

On Stability of Two Kinds of Delayed Chemical Reaction Networks ^{*}

Xiaoyu Zhang ^{*} Chuanhou Gao ^{*} Denis Dochain ^{**}

^{*} School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China (e-mail: Xiaoyu_Z@zju.edu.cn, gaohou@zju.edu.cn).

^{**} ICTEAM, UCLouvain, Bâtiment Euler, avenue Georges Lemaître 4-6, 1348 Louvain-la-Neuve, Belgium (e-mail: denis.dochain@uclouvain.be)

Abstract: For the networks that are linear conjugate to complex balanced systems, the delayed version may include two classes of networks: one class is still linear conjugate to the delayed complex balanced network, the other is not. In this paper, we prove the existence of the first class of networks, and emphasize the local asymptotic stability relative to a certain defined invariant set. For the second class of systems, we define a special subclass and derive the local asymptotic stability for the subclass. Two examples are provided to illustrate our results.

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Keywords: Delayed chemical reaction network, mass-action kinetics, linear conjugate, complex balanced system, local asymptotic stability

1. INTRODUCTION

Chemical reaction networks (CRNs) are widespread in the normal operation of nature and the realization of various biological functions by organisms. Studying their dynamic properties can guide people to produce and live more scientifically. The related research (Feinberg, 1972; Horn and Jackson, 1972) can be directly applied to systems biology, synthetic biology, industrial chemistry, ecosystems, etc. They are even used in fields that seem not related to reactions, such as medicine (Allen, 2010), electricity (Samardzija et al., 1989), and machine learning (Anderson et al., 2021). Despite these facts, there are still many challenges in understanding the interaction in reactions. Typical examples include catalytic reactions whose intermediate processes are extremely complex, such as gene regulatory networks. In these cases, if traditional network models are adopted, too many variables will be involved that make it impossible to perform dynamical analysis. A common solution is to use time delays to reduce the model complexity (Lipták and Hangos, 2019). There have been some application results, such as inducing gene switch in biological system (C. Wang and Yang, 2012), modeling transport system (G. Orosz, 2010), etc. Assuredly, the influence of time delay on the dynamical properties of the system is extremely complex. In biochemical systems, time delays are often used to produce oscillation in a stable system; conversely, they can be also used to stabilize a unstable system (Fridman, 2014).

The dynamical investigation on delayed CRNs has become active in the recent decades, such as modeling (Lipták et al., 2018b), stability analysis (Craciun et al., 2020;

Lipták et al., 2018a,b) and persistence analysis (Komatsu and Nakajima, 2019; Zhang and Gao, 2021). Following these studies, this paper continues to focus on the stability issue of delayed CRNs. As one might know, in non-delayed case, complex balanced (CB) mass action systems are quasi-thermodynamic (QTD), and thus the well-known pseudo-Helmholtz free energy can be used as the Lyapunov function to derive the local asymptotic stability of CB systems (Horn and Jackson, 1972; van der Schaft et al., 2013, 2015). The result is extended to larger set of networks that are linear conjugate to any CB system (Johnston and Siegel, 2011), referred to as $lcCB$ systems. In delayed case, Lipták et al. (2018a) showed that delayed complex balanced (DCB) systems can maintain the local asymptotic stability of the corresponding CB systems for any time delay. This motivates us to consider whether $lcCB$ systems have such a property too when time delays are introduced. We denote the systems under consideration by delayed $lcCB$ ($DlcCB$) systems.

There are two possibilities for the element in $DlcCB$ systems. One is that the system is linear conjugate to a DCB system, the other is not. We prove the first class of systems are existing, labeled by $lcDCB$ systems. Moreover, by decomposing the phase space of a $lcDCB$ system into several equivalent invariant classes, we prove there must exist a unique equilibrium relative to the invariant set. Further, we prove that each equilibrium has local asymptotic stability relative to the defined invariant set. For the second class of systems, termed $lcDCB$ systems, by defining a class of special systems, named $lcDCB_1$ systems, a subset of $lcDCB$ systems, we prove there must exist a unique equilibrium in each positive stoichiometric compatibility class. Finally, the local asymptotic stability of a $lcDCB_1$ system is also captured.

This paper is organized as follows. Section 2 gives preliminaries about CRNs and the delayed version. The local

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asymptotic stability of the $lcDCB$ system is presented in Section 3. Section 4 derives the local asymptotic stability of the $lcDCB_1$ system followed by an example as illustration.

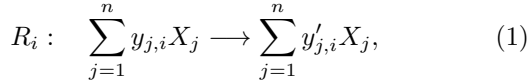
Mathematical Notation:

$\mathbb{R}^n, \mathbb{R}_{\geq 0}^n, \mathbb{R}_{> 0}^n$: n -dimensional real space; n -dimensional non-negative real space; n -dimensional positive real space.
 $\bar{\mathcal{C}}_+, \mathcal{C}_+$: $\bar{\mathcal{C}}_+ = C([- \tau, 0]; \mathbb{R}_{\geq 0}^n), \mathcal{C}_+ = C([- \tau, 0]; \mathbb{R}_{> 0}^n)$ the non-negative, positive continuous function vectors defined on the interval $[- \tau, 0]$, respectively.
 $x^{y \cdot i}$: $x^{y \cdot i} \triangleq \prod_{j=1}^n x_j^{y_j^i}$, where $x, y_i \in \mathbb{R}^n$.
 $\text{Ln}(x)$: $\text{Ln}(x) \triangleq (\ln x_1, \dots, \ln x_n)^\top$, where $x \in \mathbb{R}_{> 0}^n$.

2. PRELIMINARIES

In this section, some basic concepts about CRNs and the corresponding delayed version are given, respectively.

