CATEGORIES WITH CONJUGATION AND OCTAGONS OF WITT GROUPS

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Abstract. A categorical approach is proposed for the construction of 8-periodic chain complexes of Witt groups of categories with duality. Exactness of the resulting sequences is proved under the hypothesis that the categories are abelian and artinian, or when the endomorphism algebra of each object is semilocal and rad-adically complete.

Introduction

In the algebraic theory of quadratic forms, the special rôle played by quadratic field extensions was noticed early. Among the first observations stands out the description by Pfister [Pfi66, p. 123] of quadratic forms that become hyperbolic over a given quadratic extension, reformulated in terms of hermitian forms by Milnor–Husemoller: see [MH73, App. 2], where it appears as a 4-term exact sequence. Using Scharlau's transfer, Elman–Lam prolonged the sequence into a long 3-periodic exact sequence (an "exact triangle" or "symmetric exact hexagon", see [EL76, Th. 2.6]). Extending this early result in the framework of hermitian or skew-hermitian forms over quaternion algebras, the first author uncovered a more complex behaviour, giving rise to longer exact sequences relating forms over quaternion algebras and over quadratic subfields, see [Lew79], [Lew82] (see also [Sch85, Th. 10.3.2]). It was soon realised that exact sequences thus constructed are 8-periodic, and can therefore be folded into exact octagons, see [Lew83]. Meanwhile, Ranicki also constructed 8-periodic exact sequences of Witt groups for "twisted quadratic extensions" via L-theory: see [Ran87] and [HTW84, (0.2)].

Building on the first author's work, Parimala–Sridharan–Suresh [BFP95, Appendix 2] extended the scope of [Lew82] beyond the case of quaternion algebras, obtaining exact sequences relating Witt groups of hermitian or skew-hermitian forms over a central simple algebra and over the centraliser of a quadratic subfield. These exact sequences were further patched into exact octagons of Witt groups by Grenier-Boley–Mahmoudi [GBM05]. The latest development in this line of investigation is due to First [Fir23], who obtained an analogous exact octagon of Witt groups for Azumaya algebras over semilocal rings. His paper also discusses several important applications.

The purpose of this work is to describe a categorical mechanism that produces the octagons referred to above. Our approach is close in spirit to the method in [Lew85] making use of the periodicity of the Clifford algebra construction and Morita equivalences. The basic tool is a

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notion of *conjugation* on an additive pseudo-abelian (= idempotent-complete) category with $\frac{1}{2}$ (i.e., such that 2 is invertible in each Hom group). A conjugation is a covariant version of a duality, consisting of an endofunctor ι and a natural isomorphism $\lambda\colon \iota\iota \overset{\sim}{\Rightarrow} \mathrm{Id}$. The construction of the hermitian category of symmetric spaces from a category with duality has an analogue for a category with conjugation $(\mathscr{C},\iota,\lambda)$, which yields a *double* category with conjugation $(\mathscr{C}',\iota',\lambda')$. The categories $\mathscr{C}$ and $\mathscr{C}'$ are related by a pair of functors $F\colon \mathscr{C}\to\mathscr{C}'$ and $G\colon \mathscr{C}'\to\mathscr{C}$. The functor F is reminiscent of the construction of hyperbolic symmetric spaces in the context of categories with duality, while G is a forgetful functor. Iterating the doubling construction yields an infinite sequence of categories with conjugation $(\mathscr{C}'',\iota'',\lambda'')$, $(\mathscr{C}''',\iota''',\lambda''')$, ... and functors

$$\mathscr{C} \xrightarrow{F} \mathscr{C}' \xrightarrow{F'} \mathscr{C}'' \xrightarrow{F''} \cdots \qquad \text{and} \qquad \mathscr{C} \xleftarrow{G} \mathscr{C}' \xleftarrow{G'} \mathscr{C}'' \xleftarrow{G''} \cdots$$

We show in §1.2 that this construction is 2-periodic by defining an equivalence $(\mathscr{C},\iota,\lambda) \equiv (\mathscr{C}'',\iota'',\lambda'')$, which allows us to fold the sequence above into a 2-gon $\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{C}'$.

When $\mathscr{C}$ also carries a duality (D,δ), commuting in a natural sense with a conjugation (ι,λ), a "conjugate" duality $(\iota D,\delta_\alpha)$ on $\mathscr{C}$ and "double" dualities (D',δ') and $(\iota' D',\delta'_{\alpha'})$ on $\mathscr{C}'$ are constructed in §2. In contrast with the doubling construction, the resulting categories with duality form an 8-periodic sequence, see (2.6):

$$\begin{tikzcd}
(\mathscr{C},D,\delta) \arrow[r, "\hat F"] & (\mathscr{C}',D',\delta') \arrow[r, "\hat G"] & (\mathscr{C},\iota D,-\delta_\alpha) \arrow[r, "\check F"] & (\mathscr{C}',\iota' D',-\delta'_{\alpha'}) \arrow[d, "\check G"] \\
(\mathscr{C}',\iota' D',\delta'_{\alpha'}) \arrow[u, "\check G"] & (\mathscr{C},\iota D,\delta_\alpha) \arrow[l, "\check F"'] & (\mathscr{C}',D',-\delta') \arrow[l, "\hat G"'] & (\mathscr{C},D,-\delta) \arrow[l, "\hat F"']
\end{tikzcd} \tag{0.1}$$

The corresponding hermitian categories of symmetric spaces discussed in §3 are similarly organised, see (3.2), and we show in Theorem 3.1 that under the composition of two consecutive functors from the 8-periodic sequence the image of each hermitian category consists of hyperbolic symmetric spaces.

Defining a Witt group from the hermitian category of a category with duality $(\mathscr{C},D,\delta)$ requires an exact structure on $\mathscr{C}$, which provides the definition of metabolic symmetric spaces, see [Bal05, §1.2.4]. We consider two special cases in §4. For the *split exact structure*, in which exact sequences split, metabolic symmetric spaces are hyperbolic, and Theorem 3.1 readily yields from (0.1) an 8-periodic chain complex. Exactness requires a kind of Witt cancellation property (see Remark 4.1), which we establish under the hypothesis that the endomorphism algebra of every object in $\mathscr{C}$ is semilocal and rad-adically complete: see Theorem 4.6. This theorem does *not* cover the case discussed by First in [Fir23], which deals with the split exact category of finitely generated projective modules over Azumaya algebras over semilocal rings, but it yields the Grenier-Boley–Mahmoudi octagon [GBM05] as a special case.

When $\mathscr{C}$ is an abelian category, with its usual exact structure, our main result establishes the exactness of the octagon of Witt groups derived from (0.1) under the hypothesis that the abelian category $\mathscr{C}$ is artinian: see Theorem 4.8. This result is sufficient to account for the exact octagon from [Lew83], and also for Warshauer's exact octagon of Witt groups of asymmetric spaces in [War82]. By contrast, since characteristic 2 is ruled out from the outset in this work, our results do not cover the version of [Lew82] in characteristic 2 given by Knus–Villa in [KV01, Th. 3.8].

The sequence of categories with duality $(\mathscr{C}, D, \delta)$, $(\mathscr{C}, \iota D, -\delta_\alpha)$, $(\mathscr{C}, D, -\delta)$, $(\mathscr{C}, \iota D, \delta_\alpha)$ extracted from the 8-periodic sequence (0.1) bears some formal resemblance with the sequence of shifted dualities $T^n(\mathscr{C}, D, \delta)$ of a triangulated category with duality, which also satisfies $T^{n+2}(\mathscr{C}, D, \delta) \equiv T^n(\mathscr{C}, D, -\delta)$ for all $n \in \mathbb{Z}$ and is therefore 4-periodic, see [Bal05, 63–65]. However, a triangulation on an additive category is a very different feature than a conjugation, as witnessed by the category $\mathscr{P}_A$ of finitely generated right modules over an artinian simple ring A with involution, with the usual duality (D, δ) defined in §3.3 below and the split exact structure: it is easy to find examples of conjugations commuting with (D, δ) such that all the Witt groups obtained from the sequence $(\mathscr{P}_A, D, \delta)$, $(\mathscr{P}_A, \iota D, -\delta_\alpha)$, $(\mathscr{P}_A, D, -\delta)$, $(\mathscr{P}_A, \iota D, \delta_\alpha)$ (which are Witt groups of hermitian spaces) are nonzero, whereas the odd-indexed derived Witt groups vanish, see [BP05, Th. 4.2].

The examples of octagons worked out in detail in this work are all based on categories of modules, but even in this particular case they do not exhaust the full range of possibilities (see Remark 1.3). The formalism of conjugations extends much farther; it will hopefully be useful in geometric settings for more general exact categories with duality.

Notation. For any category $\mathscr{C}$, the notation $C \in \mathscr{C}$ means that C is an object of $\mathscr{C}$. Natural transformations of functors are denoted by double arrows. If F, F' are functors $\mathscr{C} \to \mathscr{D}$ and G, G' are functors $\mathscr{D} \to \mathscr{E}$, we write simply GF, $G'F'$ for the composition of functors. If $a\colon F \Rightarrow F'$ and $b\colon G \Rightarrow G'$ are natural transformations, we write $b \cdot a$ for the Godement product (horizontal composition) $GF \Rightarrow G'F'$; thus, for every $C \in \mathscr{C}$ the morphism $(b \cdot a)_C\colon GF(C) \to G'F'(C)$ is the diagonal of the following commutative square (assuming G, G' are covariant):

$$\begin{tikzcd} GF(C) \arrow[r, "G(a_C)"] \arrow[d, "b_{F(C)}"'] & GF'(C) \arrow[d, "b_{F'(C)}"] \\ G'F(C) \arrow[r, "G'(a_C)"] & G'F'(C) \end{tikzcd}$$

(If G, G' are contravariant, the arrows $G(a_C)$ and $G'(a_C)$ are reversed.) In particular, denoting by $\mathbb{I}_F\colon F \overset{\sim}{\Rightarrow} F$ and $\mathbb{I}_G\colon G \overset{\sim}{\Rightarrow} G$ the identity natural isomorphisms, the natural transformations $\mathbb{I}_G \cdot a\colon GF \Rightarrow GF'$ and $b \cdot \mathbb{I}_F\colon GF \Rightarrow G'F$ are defined by

$$(\mathbb{I}_G \cdot a)_C = G(a_C), \qquad (b \cdot \mathbb{I}_F)_C = b_{F(C)}.$$

1. Categories with conjugation

Throughout this section, $\mathscr{C}$ denotes a k-linear category for an arbitrary commutative ring k, and all the functors are tacitly assumed to be k-linear. Although this is not necessary for the very first definitions, we assume 2 is invertible in k and $\mathscr{C}$ is pseudo-abelian (=idempotent-complete, see [Kar78, Ch. I, Prop. 6.9]), which means that every idempotent morphism splits. The latter property holds for instance for the category of finitely-generated projective modules over an arbitrary k-algebra.

1.1. **Definitions.** A *conjugation* on a category $\mathscr{C}$ is a pair (ι, λ) consisting of a k-linear covariant functor $\iota\colon \mathscr{C} \to \mathscr{C}$ and a natural isomorphism $\lambda\colon \iota\iota \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathscr{C}}$ such that $\lambda \cdot \mathbb{I}_\iota = \mathbb{I}_\iota \cdot \lambda$, i.e. for every object C in $\mathscr{C}$,

$$\lambda_{\iota(C)} = \iota(\lambda_C)\colon \iota\iota(C) \xrightarrow{\sim} \iota(C).$$

A *morphism* of categories with conjugation $(F,\tau)\colon (\mathscr{C}_1,\iota_1,\lambda_1)\to(\mathscr{C}_2,\iota_2,\lambda_2)$ consists of a k-linear covariant functor $F\colon \mathscr{C}_1\to\mathscr{C}_2$ and a natural isomorphism $\tau\colon F\iota_1 \overset{\sim}{\Rightarrow} \iota_2 F$ such that the following diagram commutes:

$$\begin{tikzcd} \iota_2F\iota_1 \arrow[r, Rightarrow, "\mathbb{I}_{\iota_2}\cdot\tau"] \arrow[d, Rightarrow, "\tau^{-1}\cdot\mathbb{I}_{\iota_1}"'] & \iota_2\iota_2F \arrow[d, Rightarrow, "\lambda_2\cdot\mathbb{I}_F"] \\ F\iota_1\iota_1 \arrow[r, Rightarrow, "\mathbb{I}_F\cdot\lambda_1"'] & F \end{tikzcd}$$

Given a conjugation (ι,λ) on a category $\mathscr{C}$ we define the *double* category $\mathscr{C}'$: its objects are pairs (C,f) where $C\in\mathscr{C}$ and $f\colon \iota(C)\to C$ is an isomorphism in $\mathscr{C}$ such that the following diagram commutes:

$$\begin{tikzcd} \iota\iota(C) \arrow[r, "\iota(f)"] \arrow[dr, "\lambda_C"'] & \iota(C) \arrow[d, "f"] \\ & C \end{tikzcd}$$

(Note the analogy with the definition of a symmetric space in a category with duality; see §3.) Morphisms $(C,f)\to(D,g)$ in $\mathscr{C}'$ are morphisms $\varphi\colon C\to D$ in $\mathscr{C}$ such that $\varphi\, f = g\,\iota(\varphi)$.

Clearly, the double category $\mathscr{C}'$ is k-linear, with direct sum

$$(C_1,f_1)\oplus(C_2,f_2) = \bigl(C_1\oplus C_2, \bigl(\begin{smallmatrix} f_1 & 0\\ 0 & f_2\end{smallmatrix}\bigr)\bigr).$$

It is also pseudo-abelian, because idempotent morphisms $e\colon (C,f)\to(C,f)$ in $\mathscr{C}'$ are also idempotent in $\mathscr{C}$, and if $K\xrightarrow{t} C\xrightarrow{s} K$ are morphisms in $\mathscr{C}$ such that $s\,t=\mathrm{Id}_K$ and $t\,s=e$, then $\bigl(K, s\,f\,\iota(t)\bigr)\xrightarrow{t}(C,f)\xrightarrow{s}\bigl(K,s\,f\,\iota(t)\bigr)$ are morphisms in $\mathscr{C}'$ that split e.

The conjugation (ι,λ) on $\mathscr{C}$ induces a conjugation (ι',λ') on $\mathscr{C}'$ as follows: for (C,f), $(D,g)\in\mathscr{C}'$ and $\varphi\in\mathrm{Hom}_{\mathscr{C}'}\bigl((C,f),(D,g)\bigr)$,

$$\iota'(C,f)=\bigl(\iota(C),-\iota(f)\bigr),\qquad \iota'(\varphi)=\iota(\varphi)\quad\text{and}\quad \lambda'_{(C,f)}=-\lambda_C.$$

(Note the signs in the definitions of $\iota'(C,f)$ and of $\lambda'_{(C,f)}$.)

The category with conjugation $(\mathscr{C},\iota,\lambda)$ and its double $(\mathscr{C}',\iota',\lambda')$ are related by a pair of adjoint[1] functors F, G, which are described next.

For $C\in\mathscr{C}$, let

$$\Lambda_C=\begin{pmatrix}0 & \lambda_C\\ \mathrm{Id}_{\iota(C)} & 0\end{pmatrix}\colon\ \iota(C)\oplus\iota\iota(C)\to C\oplus\iota(C).$$

Then $(C\oplus\iota(C),\Lambda_C)\in\mathscr{C}'$ and we may define a k-linear functor

$$F\colon\mathscr{C}\to\mathscr{C}'\quad\text{by}\quad C\mapsto(C\oplus\iota(C),\Lambda_C),\quad \varphi\mapsto\begin{pmatrix}\varphi & 0\\ 0 & \iota(\varphi)\end{pmatrix}.$$

On the other hand, we consider the forgetful functor

$$G\colon\mathscr{C}'\to\mathscr{C}\qquad (C,f)\mapsto C,\quad \varphi\mapsto\varphi.$$

Proposition 1.1. *There are natural isomorphisms* $\mathrm{Id}_{\mathscr{C}}\oplus\iota\overset{\sim}{\Rightarrow}GF$ *and* $\mathrm{Id}_{\mathscr{C}'}\oplus\iota'\overset{\sim}{\Rightarrow}FG$.

[1]The functors F and G are adjoint to each other on each side. We omit the easy proof because this property is not used in the sequel.

Proof. The first isomorphism is clear since $GF(C) = C \oplus \iota(C)$ for $C \in \mathscr{C}$. For $C' = (C, f) \in \mathscr{C}'$, it is readily verified that $\frac{1}{2}\left(\begin{smallmatrix} \mathrm{Id}_C & -f \\ f^{-1} & \mathrm{Id}_{\iota(C)} \end{smallmatrix}\right)$ is an isomorphism $C' \oplus \iota'(C') \xrightarrow{\sim} FG(C')$. $\square$

1.2. **Periodicity.** Let $(\mathscr{C}, \iota, \lambda)$ be a category with conjugation and $(\mathscr{C}', \iota', \lambda')$ its double. Repeating the doubling construction, we obtain a new category with conjugation $(\mathscr{C}'', \iota'', \lambda'')$ and, by induction, an infinite sequence of categories with conjugations related by functors defined on the same model as F and G in §1.1:

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{C}' \underset{G'}{\overset{F'}{\rightleftarrows}} \mathscr{C}'' \underset{G''}{\overset{F''}{\rightleftarrows}} \mathscr{C}''' \underset{G'''}{\overset{F'''}{\rightleftarrows}} \cdots$$

We show in this subsection that this sequence is 2-periodic.

