>0} := \frac{1}{\Gamma(k\alpha+1)}\frac{1}{h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^h} l(t, z)\right]_{t=t_0, z=z_0}$. We here use the notation

$$\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^h} := \left(\frac{\partial}{\partial t^\alpha}\right)^k \frac{\partial^h}{\partial z^h}\,,$$

but the reader must keep in mind that $\left(\frac{\partial}{\partial t^\alpha}\right)^k \neq \frac{\partial^k}{\partial t^{k\alpha}}$ in the case of the Caputo derivative. Then, for fixed values of z_1 and z_3, the Fourier-Laplace transform $\phi_S(t, z_1, z_2, z_3\,|.)$ is a function of $z = z_2$. We rewrite here this transform $\phi_S(t, z_1, z, z_3\,|.)$ to emphasize that we consider it as a function of z_2 and we can then approximate[3] $\phi_S(t, z_1, z, z_3\,|.)$ by the following expansion around $z = 0$:

$$\phi_S(t, z_1, z, z_3\,|s, w, u, x_s, v_s) \approx \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} w_\alpha(k, h) \times z^h (t - u)^{\alpha k}, \tag{7.5}$$

where the weights in this sum for $h > 0$ are obtained as above by

$$w_\alpha(k, h) = \frac{1}{\Gamma(k\alpha + 1)h!} \left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^h} \phi_S(t, z_1, z, z_3\,|.)\right]_{t=u, z=0}. \tag{7.6}$$

The weights are then computed recursively thanks to the following proposition.

[2]cfr Section 7.1 with the computation of the moments of A_{S_t} and Section 7.2 (Second alternative) for option pricing.

[3]The approximation (7.5) is exact if $\phi_S(t, z_1, z, z_3\,|.)$ is the product of one function of t and one function of z.

Proposition 7.1. *The differential weights* (7.6) *verify the following relations*

$$\frac{1}{\Gamma(k\alpha+1)h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}\frac{\partial^{\alpha}\phi_S(t,z_1,z,z_3\,|.)}{\partial t^{\alpha}}\right]_{t=u,z=0}=\frac{\Gamma((k+1)\alpha+1)h!}{\Gamma(k\alpha+1)h!}w_{\alpha}(k+1,h) \tag{7.7}$$

$$\frac{1}{\Gamma(k\alpha+1)h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}z\phi_S(t,z_1,z,z_3\,|.)\right]_{t=u,z=0}=\frac{w_{\alpha}(k,h-1)}{h} \tag{7.8}$$

$$\frac{1}{\Gamma(k\alpha+1)h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}\frac{\partial\phi_S(t,z_1,z,z_3\,|.)}{\partial z}\right]_{t=u,z=0}=(h+1)w_{\alpha}(k,h+1) \tag{7.9}$$

$$\frac{1}{\Gamma(k\alpha+1)h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}z\frac{\partial\phi_S(t,z_1,z,z_3\,|.)}{\partial z}\right]_{t=u,z=0}=w_{\alpha}(k,h) \tag{7.10}$$

$$\frac{1}{\Gamma(k\alpha+1)h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}z^2\frac{\partial\phi_S(t,z_1,z,z_3\,|.)}{\partial z}\right]_{t=u,z=0}=\frac{2w_{\alpha}(k,h-1)}{h} \tag{7.11}$$

Proof. Equations (7.7) and (7.9) follows from the definition of the differential weights. The relation (7.8) directly results from the equality

$$\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}z\phi_S(t,z_1,z,z_3\,|.)=\frac{\partial^{k\alpha+h-1}}{\partial t^{k\alpha}\partial z^{h-1}}\phi_S(t,z_1,z,z_3\,|.)+z\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}\phi_S(t,z_1,z,z_3\,|.)\,,$$

whereas equation (7.10) comes from

$$\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}z\frac{\partial\phi_S(t,z_1,z,z_3\,|.)}{\partial z}=\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}\phi_S(t,z_1,z,z_3\,|.)+z\frac{\partial^{k\alpha+h+1}}{\partial t^{k\alpha}\partial z^{h+1}}\frac{\partial\phi_S(t,z_1,z,z_3\,|.)}{\partial z}$$

and equation (7.11) from

$$\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}z^2\frac{\partial\phi_S(t,z_1,z,z_3\,|.)}{\partial z}=2\frac{\partial^{k\alpha+h-1}}{\partial t^{k\alpha}\partial z^{h-1}}\phi_S(t,z_1,z,z_3\,|.)+z^2\frac{\partial^{k\alpha+h+1}}{\partial t^{k\alpha}\partial z^{h+1}}\frac{\partial\phi_S(t,z_1,z,z_3\,|.)}{\partial z}\,.$$

□

The next proposition hence provides an approximation for the solution of the Fourier-Laplace transform $\phi_S(t,z_1,z,z_3\,|.)$ by truncating the sum (7.5).