Consider a CRN containing n chemical species, denoted by X_1, X_2, \dots, X_n , that take part in r chemical reactions with the i -th reaction R_i ($i = 1, \dots, r$) expressed as



where the non-negative integers $y_{j,i} \in \mathbb{R}_{\geq 0}^n$ and $y'_{j,i} \in \mathbb{R}_{\geq 0}^n$ are the stoichiometric coefficient. Organize $y_i = (y_{1,i}, \dots, y_{n,i})^\top$ and $y'_i = (y'_{1,i}, \dots, y'_{n,i})^\top$, termed by complexes, then the stoichiometric subspace \mathcal{S} of the network is defined by

$$\mathcal{S} = \text{span}\{y'_i - y_i \mid \text{for all } i = 1, \dots, r\}. \quad (2)$$

And $\mathcal{S}^\perp = \{a \in \mathbb{R}^n \mid a^\top y = 0 \text{ for all } y \in \mathcal{S}\}$ denote the orthogonal complement of the stoichiometric subspace \mathcal{S} . If the mass-action kinetics is assigned to every reaction, then the reaction rate of R_k will be evaluated by

$$\delta_i(x) = \kappa_i \prod_{j=1}^n x_j^{y_{j,i}} \triangleq \kappa_i x^{y_i}, \quad (3)$$

where $x_j \in \mathbb{R}_{\geq 0}$ is the concentration of species X_j ($j = 1, \dots, n$), $x = (x_1, \dots, x_n)^\top$ represents the state, and the positive real number κ_i is the reaction rate constant. The dynamics of a mass-action system that captures the concentration evolution of each species is given by

$$\dot{x}(t) = \sum_{i=1}^r \kappa_i x(t)^{y_i} (y'_i - y_i), \quad t \geq 0. \quad (4)$$

We usually use a quadruple $\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa)$ to express a mass-action system, where $\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa$ are the set of species, complex, reactions and reaction rate constants, respectively.

A positive vector $\bar{x} \in \mathbb{R}_{> 0}^n$ is called a positive equilibrium of \mathcal{M} if it satisfies $\dot{\bar{x}} = 0$ in Eq. (4); and \bar{x} is called a *complex balanced* equilibrium if for any complex $\eta \in \mathbb{Z}_{\geq 0}^n$ in the network it satisfies

$$\sum_{i: y_i = \eta} \kappa_i \bar{x}^{y_i} = \sum_{i: y'_i = \eta} \kappa_i \bar{x}^{y'_i}. \quad (5)$$

Inclusion of time delays in the reactions will not affect the properties related to the network structure, but affect dynamical properties a lot. A delayed mass-action system shares the same stoichiometric subspace and equilibrium with the corresponding mass-action system, but has different dynamics and non-negative stoichiometric compatibility class from the latter. Lipták et al. (2018a,b) made extensive studies on delayed mass-action systems.

Now we introduce the delayed mass-action system. The dynamics of a mass-action system with time delays takes

$$\dot{x}(t) = \sum_{i=1}^r \kappa_i [x(t - \tau_i)^{y_i} y'_i - x(t)^{y_i} y_i], \quad t \geq 0. \quad (6)$$

where $\tau_i \geq 0$, $1 \leq i \leq r$ are constant time delays. Clearly, each delayed system can be denoted as $DM = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \mathbf{k}, \boldsymbol{\tau})$ where $\mathcal{S}, \mathcal{C}, \mathcal{R}$ are the set of species, complex, reactions respectively and $\mathbf{k}, \boldsymbol{\tau}$ are the vectors of reaction rate constants and time delays respectively. And the solution space of delayed system (6) is $\bar{\mathcal{C}}_+$. When $\tau_i = 0$ holds for $1 \leq i \leq r$, the system (6) will reduce to (4). Lipták et al. (2018b) gives a equivalent class decomposition of phase space $\bar{\mathcal{C}}$ called the non-negative stoichiometric compatibility class. Each equivalent class is a forward invariant set of trajectory, i.e., the trajectory x^θ starting from θ always stays in the stoichiometric compatibility class \mathcal{P}_θ containing θ . The definition of \mathcal{P}_θ for the delayed system (6) is given by

$$\mathcal{P}_\theta = \{\psi \in \bar{\mathcal{C}}_+ \mid c_a(\psi) = c_a(\theta) \text{ for all } a \in \mathcal{S}^\perp\}, \quad (7)$$

where the functional $c_a : \bar{\mathcal{C}}_+ \rightarrow \mathbb{R}$ is defined by

$$c_a(\psi) = a^\top \left[\psi(0) + \sum_{i=1}^r \left(\kappa_i \int_{-\tau_i}^0 \psi(s)^{y_i} ds \right) y_i \right]. \quad (8)$$

Lipták et al. (2018a) gives the Lyapunov functional of delayed complex balanced system with the following form $V : \bar{\mathcal{C}}_+ \rightarrow \mathbb{R}_{\geq 0}$ is given by

$$\begin{aligned} V(\psi) = & \sum_{j=1}^n (\psi_j(0)(\ln(\psi_j(0)) - \ln(\bar{x}_j) - 1) + \bar{x}_j) \\ & + \sum_{i=1}^r \kappa_i \int_{-\tau_i}^0 \{(\psi(s))^{y_i} [\ln((\psi(s))^{y_i}) - \ln(\bar{x}^{y_i}) - 1] + \bar{x}^{y_i}\} ds, \end{aligned} \quad (9)$$

They also derive the existence, uniqueness, and local asymptotic stability of equilibriums of delayed complex balanced system relative to the stoichiometric compatibility class.

3. STABILITY OF $lcDCB$ SYSTEMS

In this section, we study the local asymptotic stability of systems called $lcDCB$ systems that are linear conjugate to delayed complex balanced systems relative to some invariant set.

3.1 The existence of $lcDCB$ systems

Firstly, we give the definition of linear conjugacy which will be used throughout the paper.

Definition 1. (Johnston and Siegel (2011)). The two mass-action systems \mathcal{M} and $\tilde{\mathcal{M}}$ are linear conjugate if there exists a linear, bijective mapping: $h : \mathbb{R}_{> 0}^n \rightarrow \mathbb{R}_{> 0}^n$ such that

any trajectories Φ and $\tilde{\Phi}$ of $\mathcal{M}, \tilde{\mathcal{M}}$ satisfy $h(\Phi(x_0, t)) = \tilde{\Phi}(h(x_0), t)$ for any $x_0 \in \mathbb{R}_{>0}^n$.