We first define a pair of k-linear functors $\mathscr{C}'' \underset{\Psi}{\overset{\Theta}{\rightleftarrows}} \mathscr{C}$. To simplify notation, we write (C, f, g) for $C'' = \big((C, f), g\big) \in \mathscr{C}''$. With this notation, the map g is a morphism $\iota'(C, f) \to (C, f)$ in $\mathscr{C}'$, hence

$$g\,(-\iota(f)) = f\,\iota(g). \tag{1.1}$$

Moreover, $g\,\iota'(g) = \lambda'_{(C,f)}$, hence

$$g\,\iota(g) = -\lambda_C = -f\,\iota(f). \tag{1.2}$$

It follows that

$$g\,f^{-1}\,g\,f^{-1} = -g\,\iota(g)\,\iota(f)^{-1}\,f^{-1} = \mathrm{Id}_C\,.$$

Therefore, $\frac{1}{2}(\mathrm{Id}_C - g\,f^{-1})$ is an idempotent in $\mathrm{End}_{\mathscr{C}}(C)$. Since $\mathscr{C}$ is assumed to be pseudo-abelian, there are morphisms $\Theta(C'') \xrightarrow{t_{C''}} C \xrightarrow{s_{C''}} \Theta(C'')$ in $\mathscr{C}$ such that

$$s_{C''}\,t_{C''} = \mathrm{Id}_{\Theta(C'')} \quad \text{and} \quad t_{C''}\,s_{C''} = \tfrac{1}{2}(\mathrm{Id}_C - g\,f^{-1}). \tag{1.3}$$

Thus, $\Theta(C'')$ is "the" kernel of $\frac{1}{2}(\mathrm{Id}_C + g\,f^{-1})$. Define a functor

$$\Theta\colon \mathscr{C}'' \to \mathscr{C} \quad \text{by} \quad \begin{cases} C'' \mapsto \Theta(C''), \\ (\varphi\colon C''_1 \to C''_2) \mapsto \big(s_{C''_2}\,\varphi\,t_{C''_1}\colon \Theta(C''_1) \to \Theta(C''_2)\big). \end{cases}$$

Note that the morphisms $\iota(t_{C''})$, $\iota(s_{C''})$ satisfy

$$\iota(s_{C''})\,\iota(t_{C''}) = \mathrm{Id}_{\iota(\Theta(C''))} \qquad \text{and} \qquad \iota(t_{C''})\,\iota(s_{C''}) = \tfrac{1}{2}(\mathrm{Id}_{\iota(C)} - \iota(g)\,\iota(f)^{-1}),$$

so we may identify $\iota(\Theta(C''))$ with $\Theta(\iota(C), -\iota(f), -\iota(g)) = \Theta\iota''(C'')$ to obtain a natural isomorphism $\theta\colon \Theta\iota'' \overset{\sim}{\Rightarrow} \iota\Theta$, hence a morphism of categories with conjugation

$$(\Theta, \theta)\colon (\mathscr{C}'', \iota'', \lambda'') \to (\mathscr{C}, \iota, \lambda).$$

On the other hand, for every $C \in \mathscr{C}$, define

$$\Delta_C = \begin{pmatrix} 0 & -\lambda_C \\ \mathrm{Id}_{\iota(C)} & 0 \end{pmatrix}\colon\ \iota(C) \oplus \iota\iota(C) \to C \oplus \iota(C).$$

Computation shows that $-\Delta_C\,\iota(\Lambda_C) = \Lambda_C\,\iota(\Delta_C)$, hence Δ_C defines a morphism in $\mathscr{C}'$,

$$\Delta_C\colon\ \iota'(C \oplus \iota(C), \Lambda_C) \to (C \oplus \iota(C), \Lambda_C).$$

Moreover,

$$\Delta_C\,\iota'(\Delta_C) = \begin{pmatrix} -\lambda_C & 0 \\ 0 & -\lambda_{\iota(C)} \end{pmatrix} = \lambda'_{(C \oplus \iota(C), \Lambda_C)},$$

hence $(C \oplus \iota(C), \Lambda_C, \Delta_C) \in \mathscr{C}''$. Define a functor

$$\Psi\colon \mathscr{C} \to \mathscr{C}'' \quad \text{by} \quad C \mapsto (C \oplus \iota(C), \Lambda_C, \Delta_C), \quad \varphi \mapsto \begin{pmatrix} \varphi & 0 \\ 0 & \iota(\varphi) \end{pmatrix}.$$

For every $C \in \mathscr{C}$, let

$$\psi_C = \begin{pmatrix} -\operatorname{Id}_{\iota(C)} & 0 \\ 0 & \operatorname{Id}_{\iota\iota(C)} \end{pmatrix} : \iota(C) \oplus \iota\iota(C) \to \iota(C) \oplus \iota\iota(C).$$

The map ψ_C defines an isomorphism in $\mathscr{C}''$

$$\psi_C \colon \Psi\iota(C) \xrightarrow{\sim} \iota''\Psi(C),$$

hence a natural isomorphism $\psi\colon \Psi\iota \overset{\sim}{\Rightarrow} \iota''\Psi$. The equation $\lambda''_{\Psi(C)}\, \iota''(\psi_C) = \Psi(\lambda_C)\, \psi^{-1}_{\iota(C)}$ is readily verified for every $C \in \mathscr{C}$, hence (Ψ, ψ) is a morphism of categories with conjugation

$$(\Psi, \psi)\colon (\mathscr{C}, \iota, \lambda) \to (\mathscr{C}'', \iota'', \lambda'').$$

Theorem 1.2. *The morphisms (Θ, θ) and (Ψ, ψ) define an equivalence of categories with conjugation*

$$(\mathscr{C}, \iota, \lambda) \equiv (\mathscr{C}'', \iota'', \lambda'').$$

Moreover, the following diagrams commute (up to natural isomorphisms):

(1.4)
$$\begin{tikzcd} \mathscr{C}'' \arrow[dr, "G'"] & \\ \mathscr{C} \arrow[u, "\Psi", "\wr"'] \arrow[r, "F"'] & \mathscr{C}' \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}' \arrow[r, "F'"] \arrow[dr, "G"'] & \mathscr{C}'' \arrow[d, "\Theta", "\wr"'] \\ & \mathscr{C} \end{tikzcd}$$

Proof. For $C \in \mathscr{C}$, the maps $\left(\begin{smallmatrix} \operatorname{Id}_C \\ 0 \end{smallmatrix}\right)\colon C \to C \oplus \iota(C)$ and $(\operatorname{Id}_C, 0)\colon C \oplus \iota(C) \to C$ satisfy

$$(\operatorname{Id}_C, 0)\left(\begin{smallmatrix} \operatorname{Id}_C \\ 0 \end{smallmatrix}\right) = \operatorname{Id}_C \quad \text{and} \quad \left(\begin{smallmatrix} \operatorname{Id}_C \\ 0 \end{smallmatrix}\right)(\operatorname{Id}_C, 0) = \begin{pmatrix} \operatorname{Id}_C & 0 \\ 0 & 0 \end{pmatrix} = \tfrac{1}{2}(\operatorname{Id}_{C\oplus\iota(C)} - \Delta_C\, \Lambda_C^{-1}).$$

Therefore, we may identify C with $\Theta(C \oplus \iota(C), \Lambda_C, \Delta_C)$ to obtain a natural isomorphism $\xi\colon \operatorname{Id}_{\mathscr{C}} \overset{\sim}{\Rightarrow} \Theta\Psi$. Explicitly, for $C \in \mathscr{C}$,

(1.5) $$\xi_C \colon C \to \Theta\Psi(C) \quad \text{is given by} \quad \xi_C = s_{\Psi(C)}\left(\begin{smallmatrix} \operatorname{Id}_C \\ 0 \end{smallmatrix}\right).$$

Now, consider $C'' = (C, f, g) \in \mathscr{C}''$, and write simply X for $\Theta(C'')$. By (1.3) we have

$$t_{C''} = t_{C''} s_{C''} t_{C''} = \tfrac{1}{2}(t_{C''} - gf^{-1}t_{C''}) \quad \text{and} \quad s_{C''} = s_{C''} t_{C''} s_{C''} = \tfrac{1}{2}(s_{C''} - s_{C''}gf^{-1}),$$

hence

(1.6) $$g^{-1}t_{C''} = -f^{-1}t_{C''} \qquad \text{and} \qquad s_{C''}f = -s_{C''}g.$$

Since $\iota(fg^{-1}) = -g^{-1}f$ by (1.1), applying ι to (1.6) yields

(1.7) $$f\iota(t_{C''}) = g\iota(t_{C''}) \qquad \text{and} \qquad \iota(s_{C''})g^{-1} = \iota(s_{C''})f^{-1}.$$

From (1.6) we get $\iota(s_{C''})g^{-1}t_{C''} = -\iota(s_{C''})f^{-1}t_{C''}$, and from (1.7), $\iota(s_{C''})g^{-1}t_{C''} = \iota(s_{C''})f^{-1}t_{C''}$, hence $\iota(s_{C''})f^{-1}t_{C''} = 0$ since $\frac{1}{2} \in k$. Similarly, $s_{C''}f\iota(t_{C''}) = 0$. Computation then shows that the maps

$$(t_{C''}, f\iota(t_{C''}))\colon X \oplus \iota(X) \to C \qquad \text{and} \qquad \left(\begin{smallmatrix} s_{C''} \\ \iota(s_{C''})f^{-1} \end{smallmatrix}\right)\colon C \to X \oplus \iota(X)$$

satisfy

$$(t_{C''}, f\iota(t_{C''}))\left(\begin{smallmatrix} s_{C''} \\ \iota(s_{C''})f^{-1} \end{smallmatrix}\right) = \mathrm{Id}_C \qquad \text{and} \qquad \left(\begin{smallmatrix} s_{C''} \\ \iota(s_{C''})f^{-1} \end{smallmatrix}\right)(t_{C''}, f\iota(t_{C''})) = \mathrm{Id}_{X\oplus\iota(X)},$$

hence they define an isomorphism $C \simeq X \oplus \iota(X)$ in $\mathscr{C}$. From (1.6) and (1.7) it follows that these maps actually define an isomorphism $C'' \simeq (X \oplus \iota(X), \Lambda_X, \Delta_X)$ in $\mathscr{C}''$, hence a natural isomorphism $\eta\colon \mathrm{Id}_{\mathscr{C}''} \overset{\sim}{\Rightarrow} \Psi\Theta$. Explicitly, for $C'' \in \mathscr{C}''$,

$$\eta_{C''}\colon C'' \to \Psi\Theta(C'') \quad \text{is given by} \quad \eta_{C''} = \left(\begin{smallmatrix} s_{C''} \\ \iota(s_{C''})f^{-1} \end{smallmatrix}\right). \tag{1.8}$$

To complete the proof, we check the commutativity of the diagrams in (1.4). For $C \in \mathscr{C}$,

$$G'\Psi(C) = G'\big(C \oplus \iota(C), \Lambda_C, \Delta_C\big) = (C \oplus \iota(C), \Lambda_C) = F(C),$$

hence $G'\Psi = F$ and the left triangle in (1.4) commutes. Now, for $C' = (C, f) \in \mathscr{C}'$,

$$F'(C') = \Big(C \oplus \iota(C), \left(\begin{smallmatrix} f & 0 \\ 0 & -\iota(f) \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -\lambda_C \\ \mathrm{Id}_{\iota(C)} & 0 \end{smallmatrix}\right)\Big),$$

and $\Theta F'(C')$ is a splitting of the idempotent $\frac{1}{2}\left(\begin{smallmatrix} \mathrm{Id}_C & -f \\ -f^{-1} & \mathrm{Id}_{\iota(C)} \end{smallmatrix}\right)$, equipped with morphisms in $\mathscr{C}$ $\Theta F'(C') \xrightarrow{t_{F'(C')}} C \oplus \iota(C) \xrightarrow{s_{F'(C')}} \Theta F'(C')$ such that

$$s_{F'(C')}\, t_{F'(C')} = \mathrm{Id}_{\Theta F'(C')} \qquad \text{and} \qquad t_{F'(C')}\, s_{F'(C')} = \frac{1}{2}\begin{pmatrix} \mathrm{Id}_C & -f \\ -f^{-1} & \mathrm{Id}_{\iota(C)} \end{pmatrix}.$$

Observe that $\left(\begin{smallmatrix} \mathrm{Id}_C \\ -f^{-1} \end{smallmatrix}\right)\colon C \to C \oplus \iota(C)$ and $\frac{1}{2}\big(\mathrm{Id}_C, -f\big)\colon C \oplus \iota(C) \to C$ also satisfy

$$\frac{1}{2}\big(\mathrm{Id}_C, -f\big)\left(\begin{smallmatrix} \mathrm{Id}_C \\ -f^{-1} \end{smallmatrix}\right) = \mathrm{Id}_C \qquad \text{and} \qquad \left(\begin{smallmatrix} \mathrm{Id}_C \\ -f^{-1} \end{smallmatrix}\right)\frac{1}{2}\big(\mathrm{Id}_C, -f\big) = \frac{1}{2}\begin{pmatrix} \mathrm{Id}_C & -f \\ -f^{-1} & \mathrm{Id}_{\iota(C)} \end{pmatrix}.$$

Therefore, we may define a natural isomorphism $\zeta\colon G \overset{\sim}{\Rightarrow} \Theta F'$ as follows: for $C' = (C, f) \in \mathscr{C}'$,

$$\zeta_{C'}\colon G(C') \to \Theta F'(C') \quad \text{is given by} \quad \zeta_{C'} = s_{F'(C')}\left(\begin{smallmatrix} \mathrm{Id}_C \\ -f^{-1} \end{smallmatrix}\right). \tag{1.9}$$

Thus, the right-hand side diagram in (1.4) commutes up to a natural isomorphism. $\square$

Since $\Psi\Theta \cong \mathrm{Id}_{\mathscr{C}''}$, the diagram on the left-hand side of (1.4) yields another commutative triangle (up to a natural isomorphism):

$$\begin{tikzcd} \mathscr{C}'' \arrow[d, "\Theta"', "\wr"] \arrow[dr, "G'"] & \\ \mathscr{C} \arrow[r, "F"] & \mathscr{C}' \end{tikzcd}$$

The latter triangle and all its corresponding versions for $\mathscr{C}'$, $\mathscr{C}''$, $\mathscr{C}'''$, etc. can be pasted with the triangle on the right-hand side of (1.4) and all its corresponding versions into the following infinite diagram:

$$\begin{tikzcd}
\mathscr{C}' \arrow[r, "F'"] \arrow[dr, "G"'] & \mathscr{C}'' \arrow[r, "F''"] \arrow[d, "\Theta"] \arrow[dr, "G'"] & \mathscr{C}''' \arrow[r, "F'''"] \arrow[d, "\Theta'"] \arrow[dr, "G''"] & \mathscr{C}'''' \arrow[r, "F''''"] \arrow[d, "\Theta''"] \arrow[dr, "G'''"] & \mathscr{C}''''' \arrow[r, "F'''''"] \arrow[d, "\Theta'''"] \arrow[dr, "G''''"] & \cdots \\
& \mathscr{C} \arrow[r, "F"] & \mathscr{C}' \arrow[r, "F'"] \arrow[dr, "G"'] & \mathscr{C}'' \arrow[r, "F''"] \arrow[d, "\Theta"] \arrow[dr, "G'"] & \mathscr{C}''' \arrow[r, "F'''"] \arrow[d, "\Theta'"] \arrow[dr, "G''"] & \cdots \\
& & & \mathscr{C} \arrow[r, "F"] & \mathscr{C}' \arrow[r, "F'"] \arrow[dr, "G"'] & \cdots \\
& & & & & \ddots
\end{tikzcd}$$

Retaining only the upper line and the left-most broken diagonal yields a commutative diagram that demonstrates the periodicity of the doubling construction:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, equal] & \mathscr{C}' \arrow[r, "F'"] \arrow[d, equal] & \mathscr{C}'' \arrow[r, "F''"] \arrow[d, "\Theta", "\wr"'] & \mathscr{C}''' \arrow[r, "F'''"] \arrow[d, "\Theta'", "\wr"'] & \mathscr{C}'''' \arrow[r, "F''''"] \arrow[d, "\Theta\Theta''", "\wr"'] & \mathscr{C}''''' \arrow[r, "F'''''"] \arrow[d, "\Theta'\Theta'''", "\wr"'] & \cdots \\
\mathscr{C} \arrow[r, "F"] & \mathscr{C}' \arrow[r, "G"] & \mathscr{C} \arrow[r, "F"] & \mathscr{C}' \arrow[r, "G"] & \mathscr{C} \arrow[r, "F"] & \mathscr{C}' \arrow[r, "G"] & \cdots
\end{tikzcd} \tag{1.10}$$

1.3. **Example: Modules.** To obtain a typical example of a category with conjugation and its double, consider an arbitrary (associative) k-algebra A and a pair (ι, λ) consisting of a k-linear automorphism $\iota\colon A \to A$ and a unit $\lambda \in A^\times$ such that $\iota(\lambda) = \lambda$ and $\iota^2(a) = \lambda a \lambda^{-1}$ for all $a \in A$. Define a k-algebra

$$A' = A \oplus Aj$$

where the multiplication extends the multiplication in A by

$$j^2 = \lambda \qquad \text{and} \qquad ja = \iota(a)j \quad \text{for } a \in A.$$

In Ranicki's terminology [Ran87, §3] the pair (ι, λ) is a *structure* on A and A' is the (ι, λ)-*twisted quadratic extension* of A.

Let $\mathscr{M}_A$ be the category of right A-modules. Abusing notation, we write again ι for the endofunctor on $\mathscr{M}_A$ defined as follows: for $M \in \mathscr{M}_A$, we set

$$\iota(M) = \{{}^\iota x \mid x \in M\}$$

with the operations

$${}^\iota x + {}^\iota y = {}^\iota(x + y), \quad {}^\iota x \cdot a = {}^\iota(x\iota(a)) \qquad \text{for } x,\, y \in M \text{ and } a \in A.$$

For every morphism $\varphi\colon\ M \to N$, define $\iota(\varphi)\colon\ \iota(M) \to \iota(N)$ by

$$\iota(\varphi)({}^\iota x) = {}^\iota(\varphi(x)) \qquad \text{for } x \in M.$$

Using the ι-fixed unit $\lambda \in A$ in the definition of A', we define a natural isomorphism $\lambda\colon\ \iota\iota \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathscr{M}_A}$ as follows:

$$\lambda_M\colon\ \iota\iota(M) \to M, \quad {}^{\iota\iota}x \mapsto x\lambda \qquad \text{for every } M \in \mathscr{M}_A \text{ and every } x \in M.$$

The pair (ι, λ) is the conjugation on $\mathscr{M}_A$ derived from the structure (ι, λ) on A.

Every object (M, f) in the double category $\mathscr{M}'_A$ can be regarded as a right module over A' by defining

$$x \cdot (a_1 + a_2 j) = xa_1 + f({}^\iota x)\iota^{-1}(a_2) \qquad \text{for } x \in M \text{ and } a_1,\, a_2 \in A.$$

Under this identification, the morphisms in $\mathscr{M}'_A$ are the A'-linear maps. Conversely, every right A'-module M' can be viewed as an object $(G(M'), f) \in \mathscr{M}'_A$ where $G(M')$ is the A-module obtained from M' by restriction of scalars and

$$f\colon \iota G(M') \to G(M'), \qquad {}^\iota x \mapsto xj \qquad \text{for } x \in G(M').$$

Therefore, $\mathscr{M}'_A$ is isomorphic to the category of right A'-modules; henceforth we identify these categories: $\mathscr{M}'_A = \mathscr{M}_{A'}$. The endofunctor ι' on $\mathscr{M}_{A'}$ is then derived from the automorphism

$$\iota'\colon A' \to A', \qquad a_1 + a_2 j \mapsto \iota(a_1) - \iota(a_2)j \qquad \text{for } a_1,\, a_2 \in A,$$

just like the endofunctor ι on $\mathscr{M}_A$ is derived from the automorphism ι of A. The natural isomorphism $\lambda'\colon \iota'\iota' \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathscr{M}_{A'}}$ is given by $\lambda'_{M'} = -\lambda_{G(M')}$ for $M' \in \mathscr{M}_{A'}$, hence

$$\lambda'_{M'}({}^{\iota'\iota'}x) = -x\lambda \qquad \text{for } x \in M'.$$

Therefore, $\mathscr{M}''_A$, the double category of $\mathscr{M}'_A = \mathscr{M}_{A'}$, can be identified with the category $\mathscr{M}_{A''}$ of right modules over the k-algebra $A'' = A' \oplus A'j'$, where

$$j'^2 = -\lambda \qquad \text{and} \qquad j'a' = \iota'(a')j' \qquad \text{for } a' \in A'.$$

The conjugation (ι', λ') on $\mathscr{M}_{A'}$ is thus derived from the structure (ι', λ') on A' with the unit $\lambda' = -\lambda \in A'^{\times}$, and the k-algebra A'' is the (ι', λ')-twisted quadratic extension of A'.

Under the identification $\mathscr{M}'_A = \mathscr{M}_{A'}$, the functor $F\colon \mathscr{M}_A \to \mathscr{M}_{A'}$ is the scalar extension: for $M \in \mathscr{M}_A$, the object $(M \oplus \iota(M), \Lambda_P)$ is identified with $M \otimes_A A'$ under the map

$$(x, {}^\iota y) \mapsto x \otimes 1 + y \otimes j \qquad \text{for } x,\, y \in M.$$

Moreover, under the identification $\mathscr{M}''_A = \mathscr{M}_{A''}$, the equivalence $\mathscr{M}_A \equiv \mathscr{M}_{A''}$ afforded by the functors Θ and Ψ is a Morita equivalence. To see this, note that $j^{-1}j' \in A''$ satisfies $(j^{-1}j')^2 = 1$, hence the elements $e_+ = \frac{1}{2}(1 + j^{-1}j')$ and $e_- = \frac{1}{2}(1 - j^{-1}j')$ are orthogonal idempotents in A''. For $M'' \in \mathscr{M}_{A''}$ and $M \in \mathscr{M}_A$ we may identify $\Theta(M'') = M'' \otimes_{A''} A''e_-$ and $\Psi(M) = M \otimes_A e_- A''$. We omit the proof, as this result is not used subsequently.

Of interest for the sequel are two subcategories of $\mathscr{M}_A$ that are preserved under ι: let $\mathscr{P}_A$ (resp. $\mathscr{F}_A$) be the full subcategory of finitely generated projective modules (resp. of modules of finite length) in $\mathscr{M}_A$. Since A' is a free module of finite rank over A, for every module $P' \in \mathscr{P}_{A'}$ the A-module $G(P')$ lies in $\mathscr{P}_A$. Conversely, if $M' \in \mathscr{M}_{A'}$ is such that $G(M') \in \mathscr{P}_A$, then the isomorphism $M' \oplus \iota'(M') \simeq FG(M') = G(M') \otimes_A A'$ from Proposition 1.1 shows that M' is a direct summand of a finitely generated projective A'-module, hence $M' \in \mathscr{P}_{A'}$. Therefore, the identification $\mathscr{M}'_A = \mathscr{M}_{A'}$ restricts to $\mathscr{P}'_A = \mathscr{P}_{A'}$.