Proposition 7.2. *The Fourier-Laplace transform of the triplet* $(X_{S_t},V_{S_t},S_t-S_s)$ *is approximated around* $z=0$ *by the following sum*

$$\phi_S(t,z_1,z,z_3\,|s,w,u,x_s,v_s)\approx\sum_{k=0}^{K}\sum_{h=0}^{H}w_{\alpha}(k,h)\times z^h(t-u)^{\alpha k}, \tag{7.12}$$

where the differential weights $w_{\alpha}(k,h)$ *verify the following recursive equation for* $h>0$

$$\begin{aligned}\frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)}w_{\alpha}(k+1,h)=&-\frac{\kappa\gamma}{h}w_{\alpha}(k,h-1)\\&-\left(z_3+iz_1(r-\lambda_J m_J)-\lambda_J\left(\widehat{f}_J(z_1)-1\right)\right)w_{\alpha}(k,h)-(\kappa+i\rho\sigma z_1)w_{\alpha}(k,h)\\&-\frac{\sigma^2}{h}w_{\alpha}(k,h-1)+\frac{(h+1)(z_1^2-iz_1)}{2}w_{\alpha}(k,h+1)\,.\end{aligned} \tag{7.13}$$

For $h=0$*, the weights are obtained via*

$$\begin{aligned}\frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)}w_{\alpha}(k+1,0)=&-\left(z_3+iz_1(r-\lambda_J m_J)-\lambda_J\left(\widehat{f}_J(z_1)-1\right)\right)w_{\alpha}(k,0)\\&+\frac{(z_1^2-iz_1)}{2}w_{\alpha}(k,1)\,.\end{aligned} \tag{7.14}$$

The initial condition on the weights for initializing the recursion is given by

$$w_\alpha(0,h) = e^{-iz_1x_s}\frac{1}{h}(-v_s)^h\,.$$

Proof. To establish the recursion (7.13) for $h>0$, we apply the differential operator

$$\frac{1}{\Gamma(k\alpha+1)h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^h}\right]_{t=u,z=0}$$

on the forward Caputo PDE (7.2). Next, for $h=0$, we first notice from approximation (7.12) that

$$\phi_S(t,z_1,0,z_3\,|.) \approx \sum_{k=0}^{K}\sum_{h=0}^{H} w_\alpha(k,h)\times 0^h(t-u)^{\alpha k} = \sum_{k=0}^{K} w_\alpha(k,0)\times(t-u)^{\alpha k}\,.$$

Therefore, we see from equation (7.3) that the weight in the case $h=0$ is given by

$$w_\alpha(k,0) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\phi_S(t,z_1,0,z_2\,|.)\right]_{t=u}\,. \tag{7.15}$$

Then, similarly as in Proposition 7.1, we find the following relation

$$\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\frac{\partial\phi_S(t,z_1,0,z_3\,|.)}{\partial z}\right]_{t=u} = w_\alpha(k,1)\,,$$

and applying the operator $\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\right]_{t=u}$ on the following Caputo PDE with $z_2=0$:

$$\begin{aligned}{}_uD^\alpha\phi_S(t,z_1,0,z_3\,|.) = &-\left(z_3+iz_1(r-\lambda_Jm_J)-\lambda_J\left(\widehat{f}_J(z_1)-1\right)\right)\phi_S(t,z_1,0,z_3\,|.)\\ &+\frac{z_1^2-iz_1}{2}\frac{\partial\phi_S(t,z_1,0,z_3\,|.)}{\partial z}\,,\end{aligned}$$

leads to the announced result. Finally, since

$$\phi_S(u,z_1,z,z_3\,|.) \approx \sum_{k=0}^{K}\sum_{h=0}^{H} w_\alpha(k,h)\times z^h\,0^{\alpha k} = \sum_{h=0}^{H} w_\alpha(0,h)z^h\,,$$

and since

$$\phi_S(u,z_1,z,z_3\,|.) = e^{-iz_1x_s-zv_s} = e^{-iz_1x_s}\sum_{h=0}^{\infty}\frac{1}{h!}(-v_s)^h z^h\,,$$

we find $w_\alpha(0,h) = e^{-iz_1x_s}\frac{1}{h!}(-v_s)^h$ for the initial weight of the recursion. □

7.1. Risk management

We first use the series expansion of the Fourier-Laplace transform derived in Proposition 7.2 to compute the moments of the subdiffusive stock price A_{S_T} at time T as well two different risk measures. The moment of order k seen from time s is computed directly thanks to

$$\Phi_S(T,ik,0,0\mid s,w,u,x_s,v_s) = \mathbb{E}\left[e^{kX_{S_T}}\,\middle|\,s,w,u,x_s\right] = \mathbb{E}\left[A_{S_T}^k\,\middle|\,s,w,u,x_s\right].$$

