S-CONVEX EXTREMA, TAYLOR-TYPE EXPANSIONS AND STOCHASTIC APPROXIMATIONS

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Abstract

This paper aims to further investigate the structure of the s-convex stochastic extrema. The present study is based on a remarkable probabilistic generalization of Taylor's theorem obtained by Lin (1994). Two methods for approximating a given risk with the aid of the s-convex extremal distributions are then proposed. The goodness of these stochastic approximations is asserted using stop-loss distances.

Key words and phrases: s-convex orderings, stochastic extrema, stochastic expansions, stochastic approximations, moment spaces, stop-loss metrics

1 Introduction and Motivation

In actuarial sciences, random variables often represent financial losses incurred by an insurance company (and are usually called "risks"). In practice, the distribution function F_X associated to a given risk X possesses a bounded support, $[0, b]$ say, i.e. $F_X(b) = 1$. Actuaries often need easy-to-compute approximations of the risks they cover, typically based on a few moments of the underlying loss distribution. Several methods have been proposed in the literature, namely the Normal-Power approximation. In this paper, we examine some alternative methodologies based on s-convex stochastic extrema, that appear to be easily implemented in actuarial practice.

In the building of appropriate stochastic models intended to describe insurance business, it is often interesting to be able to compare risks. To this end, stochastic orderings, i.e. binary relations defined on sets of distribution functions, are especially helpful. These relations aim to mathematically translate for random variables the intuitive ideas of "being larger", or "being more variable"; most of those used in actuarial sciences are inspired from the Von Neumann-Morgenstern expected utility theory. In addition to these qualitative aspects, it is sometimes possible to identify extremal elements with respect to some stochastic ordering in a given class of distribution functions. These stochastic extrema allow the actuary to determine bounds on various quantities of interest (like Lundberg's coefficient or amounts of insurance premiums, for instance).

Denuit, De Vylder and Lefèvre (1999) identified in a moment space containing distribution functions with a minimum number of support points the extrema with respect to the s-convex stochastic order introduced by Denuit, Lefèvre and Shaked (1998); see also related results obtained by De Vylder and Goovaerts (1999). The aim of Section 5 is to carry on with the study of these stochastic bounds. The stochastic s-convex extrema provide convenient approximations for the distribution of a risk of which only a few moments are known (or reliable estimates of these moments are available), a situation often encountered in actuarial practice. When the distribution of a risk is fully known, the methodology is intended to make tractable computations otherwise impossible to perform. The stochastic approximations proposed in Section 6 are based on four moments, namely mean, variance, skewness and kurtosis; it is worth mentioning that higher approximations are theoretically possible, but require the aid of numerical methods. The approximations provided by these stochastic bounds are coherent with the s-convex stochastic orderings, i.e. with the common preferences of a class of decision-makers with pain functions satisfying desirable properties, as non-decreasingness (lure of gain), convexity (risk aversion), 3-convexity (related - but not equivalent - to decreasing absolute risk aversion), and so on. This new approach is illustrated in Section 7. Sections 2, 3 and 4 introduce the technical results needed for our analysis. It is worth mentioning that the presentation of these sections is entirely original, in the sense that previous results easily follow from Lin's (1994) stochastic generalization of Taylor's theorem resorting on stationary excess transforms.

Let us now introduce some notations used throughout the paper. We utilize the following classical moment matrices: for any non-negative integer k and for any sequence

$(\mu_1, \mu_2, \ldots, \mu_{2k+2})$ of real numbers, define the $(k+1) \times (k+1)$ matrices P_k, Q_k and R_k as

$$P_k = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_k & \mu_{k+1} & \cdots & \mu_{2k} \end{pmatrix}, \quad Q_k = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k+1} & \mu_{k+2} & \cdots & \mu_{2k+1} \end{pmatrix},$$

and

$$R_k = \begin{pmatrix} \mu_2 & \mu_3 & \cdots & \mu_{k+2} \\ \mu_3 & \mu_4 & \cdots & \mu_{k+3} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k+2} & \mu_{k+3} & \cdots & \mu_{2k+2} \end{pmatrix},$$

where $\mu_0 \equiv 1$. The determinant of a matrix A will be denoted as $\| A \|$, and A^t denotes the transposition of A. Henceforth, $\mathbb{R}$ denotes the real line $(-\infty, +\infty)$, $\mathbb{R}^+$ the half positive real line $[0, +\infty)$, $\mathbb{N}$ the set of the non-negative integers $\{0, 1, 2, \ldots\}$. Given a function ϕ, we denote as $\phi^{(k)}$ its kth derivative, $k \in \mathbb{N}$, with the convention that $\phi^{(0)} \equiv \phi$.

2 Taylor-type stochastic expansions

Let us define the functions ψ_k and $\psi_{k,t,+}$, $k \in \mathbb{N}$, $t \in \mathbb{R}^+$, as

$$\psi_k(x) = x^k \text{ and } \psi_{k,t,+}(x) = (x-t)_+^k,$$

where $x_+ = \max\{x, 0\}$; with the convention that $\psi_0 \equiv 1$ and $\psi_{0,t,+} = 1$ if $x > t$ and 0 otherwise. Of course, ψ_k and $\psi_{k,0,+}$ coincide on $\mathbb{R}^+$. In actuarial sciences, $E\psi_{k,t,+}(X)$ is usually referred to as the kth degree stop-loss transform of the risk X; $E\psi_k(X)$ is simply the kth moment of X. Given a distribution function F_X for a non-negative random variable X, $E\psi_{k,t,+}(X)$ is closely related to the so-called kth iterated right-tail distribution associated to F_X. To be specific, let us denote as $\overline{F}_X^{[0]}$ the survival function associated to F_X, *id est* $\overline{F}_X^{[0]} \equiv 1 - F_X$ and, for any positive integer k, let us define the kth iterated right tail of X, denoted as $\overline{F}_X^{[k]}$, as

$$\overline{F}_X^{[k]}(t) = \int_{x=t}^{+\infty} \overline{F}_X^{[k-1]}(x)dx, \ \ t \in \mathbb{R}^+.$$

For each k, either $\overline{F}_X^{[k]}(t)$ is non-negative and finite for all t or $\overline{F}_X^{[k]}(t) = +\infty$ for all t, depending only on the right-tail of F_X. Moreover, $\overline{F}_X^{[k]}(t)$ is finite if $E\psi_k(X)$ is finite. Let us mention the following well-known result relating the kth iterated right-tail distribution to the kth degree stop-loss transform.

Lemma 2.1. *The identity* $k!\overline{F}_X^{[k]}(t) = E\psi_{k,t,+}(X)$ *holds for all* $t \in \mathbb{R}^+$ *and* $k \in \mathbb{N}$.

Now, provided X has a finite expectation, consider the transformation $\mathcal{T}$ which maps F_X to the distribution function $\mathcal{T}F_X$ given by

$$\mathcal{T}F_X(t) = \int_{x=0}^{t} \frac{1 - F_X(x)}{E\psi_1(X)}dx = 1 - \frac{\overline{F}_X^{[1]}(t)}{\overline{F}_X^{[1]}(0)}, \ \ t \in \mathbb{R}^+;$$

$\mathcal{T}$ is usually referred to as the stationary excess operator and $\mathcal{T}F_X$ as the equilibrium distribution relating to F_X. Note that the expectation associated to $\mathcal{T}F_X$ is given by

$$\frac{1}{\overline{F}_X^{[1]}(0)}\int_{t=0}^{+\infty}\overline{F}_X^{[1]}(t)dt=\frac{\overline{F}_X^{[2]}(0)}{\overline{F}_X^{[1]}(0)}=\frac{E\psi_2(X)}{2E\psi_1(X)},$$

where the latter inequality follows from Lemma 2.1. Therefore the expectation of $\mathcal{T}F_X$ is finite if, and only if, F_X has a finite variance; $\mathcal{T}F_X$ is in that sense "worse behaving" than F_X. Clearly, the tranformation $\mathcal{T}$ is not one-to-one: $\mathcal{T}$ maps the Bernoulli distribution with parameter p to the unit uniform distribution for each $p \in (0,1)$. It is well-known that $\mathcal{T}F_X = F_X$ if, and only if, F_X corresponds to the exponential distribution; see e.g. Huang and Lin (1995, Theorem 1). The latter result emphasizes the special role played by the exponential distribution in the study of the stationary excess operator. The operator $\mathcal{T}$ is a standard tool in applied probability; see e.g. Nanda, Jain and Singh (1996) and the references therein. It naturally appears in the famous Beekman formula of risk theory; for further applications in actuarial sciences, see e.g. Hesselager, Wang and Willmot (1997).