Similarly, $\mathscr{M}'_A = \mathscr{M}_{A'}$ yields $\mathscr{F}'_A = \mathscr{F}_{A'}$: to see this, note that G turns every repetition-free chain of submodules in a module $M' \in \mathscr{M}_{A'}$ into a repetition-free chain of submodules in $G(M')$. Therefore, if $G(M') \in \mathscr{F}_A$, then $M' \in \mathscr{F}_{A'}$. Conversely, if $M' \in \mathscr{F}_{A'}$ then $G(M') \otimes_A A' \in \mathscr{F}_{A'}$ by Proposition 1.1, hence $G(M') \in \mathscr{F}_A$ since F turns every repetition-free chain of submodules in $G(M')$ into a repetition-free chain of submodules in $G(M') \otimes_A A'$.

Remark 1.3. The example discussed in this subsection is not the most general conjugation on a category of modules: in a general conjugation (ι,λ) on $\mathscr{M}_A$ the functor ι is an auto-equivalence of $\mathscr{M}_A$, hence Morita theory shows that, up to isomorphism, ι is given by

$$\iota(M) = M\otimes_A P \qquad \text{for some invertible } A\text{-}A\text{-bimodule } P.$$

(See [Bas68, Ch. II, §5].) The isomorphism λ is given by

$$\lambda_M\colon M\otimes_A P\otimes_A P\to M,\qquad m\otimes p\otimes q\mapsto m\,\beta(p\otimes q)$$

for some isomorphism of bimodules $\beta\colon P\otimes_A P\to A$ satisfying the associativity condition $\beta(p_1\otimes p_2)\otimes p_3 = p_1\beta(p_2\otimes p_3)$ for p_1, p_2, $p_3\in P$. According to [Bas68, Prop. 5.2, p. 73], the example discussed above is the special case where P is isomorphic to A as a left A-module. It then yields the automorphism of A denoted (abusively) by ι above because, up to isomorphism, $P=\{{}_1x_\iota \mid x\in A\}$ where $a\cdot {}_1x_\iota = {}_1ax_\iota$ and ${}_1x_\iota\cdot a = {}_1x\iota(a)_\iota$ for x, $a\in A$. The map $\beta\colon P\otimes_A P\to A$ is then given by $\beta({}_1x_\iota\otimes {}_1y_\iota) = x\iota(y)\lambda$ where $\lambda\in A^\times$ is subject to $\iota(\lambda)=\lambda$ and $\iota^2(a)=\lambda a\lambda^{-1}$ for all $a\in A$.

When A is commutative, the pairs (P,β) that yield a conjugation on $\mathscr{M}_A$ with the same action of A on P on the left and on the right are exactly the discriminant modules discussed in [Knu91, Ch. III, §3]. Since 2 is invertible in A, these discriminant modules are in one-to-one correspondence with the separable quadratic A-algebras, see [Knu91, Prop. III(4.2.4)]. Therefore, every separable quadratic A-algebra arises as A' for a suitable conjugation (ι,λ) on $\mathscr{M}_A$.

2. Conjugations and dualities

Recall from [Bal05, §1.2.1] that a *duality* on a category $\mathscr{C}$ is a pair (D,δ) consisting of a contravariant functor $D\colon\mathscr{C}\to\mathscr{C}$ and a natural isomorphism $\delta\colon \mathrm{Id}_{\mathscr{C}}\overset{\sim}{\Rightarrow} DD$ such that $(\mathbb{I}_D\cdot\delta)\,(\delta\cdot\mathbb{I}_D)=\mathbb{I}_D$, i.e., for every object C in $\mathscr{C}$,

$$\delta_{D(C)} = D(\delta_C)^{-1}\colon\ D(C)\to DDD(C).$$

Note that for any duality (D,δ), the pair $(D,-\delta)$ also is a duality.

A *morphism* of categories with duality $(\mathscr{C}_1,D_1,\delta_1)\to(\mathscr{C}_2,D_2,\delta_2)$ is a pair (R,θ) consisting of a functor $R\colon\mathscr{C}_1\to\mathscr{C}_2$ and a natural isomorphism $\theta\colon RD_1\overset{\sim}{\Rightarrow} D_2R$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, Rightarrow, "\mathbb{I}_R\cdot\delta_1"] \arrow[d, Rightarrow, "\delta_2\cdot\mathbb{I}_R"'] & RD_1D_1 \arrow[d, Rightarrow, "\theta\cdot\mathbb{I}_{D_1}"] \\ D_2D_2R \arrow[r, Rightarrow, "\mathbb{I}_{D_2}\cdot\theta"'] & D_2RD_1 \end{tikzcd}$$

Throughout this section, we only consider k-linear categories subject to the same conditions as in §1: k is a commutative ring in which 2 is invertible, and every idempotent morphism splits. All the functors we consider are k-linear. A notion of *similitude* of morphisms of categories with duality is defined as follows: morphisms $\widetilde{R}=(R,\theta)$ and $\widetilde{S}=(S,\tau)$ from $(\mathscr{C}_1,D_1,\delta_1)$ to $(\mathscr{C}_2,D_2,\delta_2)$ are said to be *similar* if there exists a natural isomorphism $\rho\colon R\overset{\sim}{\Rightarrow} S$ and a invertible scalar $\mu \in k^\times$ such that the following diagram commutes:

$$\begin{tikzcd}
RD_1 \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\rho\cdot\mathbb{I}_{D_1}"'] & D_2R \arrow[r, Rightarrow, "\mu\,\mathbb{I}_{D_2R}"] & D_2R \\
SD_1 \arrow[rr, Rightarrow, "\tau"] & & D_2S \arrow[u, Rightarrow, "\mathbb{I}_{D_2}\cdot\rho"']
\end{tikzcd}$$

i.e., for $C \in \mathscr{C}_1$,

$$D_2(\rho_C)\,\tau_C\,\rho_{D_1(C)} = \mu\,\theta_C.$$

The natural isomorphism ρ is then said to be a *similitude* $\widetilde{R} \Rightarrow \widetilde{S}$ with *multiplier* μ. Similitudes with multiplier 1 are *natural isomorphisms* of morphisms of categories with duality.

Diagram chase establishes the following multiplicativity property: for any similitude $\rho_1\colon \widetilde{R}_1 \Rightarrow \widetilde{S}_1$ of morphisms from $(\mathscr{C}_1, D_1, \delta_1)$ to $(\mathscr{C}_2, D_2, \delta_2)$ with multiplier μ_1 and any similitude $\rho_2\colon \widetilde{R}_2 \Rightarrow \widetilde{S}_2$ of morphisms from $(\mathscr{C}_2, D_2, \delta_2)$ to $(\mathscr{C}_3, D_3, \delta_3)$ with multiplier μ_2, the product $\rho_2 \cdot \rho_1$ is a similitude $\widetilde{R}_2\widetilde{R}_1 \Rightarrow \widetilde{S}_2\widetilde{S}_1$ with multiplier $\mu_1\mu_2$.

2.1. **Commuting conjugations and dualities.** A *commutation* between a conjugation (ι, λ) and a duality (D, δ) on a category $\mathscr{C}$ is a natural isomorphism

$$\alpha\colon\ \iota D \overset{\sim}{\Rightarrow} D\iota$$

such that the following diagrams commute:

$$\begin{tikzcd}
\iota \arrow[r, Rightarrow, "\mathbb{I}_\iota\cdot\delta"] \arrow[d, Rightarrow, "\delta\cdot\mathbb{I}_\iota"'] & \iota DD \arrow[d, Rightarrow, "\alpha\cdot\mathbb{I}_D"] \\
DD\iota \arrow[r, Rightarrow, "\mathbb{I}_D\cdot\alpha"'] & D\iota D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
D \arrow[r, Rightarrow, "\mathbb{I}_D\cdot\lambda"] \arrow[d, Rightarrow, "\lambda^{-1}\cdot\mathbb{I}_D"'] & D\iota\iota \arrow[d, Rightarrow, "\alpha^{-1}\cdot\mathbb{I}_\iota"] \\
\iota\iota D \arrow[r, Rightarrow, "\mathbb{I}_\iota\cdot\alpha"'] & \iota D\iota
\end{tikzcd}
\tag{2.1}$$

Note that a commutation between (ι, λ) and (D, δ) also defines a commutation between (ι, λ) and $(D, -\delta)$. Commuting conjugations and dualities on a category $\mathscr{C}$ yield new dualities on $\mathscr{C}$ and on the double category $\mathscr{C}'$ as follows:

Proposition 2.1. *Given a conjugation (ι, λ) and a duality (D, δ) with commutation α on a category $\mathscr{C}$, dualities $(\iota D, \delta_\alpha)$ and $(D\iota, \delta_{\alpha^{-1}})$ are defined by*

$$\delta_\alpha = (\mathbb{I}_\iota \cdot \alpha \cdot \mathbb{I}_D)\,(\lambda^{-1} \cdot \mathbb{I}_{DD})\,\delta\colon\ \mathrm{Id}_{\mathscr{C}} \overset{\sim}{\Rightarrow} \iota D\iota D$$

$$\delta_{\alpha^{-1}} = (\mathbb{I}_D \cdot \alpha \cdot \mathbb{I}_\iota)\,(\delta \cdot \mathbb{I}_{\iota\iota})\,\lambda^{-1}\colon\ \mathrm{Id}_{\mathscr{C}} \overset{\sim}{\Rightarrow} D\iota D\iota.$$

Moreover, $(\mathrm{Id}_{\mathscr{C}}, \alpha)$ is an isomorphism of categories with duality $(\mathscr{C}, \iota D, \delta_\alpha) \xrightarrow{\sim} (\mathscr{C}, D\iota, \delta_{\alpha^{-1}})$.

Proposition 2.2. *Given a conjugation (ι, λ) and a duality (D, δ) with commutation α on a category $\mathscr{C}$, a duality (D', δ') on the double category $\mathscr{C}'$ is defined as follows: for every $(C, f) \in \mathscr{C}'$ and every morphism φ in $\mathscr{C}'$,*

$$D'(C, f) = (D(C), -D(f)^{-1}\,\alpha_C), \qquad D'(\varphi) = D(\varphi), \qquad \delta'_{(C,f)} = \delta_C.$$

Moreover, setting $\alpha'_{(C,f)} = -\alpha_C$ for $(C, f) \in \mathscr{C}'$ defines a natural isomorphism

$$\alpha'\colon\ \iota' D' \overset{\sim}{\Rightarrow} D'\iota',$$

which is a commutation between (ι', λ') and (D', δ').

We omit the proofs of Propositions 2.1 and 2.2, which consist in lengthy but straightforward computations.

Recall from § 1.1 the functor $F\colon \mathscr{C} \to \mathscr{C}'$. Define natural isomorphisms

$$\widehat{\phi}\colon\; FD \overset{\sim}{\Rightarrow} D'F \qquad \text{and} \qquad \check{\phi}\colon\; F\iota D \overset{\sim}{\Rightarrow} \iota' D'F$$

as follows: for $C \in \mathscr{C}$,

$$\widehat{\phi}_C = \begin{pmatrix} \mathrm{Id}_{D(C)} & 0 \\ 0 & -\alpha_C \end{pmatrix} \qquad \text{and} \qquad \check{\phi}_C = \begin{pmatrix} \mathrm{Id}_{\iota D(C)} & 0 \\ 0 & \iota(\alpha_C) \end{pmatrix}.$$

Computation shows that the pairs $\widehat{F} = (F, \widehat{\phi})$ and $\check{F} = (F, \check{\phi})$ are morphisms of categories with duality

$$\widehat{F}\colon\; (\mathscr{C}, D, \pm\delta) \to (\mathscr{C}', D', \pm\delta') \qquad \text{and} \qquad \check{F}\colon\; (\mathscr{C}, \iota D, \pm\delta_\alpha) \to (\mathscr{C}', \iota' D', \pm\delta'_{\alpha'}).$$

Iterating the $\widehat{F}$ construction, we obtain an infinite sequence of morphisms of categories with duality:

$$(\mathscr{C}, D, \delta) \xrightarrow{\widehat{F}} (\mathscr{C}', D', \delta') \xrightarrow{\widehat{F}'} (\mathscr{C}'', D'', \delta'') \xrightarrow{\widehat{F}''} \cdots \tag{2.2}$$

Morphisms in the reverse direction are provided by the forgetful functor $G\colon \mathscr{C}' \to \mathscr{C}$, but they involve different dualities: it is easily checked that natural isomorphisms $\widehat{\gamma}\colon GD' \overset{\sim}{\Rightarrow} \iota DG$ and $\check{\gamma}\colon G\iota' D' \overset{\sim}{\Rightarrow} DG$ are defined by

$$\widehat{\gamma}_{(C,f)} = \alpha_C^{-1}\, D(f) \qquad \text{and} \qquad \check{\gamma}_{(C,f)} = D(f^{-1})\,\alpha_C \qquad \text{for } (C,f) \in \mathscr{C}'.$$

The pairs $\widehat{G} = (G, \widehat{\gamma})$ and $\check{G} = (G, \check{\gamma})$ are morphisms of categories with duality:

$$\widehat{G}\colon (\mathscr{C}', D', \pm\delta') \to (\mathscr{C}, \iota D, \mp\delta_\alpha) \qquad \text{and} \qquad \check{G}\colon (\mathscr{C}', \iota' D', \pm\delta'_{\alpha'}) \to (\mathscr{C}, D, \pm\delta).$$

This change of dualities (and signs) explains why the sequence (2.2) turns out to be 8-periodic, in contrast with the 2-periodicity of the sequence (1.10).

2.2. **Periodicity.** Continuing with the same notation as in the preceding section, we turn the equivalence of categories $\mathscr{C} \equiv \mathscr{C}''$ afforded by the functors Θ and Ψ of §1.2 into equivalences of categories with duality.

Natural isomorphisms $\widehat{\psi}\colon \Psi D \overset{\sim}{\Rightarrow} \iota'' D'' \Psi$ and $\check{\psi}\colon \Psi \iota D \overset{\sim}{\Rightarrow} D'' \Psi$ are defined by the following formulas, for $C \in \mathscr{C}$:

$$\widehat{\psi}_C = \begin{pmatrix} 0 & \mathrm{Id}_{\iota D(C)} \\ \iota(\alpha_C)\,\lambda_{D(C)}^{-1} & 0 \end{pmatrix} \qquad \text{and} \qquad \check{\psi}_C = \begin{pmatrix} 0 & \lambda_{D(C)} \\ -\alpha_C & 0 \end{pmatrix}.$$

The pairs $\widehat{\Psi} = (\Psi, \widehat{\psi})$ and $\check{\Psi} = (\Psi, \check{\psi})$ are morphisms of categories with duality

$$\widehat{\Psi}\colon (\mathscr{C}, D, \pm\delta) \to (\mathscr{C}'', \iota'' D'', \pm\delta''_{\alpha''}) \qquad \text{and} \qquad \check{\Psi}\colon (\mathscr{C}, \iota D, \pm\delta_\alpha) \to (\mathscr{C}'', D'', \mp\delta'').$$

Similarly, natural isomorphisms $\widehat{\theta}\colon \Theta D'' \overset{\sim}{\Rightarrow} \iota D \Theta$ and $\check{\theta}\colon \Theta \iota'' D'' \overset{\sim}{\Rightarrow} D\Theta$ are defined as follows: for $C'' = (C, f, g) \in \mathscr{C}''$,

$$\widehat{\theta}_{C''} = -\iota D(t_{C''})\,\alpha_C^{-1}\, D(f)\, t_{D''(C'')} \qquad \text{and} \qquad \check{\theta}_{C''} = D(t_{C''})\, D(f^{-1})\,\alpha_C\, t_{\iota'' D''(C'')}.$$

The pairs $\widehat{\Theta} = (\Theta, \widehat{\theta})$ and $\check{\Theta} = (\Theta, \check{\theta})$ are morphisms of categories with duality

$$\widehat{\Theta}\colon (\mathscr{C}'', D'', \pm\delta'') \to (\mathscr{C}, \iota D, \mp\delta_\alpha) \qquad \text{and} \qquad \check{\Theta}\colon (\mathscr{C}'', \iota'' D'', \pm\delta''_{\alpha''}) \to (\mathscr{C}, D, \pm\delta).$$

Theorem 2.3. *The morphisms $\widehat{\Psi}$, $\check{\Psi}$ and $\widehat{\Theta}$, $\check{\Theta}$ define equivalences of categories with duality*

$$(\mathscr{C}, D, \pm\delta) \underset{\check{\Theta}}{\overset{\widehat{\Psi}}{\rightleftarrows}} (\mathscr{C}'', \iota'' D'', \pm\delta''_{\alpha''}) \qquad \text{and} \qquad (\mathscr{C}, \iota D, \pm\delta_\alpha) \underset{\widehat{\Theta}}{\overset{\check{\Psi}}{\rightleftarrows}} (\mathscr{C}'', D'', \mp\delta'')\ .$$

Moreover, the following diagrams commute:

$$\begin{tikzcd}
(\mathscr{C}'', \iota'' D'', \pm\delta''_{\alpha''}) \arrow[dr, "\check{G}'"] & \\
(\mathscr{C}, D, \pm\delta) \arrow[u, "\widehat{\Psi}", "\wr"'] \arrow[r, "\widehat{F}"'] & (\mathscr{C}', D', \pm\delta')
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathscr{C}'', D'', \mp\delta'') \arrow[dr, "\widehat{G}'"] & \\
(\mathscr{C}, \iota D, \pm\delta_\alpha) \arrow[u, "\check{\Psi}", "\wr"'] \arrow[r, "\check{F}"'] & (\mathscr{C}', \iota' D', \pm\delta'_{\alpha'})
\end{tikzcd}
\tag{2.3}$$

and the following diagrams commute up to similitude:

$$\begin{tikzcd}
(\mathscr{C}', D', \pm\delta') \arrow[r, "\widehat{F}'"] \arrow[dr, "\widehat{G}"'] & (\mathscr{C}'', D'', \pm\delta'') \arrow[d, "\widehat{\Theta}", "\wr"'] \\
& (\mathscr{C}, \iota D, \mp\delta_\alpha)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathscr{C}', \iota' D', \pm\delta'_{\alpha'}) \arrow[r, "\check{F}'"] \arrow[dr, "\check{G}"'] & (\mathscr{C}'', \iota'' D'', \pm\delta''_{\alpha''}) \arrow[d, "\check{\Theta}", "\wr"'] \\
& (\mathscr{C}, D, \mp\delta)
\end{tikzcd}
\tag{2.4}$$

Proof. Recall from (1.5) the natural isomorphism $\xi\colon \mathrm{Id}_{\mathscr{C}} \overset{\sim}{\Rightarrow} \Theta\Psi$; computation shows that

$$(\mathbb{I}_D \cdot \xi)\,(\check{\theta} \cdot \mathbb{I}_\Psi)\,(\mathbb{I}_\Theta \cdot \widehat{\psi})\,(\xi \cdot \mathbb{I}_D) = \mathbb{I}_D \text{ and } (\mathbb{I}_{\iota D} \cdot \xi)\,(\widehat{\theta} \cdot \mathbb{I}_\Psi)\,(\mathbb{I}_\Theta \cdot \check{\psi})\,(\xi \cdot \mathbb{I}_{\iota D}) = \mathbb{I}_{\iota D}.$$

Therefore, ξ defines natural isomorphisms $(\mathrm{Id}_{\mathscr{C}}, \mathbb{I}_D) \overset{\sim}{\Rightarrow} \check{\Theta}\widehat{\Psi}$ and $(\mathrm{Id}_{\mathscr{C}}, \mathbb{I}_{\iota D}) \overset{\sim}{\Rightarrow} \widehat{\Theta}\check{\Psi}$.

Similarly, using the natural isomorphism $\eta\colon \mathrm{Id}_{\mathscr{C}''} \overset{\sim}{\Rightarrow} \Psi\Theta$ of (1.8), one checks that

$$(\mathbb{I}_{\iota'' D''} \cdot \eta)\,(\widehat{\psi} \cdot \mathbb{I}_\Theta)\,(\mathbb{I}_\Psi \cdot \check{\theta})\,(\eta \cdot \mathbb{I}_{\iota'' D''}) = \mathbb{I}_{\iota'' D''} \text{ and } (\mathbb{I}_{D''} \cdot \eta)\,(\check{\psi} \cdot \mathbb{I}_\Theta)\,(\mathbb{I}_\Psi \cdot \widehat{\theta})\,(\eta \cdot \mathbb{I}_{D''}) = \mathbb{I}_{D''},$$

hence η defines natural isomorphisms $(\mathrm{Id}_{\mathscr{C}''}, \mathbb{I}_{\iota'' D''}) \overset{\sim}{\Rightarrow} \widehat{\Psi}\check{\Theta}$ and $(\mathrm{Id}_{\mathscr{C}''}, \mathbb{I}_{D''}) \overset{\sim}{\Rightarrow} \check{\Psi}\widehat{\Theta}$.