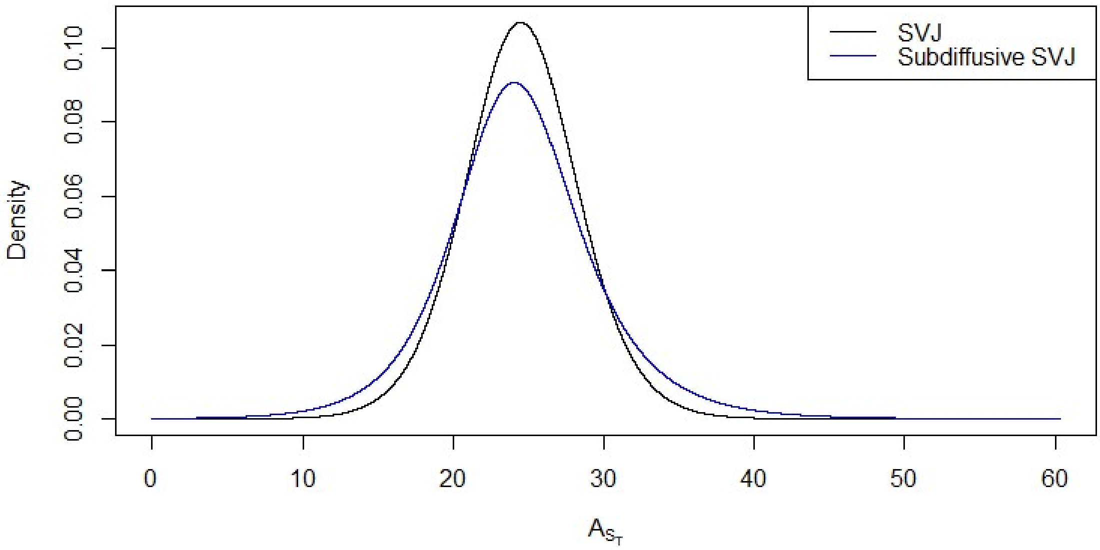
Moreover, the probability distribution of the subdiffusive asset price A_{S_T} can be obtained easily via its Fourier representation. By inverting $\phi_S(T,z_1,0,0\,|.)$, it is indeed possible to determine its density of probability $p_S(T,x,0,0\,|.)$ and then compute numerically the Value-at-Risk (VaR) and Tail Value-at-Risk. The VaR at level $\varepsilon\in(0,1)$ is defined as the shortfall value q such that

$P(A_{S_T} \leq q) = \varepsilon$, while the TVaR at level ε is defined by $\mathbb{E}\left[A_{S_T} | A_{S_T} \geq q\right]$, where q is the VaR at ε. The inversion of the Fourier transform is usually performed accurately and fast using a Discrete Fast Fourier Transform (DFFT) algorithm. Interested readers may refer to Dupret and Hainaut (2021) or Albanese *et al.* (2004) for a review. With the parameters $(\alpha, \mu, \widetilde{\kappa}, \widetilde{\gamma}, \sigma, \rho, \mu_J, \sigma_J, \lambda_J)$ estimated from the PMCMC algorithm in Table 6.2, we obtain in Table 7.1 the moments and risk measures of the ALQP share price A_{S_T}, from $s = 0$ with $T = 0.5$ years and with $x_0 = \ln(24.8)$. A comparison with the classical SVJ model based on the same parameters (and $\alpha = 1$) is also performed and the density of A_{S_T} is depicted under the two models in Figure 7.1.

Table 7.1: The first columns of the table give the first four central moments of A_{S_T} under the classical SVJ model and the subdiffusive SVJ model. The three last columns give the respective Value-at-Risk and Tail Value-at-Risk at levels $\varepsilon = 0.5\%$ and 0.995%.

	Expectation	Variance	Skewness	Kurtosis	VaR ($\varepsilon = 0.5\%$)	VaR ($\varepsilon = 0.995\%$)	TVaR ($\varepsilon = 0.5\%$)
SVJ	24.543	15.568	0.031	3.476	13.855	35.437	12.195
Subdiff. SVJ	24.517	28.959	0.322	5.156	9.254	42.225	6.124

Figure 7.1: Density of the ALQP stock price (A_{S_T}) at $T = 0.5$ with $s = 0$ and $x_0 = \ln(24.8)$ under the classical and the subdiffusive SVJ models.



We clearly see from Table 7.1 and Figure 7.1 that the subdiffusive SVJ model leads to a higher variability of A_{S_T} and to fatter left and right tails, making this model more risky.

7.2. Pricing

We now illustrate how to use our subdiffusive SVJ model to price options in illiquid markets. More precisely, we start by showing how to price a European call with maturity T and strike K whose value at time s under the subdiffusive SVJ model is given by

$$C_S(T,K,A_{S_s},S_s,U_{S_s}) = \mathbb{E}^Q\left[e^{-r(S_T-S_s)}\left(A_{S_T}-K\right)_+\Big|\,S_s,U_{S_s},A_{S_s}\right] \tag{7.16}$$

$$\iff C_S(T,K,A_{S_s},S_s,U_{S_s}) = \mathbb{E}^Q\left[e^{-r(S_T-S_s)}\left(e^{X_{S_T}}-K\right)_+\Big|\,S_s,U_{S_s},e^{X_{S_s}}\right] \tag{7.17}$$

with $X_{S_s} = \ln A_{S_s}$. In our context, conditionally to the information about the stochastic clock $S_s = w$, the subordinator $U_{S_s} = u$ and the stock value $A_{S_s} = e^{x_s}$, we can write

$$C_S(T,K,x_s,w,u) = \mathbb{E}^Q\left[e^{-r(S_T-S_s)}\left(e^{X_{S_T}}-K\right)_+\Big|\,S_s=w,U_{S_s}=u,X_{S_s}=x_s\right]. \tag{7.18}$$