The above definition is the linear conjugacy for non-delayed systems, where $x_0 \in \mathbb{R}_{>0}^n$. For systems with time delays, a generalized definition is that the dynamical equation of $D\mathcal{M}, D\tilde{\mathcal{M}}$ satisfy that $\dot{x} = Q\dot{\tilde{x}}$. Then we define the $lcDCB$ system.

Definition 2. A delayed mass-action chemical reaction system $D\mathcal{M}$ is called a $lcDCB$ system if it is linear conjugate to a delayed complex balanced system $D\tilde{\mathcal{M}}$.

Further we can derive the existence of $lcDCB$ systems for each delayed complex balanced system.

Theorem 3. For any delayed complex balanced system $D\tilde{\mathcal{M}} = \{\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{\kappa}, \tilde{\tau}\}$ and each positive diagonal matrix Q , there must exist the corresponding $lcDCB$ system.

Proof. The dynamical equation of the delayed complex balanced system $D\tilde{\mathcal{M}}$ can be described as $\dot{\tilde{x}}$. For each Q , finding a linear conjugate $lcDCB$ network is equivalent to the realization of the following delayed differential equations

$$\begin{aligned} \dot{x} &= \sum_{i=1}^r Q\tilde{\kappa}_i [(\tilde{x}(t - \tau_i))^{y_{\cdot i}} y'_{\cdot i} - (\tilde{x}(t))^{y_{\cdot i}} y_{\cdot i}] \\ &= \sum_{i=1}^r Q\tilde{\kappa}_i \prod_{j=1}^n q_j^{-y_{j i}} [x(t - \tau_i)^{y_{\cdot i}} y'_{\cdot i} - x(t)^{y_{\cdot i}} y_{\cdot i}] \end{aligned} \quad (10)$$

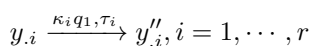
If Q is a scalar matrix ($q_1 = \dots = q_n = q$), let $\alpha_i = q\tilde{\kappa}_i \prod_{j=1}^n q_j^{-y_{j i}} = \tilde{\kappa}_i q^{1 - \sum_{j=1}^n y_{j i}}$ and $\alpha = (\alpha_1, \dots, \alpha_r)$, then (10) can be realized as a $lcDCB$ system $D\mathcal{M} = \{\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau\}$ where

$$\mathcal{S} = \tilde{\mathcal{S}}; \mathcal{C} = \tilde{\mathcal{C}}; \mathcal{R} = \tilde{\mathcal{R}}; \tau = \tilde{\tau}, \kappa = \alpha.$$

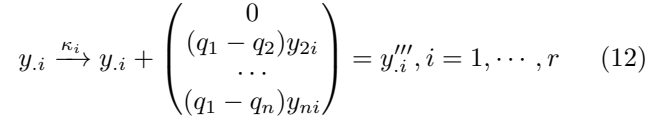
If Q is not a scalar matrix, i.e. there exists $i \neq j$, such that $q_i \neq q_j$. Without loss of generality, let $q_1 = \max\{q_j, j = 1, \dots, n\}$. Otherwise, we can adjust the order of $\tilde{\mathcal{S}}$. Denote $\kappa_i = \tilde{\kappa}_i \prod_{j=1}^n q_j^{-y_{j i}}$, (10) can be written as

$$\begin{aligned} \dot{x}_1 &= \sum_{i=1}^r \kappa_i q_1 y'_{1 i} x^{y_{\cdot i}}(t - \tau_i) - \sum_{i=1}^r \kappa_i q_1 y_{1 i} x^{y_{\cdot i}}(t), \\ &\dots \\ \dot{x}_j &= \sum_{i=1}^r \kappa_i q_j y'_{j i} x^{y_{\cdot i}}(t - \tau_i) - \sum_{i=1}^r \kappa_i q_1 y_{j i} x^{y_{\cdot i}}(t) \\ &+ \sum_{i=1}^r \kappa_i (q_1 - q_j) y_{j i} x^{y_{\cdot i}}(t) \\ &\dots \\ \dot{x}_n &= \sum_{i=1}^r \kappa_i q_n y'_{n i} x^{y_{\cdot i}}(t - \tau_i) - \sum_{i=1}^r \kappa_i q_1 y_{n i} x^{y_{\cdot i}}(t) \\ &+ \sum_{i=1}^r \kappa_i (q_1 - q_n) y_{n i} x^{y_{\cdot i}}(t) \end{aligned} \quad (11)$$

Then we introduce one network realization of (10). The first two terms of the right side of the above equation can be realized as r delayed reactions



where $y''_{\cdot i} = q_j/q_1 y'_{j i}$ and $\tau_i, \kappa_i, q_j, y_{\cdot i}, y'_{\cdot i}$ share the same meaning with those in equation (11). $\kappa_i q_1$ is the reaction rate constant of the i -th delayed reaction. The third term of the right side of equation (11) are all positive because $q_1 - q_j \geq 0$ for all $j = 1, \dots, n$. Thus it can be realized as several non-delayed reactions, for example



So the (11) can be realized as $D\mathcal{M} = \{\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau\}$ where

$$\begin{aligned} \mathcal{S} &= \tilde{\mathcal{S}}; \mathcal{C} = \{y_{\cdot i}, y''_{\cdot i}, y'''_{\cdot i}, i = 1, \dots, r\}; \\ \mathcal{R} &= \{y_{\cdot i} \xrightarrow{\kappa_i q_1, \tau_i} y''_{\cdot i}, y_{\cdot i} \xrightarrow{\kappa_i} y'''_{\cdot i}\}. \end{aligned} \quad (13)$$

Thus $\{\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau\}$ is a $lcDCB$ network corresponding to the delayed complex balanced network $D\tilde{\mathcal{M}}$ and positive diagonal matrix Q . \square

Remark 4. From the form of dynamical equation of $lcDCB$ in (10), we can obtain that the i -th delayed reaction in $lcDCB$ network and that in the corresponding complex balanced network have the same reactant complex $y_{\cdot i}$ and the delay τ_i .

