



Desingularization of a Steady Vortex Pair in the Lake Equation

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Abstract

We construct a family of steady solutions of the lake model perturbed by some small Coriolis force, that converge to a singular vortex pair. The desingularized solutions are obtained by maximization of the kinetic energy over a class of rearrangements of sign changing functions. The precise localization of the asymptotic singular vortex pair is proved to depend on the depth function and the Coriolis parameter, and it is independent on the geometry of the lake domain. We apply our result to construct a singular rotating vortex pair in a rotation invariant lake.

Keywords Lake model · Rearrangements · Vortex pair · Singular vortex · Desingularization · Asymptotic

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1 Introduction

Statement of the problem The lake equations arise from the incompressible 3D Euler equations in a regime where the typical velocity magnitude is small in comparison to the magnitude of gravity waves (small Froude number regime $\text{Fr} \ll 1$, see also [10]). Mathematically, a lake can be modeled as a planar open set $D \subseteq \mathbb{R}^2$ together with a positive depth function b . The *velocity field* $v : \mathbb{R} \times D \rightarrow \mathbb{R}^2$ and the *pressure field* $p : \mathbb{R} \times D \rightarrow \mathbb{R}$ are governed by the system of equations

$$\begin{cases} \operatorname{div}(bv) = 0 & \text{on } \mathbb{R} \times D, \\ \partial_t v + (v \cdot \nabla)v + f v^\perp = -\nabla p & \text{on } \mathbb{R} \times D, \\ bv \cdot \hat{\eta} = v & \text{on } \mathbb{R} \times \partial D. \end{cases}$$

Here $\hat{\eta} : \partial D \rightarrow \mathbb{R}^2$ is the unit outward normal to ∂D , $(v_1, v_2)^\perp = (-v_2, v_1)$, $f : D \rightarrow \mathbb{R}$ is a Coriolis parameter, and $v : \partial D \rightarrow \mathbb{R}$ is a penetration condition. When the depth

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function is constant, $b \equiv 1$, the system reduces to the incompressible 2D Euler equations. The velocity field v in the lake model may be understood as the horizontal velocity of a column water, whose total mass may vary according to the depth of the lake [10]. Global well-posedness of the lake equations has been studied by Levermore & Oliver & Titi [18], Lacave & Nguyen & Pausader [19], Munteanu [24], Huang & Chaocheng [15].

The problem we study in this article is the asymptotic behavior of steady velocity fields that are constructed as maximizers of the kinetic energy

$$E(v_\epsilon) = \frac{1}{2} \int_D |v_\epsilon|^2 d\mu, \quad d\mu = b dm = b(x) dx,$$

in the regime where the vorticity shrinks to a Dirac mass.

Let us motivate the problem we address by first recalling what is known in the easiest case where the Coriolis force is null: $f = 0$; and the topography does not vary: $b \equiv 1$. This situation is known as the 2D Euler's equations. Introducing the vorticity $\omega = \text{curl}(v)$ (and up to prescribe circulation conditions on ∂D), one may think of v as $v = \text{curl}^{-1}(\omega)$, which allows one to think off the energy as a function of ω only:

$$E(v) \equiv E(\omega).$$

Moreover, taking the curl of the evolution equation, we get the transport equation

$$\partial_t \omega + (v|\nabla \omega) = 0.$$

These observations were the starting point for Arnold [1] and Benjamin [3] to suggest that any steady solution of the 2D Euler equations should be a critical point of the energy under the following constraint: if ω solves the steady transport equation: $(\text{curl}^{-1}(\omega)|\nabla \omega)_{\mathbb{R}^2} = 0$; then for every field of Lebesgue-measure preserving diffeomorphisms field $D : \mathbb{R} \rightarrow \text{Diff}(D)$ such that $D(0) = \text{Id}_D$, the energy functional

$$E : \mathbb{R} \rightarrow \mathbb{R} : E(t) := E(\omega \circ D_t^{-1})$$

has a critical point at $t = 0$. On the other hand, any vortex constructed as $\omega_t = \omega \circ D_t^{-1}$ satisfies the infinite number of constraints:

$$\text{dm}(\{\omega_t \geq \lambda\}) = \text{dm}(\{\omega \geq \lambda\}), \quad \forall \lambda \in \mathbb{R}.$$

We say that such a field ω_t is a *dm-rearrangement* of ω . The notion of rearrangement is a relaxed condition in comparison to the identity $\omega_t = \omega \circ D_t^{-1}$, since the regularity of the transformation D becomes out of concern. In other words, the diffeomorphic-assertion on the measure-preserving transform of D may be been lost a priori. From this motivation, the following energy maximization principle has been extensively investigated by Arnold [1], Benjamin [3] and Burton [7–9]: *Find a maximizer of the energy E restricted on the set of all dm-rearrangements of a given profile function ω^* .* The asymptotic of maximizers as $\text{dm}(\{\omega_\epsilon \neq 0\}) = \epsilon^2$, $\epsilon \rightarrow 0$, has been studied by Turkington [28–30], Turkington & Friedman [14], and Elcrat & Miller [31] when ω_ϵ is non sign changing: $\omega_\epsilon \geq 0$. The authors have shown that the family of maximizers $\{\omega_\epsilon : \epsilon > 0\}$ tends to a Dirac mass as $\epsilon \rightarrow 0$, and the concentration point is chosen according to the geometry of ∂D (see also [20, 21, 23, 30] for similar results for the time-dependent problem, and [12, 27] for similar results based on other approaches).

The Case of the Lake Model The situation appears to be different in the lake model, when b is not constant. In fact, Richardson [26] has shown that the motion of the center of mass

$$Z_\epsilon = \frac{1}{\mathcal{S}_\epsilon} \int_D x \zeta_\epsilon(x) \, d\mu(x)$$

of a non sign changing potential vortex ζ_ϵ follows the evolution law

$$\frac{d}{dt} Z_\epsilon(t) = \frac{1}{2\pi} \nabla^\perp b(Z_\epsilon(t)) \log \frac{1}{\epsilon} \mathcal{S}_\epsilon + \mathcal{O}(\mathcal{S}_\epsilon),$$

with $\mathcal{S}_\epsilon = \int_D \zeta_\epsilon \, d\mu$. The above evolution equation has received a recent justification from the author & Van Schaftingen [11].

Following the lines of thought of Arnold and Benjamin, we expect a variational principle of the following form: *Steady solutions with prescribed constraints on the $d\mu$ -measure of their super level sets, can be constructed by maximization of the energy.*

Formal Statement and Results A measurable function $\zeta : D \rightarrow \mathbb{R}$ may be decomposed in a positive part $(\zeta)_+$ and a negative part $(\zeta)_-$, both measurable. Let us denote by $|D|_\mu = d\mu(D) > 0$ the positive $d\mu$ -measure of D , which we assume positive.

Let us fix a distribution function

$$D : \mathbb{R}^+ \rightarrow [0, |D|_\mu]$$

normalized as

$$\int_{\mathbb{R}^+} D(t) \, dt = 1,$$

and such that there exists $p > 1$ with

$$\int_{\mathbb{R}^+} t^p D(t) \, dt < +\infty.$$

We also fix $\tau \in [0, 1]$. We say that a family of measurable functions $\{\zeta_\epsilon : \epsilon > 0\}$ satisfies the constraint (D) if, for all $\epsilon > 0$, we have

$$\begin{aligned} d\mu(\{(\zeta_\epsilon)_+ \geq \lambda\}) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2 \lambda}{\delta \tau} \log \frac{1}{\epsilon}\right), \\ d\mu(\{(\zeta_\epsilon)_- \geq \lambda\}) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2 \lambda}{\delta(1-\tau)} \log \frac{1}{\epsilon}\right), \end{aligned} \tag{D}$$

where $\delta = \sup_{\lambda > 0} D(\lambda)$. Note that under assumption (D), we always have $d\mu(\{(\zeta_\epsilon)_+ > 0\}) \leq \epsilon^2$ and $d\mu(\{(\zeta_\epsilon)_- > 0\}) \leq \epsilon^2$. It is also easy to check that $\zeta_\epsilon \in L^p(D, d\mu)$ and

$$\int_D (\zeta_\epsilon)_+ \, d\mu = \frac{\tau}{\log \frac{1}{\epsilon}}, \quad \text{and} \quad \int_D (\zeta_\epsilon)_- \, d\mu = \frac{1-\tau}{\log \frac{1}{\epsilon}}.$$

The above growth factor $\log \frac{1}{\epsilon}$ is motivated by Richardson's formula (see also [11] for an interpretation based on a time-scaling process). The energy E_ϵ we maximize is given by

$$E(\zeta) = \frac{1}{2} \int_D (\mathfrak{K}\zeta + \Psi)\zeta \, d\mu,$$

where $flow$ is the flow that induces the Coriolis vorticity f : thus $\text{curl}(b^{-1} \nabla^\perp \Psi) = bf$; and $\mathfrak{K}\zeta$ is the flow that induces the "internal vorticity": $\text{curl}(b^{-1} \nabla^\perp \mathfrak{K}\zeta) = b\zeta$. We prove the following theorem:

Theorem 1.1. Assume that $D \subseteq \mathbb{R}^2$ is a (simply connected) bounded domain of class C^1 and assume that $b \in C(\overline{D}) \cap W_{loc}^{1,\infty}(D)$ with $\inf_D b > 0$, or $b = \phi^\alpha$ for some $\alpha > 0$ and ϕ a regularization of the distance at the boundary ∂D . Let $\Psi \in C(\overline{D}) \cap W_{loc}^{1,\infty}(D)$. There exists $\lambda_0 > 0$ depending only on τ , Ψ and b such that, for all $\lambda \in (0, \lambda_0)$, there exists a family $\{\zeta_\epsilon \in L^p(D, d\mu) : \epsilon > 0\}$ of steady solutions for the lake equations, obtained by maximization of the energy E over the class of all functions that satisfy constraint (D).

Furthermore, if $\tau > 0$, the only possible accumulation points of $\{(\zeta_\epsilon)_+ : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses δ_{x^*} with

$$\frac{\tau b(x^*)}{4\pi} + \lambda \Psi(x^*) = \sup_D \left\{ \frac{\tau b}{4\pi} + \lambda \Psi \right\},$$

and if $\tau < 1$, the only possible accumulation points of $\{(\zeta_\epsilon)_- : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses δ_{x_*} with

$$\frac{(1-\tau)b(x_*)}{4\pi} - \lambda \Psi(x_*) = \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \lambda \Psi \right\}.$$

We briefly mention that Theorem 1.1 will actually be proved for non necessarily simply connected domains, provided coherent circulation conditions are prescribed. Also, Theorem 1.1 will be proved under slightly more general regularity assumptions on D and b , and covers two situations encountered in the literature. It covers the case of a non vanishing depth function b [15, 18], and the case of a degenerated depth function that vanishes as a polynomial of some regularized distance at the boundary [4, 19]. Although Theorem 1.1 is stated for regular domains, we will be in position to deal with irregular domains as well.

In the case of a non perturbed lake ($\Psi = 0$), our energy maximizers look like singular vortex pairs as $\epsilon \rightarrow 0$, both located near a point of maximal depth b . If we add some small perturbation $\lambda \Psi$, the two parts of the pairs separate from each others, according to Ψ . The condition that the *vortex strength*

$$\int_D \zeta_\epsilon d\mu = (2\tau - 1) \frac{1}{\log \frac{1}{\epsilon}}$$

vanishes as $(\log \frac{1}{\epsilon})^{-1}$ is expected from Richardson's law. In the lake equations, this scale length is the typical scale length that does not asymptotically shrinks time (see also [11]).

Taking advantage of radial symmetries, Theorem 1.1 may be used to prove the following time-dependent result:

Theorem 1.2. Assume that $D = B(0, 1)$ and $b \in C(\overline{D}) \cap W_{loc}^{1,\infty}(D)$, $b(x) = b(|x|^2)$, is a radial function with $\inf_D b > 0$, or $b = \phi^\alpha$ for some $\alpha > 0$ and ϕ a regularization of the distance at the boundary $\partial B(0, 1)$. There exists $v_0 > 0$ depending only on $\tau \in [0, 1]$ such that, for all $v \in (0, v_0)$, there exists a family of curves $\{\zeta_\epsilon \in C(\mathbb{R}, L^p(D, d\mu)) : \epsilon > 0\}$ solving the time-dependent lake equations with the additional property that for each $t \in \mathbb{R}$, the vortex $\zeta_\epsilon(t)$ at time t is obtained from $\zeta_\epsilon(0)$ by a rotation of clockwise angle vt , and satisfies constraint (D).

Furthermore, if $\tau > 0$, the only possible accumulation points of $\{(\zeta_\epsilon(0))_+ : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses δ_{x^*} with

$$\frac{\tau b(x^*)}{4\pi} - \frac{v}{2} \int_0^{|x^*|^2} b(s) ds = \sup_{z \in D} \left\{ \frac{\tau b(z)}{4\pi} - \frac{v}{2} \int_0^{|z|^2} b(s) ds \right\},$$

and if $\tau < 1$, the only possible accumulation points of $\{(\zeta_\epsilon)_- : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses $\delta_{x_\star}$ with

$$\frac{(1-\tau)b(x_\star)}{4\pi} + \frac{\nu}{2} \int_0^{|x^\star|^2} b(s) \, ds = \sup_{z \in D} \left\{ \frac{(1-\tau)b(z)}{4\pi} + \frac{\nu}{2} \int_0^{|z|^2} b(s) \, ds \right\}.$$

Method and Insights of the Proof The condition $\operatorname{div}(bv) = 0$ together with $\operatorname{curl}(v) = b\zeta$ motivates to define $v = b^{-1}\nabla^\perp\psi$ with ψ such that

$$-\operatorname{div}(b^{-1}\nabla\psi) = b\zeta.$$

This is an elliptic equation with source term $b\zeta$, which may be solved on D , provided circulation conditions are prescribed. We emphasize that $b\zeta$ is exactly the physical vortex ω , but we are going to use the longer notation $b\zeta$ to stress relations with respect to the invariant measure $d\mu$. The function framework for the above elliptic problem is discussed in the *first section*. We prove that the solution ψ constructed above depends on ζ through an operator $\mathfrak{K}$. In [Appendix](#), we prove an integral expansion

$$\mathfrak{K}\zeta(x) = \int_D G_D(x, y)\zeta(y) \, d\mu(y) + \int_D F(x, y)\zeta(y) \, d\mu(y),$$

where G_D is the Green's function for $-\Delta$ in D with Dirichlet boundary conditions, and $F : D \times D \rightarrow \mathbb{R}$ is a bounded and continuous function. Such Green's function expansion is not known for degenerate elliptic equations (see [\[16, 22, 33\]](#) for existence of Green's function and related estimates in the case where b is non degenerated). We prove this expansion by regularity theory [\[32\]](#), under suitable assumptions on both D and b . We cover the case of non regular domains D with $b \in C(\overline{D}) \cap W_{\text{loc}}^{1,\infty}(D)$ satisfying $\inf_D b > 0$, or b satisfying the ad hoc condition

$$\inf_{\substack{q \in [1, +\infty) \\ \beta \in [0, 2)}} \limsup_{d(x, \partial D)} \left\{ \frac{1}{(\delta_q(x))^\beta} \frac{|\nabla b(x)|^2}{b(x)} \right\} < +\infty;$$

or in other words: there exists $\beta \in [0, 2)$, $q \geq 1$ and a neighborhood $\mathcal{U}$ of ∂D such that, for all $x \in \mathcal{U}$:

$$\frac{1}{(\delta_q(x))^\beta} \frac{|\nabla b(x)|^2}{b(x)} \leq C$$

for some $C > 0$. Here δ_q is the mean distance at the boundary of D of order q , as defined in [\[2\]](#); and it is well defined even for non regular domains. In comparison with [\[24\]](#), we do not rely on a Muckenhoupt condition. In a *second section*, we recall some preliminaries in rearrangements theory, and we use the work of Burton [\[6–9\]](#) to construct maximizers of the energy that are also solutions of the lake equations. In [Section 3](#), we exploit our integral representation to study asymptotic properties of energy maximizers as the vortex profile vanishes. The strategy of proof is to compare the maximal energy with the energy produces by very symmetric competitors, as one would do for the Euler equations. However, since the relevant measure is $d\mu(x) = b(x) \, dx$, we have to face some technical difficulties. We manage to prove a generalization of [Theorem 1.1](#). The proof of [Theorem 1.2](#) is done in [Section 4](#).