Now, define iteratively the sequence $\{\mathcal{T}^n F_X,\ n \in I\!N\}$ such that $\mathcal{T}^n = \mathcal{T} \circ \mathcal{T}^{n-1}$, with the convention that $\mathcal{T}^0$ is the identity. Henceforth, we denote as $X_{\{n\}}$ a random variable distributed according to $\mathcal{T}^n F_X$: obviously, $X_{\{0\}}$ and X are identically distributed. The nth degree equilibrium distribution $\mathcal{T}^n F_X$ can be cast into

$$\mathcal{T}^n F_X(t)=1-\frac{\overline{F}_X^{[n]}(t)}{\overline{F}_X^{[n]}(0)}=1-\frac{E\psi_{n,t,+}(X)}{E\psi_n(X)},\ \ t\in I\!R^+, \tag{2.1}$$

which is in accordance with Definition 1 of Hesselager *et al.* (1997). Apart from a normalization, the survival function of $X_{\{n\}}$ is thus the nth degree stop-loss transform of X. The probability density function associated to $\mathcal{T}^n F_X$ is given by

$$\frac{d}{dt}\mathcal{T}^n F_X(t)=\frac{n!\overline{F}_X^{[n-1]}(t)}{E\psi_n(X)}=\frac{nE\psi_{n-1,t,+}(X)}{E\psi_n(X)},\ \ t\in I\!R^+.$$

To end with, let us recall the following relation existing between the stop-loss transforms and the moments of the $X_{\{n\}}$'s and those of X.

Lemma 2.2. *The following formula holds true:*

$$E\psi_{k,t,+}(X_{\{n\}})=\frac{n!E\psi_{n+k,t,+}(X)}{(n+k)(n+k-1)\ldots(k+1)E\psi_n(X)},\ \ n,k\in I\!N,\ \ t\in I\!R^+.$$

Now, suppose we are interested in $E\phi(X)$ for some fixed non-linear function ϕ and some random variable X whose first two moments μ_1 and μ_2 are known. A well-known approximation of $E\phi(X)$ is furnished by the so-called delta method (proposed by Rice (1988, page 143)), namely

$$E\phi(X)\approx\phi(\mu_1)+\frac{1}{2}\sigma^2\phi^{(2)}(\mu_1), \tag{2.2}$$

where σ^2 is the variance of X, i.e. $\sigma^2 = \mu_2 - \mu_1^2$. More refined approximations are available if ϕ possesses more derivatives and X more moments. Such approximations are based on a naive Taylor expansion of ϕ around the point μ_1 yielding

$$E\phi(X) \approx \sum_{k=0}^{n} \frac{\phi^{(k)}(\mu_1)}{k!} E\psi_k(X - \mu_1); \tag{2.3}$$

the delta method (2.2) is found by taking $n = 2$. However, there is no indication about the accuracy of (2.2) and (2.3).

Let us now make such arguments more precise. The following probabilistic generalization of Taylor's theorem is known from Massey and Whitt (1993) suitably modified by Lin (1994). Define $I(F_X) = \bigcup_{n=2}^{+\infty}[0, F_X^{-1}(1-1/n)]$, where the inverse of F_X is given by $F_X^{-1}(p) = \inf\{x \in I\!R | F_X(x) \geq p\}$, $0 < p < 1$; $I(F_X)$ is the smallest interval containing both 0 and the support of F_X.

Lemma 2.3. *Assume that the inequalities $0 < E\psi_n(X) < +\infty$ hold for some positive integer n. Let ϕ be a real-valued function having a $(n-1)$th derivative $\phi^{(n-1)}$ on $I(F_X)$; $\phi^{(n-1)}$ is assumed to be absolutely continuous on $I(F_X)$ and to be the indefinite integral of a Borel-measurable function h. If $E|h(X_{\{n\}})| < +\infty$ then $E|\phi(X)| < +\infty$ and the following formula holds:*

$$E\phi(X) = \sum_{k=0}^{n-1} \frac{\phi^{(k)}(0)}{k!} E\psi_k(X) + \varrho_n(X), \tag{2.4}$$

where the remainder term is given by

$$\varrho_n(X) = \frac{Eh(X_{\{n\}})}{n!} E\psi_n(X) = \int_{t\in I\!R^+} \frac{E\psi_{n-1,t,+}(X)}{(n-1)!} h(t)dt. \tag{2.5}$$

The quality of the approximation

$$E\phi(X) \approx \sum_{k=0}^{n-1} \frac{\phi^{(k)}(0)}{k!} E\psi_k(X) \tag{2.6}$$

is thus reflected in the remainder $\varrho_n(X)$ expressible in terms of the nth degree equilibrium distribution associated to X. To apply (2.4), we need to check the validity of the condition $E|h(X_{\{n\}})| < +\infty$, which is equivalent to

$$\int_{t\in I\!R^+} |h(t)| E\psi_{n-1,t,+}(X)dt < +\infty.$$

A sufficient condition for this finiteness is

$$\int_{t\in I(F_X)} |h(t)|dt < +\infty, \tag{2.7}$$

because $E\psi_{n-1,t,+}(X) \leq E\psi_{n-1}(X) < +\infty$ for $t \in I\!R^+$. For example, if $X \leq b$ almost surely, $I(F_X) = [0, b]$ and (2.7) is fulfilled provided the function h is also bounded on $[0, b]$. Note that if $h \geq 0$ almost everywhere on $I(F_X)$, (2.4) still holds if the condition $Eh(X_{\{n\}}) < +\infty$ is replaced by the condition $E|\phi(X)| < +\infty$, in virtue of Fubini's theorem.

Example 2.4. *Let us point out two interesting particular cases of (2.4):*

(i) *if you apply this expansion to $\psi_{n,t,+}$, $h = n!\psi_{0,t,+}$ you simply get*

$$E\psi_{n,t,+}(X) = P[X_{\{n\}} > t]E\psi_n(X),$$

which is easily deduced from (2.1) together with Lemma 2.1;

(ii) *when ϕ has a nth derivative $\phi^{(n)}$ on $I(F_X)$, $h \equiv \phi^{(s)}$ and (2.5) reduces to*

$$\varrho_n(X) = \frac{E\phi^{(n)}(X_{\{n\}})}{n!}E\psi_n(X) = \int_{t\in I\!R^+} \frac{E\psi_{n-1,t,+}(X)}{(n-1)!}\phi^{(n)}(t)dt. \tag{2.8}$$

In this case, the expansion (2.4) is familiar in applied probability; it has been used e.g. in Denuit, Lefèvre and Shaked (1999a).

The present paper aims to provide stochastic approximations of X that have the same first moments as X, and thus the same approximation (2.6). The supports of these stochastic approximations are as simple as possible, given the moment constraints.

3 The s-convex stochastic orderings

For a positive integer s, let us define the class $\mathcal{M}_{s-cx}^{I\!R^+}$ of measurable functions as

$$\mathcal{M}_{s-cx}^{I\!R^+} = \left\{\pm\psi_k,\ \ k = 1, 2, \ldots, s-1; \psi_{s-1,t,+},\ \ t \in I\!R^+\right\}.$$

The s-convex order $\preceq_{s-cx}$ among (distribution functions of) random variables valued in $I\!R^+$ is the integral stochastic ordering generated by $\mathcal{M}_{s-cx}^{I\!R^+}$. To be specific, let X and Y be non-negative random variables with finite $(s-1)$th moments; X is said to be smaller than Y in the s-convex sense, denoted as $X \preceq_{s-cx} Y$, when

$$E\phi(X) \le E\phi(Y) \text{ for all } \phi \in \mathcal{M}_{s-cx}^{I\!R^+}. \tag{3.1}$$

The class $\{\preceq_{s-cx},\ \ s = 1, 2, 3, \ldots\}$ of stochastic orderings has been introduced in Denuit, Lefèvre and Shaked (1998). It is worth mentioning that $\preceq_{1-cx}$ is the usual stochastic dominance, while $\preceq_{2-cx}$ coincides with the well-known stop-loss order with equal means (also called convex order).