Since $F = G'\Psi$, the commutativity of the diagrams (2.3) is easily checked. Finally, to deal with the diagrams (2.4), we use the natural isomorphism $\zeta\colon G \overset{\sim}{\Rightarrow} \Theta F'$ of (1.9) and compute:

$$(\mathbb{I}_{\iota D} \cdot \zeta)\,(\widehat{\theta} \cdot \mathbb{I}_{F'})\,(\mathbb{I}_\Theta \cdot \widehat{\phi}')\,(\zeta \cdot \mathbb{I}_{D'}) = -2\widehat{\gamma} \text{ and } (\mathbb{I}_D \cdot \zeta)\,(\check{\theta} \cdot \mathbb{I}_{F'})\,(\mathbb{I}_\Theta \cdot \check{\phi}')\,(\zeta \cdot \mathbb{I}_{\iota' D'}) = 2\check{\gamma}.$$

Therefore, $\zeta\colon \widehat{G} \Rightarrow \widehat{\Theta}\widehat{F}'$ is a similitude with multiplier -2 and $\zeta\colon \check{G} \Rightarrow \check{\Theta}\check{F}'$ is a similitude with multiplier 2. □

Since $\widehat{\Psi}\check{\Theta} \cong (\mathrm{Id}_{\mathscr{C}''}, \mathbb{I}_{\iota'' D''})$ and $\check{\Psi}\widehat{\Theta} \cong (\mathrm{Id}_{\mathscr{C}''}, \mathbb{I}_{D''})$, the diagrams (2.3) yield diagrams that commute up to natural isomorphisms:

$$\begin{tikzcd}
(\mathscr{C}'', \iota'' D'', \pm\delta''_{\alpha''}) \arrow[d, "\check{\Theta}"', "\wr"] \arrow[dr, "\check{G}'"] & \\
(\mathscr{C}, D, \pm\delta) \arrow[r, "\widehat{F}"'] & (\mathscr{C}', D', \pm\delta')
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathscr{C}'', D'', \mp\delta'') \arrow[d, "\widehat{\Theta}"', "\wr"] \arrow[dr, "\widehat{G}'"] & \\
(\mathscr{C}, \iota D, \pm\delta_\alpha) \arrow[r, "\check{F}"'] & (\mathscr{C}', \iota' D', \pm\delta'_{\alpha'})
\end{tikzcd}$$

Pasting these diagrams with the diagrams (2.4) for the various doubles yields an infinite diagram that commutes up to similitudes:

$$\begin{tikzcd}
(\mathscr{C}, D, \delta) \arrow[r, "\widehat{F}"] & (\mathscr{C}', D', \delta') \arrow[r, "\widehat{F}'"] \arrow[dr, "\widehat{G}"'] & (\mathscr{C}'', D'', \delta'') \arrow[r, "\widehat{F}''"] \arrow[d, "\widehat{\Theta}"'] \arrow[dr, "\widehat{G}'"] & (\mathscr{C}''', D''', \delta''') \arrow[r, "\widehat{F}'''"] \arrow[d, "\widehat{\Theta}'"'] \arrow[dr, "\widehat{G}''"] & (\mathscr{C}'''', D'''', \delta'''') \cdots \arrow[d, "\widehat{\Theta}''"'] \\
 & & (\mathscr{C}, \iota D, -\delta_\alpha) \arrow[r, "\check{F}"] & (\mathscr{C}', \iota' D', -\delta'_{\alpha'}) \arrow[r, "\check{F}'"] \arrow[dr, "\check{G}"'] & (\mathscr{C}'', \iota'' D'', \delta''_{\alpha''}) \cdots \arrow[d, "\check{\Theta}"'] \\
 & & & & (\mathscr{C}, D, -\delta) \cdots
\end{tikzcd}$$

Retaining only the upper sequence and the left-most broken diagonal, we obtain a diagram that commutes up to similitudes:

(2.5)

$$\begin{tikzcd}
(\mathscr{C}, D, \delta) \arrow[r, "\widehat{F}"] \arrow[d, equal] & (\mathscr{C}', D', \delta') \arrow[r, "\widehat{F}'"] \arrow[d, equal] & (\mathscr{C}'', D'', \delta'') \arrow[r, "\widehat{F}''"] \arrow[d, "\widehat{\Theta}"'] & (\mathscr{C}''', D''', \delta''') \arrow[r, "\widehat{F}'''"] \arrow[d, "\widehat{\Theta}'"'] & (\mathscr{C}'''', D'''', \delta'''') \cdots \arrow[d, "\check{\Theta}\widehat{\Theta}''"'] \\
(\mathscr{C}, D, \delta) \arrow[r, "\widehat{F}"] & (\mathscr{C}', D', \delta') \arrow[r, "\widehat{G}"] & (\mathscr{C}, \iota D, -\delta_\alpha) \arrow[r, "\check{F}"] & (\mathscr{C}', \iota' D', -\delta'_{\alpha'}) \arrow[r, "\check{G}"] & (\mathscr{C}, D, -\delta) \cdots
\end{tikzcd}$$

The lower sequence is 8-periodic and folds into an octagon:

(2.6)
$$\begin{tikzcd}
(\mathscr{C}, D, \delta) \arrow[r, "\widehat{F}"] & (\mathscr{C}', D', \delta') \arrow[r, "\widehat{G}"] & (\mathscr{C}, \iota D, -\delta_\alpha) \arrow[d, "\check{F}"] \\
(\mathscr{C}', \iota' D', \delta'_{\alpha'}) \arrow[u, "\check{G}"] & & (\mathscr{C}', \iota' D', -\delta'_{\alpha'}) \arrow[d, "\check{G}"] \\
(\mathscr{C}, \iota D, \delta_\alpha) \arrow[u, "\check{F}"] & (\mathscr{C}', D', -\delta') \arrow[l, "\widehat{G}"'] & (\mathscr{C}, D, -\delta) \arrow[l, "\widehat{F}"']
\end{tikzcd}$$

2.3. **Example: Projective modules.** The typical example of duality arises in the context of algebras with involution or, more generally, with *antistructure* in C.T.C. Wall's terminology [Wal72], [Ran87, §1]. Let A be an associative k-algebra and (σ, ε) a pair consisting of a k-linear antiautomorphism $\sigma\colon A \to A$ and a unit $\varepsilon \in A^\times$ such that $\sigma(\varepsilon) = \varepsilon^{-1}$ and $\sigma^2(a) = \varepsilon a \varepsilon^{-1}$ for all $a \in A$. On the category $\mathscr{P}_A$ of finitely generated projective right A-modules, a contravariant endofunctor D is defined by

$$D(P) = \operatorname{Hom}_A(P, A) \qquad \text{for } P \in \mathscr{P}_A,$$

using σ to twist the natural left A-module structure on $\operatorname{Hom}_A(P, A)$ into a right A-module structure. Thus, letting $\langle \textvisiblespace, \textvisiblespace \rangle$ denote the canonical pairing between a module and its dual, we have

$$\langle \pi a, x\rangle = \sigma(a)\langle \pi, x\rangle \qquad \text{for } a \in A,\ \pi \in D(P) \text{ and } x \in P \in \mathscr{P}_A.$$

A natural isomorphism $\delta\colon \operatorname{Id}_{\mathscr{P}_A} \overset{\sim}{\Rightarrow} DD$ is defined by

$$\langle \delta_P(x), \pi\rangle = \sigma(\langle \pi, x\rangle)\,\varepsilon \qquad \text{for } \pi \in D(P) \text{ and } x \in P \in \mathscr{P}_A.$$

The pair (D, δ) is the duality on $\mathscr{P}_A$ derived from the antistructure (σ, ε).

Now, recall the conjugation (ι,λ) on $\mathscr{P}_A$ from Example 1.3, derived from a structure (ι,λ) on A. Assume there exists a unit $u\in A^\times$ such that[2]

(i) $\iota\sigma(a)\,\iota(u)=\iota(u)\,\sigma\iota(a)$ for all $a\in A$,
(ii) $\iota\sigma(\lambda)\lambda=\iota(u)u$, and
(iii) $\varepsilon\,\iota(\varepsilon)^{-1}=\sigma\iota(u)\,\iota(u)^{-1}$.

Then a commutation $\alpha\colon \iota D\overset{\sim}{\Rightarrow} D\iota$ between (ι,λ) and (D,δ) is defined by

$$\langle\alpha_P({}^\iota\pi),{}^\iota x\rangle=u\,\iota^{-1}(\langle\pi,x\rangle)\qquad\text{for }\pi\in D(P)\text{ and }x\in P\in\mathscr{P}_A.\tag{2.7}$$

In order to describe the duality $(\iota D,\delta_\alpha)$ defined in Proposition 2.1 in the same terms as the duality (D,δ), we introduce the following notation:

$$\varepsilon_\alpha=\sigma(u\lambda^{-1})\varepsilon\quad\text{and}\quad\{{}^\iota\pi,x\}=\langle\pi,x\rangle\quad\text{for }\pi\in D(P)\text{ and }x\in P\in\mathscr{P}_A.$$

Then $(\sigma\iota,\varepsilon_\alpha)$ is an antistructure on A, and $\{({}^\iota\pi)a,x\}=\sigma\iota(a)\,\{{}^\iota\pi,x\}$ for $a\in A$, ${}^\iota\pi\in\iota D(P)$ and $x\in P\in\mathscr{P}_A$. Moreover,

$$\{(\delta_\alpha)_P(x),{}^\iota\pi\}=\sigma\iota(\{{}^\iota\pi,x\})\,\varepsilon_\alpha\qquad\text{for }{}^\iota\pi\in\iota D(P)\text{ and }x\in P\in\mathscr{P}_A.$$

Therefore, the duality $(\iota D,\delta_\alpha)$ is the duality on $\mathscr{P}_A$ derived from the antistructure $(\sigma\iota,\varepsilon_\alpha)$ on A.

Next, we describe the duality (D',δ') on the double category $\mathscr{P}'_A$. Recall from Example 1.3 that $\mathscr{P}'_A$ is identified with the category $\mathscr{P}_{A'}$ of finitely generated projective right modules over the algebra $A'=A\oplus Aj$, where $j^2=\lambda$ and $ja=\iota(a)j$ for $a\in A$: each pair $P'=(P,f)\in\mathscr{P}'_A$ is viewed as the A'-module obtained by extending the A-module structure on P by the rule $xj=f({}^\iota x)$ for $x\in P$. By definition, the A'-module structure on $D'(P')=(D(P),-D(f)^{-1}\alpha_P)$ is given by $\pi j=-D(f)^{-1}\alpha_P({}^\iota\pi)$ for $\pi\in D(P)$, which means that

$$\langle\pi j,x\rangle=-\langle\alpha_P({}^\iota\pi),f^{-1}(x)\rangle=-u\,\iota^{-1}(\langle\pi,xj^{-1}\rangle).\tag{2.8}$$

Extend the antiautomorphism σ on A to an antiautomorphism σ' on A' such that ${\sigma'}^2(a')=\varepsilon a'\varepsilon^{-1}$ for all $a'\in A'$ by

$$\sigma'(a_1+a_2j)=\sigma(a_1)-uj^{-1}\sigma(a_2)\qquad\text{for }a_1,\,a_2\in A,\tag{2.9}$$

and let

$$\langle\pi,x\rangle'=\langle\pi,x\rangle+\langle\pi,xj^{-1}\rangle j\in A'\qquad\text{for }\pi\in D(P)\text{ and }x\in P.\tag{2.10}$$

Then it follows from (2.8) that $\langle\pi a',x\rangle'=\sigma'(a')\langle\pi,x\rangle'$ for $a'\in A'$, $\pi\in D(P)$ and $x\in P$, and $\langle\text{␣},\text{␣}\rangle'$ is an A'-sesquilinear pairing $D'(P')\times P'\to A'$, which identifies $D'(P')$ with $\operatorname{Hom}_{A'}(P',A')$. With this notation, the definition $\delta'_{P'}=\delta_{G(P')}$ and the conditions on u imply

$$\langle\delta'_{P'}(x),\pi\rangle'=\sigma'(\langle\pi,x\rangle')\,\varepsilon\qquad\text{for }\pi\in D'(P')\text{ and }x\in P'.$$

Therefore, the duality (D',δ') on $\mathscr{P}_{A'}$ is derived from the antistructure (σ',ε) on A'.

Defining $\{{}^{\iota'}\pi,x\}'=\langle\pi,x\rangle'$ for $\pi\in D'(P')$ and $x\in P'\in\mathscr{P}_{A'}$, we also have

$$\{\delta'_{\alpha'}(x),{}^{\iota'}\pi\}'=\sigma'\iota'(\{{}^{\iota'}\pi,x\}')\,\varepsilon_\alpha\quad\text{for }{}^{\iota'}\pi\in\iota'D'(P')\text{ and }x\in P',$$

because $\alpha'\colon\iota'D'\overset{\sim}{\Rightarrow}D'\iota'$ and $\lambda'\colon\iota'\iota'\overset{\sim}{\Rightarrow}\mathrm{Id}_{\mathscr{P}_{A'}}$ are defined by $\alpha'_{P'}=-\alpha_{G(P')}$ and $\lambda'_{P'}=-\lambda_{G(P')}$. Thus, the duality $(\iota'D',\delta'_{\alpha'})$ is derived from the antistructure $(\sigma'\iota',\varepsilon)$ on A'.

---

[2]The conditions on u are necessary and sufficient to guarantee that (2.9) defines an antiautomorphism σ' on A' such that $\sigma'(j)j\in A^\times$ and ${\sigma'}^2(j)=\varepsilon j\varepsilon^{-1}$.

3. Symmetric spaces

As in the previous sections, $\mathscr{C}$ denotes a pseudo-abelian k-linear category for an arbitrary commutative ring k in which 2 is invertible. Throughout the section, (D,δ) is a duality on $\mathscr{C}$.

3.1. **Definitions.** A *symmetric space* in the category with duality $(\mathscr{C},D,\delta)$ is a pair (C,h) consisting of an object $C\in\mathscr{C}$ and an isomorphism $h\colon C\to D(C)$ in $\mathscr{C}$ such that $D(h)\,\delta_C=h$. For $\mu\in k^\times$, a *similitude with multiplier* μ from a symmetric space (C_1,h_1) to a symmetric space (C_2,h_2) is an isomorphism $\varphi\colon C_1\xrightarrow{\sim}C_2$ in $\mathscr{C}$ such that the following diagram commutes:

$$\begin{tikzcd} C_1 \arrow[rr, "\varphi"] \arrow[d, "h_1"'] & & C_2 \arrow[d, "h_2"] \\ D(C_1) \arrow[r, "\mu\,\mathrm{Id}_{D(C_1)}"] & D(C_1) & D(C_2) \arrow[l, "D(\varphi)"'] \end{tikzcd}$$

Similitudes with multiplier 1 are *isometries*.

For $L\in\mathscr{C}$, let $H(L)$ denote the symmetric space

$$H(L)=\Bigl(L\oplus D(L),\ \bigl(\begin{smallmatrix}0 & \mathrm{Id}_{D(L)}\\ \delta_L & 0\end{smallmatrix}\bigr)\Bigr).$$

By definition, a symmetric space (C,h) is *hyperbolic* if it is isometric to $H(L)$ for some $L\in\mathscr{C}$. Note that for every $\mu\in k^\times$ the morphism $\bigl(\begin{smallmatrix}\mu\,\mathrm{Id}_L & 0\\ 0 & \mathrm{Id}_{D(L)}\end{smallmatrix}\bigr)$ is a similitude $H(L)\to H(L)$ with multiplier μ. Therefore, hyperbolicity is preserved under similitude.

3.2. **Hermitian categories.** Let $\mathscr{S}(\mathscr{C},D,\delta)$ denote the category whose objects are symmetric spaces in $(\mathscr{C},D,\delta)$ and morphisms are isometries. We refer to $\mathscr{S}(\mathscr{C},D,\delta)$ as the *hermitian category* of the category with duality $(\mathscr{C},D,\delta)$. Every morphism of categories with duality $\widetilde{R}=(R,\theta)\colon(\mathscr{C}_1,D_1,\delta_1)\to(\mathscr{C}_2,D_2,\delta_2)$ induces a functor

$$\mathscr{S}(\widetilde{R})\colon\mathscr{S}(\mathscr{C}_1,D_1,\delta_1)\to\mathscr{S}(\mathscr{C}_2,D_2,\delta_2)$$

mapping $(C,h)\in\mathscr{S}(\mathscr{C}_1,D_1,\delta_1)$ to

$$\mathscr{S}(\widetilde{R})(C,h)=\bigl(R(C),\theta_C\,R(h)\bigr),$$

see [Knu91, II(2.6)] or [Bal05, Def. 11]. It readily follows from the definitions that $\mathscr{S}(\widetilde{R})$ maps hyperbolic symmetric spaces to hyperbolic symmetric spaces, because for every $L\in\mathscr{C}_1$ the morphism $\bigl(\begin{smallmatrix}\mathrm{Id}_{R(L)} & 0\\ 0 & \theta_L\end{smallmatrix}\bigr)\colon R(L)\oplus RD_1(L)\to R(L)\oplus D_2R(L)$ is an isometry from $\mathscr{S}(\widetilde{R})\bigl(H(L)\bigr)$ to $HR(L)$.