2 The Stream Function

2.1 Lakes

By a *lake* we mean a couple (D, b) where $D \subseteq \mathbb{R}^2$ is a bounded open set and $b \in C(\overline{D})$ is a non negative continuous function. We ask the following additional conditions:

- (1) $D = \overline{D_0} \setminus \bigcup_{i=0}^m \mathcal{C}_i$ for a bounded connected open set $D_0 \subseteq \mathbb{R}^2$ and disjoint compact simply connected subsets $\mathcal{C}_i \subset \overline{D_0}$, $i > 0$, and the convention that $\mathcal{C}_0 = \partial D_0$;
- (2) for all $i = 0, \dots, m$, we have either $\text{cap}(\mathcal{C}_i) > 0$, or b admits a Dini continuous extension by 0 on $D \cup \mathcal{C}_i$;
- (3) for all compact subset $K \subset D$, we have $\inf_K b > 0$.

In the above definition, each $\partial \mathcal{C}_i$ represents a shore. We allow $\mathcal{C}_i$ to be the boundary of a closed connected open set (in which case we represent an island), $\mathcal{C}_i$ a (non closed) curve, but we also allow $\mathcal{C}_i$ to be reduced to a point shore, provided the depth function b is sufficiently regular in a neighborhood of $\mathcal{C}_i$.

2.2 Basic Functional Framework

In this section we introduce a functional framework for the stream function. We write $W^{1,2}(D)$ the standard Sobolev space of square integrable functions on D whose distributional derivatives exist as square integrable functions on D , endowed with its usual scalar product

$$(\phi|\psi)_{W^{1,2}} = \int_D \phi \psi \, \text{dm} + \int_D (\nabla \phi | \nabla \psi)_{\mathbb{R}^2} \, \text{dm}.$$

Here dm denotes the Lebesgue measure on D . We denote by $\mathcal{H}$ the set of those functions $\phi \in W^{1,2}(D)$ such that $b^{-1}|\nabla \phi|^2 \in L^1(D, \text{dm})$, endowed with the scalar product

$$(\phi|\psi)_{\mathcal{H}} = (\phi|\psi)_{W^{1,2}} + \int_D (\nabla \phi | \nabla \psi)_{\mathbb{R}^2} \frac{\text{dm}}{b}.$$

Since b is assumed to be positive on compact subsets of D , we have the set inclusion $C_c^1(D) \subseteq \mathcal{H}$. We denote by $\mathcal{H}_0 = \overline{C_c^1(D)}$ the closure of $C_c^1(D)$ in $\mathcal{H}$.

Proposition 2.1. *The quadratic form*

$$(\phi, \psi) \in \mathcal{H}_0 \mapsto (\phi|\psi)_{\mathcal{H}_0} = \int_D (\nabla \phi | \nabla \psi)_{\mathbb{R}^2} \frac{\text{dm}}{b}$$

defines a scalar product on $\mathcal{H}_0$ whose induced norm is equivalent to the norm induced by $\mathcal{H}$.

Proof The fact that it defines a scalar product is straightforward. For the equivalence between the norms, one first observe that

$$\|\phi\|_{\mathcal{H}_0}^2 \leq \|\phi\|_{\mathcal{H}}^2$$

for all $\phi \in \mathcal{H}_0$. For a converse inequality, one may write

$$\|\phi\|_{\mathcal{H}_0}^2 \geq \frac{1}{2} \|\phi\|_{\mathcal{H}_0}^2 + \frac{1}{2(\sup_D b)} \int_D |\nabla \phi|^2 \, \text{dm}.$$

Since D is a bounded set, we may apply the standard Poincaré's inequality to obtain some constant $C_2 > 0$ such that

$$C_2 \int_D |\nabla \phi|^2 \, dm \geq \|\phi\|_{W^{1,2}}^2,$$

and thus we have

$$\|\phi\|_{\mathcal{H}_0}^2 \geq C_3 (\|\phi\|_{\mathcal{H}_0}^2 + \|\phi\|_{W^{1,2}}^2) = C_3 \|\phi\|_{\mathcal{H}}^2. \quad \square$$

2.3 Circulations

In this section we show how we are going to solve a problem of the form

$$\begin{cases} -\operatorname{div}(b^{-1} \nabla \phi) = 0, \\ \oint_{\partial \mathcal{C}_i} (b^{-1} \nabla^\perp \phi | \tau_i)_{\mathbb{R}^2} = c_i. \end{cases} \quad (\text{C})$$

Here above, the real numbers $c_0, \dots, c_m \in \mathbb{R}$ are fixed, and τ_i denotes the tangent vector field associated with $\mathcal{C}_i$ with clockwise orientation for $i = 1, \dots, m$, counterclockwise orientation for $\mathcal{C}_0$; in other words: $\tau_i = \hat{\eta}_i^\perp$ where $\hat{\eta}_i$ is the unit normal vector field outward to the boundary of ∂D_i , and $(v_1, v_2)^\perp = (-v_2, v_1)$. Each quantity

$$\oint_{\partial \mathcal{C}_i} (b^{-1} \nabla^\perp \phi | \tau_i)_{\mathbb{R}^2}$$

represents the *circulation of the velocity field* $b^{-1} \nabla^\perp \phi$ along $\partial \mathcal{C}_i$. The first equation

$$\operatorname{curl}(b^{-1} \nabla^\perp \phi) = -\operatorname{div}(b^{-1} \nabla \phi) = 0$$

means that the curl of the vector field $b^{-1} \nabla^\perp \phi$ is null, that is, no vortex is produced by this velocity field. Observe that from the circulation conditions we obtain

$$\begin{aligned} \sum_{i=0}^m \oint_{\partial \mathcal{C}_i} (b^{-1} \nabla^\perp \phi | \tau_i)_{\mathbb{R}^2} &= \sum_{i=0}^m \int_{\partial \mathcal{C}_i} (b^{-1} \nabla \phi | -\eta_i)_{\mathbb{R}^2} \\ &= - \int_{\partial D} (b^{-1} \nabla \phi | \eta)_{\mathbb{R}^2} = \int_D -\operatorname{div}(b^{-1} \nabla \phi), \end{aligned}$$

where η is the unit normal outward vector on D , and η_i its restriction on $\partial \mathcal{C}_i$. For problem (C), this leads to the consistency condition

$$\sum_{i=0}^m c_i = \int_D \operatorname{curl}(b^{-1} \nabla^\perp \phi) \, dm = 0.$$

Since we work in an a priori rough setting (the normal vector τ_i may not be defined), we understand problem (C) in the weak sense. Let us assume that, for all $i = 0, \dots, m$, we have constructed a function $\psi_i \in \mathcal{H}$ that solves

$$\begin{cases} -\operatorname{div}(b^{-1} \nabla \psi_i) = 0, \\ \psi_i = \delta_{ij} \quad \text{on } \partial \mathcal{C}_j, \forall j \in \{0, \dots, m\}. \end{cases} \quad (\text{T})$$

Here δ_{ij} denotes the Kronecher's symbol. Then we would have

$$c_j = \int_{\partial \mathcal{C}_j} (b^{-1} \nabla \phi | -\eta_j)_{\mathbb{R}^2} = - \int_{\partial D} (\psi_j b^{-1} \nabla \phi | \eta)_{\mathbb{R}^2} = - \int_D \operatorname{div}(\psi_j b^{-1} \nabla \phi),$$

and therefore, if $\phi \in \mathcal{H}$ is a solution of problem (C):

$$c_j = - \int_D (\nabla \psi_j | \nabla \phi)_{\mathbb{R}^2} \frac{dm}{b}.$$

It is therefore natural to look for a solution $\phi \in \mathcal{H}$ of problem (C) as a linear combination

$$\phi = \sum_{i=0}^m \alpha_i \psi_i,$$

where the coefficients $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ should be chosen in such a way that

$$-c_j = \sum_{i=0}^m \alpha_i \int_D (\nabla \psi_j | \nabla \psi_i)_{\mathbb{R}^2} \frac{dm}{b}, \quad \text{for all } j \in \{0, \dots, m\}.$$

Such a family $\psi_1, \dots, \psi_m$ would thus be useful to construct weak solutions of the circulation problem. However, one should take into account that D may be irregular, so that the notion of trace needed to define $\psi_1, \dots, \psi_m$, may be hard to define. In this section, we begin by proving that problem (T) admits a unique solution in a certain weak sense, and this solution is bounded. Once we have proved the existence of the functions $\psi_1, \dots, \psi_m$, we focus on the existence of a solution for our weak formulation of problem (C).

Proposition 2.2. *For all $i \in \{0, \dots, m\}$, there exists a unique function $\psi_i \in \mathcal{H}$ that satisfies*

$$\int_D (\nabla \psi_i | \nabla \phi)_{\mathbb{R}^2} \frac{dm}{b} = 0, \quad \text{for all } \phi \in \mathcal{H}_0,$$

and such that there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of functions from $C^1(\overline{D})$ converging in $\mathcal{H}$ to ψ_i ; with $\varphi_n = \delta_{ij}$ on a neighborhood of $\partial \mathcal{C}_j$, for all $j \in \{0, \dots, m\}$. Moreover, the solution ψ_i satisfies $0 \leq \psi_i \leq 1$ on D .

This kind of existence result is standard [Lemma 4, 18] and it is usually proved by using Schauder's theory. Here we propose an original proof based on purely functional analysis methods, that may be applied for non regular domains.

Proof Let us consider the set

$$\Gamma_i = \{\varphi \in C^1(\overline{D}) : \varphi = \delta_{ij} \text{ on a neighborhood of } \partial \mathcal{C}_j, \text{ for all } j \in \{0, \dots, m\}\}.$$

We are going to work in the quotient space $H = \mathcal{H} / \sim$, where $\sim$ is the equivalence relation

$$\phi_1 \sim \phi_2 \iff \int_D |\nabla(\phi_1 - \phi_2)|^2 \frac{dm}{b} = 0.$$

The equivalence class of a function $\phi \in \mathcal{H}$ is simply denoted by $[\phi]$. The quotient H is then endowed with the well-defined scalar product

$$([\phi_1] | [\phi_2])_H := \int_D (\nabla \phi_1 | \nabla \phi_2)_{\mathbb{R}^2} \frac{dm}{b}.$$

We claim that the set

$$\Pi_i = \{[\phi] : \phi \in \overline{\Gamma_i}\}$$

is a complete and convex subset of H . The set Γ_i itself is convex, and so is its closure $\overline{\Gamma_i}$ in $\mathcal{H}$. Thus Π_i is convex as well. For the completeness, the key point is to observe that the equivalence class $[\phi]$ of some $\phi \in \mathcal{H}_0$ only contains ϕ . Hence the equivalence class of some $[\phi]$ with $\phi \in \overline{\Gamma_i}$ only contains ϕ . Hence completeness of Π_i is consequence of completeness of $\mathcal{H}_0$. According to the convex projection theorem [Theorem 5.2, 5], there exists a unique

$\psi_i \in \overline{\Gamma_i}$ such that $[\psi_i] \in \Pi_i$ is the projection of $[0] \in H$ onto Π_i . For all $t \in \mathbb{R}$ and for all $\varphi \in \mathcal{H}_0$, we have $\psi_i + t\varphi \in \overline{\Gamma_i}$ and in particular, we must have

$$\|[\psi_i + t\varphi]\|_H^2 \geq \|[\psi_i]\|_H^2,$$

which also reads as

$$|t| \int_D |\nabla \varphi|^2 \frac{dm}{b} + 2 \frac{t}{|t|} \int_D (\nabla \varphi | \nabla \psi_i)_{\mathbb{R}^2} \frac{dm}{b} \geq 0.$$

Letting $t \rightarrow 0$ yields the identity

$$\int_D (\nabla \varphi | \nabla \psi_i)_{\mathbb{R}^2} \frac{dm}{b} = 0,$$

for all $\varphi \in \mathcal{H}_0$.

Finally, the inequalities $0 \leq \psi_i \leq 1$ are proved by an adaptation of weak maximum principle techniques [Theorem 8.1, 32]. Let us illustrate this by proving $\psi_i \leq 1$. The lower bound $\psi_i \geq 0$ will be proved using similar arguments. By construction of ψ_i , the positive part $(\psi_i - 1)_+$ belongs to $\mathcal{H}_0$. In particular, we have

$$0 = \int_D (\nabla \psi_i | \nabla (\psi_i - 1)_+)_{\mathbb{R}^2} \frac{dm}{b} = \int_D |\nabla (\psi_i - 1)_+|^2 \frac{dm}{b}.$$

Thus $\nabla (\psi_i - 1)_+ = 0$ on D . This is possible only if $(\psi_i - 1)_+ = 0$ on D , and thus $\psi_i \leq 1$. $\square$

Now we turn to the existence of solution of problem (C). We recall that by a weak solution of problem (C), we mean a linear combination

$$\phi = \sum_{i=0}^m \alpha_i \psi_i,$$

with $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ such that

$$-c_j = \sum_{i=0}^m \alpha_i \int_D (\nabla \psi_j | \nabla \psi_i)_{\mathbb{R}^2} \frac{dm}{b}.$$

We require a first lemma:

Lemma 2.1. *A linear combination*

$$\phi = \sum_{i=0}^m \alpha_i \psi_i$$

with $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ belongs to $\mathcal{H}_0$ if, and only if, we have $\alpha_i = 0$ for all $i = 0, \dots, m$. In particular,

$$\sum_{i=0}^m \psi_i = 1.$$

This lemma strongly uses the regularity assumptions on b near a point shore.

Proof If we have $\alpha_i = 0$ for all $i \in \{0, \dots, m\}$, then the linear combination is null and belongs to $\mathcal{H}_0$. From this it is easy to see that $1 - \sum_{i=0}^m \psi_i$ belongs to $\mathcal{H}_0$ by construction, and therefore $\sum_{i=0}^m \psi_i = 1$. For the converse, let us assume that $\sum_{i=0}^m \alpha_i \psi_i$ belongs to $\mathcal{H}_0$. By construction of each ψ_i and by definition of $\mathcal{H}_0$ there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$

of smooth functions $\varphi_n \in C^1(\overline{D})$ and $\varphi_n = \alpha_i$ on a neighborhood of $\mathcal{C}_i$; and the sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges in $\mathcal{H}$ to 0. Up to a subtraction by α_0 , we have found a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of $C_c^1(D)$ converging in $\mathcal{H}$ to $-\alpha_0$. By completeness of $W_0^{1,2}(\mathbb{R}^2)$ we have $\alpha_0 = 0$ and the convergence also occurs in $W_0^{1,2}(\mathbb{R}^2)$. Up to passing to a subsequence, we may assume that the convergence occurs quasi-everywhere on $\mathbb{R}^2$ [Section 7.2, 34]. Let $i \in \{0, \dots, m\}$ be such that $\text{cap}(\mathcal{C}_i) > 0$. Then we must have $\alpha_i = -\alpha_0$. Assume that there exists $i \in \{0, \dots, m\}$ such that $\text{cap}(\mathcal{C}_i) = 0$. Since $\mathcal{C}_i$ is connected and has null capacity, Pólya-Szegő's inequality [Theorem 8.3.14, 34] implies that $\mathcal{C}_i = \{p_i\}$. By assumption, the depth function b may be extended by 0 on $D \cup \{p_i\}$, and the extension (still denoted b) is Dini continuous at p_i . In particular, we have for all $\varphi \in C^1(\mathbb{R}^2)$ supported in D :

$$\begin{aligned} |\varphi(p_i)| &= \left| \int_D (\nabla \varphi(x) | \frac{p_i - x}{|p_i - x|^2})_{\mathbb{R}^2} \frac{\text{dm}(x)}{b(x)} \right| \\ &\leq \left(\int_D \frac{b(x)}{|p_i - x|^2} \text{dm}(x) \right)^{\frac{1}{2}} \left(\int_D |\nabla \varphi|^2 \frac{\text{dm}}{b} \right)^{\frac{1}{2}}. \end{aligned}$$

There exists a constant $C_4(b, p_i) > 0$ such that for all $\varphi \in C^1(\mathbb{R}^2)$ supported in D , we have

$$|\varphi(p_i)| \leq C_4 \left(\int_D |\nabla \varphi|^2 \frac{\text{dm}}{b} \right)^{\frac{1}{2}} \leq C_4 \|\varphi\|_{\mathcal{H}}.$$

Applying this inequality on the sequence $(\varphi_n)_{n \in \mathbb{N}}$ yields

$$|\alpha_i| = \lim_{n \rightarrow +\infty} |\varphi_n(p_i)| = -\alpha_0 = 0.$$

□

Theorem 2.1. *The symmetric matrix $\mathcal{A} \in \mathbb{R}^{(m+1) \times (m+1)}$ defined by*

$$\mathcal{A}_{ij} = \int_D (\nabla \psi_i | \nabla \psi_j)_{\mathbb{R}^2} \frac{\text{dm}}{b}$$

has kernel spanned by $(1, \dots, 1) \in \mathbb{R}^{m+1}$ and non negative eigenvalues. In particular, for any choice $c_0, \dots, c_m \in \mathbb{R}$ such that

$$\sum_{i=0}^m c_i = 0,$$

there exists a unique weak solution $\phi \in \mathcal{H}$ for problem (C) of the form $\phi = \sum_{i=1}^m \alpha_i \psi_i$.