Let us now prove the following result which extends Theorem 6.1 in Denuit, Lefèvre and Shaked (1998) and illustrates the usefulness of the stationary excess operator $\mathcal{T}$ in the study of the s-convex ordering.

Proposition 3.1. *Given a pair of random variables X and Y such that $E\psi_k(X) = E\psi_k(Y)$ for $k = 1, 2, \ldots, s-1$, the following chain of equivalences holds:*

$$\begin{aligned} X \preceq_{s-cx} Y \quad &\Leftrightarrow \quad X_{\{1\}} \preceq_{(s-1)-cx} Y_{\{1\}} \\ &\Leftrightarrow \quad X_{\{2\}} \preceq_{(s-2)-cx} Y_{\{2\}} \\ &\ \ \vdots \\ &\Leftrightarrow \quad X_{\{s-1\}} \preceq_{1-cx} Y_{\{s-1\}}. \end{aligned}$$

Proof. First of all, note that the equality of the $s-j-1$ first moments of $X_{\{j\}}$ and $Y_{\{j\}}$ follows from Lemma 2.2. Now, the implication

$$X \preceq_{s-cx} Y \Rightarrow X_{\{1\}} \preceq_{(s-1)-cx} Y_{\{1\}} \tag{3.2}$$

comes from Lemma 2.2 which ensures that

$$E\psi_{s-2,t,+}(X_{\{1\}}) = \frac{E\psi_{s-1,t,+}(X)}{(s-1)E\psi_1(X)} \leq \frac{E\psi_{s-1,t,+}(Y)}{(s-1)E\psi_1(Y)} = E\psi_{s-2,t,+}(Y_{\{1\}}).$$

Iterating (3.2) yields the announced chain of implications. To end with, we deduce from Lemma 2.1 that

$$X \preceq_{s-cx} Y \Leftrightarrow \overline{F}_X^{[s-1]}(t) \leq \overline{F}_Y^{[s-1]}(t) \text{ for all } t \in I\!R^+,$$

so that (2.1) yields

$$X \preceq_{s-cx} Y \Leftrightarrow X_{\{s-1\}} \preceq_{1-cx} Y_{\{s-1\}};$$

in other words, X precedes Y in the s-convex sense if, and only if, the associated $(s-1)$th degree equilibrium distributions are ordered in the stochastic dominance. □

Considering Proposition 3.1, we see that the stationary excess operator $\mathcal{T}$ attenuates the s-convex inequalities.

When $X \preceq_{s-cx} Y$, the inequality (3.1) holds for a much larger class of functions. Let us consider the class

$$\mathcal{U}_{s-cx}^{I\!R^+} = \{\phi : I\!R^+ \to I\!R | \phi^{(s)} \geq 0\};$$

i.e. $\mathcal{U}_{s-cx}^{I\!R^+}$ consists in all the functions with a non-negative sth derivative. Moreover, let $\overline{\mathcal{U}}_{s-cx}^{I\!R^+}$, $s \geq 2$, be the class of the s-convex functions ϕ, that is, the class of all the functions ϕ being $s-2$ times continuously differentiable such that $\phi^{(s-2)}$ is convex; $\overline{\mathcal{U}}_{1-cx}^{I\!R^+}$ is the class of the non-decreasing functions with domain $I\!R^+$. Obviously $\mathcal{U}_{s-cx}^{I\!R^+} \subset \overline{\mathcal{U}}_{s-cx}^{I\!R^+}$, and it can be shown that $\overline{\mathcal{U}}_{s-cx}^{I\!R^+}$ is the closure of $\mathcal{M}_{s-cx}^{I\!R^+}$ in the topology of uniform convergence. Using Lemma 2.3, we are now in a position to provide an elementary proof for Theorems 3.3 and 3.5 in Denuit, Lefèvre and Shaked (1998).

Proposition 3.2. *Given two risks X and Y with finite first $(s-1)$th moments, we have that*

$$\begin{aligned} X \preceq_{s-cx} Y \quad &\Leftrightarrow \quad E\phi(X) \leq E\phi(Y) \textit{ for all } \phi \in \mathcal{U}_{s-cx}^{I\!R^+}, &(3.3)\\ &\Leftrightarrow \quad E\phi(X) \leq E\phi(Y) \textit{ for all } \phi \in \overline{\mathcal{U}}_{s-cx}^{I\!R^+}, &(3.4)\end{aligned}$$

provided the expectations exist.

Proof. (3.3) follows from (2.4) combining with (2.8). Let us now turn to (3.4). The implication "$\Leftarrow$" is obvious since $\mathcal{M}_{s-cx}^{I\!R^+} \subset \overline{\mathcal{U}}_{s-cx}^{I\!R^+}$. To get the converse, we proceed as follows. In virtue of Proposition 3.1, we have that

$$\begin{aligned} X \preceq_{s-cx} Y \quad &\Rightarrow \quad X_{\{s-2\}} \preceq_{2-cx} Y_{\{s-2\}} \\ &\Rightarrow \quad E\phi^{(s-2)}(X_{\{s-2\}}) \leq E\phi^{(s-2)}(Y_{\{s-2\}}) \text{ for all } \phi \in \overline{\mathcal{U}}_{s-cx}^{I\!R^+} \\ &\Rightarrow \quad E\phi(X) \leq E\phi(Y) \text{ for all } \phi \in \overline{\mathcal{U}}_{s-cx}^{I\!R^+},\end{aligned}$$

provided the expectations exist, where the last equivalence follows from Lemma 2.3 with $n = s - 2$. $\square$

To end with, let us recall a useful sufficient (but not necessary) condition for $\preceq_{s-cx}$ to hold between two risks with identical first $s-1$ moments. Define the number of sign changes of the function ϕ on $I\!R^+$ by

$$S^-(\phi) = \sup S^-[\phi(x_1), \phi(x_2), \ldots, \phi(x_n)],$$

where $S^-[\phi(x_1), \phi(x_2), \ldots, \phi(x_n)]$ is the number of sign changes of the indicated sequence, zero terms being discarded, and the supremum is extended over all $x_1 < x_2 < \cdots < x_n$, $x_i \in I\!R^+$, $n < \infty$. Let X and Y be two non-negative random variables with the same $s-1$ first moments. Denote by F_X and F_Y the distribution functions of X and Y, respectively. In Denuit, Lefèvre and Shaked (1998) it is shown that

$$S^-(F_X - F_Y) = s - 1 \Rightarrow X \preceq_{s-cx} Y, \tag{3.5}$$

provided the last sign of $F_X - F_Y$ is a "+". Given any pair of random variables X and Y such that $E\psi_k(X) = E\psi_k(Y)$ for $k = 1, 2, \ldots, s-1$, it is well-known that $S^-(F_X - F_Y) \geq s - 1$. The sufficient condition (3.5) for a ranking in the s-convex sense thus boils down to require that the difference of the distribution functions exhibits the minimum number of crossings.

Let us now introduce a natural companion of the s-convex order, namely the $[s;j]$-convex order, where $j \in I\!N$ satisfies $j \leq s-2$; it is the integral stochastic ordering among (distribution functions of) non-negative random variables X and Y with finite $(s-1)$th moments generated by

$$\mathcal{M}^{I\!R^+}_{[s;j]-cx} = \left\{ \pm\psi_k, \;\; k = 1, 2, \ldots, j; \;\; \psi_k, \;\; k = j+1, j+2, \ldots, s-1; \;\; \psi_{s-1,t,+}, \;\; t \in I\!R^+ \right\},$$

with the understanding that $\{\pm\psi_k, \;\; k = 1, 2, \ldots, j\} = \emptyset$ if $j = 0$. Henceforth, given two random variables X and Y with finite $(s-1)$th moments, we write $X \preceq_{[s;j]-cx} Y$ when X precedes Y in the $[s;j]$-convex order, i.e. when

$$E\phi(X) \leq E\phi(Y) \text{ for all } \phi \in \mathcal{M}^{I\!R^+}_{[s;j]-cx}. \tag{3.6}$$

It is worth mentioning that $\preceq_{[1;0]-cx}$ is equivalent to $\preceq_{1-cx}$ (they thus both reduce to stochastic dominance), while $\preceq_{[2;0]-cx}$ is the celebrated stop-loss order. The relation $\preceq_{[s;0]-cx}$ is known as the s-increasing convex order or as the stop-loss order of degree $s-1$. It can be shown that $\preceq_{s-cx}$ and $\preceq_{[s;j]-cx}$ are related through

$$X \preceq_{s-cx} Y \Leftrightarrow \begin{cases} E\psi_k(X) = E\psi_k(Y) \text{ for } k = j+1, j+2, \ldots, s-1, \\ X \preceq_{[s;j]-cx} Y; \end{cases} \tag{3.7}$$

in other words, $\preceq_{s-cx}$ is a strengthening of $\preceq_{[s;j]-cx}$ obtained by imposing the equality of the $(j+1)$th, $(j+2)$th, …, $(s-1)$th moments of the random variables to compare.