If $\rho\colon\widetilde{R}\overset{\sim}{\Rightarrow}\widetilde{S}$ is a similitude with multiplier $\mu\in k^\times$ between morphisms of categories with duality $\widetilde{R},\widetilde{S}\colon(\mathscr{C}_1,D_1,\delta_1)\to(\mathscr{C}_2,D_2,\delta_2)$, then for every symmetric space $(C,h)\in\mathscr{S}(\mathscr{C}_1,D_1,h_1)$ the isomorphism $\rho_C\colon R(C)\to S(C)$ in $\mathscr{C}_2$ defines a similitude with multiplier μ

$$\mathscr{S}(\rho_C)\colon\mathscr{S}(\widetilde{R})(C,h)\to\mathscr{S}(\widetilde{S})(C,h).$$

Letting $\mathscr{S}(\rho)_{(C,h)}=\mathscr{S}(\rho_C)$, we also call $\mathscr{S}(\rho)\colon\mathscr{S}(\widetilde{R})\overset{\sim}{\Rightarrow}\mathscr{S}(\widetilde{S})$ a *similitude* with multiplier μ. In particular, equivalences of categories with duality yield equivalences of the corresponding hermitian categories, which preserve hyperbolicity. This applies to the morphisms from §2.2, which thus define equivalences

$$\mathscr{S}(\mathscr{C}, D, \pm\delta) \underset{\mathscr{S}(\breve{\Theta})}{\overset{\mathscr{S}(\hat{\Psi})}{\rightleftarrows}} \mathscr{S}(\mathscr{C}'', \iota'' D'', \pm\delta''_{\alpha''}) \quad \text{and} \quad \mathscr{S}(\mathscr{C}, \iota D, \pm\delta_\alpha) \underset{\mathscr{S}(\hat{\Theta})}{\overset{\mathscr{S}(\check{\Psi})}{\rightleftarrows}} \mathscr{S}(\mathscr{C}'', D'', \mp\delta'')\ .$$

Applying the $\mathscr{S}$ construction to the diagram (2.5), we obtain a diagram that commutes up to similitudes:

$$\begin{tikzcd}
\mathscr{S}(\mathscr{C}, D, \delta) \arrow[r, "\mathscr{S}(\widehat{F})"] \arrow[d, equal] & \mathscr{S}(\mathscr{C}', D', \delta') \arrow[r, "\mathscr{S}(\widehat{F}')"] \arrow[d, equal] & \mathscr{S}(\mathscr{C}'', D'', \delta'') \arrow[r, "\mathscr{S}(\widehat{F}'')"] \arrow[d, "\mathscr{S}(\widehat{\Theta})"'] & \mathscr{S}(\mathscr{C}''', D''', \delta''') \arrow[r, "\mathscr{S}(\widehat{F}''')"] \arrow[d, "\mathscr{S}(\widehat{\Theta}')"'] & \cdots \\
\mathscr{S}(\mathscr{C}, D, \delta) \arrow[r, "\mathscr{S}(\widehat{F})"] & \mathscr{S}(\mathscr{C}', D', \delta') \arrow[r, "\mathscr{S}(\widehat{G})"] & \mathscr{S}(\mathscr{C}, \iota D, -\delta_\alpha) \arrow[r, "\mathscr{S}(\check{F})"] & \mathscr{S}(\mathscr{C}', \iota' D', -\delta'_{\alpha'}) \arrow[r, "\mathscr{S}(\check{G})"] & \cdots
\end{tikzcd} \tag{3.1}$$

The 8-periodic lower sequence folds into an oriented octagon of hermitian categories:

$$\begin{tikzcd}
\mathscr{S}(\mathscr{C}, D, \delta) \arrow[r, "\mathscr{S}(\widehat{F})"] & \mathscr{S}(\mathscr{C}', D', \delta') \arrow[r, "\mathscr{S}(\widehat{G})"] & \mathscr{S}(\mathscr{C}, \iota D, -\delta_\alpha) \arrow[d, "\mathscr{S}(\check{F})"] \\
\mathscr{S}(\mathscr{C}', \iota' D', \delta'_{\alpha'}) \arrow[u, "\mathscr{S}(\check{G})"] & & \mathscr{S}(\mathscr{C}', \iota' D', -\delta'_{\alpha'}) \arrow[d, "\mathscr{S}(\check{G})"] \\
\mathscr{S}(\mathscr{C}, \iota D, \delta_\alpha) \arrow[u, "\mathscr{S}(\check{F})"] & \mathscr{S}(\mathscr{C}', D', -\delta') \arrow[l, "\mathscr{S}(\widehat{G})"'] & \mathscr{S}(\mathscr{C}, D, -\delta) \arrow[l, "\mathscr{S}(\widehat{F})"']
\end{tikzcd} \tag{3.2}$$

The main result of this section is as follows:

Theorem 3.1. *Under the composition of two consecutive functors in the octagon* (3.2)*, the image of each category consists of hyperbolic symmetric spaces.*

Since hyperbolicity is preserved by similitude and by $\mathscr{S}(\widehat{\Theta})$ and $\mathscr{S}(\breve{\Theta})$, we may substitute in the theorem the functors in the upper row of (3.1) for the functors $\mathscr{S}(\widehat{F})$, $\mathscr{S}(\widehat{G})$, $\mathscr{S}(\check{F})$, $\mathscr{S}(\check{G})$ from (3.2). Moreover, since $\widehat{F}^{(n+1)} = \big(\widehat{F}^{(n)}\big)'$, we may substitute $\widehat{F}$ for $\widehat{F}^{(n)}$ and consider only the composition of $\mathscr{S}(\widehat{F})$ and $\mathscr{S}(\widehat{F}')$. Therefore, the theorem follows from the following proposition:

Proposition 3.2. *The composition $\mathscr{S}(\widehat{F}')\mathscr{S}(\widehat{F}) = \mathscr{S}(\widehat{F}'\widehat{F})$ maps every symmetric space in $\mathscr{S}(\mathscr{C}, D, \delta)$ to a hyperbolic symmetric space in $\mathscr{S}(\mathscr{C}'', D'', \delta'')$.*

The following lemma characterises the symmetric spaces in the image of $\mathscr{S}(\widehat{F})$. It is to be used in the proofs of Proposition 3.2 and Theorems 4.6 and 4.8.

Lemma 3.3. *For a symmetric space $(C', h') \in \mathscr{S}(\mathscr{C}', D', \delta')$, the following conditions are equivalent:*

(a) *$(C', h') \simeq \mathscr{S}(\widehat{F})(X, h)$ for some symmetric space $(X, h) \in \mathscr{S}(\mathscr{C}, D, \delta)$;*
(b) *there exists an isometry $\theta' \colon \big(\iota'(C'), \alpha'_{C'}\, \iota'(h')\big) \to (C', h')$ in $\mathscr{S}(\mathscr{C}', D', \delta')$ such that $\theta'\, \iota'(\theta') = \lambda'_{C'}$.*

Proof. (a) $\Rightarrow$ (b) For every symmetric space $(X, h) \in \mathscr{S}(\mathscr{C}, D, \delta)$, the map $\theta' := \left(\begin{smallmatrix} 0 & \lambda_X \\ -\mathrm{Id}_{\iota(X)} & 0 \end{smallmatrix}\right)$ defines an isometry $\big(\iota' F(X), \alpha'_{F(X)}\, \iota'(\widehat{\phi}_X\, F(h))\big) \to \big(F(X), \widehat{\phi}_X\, F(h)\big)$ such that $\theta'\, \iota'(\theta') = \lambda'_{F(X)}$.

(b) $\Rightarrow$ (a) Let $C' = (C, f)$. For θ' in (b), the pair $C'' := (C', \theta')$ is an object in $\mathscr{C}''$. The functor $\Theta\colon \mathscr{C}'' \to \mathscr{C}$ from §1.2 yields an object $\Theta(C'') \in \mathscr{C}$, which we denote simply by X, and morphisms $X \xrightarrow{t} C \xrightarrow{s} X$ in $\mathscr{C}$ such that

$$s\,t = \mathrm{Id}_X \qquad \text{and} \qquad t\,s = \tfrac{1}{2}(\mathrm{Id}_C - G(\theta')\,f^{-1}).$$

Note that θ' is an isomorphism in $\mathscr{C}'$, hence $f\,\iota G(\theta') = -G(\theta')\,\iota(f)$, and therefore

$$f^{-1}\,G(\theta') = G(\theta')\,f^{-1} \qquad \text{and} \qquad f\,\iota(t\,s)\,f^{-1} = \tfrac{1}{2}(\mathrm{Id}_C + G(\theta')\,f^{-1}) = \mathrm{Id}_C - t\,s. \tag{3.3}$$

Define $h\colon X \to D(X)$ by $h = D(t)\,G(h')\,t$. Since $h' = D'(h')\,\delta'_{C'}$ and $G(\delta'_{C'}) = \delta_C$, it follows that $h = D(h)\,\delta_X$, but we need to show that h is an isomorphism to be able to consider $(X, h) \in \mathscr{S}(\mathscr{C}, D, \delta)$.

Recall from (1.8) the natural isomorphism $\eta\colon \mathrm{Id}_{\mathscr{C}''} \overset{\sim}{\Rightarrow} \Psi\Theta$, which yields an isomorphism $\eta_{C''} = \binom{s}{\iota(s) f^{-1}}\colon C'' \to \Psi(X)$, hence also an isomorphism $G'(\eta_{C''})\colon C' \to G'\Psi(X) = F(X)$. We claim that

$$h' = D'G'(\eta_{C''})\,\widehat{\phi}_X\,F(h)\,G'(\eta_{C''}). \tag{3.4}$$

This equation shows at once that h is an isomorphism and that $G'(\eta_{C''})$ is an isometry $(C', h') \to \mathscr{S}(\widehat{F})(X, h)$, completing the proof of the lemma.

To prove (3.4), observe that the isometry θ' induces an isometry $G(\theta')\colon (\iota(C), -\alpha_C\,\iota G(h')) \to (C, G(h'))$ in $\mathscr{S}(\mathscr{C}, D, \delta)$, and that the condition that h' is an isometry in $\mathscr{S}(\mathscr{C}', D', \delta')$ implies that f^{-1} is an isometry $(C, G(h')) \to (\iota(C), -\alpha_C\,\iota G(h'))$ in $\mathscr{S}(\mathscr{C}, D, \delta)$. Therefore, $G(\theta')\,f^{-1}$ is an isometry $(C, G(h')) \to (C, G(h'))$, and it follows that

$$D(t\,s)\,G(h')\,t\,s = G(h')\,t\,s. \tag{3.5}$$

From (3.3) and (3.5), we obtain

$$\iota(D(t\,s)\,G(h')\,t\,s)\,f^{-1} = \iota G(h')\,f^{-1}\,(\mathrm{Id}_C - t\,s).$$

Therefore, since $f^{-1}\colon (C, G(h')) \to (\iota(C), -\alpha_C\,\iota G(h'))$ is an isometry,

$$D(f^{-1})\,\alpha_C\,\iota(D(t\,s)\,G(h')\,t\,s)\,f^{-1} = -G(h')\,(\mathrm{Id}_C - t\,s). \tag{3.6}$$

Now, by definition of $\eta_{C''}$ we have

$$D'G'(\eta_{C''})\,\widehat{\phi}_X\,F(h)\,G'(\eta_{C''}) = D(t\,s)\,G(h')\,t\,s - D(f^{-1})\,\alpha_C\,\iota(D(t\,s)\,G(h')\,t\,s)\,f^{-1}.$$

Therefore, (3.4) follows from (3.5) and (3.6). $\square$

Proof of Proposition 3.2. Let $(C', h') \in \mathscr{S}(\mathscr{C}', D', \delta')$ be such that $(C', h') = \mathscr{S}(\widehat{F})(X, h)$ for some $(X, h) \in \mathscr{S}(\mathscr{C}, D, \delta)$. Lemma 3.3 yields an isometry $\theta'\colon \big(\iota'(C'), \alpha'_{C'}\,\iota'(h')\big) \to (C', h')$ such that $\theta'\,\iota'(\theta') = \lambda'_{C'}$, hence an object $C'' = (C', \theta') \in \mathscr{C}''$. In $\mathscr{C}''$, the morphism

$$\begin{pmatrix} \mathrm{Id}_{C'} & \theta' \\ \tfrac{1}{2}h' & -\tfrac{1}{2}h'\,\theta' \end{pmatrix} \colon F'(C') \to C'' \oplus D''(C'')$$

is an isometry $\mathscr{S}(\widehat{F}')(C', h') \to H(C'')$. $\square$

3.3. **Example: Hermitian spaces.** For the typical dualities on categories of projective modules discussed in §2.3, the symmetric spaces are (slightly generalised) hermitian spaces. Let (σ,ε) be an antistructure on a k-algebra A and let (D,δ) be the derived duality on $\mathscr{P}_A$. To every $(P,h)\in\mathcal{S}(\mathscr{P}_A,D,\delta)$ we attach a map $P\times P\to A$, for which we use the same notation h, defined by

$$h(x,y)=\langle h(x),y\rangle \quad \text{for } x,\,y\in P.$$

It readily follows from the definitions that h is a nonsingular sesquilinear map for the antiautomorphism σ on A, and the property $h=D(h)\,\delta_P$ means that

$$h(y,x)=\sigma\big(h(x,y)\big)\,\varepsilon \qquad \text{for } x,\,y\in P.$$

In C.T.C. Wall's terminology [Wal70] the map $h\colon P\times P\to A$ is a nonsingular (σ,ε)-*reflexive form*. When ε lies in the center of A, then σ is an involution and $\mathcal{S}(\mathscr{P}_A,D,\delta)$ is the category of ε-hermitian A-modules as defined in [Knu91, p. 12] (and denoted there by $\mathfrak{H}^\varepsilon(A)$). Abusing terminology, we also call (σ,ε)-*hermitian modules* the symmetric spaces in $\mathcal{S}(\mathscr{P}_A,D,\delta)$ when (D,δ) is derived from the antistructure (σ,ε), and call (σ,ε)-*hermitian forms* C.T.C. Wall's (σ,ε)-reflexive forms. We write

$$\mathcal{S}(\mathscr{P}_A,D,\delta)=\mathscr{H}^\varepsilon(A,\sigma) \qquad\text{and}\qquad \mathcal{S}(\mathscr{P}_A,D,-\delta)=\mathscr{H}^{-\varepsilon}(A,\sigma).$$

Similarly, given a commutation α between (ι,λ) and (D,δ) as in §2.3, we attach to every symmetric space $(P,h)\in\mathcal{S}(\mathscr{P}_A,\iota D,\delta_\alpha)$ a map

$$h\colon P\times P\to A \quad\text{defined by}\quad h(x,y)=\{h(x),y\}\quad \text{for } x,\,y\in P.$$

This map is a nonsingular $(\sigma\iota,\varepsilon_\alpha)$-hermitian form, hence

$$\mathcal{S}(\mathscr{P}_A,\iota D,\delta_\alpha)=\mathscr{H}^{\varepsilon_\alpha}(A,\sigma\iota) \qquad\text{and}\qquad \mathcal{S}(\mathscr{P}_A,\iota D,-\delta_\alpha)=\mathscr{H}^{-\varepsilon_\alpha}(A,\sigma\iota).$$

Likewise, for the dualities $(D',\pm\delta')$ and $(\iota'D',\pm\delta'_{\alpha'})$ derived from $(\sigma',\pm\varepsilon')$ and $(\sigma'\iota',\pm\varepsilon_\alpha)$,

$$\mathcal{S}(\mathscr{P}_{A'},D',\pm\delta')=\mathscr{H}^{\pm\varepsilon}(A',\sigma') \qquad\text{and}\qquad \mathcal{S}(\mathscr{P}_{A'},\iota'D',\pm\delta'_{\alpha'})=\mathscr{H}^{\pm\varepsilon_\alpha}(A',\sigma'\iota').$$

The functors

$$\mathcal{S}(\widehat{F})\colon \mathscr{H}^{\pm\varepsilon}(A,\sigma)\to\mathscr{H}^{\pm\varepsilon}(A',\sigma') \quad\text{and}\quad \mathcal{S}(\check{F})\colon \mathscr{H}^{\pm\varepsilon_\alpha}(A,\sigma\iota)\to\mathscr{H}^{\pm\varepsilon_\alpha}(A',\sigma'\iota')$$

are given by scalar extensions, mapping every $(\sigma,\pm\varepsilon)$- or $(\sigma\iota,\pm\varepsilon_\alpha)$-hermitian space (P,h) to $(P\otimes_A A',h')$ defined by

$$\begin{aligned} h'(x_1\otimes 1+x_2\otimes j,\, y_1\otimes 1+y_2\otimes j) &= h(x_1,y_1)+\sigma'(j)h(x_2,y_2)j+h(x_1,y_2)j+\sigma'(j)h(x_2,y_1)\\ &= h(x_1,y_1)-u\,\iota^{-1}(h(x_2,y_2))+\big(h(x_1,y_2)-u\,\iota^{-1}(h(x_2,y_1))z^{-1}\big)j \end{aligned}$$

for $x_1,\,x_2,\,y_1,\,y_2\in P$.

On the other hand, the functors

$$\mathcal{S}(\widehat{G})\colon \mathscr{H}^{\pm\varepsilon}(A',\sigma')\to\mathscr{H}^{\mp\varepsilon_\alpha}(A,\sigma\iota) \quad\text{and}\quad \mathcal{S}(\check{G})\colon \mathscr{H}^{\pm\varepsilon_\alpha}(A',\sigma'\iota')\to\mathscr{H}^{\pm\varepsilon}(A,\sigma)$$

are given by transfer maps. To make this clear, consider the map

$$\mathsf{tr}\colon A'\to A \quad\text{defined by}\quad \mathsf{tr}(a_1+a_2j)=a_1 \quad\text{for } a_1,\,a_2\in A.$$

It satisfies $\mathsf{tr}(ja)=\iota\big(\mathsf{tr}(aj)\big)$, $\mathsf{tr}\big(\iota'(a)\big)=\iota\big(\mathsf{tr}(a)\big)$ and $\mathsf{tr}\big(\sigma'(a)\big)=\sigma\big(\mathsf{tr}(a)\big)$ for all $a\in A'$. Moreover, for $P'\in\mathscr{P}_{A'}$, the identifications $GD'(P')=DG'(P')$ and $D'(P')=\operatorname{Hom}_{A'}(P',A')$ from (2.10) show that

$$\langle\pi,x\rangle=\mathsf{tr}(\langle\pi,x\rangle') \qquad \text{for } \pi\in D'(P') \text{ and } x\in P'.$$

Let $(P',h') \in \mathcal{S}(\mathscr{P}_{A'}, D', \pm\delta')$ and $h = \widehat{\gamma}_{P'}\, G(h')$, so $(G(P'),h)$ is the image of (P',h') under $\mathcal{S}(\widehat{G})$. Let also sesquilinear maps h' and h be defined by

$$h'(x,y) = \langle h'(x), y\rangle' \quad \text{and} \quad h(x,y) = \{h(x), y\} \quad \text{for } x,\, y \in P'.$$

The definition of $\alpha_{G(P')}$ in (2.7) yields

$$\langle \alpha_{G(P')}(h(x)), {}^{\iota}y\rangle = u\, \iota^{-1}\big(h(x,y)\big) \qquad \text{for } x,\, y \in P'.$$

On the other hand, from the definition of $\widehat{\gamma}_{P'}$ it follows that

$$\langle \alpha_{G(P')}(h(x)), {}^{\iota}y\rangle = \langle h'(x), yj\rangle = \mathsf{tr}(\langle h'(x), yj\rangle').$$

Therefore,

$$h(x,y) = \iota(u)^{-1}\iota\big(\mathsf{tr}(h'(x,y)j)\big) = \iota(u)^{-1}\mathsf{tr}(j\, h'(x,y)) \quad \text{for } x,\, y \in P'.$$

Similarly, for $(P',h') \in \mathcal{S}(\mathscr{P}_{A'}, \iota' D', \pm\delta'_{\alpha'})$ and $(G(P'),h)$ with $h = \check{\gamma}_{P'}\, G(h')$ the image of (P',h') under $\mathcal{S}(\check{G})$, the associated $(\sigma,\pm\varepsilon)$-hermitian and $(\sigma'\iota', \pm\varepsilon_\alpha)$-hermitian forms h and h' are related by

$$h(x,y) = u\, \mathsf{tr}(j^{-1}h'(x,y)) \qquad \text{for } x,\, y \in P'.$$

The functors $\mathcal{S}(\widehat{G})$ and $\mathcal{S}(\check{G})$ are thus given by the transfers $\widehat{\mathsf{tr}}$ and $\check{\mathsf{tr}}$ defined by

$$\widehat{\mathsf{tr}}(h')(x,y) = \iota(u)^{-1}\mathsf{tr}(j\, h'(x,y)) \qquad \text{for } (P',h') \in \mathscr{H}^{\pm\varepsilon}(A',\sigma') \text{ and } x,\, y \in P'$$

and

$$\check{\mathsf{tr}}(h')(x,y) = u\, \mathsf{tr}(j^{-1}h'(x,y)) \qquad \text{for } (P',h') \in \mathscr{H}^{\pm\varepsilon_\alpha}(A',\sigma'\iota') \text{ and } x,\, y \in P'.$$

Writing simply ext for the scalar extension functors $\mathcal{S}(\widehat{F})$ and $\mathcal{S}(\check{F})$, the octagon (3.2) takes the form

$$\begin{tikzcd}
\mathscr{H}^{\varepsilon}(A,\sigma) \arrow[r, "\mathsf{ext}"] & \mathscr{H}^{\varepsilon}(A',\sigma') \arrow[r, "\widehat{\mathsf{tr}}"] & \mathscr{H}^{-\varepsilon_\alpha}(A,\sigma\iota) \arrow[d, "\mathsf{ext}"] \\
\mathscr{H}^{\varepsilon_\alpha}(A',\sigma'\iota') \arrow[u, "\check{\mathsf{tr}}"] & & \mathscr{H}^{-\varepsilon_\alpha}(A',\sigma'\iota') \arrow[d, "\check{\mathsf{tr}}"] \\
\mathscr{H}^{\varepsilon_\alpha}(A,\sigma\iota) \arrow[u, "\mathsf{ext}"] & \mathscr{H}^{-\varepsilon}(A',\sigma') \arrow[l, "\widehat{\mathsf{tr}}"'] & \mathscr{H}^{-\varepsilon}(A,\sigma) \arrow[l, "\mathsf{ext}"']
\end{tikzcd} \tag{3.7}$$