3.2 The stability of $lcDCB$ systems

This subsection derives the stability of $lcDCB$ systems through the Lyapunov second method.

Theorem 5. $D\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau)$ is a $lcDCB$ system, then all positive equilibria of the system $D\mathcal{M}$ are stable.

Proof. The corresponding delayed complex balanced system of a $lcDCB$ system can be expressed by $D\tilde{\mathcal{M}} = \{\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{\kappa}, \tilde{\tau}\}$. Now we consider the following candidate Lyapunov-Krasovskii functional $V_L : \mathcal{C}_+ \rightarrow \mathbb{R}_{\geq 0}$ is given by

$$\begin{aligned} V_L(\psi) &= \sum_{j=1}^n q_j^{-1} (\psi_j(0) (\ln(\psi_j(0)) - \ln(x_j^*) - 1) + x_j^*) \\ &+ \sum_{i=1}^r \kappa_i \int_{- \tau_i}^0 \left\{ (\psi(s))^{y_{\cdot i}} \left[\ln \left(\frac{\psi(s)^{y_{\cdot i}}}{x^{* y_{\cdot i}}} \right) - 1 \right] + x^{* y_{\cdot i}} \right\} ds, \end{aligned}$$

where $\kappa_i = \tilde{\kappa}_i \prod_{j=1}^n q_j^{-y_{j i}}$, and q_j share the same meaning with that in Definition 10. From the inequality: for arbitrary $c_1 > 0$, there exist $c_2 > 0$ that

$$x[\ln x - \ln c_1 - 1] + c_1 \geq c_2 \ln [1 + (x - c_1)^2] \geq 0$$

Thus $V_L \geq 0$ and $V_L(\psi) = 0$ iff ψ is a positive equilibrium of $lcDCB$ system. The following part devoted to deriving that V_L is also dissipative.

$$\begin{aligned} \dot{V}_L(x(t)) &= Q^{-1} \text{Ln} \left(\frac{x(t)}{x^*} \right) \dot{x} \\ &+ \sum_{i=1}^r \tilde{\kappa}_i \prod_{j=1}^n q_j^{-y_{j i}} x(t)^{y_{\cdot i}} \left(\ln \left(\left\{ \frac{x(t)}{x^*} \right\}^{y_{\cdot i}} \right) - 1 \right) \\ &- \sum_{i=1}^r \tilde{\kappa}_i \prod_{j=1}^n q_j^{-y_{j i}} x(t - \tau_i)^{y_{\cdot i}} \left(\ln \left(\left\{ \frac{x(t - \tau_i)}{x^*} \right\}^{y_{\cdot i}} \right) - 1 \right) \\ &= \text{Ln} \left(\frac{\tilde{x}(t)}{\tilde{x}^*} \right) \dot{\tilde{x}} + \sum_{i=1}^r \tilde{\kappa}_i \tilde{x}(t)^{y_{\cdot i}} \left(\ln \left(\left\{ \frac{\tilde{x}(t)}{\tilde{x}^*} \right\}^{y_{\cdot i}} \right) - 1 \right) \end{aligned}$$

$$\begin{aligned}
 & - \sum_{i=1}^r \tilde{\kappa}_i \tilde{x}(t - \tau_i)^{y_i} \left(\ln \left(\left\{ \frac{\tilde{x}(t - \tau_i)}{\tilde{x}^*} \right\}^{y_i} \right) - 1 \right) \\
 & = \dot{V}(\tilde{x}(t)) \leq 0.
 \end{aligned} \tag{14}$$

The last equation of (3.2) is obtained by using the dissipativeness of the Lyapunov-Krasovskii functional V along each trajectory $\tilde{x}(t)$ of the complex balanced system DM and $\dot{V}(\tilde{x}^*) = 0, \tilde{x}^* \in \mathcal{E}_+$ if and only if \tilde{x}^* is an equilibrium of the delayed complex balanced system. Thus (3.2) reveals that V_L is dissipative along each trajectory $x(t)$ of the $lcDCB$ system DM and

$$\dot{V}_L(x^*) = 0 \iff \dot{V}(\tilde{x}^*) = 0 \tag{15}$$

\tilde{x}^* must be a positive equilibrium of $D\tilde{M}$. Thus x^* is a positive equilibrium of the $lcDCB$ system DM . Hence, any positive equilibrium of the $lcDCB$ system is stable.

But usually the local asymptotic stability of the $lcDCB$ system does not hold relative to the invariant class—chemical stoichiometric compatibility class \mathcal{P}_θ defined in (7) because of the degenerate equilibrium points. So we re-decompose the solution space \mathcal{E}_+ of $lcDCB$ networks:

Lemma 6. $DM = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau)$ is a $lcDCB$ system, and its corresponding delayed complex balanced system is $D\tilde{M} = (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{\kappa}, \tilde{\tau})$. $\mathcal{S}, \tilde{\mathcal{S}}$ are the stoichiometric subspace of $DM, D\tilde{M}$ respectively. Then the following set is the invariant set of each trajectory of $lcDCB$ networks.

$$\mathcal{H}_\theta = \{ \psi \in \tilde{\mathcal{E}}_+ \mid h_a(\psi) = h_a(\theta) \text{ for all } a \in (Q^{-1})^\top \tilde{\mathcal{S}}^\perp \}, \tag{16}$$

where the functional $h_a : \tilde{\mathcal{E}}_+ \rightarrow \mathbb{R}$ is defined by

$$h_a(\psi) = a^\top \left[\psi(0) + \sum_{i=1}^{\tilde{r}} \left(\tilde{\kappa}_i \prod_{j=1}^n q_j^{-y_{ji}} \int_{-\tau_i}^0 \psi(s)^{y_i} ds \right) Q y_i \right] \tag{17}$$

Proof. Consider any trajectory $x(t)$ of a $lcDCB$ system, then we study the change of the value of h_a along the trajectory $x(t)$.

$$\frac{dh_a(x(t))}{dt} = a^\top \left(\sum_{i=1}^{\tilde{r}} \tilde{\kappa}_i \tilde{x}^{y_i} (t - \tau_i) Q (\tilde{y}'_i - \tilde{y}_i) \right) = 0 \tag{18}$$

Thus we conclude the result. \square

Now we consider the situation of equilibriums of $lcDCB$ system relative to the above invariant set.