Proof According to Lemma 2.1, we already know that $(1, \dots, 1)$ belongs to the kernel $\text{Ker}(\mathcal{A})$ of $\mathcal{A}$. Let $(\alpha_0, \dots, \alpha_m)$ be another element of $\text{Ker}(\mathcal{A})$. We compute

$$\int_D \left| \nabla \sum_{i=0}^m \alpha_i \psi_i \right|^2 \frac{\text{dm}}{b} = (\alpha_0, \dots, \alpha_m) \mathcal{A} (\alpha_0, \dots, \alpha_m)^{\dagger} = 0,$$

where the $M^{\dagger}$ denotes the transpose of M . In particular, the linear combination $\sum_{i=0}^m \alpha_i \psi_i$ is a constant function. According to Lemma 2.1, we have $\alpha_0 = \alpha_i$ for all $i \in \{0, \dots, m\}$, and

thus $\text{Ker}(\mathcal{A})$ is spanned by $(1, \dots, 1)$. Now if $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{A}$, with eigenvector $\alpha = (\alpha_0, \dots, \alpha_m)$, the same computation shows that

$$0 \leq \int_D \left| \nabla \sum_{i=0}^m \alpha_i \psi_i \right|^2 \frac{dm}{b} = \lambda |\alpha|^2,$$

and thus $\lambda \geq 0$. Finally, since $\mathcal{A}$ is symmetric, its range is the orthogonal space of its kernel in $\mathbb{R}^{m+1}$, and the restriction of $\mathcal{A}$ to its range defines a linear isomorphism. The conclusion follows. $\square$

2.4 Construction of the Stream Function

In this section, we show how to construct a stream function $\psi \in \mathcal{H}$ that solves

$$\begin{cases} -\text{div}(b^{-1} \nabla \psi) = \omega, \\ \oint_{\partial \mathcal{C}_i} (b^{-1} \nabla^\perp \psi | \tau_i)_{\mathbb{R}^2} = c_i, \end{cases}$$

for given circulations $c_0, \dots, c_m$ that satisfy

$$\sum_{i=0}^m c_i = \int_D \omega \, dm.$$

The function ω is the physical vortex associated to the velocity field $b^{-1} \nabla^\perp \psi$. As mentioned in the introduction, it will be suitable to work with the potential vortex $\zeta = b^{-1} \omega$ instead, so we change our notations and we solve in $\mathcal{H}$:

$$\begin{cases} -\text{div}(b^{-1} \nabla \psi) = b \zeta, \\ \oint_{\partial \mathcal{C}_i} (b^{-1} \nabla^\perp \psi | \tau_i)_{\mathbb{R}^2} = c_i. \end{cases} \quad (\text{P})$$

The functional space of interest for the vortex data ζ is the Lebesgue space $L^p(D, d\mu)$, where $p > 1$ and the measure $d\mu(x) = b(x) \, dm(x)$ is the invariant measure associated to the lake (D, b) . Observe that since b is a bounded function, we automatically have $(b\zeta) \in L^p(D, dm)$.

Let us explain in what sense we understand the weak form of the problem (P). Assume that $\psi \in \mathcal{H}$ solves the above problem, and let $\phi \in C_c^1(D)$ be a smooth test function. By the divergence theorem we have formally, if ψ is a solution of problem (P):

$$\int_D \phi \zeta \, d\mu = - \int_D \phi \text{div}(b^{-1} \nabla \psi) \, dm = \int_D (\nabla \psi | \nabla \phi)_{\mathbb{R}^2} \frac{dm}{b}.$$

Since we have continuous embeddings

$$\begin{aligned} (\mathcal{H}, \|\cdot\|_{\mathcal{H}}) &\hookrightarrow (W^{1,2}(D), \|\cdot\|_{W^{1,2}}) \\ &\hookrightarrow (L^{p'}(D, dm), \|\cdot\|_{L^{p'}_{dm}}) \hookrightarrow (L^{p'}(D, d\mu), \|\cdot\|_{L^{p'}_{d\mu}}), \end{aligned}$$

a weak formulation of the elliptic equation in problem (P) would read as

$$\int_D \phi \zeta \, d\mu = (\psi | \phi)_{\mathcal{H}_0}, \quad \text{for all } \phi \in \mathcal{H}_0.$$

Let us now turn to the interpretation of the circulation conditions. As in the previous section, we compute for all $j \in \{0, \dots, m\}$, at least formally:

$$\begin{aligned} c_j &= - \int_{\mathcal{C}_j} (b^{-1} \nabla \psi | - \eta_j)_{\mathbb{R}^2} = - \int_{\partial D} (\psi_j b^{-1} \nabla \psi | - \eta)_{\mathbb{R}^2} \\ &= - \int_D \operatorname{div}(\psi_j b^{-1} \nabla \psi), \end{aligned}$$

so that

$$-c_j + \int_D \psi_j \zeta \, d\mu = \int_D (\nabla \psi_j | \nabla \psi)_{\mathbb{R}^2} \frac{d\mu}{b}.$$

Accordingly, we propose the following weak formulation of problem (P).

Definition 2.1. A function $\psi \in \mathcal{H}$ is a weak solution of problem (P) if $\psi = u + \mathfrak{H}(\zeta)$ for $u \in \mathcal{H}_0$ such that

$$(u|\phi)_{\mathcal{H}_0} = \int_D \phi \zeta \, d\mu, \quad \text{for all } \phi \in \mathcal{H}_0,$$

and

$$\mathfrak{H}(\zeta) = \sum_{i=0}^m \alpha_i (\psi_i - c_i)$$

for $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ such that $\alpha_0 = 0$ and for all $j \in \{0, \dots, m\}$:

$$-c_j + \int_D \psi_j \zeta \, d\mu = \sum_{i=0}^m \alpha_i \int_D (\nabla \psi_i | \nabla \psi_j)_{\mathbb{R}^2} \frac{d\mu}{b}.$$

Theorem 2.2. Given $\zeta \in L^p(D, d\mu)$ for $p > 1$ and prescribed circulations $c_0, \dots, c_m \in \mathbb{R}$ with

$$\sum_{i=0}^m c_i = \int_D \zeta \, d\mu,$$

there exists a unique linear combination

$$\mathfrak{H}(\zeta) = \sum_{i=1}^m \alpha_i (\psi_i - c_i) \in \mathcal{H},$$

where ψ_i is given by Proposition 2.2; and a unique $\mathfrak{K}(\zeta) \in \mathcal{H}_0$, such that the combined flow $\mathfrak{K}(\zeta) + \mathfrak{H}(\zeta) \in \mathcal{H}$ solves problem (P) in the sense of Definition 2.1.

Proof Let us consider the functional

$$\Phi : \mathcal{H}_0 \rightarrow \mathbb{R} : \phi \mapsto \frac{1}{2} \|\phi\|_{\mathcal{H}_0}^2 - \int_D \phi \zeta \, d\mu.$$

The functional Φ is strictly convex, hence it admits at most one minimizer. The existence of the minimizer follows from the lower semi-continuity of the norm and the compact embedding

$$(\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0}) \hookrightarrow (L^{p'}(D, d\mu), \|\cdot\|_{L^{p'}_{d\mu}}).$$

We define $\mathfrak{K}(\zeta)$ as the unique minimizer of Φ . From this variational principle we conclude that

$$(\mathfrak{K}(\zeta)|\phi)_{\mathcal{H}_0} = \int_D \phi \zeta \, d\mu, \quad \text{for all } \phi \in \mathcal{H}_0,$$

According to Theorem 2.1, the required coefficients $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ are uniquely determined if we impose $\alpha_0 = 0$. We define

$$\mathfrak{H}(\zeta) = \sum_{i=1}^m \alpha_i (\psi_i - c_i).$$

Now by construction of the static flows ψ_i for $i = 0, \dots, m$, the function $\mathfrak{K}(\zeta) + \mathfrak{H}(\zeta) \in \mathcal{H}$ is a weak solution of problem (P) in the sense of Definition 2.1. The uniqueness follows from Lemma 2.1. $\square$

The choice of α_0 in our definition of weak solution is motivated by the following result:

Corollary 2.1 (Energy equivalence). *For all $\zeta \in L^p(D, d\mu)$, we have*

$$\int_D (\mathfrak{K}(\zeta) + \mathfrak{H}(\zeta)) \zeta d\mu = \int_D \|\nabla(\mathfrak{K}(\zeta) + \mathfrak{H}(\zeta))\|_{\mathbb{R}^2}^2 \frac{d\mathbf{m}}{b}.$$

Below are listed basic properties of the stream function.

Corollary 2.2 (Symmetry). *For all $\zeta_1, \zeta_2 \in L^p(D, d\mu)$, we have*

$$\int_D \zeta_1 \mathfrak{K}(\zeta_2) d\mu = \int_D \zeta_2 \mathfrak{K}(\zeta_1) d\mu$$

and similarly

$$\int_D \zeta_1 \mathfrak{H}(\zeta_2) d\mu = \int_D \zeta_2 \mathfrak{H}(\zeta_1) d\mu.$$

Corollary 2.3 (Linearity). *For all $\zeta_1, \zeta_2 \in L^p(D, d\mu)$ and for all $\alpha \in \mathbb{R}$, we have*

$$\mathfrak{K}(\zeta_1 + \alpha \zeta_2) = \mathfrak{K}(\zeta_1) + \alpha \mathfrak{K}(\zeta_2).$$

Corollary 2.4 (Positivity). *For all $\zeta \in L^p(D, d\mu)$ with $\zeta \geq 0$, we have $\mathfrak{K}(\zeta) \geq 0$.*

Corollary 2.5 (Boundedness). *There exists constants $C, C' > 0$ such that For all $\zeta \in L^p(D, d\mu)$ with $\zeta \geq 0$, we have*

$$\|\mathfrak{K}(\zeta)\|_{\mathcal{H}_0} \leq C \|\zeta\|_{L^p_{d\mu}},$$

and

$$\|\mathfrak{H}(\zeta)\|_{\mathcal{H}} \leq C' \|\zeta\|_{L^1_{d\mu}}.$$

Proposition 2.3. *If a sequence $(\zeta_n)_{n \in \mathbb{N}}$ of $L^p(D, d\mu)$ functions is bounded in $L^p(D, d\mu)$ and weakly converges to some $\zeta \in L^p(D, d\mu)$, then the sequences $(\mathfrak{K}(\zeta_n))_{n \in \mathbb{N}}$ and $(\mathfrak{H}(\zeta_n))_{n \in \mathbb{N}}$ both converge strongly, respectively to $\mathfrak{K}(\zeta)$ in $\mathcal{H}_0$ and $\mathfrak{H}(\zeta)$ in $\mathcal{H}$.*

Proof We first prove the claim for the operator $\mathfrak{K}$. Since we have a compact embedding $(\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0}) \hookrightarrow (L^{p'}(D, d\mu), \|\cdot\|_{L^{p'}_{d\mu}})$, it is sufficient to prove that every accumulation point of the sequence $(\mathfrak{K}(\zeta_n))_{n \in \mathbb{N}}$ in the sense of strong topology of $L^{p'}(D, d\mu)$ equals $\mathfrak{K}(\zeta)$, and that the convergence also occurs in $(\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0})$. Let then $u \in L^{p'}(D, d\mu)$ be an accumulation point of $(\mathfrak{K}(\zeta_n))_{n \in \mathbb{N}}$ in $L^{p'}(D, d\mu)$. There exists a subsequence $(\mathfrak{K}(\zeta_{n_k}))_{k \in \mathbb{N}}$

converging in $L^{p'}(D, d\mu)$ to $u \in L^{p'}(D, d\mu)$. In particular, for all test function $\phi \in C_c^1(D)$, we have

$$\begin{aligned} \int_D \phi u \, d\mu &= \lim_{k \rightarrow +\infty} \int_D \phi \mathfrak{K}(\zeta_{n_k}) \, d\mu \\ &= \lim_{k \rightarrow +\infty} \int_D \mathfrak{K}(\phi) \zeta_{n_k} \, d\mu = \int_D \mathfrak{K}(\phi) \zeta \, d\mu = \int_D \phi \mathfrak{K}(\zeta) \, d\mu. \end{aligned}$$

Therefore $u = \mathfrak{K}(\zeta)$ in $L^p(D, d\mu)$, that is, almost-everywhere on D . On the other hand, we also have

$$\|\mathfrak{K}(\zeta) - \mathfrak{K}(\zeta_{n_k})\|_{\mathcal{H}_0}^2 = \int_D \zeta \mathfrak{K}(\zeta) \, d\mu + \int_D \zeta_{n_k} \mathfrak{K}(\zeta_{n_k}) \, d\mu - 2 \int_D \zeta_{n_k} \mathfrak{K}(\zeta) \, d\mu.$$

The right side converges to 0, and therefore $(\mathfrak{K}(\zeta_{n_k}))_{k \in \mathbb{N}}$ converges strongly in $(\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0})$ to the function $\mathfrak{K}(\zeta)$.

Let us now prove the similar statement for the operator $\mathfrak{H}$. By construction, the coefficients $\alpha_0, \dots, \alpha_m$ given by Theorem 2.1 depends on the vortex function ζ only through the perturbed circulations

$$-c_j + \int_D \psi_j \zeta \, d\mu.$$

In particular, weak convergence of the sequence $(\zeta_n)_{n \in \mathbb{N}}$ implies strong convergence in $\mathbb{R}$ of each perturbed circulations, and so strong convergence in $\mathbb{R}$ of each coefficient α_j . Therefore the sequence $(\mathfrak{H}(\zeta_n))_{n \in \mathbb{N}}$ strongly converges in $\mathcal{H}$ to some $u \in \mathcal{H}$ and by uniqueness, we must have the identity $u = \mathfrak{H}(\zeta)$. $\square$

2.5 Continuous Lakes

Let us write $G_D : D \times D \rightarrow \mathbb{R}$ the Green's function for the Laplace's operator $-\Delta$ with Dirichlet's boundary conditions on ∂D . We extend G_D by 0 on $\overline{D} \times \overline{D}$. The following lemma ensures that this extension makes sense, at least in the sense of weak convergence in every $W_0^{1,q}(D)$, $q \in (1, 2)$:

Lemma 2.2. *The function $[y \in \overline{D} \mapsto G_D(\cdot, y)]$ is continuous in the sense of weak convergence in $W_0^{1,q}(D)$, for any $q \in (1, 2)$.*

Proof Let $q \in (1, 2)$ be fixed. The fact that $G_D(\cdot, y)$ is uniformly bounded in $W_0^{1,q}(D)$ as y varies in $\overline{D}$ follows from standard argument. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence of points in D converging to some $y \in \overline{D}$. The sequence $(G_D(\cdot, y_n))_{n \in \mathbb{N}}$ admits at least one accumulation point $g \in W_0^{1,q}(D)$ in the sense of weak convergence in $W_0^{1,q}(D)$. We are done if we prove $g = G_D(\cdot, y)$, since the weak convergence on a bounded set defines a metrizable topology. Let $L : W_0^{1,q}(D) \rightarrow \mathbb{R}$ be any non null continuous linear functional, where the norm of $W_0^{1,q}(D)$ is understood as the gradient norm in $L^q(D)$. According to James' representation theorem [Proposition 5.2.3, 34], there exists a (unique) $u \in W_0^{1,q}(D)$ such that $\|u\|_{W_0^{1,q}} = 1$ and

$$\begin{aligned} L(v) &= \|L\|_{(W_0^{1,q})^*} \int_D (\nabla v | \nabla u)_{\mathbb{R}^2} |\nabla u|^{q-2} \\ &= \|L\|_{(W_0^{1,q})^*} \int_D (\nabla v | \nabla \frac{|u|^{q-1}}{q-1})_{\mathbb{R}^2}. \end{aligned}$$

Now it is readily checked that there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of $C_c^1(D)$ functions converging in $W_0^{1, \frac{q}{q-1}}(D)$ to $|u|^{q-1}$, so that

$$L(G_D(\cdot, y) - g) = \lim_{n \rightarrow +\infty} \|L\|_{(W_0^{1,q})^*} \int_D (\nabla(G_D(\cdot, y) - g) | \nabla \varphi_n)_{\mathbb{R}^2} = 0.$$

In the last line we have used the definition of the Green's function together with the definition of the weak limit g . This shows that every element of $(W_0^{1,q}(D))^*$ vanishes on $G_D(\cdot, y) - g$. This forces $G_D(\cdot, y) = g$, and therefore the sequence $(G_D(\cdot, y_n))_{n \in \mathbb{N}}$ converges weakly to $G_D(\cdot, y)$. $\square$

As a matter of fact, we mention that the uniform boundedness argument for the Green's function that we rely on in the previous lemma may be proved through at least three different approaches: through regularity and a duality argument in the spirit of [22] where the characterization of the dual of $L^\infty(D)$ as a set of finitely additive measures; from complex analysis through the Riemann conformal mapping theorem; or from direct measure geometric like argument [33]. We do not enter into details.