3.4. **Example: The Grenier-Boley–Mahmoudi–First octagon.** We spell out in this subsection the relation between (3.7) and the octagon from [GBM05] and [Fir23]. Let A be a k-algebra with a k-linear involution σ and let $\varepsilon \in A^\times$ be a central unit such that $\sigma(\varepsilon) = \varepsilon^{-1}$. For every unit $\xi \in A^\times$ we write $\mathrm{Int}(\xi)$ for the inner automorphism mapping $a \in A$ to $\xi a \xi^{-1}$. Assume A contains units ζ, η such that

$$\zeta^2 \in k, \qquad \sigma(\zeta) = -\zeta, \qquad \sigma(\eta) = -\eta \quad \text{and} \quad \zeta\eta = -\eta\zeta.$$

Let $B = C_A(\zeta)$ be the centraliser of ζ and let

$$\lambda_B = \eta^2 \in B, \qquad \iota_B = \mathrm{Int}(\eta)|_B, \qquad \tau_1 = \sigma|_B \quad \text{and} \quad \tau_2 = \tau_1\, \iota_B.$$

The pair (ι_B, λ_B) is a structure on B in the sense of §1.3, and the pair (τ_1, ε) is an antistructure on B. The unit $\lambda_B \in B^\times$ satisfies the conditions (i)—(iii) on u in §2.3, hence it defines as in (2.7) a commutation α_B between the conjugation derived from (ι_B, λ_B) and the duality on $\mathscr{P}_B$ derived from (τ_1, ε). The decomposition

$$a = \tfrac{1}{2}(a + \zeta a \zeta^{-1}) + \tfrac{1}{2}(a - \zeta a \zeta^{-1}) \qquad \text{for } a \in A$$

yields $A = B \oplus B\eta$, hence A is the (ι_B, λ_B)-twisted quadratic extension B' of B and σ is the extension τ_1' of τ_1 to $B' = A$. Moreover, $\mathrm{Int}(\eta\zeta)$ is the extension ι_B' of ι_B to A, and $\varepsilon_{\alpha_B} = \varepsilon$. With these data, the octagon (3.7) becomes

$$\begin{tikzcd}
\mathscr{H}^{\varepsilon}(B,\tau_1) \arrow[r, "\mathsf{ext}"] & \mathscr{H}^{\varepsilon}(A,\sigma) \arrow[r, "\hat{\mathsf{tr}}"] & \mathscr{H}^{-\varepsilon}(B,\tau_2) \arrow[d, "\mathsf{ext}"] \\
\mathscr{H}^{\varepsilon}(A,\sigma\ \mathrm{Int}(\eta\zeta)) \arrow[u, "\check{\mathsf{tr}}"] & & \mathscr{H}^{-\varepsilon}(A,\sigma\ \mathrm{Int}(\eta\zeta)) \arrow[d, "\check{\mathsf{tr}}"] \\
\mathscr{H}^{\varepsilon}(B,\tau_2) \arrow[u, "\mathsf{ext}"] & \mathscr{H}^{-\varepsilon}(A,\sigma) \arrow[l, "\hat{\mathsf{tr}}"'] & \mathscr{H}^{-\varepsilon}(B,\tau_1) \arrow[l, "\mathsf{ext}"']
\end{tikzcd} \tag{3.8}$$

This octagon is only superficially different from the one in [Fir23, (3.1)], which has $\mathscr{H}^{\mp\varepsilon}(B,\tau_1)$ for $\mathscr{H}^{\pm\varepsilon}(B,\tau_1)$ and $\mathscr{H}^{\pm\varepsilon}(A,\sigma)$ for $\mathscr{H}^{\mp\varepsilon}(A,\sigma\ \mathrm{Int}(\eta\zeta))$. The differences are resolved by scaling: if $(Q,g) \in \mathscr{H}^{\pm\varepsilon}(B,\tau_1)$, then $(Q,\zeta g) \in \mathscr{H}^{\mp\varepsilon}(B,\tau_1)$, and if $(P,h) \in \mathscr{H}^{\mp\varepsilon}(A,\sigma\ \mathrm{Int}(\eta\zeta))$, then $(P,\zeta\eta h) \in \mathscr{H}^{\pm\varepsilon}(A,\sigma)$.

Note that, by the very nature of its periodic construction, the same octagon can still be obtained with a shift of one edge by starting with the algebra A. To see this explicitly, let $\iota_A = \mathrm{Id}_A$ and $\lambda_A = \zeta^2 \in k^\times$, and set $K = k \oplus kj$ with $j^2 = \lambda_A$, a quadratic étale k-algebra with nontrivial automorphism $\mathsf{can}_{K/k}$. The (ι_A,λ_A)-twisted quadratic extension of A is $A' = A \otimes_k K$, with the automorphism $\iota_A' = \mathrm{Id}_A \otimes \mathsf{can}_{K/k}$. The scalar λ_A satisfies the conditions (i)—(iii) on u in §2.3, hence it yields a commutation α_A between the conjugation on $\mathscr{P}_A$ derived from (ι_A,λ_A) and the duality derived from (σ,ε). With this choice, $\varepsilon_{\alpha_A} = \varepsilon$ and the involutions σ' and $\sigma'\iota_A'$ on A' are $\sigma \otimes \mathsf{can}_{K/k}$ and $\sigma \otimes \mathrm{Id}_K$ respectively. In order to relate the resulting octagon (3.7) with (3.8) or [Fir23, (3.1)], observe that $e = \frac{1}{2}(1 + \zeta \otimes j^{-1}) \in A'$ is an idempotent centralised by B. Mapping $b \in B$ to $e(b \otimes 1)e$ identifies B with $eA'e \simeq \mathrm{End}_{A'}(eA')$. Following [Knu91, I(9.1)], Morita theory yields an equivalence of categories $\mathscr{P}_B \xrightarrow{\sim} \mathscr{P}_{A'}$ mapping $P \in \mathscr{P}_B$ to $P \otimes_B eA'$. Moreover, equivalences of categories

$$\mathsf{M}_1 \colon \mathscr{H}^{\pm\varepsilon}(B,\tau_1) \xrightarrow{\sim} \mathscr{H}^{\pm\varepsilon}(A',\sigma') \qquad \text{and} \qquad \mathsf{M}_2 \colon \mathscr{H}^{\pm\varepsilon}(B,\tau_2) \xrightarrow{\sim} \mathscr{H}^{\mp\varepsilon}(A',\sigma'\iota')$$

can be obtained by hermitian form composition with the maps h_1, $h_2 \colon eA' \times eA' \to A'$ defined by

$$h_1(x,y) = \sigma'(x)y \qquad \text{and} \qquad h_2(x,y) = \sigma'\iota'(x)\eta y \qquad \text{for } x,\, y \in eA'.$$

The map h_1 is a nonsingular hermitian form over (A',σ') and h_2 is a nonsingular skew-hermitian form over $(A',\sigma'\iota_A')$.

The following diagram illustrates the relation between First's octagon in [Fir23, (3.1)] (consisting of the outer maps) and the octagon of (3.7) (of inner maps) with the data above. All the triangles commute up to similitude.

$$
\begin{tikzcd}
 & & \mathscr{H}^{\varepsilon}(B,\tau_1) \arrow[d, "\mathsf{M}_1"] \arrow[dr, "\rho_1^{(\varepsilon)}"] & & \\
 & \mathscr{H}^{\varepsilon}(A,\sigma) \arrow[ur, "\pi_1^{(\varepsilon)}"] \arrow[r, "\mathsf{ext}"] & \mathscr{H}^{\varepsilon}(A',\sigma') \arrow[r, "\hat{\mathsf{tr}}"] & \mathscr{H}^{-\varepsilon}(A,\sigma) \arrow[d, "\mathsf{ext}"] \arrow[dr, "\pi_2^{(-\varepsilon)}"] & \\
\mathscr{H}^{-\varepsilon}(B,\tau_2) \arrow[ur, "\rho_2^{(-\varepsilon)}"] \arrow[r, "\mathsf{M}_2"] & \mathscr{H}^{\varepsilon}(A',\sigma'\iota') \arrow[u, "\check{\mathsf{tr}}"] & & \mathscr{H}^{-\varepsilon}(A',\sigma'\iota') \arrow[d, "\check{\mathsf{tr}}"] & \mathscr{H}^{\varepsilon}(B,\tau_2) \arrow[l, "\mathsf{M}_2"'] \arrow[dl, "\rho_2^{(\varepsilon)}"] \\
 & \mathscr{H}^{\varepsilon}(A,\sigma) \arrow[u, "\mathsf{ext}"] \arrow[ul, "\pi_2^{(\varepsilon)}"] & \mathscr{H}^{-\varepsilon}(A',\sigma') \arrow[l, "\hat{\mathsf{tr}}"'] & \mathscr{H}^{-\varepsilon}(A,\sigma) \arrow[l, "\mathsf{ext}"'] \arrow[dl, "\pi_1^{(-\varepsilon)}"] & \\
 & & \mathscr{H}^{-\varepsilon}(B,\tau_1) \arrow[u, "\mathsf{M}_1"] \arrow[ul, "\rho_1^{(-\varepsilon)}"] & &
\end{tikzcd}
$$

4. Witt groups

Throughout this section, $\mathscr{C}$ is, as in the preceding sections, a pseudo-abelian k-linear category where k is a commutative ring in which 2 is invertible. We assume moreover that $\mathscr{C}$ is essentially small, which means that the isomorphism classes of objects in $\mathscr{C}$ form a set. If $\mathscr{C}$ carries a conjugation (ι,λ) or a duality (D,δ), it follows that the categories $\mathscr{C}'$ and $\mathcal{S}(\mathscr{C},D,\delta)$ also are essentially small. If $(\mathscr{C},D,\delta)$ is an *exact* category with duality in the sense of [Bal05, §1.2.2], then *metabolic* symmetric spaces are defined, and a classical construction due to Knebusch [Bal05, Def. 27] yields the *Witt group* $W(\mathscr{C},D,\delta)$: this group is the quotient of the monoid of isometry classes of symmetric spaces in $\mathcal{S}(\mathscr{C},D,\delta)$ by the submonoid of isometry classes of metabolic symmetric spaces. Thus, the definition of $W(\mathscr{C},D,\delta)$ depends on additional structure on the category with duality $(\mathscr{C},D,\delta)$.

If $(\mathscr{C}_1,D_1,\delta_1)$ and $(\mathscr{C}_2,D_2,\delta_2)$ are exact categories with duality and $\widetilde{R}=(R,\theta)\colon(\mathscr{C}_1,D_1,\delta_1)\to(\mathscr{C}_2,D_2,\delta_2)$ is a morphism of categories with duality with R an exact functor, then it follows from the definitions that the functor $\mathcal{S}(\widetilde{R})\colon\mathcal{S}(\mathscr{C}_1,D_1,\delta_1)\to\mathcal{S}(\mathscr{C}_2,D_2,\delta_2)$ induces a group homomorphism $W(\widetilde{R})\colon W(\mathscr{C}_1,D_1,\delta_1)\to W(\mathscr{C}_2,D_2,\delta_2)$; and if two morphisms $\widetilde{R},\widetilde{S}\colon(\mathscr{C}_1,D_1,\delta_1)\to(\mathscr{C}_2,D_2,\delta_2)$ preserving the exact structures are related by a similitude $\rho\colon\widetilde{R}\overset{\sim}{\Rightarrow}\widetilde{S}$ with multiplier $\mu\in k^{\times}$, then the similitude $\mathcal{S}(\rho)\colon\mathcal{S}(\widetilde{R})\overset{\sim}{\Rightarrow}\mathcal{S}(\widetilde{S})$ defines an isomorphism $\langle\mu\rangle$ mapping the Witt class of $(C,h)\in\mathcal{S}(C_2,D_2,\delta_2)$ to the Witt class of $(C,\mu h)$, which fits in the following commutative diagram:

$$
\begin{tikzcd}
W(\mathscr{C}_1,D_1,\delta_1) \arrow[r, "W(\widetilde{R})"] \arrow[dr, "W(\widetilde{S})"'] & W(\mathscr{C}_2,D_2,\delta_2) \arrow[d, "\langle\mu\rangle"] \\
 & W(\mathscr{C}_2,D_2,\delta_2)
\end{tikzcd}
$$

In this section, we consider two cases where Knebusch's construction turns the octagon (3.2) of hermitian categories into an 8-periodic chain complex of Witt groups: the split exact structure and the usual exact structure in abelian categories. In each case, the crucial property that determines the exactness of the resulting octagon of Witt groups is the following:

Property (P). Let $(C,h) \in \mathcal{S}(\mathscr{C},D,\delta)$ be a symmetric space. If $\mathcal{S}(\widehat{F})(C,h)$ is metabolic, then there exists an isometry $\theta\colon \big(\iota(C),\alpha_C\,\iota(h)\big) \to (C,h)$ such that $\theta\,\iota(\theta) = \lambda_C$.

When this property holds for representatives of each Witt class in the Witt groups involved in the octagon, exactness follows from Lemma 3.3.

Remark 4.1. Note that $\mathcal{S}(\widehat{F})(C,h) = (F(C),\widehat{\phi}_C\,F(h))$ with $(GF(C),G(\widehat{\phi}_C\,F(h))) = (C,h) \perp (\iota(C),-\alpha_C\,\iota(h))$. When Witt cancellation holds in $\mathcal{S}(\mathscr{C},D,\delta)$, there exists an isometry from $(\iota(C),\alpha_C\iota(h))$ to (C,h) if and only if $(C,h) \perp (\iota(C),-\alpha_C\,\iota(h)) \simeq (C,h) \perp (C,-h)$, i.e., $G\mathcal{S}(\widehat{F})(C,h) \simeq H(C)$.

4.1. **The split exact structure.** In this case, the exact sequences are the split sequences, hence every additive functor preserves the exact structure. Since 2 is invertible in k, the metabolic spaces are the hyperbolic spaces, see [Bal05, Example 21]. Therefore, the Witt group $W(\mathscr{C},D,\delta)$ is the usual Witt group as defined in [Sch85, Ch. 7, §2]. When $\mathscr{C}$ carries a conjugation (ι,λ) and a commuting duality (D,δ), the functors in the octagon (3.2) induce Witt group homomorphisms since they carry hyperbolic spaces to hyperbolic spaces, and Theorem 3.1 shows that they form an 8-periodic chain complex of Witt groups:

$$\begin{tikzcd}
W(\mathscr{C},D,\delta) \arrow[r, "W(\widehat{F})"] & W(\mathscr{C}',D',\delta') \arrow[r, "W(\widehat{G})"] & W(\mathscr{C},\iota D,-\delta_\alpha) \arrow[d, "W(\check{F})"] \\
W(\mathscr{C}',\iota' D',\delta'_{\alpha'}) \arrow[u, "W(\check{G})"] & & W(\mathscr{C}',\iota' D',-\delta'_{\alpha'}) \arrow[d, "W(\check{G})"] \\
W(\mathscr{C},\iota D,\delta_\alpha) \arrow[u, "W(\check{F})"] & W(\mathscr{C}',D',-\delta') \arrow[l, "W(\widehat{G})"'] & W(\mathscr{C},D,-\delta) \arrow[l, "W(\widehat{F})"']
\end{tikzcd} \tag{4.1}$$

In this subsection, we provide conditions under which this chain complex is an exact sequence.

For every symmetric space $(C,h) \in \mathcal{S}(\mathscr{C},D,\delta)$ the endomorphism k-algebra $\mathrm{End}_{\mathscr{C}}(C)$ carries an involution σ_h (i.e., an anti-automorphism of period 2) defined by

$$\sigma_h(\varphi) = h^{-1}\,D(\varphi)\,h \qquad \text{for } \varphi \in \mathrm{End}_{\mathscr{C}}(C).$$

The symmetric space (C,h) is said to be *split anisotropic* if there is no nonzero direct summand $t\colon L \to C$ such that $D(t)\,h\,t = 0$ or, equivalently since $\mathscr{C}$ is pseudo-abelian, if there is no nonzero idempotent $e \in \mathrm{End}_{\mathscr{C}}(C)$ such that $\sigma_h(e)\,e = 0$. Recall that $\mathrm{End}_{\mathscr{C}}(C)$ is said to be *semilocal* if factoring out its Jacobson radical R yields a k-algebra $\mathrm{End}_{\mathscr{C}}(C)/R$ that is right artinian; it is *rad-adically complete* if it is complete for the R-adic topology (in which $(R^n)_{n\in\mathbb{N}}$ is a basis of neighborhoods of 0), see [Knu91, II(4.5)] or [Lam91, §21]. For example, right artinian algebras are semilocal and rad-adically complete since their Jacobson radical is nilpotent, see [Lam91, (4.12)].

The next lemma demonstrates how these properties are used in the proof of Property (P).

Lemma 4.2. *Let $(C,h) \in \mathcal{S}(\mathscr{C},D,\delta)$ be a split anisotropic symmetric space. If $\mathrm{End}_{\mathscr{C}}(C)$ is semilocal and rad-adically complete, then every $\varphi \in \mathrm{End}_{\mathscr{C}}(C)$ such that $\sigma_h(\varphi) = \mathrm{Id}_C - \varphi$ is invertible.*

Proof. Let R be the Jacobson radical of $\mathrm{End}_{\mathscr{C}}(C)$ and $\overline{\mathrm{End}_{\mathscr{C}}(C)} = \mathrm{End}_{\mathscr{C}}(C)/R$. We write $\overline{\varphi}$ for the image of $\varphi \in \mathrm{End}_{\mathscr{C}}(C)$ in $\overline{\mathrm{End}_{\mathscr{C}}(C)}$. Properties of the Jacobson radical imply that φ is invertible if $\overline{\varphi}$ is invertible (see [Knu91, II(4.2.2)] or [Lam91, (4.8)]), and that idempotents in $\overline{\mathrm{End}_{\mathscr{C}}(C)}$ lift to idempotents in $\mathrm{End}_{\mathscr{C}}(C)$ when $\mathrm{End}_{\mathscr{C}}(C)$ is rad-adically complete (see [Knu91, II(4.5.4)] or [Lam91, (21.31)]). Since σ_h is an anti-automorphism, it preserves R and induces an involution $\overline{\sigma_h}$ on $\overline{\mathrm{End}_{\mathscr{C}}(C)}$.

As a first step in the proof, we show that if $\varepsilon \in \overline{\mathrm{End}_{\mathscr{C}}(C)}$ is an idempotent such that $\overline{\sigma_h}(\varepsilon)\varepsilon = 0$, then $\varepsilon = 0$. This follows from the hypothesis that (C, h) is split anisotropic by the reduction theory in [Knu91, II.4] or [Sch85, Ch. 7, §4]; the following is an easy direct proof.

Let $e_0 \in \mathrm{End}_{\mathscr{C}}(C)$ be an idempotent such that $\overline{e_0} = \varepsilon$. We construct a sequence of idempotents $(e_n)_{n\in\mathbb{N}}$ in $\mathrm{End}_{\mathscr{C}}(C)$ such that $e_{n+1} \equiv e_n \bmod R^{n+1}$ and $\sigma_h(e_n)\, e_n \in R^{n+1}$ for all $n \in \mathbb{N}$: if e_n is given, then $\mathrm{Id}_C + \frac{1}{2}\sigma_h(e_n)\, e_n$ is invertible since $\overline{\mathrm{Id}_C + \frac{1}{2}\sigma_h(e_n)e_n} = \overline{\mathrm{Id}_C}$. We may therefore set

$$e_{n+1} = (\mathrm{Id}_C + \tfrac{1}{2}\sigma_h(e_n)\, e_n)^{-1}\, e_n\, (\mathrm{Id}_C + \tfrac{1}{2}\sigma_h(e_n)\, e_n).$$

This is clearly an idempotent, and $e_{n+1} \equiv e_n \bmod R^{n+1}$ since $\frac{1}{2}\sigma_h(e_n)\, e_n \in R^{n+1}$. This last property also implies

$$\begin{aligned} e_{n+1} &\equiv (\mathrm{Id}_C - \tfrac{1}{2}\sigma_h(e_n)\, e_n)\, e_n\, (\mathrm{Id}_C + \tfrac{1}{2}\sigma_h(e_n)\, e_n) \\ &\equiv e_n - \tfrac{1}{2}\sigma_h(e_n)\, e_n + \tfrac{1}{2}e_n\, \sigma_h(e_n)\, e_n \bmod R^{n+2}, \end{aligned}$$

whence $\sigma_h(e_{n+1})\, e_{n+1} \in R^{n+2}$. Since $\mathrm{End}_{\mathscr{C}}(C)$ is assumed to be R-adically complete, there exists $e_\infty \in \mathrm{End}_{\mathscr{C}}(C)$ such that $e_\infty \equiv e_n \bmod R^{n+1}$ for all $n \in \mathbb{N}$. This endomorphism is idempotent and $\sigma_h(e_\infty)\, e_\infty = 0$. But (C, h) is assumed to be split anisotropic, hence $e_\infty = 0$ and therefore $\varepsilon = 0$.