Lemma 7. For a $lcDCB$ network denoted as $DM = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau)$, the positive equilibrium in each invariant set \mathcal{H}_θ of DM defined as (16) is unique.

Proof. \mathcal{H}_θ defined in (16) is a arbitrary invariant set of DM . The corresponding delayed complex balanced system of DM is denoted as $D\tilde{M} = (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{\kappa}, \tilde{\tau})$ and $\mathcal{P}_{\tilde{\theta}}$ is the chemical stoichiometric compatibility class of $D\tilde{M}$ containing $\tilde{\theta}$ where $\theta = Q\tilde{\theta}$. Denoting the unique positive equilibrium of $\mathcal{P}_{\tilde{\theta}}$ as \tilde{x}^* , we claim that $x^* = Q\tilde{x}^*$ must be the unique positive equilibrium of the invariant set \mathcal{H}_θ . In order to derive this result, we just need to verify the values of $h_a(\theta), h_a(x^*)$ for each a . \tilde{x}^* is in $\mathcal{P}_{\tilde{\theta}}$, thus $c_{\tilde{a}}(\tilde{x}^*) = c_{\tilde{a}}(\tilde{\theta})$ where $\tilde{a} \in \tilde{\mathcal{S}}^\perp$. Also,

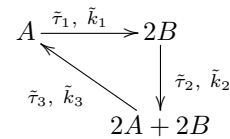
$$\begin{aligned}
 h_a(\theta) &= \tilde{a}^\top Q^{-1} \left[Q\theta(0) + \sum_{i=1}^r \left(\tilde{\kappa}_i \int_{-\tau_i}^0 \tilde{\theta}(s)^{y_i} ds \right) Q y_i \right] \\
 &= c_{\tilde{a}}(\tilde{\theta})
 \end{aligned}$$

and $h_a(x^*) = c_{\tilde{a}}(\tilde{x}^*)$. Thus $h_a(x^*) = h_a(\theta)$, i.e., x^* is in the invariant set \mathcal{H}_θ . Further, $x^* = D\tilde{x}^*$ is the positive equilibrium of $h_a(\theta)$. Similarly, if $\mathcal{H}(\theta)$ has another positive equilibrium except x^* , the stoichiometric compatibility class $\mathcal{P}_{\tilde{\theta}}$ of system $D\tilde{M}$ also has more than one equilibrium. This is obviously contradiction to the uniqueness of positive equilibrium of the delayed complex balanced system. Thus we derive the uniqueness and existence of equilibrium in $lcDCB$ systems relative to \mathcal{H}_θ . \square

Theorem 8. $DM = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau)$ is a $lcDCB$ network, then each positive equilibrium x^* is local asymptotic stability relative to the invariant set \mathcal{H}_θ containing x^* .

Proof. This result can be derived from Theorem 5, Lemma 7. \square

Example 1. Consider the following delayed complex balanced system $D\tilde{M}$:



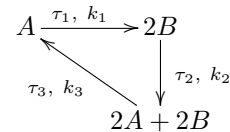
By choosing $\tilde{k}_1 = \tilde{k}_2 = \tilde{k}_3 = 1$, the dynamical equation of $D\tilde{M}$ can be written as:

$$\begin{aligned}
 \dot{\tilde{x}}_A &= -\tilde{x}_A(t) + 2\tilde{x}_B^2(t - \tilde{\tau}_2) + \tilde{x}_A^2 \tilde{x}_B^2(t - \tilde{\tau}_3) - 2\tilde{x}_A^2 \tilde{x}_B^2(t); \\
 \dot{\tilde{x}}_B &= 2\tilde{x}_A(t - \tilde{\tau}_1) + 2\tilde{x}_B^2(t - \tilde{\tau}_2) - 2\tilde{x}_B^2(t) - 2\tilde{x}_A^2 \tilde{x}_B^2(t)
 \end{aligned} \tag{19}$$

If $Q = \text{diag}(2, 2)$ is a scalar matrix, the dynamical equation of the corresponding $lcDCB$ should be:

$$\begin{aligned}
 \dot{x}_A &= -x_A(t) + x_B^2(t - \tilde{\tau}_2) + \frac{1}{8} x_A^2 x_B^2(t - \tilde{\tau}_3) - \frac{1}{4} x_A^2 x_B^2(t) \\
 \dot{x}_B &= 2x_A(t - \tilde{\tau}_1) + x_B^2(t - \tilde{\tau}_2) - x_B^2(t) - \frac{1}{4} \tilde{x}_A^2 \tilde{x}_B^2(t)
 \end{aligned} \tag{20}$$

Then one $lcDCB$ realization as described in Theorem 3 is

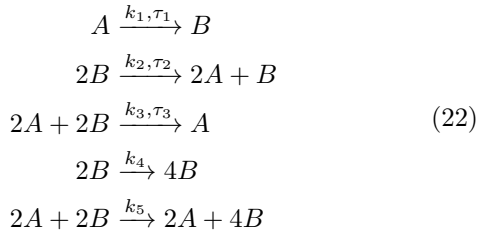


where $k_1 = 1, k_2 = \frac{1}{2}, k_3 = \frac{1}{8}$ and $\tau_i = \tilde{\tau}_i$.