Combining (2.4) together with (2.8) yields the following result.

Proposition 3.3. *Given two risks X and Y with finite first $(s-1)$th moments, we have that*

$$X \preceq_{[s;j]-cx} Y \Leftrightarrow E\phi(X) \le E\phi(Y) \text{ for all } \phi \in \mathcal{U}^{I\!R^+}_{[s;j]-cx} = \bigcap_{k=j+1}^{s} \mathcal{U}^{I\!R^+}_{k-cx},$$

provided the expectations exist.

Henceforth, we use $\preceq_{[s;s-2]}$, that is, we relax the equality of the $(s-1)$th moments to an inequality compared to $\preceq_{s-cx}$. Let us define $\overline{\mathcal{U}}^{I\!R^+}_{[s;s-2]-cx}$ as the class of all the functions ϕ such that $\phi^{(s-2)}$ is non-decreasing and convex. We then have the following result.

Proposition 3.4. *Given two risks X and Y with finite first $(s-1)$th moments, we have that*

$$\begin{aligned} X \preceq_{[s;s-2]-cx} Y &\Leftrightarrow E\phi(X) \le E\phi(Y) \text{ for all } \phi \in \mathcal{U}^{I\!R^+}_{[s;s-2]-cx} = \mathcal{U}^{I\!R^+}_{(s-1)-cx} \cap \mathcal{U}^{I\!R^+}_{s-cx}, \\ &\Leftrightarrow E\phi(X) \le E\phi(Y) \text{ for all } \phi \in \overline{\mathcal{U}}^{I\!R^+}_{[s;s-2]-cx} = \overline{\mathcal{U}}^{I\!R^+}_{(s-1)-cx} \cap \overline{\mathcal{U}}^{I\!R^+}_{s-cx}, \end{aligned}$$

provided the expectations exist.

Proof. The first equivalence is a direct corollary of Proposition 3.3. To prove the second one, it suffices to see that

$$\begin{aligned} X \preceq_{[s;s-2]-cx} Y &\Rightarrow X_{\{s-2\}} \preceq_{[2;0]-cx} Y_{\{s-2\}} \\ &\Rightarrow E\phi^{(s-2)}(X_{\{s-2\}}) \le E\phi^{(s-2)}(Y_{\{s-2\}}) \text{ for all } \phi \in \overline{\mathcal{U}}^{I\!R^+}_{[s;s-2]-cx} \\ &\Rightarrow E\phi(X) \le E\phi(Y) \text{ for all } \phi \in \overline{\mathcal{U}}^{I\!R^+}_{[s;s-2]-cx}, \end{aligned}$$

provided the expectations exist, where the last implication follows from (2.4) with $n = s-2$ together with the fact that $E\psi_{s-2}(X) = E\psi_{s-2}(Y)$. Since $\mathcal{M}^{I\!R^+}_{[s;s-2]-cx} \subset \overline{\mathcal{U}}^{I\!R^+}_{[s;s-2]-cx}$, we finally get the announced equivalence. □

The order $\preceq_{[s;s-2]-cx}$ has been examined by Kaas and Hesselager (1995). In the vein of (3.5), these authors obtained an interesting sufficient condition of crossing-type. Namely, they proved that given two risks X and Y such that $E\psi_k(X) = E\psi_k(Y)$ for $k = 1, 2, \ldots, s-2$ and $E\psi_{s-1}(X) \le E\psi_{s-1}(Y)$,

$$s-2 \le S^-(F_X - F_Y) \le s-1 \Rightarrow X \preceq_{[s,s-2]-cx} Y \tag{3.8}$$

provided the last sign of $F_X - F_Y$ is a "+".

4 Moment spaces

Any comparison of risks in the $\preceq_{s-cx}$- or $\preceq_{[s;s-2]-cx}$-sense requires the equality of some moments. It is therefore clear that the characteristics of moment spaces, i.e. classes of (distribution functions of) random variables sharing the same first moments, will play a central role in the theory of these stochastic order relations.

Let $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ be the moment space associated with the moment sequence $(\mu_1, \mu_2, \ldots, \mu_{s-1})$, i.e. the class of all the (distribution functions of the) risks valued in $[0,b]$ and satisfying the moment constraints $E\psi_k(X) = \mu_k$ for $k = 1, 2, \ldots, s-1$. From (3.1) (resp. (3.6)), it is clearly seen that $\preceq_{s-cx}$ (resp. $\preceq_{[s;s-2]-cx}$) can only be used to compare elements of the same moment space $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ (resp. $\mathcal{B}_{s-1}([0,b];\boldsymbol{\mu})$). It is also interesting to note that in virtue of (3.7), $\preceq_{s-cx}$ and $\preceq_{[s;s-2]-cx}$ are equivalent within $\mathcal{B}_s([0,b];\boldsymbol{\mu})$. Because of the importance of the moment spaces in the theory of the s-convex orders, we recall hereafter some of their basic features. For more details, the reader is referred e.g. to De Vylder (1996, Chapter II.3).

The class $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ is a Krein-Milman space, i.e. it is the closed convex hull of its extremal points. The extreme points of $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ are distribution functions F that cannot be expressed as a linear combination of two distinct elements of $\mathcal{B}_s([0,b];\boldsymbol{\mu})$, i.e. such that

$$F = \pi F_1 + (1-\pi)F_2, \ \ 0 < \pi < 1 \Rightarrow F = F_1 = F_2.$$

Henceforth, we denote as $\mathcal{E}_s([0,b];\boldsymbol{\mu})$ the set of all the extreme points of $\mathcal{B}_s([0,b];\boldsymbol{\mu})$. The extreme points of $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ are those having at most s points of support; $\mathcal{E}_s([0,b];\boldsymbol{\mu})$ is compact. $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ appears thus as the closure of the convex hull of $\mathcal{E}_s([0,b];\boldsymbol{\mu})$, i.e. as the closure of the family of all convex combinations $\pi_1 F_1 + \pi_2 F_2 + \ldots + \pi_n F_n$, $0 \leq \pi_1, \pi_2, \ldots, \pi_n \leq 1$, $\sum_{i=1}^n \pi_i = 1$, of points $F_i \in \mathcal{E}_s([0,b];\boldsymbol{\mu})$.

The elements of $\mathcal{E}_s([0,b];\boldsymbol{\mu})$ play an important role when extrema on $E\phi(X)$ for $X \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$ are desired. Indeed, if ϕ is continuous on $[0,b]$, Taylor (1977) proved that the solutions of the optimization problems

$$X_{\inf} = \arg \inf_{X \in \mathcal{B}_s([0,b];\boldsymbol{\mu})} E\phi(X) \text{ or } X_{\sup} = \arg \sup_{X \in \mathcal{B}_s([0,b];\boldsymbol{\mu})} E\phi(X) \tag{4.1}$$

belongs to $\mathcal{E}_s([0,b];\boldsymbol{\mu})$.