If we know the conditional bivariate pdf of the pair $(X_{S_T}, S_T - S_s)$, which is defined similarly as previously by

$$p_S(T,x,\tau \mid s,w,u,x_s) := \frac{\partial^2}{\partial x \partial \tau} P(X_{S_T} \le x, S_T - S_s \le \tau \mid S_s = w, U_{S_s} = u, X_{S_s} = x_s)\,,$$

we can rewrite the price at time s of the European call by

$$C_S(T,K,x_s,w,u) = \int_0^\infty \int_{-\infty}^\infty e^{-r\tau}\left(e^x - K\right)_+ p_S(T,x,\tau \mid s,w,u,x_s)\, dx\, d\tau\,. \tag{7.19}$$

From this point, we can compute numerically this European call price using several methods. The first one is to use classical Monte-Carlo simulations. However, these simulations are computer-intensive and we hence rather consider the two following alternatives.

- **First Alternative :**

From the independence between $S_T - S_s$ and X_{S_T}, we can rewrite the price (7.19) by

$$\begin{aligned} C_S(T,K,x_s,w,u) &= \int_0^\infty \int_{-\infty}^\infty e^{-r\tau}\left(e^x - K\right)_+ p(\tau,x \mid 0,x_s)\, g(T,\tau \mid s,w,u)\, dx\, d\tau \\ &= \int_0^\infty C(\tau,K,x_s)\, g(T,\tau \mid s,w,u)\, d\tau \end{aligned} \tag{7.20}$$

where $C(\tau,K,x_s)$ is the call price with maturity τ under the classical SVJ model (2.1)-(2.2) given by

$$C(\tau,K,x_s) = \mathbb{E}^Q\left[e^{-r\tau}(A_\tau - K)_+ \big|\, A_0 = e^{x_s}\right]$$

This European call is traditionally computed in a very efficient way using Fourier methods, see Gatheral (2011). This method works well when the conditional density $g(T,\tau\,|.)$ of the inverse subordinator can be obtained easily, which is the case with the α-stable subordinator U_t. Indeed, we have from Hainaut (2021) that the probability density function of $S_T - S_s$, conditionally to $S_s = w$ and $U_{S_s} = u$, can be represented as an infinite sum

$$g(T,\tau \mid s,w,u) = \frac{1}{\alpha\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\Gamma(\alpha k+1)\sin(\pi k\alpha)}{k!}\frac{\tau^{k-1}}{(T-u)^{\alpha k}}. \tag{7.21}$$

However, when the conditional density $g(T,\tau\,|.)$ is hard to derive, we rather use the second alternative.

- **Second Alternative** :

We now consider the call price (7.19) as a function of the log-strike $k := \ln K$. Since this function is not integrable, its Fourier transform is not defined. However, if we fix $\varepsilon > 1$, then the function

$$c_S(k) := e^{\varepsilon k}\, \mathbb{E}^Q \left[e^{-r(S_T - S_s)} \left(e^{X_{S_T}} - e^k \right)_+ \middle| S_s = w, U_{S_s} = u, X_{S_s} = x_s \right]$$

is square integrable and its Fourier transform exists. We denote it by $\widehat{c}(\omega) := \int_{-\infty}^{+\infty} e^{-i\omega k} c(k) dk$. We then have

$$\begin{aligned}
\widehat{c}(\omega) &= \int_{-\infty}^{\infty} e^{(\varepsilon - i\omega)k}\, C_S(T, e^k, x_s, w, u\,|.)\, dk \\
&= \int_{-\infty}^{\infty} e^{(\varepsilon - i\omega)k} \int_0^{+\infty} \int_k^{+\infty} e^{-r\tau}(e^x - e^k)\, p_S(T, x, \tau\,|.)\, dx d\tau dk \\
&= \int_0^{\infty} e^{-r\tau} \int_{-\infty}^{+\infty} p_S(T, x, \tau\,|.) \left(e^x \int_{-\infty}^{x} e^{(\varepsilon - i\omega)k} dk - \int_{-\infty}^{x} e^{(\varepsilon - i\omega + 1)k} dk \right) dx d\tau \\
&= \left(\frac{1}{\varepsilon - i\omega} - \frac{1}{\varepsilon - i\omega + 1} \right) \int_0^{\infty} \int_{-\infty}^{+\infty} e^{-r\tau + (\varepsilon - i\omega + 1)x}\, p_S(T, x, \tau\,|.)\, dx d\tau \\
&= \frac{1}{\varepsilon^2 + \varepsilon - i(2\omega\varepsilon + \omega) - \omega^2}\, \phi_S(T, \omega + (\varepsilon + 1)i, 0, \tau\,|.)\,.
\end{aligned}$$

Then we can invert the Fourier transform to obtain the call price

$$C_S(T, e^k, x_s, w, u\,|.) = \frac{e^{-\varepsilon k}}{\pi} \int_0^{+\infty} e^{i\omega k}\, \frac{\phi_S(T, \omega + (\varepsilon + 1)i, 0, \tau\,|.)}{\varepsilon^2 + \varepsilon - i(2\omega\varepsilon + \omega) - \omega^2}\, d\omega \tag{7.22}$$

The Fourier-Laplace transform $\phi_S(T, \omega + (\varepsilon + 1)i, 0, \tau\,|.)$ is evaluated as above using the approximation of Proposition 7.2 and the inversion of this transform can be performed efficiently using a two-dimensional DFFT algorithm, cfr *e.g.* Hainaut (2021).