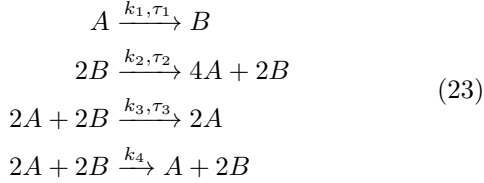
If $Q = \text{diag}(2, 1)$ is not a scalar matrix, the dynamical equation of the $lcDCB$ determined by $D\tilde{M}$ and Q is

$$\begin{aligned}
 \dot{x}_A &= -x_A(t) + 4x_B^2(t - \tilde{\tau}_2) + \frac{1}{2} x_A^2 x_B^2(t - \tilde{\tau}_3) - x_A^2 x_B^2(t) \\
 \dot{x}_B &= x_A(t - \tilde{\tau}_1) + 2x_B^2(t - \tilde{\tau}_2) - 2x_B^2(t) - \frac{1}{2} \tilde{x}_A^2 \tilde{x}_B^2(t)
 \end{aligned} \tag{21}$$

The following delayed system is the corresponding $lcDCB$ realization shown in Theorem 3



where $k_1 = 1, k_2 = 2, k_3 = \frac{1}{2}, k_4 = 1, k_5 = \frac{1}{4}$, $\tau_i = \tilde{\tau}_i, i = 1, 2, 3$ and $\tau_4 = 0, \tau_5 = 0$. Note that the above realization is not unique, the dynamical equation of the following delayed ℓ cDCB system is also (21).



where $k_1 = 1, k_2 = 1, k_3 = \frac{1}{4}, k_4 = \frac{1}{2}$, $\tau_i = \tilde{\tau}_i, i = 1, 2, 3$ and $\tau_4 = 0$.

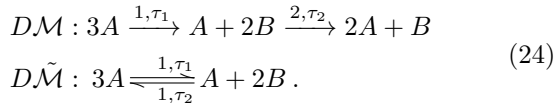
Thus from Theorem 8, above delayed ℓ cDCB systems are all local asymptotic stability relative to the invariant class.

4. STABILITY OF $\bar{\ell}$ cDCB₁ SYSTEMS

When time delays are introduced to non-delayed systems \mathcal{M} linear conjugate to complex balanced systems $\tilde{\mathcal{M}}$, the delayed version of some system \mathcal{M} denoted as $D\mathcal{M}$ is no longer conjugate to the delayed complex balanced system $D\tilde{\mathcal{M}}$. In this section, we study the local asymptotic stability of a special case of $D\mathcal{M}$ called the $\bar{\ell}$ cDCB₁ system.

4.1 Problem statement

We illustrate our motivation through the following two delayed systems



The above two networks with no time delay ($\tau_i = 0, i = 1, 2$) denoted as $\mathcal{M}, \tilde{\mathcal{M}}$ share the same dynamical equation:

$$\begin{cases} \dot{x}_A(t) = -2x_A^3 + 2x_A x_B^2, \\ \dot{x}_B(t) = 2x_A^3 - 2x_A x_B^2. \end{cases} \tag{25}$$

Note that the network structure of system $\tilde{\mathcal{M}}$ is weakly reversible and zero deficiency, thus $\tilde{\mathcal{M}}$ is complex balanced system (Feinberg, 1972). The local asymptotic stability of $\tilde{\mathcal{M}}$ can be derived by the Lyapunov function—Pseudo-Helmholtz free energy function. Although \mathcal{M} is not a complex balanced system even not a weakly reversible system, but the local asymptotic stability of \mathcal{M} can be also derived using the dynamic equivalence.

But when time delays are introduced into reactions, the situation will be completely different. The delayed systems $D\mathcal{M}$ and $D\tilde{\mathcal{M}}$ have completely different dynamics and are not linear conjugate to each other. The dynamical equation of delayed system $D\mathcal{M}$ can be written as

$$\begin{cases} \dot{x}_A(t) = x_A^3(t - \tau_1) - 3x_A^3(t) + 4x_A x_B^2(t - \tau_2) - 2x_A x_B^2(t) \\ \dot{x}_B(t) = 2x_A^3(t - \tau_1) + 2x_A x_B^2(t - \tau_2) - 4x_A x_B^2(t); \end{cases} \tag{26}$$

But the dynamics of delayed complex balanced network $D\tilde{\mathcal{M}}$ is

$$\begin{cases} \dot{x}_A(t) = x_A^3(t - \tau_1) - 3x_A^3(t) + 3x_A x_B^2(t - \tau_2) - x_A x_B^2(t) \\ \dot{x}_B(t) = 2x_A^3(t - \tau_1) - 2x_A x_B^2(t); \end{cases}$$

$D\mathcal{M}$ is not a ℓ cDCB network that we introduced in section 3. In this section, we focus on the local asymptotic stability of this kind of delayed networks like $D\mathcal{M}$ which are not ℓ cDCB networks but its non-delayed system \mathcal{M} shares the same dynamical equation with a complex balanced network $\tilde{\mathcal{M}}$.

4.2 Local asymptotic stability of $\bar{\ell}$ cDCB₁ network

This subsection obtains the local asymptotic stability of one type of delayed systems which is a generalization of the system $D\mathcal{M}$ proposed in subsection 4.1.

Definition 9. A delayed system $D\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau)$ is called a $\bar{\ell}$ cDCB₁ system if there exists a complex balanced mass-action system $D\tilde{\mathcal{M}} = (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{\kappa}, \tilde{\tau})$ such that

- $\mathcal{S} = \tilde{\mathcal{S}}$;
- $\text{card}(\tilde{\mathcal{R}}) = \text{card}(\mathcal{R})$; And for i -th reaction of two systems, there exist $y_i = \tilde{y}_i$ and the reaction vectors satisfy $v_i = b_i \tilde{v}_i$, where $b_i = \frac{\tilde{\kappa}_i}{\kappa_i} \leq 1$ is a positive constant.