Of course, not all the sequences $(\mu_1, \mu_2, \ldots, \mu_{s-1})$ of real numbers are moment sequences, that is, there does not always exist a random variable X valued in $[0,b]$ such that $E\psi_k(X) = \mu_k$ for $k = 1, 2, \ldots, s-1$. For instance, there are no random variables valued in $[0,10]$ matching $\mu_1 = 2$ and $\mu_2 = 3$. Moreover, some moments sequences correspond to very particular random variables. For instance, $\mu_1 = 2$ and $\mu_2 = 4$ correspond to a random variable equal to 2 almost surely. Therefore, we recall hereafter a classical result about the admissibility of moment sequences; for an elementary proof, the reader is referred for instance to Denuit, Lefèvre and Shaked (1999b). Henceforth, we denote as supp(F) the support of the distribution function F.

Proposition 4.1. **(i)** *Let $(\mu_1, \mu_2, \ldots, \mu_{2s+1})$ be a sequence of non-negative real numbers; $\mathcal{B}_{2s+2}([0,b];\boldsymbol{\mu})$ is not void and contains elements F such that*

$$\#\left(supp(F) - \{0\}\right) \geq s+1 \text{ and } \#\left(supp(F) - \{b\}\right) \geq s+1$$

if, and only if,

$$\| Q_k \| > 0, \ \ \| bP_k - Q_k \| > 0, \ \ \| P_k \| > 0 \text{ and } \| bQ_{k-1} - R_{k-1} \| > 0 \text{ for } k = 0, 1, \ldots, s.$$

(ii) *Let* $(\mu_1, \mu_2, \ldots, \mu_{2s})$ *be a sequence of non-negative real numbers;* $\mathcal{B}_{2s+1}([0, b]; \boldsymbol{\mu})$ *is not void and contains elements* F *such that*

$$\#supp(F) \geq s + 1 \text{ and } \# (supp(F) - \{0, b\}) \geq s$$

if, and only if,

$$\| P_k \| > 0 \text{ and } \| bQ_{k-1} - R_{k-1} \| > 0 \text{ for } k = 0, 1, \ldots, s,$$

and

$$\| Q_k \| > 0 \text{ and } \| bP_k - Q_k \| > 0 \text{ for } k = 0, 1, \ldots, s - 1.$$

To be precise, the strict inequalities in (i) and (ii) determine the topological interior of the set of all the admissible moments sequences. As an illustration, the sequence $(\mu_1, \mu_2, \mu_3, \mu_4)$ of real numbers satisfies the conditions of Proposition 4.1 if, and only if,

$$0 < \mu_1 < b$$

$$\mu_1^2 < \mu_2 < b\mu_1$$

$$\frac{\mu_2^2}{\mu_1} < \mu_3 < \frac{b\mu_2(b - \mu_1) - (b\mu_1 - \mu_2)^2}{b - \mu_1}$$

and

$$\frac{\mu_3^2 + \mu_2^3 - 2\mu_1\mu_2\mu_3}{\mu_2 - \mu_1^2} < \mu_4 < \frac{(b\mu_1 - \mu_2)b\mu_3 - (b\mu_2 - \mu_3)^2}{b\mu_1 - \mu_2}.$$

5 S-convex extrema

In a moment space $\mathcal{B}_s([0, b]; \boldsymbol{\mu})$, the random variables $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ are said to be the s-convex extrema if, and only if, the stochastic inequalities

$$X_{\min}^{(s)} \preceq_{s-cx} X \preceq_{s-cx} X_{\max}^{(s)}$$

hold for all $X \in \mathcal{B}_s([0, b]; \boldsymbol{\mu})$. Given a sequence of moments $(\mu_1, \mu_2, \ldots, \mu_{s-1})$ satisfying Proposition 4.1, the extrema $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ in $\mathcal{B}_s([0, b]; \boldsymbol{\mu})$ exist and are unique, as shown by Denuit, De Vylder and Lefèvre (1999). The s-convex stochastic extrema in $\mathcal{B}_s([0, b]; \boldsymbol{\mu})$ are the elements of $\mathcal{E}_s([0, b]; \boldsymbol{\mu})$ with the minimal number of support points, according to the terms of Proposition 4.1. This assertion follows from (3.5). As a consequence, if you find in $\mathcal{B}_{2s}([0, b]; \boldsymbol{\mu})$ a random variable X with exactly s points of support in $]0, b[$ or with exactly $s + 1$ points of support, namely 0, b and $s - 1$ points in $]0, b[$, then (3.5) must hold for any $Y \in \mathcal{B}_{2s}([0, b]; \boldsymbol{\mu})$. This identifies the nature of the support of the extrema in the $2s$-convex sense. Similarly, if you find in $\mathcal{B}_{2s+1}([0, b]; \boldsymbol{\mu})$ a random variable X with exactly $s + 1$ points of support, namely either 0 or b together with s points in $]0, b[$, then (3.5) must hold for any $Y \in \mathcal{B}_{2s+1}([0, b]; \boldsymbol{\mu})$. As above, this specifies the nature of the support of the extrema in the $(2s + 1)$-convex sense. Such random variables exist when the moments sequence defining $\mathcal{B}_s([0, b]; \boldsymbol{\mu})$ satisfies the conditions of Proposition 4.1. An effective method for deriving these stochastic extrema is given next:

1. if s is even, $s = 2k$ say:

   (a) the support of $X_{\min}^{(2k)}$ contains k interior points $x_1, x_2, \dots, x_k$, $0 < x_1 < x_2 < \dots < x_k < b$, which are the roots of the equation $||\underline{D}_{2k}(x)|| = 0$, where

$$\underline{D}_{2k}(x) = \begin{pmatrix} 1 & x & x^2 & \dots & x^k \\ \mu_0 & \mu_1 & \mu_2 & \dots & \mu_k \\ \mu_1 & \mu_2 & \mu_3 & \dots & \mu_{k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{k-1} & \mu_k & \mu_{k+1} & \dots & \mu_{2k-1} \end{pmatrix};$$

   (b) the support of $X_{\max}^{(2k)}$ contains the points 0, b and $k-1$ interior points, $x_1, x_2, \dots, x_{k-1}$, $0 < x_1 < x_2 < \dots < x_{k-1} < b$, the x_i's being the roots of the equation $||\overline{D}_{2k}(x)|| = 0$, where

$$\overline{D}_{2k}(x) = \begin{pmatrix} 1 & x & x^2 & \dots & x^{k-1} \\ \mu_2 - b\mu_1 & \mu_3 - b\mu_2 & \mu_4 - b\mu_3 & \dots & \mu_{k+1} - b\mu_k \\ \mu_3 - b\mu_2 & \mu_4 - b\mu_3 & \mu_5 - b\mu_4 & \dots & \mu_{k+2} - b\mu_{k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_k - b\mu_{k-1} & \mu_{k+1} - b\mu_k & \mu_{k+2} - b\mu_{k+1} & \dots & \mu_{2k-1} - b\mu_{2k-2} \end{pmatrix};$$

2. if s is odd, $s = 2k+1$, say:

   (a) the support of $X_{\min}^{(2k+1)}$ contains 0 and k interior points $x_1, x_2, \dots, x_k$, $0 < x_1 < x_2 < \dots < x_k$, the x_i's being the roots of the equation $||\underline{D}_{2k+1}(x)|| = 0$, where

$$\underline{D}_{2k+1}(x) = \begin{pmatrix} 1 & x & x^2 & \dots & x^k \\ \mu_1 & \mu_2 & \mu_3 & \dots & \mu_{k+1} \\ \mu_2 & \mu_3 & \mu_4 & \dots & \mu_{k+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_k & \mu_{k+1} & \mu_{k+2} & \dots & \mu_{2k} \end{pmatrix};$$

   (b) the support of $X_{\max}^{(2k+1)}$ contains b and k interior points $x_1, x_2, \dots, x_k$, $0 < x_1 < x_2 < \dots < x_k < b$, the x_i's being the roots of the equation $||\overline{D}_{2k+1}(x)|| = 0$, where

$$\overline{D}_{2k+1}(x) = \begin{pmatrix} 1 & x & x^2 & \dots & x^s \\ \mu_1 - b & \mu_2 - b\mu_1 & \mu_3 - b\mu_2 & \dots & \mu_{k+1} - b\mu_k \\ \mu_2 - b\mu_1 & \mu_3 - b\mu_2 & \mu_4 - b\mu_3 & \dots & \mu_{k+2} - b\mu_{k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_k - b\mu_{k-1} & \mu_{k+1} - b\mu_k & \mu_{k+2} - b\mu_{k+1} & \dots & \mu_{2k} - b\mu_{2k-1} \end{pmatrix}.$$