Contrary to risk management, pricing is done under the risk-neutral measure Q and we therefore need to find the corresponding parameters κ^* and γ^* under Q using the relation (5.9). Without loss of generality, we consider for this paper no volatility risk premium, *i.e.* $\theta_2 = 0$. For a more refined estimation of θ_2 based on realized volatility measures, we refer to Bollerslev *et al.* (2011). The risk-free rate r is chosen as the EONIA rate, equal to -0.49% in November 2021. Since both of the above alternatives lead to the same price of European calls, we use the first method based on the conditional density $g(T, \tau\,|.)$, which is more direct in the case of α-stable subordinators. We then compare for different maturities T the call price $C_S(T, K, x_0)$ under the subdiffusive SVJ model with the call price $C(T, K, x_0)$ under the classical SVJ model (both at time $s = 0$). We also fix $K = e^{x_0} = 24.8$,which leads to Figure 7.2. We clearly see an increased concavity of the curve of call prices when considering the subdiffusive model. Indeed, the lower the α, the more concave is this curve. Note that this methodology can be extended to price other financial products such as American options.

7.3. Hedging

We end this paper by a brief discussion on the hedging of options in fractional (illiquid) markets. We propose a simple Delta hedging strategy made possible by the fact that we can evaluate the call's price $C_S(T, K, A_{S_t}, S_t, U_{S_t})$ from time $t \geq 0$.

We consider a short position in $C_S(T, K, A_{S_t}, S_t, U_{S_t})$ at time t. We wish to hedge this position with

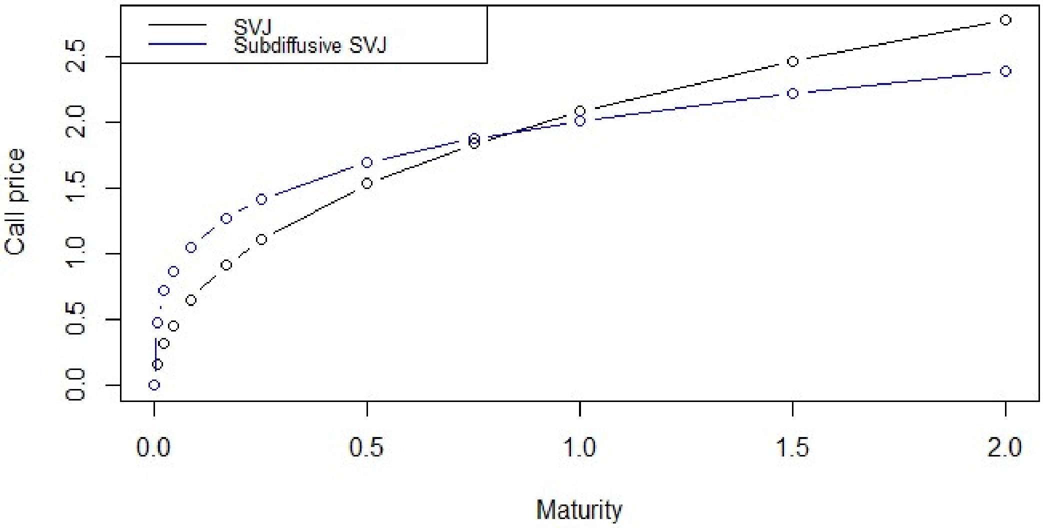


Figure 7.2: Call option prices on ALQP under the classical and the subdiffusive SVJ models for various maturities with $K = 24.8$.

Δ_S shares between t and $t+\delta$, partly funded by borrowing. We denote by $\delta A_{S_t} = A_{S_{t+\delta}} - A_{S_t}$ the variation of stock prices, by $\delta S_t = S_{t+\delta} - S_t$ the increment of the stochastic clock and by $\delta U_{S_t} = U_{S_{t+\delta}} - U_{S_t}$ the increment of the subordinator. The profit and loss (P&L) account over the period $[t, t+\delta]$ then amounts to

$$\begin{aligned} P\&L = &-\Big(C_S(T,K,A_{S_{t+\delta}},S_{t+\delta},U_{S_{t+\delta}}) - C_S(T,K,A_{S_t},S_t,U_{S_t})\Big) \\ &+ rC_S(T,K,A_{S_t},S_t,U_{S_t})\,\delta S_t + \Delta_S\left(\delta A_{S_t} - rA_{S_t}\,\delta S_t\right). \end{aligned}$$

We expand this P&L in powers of δA_{S_t} and δ with a first-order Taylor development of the call price

$$C_S(T,K,A_{S_{t+\delta}},S_{t+\delta},U_{S_{t+\delta}}) \approx C_S(.) + \frac{\partial C_S(.)}{\partial A_{S_t}}\delta A_{S_t} + \frac{\partial C_S(.)}{\partial S_t}\delta S_t + \frac{\partial C_S(.)}{\partial U_{S_t}}\delta U_{S_t}\,.$$