Remark 10. $D\mathcal{M} = \{\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau\}$ is a $\bar{\ell}$ cDCB₁ system with

$$\mathcal{R} = \{y_i \xrightarrow{\kappa_i, \tau_i} y'_i\} = \{y_i \xrightarrow{\kappa_i, \tau_i} y_i + v_i\}.$$

The corresponding complex balanced network of $D\mathcal{M}$ is $D\tilde{\mathcal{M}} = (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{\kappa}, \tilde{\tau})$. $\tilde{\mathcal{R}}$ in system $D\tilde{\mathcal{M}}$ is expressed by:

$$\tilde{\mathcal{R}} = \{\tilde{y}_i \xrightarrow{\tilde{\kappa}_i, \tilde{\tau}_i} \tilde{y}'_i\} = \{y_i \xrightarrow{\tilde{\kappa}_i, \tilde{\tau}_i} y_i + \tilde{v}_i\}.$$

The dynamical equation of the delayed system $D\mathcal{M}$

$$D\mathcal{M} : \dot{x}(t) = \sum_{i=1}^r \kappa_i [(x(t - \tau_i))^{y_i} (y_i + b_i \tilde{v}_i) - (x(t))^{y_i} y_i]$$

is different from the dynamical equation of $D\tilde{\mathcal{M}}$ even if choosing $\tilde{\tau}_i = \tau_i$ for each $i = 1, \dots, r$.

$$D\tilde{\mathcal{M}} : \dot{x}(t) = \sum_{i=1}^r \tilde{\kappa}_i [(x(t - \tau_i))^{y_i} (y_i + \tilde{v}_i) - (x(t))^{y_i} y_i].$$

Also, two networks are not linear conjugate.

Now we give the local asymptotic stability of the delayed system ℓ cDCB₁ by using the Lyapunov second method.

Lemma 11. (existence and uniqueness). For a $\bar{\ell}$ cDCB₁ system $D\mathcal{M} = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau)$, each stoichiometric compatibility class of $D\mathcal{M}$ contains a unique positive equilibrium.

Proof. \mathcal{P}_θ defined as (7) denotes arbitrary stoichiometric compatibility class of the ℓ cDCB₁ system $D\mathcal{M}$. $\mathbf{a} = \{a_1, \dots, a_{n-s}\}$ is a set of basis of \mathcal{S}^\perp where s denotes the dimension of \mathcal{S} . If $n = s$, this lemma holds obviously. If $s < n$, let $c_{a_i}(\theta) = M_i$. The stoichiometric compatibility class \mathcal{P}_θ can be expressed as

$$\mathcal{P}_\theta = \{\psi \in \mathcal{E}_+^\perp \mid c_{a_i}(\psi) = M_i \text{ for all } a_i \in \mathbf{a}\},$$

Now we consider the delayed complex balanced system $D\tilde{\mathcal{M}} = (\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{\kappa}, \tilde{\tau})$, where the time delay of i -th reaction $\tilde{\tau}_i = \tau_i/b_i$ and $\tilde{\mathcal{S}}, \tilde{\mathcal{C}}, \tilde{\mathcal{R}}, \tilde{\kappa}$ share the same meaning with

Definition 9. Also, Definition 9 reveals that stoichiometric subspaces of systems DM and $D\tilde{M}$ are the same, thus \mathcal{S}^\perp are also same. In this case, for the same conserved quantity $M_i, i = 1, \dots, n - s$, the corresponding stoichiometric compatibility class \mathcal{P}_θ of the system DM and $\tilde{\mathcal{P}}_\theta$ of the system $D\tilde{M}$ contain same constant points $\psi \in \mathcal{C}_+$, which can be seen that

$$\begin{aligned} c_{a_i}(\psi) &= \psi + \sum_{i=1}^r \kappa_i \psi^{y_i} \tau_i y_i = \psi + \sum_{i=1}^r \tilde{\kappa}_i / b_i \psi^{y_i} \tau_i y_i \\ &= \psi + \sum_{i=1}^r \tilde{\kappa}_i \psi^{y_i} \tilde{\tau}_i y_i = \tilde{c}_{a_i}(\psi) = M_i \end{aligned}$$

From the fact that each positive stoichiometric compatibility class in the delayed complex balanced system $D\tilde{M}$ contains and only contains one equilibrium. We can conclude the existence and uniqueness of positive equilibrium in a $\bar{\ell}cDCB_1$ network. \square

Theorem 12. If $DM = (\mathcal{S}, \mathcal{C}, \mathcal{R}, \kappa, \tau)$ is a delayed $\bar{\ell}cDCB_1$ network, each positive equilibrium of DM is local asymptotic stability.

Proof. The following part is devoted to proving that the pseudo-Helmholtz functional proposed in (9) also can be the Lyapunov functional of the system DM denoted by $V_D(x(t))$. The $\dot{V}_D(x(t))$ can be divided into two parts ($\dot{V}_D(x(t)) = A + B$):

$$\begin{aligned} A &= \text{Ln}\left(\frac{x(t)}{\bar{x}}\right) \sum_{i=1}^r \tilde{\kappa}_i [x(t - \tau_i)^{y_i} (y_i + \tilde{v}_i) - x(t)^{y_i} y_i] \\ &\quad + \sum_{i=1}^r \tilde{\kappa}_i x(t)^{y_i} \left[\ln \frac{x(t)}{\bar{x}} - 1 \right] \\ &\quad - \sum_{i=1}^r \tilde{\kappa}_i x(t - \tau_i)^{y_i} \left[\ln \left(\frac{x(t - \tau_i)^{y_i}}{\bar{x}} \right) - 1 \right] \\ B &= \text{Ln}\left(\frac{x(t)}{\bar{x}}\right) \sum_{i=1}^r (\kappa_i - \tilde{\kappa}_i) [x(t - \tau_i)^{y_i} y_i - x(t)^{y_i} y_i] \\ &\quad + \sum_{i=1}^r (\kappa_i - \tilde{\kappa}_i) x(t)^{y_i} \left[\ln \left(\frac{x(t)}{\bar{x}} \right) - 1 \right] \\ &\quad - \sum_{i=1}^r (\kappa_i - \tilde{\kappa}_i) x(t - \tau_i)^{y_i} \left[\ln \left(\frac{x(t - \tau_i)^{y_i}}{\bar{x}} \right) - 1 \right] \end{aligned}$$

Actually, A is the derivative of the Lyapunov functional of the delayed complex balanced system $D\tilde{M}$ with respect to time. Therefore, from the dissipation of the delayed complex balanced network, we know that $A \leq 0$ and equality holds iff x is a equilibrium of the DM and $D\tilde{M}$.