Definition 2.2. A lake (D, b) is said to be *continuous* if

- (1) the function $b \in C(\overline{D})$ admits weak derivatives,
- (2) there exists $\ell > 2$ such that function

$$y \in D \mapsto \int_D g(\cdot, y)^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \mathrm{d}m$$

is bounded on $\overline{D}$,

- (3) for all $y_\star \in \overline{D}$, we have

$$\lim_{y \rightarrow y_\star} \int_D |g(\cdot, y) - g(\cdot, y_\star)|^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \mathrm{d}m = 0.$$

Note that the fact that b is positive on compact subsets implies that $b \in W_{\mathrm{loc}}^{1,\ell}(D)$, for some $\ell > 2$. In particular we must have $C^{0,\alpha}(K)$ for all compact subsets $K \Subset D$, and for some $\alpha > 0$ depending on K . The main theorem we are going to use on continuous lakes is the following:

Theorem 2.3. *Let (D, b) be a continuous lake. There exists a bounded measurable function $R : D \times D \rightarrow \mathbb{R}$ such that, for all $\zeta \in L^p(D, \mathrm{d}\mu)$, $p > 1$, we have almost-everywhere*

$$\mathfrak{K}\zeta(x) = b(x) \int_D G_D(x, y) \zeta(y) \mathrm{d}\mu(y) + \int_D R(x, y) \zeta(y) \mathrm{d}\mu(y).$$

Furthermore, we have $R(\cdot, y) \in \mathcal{H}_0$ for all $y \in D$, with for all $\varphi \in \mathcal{H}_0$:

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{\mathrm{d}m}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{\mathrm{d}m}{b}.$$

In particular, there exists a bounded measurable function $F : D \times D \rightarrow \mathbb{R}$ such that

$$\mathfrak{K}\zeta(x) + \mathfrak{J}\zeta(x) = b(x) \int_D G_D(x, y) \zeta(y) \mathrm{d}\mu(y) + \int_D F(x, y) \zeta(y) \mathrm{d}\mu(y).$$

The proof of Theorem 2.3 is based on standard ideas from regularity theory. The so-constructed function R is measurable on the product space $D \times D$, which will allow us to manipulate it through Fubini's theorem. In fact, we can prove that R is continuous on $D \times D$, but this extra assumption will not be used in the text. The details on the construction of R are done in the [Appendix](#). Note that the existence and boundedness of the function F simply follows from the one of R and the construction of $\mathfrak{H}\zeta$.

Below we give examples of continuous lakes (D, b) .

Example 1. If $b \in W^{1,\infty}(D)$ with $\inf_D b > 0$, then (D, b) is a continuous lake. Here we rely on the property that $G_D(\cdot, y)$ converges weakly in $W_0^{1,q}(D)$ to $G_D(\cdot, y_*)$ as y converges to y_* ; then we use Rellich-Kondrashov's theorem to obtain strong convergence in $L^p(D, dm)$ for all $p \in [1, +\infty)$.

Example 2. If $\delta : \overline{D} \rightarrow \mathbb{R}$ is a regularization of the distance at the boundary, then any function $b : D \rightarrow \mathbb{R}$ that satisfies

$$\limsup_{\delta(x) \rightarrow 0} \left\{ \frac{1}{\delta(x)^\beta} \frac{|\nabla b(x)|^2}{b(x)} \right\} < +\infty$$

for some small exponent $\beta \in [0, 2)$, yields to a continuous lake (D, b) . This follows from the Hardy's inequality for regular domains D . This family of examples covers the cases of $b = \delta^\alpha$, for sufficiently well behaving $\alpha \in C^1(\overline{D})$. Indeed we have

$$\frac{|\nabla b|^2}{b} \sim \delta^\alpha |\log \delta|^2 |\nabla \alpha|^2 + \delta^{\alpha-2} \alpha |\nabla \delta|^2.$$

If $\inf_D \alpha > 0$, then $\alpha - 2 > -2$ uniformly on D , and the Hardy's inequality may be applied. If, for example, $\alpha = \delta^p$ for some $p > 1$, then we would have

$$\frac{|\nabla b|^2}{b} \sim \delta^{\delta^p + 2(p-1)} |\log \delta|^2 + \delta^{\delta^p - 2 + p} |\nabla \delta|^2$$

and again $\delta^p - 2 + p > -1$ uniformly on D . Observe that the special case (D, b) with $D = B(0, 1)$ and $b(x) = 1 - |x|$, does not satisfy the $\mathcal{A}_2$ Muckenhoupt condition.

Example 3. Let us consider $D = [-1, 1]^2 \setminus \{(0, y) : y \leq 0\}$ the square of side length 2 centered on 0, without one half inner vertical median. On this set we consider the function

$$b : D \rightarrow \mathbb{R}^+ : b(x_1, x_2) = |x_1| + x_2 \chi_{\{x_2 > 0\}}.$$

So the lake (D, b) has a line shore. We compute

$$\frac{|\nabla b(x_1, x_2)|^2}{b(x_1, x_2)} \leq \frac{2}{|x_1| + x_2 \chi_{\{x_2 > 0\}}}.$$

By linearity of the integral, we do not lose in generality in splitting D into three disjoint parts:

$$\begin{aligned} D_1 &:= \{(x_1, x_2) \in D : |x_1| < 1, 0 < x_2 < 1\}, \\ D_2 &:= \{(x_1, x_2) \in D : (x_1 \geq 1, x_2 > 0) \text{ or } (x_2 \geq 1)\}, \\ D_3 &:= \{(x_1, x_2) \in D : x_2 \leq 0\}. \end{aligned}$$

On D_2 the function $b^{-1}|\nabla b|^2$ is uniformly bounded, so one may rely on the first example to control the contribution from D_2 . On D_1 we always have

$$\frac{|\nabla b(x_1, x_2)|^2}{b(x_1, x_2)} \leq \frac{2}{\sqrt{x_1^2 + x_2^2}},$$

and therefore it belongs to $L^q(D_1)$ for all $q \in [1, 2)$. The conclusion on D_1 then follows as in the previous example. For the last part, an elementary computation shows that for all $(x_1, x_2) \in D_3$, we have

$$\frac{|\nabla b(x_1, x_2)|^2}{b(x_1, x_2)} \leq \frac{2}{|x_1|} \leq C_5 \frac{1}{d_p(x_1, x_2)},$$

where d_p is the mean distance at the boundary ∂D of order $p > 1$, defined for all $x \in D$ by

$$(d_p(x))^{-1} = 2 \left(\int_{\mathbb{P}_2} \frac{1}{(\rho_v(x))^p} d\sigma(v) \right)^{\frac{1}{p}},$$

where $\mathbb{P}_2$ denotes the 2-dimensional projective plane endowed with its Haar measure $d\sigma$, and $\rho_v(x)$ is the least distance at x to the boundary in the direction v [Chapter 3, 2]. One may then use the Hardy's inequality on general domains [Theorem 3.3.2, 2] to obtain the conclusion on D_3 .

3 Maximization of the Energy

Let us consider an energy functional of the form

$$E : L^p(D, d\mu) \rightarrow \mathbb{R} : E(\zeta) = \frac{1}{2} \int_D \zeta(\mathfrak{K} + \mathfrak{H})(\zeta) d\mu + \int_D \Psi \zeta d\mu.$$

The flow $\Psi \in \mathcal{H}$ is independent of the vortex ζ . The assumption $\Psi \in \mathcal{H}$ is of physical importance, since it means that the associated velocity $b^{-1} \nabla \Psi$ brings a finite contribution to the kinetic energy of the system.

3.1 Rearrangement of Functions

Let $\zeta \in L^p(D, d\mu)$ be a given function. We recall that by a $d\mu$ -rearrangement of ζ , we mean a function $\tilde{\zeta} \in L^p(D, d\mu)$ such that, for all $\lambda \in \mathbb{R}$, we have

$$d\mu(\{\zeta \geq \lambda\}) = d\mu(\{\tilde{\zeta} \geq \lambda\}).$$

This defines an equivalence relation on $L^p(D, d\mu)$. The set of all $d\mu$ -rearrangements of a given function $\zeta \in L^p(D, d\mu)$ will be denoted by $\text{Rearg}(\zeta)$. Observe that for $\zeta \in L^p(D, d\mu)$ and $\xi \in \text{Rearg}(\zeta)$, we have $(\xi)_+ \in \text{Rearg}((\zeta)_+)$ and similarly $(\xi)_- \in \text{Rearg}((\zeta)_-)$. It then follows from the Cavalieri's principle (see also [25]) that for all $\tilde{\zeta} \in \text{Rearg}(\zeta)$, we have

$$\left(\int_D |\xi|^q d\mu \right)^{\frac{1}{q}} = \left(\int_D |\zeta|^q d\mu \right)^{\frac{1}{q}},$$

for any $q \in [1, +\infty)$. In particular, for $p \in (1, +\infty)$, the set $\text{Rearg}(\zeta)$ is closed with respect to the strong topology of $L^p(D, d\mu)$, and relatively compact with respect to the weak topology.

The following proposition provides a way to construct “well behaving” rearrangements. The proof is standard in symmetrization theory, but we recall it for completeness.

Proposition 3.1. *For all $x \in D$, there exists a function*

$$x_{\sharp}^{\cdot} : L^1(D, d\mu) \rightarrow L^1(D, d\mu)$$

such that for all positive function $\zeta \in L^1(D, d\mu)$ we have

- (1) $x\sharp\zeta \in \text{Rearg}(\zeta)$;
- (2) the superlevel sets of $x\sharp\zeta$ are balls in D centered on x .

Proof For all $\lambda \in \mathbb{R}$, we define

$$r_\lambda = \sup \left\{ r \geq 0 : d\mu(D \cap \overline{B(x, r)}) \leq d\mu(\{\zeta \geq \lambda\}) \right\}.$$

We have $r_{\lambda_1} \leq r_{\lambda_2}$ as soon as $\lambda_1 \geq \lambda_2$. We then define the function

$$x\sharp\zeta : D \rightarrow \mathbb{R}^+ : x\sharp\zeta(y) = \sup \left\{ \lambda \in \mathbb{R} : y \in D \cap \overline{B(x, r_\lambda)} \right\}.$$

It then follows from the definition of $(x\sharp\zeta)$ that for all $\lambda \in \mathbb{R}$:

$$\{x\sharp\zeta \geq \lambda\} = D \cap \overline{B\left(x, \inf_{s < \lambda} r_s\right)} = D \cap \overline{B(x, r_\lambda)}.$$

On the other hand, we have

$$d\mu(D \cap \overline{B(x, r_\lambda)}) = d\mu(\{\zeta \geq \lambda\}),$$

and therefore $x\sharp\zeta$ is a $d\mu$ -rearrangement of ζ whose superlevel sets are balls centered on x . $\square$

Proposition 3.2. *The energy functional E is strictly convex and for all $\zeta \in L^p(D, d\mu)$, the function $(\mathfrak{K} + \mathfrak{H})(\zeta) + \Psi$ belongs to the subgradient of E at point ζ .*

Proof The third contribution

$$\zeta \in L^p(D, d\mu) \mapsto \int_D \Psi \zeta \, d\mu$$

is linear, hence convex. The first contribution

$$\zeta \in L^p(D, d\mu) \mapsto \frac{1}{2} \int_D \zeta \mathfrak{K}(\zeta) \, d\mu = \frac{1}{2} \|\mathfrak{K}(\zeta)\|_{\mathcal{H}_0}^2$$

is strictly convex. It remains to prove that the second contribution

$$\zeta \in L^p(D, d\mu) \mapsto \frac{1}{2} \int_D \zeta \mathfrak{H}(\zeta) \, d\mu$$

is convex. Let $\zeta_1, \zeta_2 \in L^p(D, d\mu)$ and $t \in [0, 1]$. A computation shows that, if $\zeta = \zeta_1 - \zeta_2$, we have

$$\int_D \zeta \mathfrak{H}(\zeta) \, d\mu = \sum_{i=0}^m \alpha_i \int_D \zeta (\psi_i - c_i) \, d\mu = (\alpha_0, \dots, \alpha_m) \mathcal{A}(\alpha_0, \dots, \alpha_m)^\dagger,$$

where $\mathcal{A}$ is the matrix constructed in Theorem 2.1. According to Theorem 2.1, the matrix $\mathcal{A}$ has non negative eigenvalues, and therefore the above quantity is non negative as well. In particular, we have

$$\begin{aligned} & \frac{1}{2} \int_D (t\zeta_1 + (1-t)\zeta_2) \mathfrak{H}(t\zeta_1 + (1-t)\zeta_2) \, d\mu \\ & \leq t \frac{1}{2} \int_D \zeta_1 \mathfrak{H}(\zeta_1) \, d\mu + (1-t) \frac{1}{2} \int_D \zeta_2 \mathfrak{H}(\zeta_2) \, d\mu. \end{aligned}$$

For the last claim, we have to prove that, for all $\tilde{\zeta} \in L^p(D, d\mu)$, we have

$$E(\tilde{\zeta}) - E(\zeta) \geq \int_D (\tilde{\zeta} - \zeta)(\mathfrak{K}(\zeta) + \mathfrak{H}(\zeta) + \Psi) d\mu.$$

By linearity, it is sufficient to check that

$$\int_D \tilde{\zeta}(\mathfrak{K} + \mathfrak{H})(\tilde{\zeta}) d\mu - \int_D \zeta(\mathfrak{K} + \mathfrak{H})(\zeta) d\mu \geq 2 \int_D (\tilde{\zeta} - \zeta)(\mathfrak{K} + \mathfrak{H})(\zeta) d\mu.$$

Here we have used the fact that the eigenvalues of $\mathcal{A}$ are non negative. $\square$

3.2 Energy Maximization and Pressure Field

In this section we show that E admits a maximizer over $\text{Rearg}(\zeta)$, and that such maximizer leads to solution of the steady lake equations.

Proposition 3.3. *Let $\zeta \in L^p(D, d\mu)$, $p > 1$. The restriction of the energy E on $\text{Rearg}(\zeta)$ admits at least one maximizer $\tilde{\zeta}$. If $\zeta u \in L^1_{\text{loc}}(D, \mathbb{R}^2)$ is defined by*

$$u = b^{-1} \nabla^\perp (\mathfrak{K}(\tilde{\zeta}) + \mathfrak{H}(\tilde{\zeta}) + \Psi),$$

then for all $\phi \in C^1_c(D)$, we have

$$\int_D (\tilde{\zeta} u | \nabla \phi)_{\mathbb{R}^2} d\mu = 0.$$

The proof strongly relies on an existence theorem due to Burton [Theorem A, 7], which has been proved in the case of sign changing vortices by an adaptation of the bathtub principle [17].