In view of equations (7.1)-(7.2), the call price does not depend explicitly upon S_t (the main driving factor is $U_{S_t} = u$) and hence, $\frac{\partial C_S(.)}{\partial S_t} = 0$. The first order approximation then becomes

$$P\&L \approx -\left(\frac{\partial C_S(.)}{\partial A_{S_t}}\delta A_{S_t} + \frac{\partial C_S(.)}{\partial U_{S_t}}\delta U_{S_t}\right) + rC_S(A_{S_t},S_t,U_{S_t})\,\delta S_t + \Delta_S\left(\delta A_{S_t} - rA_{S_t}\,\delta S_t\right).$$

Choosing $\Delta_S = \frac{\partial C_S(T,K,A_{S_t},S_t,U_{S_t})}{\partial A_{S_t}}$ cancels the first order term in δA_{S_t} and removes at first order the dependence between the $P\&L$ and the underlying asset. Contrary to the B&S case, the first order approximation of the P&L is still exposed the the random fluctuation of the subordinator. The Delta can then be computed numerically via equation (7.20) :

$$\Delta_S = \int_0^\infty \frac{\partial C(\tau,K,A_{S_t})}{\partial A_{S_t}}\, g(T,\tau \mid s,w,u)\, d\tau\,,$$

where $\frac{\partial C(\tau,K,A_{S_t})}{\partial A_{S_t}}$ is the Delta in the non-fractional SVJ model, as obtained in Gatheral (2011).

8. Conclusion

This paper proposes a new procedure for modeling illiquidity in financial markets. More precisely, we introduce the subdiffusive stochastic volatility jump model as a model of illiquid asset prices. It is built from the classical SVJ model that we time-changed by an inverse subordinator S_t, controlling the periods of constant values in the asset price and the long-memory property of the log-returns. This subdiffusive SVJ model allows to reproduce the motionless periods observed in the price of thinly-traded assets as well as the well-known volatility clustering and jump effects. The generated sample paths of this model are clearly more in line with the observed prices in emerging markets and of small capitalization stocks, in comparison with existing subdiffusive models. As the usual model of subdiffusion is developed in terms of FFPE, we show how to derive the FFPE satisfied by the conditional bivariate pdf of the time-changed log-stock price with its inverse subordinator, seen from time s. However, the Fourier-Laplace transform of this pdf also admits a representation in terms of forward fractional PDE, which is far more easy to solve and on which we hence focus throughout the paper. These fractional PDE are valid for a whole class of subordinators with no drift (encompassing the α, Poisson or Gamma subordinators) and they can also be used at any subsequent time $s \geq 0$. This results from the fact that the derivative in the forward fractional PDE is the fractional D-C derivative, which allows for a lot of flexibility. We also show that this model admits a risk-neutral measure Q, leading to an arbitrage-free and incomplete market. The subdiffusive SVJ model is hence suitable for pricing various financial products (such as European or American options) and leads to a very different structure of option prices compared with the classical SVJ model. Moreover, a Delta hedging strategy is also derived in the last section of the paper thanks to the conditional PDE describing the subdiffusive assets' behavior. Finally, the Fourier-Laplace representation of the subdiffusive SVJ model makes it convenient for risk management.

However, if subdiffusions appear as good candidates for modeling illiquidity, options pricing in this framework remains a challenging task for two reasons. First, having no option quotes available in illiquid markets, we are forced to use the observed log-return data in order to estimate the parameters of our model. We hence provide an econometric estimation of the model parameters based on the PMCMC procedure (with boostrap filter). This method works well in our fractional case and gives reliable and consistent results. Secondly, for option pricing and risk management, we first have to solve numerically the forward fractional PDE satisfied by the Fourier-Laplace transform (4.13). We use the series expansion of Proposition 7.2 since this approximation is the most adapted for solving our fractional PDE at $z_2 = 0$. Finally, a DFFT algorithm is applied to find the corresponding pdf associated with the obtained solution of the Fourier-Laplace transform ϕ_S and hence price options on subdiffusive assets. Note that throughout the practical part of the paper, we have considered an α-stable process as subordinator and a Poisson process with normally distributed jump size for the jump process. However, since equation (4.13) is quite general, many other processes can be considered so as to fit the subdiffusive market as best as possible.

Data availability statement The datasets and code generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.
Disclosure statement The authors declare that they have no conflict of interest.