Now, suppose $\varphi \in \mathrm{End}_{\mathscr{C}}(C)$ is such that $\sigma_h(\varphi) = \mathrm{Id}_C - \varphi$, hence $\overline{\sigma_h}(\overline{\varphi}) = \overline{\mathrm{Id}_C} - \overline{\varphi}$. The Wedderburn–Artin structure theorem (see [Lam91, (3.5)]) shows that $\overline{\mathrm{End}_{\mathscr{C}}(C)}$ is isomorphic to a direct product of matrix algebras over division algebras. Therefore, $\overline{\varphi}$ is invertible unless there exists a nonzero idempotent $\varepsilon \in \overline{\mathrm{End}_{\mathscr{C}}(C)}$ such that $\overline{\varphi}\varepsilon = 0$. But if $\overline{\varphi}\,\varepsilon = 0$, then

$$\overline{\sigma_h}(\varepsilon)\varepsilon = \overline{\sigma_h}(\varepsilon)\,(\overline{\mathrm{Id}_C} - \overline{\varphi})\,\varepsilon = \overline{\sigma_h}(\overline{\varphi}\,\varepsilon)\varepsilon = 0.$$

We just saw that this implies $\varepsilon = 0$, hence $\overline{\varphi}$ is invertible and φ is invertible. □

Proposition 4.3. *Property (P) holds for $(C, h) \in \mathscr{S}(\mathscr{C}, D, \delta)$ if (C, h) is split anisotropic and $\mathrm{End}_{\mathscr{C}}(C)$ is semilocal and rad-adically complete.*

Proof. Let $\mathscr{S}(\widehat{F})(C, h) = (F(C), h')$ where $h' = \widehat{\phi}_C\, F(h) = \left(\begin{smallmatrix} h & 0 \\ 0 & -\alpha_C\,\iota(h)\end{smallmatrix}\right)$. Assume $(F(C), h') \simeq H(L')$ for some $L' \in \mathscr{C}'$, hence $F(C)$ may be identified with $L' \oplus D'(L')$. Projection on L' yields an idempotent $e \in \mathrm{End}_{\mathscr{C}'}(F(C))$ such that $\mathrm{Id}_{F(C)} - e$ is the projection on $D'(L')$. The involution adjoint to the hyperbolic form on $L' \oplus D'(L')$ interchanges the projections on L' and on $D'(L')$, hence $\sigma_{h'}(e) = \mathrm{Id}_{F(C)} - e$. Viewing $\mathrm{End}_{\mathscr{C}'}(F(C)) \subset \mathrm{End}_{\mathscr{C}}(C \oplus \iota(C))$, we may write $e = \left(\begin{smallmatrix} \varphi_0 & \varphi_1 \\ \varphi_2 & \varphi_3\end{smallmatrix}\right)$ for some $\varphi_0 \in \mathrm{End}_{\mathscr{C}}(C)$, $\varphi_1 \in \mathrm{Hom}_{\mathscr{C}}(\iota(C), C)$, $\varphi_2 \in \mathrm{Hom}_{\mathscr{C}}(C, \iota(C))$ and $\varphi_3 \in \mathrm{End}_{\mathscr{C}}(\iota(C))$. Because e is a morphism in $\mathscr{C}'$, we have $\varphi_2 = \iota(\varphi_1)\,\lambda_C^{-1}$ and $\varphi_3 = \iota(\varphi_0)$. The equation $e^2 = e$ then yields

$$\varphi_0^2 + \varphi_1\,\iota(\varphi_1)\,\lambda_C^{-1} = \varphi_0 \quad\text{and}\quad \varphi_0\,\varphi_1 + \varphi_1\,\iota(\varphi_0) = \varphi_1. \tag{4.2}$$

On the other hand, the condition $\sigma_{h'}(e) = \mathrm{Id}_{F(C)} - e$ yields

$$\sigma_h(\varphi_0) = \mathrm{Id}_C - \varphi_0 \quad\text{and}\quad h^{-1}\,D(\iota(\varphi_1)\,\lambda_C^{-1})\,\alpha_C\,\iota(h) = \varphi_1. \tag{4.3}$$

Using the first equation in (4.3), we may rewrite the equations in (4.2) as

$$\varphi_1\,\iota(\varphi_1)\,\lambda_C^{-1} = \sigma_h(\varphi_0)\,\varphi_0 \quad\text{and}\quad \varphi_1\,\iota(\varphi_0) = \sigma_h(\varphi_0)\,\varphi_1. \tag{4.4}$$

Moreover, the first equation in (4.3) shows by Lemma 4.2 that φ_0 is invertible, hence it follows from the first equation in (4.4) that φ_1 is also invertible. We may therefore define an isomorphism $\theta\colon \iota(C) \to C$ by $\theta = \sigma_h(\varphi_0)^{-1}\,\varphi_1$. The equations in (4.4) yield

$$\theta = \varphi_0\,\lambda_C\,\iota(\varphi_1)^{-1} = \varphi_1\,\iota(\varphi_0)^{-1}.$$

Therefore, $\theta\,\iota(\theta) = \varphi_0\,\lambda_C\,\iota(\varphi_1)^{-1}\,\iota(\varphi_1)\,\iota\iota(\varphi_0)^{-1} = \lambda_C$. Moreover, the second equation in (4.3) yields

$$\alpha_C\,\iota(h) = D(\lambda_C\,\iota(\varphi_1)^{-1})\,h\,\varphi_1 = D(\varphi_0^{-1}\,\theta)\,h\,\varphi_1.$$

But, by definition, $\theta = \sigma_h(\varphi_0)^{-1}\,\varphi_1 = h^{-1}\,D(\varphi_0)^{-1}\,h\,\varphi_1$, hence we obtain by substituting in the last equation

$$\alpha_C\,\iota(h) = D(\theta)\,h\,\theta.$$

This shows that θ is an isometry $(\iota(C), \alpha_C\,\iota(h)) \to (C,h)$ and completes the proof. □

Next, we show that the hypotheses on endomorphism algebras in Proposition 4.3 pass on to the double category.

Proposition 4.4. *Let $C' = (C, f) \in \mathscr{C}'$. If $\mathrm{End}_{\mathscr{C}}(C)$ is semilocal and rad-adically complete, then $\mathrm{End}_{\mathscr{C}'}(C')$ also is semilocal and rad-adically complete.*

Proof. Let $\tau\colon \mathrm{End}_{\mathscr{C}}(C) \to \mathrm{End}_{\mathscr{C}}(C)$ be the automorphism defined by

$$\tau(\varphi) = f\,\iota(\varphi)\,f^{-1} \qquad \text{for } \varphi \in \mathrm{End}_{\mathscr{C}}(C).$$

It satisfies $\tau^2(\varphi) = \varphi$ for all $\varphi \in \mathrm{End}_{\mathscr{C}}(C)$ and

$$\mathrm{End}_{\mathscr{C}'}(C') = \{\varphi \in \mathrm{End}_{\mathscr{C}}(C) \mid \tau(\varphi) = \varphi\}.$$

Therefore, the proposition follows from (b) in the next ring-theoretic result:

Lemma 4.5. *Let A be a k-algebra and $\tau\colon A \to A$ an automorphism such that $\tau^2 = \mathrm{Id}_A$, and set $B = \{a \in A \mid \tau(a) = a\}$. Write $R(A)$ (resp. $R(B)$) for the Jacobson radical of A (resp. B).*

(a) *If A is right artinian and $R(A) = 0$, then B also is right artinian and $R(B) = 0$.*
(b) *If A is semilocal, then $R(B) = B \cap R(A)$ and B is semilocal. If moreover A is rad-adically complete, then B also is rad-adically complete.*

Proof. (a) Setting $E = \{a \in A \mid \tau(a) = -a\}$ yields a k-module decomposition $A = B \oplus E$. The k-module E is a B-bimodule, hence for every right ideal $I \subset B$

$$I \cdot A = I \oplus I \cdot E \qquad\text{and}\qquad I = (I \cdot A) \cap B.$$

If $(I_n)_{n\in\mathbb{N}}$ is a descending chain of right ideals in B, then there is an integer N such that $I_n \cdot A = I_N \cdot A$ for all $n \geq N$, hence

$$I_n = (I_n \cdot A) \cap B = (I_N \cdot A) \cap B = I_N \quad \text{for all } n \geq N.$$

Thus, B is right artinian.

In order to show that $R(B) = 0$, we show that the right ideal $R(B) \oplus R(B) \cdot E$ in A is a nilideal. This ideal is therefore contained in $R(A)$ by [Lam91, (4.11)], hence it vanishes since $R(A) = 0$. This is thus sufficient to prove that $R(B) = 0$.

For the rest of the proof, fix $r \in R(B)$ and $s \in R(B) \cdot E$, and let $M(d)$ denote the set of all products of $2d-1$ factors r, s in A, i.e., all monomials of total degree $2d-1$ in r,

s. By induction on d, we show below that $M(d) \subset A \cdot R(B)^d \cdot A$ for all $d \geq 1$, hence that $(r+s)^{2d-1} \in A \cdot R(B)^d \cdot A$. We know from the first part of the proof that B is right artinian, hence $R(B)$ is nilpotent by [Lam91, (4.12)]. If the integer N is such that $R(B)^N = 0$, it will follow that $(r+s)^{2N-1} = 0$ for all $r+s \in R(B) \oplus R(B) \cdot E$.

For all integers $d \geq 1$ and $i \in \{-d+1, \ldots, d\}$, let

$$M(d,i) = \begin{cases} M(d) \cap A \cdot R(B)^{2d-1} & \text{if } i = -d+1, \\ M(d) \cap A \cdot R(B)^{d-i} \cap \big(\bigcup_{j\in\mathbb{N}} A \cdot R(B)^{d-1+i} \cdot E \cdot R(B)^j\big) & \text{if } i > -d+1. \end{cases}$$

Thus, $M(1,0) = \{r\}$ and $M(1,1) = \{s\}$, and $M(1) = M(1,0) \cup M(1,1)$. It is clear from the definitions that $M(d,i)r^2 \subset M(d+1, i-1)$ for all d, i. Moreover, since $s \in R(B) \cdot E$ we have $R(B)^j s \subset R(B)^{j+1} \cdot E$ for all $j \geq 0$, and since $\tau(s) = -s$ it follows that $E \cdot R(B)^j s \subset B$ for all $j \geq 0$. Therefore,

$$M(d,i)rs \subset M(d+1, 2-i) \quad \text{and} \quad M(d,i)sr \subset M(d+1, 1-i) \quad \text{for all } d,\, i.$$

Also, from $E \cdot R(B)^j s \subset B$ and $s \in R(B) \cdot E$ it follows that $s^2 \in R(B)$ and $(E \cdot R(B)^j s)s \subset R(B) \cdot E$ for all $j \geq 0$, hence $M(d,i)s^2 \subset M(d+1, i)$ for all d, i. Since $M(1) = M(1,0) \cup M(1,1)$ and

$$M(d+1) = M(d)r^2 \cup M(d)rs \cup M(d)sr \cup M(d)s^2,$$

induction on d yields

$$M(d) = \bigcup_{i=-d+1}^{d} M(d,i) \qquad \text{for all } d \geq 1.$$

By definition, $M(d,i) \subset A \cdot R(B)^d \cdot A$, because $d - i \geq d$ if $i \leq 0$ and $d - 1 + i \geq d$ if $i \geq 1$. Therefore, $M(d) \subset A \cdot R(B)^d \cdot A$ and the proof of (a) is complete.

(b) Now, assume A is semilocal and write $\overline{A}$ for $A/R(A)$, so $\overline{A}$ is right artinian and $R(\overline{A}) = 0$. For every $a \in A$, write $\overline{a}$ for the image of a in $\overline{A}$. The automorphism τ preserves $R(A)$ and induces an automorphism $\overline{\tau}$ of $\overline{A}$. If $\overline{\tau}(\overline{a}) = \overline{a}$, then $\overline{a} = \frac{1}{2}\overline{(a + \tau(a))}$, hence the k-subalgebra of $\overline{A}$ fixed under $\overline{\tau}$ is $B/B \cap R(A)$. Part (a) of the proof shows that $B/B \cap R(A)$ is right artinian and $R(B/B \cap R(A)) = 0$. But $B \cap R(A) \subset R(B)$ by [Lam91, (5.6)] and $R(B/B \cap R(A)) = R(B)/B \cap R(A)$ by [Lam91, (4.6)], hence $R(B) = B \cap R(A)$. Therefore $B/B \cap R(A) = B/R(B)$, hence B is semilocal.

If A is rad-adically complete, then it is clear from the definitions that B is $(B \cap R(A))$-adically complete. But $B \cap R(A) = R(B)$, hence B is rad-adically complete. This completes the proofs of Lemma 4.5 and Proposition 4.4. □

Theorem 4.6. *Assume that* $\mathrm{End}_{\mathscr{C}}(C)$ *is semilocal and rad-adically complete for every object* $C \in \mathscr{C}$. *Then the octagon of Witt groups* (4.1) *is exact.*

Proof. By Proposition 4.4, the hypothesis on endomorphism algebras also holds for the double category $\mathscr{C}'$, hence we may use the same shifting argument as in the proof of Theorem 3.1 to see that it suffices to prove the exactness of

$$W(\mathscr{C}, D, \delta) \xrightarrow{W(\widehat{F})} W(\mathscr{C}', D', \delta') \xrightarrow{W(\widehat{F}')} W(\mathscr{C}'', D'', \delta'').$$

Theorem 3.1 yields $W(\widehat{F}')\,W(\widehat{F}) = 0$. To prove that the kernel of $W(\widehat{F}')$ lies in the image of $W(\widehat{F})$, pick a symmetric space $(C', h') \in \mathcal{S}(\mathscr{C}', D', \delta')$ such that $W(\widehat{F}')(C', h') = 0$. If

(C',h') is split isotropic, consider a direct summand $L \xrightarrow{t} C' \xrightarrow{s} L$ such that $s\,t = \mathrm{Id}_L$ and ${h'}^{-1}\,D'(t)\,h' = 0$. Substituting $s - \frac{1}{2} s\,{h'}^{-1}\,D'(t\,s)\,h'$ for s, we may assume $s\,{h'}^{-1}\,D'(s) = 0$. The idempotent $e = t\,s \in \mathrm{End}_{\mathscr{C}'}(C')$ then satisfies $\sigma_{h'}(e)\,e = 0 = e\,\sigma_{h'}(e)$, hence $e + \sigma_{h'}(e)$ also is an idempotent. Since $\mathscr{C}'$ is pseudo-abelian we may consider a splitting of $\mathrm{Id}_{C'} - e - \sigma_{h'}(e)$, i.e., morphisms $M \xrightarrow{t_M} C' \xrightarrow{s_M} M$ in $\mathscr{C}'$ such that $s_M\,t_M = \mathrm{Id}_M$ and $t_M\,s_M = \mathrm{Id}_{C'} - e - \sigma_{h'}(e)$. Then $(t, {h'}^{-1}\,D'(s), t_M)\colon L \oplus D'(L) \oplus M \to C'$ is an isomorphism with inverse $\left(\begin{smallmatrix} s \\ D'(t)\,h' \\ s_M \end{smallmatrix}\right)$ such that

$$
\begin{pmatrix} D'(t) \\ D'({h'}^{-1}\,D'(s)) \\ D'(t_M) \end{pmatrix} h' \begin{pmatrix} t & {h'}^{-1}\,D'(s) & t_M \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id}_{D'(L)} & 0 \\ \delta'_{D'(L)} & 0 & 0 \\ 0 & 0 & D'(t_M)\,h'\,t_M \end{pmatrix}. \tag{4.5}
$$

Since the left side is an isomorphism, it follows that $D'(t_M)\,h'\,t_M$ is an isomorphism. Letting $g = D'(t_M)\,h'\,t_M$, we have $(M,g) \in \mathscr{S}(\mathscr{C}', D', \delta')$, and (4.5) shows that $(t, {h'}^{-1}\,D'(s), t_M)$ is an isometry $H(L) \perp (M,g) \simeq (C',h')$, hence (C',h') is Witt-equivalent to (M,g). If (M,g) is again split isotropic, we may repeat the procedure to obtain another isometry $(C',h') \simeq H(L) \perp H(L_1) \perp (M_1,g_1)$. But since $\mathrm{End}_{\mathscr{C}'}(C')$ is semilocal, [Knu91, Prop. II(5.2.6)][3] shows that C' has no direct sum decomposition with infinitely many nonzero summands, hence the procedure terminates in a finite number of steps and yields a split anisotropic symmetric space that is Witt-equivalent to (C',h'). Therefore, we may assume (C',h') is split anisotropic. Witt cancellation holds in $\mathscr{S}(\mathscr{C}', D', \delta')$ by [Knu91, Th. II(6.6.1)], hence $W(\widehat{F}')(C',h') = 0$ means that $\mathscr{S}(\widehat{F}')(C',h')$ is hyperbolic. Proposition 4.3 then yields an isometry $\theta\colon (\iota'(C'), -\alpha'_{C'}\,\iota'(h')) \to (C',h')$ such that $\theta\,\iota'(\theta) = \lambda'_{C'}$, and Lemma 3.3 shows that there exists a symmetric space $(X,h) \in \mathscr{S}(\mathscr{C}, D, \delta)$ such that $\mathscr{S}(\widehat{F})(X,h) = (C',h')$. Thus, the Witt class of (C',h') is the image of the Witt class of (X,h) under $W(\widehat{F})$. □

Theorem 4.6 applies for instance in the case where $\mathscr{C} = \mathscr{P}_A$, with A a semilocal rad-adically complete k-algebra. To see this, note that each $P \in \mathscr{P}_A$ is a direct summand in a finitely generated free A-module, hence $\mathrm{End}_A(P) \simeq e\,M_n(A)\,e$ for some integer n and some idempotent e in the matrix algebra $M_n(A)$. If A is semilocal and rad-adically complete, then $M_n(A)$ has the same properties since $R\big(M_n(A)\big) = M_n\big(R(A)\big)$, see [Lam91, p. 61 and (20.4)]. Moreover, $e\,M_n(A)\,e$ also is semilocal and rad-adically complete because $R(e\,M_n(A)\,e) = e\,R\big(M_n(A)\big)\,e = R\big(M_n(A)\big) \cap (e\,M_n(A)\,e)$, see [Lam91, (21.10)]. In particular, the octagons of Witt groups derived from (3.7) or (3.8) are exact when A is a right artinian k-algebra, hence Theorem 4.6 extends the Grenier-Boley–Mahmoudi results in [GBM05] and Ranicki's results in [HTW84, (0.2)] for the case where the characteristic is different from 2. (For A right artinian and semisimple, the quadratic L-groups $L_n(A,\sigma,\varepsilon)$ with n even are Witt groups, see [Ran87, §1]. Note that Ranicki does not assume 2 is invertible.)

4.2. **Abelian categories.** In this subsection, we assume that $\mathscr{C}$ is an (essentially small) abelian category, which we endow with its usual exact structure, and we only consider dualities (D,δ) with D exact. For any symmetric space $(C,h) \in \mathscr{S}(\mathscr{C}, D, \delta)$ and any subobject $\varphi\colon L \hookrightarrow C$, the *orthogonal subobject* is defined as $L^\perp := \ker(D(\varphi)\,h)$. The space (C,h) is said to be *isotropic* if there is a nonzero subobject $L \subset C$ such that $L \subset L^\perp$, and *anisotropic* otherwise. (Note that L is not required to be a direct summand.) The space (C,h) is said to be *metabolic* (*weakly*

[3] Knus assumes that the category is abelian and the endomorphism algebra of every object is right artinian, but inspection of the proof shows that all the arguments go through without change under our weaker hypotheses.

hyperbolic in [Sch85, Def. 7.5.4]) if there is a subobject $L \subset C$ such that $L = L^{\perp}$. The resulting Witt group $W(\mathscr{C}, D, \delta)$ is called *weak Witt group* in [Sch85, Def. 7.5.4].