Explicit expressions of these extrema are available for s up to 5; they are listed in Tables 5.1-5.2, where we use the notation

$$\varrho = \left((\mu_1 - b)(\mu_4 - b\mu_3) - (\mu_2 - b\mu_1)(\mu_3 - b\mu_2)\right)^2 - 4\big((\mu_1 - b)(\mu_3 - b\mu_2) - (\mu_2 - b\mu_1)^2\big)\big((\mu_2 - b\mu_1)(\mu_4 - b\mu_3) - (\mu_3 - b\mu_2)^2\big).$$

s	Support points	Probability masses
1	0	1
2	μ_1	1
3	0 $\frac{\mu_2}{\mu_1}$	$\frac{\mu_2-\mu_1^2}{\mu_2}$ $\frac{\mu_1^2}{\mu_2}$
4	$r_+ = \frac{\mu_3-\mu_1\mu_2+\sqrt{(\mu_3-\mu_1\mu_2)^2-4(\mu_2-\mu_1^2)(\mu_1\mu_3-\mu_2^2)}}{2(\mu_2-\mu_1^2)}$ $r_- = \frac{\mu_3-\mu_1\mu_2-\sqrt{(\mu_3-\mu_1\mu_2)^2-4(\mu_2-\mu_1^2)(\mu_1\mu_3-\mu_2^2)}}{2(\mu_2-\mu_1^2)}$	$\frac{\mu_1-r_-}{r_+-r_-}$ $1-\frac{\mu_1-r_-}{r_+-r_-}$
5	0 $t_+ = \frac{\mu_1\mu_4-\mu_2\mu_3+\sqrt{(\mu_1\mu_4-\mu_2\mu_3)^2-4(\mu_1\mu_3-\mu_2^2)(\mu_2\mu_4-\mu_3^2)}}{2(\mu_1\mu_3-\mu_2^2)}$ $t_- = \frac{\mu_1\mu_4-\mu_2\mu_3-\sqrt{(\mu_1\mu_4-\mu_2\mu_3)^2-4(\mu_1\mu_3-\mu_2^2)(\mu_2\mu_4-\mu_3^2)}}{2(\mu_1\mu_3-\mu_2^2)}$	$1-q_+-q_-$ $q_+ = \frac{\mu_2-t_-\mu_1}{t_+(t_+-t_-)}$ $q_- = \frac{\mu_2-t_+\mu_1}{t_-(t_--t_+)}$

Table 5.1: Probability distribution of $X_{\min}^{(s)} \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$ for $s=1$ to 5.

For higher values of s, one must resort to numerical computations to find the atoms of the s-convex bounds (which boils down to solve a polynomial for its roots).

To end with, it is worth mentioning that the fact that the s-convex stochastic extrema in $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ are the elements of $\mathcal{E}_s([0,b];\boldsymbol{\mu})$ with the minimum number of support points is interesting when compared to Taylor's result (4.1). A possible conjecture could be that imposing further regularity conditions on ϕ (namely higher differentiability) limits the search for the solution of (4.1) to the elements of $\mathcal{E}_s([0,b];\boldsymbol{\mu})$ with less support points. If, as in Taylor's case, ϕ is just continuous then the search for $X_{\inf}$ and $X_{\sup}$ has to be performed over the whole $\mathcal{E}_s([0,b];\boldsymbol{\mu})$.

Now, assume you are given a sequence $(\mu_1,\mu_2,\ldots,\mu_{s-1})$ satisfying the conditions of Proposition 4.1 for some $b>0$. We would like to characterize all the reals ζ such that $(\mu_1,\mu_2,\ldots,\mu_{s-1},\zeta)$ still fulfills the condition of this proposition. An answer is given in the next result.

Proposition 5.1. *Given a sequence of moments $(\mu_1,\mu_2,\ldots,\mu_{s-1})$ that satisfies the conditions of Proposition 4.1 for some $b>0$, the sequence $(\mu_1,\mu_2,\ldots,\mu_{s-1},\zeta)$ still fulfills these conditions if, and only if, $\zeta\in]\underline{\mu}_s,\overline{\mu}_s[$, where*

$$\underline{\mu}_s = E\psi_s\left(X_{\min}^{(s)}\right) \text{ and } \overline{\mu}_s = E\psi_s\left(X_{\max}^{(s)}\right). \tag{5.1}$$

Proof. This condition is obviously necessary since the function ψ_s is s-convex. In order to show that this condition is also sufficient, one possibility consists in showing that for each value of ζ in $]\underline{\mu}_s,\overline{\mu}_s[$, there exist an element X of $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ such that $EX^s=\mu_s$. For this purpose, define X as a mixture with weight α of the stochastic s-convex extrema in $\mathcal{B}_s([0,b];\boldsymbol{\mu})$, i.e.

$$X = \begin{cases} X_{\min}^{(s)} & \text{with probability } \alpha, \\ X_{\max}^{(s)} & \text{with probability } 1-\alpha. \end{cases}$$

s	Support points	Probability masses
1	b	1
2	0 b	$\frac{b-\mu_1}{b}$ $\frac{\mu_1}{b}$
3	$\frac{b\mu_1-\mu_2}{b-\mu_1}$ b	$\frac{(b-\mu_1)^2}{(b-\mu_1)^2+\mu_2-\mu_1^2}$ $\frac{\mu_2-\mu_1^2}{(b-\mu_1)^2+\mu_2-\mu_1^2}$
4	0 $\frac{\mu_3-b\mu_2}{\mu_2-b\mu_1}$ b	$1-p_1-p_2$ $p_1=\frac{(\mu_2-b\mu_1)^3}{(\mu_3-b\mu_2)(\mu_3-2b\mu_2+b^2\mu_1)}$ $p_2=\frac{\mu_1\mu_3-\mu_2^2}{b(\mu_3-2b\mu_2+b^2\mu_1)}$
5	$z_+=\frac{(\mu_1-b)(\mu_4-b\mu_3)-(\mu_2-b\mu_1)(\mu_3-b\mu_2)+\sqrt{\varrho}}{2((\mu_1-b)(\mu_3-b\mu_2)-(\mu_2-b\mu_1)^2)}$ $z_-=\frac{(\mu_1-b)(\mu_4-b\mu_3)-(\mu_2-b\mu_1)(\mu_3-b\mu_2)-\sqrt{\varrho}}{2((\mu_1-b)(\mu_3-b\mu_2)-(\mu_2-b\mu_1)^2)}$ b	$p_+=\frac{\mu_2-(b+z_-)\mu_1+bz_-}{(z_+-z_-)(z_+-b)}$ $p_-=\frac{\mu_2-(b+z_+)\mu_1+bz_+}{(z_--z_+)(z_--b)}$ $1-p_+-p_-$

Table 5.2: Probability distribution of $X_{\max}^{(s)} \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$ for $s=1$ to 5.

Obviously, $X \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$. Now it is easy to determine α in order that $E\psi_s(X)=\zeta$: it suffices to set

$$\alpha = \frac{\overline{\mu}_s-\zeta}{\overline{\mu}_s-\underline{\mu}_s}.$$

To end with, note that if there is an equality in (5.1), then ζ is the sth moment of either $X_{\min}^{(s)}$ or $X_{\max}^{(s)}$. Since for any $X \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$, the implications

$$E\psi_s(X)=\underline{\mu}_s \Rightarrow X =_d X_{\min}^{(s)}$$

and

$$E\psi_s(X)=\overline{\mu}_s \Rightarrow X =_d X_{\max}^{(s)}$$

both hold, the requirement concerning the minimal number of support points in Proposition 4.1 cannot be fulfilled. This ends the proof. □

Let $(\mu_1,\mu_2,\ldots,\mu_s)$ be a moments sequence satisfying the conditions of Proposition 4.1 and consider the stochastic minima $X_{\min}^{(s)}$ and $X_{\min}^{(s+1)}$, in $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ and in $\mathcal{B}_{s+1}([0,b];\boldsymbol{\mu})$, respectively. We would like to know which stochastic order relation, if any, holds between $X_{\min}^{(s)}$ and $X_{\min}^{(s+1)}$. Since $X_{\min}^{(s+1)} \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$, we obviously have that $X_{\min}^{(s)} \preceq_{s-cx} X_{\min}^{(s+1)}$. Of course, $X_{\min}^{(s)}$ and $X_{\min}^{(s+1)}$ cannot be compared in the $(s+1)$-convex sense since $E\psi_s\left(X_{\min}^{(s)}\right) < E\psi_s\left(X_{\min}^{(s+1)}\right)$ in virtue of Proposition 5.1. Nevertheless, we now prove that they can be compared in the $[s+1;s-1]$-convex sense.