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9. Appendix

9.1. Proof of Proposition 2.1

Let us shorten $p(t, x, v \mid s, x_s, v_s)$ by $p(t, x, v\,|.)$. Being independent from the Brownian motions $(W_{j,t})_{t\geq 0,\, j=1,2}$, we omit in a first step the jump term JdN_t from equation (2.1). The pdf's, $p(t + \Delta t, x, v\,|.)$ at time $t + \Delta t$ and $p(t, x, v\,|.)$ at time t, are then related to the cross moments of X_t and V_t by the bivariate version of the Moyal expansion developed in Hainaut (2020) :

$$p(t+\Delta t, x, v\,|.) - p(t, x, v\,|.) = \sum_{n=1}^{\infty}\sum_{j=0}^{n} \frac{(-1)^n}{j!n-j!}\frac{\partial^j}{\partial x^j}\frac{\partial^{n-j}}{\partial v^{n-j}}\left(M\left(j, n-j \mid x, v\right) p(t, x, v\,|.)\right), \quad (9.1)$$

where $M\left(j, n-j \mid x, v\right)$ is the cross moment of variations given by

$$M\left(j, n-j \mid x, v\right) = \mathbb{E}\left(\left(X_{t+\Delta t} - X_t\right)^j \left(V_{t+\Delta t} - V_t\right)^{n-j} \mid X_t = x, V_t = v\right).$$

For a small step of time Δt, most of terms in the Moyal expansion are of order $\mathcal{O}\left(\Delta t^2\right)$, excepted the following moments

$$\begin{aligned} M\left(1, 0 \mid x, v\right) &= \mathbb{E}\left(\left(X_{t+\Delta t} - X_t\right)^1 \left(V_{t+\Delta t} - V_t\right)^0 \mid X_t = x, V_t = v\right) \\ &= (r - \lambda_J m_J - \frac{1}{2}v)\,\Delta t + \mathcal{O}\left(\Delta t^2\right), \\ M\left(2, 0 \mid x, v\right) &= \mathbb{E}\left(\left(X_{t+\Delta t} - X_t\right)^2 \left(V_{t+\Delta t} - V_t\right)^0 \mid X_t = x, V_t = v\right) \\ &= v\Delta t + \mathcal{O}\left(\Delta t^2\right). \end{aligned}$$

On the other hand,

$$\begin{aligned}
M\left(0,1 \mid x,v\right) &= \mathbb{E}\left(\left(X_{t+\Delta t}-X_t\right)^0\left(V_{t+\Delta t}-V_t\right)^1 \mid X_t=x, V_t=v\right)\\
&= \kappa\left(\gamma-v\right)\Delta t+\mathcal{O}\left(\Delta t^2\right),\\
M\left(0,2 \mid x,v\right) &= \mathbb{E}\left(\left(X_{t+\Delta t}-X_t\right)^0\left(V_{t+\Delta t}-V_t\right)^2 \mid X_t=x, V_t=v\right)\\
&= \sigma^2 v\Delta t+\mathcal{O}\left(\Delta t^2\right).
\end{aligned}$$

The cross moments are

$$\begin{aligned}
M\left(1,1 \mid x,v\right) &= \mathbb{E}\left(\left(X_{t+\Delta t}-X_t\right)^1\left(V_{t+\Delta t}-V_t\right)^1 \mid X_t=x, V_t=v\right)\\
&= \rho\sigma v\Delta t+\mathcal{O}\left(\Delta t^2\right).
\end{aligned}$$

For i and j such that $i+j\geq 3$, we have

$$M\left(i,j \mid x,v\right)=\mathcal{O}\left(\Delta t^{j/2}\right).$$

Injecting these expansions in Equation (9.1) gives us

$$\begin{aligned}
\frac{p(t+\Delta t,x,v\,|.)-p(t,x,v\,|.)}{\Delta t} &= -\frac{\partial}{\partial x}\left[(r-\lambda_J m_J-\frac{1}{2}v)\,p(t,x,v\,|.)\right]-\frac{\partial}{\partial v}[\kappa(\gamma-v)p(t,x,v\,|.)]\\
&+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[v\,p(t,x,v\,|.)\right]+\frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\sigma^2 v\,p(t,x,v\,|.)\right]+\frac{\partial^2}{\partial x\partial v}\left[\rho\sigma v\,p(t,x,v\,|.)\right]+\mathcal{O}(\Delta t).
\end{aligned}$$

Letting Δt tend to zero then leads to the announced result for the continuous part of the SVJ model. We now turn to the jump term JdN_t of equation (2.1). We again consider the interval $[t,t+\Delta t]$ and we assume that the cumulative distribution function (cdf) of the bivariate pdf at time t, denoted $\mathcal{P}(t,x,v\,|.)$, is known. Hence, we have that $\mathcal{P}(t+\Delta t,x,v\,|.)$ can take the following values

$$\mathcal{P}(t+\Delta t,x,v\,|.)=\begin{cases}\lambda_J\,\Delta t\,\int_{\mathbb{R}}\mathcal{P}\left(t,x-j,v\,|.\right)f_J(j)\,dj & \text{if a jump occurs}\\ (1-\lambda_J\,\Delta t)\,\mathcal{P}(t,x,v\,|.) & \text{if no jumps occurs}\end{cases}$$

Hence, we obtain

$$\mathcal{P}(t+\Delta t,x,v\,|.)=\lambda_J\Delta t\int_{\mathbb{R}}\mathcal{P}\left(t,x-j,v\,|.\right)f_J(j)\,dj+(1-\lambda_J\Delta t)\,\mathcal{P}(t,x,v\,|.).$$