Proof By [Theorem A, 7], there exists a maximizer $\tilde{\zeta} \in \text{Rearg}(\zeta)$ of E over $\text{Rearg}(\tilde{\zeta})$, and a non decreasing function $G : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\tilde{\zeta} = G(\mathfrak{K}(\tilde{\zeta}) + \mathfrak{H}(\tilde{\zeta}) + \Psi).$$

Let us write for short $\psi = \mathfrak{K}(\tilde{\zeta}) + \mathfrak{H}(\tilde{\zeta}) + \Psi$, and assume that

$$\tilde{\zeta} b^{-1} \nabla^\perp \psi \in L^1_{\text{loc}}(D, \mathbb{R}^2).$$

We define, for all $n \in \mathbb{N}$:

$$[G]_n : \mathbb{R} \rightarrow \mathbb{R} : [G]_n(t) = \min \{n, \max \{G(t), -n\}\}.$$

For all $n \in \mathbb{N}$, the function $[G]_n$ is non decreasing and bounded. We define similarly the truncation at $k \in \mathbb{N}$ of the function ψ , so that:

$$[\psi]_k \in \mathcal{H}, \quad \nabla [\psi]_k = \chi_{\{|\psi| \leq k\}} \nabla \psi.$$

The function

$$F_{n,k} : [-n, +\infty) \rightarrow \mathbb{R} : \int_{-k}^t [G]_n(s) ds$$

is Lipschitz continuous. The function $F_{n,k} \circ [\psi]_k$ then belongs to $W^{1,2}(D)$ with

$$\nabla(F_{n,k} \circ [\psi]_k) = ([G]_n \circ [\psi]_k) \nabla [\psi]_k.$$

As a result, it follows from the divergence theorem that for all $\phi \in C_c^1(D)$, we have

$$\int_D (\nabla^\perp [\psi]_k | \nabla \phi)_{\mathbb{R}^2} ([G]_n \circ [\psi]_k) \, d\mu = 0.$$

For all $n \in \mathbb{N}$ we have

$$\left| (\nabla^\perp [\psi]_k | \nabla \phi)_{\mathbb{R}^2} ([G]_n \circ [\psi]_k) \right| \leq \left| (\nabla^\perp \psi | \nabla \phi)_{\mathbb{R}^2} \tilde{\zeta} \right|,$$

and the latter belongs to $L^1(D, \mu)$ by Hölder's inequality. On the other hand, we have almost-everywhere on D :

$$\lim_{k \rightarrow +\infty} (\nabla^\perp [\psi]_k(x) | \nabla \phi(x))_{\mathbb{R}^2} [G]_n([\psi]_k(x)) = (\nabla^\perp \psi(x) | \nabla \phi(x))_{\mathbb{R}^2} [G]_n(\psi(x)),$$

because $\nabla \psi = 0$ almost-everywhere on a set of the form $\{\psi = \alpha\}$. Since G is monotone, the set of discontinuities of G is at most countable, hence we miss at most a countable union of negligible sets in D . According to the Lebesgue's dominated convergence theorem, we obtain

$$\int_D (\tilde{\zeta} \nabla^\perp \psi | \nabla \phi)_{\mathbb{R}^2} \, d\mu = 0,$$

which is equivalent to the conclusion. $\square$

4 Asymptotic Behavior of Maximizers

In this section, we turn our attention to the asymptotic behavior of a maximizing family $\{\zeta_\epsilon \in L^p(D, d\mu) : \epsilon > 0\}$ with related energies

$$E_\epsilon(\zeta) = \frac{1}{2} \int_D \zeta \mathfrak{K}(\zeta) + \zeta \mathfrak{H}_\epsilon(\zeta) \, d\mu + \int_D \lambda_\epsilon \Psi \zeta \, d\mu.$$

We assume that the vortex profiles obey the identities

$$\tau \mathcal{S}_\epsilon = \int_D (\zeta_\epsilon)_+ \, d\mu, \quad (1 - \tau) \mathcal{S}_\epsilon = \int_D (\zeta_\epsilon)_- \, d\mu$$

for some $\tau \in [0, 1]$ and $\mathcal{S}_\epsilon > 0$. We make the following ($d\mu$ -rearrangement invariant) extra assumption:

$$\sup_{\epsilon > 0} \left\{ \frac{\|(\zeta_\epsilon)_+\|_{L^p_{d\mu}} \epsilon^{2(1-\frac{1}{p})}}{\tau \mathcal{S}_\epsilon} + \frac{\|(\zeta_\epsilon)_-\|_{L^p_{d\mu}} \epsilon^{2(1-\frac{1}{p})}}{(1-\tau) \mathcal{S}_\epsilon} \right\} < +\infty.$$

(Here above, the convention $\frac{0}{0} = 0$ is used.) This extra condition is motivated by the fact that any family that satisfies constraint (D) (page 4) as in the introduction also satisfies the above control condition.

More precisely, let us denote $|D|_\mu = \mu(D)$ the measure of D with respect to μ . Let's be given a distribution function

$$D : \mathbb{R}^+ \rightarrow [0, |D|_\mu]$$

normalized as

$$\int_{\mathbb{R}^+} D(t) \, dt = 1,$$

and such that there exists $p > 1$ with

$$\int_{\mathbb{R}^+} t^p D(t) \, dt < +\infty.$$

We are also given, for all $\epsilon > 0$, a reference profile $\tilde{\zeta}_\epsilon$ such that

$$\begin{aligned} d\mu \left(\left\{ \left(\tilde{\zeta}_\epsilon \right)_+ \geq \lambda \right\} \right) &= \frac{\epsilon^2}{\delta} D \left(\frac{\epsilon^2 \lambda}{\delta \tau} \log \frac{1}{\epsilon} \right), \\ d\mu \left(\left\{ \left(\tilde{\zeta}_\epsilon \right)_- \geq \lambda \right\} \right) &= \frac{\epsilon^2}{\delta} D \left(\frac{\epsilon^2 \lambda}{\delta(1-\tau)} \log \frac{1}{\epsilon} \right), \end{aligned}$$

where $\delta = \sup_{\lambda > 0} D(\lambda)$. The existence of a family of maximizers $\{\zeta_\epsilon : \epsilon > 0\}$ as above always exists, according to the axiom of choice, and the condition we impose on the $L^p(D, d\mu)$ -norms is satisfied by every family constructed by some scaling process.

The circulations $c_{\epsilon;0}, \dots, c_{\epsilon;m} \in \mathbb{R}$ that come into play in the definition of operator $\mathfrak{H}_\epsilon$ are assumed to depend on $\epsilon > 0$ through

$$c_{\epsilon;i} = c_i(2\tau - 1)\mathcal{S}_\epsilon$$

for $c_0, \dots, c_m \in \mathbb{R}$ independent of $\epsilon > 0$, and

$$\sum_{i=0}^m c_i = 1.$$

We also assume that we have $\lambda_\epsilon \geq 0$ with

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} = 1.$$

On the lake (D, b) we make the assumption that (D, b) is continuous, in the sense of Definition 2.2. We also make the following assumption that the external flow Ψ is continuous on $\overline{D}$, and sufficiently small so that:

(1) there exists $\varsigma > 0$ such that, for all sufficiently small $\epsilon > 0$, if $\tau > 0$:

$$\sup_D \left\{ \frac{4\pi\lambda_\epsilon|\Psi|}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \leq \frac{\tau}{2} - \varsigma;$$

and if $\tau < 1$:

$$\sup_D \left\{ \frac{4\pi\lambda_\epsilon|\Psi|}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \leq \frac{(1-\tau)}{2} - \varsigma.$$

If $\tau > 0$ (resp. $\tau < 1$), then the above conditions ensure that the function

$$\phi : \overline{D} \rightarrow \mathbb{R} : \phi(x) = \frac{\tau b(x)}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi(x)$$

(resp. $b/(4\pi) - \phi$) never reaches a maximal value on a shore, that is: on some point x_* where $b(x_*) = 0$. This qualitative information will be used several times, but we will also require the quantitative estimate to show that the positive and the negative parts of the vortex remain concentrated.

Finally, we write $d\zeta(x) = \frac{\zeta}{\|\zeta\|_{L^1_{d\mu}}} d\mu$ for all non null function $\zeta \in L^p(D, d\mu)$, and $d0 = 0$.

4.1 Leading Partial Flows

The leading partial flows induced by some $\zeta \in L^p(D, d\mu)$ are defined for all $x \in D$ by

$$\mathcal{T}_\zeta^+(x) = \frac{b(x)}{4\pi} \int_D \log \frac{\text{diam}(D)}{|x-y|} (\zeta)_+(y) d\mu(y) + \lambda_\epsilon \Psi(x),$$

and

$$\mathcal{T}_{\zeta}^{-}(x) = \frac{b(x)}{4\pi} \int_D \log \frac{\text{diam}(D)}{|x-y|} (\zeta)_{-}(y) \, d\mu(y) - \lambda_{\epsilon} \Psi(x).$$

Proposition 4.1. *Let*

$$G_D(x, y) = \frac{1}{2\pi} \log \frac{\text{diam}(D)}{|x-y|} - H(x, y)$$

be the Green's function for the Laplace's operator $-\Delta$ on D with Dirichlet boundary conditions, let F be given as in Theorem 2.3. Then we have for all $\zeta \in L^p(D, d\mu)$ and for all $\epsilon > 0$:

$$\begin{aligned} E_{\epsilon}(\zeta) &= \tau \mathcal{S}_{\epsilon} \int_D \mathcal{T}_{\zeta}^{+} \, d(\zeta)_{+} + (1-\tau) \mathcal{S}_{\epsilon} \int_D \mathcal{T}_{\zeta}^{-} \, d(\zeta)_{-} \\ &\quad + \frac{1}{2} \iint_{D \times D} (F(x, y) - b(x)H(x, y)) \zeta(x) \zeta(y) \, d\mu \otimes d\mu(x, y) \\ &\quad - \iint_{D \times D} \frac{b(x) + b(y)}{4\pi} \log \frac{\text{diam}(D)}{|x-y|} (\zeta)_{+}(x) (\zeta)_{-}(y) \, d\mu \otimes d\mu(x, y). \end{aligned}$$

In particular, there exists a constant $C > 0$ such that

$$E_{\epsilon}(\zeta) \leq \tau \mathcal{S}_{\epsilon} \int_D \mathcal{T}_{\zeta}^{+} \, d(\zeta)_{+} + (1-\tau) \mathcal{S}_{\epsilon} \int_D \mathcal{T}_{\zeta}^{-} \, d(\zeta)_{-} + C \mathcal{S}_{\epsilon}^2.$$

The proof is straightforward from the fact that (D, b) is a continuous lake, by using Theorem 2.3. and the properties of H , R , $\mathfrak{H}$, and the positivity of $\mathfrak{K}$. We omit it.

Proposition 4.2. *There exists a constant $C > 0$ such that for all $\epsilon > 0$ and for all $\zeta \in \text{Rearg}(\zeta_{\epsilon})$:*

$$\mathcal{T}_{\zeta}^{+}(x) \leq \frac{\tau b(x)}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_{\epsilon} + \lambda_{\epsilon} \Psi(x) + C \tau \mathcal{S}_{\epsilon}$$

and

$$\mathcal{T}_{\zeta}^{-}(x) \leq \frac{(1-\tau)b(x)}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_{\epsilon} - \lambda_{\epsilon} \Psi(x) + C(1-\tau) \mathcal{S}_{\epsilon}.$$

Proof In both cases, it is sufficient to control the quantity

$$\int_D \log \frac{\text{diam}(D)}{|x-y|} (\zeta_{\epsilon})_{+} \, d\mu$$

by a suitable upper bound. Define

$$M = \sup_{\epsilon > 0} \left\{ \frac{\|(\zeta_{\epsilon})_{+}\|_{L^p_{d\mu}} \epsilon^{2(1-\frac{1}{p})}}{\tau \mathcal{S}_{\epsilon}} + \frac{\|(\zeta_{\epsilon})_{-}\|_{L^p_{d\mu}} \epsilon^{2(1-\frac{1}{p})}}{(1-\tau) \mathcal{S}_{\epsilon}} \right\} < +\infty,$$

so that

$$\sup_{\epsilon > 0} \left\{ \frac{\|(\zeta_{\epsilon})_{+}\|_{L^p_{d\mu}} \epsilon^{2(1-\frac{1}{p})}}{\mathcal{S}_{\epsilon}} \right\} \leq \tau M < +\infty.$$

In the above condition, one may replace ζ_ϵ by any of its $d\mu$ -rearrangement, because $d\mu$ -rearrangements preserve every $L^q(D, d\mu)$ -norms, $q \in [1, +\infty]$. From this we compute, for all $\zeta \in \text{Rearg}(\zeta_\epsilon)$:

$$\begin{aligned} \int_D \log \frac{\epsilon}{|x-y|} (\zeta)_+(y) d\mu(y) &\leq \int_D \left(\log \frac{\epsilon}{|x-y|} \right)_+ (\zeta)_+(y) d\mu(y) \\ &\leq \|(\zeta)_+\|_{L^p_{d\mu}} \left(\sup_D b \right)^{\frac{1}{p}} \left(\int_D \left(\log \frac{\epsilon}{|x-y|} \right)_+^{p'} d\mu(y) \right)^{\frac{1}{p'}} \\ &\leq \|(\zeta)_+\|_{L^p_{d\mu}} \left(\sup_D b \right)^{\frac{1}{p}} \left(\int_{B(0,\epsilon)} \left(\log \frac{\epsilon}{|y|} \right)_+^{p'} d\mu(y) \right)^{\frac{1}{p'}} \\ &\leq \|(\zeta)_+\|_{L^p_{d\mu}} \left(\sup_D b \right)^{\frac{1}{p}} \epsilon^{\frac{2}{p'}} \left(\int_{B(0,1)} \left(\log \frac{1}{|y|} \right)_+^{p'} d\mu(y) \right)^{\frac{1}{p'}}. \end{aligned}$$

Using the definition of M , there exists a constant $C_6 > 0$ such that

$$\int_D \log \frac{\epsilon}{|x-y|} (\zeta)_+(y) d\mu(y) \leq C_6 \mathcal{S}_\epsilon.$$

The conclusion now follows from the definitions of $\mathcal{T}_\zeta^+$ and $\mathcal{T}_\zeta^-$. $\square$

The following lemma will be used several times in the text.

Lemma 4.1. *Let $x_1, x_2 \in \overline{D}$ be such that $b(x_1) > 0$ and $b(x_2) > 0$, and let $\mathcal{U}_1, \mathcal{U}_2 \subseteq D$ be open sets containing respectively x_1 and x_2 . There exists approximation families $\{x_\epsilon^1 \in \mathcal{U}_1 : \epsilon > 0\}$ and $\{x_\epsilon^2 \in \mathcal{U}_2 : \epsilon > 0\}$ such that*

$$\lim_{\epsilon \rightarrow 0} d(x_1, x_\epsilon^1) = 0, \quad \lim_{\epsilon \rightarrow 0} d(x_2, x_\epsilon^2) = 0,$$

and the following conditions hold for sufficiently small $\epsilon > 0$:

- there exists $r_1 > 0$ such that $\epsilon^2 \leq d\mu(B(x_\epsilon^1, r_1\epsilon))$;
- there exists $r_2 > 0$ such that $\epsilon^2 \leq d\mu(B(x_\epsilon^2, r_2\epsilon))$;
- $B(x^1, r_1\epsilon) \subseteq \mathcal{U}_1$ and $B(x^2, r_2\epsilon) \subseteq \mathcal{U}_2$;
- $d(B(x^1, r_1\epsilon), \partial D) \geq \frac{1}{\log \frac{1}{\epsilon}}$ and $d(B(x^2, r_2\epsilon), \partial D) \geq \frac{1}{\log \frac{1}{\epsilon}}$;
- $d(B(x^1, r_1\epsilon), B(x^2, r_2\epsilon)) \geq \frac{1}{\log \frac{1}{\epsilon}}$.

We omit the proof of Lemma 4.1, which may be done by geometric arguments. Note that since we do not claim anything on the regularity of ∂D at points x_1 and x_2 , the rate of convergence of the families $\{x_\epsilon^1 : \epsilon > 0\}$ and $\{x_\epsilon^2 : \epsilon > 0\}$ to their respective limit is not explicitly known.

Proposition 4.3. *For all $\epsilon > 0$ sufficiently small, there exists $\Theta_\epsilon \leq 0$ with $\lim_{\epsilon \rightarrow 0} \Theta_\epsilon = 0$ and such that there holds*

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} + (1-\tau) \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ \leq E_\epsilon(\zeta_\epsilon) - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon}. \end{aligned}$$

Proof Define the function

$$\phi : \overline{D} \rightarrow \mathbb{R} : \phi(z) = \frac{\tau b(z)}{4\pi} + \Psi(z).$$

The function ϕ reaches a maximal value at some point $x_1 \in \{b > 0\}$, and the function $\frac{b}{4\pi} - \phi$ reaches a maximal value at some point $x_2 \in \{b > 0\}$. There exists $\eta_1, \eta_2 > 0$ such that the sets $\mathcal{U}_1 = \{b > \eta_1\}$ and $\mathcal{U}_2 = \{b > \eta_2\}$ contain respectively x_1 and x_2 . Let $\{x_\epsilon^1 \in \mathcal{U}_1 : \epsilon > 0\}$ and $\{x_\epsilon^2 \in \mathcal{U}_2 : \epsilon > 0\}$ be given by Lemma 4.1, and $r_1, r_2 > 0$ the associated radii. For all $\epsilon > 0$ sufficiently small, we compare the maximal energy with the energy produced by symmetrized pairs $x_\epsilon^1 \sharp(\zeta_\epsilon)_+$ and $x_\epsilon^2 \sharp(\zeta_\epsilon)_-$. For sufficiently small $\epsilon > 0$, the function $\xi_\epsilon = (x_\epsilon^1 \sharp(\zeta_\epsilon)_+ - x_\epsilon^2 \sharp(\zeta_\epsilon)_-)$ is a $d\mu$ -rearrangement of ζ_ϵ , and

$$\{x_\epsilon^1 \sharp(\zeta_\epsilon)_+ > 0\} \subseteq B(x_\epsilon^1, r_1\epsilon), \quad \{x_\epsilon^2 \sharp(\zeta_\epsilon)_- > 0\} \subseteq B(x_\epsilon^2, r_2\epsilon).$$

Now we compute the energy E_ϵ produced by the above competitor ξ_ϵ using Proposition 4.1:

$$\begin{aligned} E_\epsilon(\xi_\epsilon) &= \int_D \mathcal{T}_{x_\epsilon^1 \sharp(\zeta_\epsilon)_+}^+ x_\epsilon^1 \sharp(\zeta_\epsilon)_+ d\mu + \int_D \mathcal{T}_{x_\epsilon^2 \sharp(\zeta_\epsilon)_-}^- x_\epsilon^2 \sharp(\zeta_\epsilon)_- d\mu \\ &\quad + \frac{1}{2} \iint_{D \times D} (F(x, y) - b(x)H(x, y)) \xi_\epsilon(x) \xi_\epsilon(y) d\mu \otimes d\mu(x, y) \\ &\quad - \iint_{D \times D} \frac{b(x) + b(y)}{4\pi} \log \frac{\text{diam}(D)}{|x - y|} x_\epsilon^1 \sharp(\zeta_\epsilon)_+(x) x_\epsilon^2 \sharp(\zeta_\epsilon)_-(y) d\mu \otimes d\mu(x, y). \end{aligned}$$