In every conjugation (ι, λ) we consider, the functor ι is assumed to be exact. The double category $\mathscr{C}'$ is then abelian, and the functors F, G, Φ, Ψ are exact. Therefore, given a conjugation (ι, λ) and a commuting duality (D, δ), the functors in the octagon (3.2) induce Witt group homomorphisms, which again yield the 8-periodic diagram (4.1). Hyperbolic spaces are clearly metabolic, hence Theorem 3.1 again shows that this diagram is a chain complex. The goal of this subsection is to show that the homology of this chain complex vanishes when $\mathscr{C}$ is an artinian category.

Lemma 4.7. *Property (P) holds for anisotropic spaces in $\mathscr{S}(\mathscr{C}, D, \delta)$.*

Proof. Let $(C, h) \in \mathscr{S}(\mathscr{C}, D, \delta)$ be an anisotropic symmetric space. Let $\mathscr{S}(\widehat{F})(C, h) = (F(C), h')$ with $h' = \widehat{\phi}_C\, F(h)$ and let $\varphi' \colon L' \hookrightarrow F(C)$ be a subobject such that $L' = L'^{\perp}$. Write $\varphi = G(\varphi')$ and $L = G(L')$, so $\varphi\colon L \to C \oplus \iota(C)$ is a morphism in $\mathscr{C}$. Write also ϵ_1, ϵ_2, π_1, π_2 for the canonical morphisms

$$\epsilon_1 \colon C \to C \oplus \iota(C), \quad \epsilon_2 \colon \iota(C) \to C \oplus \iota(C), \quad \pi_1 \colon C \oplus \iota(C) \to C, \quad \pi_2 \colon C \oplus \iota(C) \to \iota(C).$$

Claim: $\pi_2\, \varphi \colon L \to \iota(C)$ is an isomorphism.

We show that $\pi_2\,\varphi$ is a monomorphism and an epimorphism. If $\psi\colon M \to L$ is a morphism such that $\pi_2\,\varphi\,\psi = 0$, then $\varphi\,\psi$ factors through the kernel of π_2, which is C, hence there exists $\psi_1\colon M \to C$ such that $\varphi\,\psi = \epsilon_1\,\psi_1$. Since $D'(\varphi')\,h'\,\varphi' = 0$, it follows that

$$D(\psi_1)\,D(\epsilon_1)\begin{pmatrix} h & 0 \\ 0 & -\alpha_C\,\iota(h)\end{pmatrix}\epsilon_1\,\psi_1 = D(\varphi\,\psi)\,G(h')\,\varphi\,\psi = 0.$$

But $D(\epsilon_1)\,G(h')\,\epsilon_1 = h$ and (C, h) is anisotropic, hence $\psi_1 = 0$ and $\varphi\,\psi = 0$. But φ is a monomorphism, hence $\psi = 0$. This shows that $\pi_2\,\psi$ is a monomorphism.

Similarly, if $\psi\colon M \to D\iota(C)$ is a morphism such that $D(\pi_2\,\varphi)\,\psi = 0$, then $G(h')^{-1}\,D(\pi_2)\,\psi$ factors through $\ker(D(\varphi)\,G(h')) = L$, hence $\psi = 0$ because $(\iota(C), -\alpha_C\,\iota(h))$ is anisotropic and $D(\pi_2)$ is a monomorphism. Therefore, $D(\pi_2\,\varphi)$ is a monomorphism, hence $\pi_2\,\varphi$ is an epimorphism because D is an exact functor.

The claim is thus proved, and me may consider $\theta = (\pi_1\,\varphi)\,(\pi_2\,\varphi)^{-1} \colon \iota(C) \to C$. Let $L' = (L, \ell)$ for some $\ell\colon \iota(L) \to L$. Since φ' is a morphism in $\mathscr{C}'$, there are commutative diagrams

$$\begin{tikzcd} \iota(L) \arrow[r, "\iota(\varphi)"] \arrow[d, "\ell"'] & \iota(C) \oplus \iota\iota(C) \arrow[r, "\iota(\pi_1)"] \arrow[d, "\Lambda_C"'] & \iota(C) \arrow[d, equal] \\ L \arrow[r, "\varphi"] & C \oplus \iota(C) \arrow[r, "\pi_2"] & \iota(C) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \iota(L) \arrow[r, "\iota(\varphi)"] \arrow[d, "\ell"'] & \iota(C) \oplus \iota\iota(C) \arrow[r, "\iota(\pi_2)"] \arrow[d, "\Lambda_C"'] & \iota\iota(C) \arrow[d, "\lambda_C"] \\ L \arrow[r, "\varphi"] & C \oplus \iota(C) \arrow[r, "\pi_1"] & C \end{tikzcd}$$

It follows that $\ell = (\pi_2\,\varphi)^{-1}\,\iota(\pi_1\,\varphi)$ and $(\pi_1\,\varphi)\,\ell = \lambda_C\,\iota(\pi_2\,\varphi)$, hence $\theta\,\iota(\theta) = \lambda_C$. Since λ_C is an isomorphism, we see that θ is invertible on the right. But taking the image of each side under ι and using $\lambda_C\,\iota\iota(\theta) = \theta\,\iota(\lambda_C)$ we obtain $\iota(\theta)\,\lambda_C^{-1}\,\theta = \mathrm{Id}_{\iota(C)}$, hence θ is also invertible on the left and is therefore an isomorphism.

To compute $D(\theta)\,h\,\theta$, we use

$$G(h') = D(\pi_1)\,h\,\pi_1 - D(\pi_2)\,(\alpha_C\,\iota(h))\,\pi_2.$$

Since $D(\varphi)\, G(h')\, \varphi = 0$, this equation yields

$$D(\pi_1\, \varphi)\, h\, (\pi_1\, \varphi) = D(\pi_2\, \varphi)\, \alpha_C\, \iota(h)\, (\pi_2\, \varphi),$$

hence $D(\theta)\, h\, \theta = \alpha_C\, \iota(h)$. □

Recall that an abelian category $\mathscr{C}$ is said to be *artinian* (resp. *noetherian*) if the descending (resp. ascending) chain condition for subobjects holds for every $C \in \mathscr{C}$. These properties are inherited by the double category when $\mathscr{C}$ carries a conjugation, since the forgetful functor G carries subobjects to subobjects. If $\mathscr{C}$ carries a duality, the artinian and the noetherian conditions are equivalent, and imply that each object has finite length.

Theorem 4.8. *If $\mathscr{C}$ is an artinian abelian category, the octagon of Witt groups* (4.1) *is exact.*

Proof. By the same shifting argument as in the proof of Theorem 3.1, we are reduced to proving the exactness of the sequence

$$W(\mathscr{C}, D, \delta) \xrightarrow{W(\widehat{F})} W(\mathscr{C}', D', \delta') \xrightarrow{W(\widehat{F}')} W(\mathscr{C}'', D'', \delta'').$$

We already know from Theorem 3.1 that this is a zero-sequence. Let $(C', h') \in \mathcal{S}(\mathscr{C}', D', \delta')$ be a symmetric space whose Witt equivalence class is in the kernel of $W(\widehat{F}')$. Since $\mathscr{C}'$ is artinian, isotropic reduction (also known as sublagrangian reduction [Bal05, §1.2.5]) yields an anisotropic symmetric space Witt-equivalent to (C', h'), see [SW19, Cor. 2.12] or [Sch85, Th. 7.5.5]. We may therefore assume (C', h') is anisotropic. Since the Witt class of (C', h') lies in the kernel of $W(\widehat{F}')$, it follows from [You97, Th. (4.9)] that $\mathcal{S}(\widehat{F}')(C', h')$ is metabolic. Lemma 4.7 shows that (C', h') satisfies Property (P), hence Lemma 3.3 yields a symmetric space $(X, h) \in \mathcal{S}(\mathscr{C}, D, \delta)$ whose Witt equivalence class is mapped under $W(\widehat{F})$ to the Witt equivalence class of (C', h'). □

4.3. **Example: The Warshauer octagon.** Let $R = k[t, t^{-1}]$ be the ring of Laurent polynomials in one indeterminate over a field k of characteristic different from 2, and let $\mathscr{F}_R$ be the category of right R-modules of finite length, i.e., of finite-dimensional k-vector spaces with an invertible operator (given by the action of t). Consider the structure (Id_R, t) on R (in the sense of §1.3); the corresponding (Id_R, t)-twisted quadratic extension of R is $R' = k[u, u^{-1}]$ where $u^2 = t$, and the conjugation (ι, λ) on $\mathscr{F}_R$ derived from (Id_R, t) has $\iota = \mathrm{Id}_{\mathscr{F}_R}$ and $\lambda_X \colon X \to X$ is multiplication by t for $X \in \mathscr{F}_R$. As in §1.3, we endow R' with the structure $(\iota', -t)$, where $\iota'(u) = -u$, and identify $\mathscr{F}'_R = \mathscr{F}_{R'}$.

To define a duality on $\mathscr{F}_R$, fix an element $\mu \in k^\times$ and let

$$D(X) = \mathrm{Hom}_k(X, k) \qquad \text{for } X \in \mathscr{F}_R,$$

endowed with the R-module action defined by the condition

$$\langle \pi t, x\rangle = \langle \pi, x t^{-1}\rangle \mu^2 \qquad \text{for } \pi \in D(X) \text{ and } x \in X,$$

where $\langle \llcorner, \llcorner \rangle$ is the canonical pairing $\mathrm{Hom}_k(X, k) \times X \to k$. With $\delta \colon \mathrm{Id}_{\mathscr{F}_R} \overset{\sim}{\Rightarrow} DD$ the usual natural isomorphism given by $\langle \delta_X(x), \pi\rangle = \langle \pi, x\rangle$ for $x \in X$ and $\pi \in D(X)$, the pair (D, δ) is a duality[4] on $\mathscr{F}_R$. It is commuting with the conjugation $(\mathrm{Id}_{\mathscr{F}_R}, \lambda)$ under the natural isomorphism $\alpha \colon D \overset{\sim}{\Rightarrow} D$ given by

$$\alpha_X \colon D(X) \to D(X), \qquad \pi \mapsto \pi\mu \text{ for } \pi \in D(X).$$

[4]The usual duality on the category of modules of finite length is different, see [Sch85, Ch. 6, §1].

The construction in Proposition 2.1 yields a new duality (D,δ_α) where

$$\langle(\delta_\alpha)_X(x),\pi\rangle=\langle\pi,xt^{-1}\rangle\mu\qquad\text{for } x\in X \text{ and } \pi\in D(X).$$

On the other hand, the duality (D',δ') on $\mathscr{F}_{R'}$ is given by $D'(X')=\operatorname{Hom}_k(X',k)$ with the R'-module action given by

$$\langle\pi u,x\rangle=-\langle\pi,xu^{-1}\rangle\mu\qquad\text{for } \pi\in D'(X') \text{ and } x\in X',$$

and $\delta'_{X'}=\delta_{G(X')}$ for $X'=(X,f)\in\mathscr{F}'_R$. Since the automorphism ι' on R' maps u to $-u$, for $X'\in\mathscr{F}_{R'}$ the R'-module action on $\iota'D'(X')=\operatorname{Hom}_k(X',k)$ is given by

$$\langle\pi u,x\rangle=\langle\pi,xu^{-1}\rangle\mu\qquad\text{for } \pi\in D'(X') \text{ and } x\in X'.$$

Moreover, $\alpha'_{X'}=-\alpha_{G(X')}$ and $\lambda'_{X'}=-\lambda_{G(X')}$, hence the duality $(\iota'D',\delta'_{\alpha'})$ satisfies

$$\langle(\delta'_{\alpha'})_{X'}(x),\,\pi\rangle=\langle\pi,\,xt^{-1}\rangle\mu\qquad\text{for } \pi\in\iota'D'(X') \text{ and } x\in X'.$$

For $(X,h)\in\mathcal{S}(\mathscr{F}_R,D,\delta)$, define

$$b\colon X\times X\to k\qquad\text{by}\quad b(x,y)=\langle h(x),y\rangle\quad\text{for } x,\,y\in X.$$

The map b is a nonsingular k-bilinear form and the R-linearity of h means that

$$b(xt,\,yt)=b(x,\,y)\mu^2\qquad\text{for } x,\,y\in X.$$

Thus, multiplication by t is a similitude with multiplier μ^2. Writing $\mathscr{B}^\pm(k,\mu^2)$ for the category of triples (X,b,ℓ) where X is a finite-dimensional k-vector space, $b\colon X\times X\to k$ is a nonsingular (±1)-symmetric bilinear form and $\ell\in\operatorname{End}_k(X)$ is a similitude with multiplier μ^2, we thus have

$$\mathcal{S}(\mathscr{F}_R,D,\pm\delta)=\mathscr{B}^\pm(k,\mu^2).\tag{4.6}$$

When $\mu=\pm1$, the objects in this category are the inner product spaces with isometry considered by Milnor [Mil69].

With the commutation α between $(\mathrm{Id}_{\mathscr{F}_R},\lambda)$ and (D,δ) given above we may similarly define for every $(X,h)\in\mathcal{S}(\mathscr{F}_R,D,\pm\delta_\alpha)$ the map

$$b\colon X\times X\to k\qquad\text{by}\quad b(x,y)=\langle h(x),y\rangle\quad\text{for } x,\,y\in X.$$

Now, the form b is k-bilinear but not symmetric: the property $h=D(h)\,(\delta_\alpha)_X$ yields

$$b(xt,\,y)=\pm b(y,\,x)\mu\qquad\text{for all } x,\,y\in X,$$

from which it follows that multiplication by t is a similitude with multiplier μ^2. To obtain an alternative description of this category, consider the category $\mathscr{A}(k)$ whose objects are pairs (X,a) where X is a finite-dimensional k-vector space and $a\colon X\times X\to k$ is a nonsingular bilinear form without symmetry requirement, and whose morphisms are isometries. (The objects in $\mathscr{A}(k)$ are the *asymmetric inner product spaces over* k in Warshauer's terminology from [War82].) Recall from [War82, Cor. I.4.2] that to each $(X,a)\in\mathscr{A}(k)$ is attached a bijective *symmetry operator* $s\in\operatorname{End}_k(X)$ defined by the property that $a(x,y)=a(y,s(x))$ for $x,\,y\in X$. There are isomorphisms of categories

$$\mathcal{S}(\mathscr{F}_R,D,\delta_\alpha)=\mathscr{A}(k)\qquad\text{and}\qquad\mathcal{S}(\mathscr{F}_R,D,-\delta_\alpha)=\mathscr{A}(k)\tag{4.7}$$

under which $(X,h)\in\mathcal{S}(\mathscr{F}_R,D,\pm\delta_\alpha)$ is identified with $(X,b)\in\mathscr{A}(k)$. Conversely, for each $(X,a)\in\mathscr{A}(k)$ we define $h\colon X\to D(X)$ by $h(x)=a(x,\text{␣})$ for $x\in X$ and identify (X,a) with $(X,h)\in\mathcal{S}(\mathscr{F}_R,D,\pm\delta_\alpha)$ in which the action of t is given by $xt=\pm s^{-1}(x)\mu$ for $x\in X$.

Under the identification $\mathscr{F}'_R = \mathscr{F}_{R'}$, and with the duality (D', δ') defined above, we may identify by the same construction as in (4.6)

$$\mathcal{S}(\mathscr{F}_{R'}, D', \pm\delta') = \mathscr{B}^{\pm}(k, -\mu). \tag{4.8}$$

For every symmetric space $(X', h') \in \mathcal{S}(\mathscr{F}_{R'}, \iota' D', \pm\delta'_{\alpha'})$ the associated k-bilinear map $b' \colon X' \times X' \to k$ defined by $b'(x, y) = \langle h'(x),\, y\rangle$ for $x,\, y \in X'$ satisfies

$$b'(xu, yu) = b'(x, y)\mu \qquad \text{and} \qquad b'(xt, y) = \pm b'(y, x)\mu \quad \text{for } x,\, y \in X'.$$

Since $u^2 = t$, these equations imply

$$b'(yu, x) = \pm b'(xt, yu)\mu^{-1} = \pm b'(xu, y) \qquad \text{for } x,\, y \in X',$$

hence the map $b \colon X' \times X' \to k$ defined by $b(x, y) = b'(xu, y)$ for $x,\, y \in X'$ is a nonsingular (± 1)-symmetric bilinear form on X' such that $b(xu, yu) = b(x, y)\mu$ for $x,\, y \in X'$. The map $h' \colon X' \to \iota' D'(X')$ is uniquely determined by b since $\langle h'(x), y\rangle = b(xu^{-1}, y)$ for $x,\, y \in X'$. Therefore, by mapping (X', h') to (X', b) we may identify

$$\mathcal{S}(\mathscr{F}_{R'}, \iota' D', \delta'_{\alpha'}) = \mathscr{B}^{\pm}(k, \mu). \tag{4.9}$$

With the identifications (4.6) and (4.8) above, the functor $\mathcal{S}(\widehat{F}) \colon \mathscr{B}^{\pm}(k, \mu^2) \to \mathscr{B}^{\pm}(k, -\mu)$ is given by

$$\mathcal{S}(\widehat{F})(X, b, \ell) = (X \oplus X, b \perp \langle -\mu\rangle b, \ell'), \quad \text{where } \ell'(x_1, x_2) = (\ell(x_2), x_1) \text{ for } x_1,\, x_2 \in X.$$

It gives rise to the maps denoted by $I_{\pm 1}$ in [War82, p. 109].

The functor $\mathcal{S}(\check{F}) \colon \mathscr{A}(k) \to \mathscr{B}^{\pm}(k, \mu)$ takes two different forms under the identifications (4.7) and (4.9), depending on whether $\mathscr{A}(k)$ is viewed as $\mathcal{S}(\mathscr{F}_R, D, \delta_\alpha)$ or $\mathcal{S}(\mathscr{F}_R, D, -\delta_\alpha)$: for $(X, a) \in \mathscr{A}(k)$,

$$\mathcal{S}(\check{F})_{\pm}(X, a) = (X \oplus X, a_{\pm}, \ell_{\pm})$$

where for $x_1,\, x_2,\, y_1,\, y_2 \in X$,

$$a_{\pm}\big((x_1, x_2), (y_1, y_2)\big) = \mu\big(a(x_1, y_2) \pm a(y_1, x_2)\big) \quad \text{and} \quad \ell_{\pm}(x_1, x_2) = (\pm s^{-1}(x_2)\mu, x_1).$$

These functors give rise to the maps $m_{\pm 1}$ in [War82, p. 108] up to similitude: the definition of $a_{\pm}$ in [War82, p. 108] lacks the factor μ, which does not affect the main results.

By contrast, the functor $\mathcal{S}(\widehat{G}) \colon \mathscr{B}^{\pm}(k, -\mu) \to \mathscr{A}(k)$ has a uniform description: for $(X, b', \ell') \in \mathscr{B}^{\pm}(k, -\mu)$,

$$\mathcal{S}(\widehat{G})(X, b', \ell') = (X, b) \qquad \text{with} \quad b(x, y) = \mu^{-1} b'(x, \ell(y)) \text{ for } x,\, y \in X.$$

It yields the maps $d_{\pm 1}$ in [War82, p. 109]. Finally, the functor $\mathcal{S}(\check{G}) \colon \mathscr{B}^{\pm}(k, \mu) \to \mathscr{B}^{\pm}(k, \mu^2)$ yields the "squaring map" of [War82, p. 108]: for $(X, b, \ell) \in \mathscr{B}^{\pm}(k, \mu)$,

$$\mathcal{S}(\check{G})(X, b, \ell) = (X, b, \ell^2).$$

The octagon (3.2) takes the form

$$\begin{tikzcd}
\mathscr{B}^+(k,\mu^2) \arrow[r, "\mathcal{S}(\hat{F})"] & \mathscr{B}^+(k,-\mu) \arrow[r, "\mathcal{S}(\hat{G})"] & \mathscr{A}(k) \arrow[d, "\mathcal{S}(\check{F})_-"] \\
\mathscr{B}^+(k,\mu) \arrow[u, "\mathcal{S}(\check{G})"] & & \mathscr{B}^-(k,\mu) \arrow[d, "\mathcal{S}(\check{G})"] \\
\mathscr{A}(k) \arrow[u, "\mathcal{S}(\check{F})_+"] & \mathscr{B}^-(k,-\mu) \arrow[l, "\mathcal{S}(\hat{G})"'] & \mathscr{B}^-(k,\mu^2) \arrow[l, "\mathcal{S}(\hat{F})"']
\end{tikzcd}$$

Theorem 4.8 shows that the corresponding octagon of Witt groups is exact.

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