Proposition 5.2. *Let $(\mu_1,\mu_2,\ldots,\mu_s)$ be a moment sequence satisfying the conditions of Proposition 4.1. Let $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ be the s-convex extrema in $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ and let $X_{\min}^{(s+1)}$ and $X_{\max}^{(s+1)}$ be the $(s+1)$-convex extrema in $\mathcal{B}_{s+1}([0,b];\boldsymbol{\mu})$. Then, the stochastic inequalities*

$$X_{\min}^{(s)} \preceq_{[s+1;s-1]-cx} X_{\min}^{(s+1)} \text{ and } X_{\max}^{(s+1)} \preceq_{[s+1;s-1]-cx} X_{\max}^{(s)}$$

hold and the support points of $X_{\min}^{(s)}$ (resp. $X_{\max}^{(s)}$) and $X_{\min}^{(s+1)}$ (resp. $X_{\max}^{(s+1)}$) are interwoven.

Proof. Let us show that $X_{\min}^{(2s)} \preceq_{[2s+1;2s-1]-cx} X_{\min}^{(2s+1)}$. First note that

$$S^{-}\left(F_{X_{\min}^{(2s)}} - F_{X_{\min}^{(2s+1)}}\right) \geq 2s-1 \tag{5.2}$$

since $X_{\min}^{(2s)}$ and $X_{\min}^{(2s+1)}$ share the same first $2s-1$ moments. Now, let us denote by $x_1, x_2, \ldots, x_s \in]0,b[$ (resp. 0 and $y_2, \ldots, y_{s+1} \in]0,b[$) the s (resp. $s+1$) support points of $X_{\min}^{(2s)}$ (resp. of $X_{\min}^{(2s+1)}$). It is easily seen, because of the fixed number of support points, that

$$S^{-}\left(F_{X_{\min}^{(2s)}} - F_{X_{\min}^{(2s+1)}}\right) \leq 2s-1 \tag{5.3}$$

whence, by combining (5.2) and (5.3),

$$S^{-}\left(F_{X_{\min}^{(2s)}} - F_{X_{\min}^{(2s+1)}}\right) = 2s-1$$

finally follows. Moreover the atoms then have to satisfy

$$0 < x_1 < y_2 < x_2 < \ldots < x_s < y_{s+1}$$

in order to fulfill requirement (5.2). The announced result finally follows from (3.8). The proof is similar for the other cases and is therefore omitted. □

Let us now examine s-convex extrema in moment spaces differing only from the last elements of their generating moments sequences; an interesting stochastic monotonicity property is obtained in Proposition 5.3.

Proposition 5.3. *Let $(\mu_1, \mu_2, \ldots, \mu_{s-1})$ and $(\nu_1, \nu_2, \ldots, \nu_{s-1})$ be two sequences of moments such that*

(i) $\mu_k = \nu_k$ *for* $k = 1, 2, \ldots, s-2$;

(ii) $\mu_{s-1} \leq \nu_{s-1}$;

(iii) *they both satisfy the conditions of Proposition 4.1.*

Then, if you denote as X_1 and X_2 the stochastic s-convex maxima (or the stochastic s-convex minima) in $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ and in $\mathcal{B}_s([0,b];\boldsymbol{\nu})$, respectively, we have that $X_1 \preceq_{(s-1)-cx} X_2$.

Proof. It suffices to note that the structure of the supports of the X_i's, together with the equality of their first $s-2$ moments yield $S^{-}(F_{X_1} - F_{X_2}) = s-2$ hence the result from (3.5). □

6 S-convex approximations

6.1 Extremal approximations

In order to approximate an element X of a moment space $\mathcal{B}_s([0,b];\boldsymbol{\mu})$, it seems natural, considering the previous sections, to use the s-convex minimum $X_{\min}^{(s)}$ or the s-convex maximum $X_{\max}^{(s)}$. As shown by Denuit, De Vylder and Lefèvre (1999), this provides accurate margins on quantity of actuarial interest that can be cast into $E\phi(X)$, with ϕ s-convex. The approximations of X by either $X_{\min}^{(s)}$ or $X_{\max}^{(s)}$ are referred to as the extremal approximations in the remainder of the paper.

A goodness of fit measure of the extremal approximations can be obtained from Denuit, Lefèvre and Shaked (1999b), where bounds on $P[\inf_i |X - x_i| > \epsilon]$ (where the x_i's represent the atoms of one of the s-convex stochastic extrema and X an arbitrary element of $\mathcal{B}_s([0,b];\boldsymbol{\mu})$) are provided in terms of the moment matrices P_k, Q_k and R_k. Here, we resort on stop-loss distances to evaluate the closeness between $X \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$ and its extremal approximations. For a presentation of probability metrics, we refer the interested reader e.g. to Rachev (1991). In actuarial sciences, stop-loss metrics have been mainly applied to evaluate the distance between the individual and the collective models in risk theory. Nevertheless, this concept seems to possess many other useful applications.

Rachev and Rüschendorf (1990) defined the stop-loss metric of order s as follows: given two risks X and Y, the stop-loss distance of degree s between them, denoted as $d_{s-s\ell}(X,Y)$ is given by

$$d_{s-s\ell}(X,Y) = \frac{1}{s!} \sup_{t \in I\!R^+} |E\psi_{s,t,+}(X) - E\psi_{s,t,+}(Y)| \, ;$$

$d_{0-s\ell}$ is the well-known Kolmogorov distance, while $d_{1-s\ell}$ is the classical stop-loss distance introduced by Gerber (1981).

Lemma 6.1. **(i)** *Let X, Y and Z all belong to $\mathcal{B}_s([0,b];\boldsymbol{\mu})$; we then have that*

$$X \preceq_{s-cx} Y \preceq_{s-cx} Z \Rightarrow d_{s-s\ell}(X,Y) \leq d_{s-s\ell}(X,Z).$$

The latter inequalities still hold when $X \preceq_{[s;s-2]-cx} Y \preceq_{[s;s-2]-cx} Z$.

[ii] Given two risks X and Y with finite sth moments,

$$d_{s-s\ell}(X,Y) < +\infty \Rightarrow E\psi_k(X) = E\psi_k(Y) \text{ for } k = 1,2,\ldots,s-1.$$

Proof. For (i), it suffices to note that the functions $\psi_{s,t,+} \in \overline{\mathcal{U}}_{k-cx}^{I\!R^+}$ for $k = 1,2,\ldots,s+1$, for all t. (ii) is Proposition 2.6(a) in Rachev and Rüschendorf (1990). $\square$

From Lemma 6.1(ii), $d_{s-s\ell}$ seems particularly suitable for working in moment spaces.

Now, let us examine the distance between an element X of $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ and its extremal approximations $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$.