Computing the derivative of the cdf with respect to t, we find

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{P}(t,x,v\,|.) &= \lim_{\Delta t\to 0}\frac{\mathcal{P}(t+\Delta t,x,v\,|.)-\mathcal{P}(t,x,v\,|.)}{\Delta t}\\
&= \lambda_J\int_{\mathbb{R}}\mathcal{P}\left(t,x-j,v\,|.\right)f_J(j)\,dj-\lambda_J\,\mathcal{P}(t,x,v\,|.).
\end{aligned}$$

And finally, we have directly from the preceding equation that the pdf of the jump term satisfies the following forward equation

$$\frac{\partial}{\partial t}\,p(t,x,v\,|.)=\lambda_J\int_{\mathbb{R}}p(t,x-j,v\,|.)f_J(j)\,dj-\lambda_J\,p(t,x,v\,|.).$$

□

9.2. PMCMC algorithm

The PMCMC method generates a sample from $f(\vartheta \,|\, x)$ by creating a Markov chain with the same distribution as the parameters posterior one. Once the Markov chain has reached stationarity after a transient phase, called *burn-in* period, samples from the posterior distribution are simulated. Standard MCMC algorithm requires a pointwise estimate of $f(x \,|\, \vartheta)$, which is not available in our model. Instead, $f(x \,|\, \vartheta)$ is approached by its estimate (6.13), yield by the particle filter of the previous section.

The construction of the Markov chain consists of two steps, repeated iteratively for $k = 1, \ldots, K$. At the beginning of the $(k+1)^{th}$ iteration, we propose a candidate parameter $\vartheta^{'}$ from a proposal distribution $q(\vartheta^{'} \,|\, \vartheta_k)$ given the previous state of the Markov chain, ϑ_k. The proposal distribution must have a support that covers the target distribution. In the second step, we determine if we update the state $\vartheta^{'}$. For this purpose, the acceptance probability of the Metropolis-Hasting algorithm described in Algorithm 2, is computed as follows

$$\rho(\vartheta^{'}, \vartheta_k) = \min \left\{ \frac{f(\vartheta^{'} \,|\, x)}{f(\vartheta_k \,|\, x)} \frac{q(\vartheta_k \,|\, \vartheta^{'})}{q(\vartheta^{'} \,|\, \vartheta_k)}, 1 \right\}. \tag{9.2}$$

This determines the probability that we assign the candidate parameter as the next state of the Markov chain, $\vartheta^{'} \to \vartheta_{k+1}$. Intuitively, if we disregard the influence of the proposal distribution $q(\cdot, \cdot)$, a candidate is accepted if it increases the posterior likelihood $f(\vartheta^{'} \,|\, x) > f(\vartheta_k \,|\, x)$. The presence of $q(\cdot, \cdot)$ in (9.2) allows a small decrease in the posterior likelihood, such as to explore the space of parameters χ.

In numerical applications, the transition distribution $q(\vartheta^{'} \,|\, \vartheta_k)$ is assumed Gaussian, $\vartheta^{'} \sim N(\vartheta_k, \sigma_\vartheta\, I_m)$ where I_m is the identity matrix. As this distribution is symmetric, we have that $q(\vartheta_k \,|\, \vartheta^{'}) = q(\vartheta^{'} \,|\, \vartheta_k)$ and the acceptance probability simplifies to

$$\rho(\vartheta^{'}, \vartheta_k) = \min \left\{ \frac{f(x \,|\, \vartheta^{'})}{f(x \,|\, \vartheta_k)} \frac{f(\vartheta^{'})}{f(\vartheta_k)}, 1 \right\} .$$

The resulting samples $\vartheta_{1:K}$ (K iterations after the burn-in period) serves next to build the empirical distribution of $f(\vartheta \,|\, x)$, which is defined by

$$f(\vartheta \,|\, x) = \frac{1}{K} \sum_{k=1}^{K} \delta_{\vartheta_k}(d\vartheta)\,,$$

where $\delta_{\vartheta_k}(d\vartheta)$ are the Dirac atoms located at $\vartheta = \vartheta_k$, with equal weights. The estimate of parameters with respect to the posterior distribution is then

$$\widehat{\Theta} = \mathbb{E}(\Theta \,|\, x) \approx \frac{1}{K} \sum_{k=1}^{K} f(\vartheta_k \,|\, x) \vartheta_k\,. \tag{9.3}$$

Algorithm 2 Metropolis Hastings algorithm.

Main procedure :

for $j = 1$ to n **do**

Simulate $\vartheta' \sim q(\vartheta' \mid \vartheta_t)$

Take

$$\Theta_{t+1} = \begin{cases} \vartheta' & \text{with probability } \rho(\vartheta_t, \vartheta') \\ \vartheta_t & \text{with probability } 1 - \rho(\vartheta_t, \vartheta') \end{cases}$$

where $\rho(\vartheta_t, \vartheta')$ is the acceptance probability

$$\rho(\vartheta_t, \vartheta') = \min \left\{ \frac{f(\vartheta' \mid x)}{f(\vartheta_k \mid x)} \frac{q(\vartheta_k \mid \vartheta')}{q(\vartheta' \mid \vartheta_k)}, 1 \right\}$$

end for