Now we consider the part B , let $\bar{\kappa}_i = \kappa_i - \tilde{\kappa}_i$

$$\begin{aligned} B &= \sum_{i=1}^r \bar{\kappa}_i \left[x(t - \tau_i)^{y_i} \left(\ln \left(\frac{x(t)}{\bar{x}} \right)^{y_i} - \ln \left(\frac{x(t - \tau_i)}{\bar{x}} \right)^{y_i} \right) \right] \\ &\quad - \sum_{i=1}^r \bar{\kappa}_i [x(t)^{y_i} - x(t - \tau_i)^{y_i}] \end{aligned}$$

From the inequation $e^a(b - a) \leq e^b - e^a$ (equality holds iff $a = b$), for each i

$$\begin{aligned} x(t - \tau_i)^{y_i} \left(\ln \left(\frac{x(t)}{\bar{x}} \right)^{y_i} - \ln \left(\frac{x(t - \tau_i)}{\bar{x}} \right)^{y_i} \right) \\ \leq x(t)^{y_i} - x(t - \tau_i)^{y_i}. \end{aligned}$$

Thus combining above two equations, it is obviously that the $B \leq 0$ with equality iff $x(t) = x(t - \tau_i)$ for any τ_i .

Consequently, $\dot{V}_D(x(t)) = A + B \leq 0$. And $\dot{V}_D(x^*) = 0$ iff x^* is a constant positive equilibrium. Thus each equilibrium of the system DM is locally asymptotically stable. \square

Example 2. Now we reconsider the system DM in the equation (24), $D\tilde{M}$ is the corresponding complex balanced network by taking

$$b_1 = 1, b_2 = \frac{1}{2}; v_{.1} = \tilde{v}_{.1} = (-2, 2)^T, v_{.2} = \frac{1}{2} \tilde{v}_{.2} = (1, -1)^T.$$

Thus from theorem 12, each positive equilibrium of DM is locally asymptotically stable. Through the numerical analysis of the DM with different time delays and initial points, the local asymptotic stability of the equilibrium point is further intuitively obtained (See Fig. 1).

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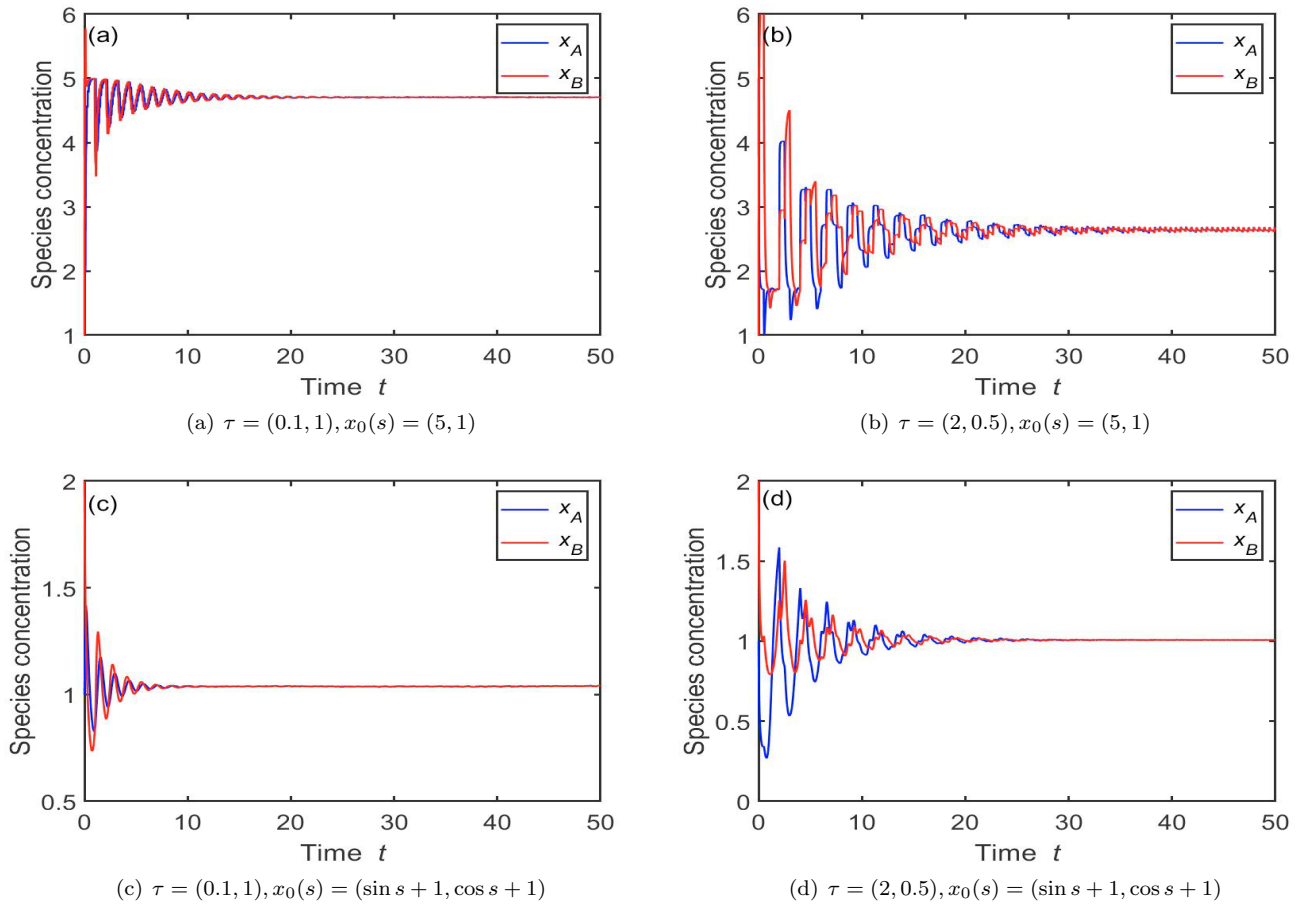


Fig. 1. The evolution behavior of Example 2 with different initial datas and different time delays.

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