We recall that F is bounded, and H is symmetric, positive, continuous and satisfies

$$H(x, y) \leq \frac{1}{2\pi} \log \frac{\text{diam}(D)}{d(y, \partial D)}.$$

Because the diameter of $\{x_\epsilon^1 \sharp(\zeta_\epsilon)_+ > 0\}$ is smaller than $r_1\epsilon$, we have

$$\begin{aligned} &\iint_{D \times D} \frac{b(x)}{4\pi} \log \frac{\text{diam}(D)}{|x - y|} x_\epsilon^1 \sharp(\zeta_\epsilon)_+ x_\epsilon^1 \sharp(\zeta_\epsilon)_+ d\mu d\mu \\ &\geq \tau \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \int_D \frac{b}{4\pi} x_\epsilon^1 \sharp(\zeta_\epsilon)_+ d\mu - C_7 \mathcal{S}_\epsilon^2. \end{aligned}$$

for some constant $C_7 > 0$. Similarly, there exists $C_8 > 0$ such that

$$\begin{aligned} &\iint_{D \times D} \frac{b(x)}{4\pi} \log \frac{\text{diam}(D)}{|x - y|} x_\epsilon^2 \sharp(\zeta_\epsilon)_- x_\epsilon^2 \sharp(\zeta_\epsilon)_- d\mu d\mu \\ &\geq (1 - \tau) \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \int_D \frac{b}{4\pi} x_\epsilon^2 \sharp(\zeta_\epsilon)_- d\mu - C_9 \mathcal{S}_\epsilon^2. \end{aligned}$$

By construction (Lemma 4.1), we also have

$$d\left(B(x_\epsilon^1, r_1\epsilon), B(x_\epsilon^2, r_2\epsilon)\right) \geq \frac{1}{\log \frac{1}{\epsilon}},$$

so that there exists $C_{10} > 0$ with

$$\iint_{D \times D} \frac{b(x)}{4\pi} \log \frac{\text{diam}(D)}{|x - y|} x_\epsilon^1 \sharp(\zeta_\epsilon)_+ x_\epsilon^2 \sharp(\zeta_\epsilon)_- d\mu d\mu \leq \tau(1 - \tau) C_{10} \mathcal{S}_\epsilon^2 \left(1 + \log \log \frac{1}{\epsilon}\right).$$

Recalling the conditions (Lemma 4.1) that

$$d\left(B(x_\epsilon^1, r_1\epsilon), \partial D\right) \geq \frac{1}{\log \frac{1}{\epsilon}}, \quad d\left(B(x_\epsilon^2, r_2\epsilon), \partial D\right) \geq \frac{1}{\log \frac{1}{\epsilon}},$$

the upper bound for H and the boundedness of R , we obtain

$$E_{\epsilon}(x_{\epsilon}^1 \sharp(\zeta_{\epsilon})_+ - x_{\epsilon}^2 \sharp(\zeta_{\epsilon})_-) \geq T_1 + T_2 + T_3,$$

with

$$\begin{aligned} T_1 &= S_{\epsilon} \log \frac{1}{\epsilon} \int_D \frac{\tau b}{4\pi} x_{\epsilon}^1 \sharp(\zeta_{\epsilon})_+ d\mu + \int_D x_{\epsilon}^1 \sharp(\zeta_{\epsilon})_+ \lambda_{\epsilon} \Psi d\mu, \\ T_2 &= S_{\epsilon} \log \frac{1}{\epsilon} \int_D \frac{(1-\tau)b}{4\pi} x_{\epsilon}^2 \sharp(\zeta_{\epsilon})_- d\mu - \int_D x_{\epsilon}^2 \sharp(\zeta_{\epsilon})_- \lambda_{\epsilon} \Psi d\mu \end{aligned}$$

and

$$T_3 = -C_{10} S_{\epsilon}^2 \left(1 + \log \log \frac{1}{\epsilon} \right).$$

From this we conclude that

$$\begin{aligned} E_{\epsilon}(x_{\epsilon}^1 \sharp(\zeta_{\epsilon})_+ - x_{\epsilon}^2 \sharp(\zeta_{\epsilon})_-) &\geq \tau S_{\epsilon}^2 \log \frac{1}{\epsilon} \inf_{B(x_{\epsilon}^1, r_1 \epsilon)} \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\} \\ &\quad + (1-\tau) S_{\epsilon}^2 \log \frac{1}{\epsilon} \inf_{B(x_{\epsilon}^2, r_2 \epsilon)} \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\} \\ &\quad - C_{10} S_{\epsilon}^2 \left(1 + \log \log \frac{1}{\epsilon} \right). \end{aligned}$$

By definition of the family $\{\zeta_{\epsilon} : \epsilon > 0\}$, we have

$$E_{\epsilon}(\xi_{\epsilon}) = E_{\epsilon}(x_{\epsilon}^1 \sharp(\zeta_{\epsilon})_+ - x_{\epsilon}^2 \sharp(\zeta_{\epsilon})_-) \leq E_{\epsilon}((\zeta_{\epsilon})_+ - (\zeta_{\epsilon})_-) = E_{\epsilon}(\zeta_{\epsilon}),$$

and therefore we have

$$\begin{aligned} S_{\epsilon}^2 \log \frac{1}{\epsilon} \left(\tau \inf_{B(x_{\epsilon}^1, r_1 \epsilon)} \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\} + (1-\tau) \inf_{B(x_{\epsilon}^2, r_2 \epsilon)} \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ \leq E_{\epsilon}(\zeta_{\epsilon}) + C_{10} \left(1 + \log \log \frac{1}{\epsilon} \right). \end{aligned}$$

We define the error term

$$\Theta_{\epsilon} = \tau E_1 + (1-\tau) E_2 - C_{10} \frac{1 + \log \log \frac{1}{\epsilon}}{\log \frac{1}{\epsilon}},$$

where

$$E_1 = \inf_{B(x_{\epsilon}^1, r_1 \epsilon)} \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\} - \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\}$$

and

$$E_2 = \inf_{B(x_{\epsilon}^2, r_2 \epsilon)} \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\} - \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\}.$$

By definition, we directly have $\Theta_{\epsilon} \leq 0$, and

$$\begin{aligned} S_{\epsilon}^2 \log \frac{1}{\epsilon} \left(\tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\} + (1-\tau) \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ \leq E_{\epsilon}(\zeta_{\epsilon}) - \Theta_{\epsilon} S_{\epsilon}^2 \log \frac{1}{\epsilon}. \end{aligned}$$

Finally, the uniform continuity of b and Ψ on the compact set $\overline{D}$, shows that we have $\lim_{\epsilon \rightarrow 0} \Theta_{\epsilon} = 0$. $\square$

In this section we show that the main part of the vortex core is located in an area of the domain D where the leading partial flows are large. The truncation process was already used by Turkington in the context of time-dependent solutions of the Euler equations [30] (see also [28, 29]).

Corollary 4.1. *Let $\{\Theta_\epsilon \in \mathbb{R} : \epsilon > 0\}$ be a family of real numbers satisfying the claim of Proposition 4.3. There exists a constant $C > 0$ such that for all $\kappa > 0$, we have*

$$\tau \mathcal{S}_\epsilon \int_{D \setminus D_\epsilon^\kappa} d(\zeta_\epsilon)_+ \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C}{\log \frac{1}{\epsilon}} \right).$$

with

$$D_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^+(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ - \kappa \frac{\sup_D b}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon \right\};$$

and similarly

$$(1 - \tau) \mathcal{S}_\epsilon \int_{D \setminus U_\epsilon^\kappa} d(\zeta_\epsilon)_- \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C}{\log \frac{1}{\epsilon}} \right).$$

with

$$U_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^-(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^- d(\zeta_\epsilon)_- - \kappa \frac{\sup_D b}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon \right\}.$$

Proof It is sufficient to prove the claim for $\mathcal{T}_{\zeta_\epsilon}^+$ and $\tau > 0$. By assumption, we have for all sufficiently small $\epsilon > 0$:

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} + (1 - \tau) \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ \leq E_\epsilon(\zeta_\epsilon) - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon}. \end{aligned}$$

According to Proposition 4.1, there exists $C_{11} > 0$ such that

$$E_\epsilon(\zeta_\epsilon) \leq \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ + (1 - \tau) \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^- d(\zeta_\epsilon)_- + C_{11} \mathcal{S}_\epsilon^2.$$

From this we conclude the inequality

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} + (1 - \tau) \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right) \\ \leq \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ + (1 - \tau) \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^- d(\zeta_\epsilon)_- \\ - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} + C_{11} \mathcal{S}_\epsilon^2. \end{aligned}$$

On the other hand, an application of Proposition 4.2 for $\mathcal{T}_{\zeta_\epsilon}^-$ yields

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \\ \leq \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} + C_{12} \mathcal{S}_\epsilon^2. \end{aligned}$$

An application of Proposition 4.2 for $\mathcal{T}_{\zeta_\epsilon}^+$ yields in turn

$$\mathcal{T}_{\zeta_\epsilon}^+ \leq \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} + C_{13} \mathcal{S}_\epsilon.$$

On D we thus have

$$\tau \mathcal{S}_\epsilon \mathcal{T}_{\zeta_\epsilon}^+ - \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ \leq \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} + C_{14} \mathcal{S}_\epsilon^2.$$

Now we compute

$$\begin{aligned} & \int_{D \setminus D_\epsilon^\kappa} \left(\int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ - \mathcal{T}_{\zeta_\epsilon}^+ \right) d(\zeta_\epsilon)_+ \\ &= \int_{D_\epsilon^\kappa} \left(\mathcal{T}_{\zeta_\epsilon}^+ - \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ \right) d(\zeta_\epsilon)_+. \end{aligned}$$

Using the definition of D_ϵ^κ we obtain

$$\tau \mathcal{S}_\epsilon \int_{D \setminus D_\epsilon^\kappa} d(\zeta_\epsilon)_+ \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C_{14}}{\log \frac{1}{\epsilon}} \right). \quad \square$$

4.2 Concentration of the truncated vortices

In this section we prove that most of the vortex is located in two small balls: one containing the positive part of the pair, and the other the negative part of the pair.

Theorem 4.1. *Let $\{\Theta_\epsilon : \epsilon > 0\}$ be a family of real numbers as in Proposition 4.3, and let*

$$D_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^+(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ - \kappa \frac{\sup_D b}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon \right\}$$

and

$$U_\epsilon^\kappa = \left\{ x \in D : \mathcal{T}_{\zeta_\epsilon}^-(x) \geq \int_D \mathcal{T}_{\zeta_\epsilon}^- d(\zeta_\epsilon)_- - \kappa \frac{\sup_D b}{4\pi} \log \frac{1}{\epsilon} \mathcal{S}_\epsilon \right\}.$$

There exists $\sigma, \kappa > 0$ such that for all sufficiently small $\epsilon > 0$,

$$\text{diam}(D_\epsilon^\kappa) \leq \epsilon^\sigma$$

and

$$\text{diam}(U_\epsilon^\kappa) \leq \epsilon^\sigma.$$

Proof It is sufficient to prove the claim for $(\zeta_\epsilon)_+$ and $\tau > 0$. Recall that we always assume that there exists $\varsigma > 0$ such that, for all sufficiently small $\epsilon > 0$:

$$\sup_D \left\{ \frac{4\pi \lambda_\epsilon |\Psi|}{(\sup_D b) \mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \leq \frac{\tau}{2} - \varsigma.$$

In particular, there exists $s, \delta, \eta \in (0, 1)$ close to 1 and $\kappa > 0$ close to 0 such that for all sufficiently small $\epsilon > 0$:

$$(1 - \delta)\tau + (1 - \eta) \frac{(1 - \tau)^2}{\tau} + \kappa + \sup_D \left\{ \frac{4\pi \lambda_\epsilon |\Psi|}{(\sup_D b) \mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} < \frac{s\tau}{2}.$$

Let $\{\Theta_\epsilon \in \mathbb{R} : \epsilon > 0\}$ be a family of real numbers as in Proposition 4.3. Let $C > 0$ and D_ϵ^+ be given as in Corollary 4.1. First pick points $x^*, y^* \in D$ such that

$$\frac{\tau b(x^*)}{4\pi} > \frac{\delta + 1}{2} \sup_D \frac{\tau b}{4\pi}$$

and $b(y_*) > 0$, $y^* \neq x^*$ and

$$\frac{(1-\tau)b(y^*)}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi(y^*) > \frac{\eta + 1}{2} \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\}.$$

Such points exist by continuity. Let us compute the energy produced by the competitor $\tilde{\zeta}_\epsilon = x^* \sharp(\zeta_\epsilon)_+ - y^* \sharp(\zeta_\epsilon)_-$. For sufficiently small $\epsilon > 0$, the function $\tilde{\zeta}$ belongs to $\text{Rearg}(\zeta_\epsilon)$ because $y^* \neq x^*$. Thus for sufficiently small $\epsilon > 0$ we have

$$E_\epsilon(x^* \sharp(\zeta_\epsilon)_+ - y^* \sharp(\zeta_\epsilon)_-) \leq E_\epsilon(\zeta_\epsilon).$$

According to Proposition 4.1 and Proposition 4.2, we then have

$$\begin{aligned} E_\epsilon(\zeta_\epsilon) &\leq \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ \\ &+ (1-\tau) \log \frac{1}{\epsilon} \mathcal{S}_\epsilon^2 \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} + C_{15} \mathcal{S}_\epsilon^2. \end{aligned}$$

On the other hand, because $x^* \neq y^*$ with $d(x^*, \partial D) > 0$ and $d(y^*, \partial D) > 0$, there exists $C_{16} > 0$ such that for all sufficiently small $\epsilon > 0$, we have

$$\begin{aligned} \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\frac{\delta \tau (\sup_D b)}{4\pi} - \sup_D \left\{ \frac{\lambda_\epsilon |\Psi|}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \right) &+ (1-\tau) \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \eta \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon \Psi}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \\ &\leq E_\epsilon(x^* \sharp(\zeta_\epsilon)_+ - y^* \sharp(\zeta_\epsilon)_-) + C_{16} \mathcal{S}_\epsilon^2. \end{aligned}$$

From this we conclude that, for all sufficiently small $\epsilon > 0$:

$$\begin{aligned} \tau \mathcal{S}_\epsilon \int_D \mathcal{T}_{\zeta_\epsilon}^+ d(\zeta_\epsilon)_+ &\geq \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\frac{\delta \tau (\sup_D b)}{4\pi} - \sup_D \left\{ \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \right) \\ &+ (\eta - 1)(1-\tau) \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} - C_{17} \mathcal{S}_\epsilon^2 \end{aligned}$$

Recalling the definition of D_ϵ^κ in Corollary 4.1, we have for all $x \in D_\epsilon^\kappa$:

$$\begin{aligned} \tau \mathcal{S}_\epsilon \mathcal{T}_{\zeta_\epsilon}^+(x) &\geq \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\frac{\delta \tau (\sup_D b)}{4\pi} - \sup_D \left\{ \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \right) \\ &+ (\eta - 1)(1-\tau) \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \\ &- C_{18} \mathcal{S}_\epsilon^2 - \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \kappa \frac{\sup_D b}{4\pi}. \end{aligned}$$

Now by definition of $\mathcal{T}_{\zeta_\epsilon}^+$, we have

$$\mathcal{T}_{\zeta_\epsilon}^+(x) \leq \frac{\sup_D b}{4\pi} \int_D \log \frac{\text{diam}(D)}{|x-y|} (\zeta_\epsilon)_+(y) d\mu(y) + \sup_D \{ \lambda_\epsilon |\Psi| \},$$

and therefore, for all $x \in D_\epsilon^\kappa$ and sufficiently small $\epsilon > 0$:

$$\begin{aligned} & \tau^2 \mathcal{S}_\epsilon^2 \frac{\sup_D b}{4\pi} \int_D \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) \\ & \geq \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\frac{(\delta-1)\tau(\sup_D b)}{4\pi} - \sup_D \left\{ \frac{2\lambda_\epsilon |\Psi|}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \right\} \right) \\ & + (\eta-1)(1-\tau) \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \\ & - C_{19} \mathcal{S}_\epsilon^2 - \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \kappa \frac{\sup_D b}{4\pi}. \end{aligned}$$

Regrouping terms, we obtain

$$\begin{aligned} & \tau^2 \mathcal{S}_\epsilon^2 \int_D \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) \\ & \geq -C_{20} \mathcal{S}_\epsilon^2 - \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left((1-\delta)\tau^2 + (1-\eta)(1-\tau)^2 + \kappa\tau \right) \\ & - \tau \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \sup_D \left\{ \frac{8\pi\lambda_\epsilon}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \end{aligned}$$

Since $\tau > 0$, we obtain in particular

$$\begin{aligned} & \tau \mathcal{S}_\epsilon \int_D \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) \\ & \geq -C_{21} \mathcal{S}_\epsilon - \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \left((1-\delta)\tau + (1-\eta) \frac{(1-\tau)^2}{\tau} + \kappa \right) \\ & - \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \sup_D \left\{ \frac{8\pi\lambda_\epsilon}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} \end{aligned}$$

Using the fact that

$$(1-\delta)\tau + (1-\eta) \frac{(1-\tau)^2}{\tau} + \kappa + \sup_D \left\{ \frac{8\pi\lambda_\epsilon}{(\sup_D b)\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} |\Psi| \right\} < \frac{s\tau}{2},$$

we obtain the strict lower bound

$$\tau \mathcal{S}_\epsilon \int_D \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) > -C_{22} \mathcal{S}_\epsilon - \frac{s\tau}{2} \mathcal{S}_\epsilon \log \frac{1}{\epsilon}.$$

Fix $R \geq 1$. Then using the same computation than the one in the proof of Proposition 4.2, we infer the existence of some constant $C_{23} > 0$ such that, for all $x \in D$:

$$\int_{B(x, R\epsilon)} \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) \leq C_{23}.$$

Injecting this upper bound in the above computation, we obtain for all $x \in D_\epsilon^\kappa$ and for all $R \geq 1$:

$$\tau \mathcal{S}_\epsilon \int_{D \setminus B(x, R\epsilon)} \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) > -C_{24} \mathcal{S}_\epsilon - \frac{s\tau}{2} \mathcal{S}_\epsilon \log \frac{1}{\epsilon}.$$

Thus for all $R > 1$ and for all $x \in D_\epsilon^\kappa$, we have

$$\tau \mathcal{S}_\epsilon \int_{D \setminus B(x, R\epsilon)} \log \frac{\epsilon}{|x-y|} \, d(\zeta_\epsilon)_+(y) < \frac{C_{25} \mathcal{S}_\epsilon + \frac{s\tau}{2} \mathcal{S}_\epsilon \log \frac{1}{\epsilon}}{\log(R)}.$$

Choose $R = \epsilon^{-s}$. Then we have for all $x \in D_\epsilon^\kappa$, and for all sufficiently small $\epsilon > 0$:

$$\tau \mathcal{S}_\epsilon \int_{D \setminus B(x, R\epsilon)} d(\zeta_\epsilon)_+(y) < \frac{1}{2} \tau \mathcal{S}_\epsilon.$$

Choosing $x_1, x_2 \in D_\epsilon^\kappa$, we now observe that if $d(x_1, x_2) \geq R\epsilon = \epsilon^{1-s}$, then we would have

$$\tau \mathcal{S}_\epsilon \leq \tau \mathcal{S}_\epsilon \int_{D \setminus B(x_1, R\epsilon)} d(\zeta_\epsilon)_+(y) + \tau \mathcal{S}_\epsilon \int_{D \setminus B(x_2, R\epsilon)} d(\zeta_\epsilon)_+(y) < \tau \mathcal{S}_\epsilon,$$

which would be a contradiction. Thus any two point x_1, x_2 in D_ϵ^κ satisfy $d(x_1, x_2) \leq \epsilon^{1-s}$. The claim now follows from Corollary 4.1. $\square$

From the following result, one may conclude Theorem 1.1, page 4:

Theorem 4.2. *If $\tau > 0$, then every accumulation points as $\epsilon \rightarrow 0$ of the family of probability measures $\{d(\zeta_\epsilon)_+ : \epsilon > 0\}$, in the sense of vague convergence of measures, is a Dirac mass δ_{x^*} with*

$$\frac{\tau b(x^*)}{4\pi} + \Psi(x^*) = \sup_D \left\{ \frac{\tau b}{4\pi} + \Psi \right\}.$$

Similarly, if $\tau < 1$, then every accumulation points as $\epsilon \rightarrow 0$ of the family of probability measures $\{d(\zeta_\epsilon)_- : \epsilon > 0\}$, in the sense of vague convergence of measures, is a Dirac mass δ_{x_} with*

$$\frac{(1-\tau)b(x_*)}{4\pi} - \Psi(x_*) = \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \Psi \right\}.$$

Proof It is sufficient to prove the claim for the positive parts $(\zeta_\epsilon)_+$ with $\tau > 0$. According to Proposition 4.3, there exists $\{\Theta_\epsilon : \epsilon > 0\}$ such that, for sufficiently small $\epsilon > 0$:

$$\begin{aligned} \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \left(\tau \sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi + (1-\tau) \sup_D \left\{ \frac{(1-\tau)b}{4\pi} - \frac{\lambda_\epsilon}{\mathcal{S}_\epsilon \log \frac{1}{\epsilon}} \Psi \right\} \right) \right. \\ \left. \leq E_\epsilon(\zeta_\epsilon) - \Theta_\epsilon \mathcal{S}_\epsilon^2 \log \frac{1}{\epsilon} \right) \end{aligned}$$

According to Theorem 4.1, there exists $\sigma, \kappa > 0$ such that for all sufficiently small $\epsilon > 0$, there exists a ball $B_\epsilon^+ = B(x_\epsilon, \epsilon^\sigma)$ with $x_\epsilon \in D$ and such that

$$\int_{B_\epsilon^+} (\zeta_\epsilon)_+ d\mu \leq \frac{1}{\kappa} \frac{4\pi}{\sup_D b} \mathcal{S}_\epsilon \left(\Theta_\epsilon + \frac{C}{\log \frac{1}{\epsilon}} \right).$$

Let $x^* \in \overline{D}$ be an accumulation point for $\{x_\epsilon : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, and $(\epsilon_n)_{n \in \mathbb{N}}$ the corresponding sequence, decreasing to 0. For all $\varphi \in C(\overline{D})$, we have from the construction of x_ϵ :

$$\lim_{n \rightarrow +\infty} \left| \varphi(x^*) - \int_D \varphi d(\zeta_{\epsilon_n})_+ \right| = 0.$$

Therefore the sequence of probability measures $((\zeta_{\epsilon_n})_+)_{n \in \mathbb{N}}$ converges in the sense of vague convergence to δ_{x^*} . Now for the last part of the claim, we apply Proposition 4.2 and the Green's function expansion to obtain

$$\begin{aligned} E_{\epsilon}(\zeta_{\epsilon}) &\leq \tau S_{\epsilon} \int_D \left(\frac{\tau b}{4\pi} \log \frac{1}{\epsilon} S_{\epsilon} + \lambda_{\epsilon} \Psi \right) d(\zeta_{\epsilon})_+ \\ &\quad + (1 - \tau) S_{\epsilon}^2 \log \frac{1}{\epsilon} \sup_D \left\{ \frac{(1 - \tau)b}{4\pi} - \frac{\lambda_{\epsilon}}{\log \frac{1}{\epsilon} S_{\epsilon}} \Psi \right\} + C_{26} S_{\epsilon}^2. \end{aligned}$$

This gives the inequality

$$\sup_D \left\{ \frac{\tau b}{4\pi} + \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right\} - \int_D \left(\frac{\tau b}{4\pi} + \frac{\lambda_{\epsilon}}{S_{\epsilon} \log \frac{1}{\epsilon}} \Psi \right) d(\zeta_{\epsilon})_+ \leq \frac{1}{\tau} \left(|\Theta_{\epsilon}| + \frac{C_{27}}{\log \frac{1}{\epsilon}} \right).$$

Letting $\epsilon \rightarrow 0$ and relying on compactness of $\overline{D}$ and continuity of b and Ψ , we obtain

$$\sup_D \left\{ \frac{\tau b}{4\pi} + \Psi \right\} = \lim_{\epsilon \rightarrow 0} \int_D \left(\frac{\tau b}{4\pi} + \Psi \right) d(\zeta_{\epsilon})_+. \quad \square$$

5 Construction of a Rotating Singular Pair

In this section we apply Theorem 4.2 to construct a rotating singular pair in a lake $(B(0, 1), b)$ invariant under rotations. On $D = B(0, 1)$ we will consider a continuous radial depth function $b : D \rightarrow \mathbb{R}$ with the property that (D, b) is continuous. Our computations may also be done in the case where D is an annulus, the only required change to bring would be the introduction of prescribed circulations on the inner boundary.

Given $|D|_{\mu} = d\mu(D) > 0$ the measure of the domain D according to the measure $d\mu$, let us fix a distribution function

$$D : \mathbb{R}^+ \rightarrow [0, d\mu_D]$$

normalized as

$$\int_{\mathbb{R}^+} D(t) dt = 1,$$

and such that there exists $p > 1$ with

$$\int_{\mathbb{R}^+} t^p D(t) dt < +\infty.$$

We also fix $\tau \in [0, 1]$. We say that a family of measurable functions $\{\zeta_{\epsilon} : \epsilon > 0\}$ satisfies the constraints (D)

$$\begin{aligned} d\mu(\{(\zeta_{\epsilon})_+ \geq \lambda\}) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2 \lambda}{\delta \tau} \log \frac{1}{\epsilon}\right), \\ d\mu(\{(\zeta_{\epsilon})_- \geq \lambda\}) &= \frac{\epsilon^2}{\delta} D\left(\frac{\epsilon^2 \lambda}{\delta(1-\tau)} \log \frac{1}{\epsilon}\right), \end{aligned} \quad (D)$$

where $\delta = \sup_{\lambda > 0} D(\lambda)$. An adaptation of the proof of Proposition 3.1 shows that such family always exists. A straightforward computation also shows that such a family satisfies

$$\begin{aligned} \mu(\{(\zeta_{\epsilon})_+ > 0\}) &\leq \epsilon^2, & \mu(\{(\zeta_{\epsilon})_- > 0\}) &\leq \epsilon^2, \\ \int_D (\zeta_{\epsilon})_+ d\mu &= \frac{\tau}{\log \frac{1}{\epsilon}}, & \int_D (\zeta_{\epsilon})_- d\mu &= \frac{1 - \tau}{\log \frac{1}{\epsilon}}, \end{aligned}$$

and finally

$$\sup_{\epsilon > 0} \left\{ \log \frac{1}{\epsilon} \epsilon^{2(1-\frac{1}{p})} \left(\|(\zeta_\epsilon)_+\|_{L^p_{d\mu}} + \|(\zeta_\epsilon)_-\|_{L^p_{d\mu}} \right) \right\} < +\infty.$$

We are thus in position to apply Theorem 4.2. Exploiting the symmetry of the domain, we construct a rotating vortex pair around the origin $(0, 0)$:

Theorem 5.1 (Theorem 1.2, p. 5). *Assume that $D = B(0, 1)$ and $b \in C(\overline{D}) \cap W^{1,\infty}_{loc}(D)$ is a radial function: $b(x) = b(|x|^2)$, such that (D, b) is a continuous lake.*

There exists $v_0 > 0$ depending only on $\tau \in [0, 1]$ such that, for all $v \in [0, v_0]$, there exists a family $\{(\zeta_\epsilon, u_\epsilon) \in C(\mathbb{R}, L^p(D, d\mu)) \times C(\mathbb{R}, L^2(D, d\mu)) : \epsilon > 0\}$ that solves the weak form of

$$\begin{cases} \operatorname{div}(bu_\epsilon) = 0 \\ \operatorname{curl}(u_\epsilon) = b\zeta_\epsilon \\ \partial_t \zeta_\epsilon + (u_\epsilon | \nabla \zeta_\epsilon)_{\mathbb{R}^2} = 0; \end{cases}$$

the family $\{\zeta_\epsilon : \epsilon > 0\}$ satisfies constraint (D), and for each $t \in \mathbb{R}$, the potential vortex $\zeta_\epsilon(t)$ at time t is obtained from $\zeta_\epsilon(0)$ by a rotation of clockwise angle vt .

Furthermore, if $\tau > 0$, the only possible accumulation points of $\{(\zeta_\epsilon(0))_+ : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses $\delta_{x^\star}$ with

$$\frac{\tau b(x^\star)}{4\pi} - \frac{v}{2} \int_0^{|x^\star|^2} b(s) ds = \sup_{z \in D} \left\{ \frac{\tau b(z)}{4\pi} - \frac{v}{2} \int_0^{|z|^2} b(s) ds \right\},$$

and if $\tau < 1$, the only possible accumulation points of $\{(\zeta_\epsilon)_- : \epsilon > 0\}$ as $\epsilon \rightarrow 0$, in the sense of convergence of probability measures on D , are Dirac masses $\delta_{x_\star}$ with

$$\frac{(1-\tau)b(x_\star)}{4\pi} + \frac{v}{2} \int_0^{|x_\star|^2} b(s) ds = \sup_{z \in D} \left\{ \frac{(1-\tau)b(z)}{4\pi} + \frac{v}{2} \int_0^{|z|^2} b(s) ds \right\}.$$

Since the boundary of $D = B(0, 1)$ is connected, the operator $\nabla \mathfrak{H}$ is null, so that a weak formulation of the time-dependent evolution equation

$$\partial_t \zeta_t + (u_t | \nabla \zeta_t)_{\mathbb{R}^2} = 0$$

reads as

$$\int_{\mathbb{R}} \int_D \partial_t (\varphi(t, \cdot)) \zeta_\epsilon(t) d\mu dt + \int_{\mathbb{R}} \int_D (\nabla^\perp \mathfrak{R}(\zeta_\epsilon(t)) | \nabla \varphi(t, \cdot))_{\mathbb{R}^2} \zeta_\epsilon dm dt = 0,$$

for all $\varphi \in C_c^1(\mathbb{R} \times D)$.

Proof of Theorem 5.11 Define for all $x \in \overline{B(0, 1)}$:

$$\Psi(x) = \frac{v}{2} \int_0^{|x|^2} b(s) ds,$$

with $v \in \mathbb{R}$ such that

$$|v| < \frac{\min\{\tau, (1-\tau)\}}{4\pi}.$$

In particular, this implies that

$$\sup_D \left\{ \frac{4\pi |\Psi|}{(\sup_D b)} \right\} < \frac{\min\{\tau, 1-\tau\}}{2}.$$

For all $\epsilon > 0$, consider the energy

$$E_\epsilon : L^p(D, d\mu) \rightarrow \mathbb{R} : E_\epsilon(\zeta) = \frac{1}{2} \int_D \zeta \mathfrak{K} \zeta d\mu + \mathcal{S}_\epsilon \log \frac{1}{\epsilon} \int_D \Psi \zeta d\mu.$$

According to Theorem 4.2, there exists a family $\{\zeta_\epsilon : \epsilon > 0\}$ of solutions of the steady lake equation with Coriolis external flow Ψ , that maximizes E_ϵ over their set of own $d\mu$ -rearrangements. Thus for all $\phi \in C_c^1(D)$, we have

$$\int_D \zeta_\epsilon (\nabla^\perp (\mathfrak{K} \zeta_\epsilon + \Psi) |\nabla \phi|)_{\mathbb{R}^2} d\mu = 0,$$

or also, using the definition of Ψ :

$$\int_D \zeta_\epsilon (\nabla^\perp \mathfrak{K} \zeta_\epsilon - \nu x^\perp b(x) |\nabla \phi|)_{\mathbb{R}^2} d\mu = 0.$$

For all $\epsilon > 0$ and for all $t \in \mathbb{R}$, let us define $\tilde{\zeta}_\epsilon(t) = \zeta_\epsilon \circ R_{t\nu}$, where R_α is the notation of angle α in the plane $\mathbb{R}^2$. We have $\tilde{\zeta}_\epsilon \in C(\mathbb{R}, \text{Rearg}(\zeta_\epsilon))$, for all $\epsilon > 0$. Furthermore, we have $\mathfrak{K} \tilde{\zeta}_\epsilon(t) = \mathfrak{K} \zeta_\epsilon \circ R_{t\nu}$. A change of variables in space shows that

$$\begin{aligned} & \int_{\mathbb{R}} \int_D (\nabla^\perp \mathfrak{K} \tilde{\zeta}_\epsilon(t)(x) |\nabla \varphi(t, x)|)_{\mathbb{R}^2} \tilde{\zeta}_\epsilon(t, x) d\mu(x) dt \\ &= \int_{\mathbb{R}} \int_D (\nabla^\perp \mathfrak{K} \zeta_\epsilon(x) |\nabla_x (\varphi(t, \cdot) \circ R_{t\nu}^{-1})(x)|)_{\mathbb{R}^2} \zeta_\epsilon d\mu(x) dt \\ &= -\nu \int_{\mathbb{R}} \int_D (x^\perp |\nabla (\varphi(t, \cdot) \circ R_{t\nu}^{-1})(x)|)_{\mathbb{R}^2} \zeta_\epsilon(x) d\mu(x) dt. \end{aligned}$$

Another change of variable in space and an integration by parts in time also give

$$\begin{aligned} \int_{\mathbb{R}} \int_D \partial_t \varphi(t, x) \tilde{\zeta}_\epsilon(t, x) d\mu(x) dt &= \int_D \int_{\mathbb{R}} \zeta_\epsilon(x) (\partial_t \varphi)(t, R_{-t\nu}(x)) dt d\mu \\ &= \nu \int_D \int_{\mathbb{R}} \zeta_\epsilon(x) (x^\perp |\nabla_x (\varphi \circ R_{t\nu}^{-1})(x)|)_{\mathbb{R}^2} dt d\mu(x). \end{aligned}$$

Hence we have

$$\int_{\mathbb{R}} \int_D \partial_t \varphi(t, x) \tilde{\zeta}_\epsilon(t, x) d\mu(x) dt + \int_{\mathbb{R}} \int_D (\nabla^\perp \mathfrak{K} \tilde{\zeta}_\epsilon(t)(x) |\nabla \varphi(t, x)|)_{\mathbb{R}^2} \tilde{\zeta}_\epsilon(t, x) d\mu(x) dt = 0,$$

for all test function $\varphi \in C_c^1(\mathbb{R} \times D)$ and for all $\epsilon > 0$. $\square$

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Appendix: Green's Function Expansion

In this section we prove our Green's function expansion, Theorem 2.3; that is: we would like to prove that the operator $\mathfrak{K}$ admits the integral representation

$$\mathfrak{K} \zeta(x) = b(x) \int_D G_D(x, y) \zeta(y) d\mu(y) + \int_D R(x, y) \zeta(y) d\mu(y)$$

for some $R : D \times D \rightarrow \mathbb{R}$ bounded and measurable, and G_D the Green's function for the Laplace's operator $-\Delta$ on D with Dirichlet boundary conditions.

Before doing the proof, let us sketch the general idea. For fixed $y \in D$, consider the ansatz

$$g^\sharp : D \rightarrow \mathbb{R} : g^\sharp(x) = b(x)G_D(x, y).$$

Then we have at a formal level

$$\nabla g^\sharp = b \nabla G_D(\cdot, y) + G_D(\cdot, y) \nabla b,$$

and therefore

$$\begin{aligned} -\operatorname{div} \left(b^{-1} \nabla g^\sharp \right) &= -\Delta G_D(\cdot, y) - \operatorname{div} \left(G_D(\cdot, y) b^{-1} \nabla b \right) \\ &= \delta_y - \operatorname{div} \left(G_D(\cdot, y) b^{-1} \nabla b \right). \end{aligned}$$

It is then natural to look for $R(\cdot, y) \in \mathcal{H}_0$ such that

$$-\operatorname{div} \left(b^{-1} \nabla R(\cdot, y) \right) = \operatorname{div} \left(G_D(\cdot, y) b^{-1} \nabla b \right),$$

whose weak form is

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b}, \quad \forall \varphi \in \mathcal{H}_0.$$

Proposition A.1. *Let (D, b) be a continuous lake. For all $y \in D$, there exists a unique function $R(\cdot, y) \in \mathcal{H}_0$ such that*

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b}, \quad \forall \varphi \in \mathcal{H}_0.$$

Moreover, the function $R : D \times D \rightarrow \mathbb{R} : (x, y) \mapsto R(x, y)$ is measurable and bounded on $D \times D$.

We mimic the proof of [Theorem 8.8, 32].

Proof We recall that since (D, b) is continuous, by Definition 2.2, there exists $\ell > 2$ such that function

$$y \in D \mapsto \int_D g(\cdot, y)^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} dm$$

is bounded on D , and for all $y_\star \in D$, we have

$$\lim_{y \rightarrow y_\star} \int_D |g(\cdot, y) - g(\cdot, y_\star)|^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} dm = 0.$$

First define

$$\kappa = 2 \sup_{y \in D} \left\{ \int_D G_D(\cdot, y)^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} dm \right\}^{\frac{1}{\ell}}.$$

From the Hölder's inequality, since D is bounded by assumption that (D, b) is a lake, we have

$$\left| \int_D G_D(\cdot, y)^2 \frac{|\nabla b|^2}{b} dm \right|^{\frac{1}{2}} \leq C_1 \kappa,$$

for some $C_1 > 0$, and therefore the linear functional

$$\varphi \in \mathcal{H}_0 \mapsto - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b}$$

is continuous on $\mathcal{H}_0$. According to Riesz's representation theorem, there exists a unique function $R(\cdot, y) \in \mathcal{H}_0$ in $\mathcal{H}_0$ such that

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b}, \quad \forall \varphi \in \mathcal{H}_0,$$

and $\|R(\cdot, y)\|_{\mathcal{H}_0} \leq C_1 \kappa$. Let us now prove that $R(\cdot, y)$ is bounded independently of $y \in D$. For all $\beta \geq 1$ and for all $N > \kappa$, define the auxiliary functions

$$F : \mathbb{R}^+ \rightarrow \mathbb{R} : F(t) = \max \{N, t^\beta\},$$

and

$$G : \mathbb{R} \rightarrow \mathbb{R} : G(t) = \int_{\kappa}^{\max\{t, \kappa\}} |F'(s)|^2 ds,$$

so that for all $t \in \mathbb{R}$, we have

$$G'(t) = \chi_{t \geq \kappa} |F'(t)|^2 = \chi_{[\kappa, N]}(t) \beta^2 t^{2(\beta-1)}.$$

Thus G is derivable with bounded derivative, and $G(t) = 0$ for all $t \leq \kappa$. Define the positive function $u = \max \{R(\cdot, y), \kappa\}$. Then the function $G \circ u$ belongs to $\mathcal{H}_0$ with

$$\nabla(G \circ u) = \nabla u (G' \circ u) = \chi_{\{R(\cdot, y) \geq \kappa\}} \nabla R(\cdot, y) (G' \circ u).$$

From this it follows that

$$\int_D (\nabla u | \nabla(G \circ u))_{\mathbb{R}^2} \frac{dm}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla u)_{\mathbb{R}^2} (G' \circ u) \frac{dm}{b}.$$

Since $(\nabla u | \nabla(G \circ u))_{\mathbb{R}^2} = |\nabla(F \circ u)|^2$ is positive, one may apply Schwartz' and Young's inequalities to get

$$\int_D (\nabla u | \nabla(G \circ u))_{\mathbb{R}^2} \frac{dm}{b} \leq 4 \int_D G_D(\cdot, y)^2 \frac{|\nabla b|^2}{b} (G' \circ u) dm.$$

Recalling that $u \geq \kappa \geq 0$ by construction, we have

$$\int_D (\nabla u | \nabla(G \circ u))_{\mathbb{R}^2} \frac{dm}{b} \leq \frac{4}{\kappa^2} \int_D G_D(\cdot, y)^2 \frac{|\nabla b|^2}{b} |u|^2 (G' \circ u) dm,$$

or also, in terms of the auxiliary function F :

$$\int_D |\nabla(F \circ u)|^2 \frac{dm}{b} \leq \frac{4}{\kappa^2} \int_D G_D(\cdot, y)^2 \frac{|\nabla b|^2}{b} |u(F' \circ u)|^2 dm.$$

By Hölder's inequality

$$\int_D |\nabla(F \circ u)|^2 \frac{dm}{b} \leq \frac{4}{\kappa^2} \left(\int_D G_D(\cdot, y)^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} dm \right)^{\frac{2}{\ell}} \|u(F' \circ u)\|_{L_{dm}^{\frac{2\ell}{\ell-2}}}^2.$$

By construction of κ , we have

$$\int_D |\nabla(F \circ u)|^2 \frac{dm}{b} \leq \|u(F' \circ u)\|_{L_{dm}^{\frac{2\ell}{\ell-2}}}^2.$$

Since b is bounded and D is bounded, one may use Hölder's inequality and Sobolev's inequality (because $F \circ u \in W_0^{1,2}(D)$) to show that there exists some constant $C_2 > 0$ such that

$$\|F \circ u\|_{L_{\text{dm}}^{\frac{2\ell}{\ell-2}}} \leq C_2 \|u(F' \circ u)\|_{L_{\text{dm}}^{\frac{2\ell}{\ell-2}}}.$$

Using the definition of F and letting $N \rightarrow +\infty$, we obtain for all $\beta \geq 1$:

$$\|u^\beta\|_{L_{\text{dm}}^{\frac{2\ell}{\ell-2}}} \leq \beta C_2 \|u^\beta\|_{L_{\text{dm}}^{\frac{2\ell}{\ell-2}}}.$$

One may now use the technique of iteration of norms to prove that there exists some constant $C_3 > 0$ such that

$$\|u\|_{L^\infty} \leq C_3 \|u\|_{L_{\text{dm}}^{\frac{2\ell}{\ell-2}}} \leq C_4 (\kappa + \|\mathbf{R}(\cdot, y)\|_{\mathcal{H}_0}) \leq C_5 \kappa.$$

This implies, by construction of u , that $|\mathbf{R}(\cdot, y)| \leq \max\{1, C_5\}\kappa$. Observe that by symmetry of the absolute value, this estimate also holds for $-\mathbf{R}(\cdot, y)$, so that

$$\|\mathbf{R}(\cdot, y)\|_{L^\infty} \leq C_6 \left(\int_D G_D(\cdot, y)^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \text{dm} \right)^{\frac{1}{\ell}},$$

for all $y \in D$. Because the lake (D, b) is continuous, the function

$$\mathbf{R} : D \times D \rightarrow \mathbb{R} : (x, y) \mapsto \mathbf{R}(x, y)$$

is uniformly bounded. Furthermore, for fixed $y \in D$, the function $[x \in D \mapsto \mathbf{R}(x, y)]$ is measurable (by construction) and for fixed $x \in D$, the function $[y \in D \mapsto \mathbf{R}(x, y)]$ is continuous. Indeed, by linearity of the elliptic equation and according to the above estimate, we have for all points $y, y^* \in D$:

$$\begin{aligned} |\mathbf{R}(x, y) - \mathbf{R}(x, y^*)| &\leq \sup_{z \in D} |\mathbf{R}(z, y) - \mathbf{R}(z, y^*)| \\ &\leq C_7 \left(\int_D |G_D(\cdot, y) - G_D(\cdot, y^*)|^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \text{dm} \right)^{\frac{1}{\ell}}, \end{aligned}$$

which goes to 0 as $y \rightarrow y^*$. In particular, the function $\mathbf{R}$ is measurable on $D \times D$. $\square$

Remark A.1. Let $\varphi \in C_c^1(D)$ and consider the solution of the problem

$$-\text{div}(b^{-1} \nabla \psi) = -\text{div}(\varphi b^{-1} \nabla b).$$

Then using standard regularity theory [Theorem 8.22, 32], one can prove that the solution ψ_φ of this problem is continuous. In particular, if we let φ goes to $G_D(\cdot, y)$ for some fixed $y \in D$, in such a way that

$$\left(\int_D |\varphi - G_D(\cdot, y)|^\ell \left(\frac{|\nabla b|^2}{b} \right)^{\frac{\ell}{2}} \text{dm} \right)^{\frac{1}{\ell}} \rightarrow 0, \quad \varphi \rightarrow G_D(\cdot, y),$$

then we obtain that $\psi_\varphi \rightarrow \mathbf{R}(\cdot, y)$ uniformly, and the function $\mathbf{R}(\cdot, y)$ is continuous on D . Now using the uniform continuity of the function $\mathbf{R}(x, \cdot)$, for fixed $x \in D$, we obtain the continuity of the function $\mathbf{R}$ on the product $D \times D$.

We are now ready to prove Theorem 2.3, which we recall now:

Theorem (Theorem 2.3, page 18). *Let (D, b) be a continuous lake. There exists a bounded measurable function $R : D \times D \rightarrow \mathbb{R}$ such that, for all $\zeta \in L^p(D, d\mu)$, $p > 1$, we have almost-everywhere*

$$\mathfrak{R}\zeta(x) = b(x) \int_D G_D(x, y) \zeta(y) d\mu(y) + \int_D R(x, y) \zeta(y) d\mu(y).$$

Furthermore, we have $R(\cdot, y) \in \mathcal{H}_0$ for all $y \in D$, with for all $\varphi \in \mathcal{H}_0$:

$$\int_D (\nabla R(\cdot, y) | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b} = - \int_D (G_D(\cdot, y) \nabla b | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b}.$$

Proof Let R be as given by Proposition A.1. Then R is bounded and measurable. Without loss of generality, one may assume that $\zeta \in L^\infty(D)$. Define

$$\psi : D \rightarrow \mathbb{R} : \psi(x) = \int_D G_D(x, y) \zeta(y) d\mu(y).$$

The function ψ belongs to $W_0^{1,2}(D) \cap L^\infty(D)$. Let $\varphi \in C_c^1(D)$, and let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence of $C_c^1(D)$ functions converging in $W_0^{1,2}(D)$ and uniformly to the function ψ . We have

$$\int_D b \psi_n \nabla \varphi dm = - \int_D (\nabla b) \psi_n \varphi dm - \int_D b \varphi \nabla \psi_n dm,$$

and letting $n \rightarrow +\infty$ yields the chain rule formula

$$\nabla(b\psi) = b \nabla \psi + \psi \nabla b.$$

On the other hand, for all $y \in D$, the function

$$x \in D \mapsto \nabla_x R(x, y) = \begin{pmatrix} \limsup_{t \rightarrow 0} \frac{R(x+(t,0),y) - R(x,y)}{t} \\ \limsup_{t \rightarrow 0} \frac{R(x+(0,t),y) - R(x,y)}{t} \end{pmatrix}$$

almost-everywhere equals $\nabla R(\cdot, y)$ [Theorem 2, section 4.9.2, 13]. Hence the function $(x, y) \mapsto \nabla_x R(x, y)$ is Lebesgue-integrable on the product $D \times D$ (endowed with the Lebesgue's measure), by Tonelli's theorem and Proposition A.1. From this it follows that the ansatz function

$$\phi : D \rightarrow \mathbb{R} : \phi(x) = b(x) \psi(x) + \int_D R(x, y) \zeta(y) d\mu(y)$$

is weakly differentiable with weak gradient given by

$$b \nabla \psi + \psi \nabla b + \int_D \nabla_x R(x, y) \zeta(y) d\mu(y).$$

Now we compute from the definitions

$$\int_D (\nabla \phi | \nabla \varphi)_{\mathbb{R}^2} \frac{dm}{b} = \int_D \varphi \zeta d\mu,$$

so that $\phi \in W^{1,2}(D)$ is a weak solution for the same problem as $\mathfrak{K}(\zeta)$. It remains to prove that ϕ belongs to $\mathcal{H}_0$. First, let us prove that it belongs to $\mathcal{H}$. Since it already belongs to $W^{1,2}(D)$, it remains to observe that

$$\frac{|\nabla \phi|^2}{b} \leq C_1 \left(b |\nabla \psi|^2 + |\psi|^2 \frac{|\nabla b|^2}{b} + b^{-1} \left| \int_D \nabla_x R(x, y) \zeta(y) d\mu(y) \right|^2 \right)$$

in integrable on D . The last term is integrable, according to Proposition A.1. The first term is integrable, by definition of ψ . The second term is integrable, according to Hölder's inequality and because (D, b) is a continuous lake. Therefore $\phi \in \mathcal{H}$. Let us now prove that $b\psi \in \mathcal{H}$ belongs to $\mathcal{H}_0$. Taking back our sequence $(\varphi_n)_{n \in \mathbb{N}}$, we use Hölder's inequality to observe that the sequence $(b\varphi_n)_{n \in \mathbb{N}}$ converges to $b\psi$ in $\mathcal{H}$. Since it belongs to $\mathcal{H}_0$, we have $b\psi \in \mathcal{H}_0$. Now ϕ belongs to $\mathcal{H}_0$. To see this, let us exploit the Hilbert structure of $\mathcal{H}$ and consider a function $u \in (\mathcal{H}_0)^\perp$ in the orthogonal space of $\mathcal{H}_0 \subseteq \mathcal{H}$. Then we have, because $R(\cdot, y) \in \mathcal{H}_0$ and by Fubini's theorem:

$$(\phi|u)_{\mathcal{H}} = \int_D (R(\cdot, y)|u)_{\mathcal{H}} \zeta(y) d\mu(y) = 0,$$

so that $\phi \in ((\mathcal{H}_0)^\perp)^\perp = \mathcal{H}_0$. This yields $\phi = \mathfrak{K}(\zeta)$ whenever $\zeta \in L^\infty(D)$. Now both $\mathfrak{K}$ and the integral representation for ϕ are stable under limits in $L^p(D, d\mu)$, hence the conclusion. $\square$

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