Proposition 6.2. *For any $X \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$,*

$$d_{s-s\ell}(X, X_{\min}^{(s)}) \leq \frac{1}{s!}\left(E\psi_s(X) - \underline{\mu}_s\right)$$

and

$$d_{s-s\ell}(X, X_{\max}^{(s)}) \leq \frac{1}{s!}\left(\overline{\mu}_s - E\psi_s(X)\right).$$

Proof. In virtue of Lemma 2.1,

$$d_{s-s\ell}(X, X_{\min}^{(s)}) = \sup_{t\in I\!R^+}\left(\overline{F}_X^{[s]}(t) - \overline{F}_{X_{\min}^{(s)}}^{[s]}(t)\right).$$

Now, the function

$$t \mapsto \overline{F}_X^{[s]}(t) - \overline{F}_{X_{\min}^{(s)}}^{[s]}(t)$$

is non-increasing, since its first derivative equals

$$\overline{F}_{X_{\min}^{(s)}}^{[s-1]}(t) - \overline{F}_X^{[s-1]}(t) \leq 0 \text{ for all } t \in I\!R^+$$

in virtue of Proposition 3.1. Therefore,

$$\overline{F}_X^{[s]}(t) - \overline{F}_{X_{\min}^{(s)}}^{[s]}(t) \leq \overline{F}_X^{[s]}(0) - \overline{F}_{X_{\min}^{(s)}}^{[s]}(0) = \frac{1}{s!}\left(E\psi_s(X) - \underline{\mu}_s\right),$$

yielding the announced result. The reasoning for $X_{\max}^{(s)}$ is similar. □

If $E\psi_s(X)$ is unknown then for any element X of $\mathcal{B}_s([0,b];\boldsymbol{\mu})$, we deduce from Lemma 6.1(i) that

$$\max\left\{d_{s-s\ell}(X, X_{\min}^{(s)}), d_{s-s\ell}(X_{\max}^{(s)}, X)\right\} \leq \frac{1}{s!}\left(\overline{\mu}_s - \underline{\mu}_s\right).$$

Explicit expressions of $\underline{\mu}_s$ and of $\overline{\mu}_s$ up to 5 are listed in Table 6.1, in the notations of Tables 5.1-5.2. The quantity $\overline{\mu}_s - \underline{\mu}_s$ can be considered as the sth stop-loss width of $\mathcal{B}_s([0,b];\boldsymbol{\mu})$,

s	$\underline{\mu}_s$	$\overline{\mu}_s$
2	μ_1^2	$b\mu_1$
3	$\frac{\mu_2^2}{\mu_1}$	$\left(\frac{b\mu_1-\mu_2}{b-\mu_1}\right)^3 \frac{(b-\mu_1)^2}{(b-\mu_1)^2+\sigma^2} + b^3 \frac{\sigma^2}{(b-\mu_1)^2+\sigma^2}$
4	$\left(1 - \frac{\mu_1-r_-}{r_+-r_-}\right) r_-^4 + \frac{\mu_1-r_-}{r_+-r_-} r_+^4$	$p_1 \left(\frac{\mu_3-b\mu_2}{\mu_2-b\mu_1}\right)^4 + p_2 b^4$
5	$q_+t_+^5 + q_-t_-^5$	$p_+z_+^5 + p_-z_-^5 + (1-p_+-p_-)b^5$

Table 6.1: $\underline{\mu}_s$ and $\overline{\mu}_s$ for $s = 2$ to 5.

6.2 Canonical approximations

Denuit, Lefèvre and Shaked (1999a) noticed that the s-convex stochastic extrema play a central role in the theory of canonical moments. Let $X \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$. The sth canonical moment of X, denoted as c_s, is equal to

$$c_s = \frac{E\psi_s(X) - \underline{\mu}_s}{\overline{\mu}_s - \underline{\mu}_s}.$$

Thus, c_s is simply the position of the sth moment of X relative to its possible range. The recent monograph by Dette and Studden (1997) provides a detailed study of the canonical moments with applications in optimal experimental designs, approximation theory and random walks.

Considering the results given in Subsection 6.1, c_s gives a good indication of the "position" of X in $\mathcal{B}_s([0,b];\boldsymbol{\mu})$ with respect to the extrema $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$. Therefore, we might expect that a satisfactory approximation of $X \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$ is furnished by a convex combination of the stochastic extrema in $\mathcal{B}_{s-1}([0,b];\boldsymbol{\mu})$ with weights dependending on the $(s-1)$th canonical moment c_{s-1}, i.e. we use the mixture

$$\widetilde{X}_s = \begin{cases} X_{\min}^{(s-1)} \text{ with probability } 1-c_{s-1} \\ X_{\max}^{(s-1)} \text{ with probability } c_{s-1} \end{cases} \tag{6.1}$$

in order to approximate X. In the sequel, we refer to $\widetilde{X}_s$ as the sth canonical approximation of X. It is easily seen that $\widetilde{X}_s \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$, so that the stochastic inequalities

$$X_{\min}^{(s)} \preceq_{s-cx} \widetilde{X}_s \preceq_{s-cx} X_{\max}^{(s)}$$

hold, but no s-convex relation in general exists between X and $\widetilde{X}_s$. However, for $s = 4$, 5 or 6, we may reasonably expect X and $\widetilde{X}_s$ to be "close".

It is possible to get an upper bound for the stop-loss distance of degree $s-1$ between X and its sth canonical approximation $\widetilde{X}_s$; this is indicated in Proposition 6.3.

Proposition 6.3. *Let $X \in \mathcal{B}_s([0,b];\boldsymbol{\mu})$ and consider its sth canonical approximation $\widetilde{X}_s$. Then,*

$$d_{(s-1)-s\ell}(X,\widetilde{X}_s) \leq \frac{2}{s!}\frac{(\overline{\mu}_{s-1}-\mu_{s-1})(\mu_{s-1}-\underline{\mu}_{s-1})}{\overline{\mu}_{s-1}-\underline{\mu}_{s-1}}.$$

Proof. we have that

$$\begin{aligned} d_{(s-1)-s\ell}(X,\widetilde{X}_s) &= \frac{1}{s!}\sup_{t\in I\!R^+}\left|E\psi_{s-1,t,+}(X)-(1-c_{s-1})E\psi_{s-1,t,+}(X_{\min}^{(s-1)})-c_{s-1}E\psi_{s-1,t,+}(X_{\max}^{(s-1)})\right| \\ &\leq \frac{1}{s!}(1-c_{s-1})d_{(s-1)-s\ell}(X,X_{\min}^{(s-1)})+\frac{1}{s!}c_{s-1}d_{(s-1)-s\ell}(X,X_{\max}^{(s-1)}). \end{aligned}$$

Now, by Proposition 6.2, we get

$$\begin{aligned} d_{(s-1)-s\ell}(X,\widetilde{X}_s) &\leq \frac{1}{s!}(1-c_{s-1})(\mu_{s-1}-\underline{\mu}_{s-1})+\frac{1}{s!}c_{s-1}(\overline{\mu}_{s-1}-\mu_{s-1}) \\ &= \frac{2}{s!}\frac{(\overline{\mu}_{s-1}-\mu_{s-1})(\mu_{s-1}-\underline{\mu}_{s-1})}{\overline{\mu}_{s-1}-\underline{\mu}_{s-1}}, \end{aligned}$$

which ends the proof. □

7 Numerical illustrations

Let us now consider the example discussed in Denuit, De Vylder and Lefèvre (1999). Assume that X is the remaining lifetime of a x-aged male insured people and let ω be the ultimate age of the life table. Putting $\omega_x \equiv \omega - x$, X is thus a random variable with support $[0, \omega_x]$. Then, $\overline{A}_x \equiv Ev^X$ is the whole life insurance premium, where v be the discount rate corresponding to the yearly interest rate $i > 0$, that is $v = (1+i)^{-1}$. For any $s \in I\!N$, the function

$$\phi_s \ : \ [0, \omega_x] \to [0, (-1)^s] \ ; \ t \mapsto \phi_s(t) = (-1)^s v^t \tag{7.1}$$

belongs to $\mathcal{U}_{s-cx}^{[0,\omega_x]}$ (simply because its derivative of degree s is equal to $(-\ln(v))^s v^t$ which is obviously non-negative). Therefore, lower and upper bounds $\overline{A}_{x;\min}^{(s)}$ and $\overline{A}_{x;\max}^{(s)}$ on the premium $\overline{A}_x$ can be obtained with the aid of the extremal approximations of X. These bounds have been explicitly obtained by Denuit, De Vylder and Lefèvre (1999) for $s = 2, 3, 4, 5$. These are listed in Table 7.1. The canonical approximations $Ev^{\widetilde{X}_s}$ of $\overline{A}_x$ are then easily deduced with the aid of Table 6.1.

s	Lower bound $\overline{A}_{x;\min}^{(s)}$	Upper bound $\overline{A}_{x;\max}^{(s)}$
2	v^{μ_1}	$\frac{\omega_x-\mu_1}{\omega_x} + v^{\omega_x}\frac{\mu_1}{\omega_x}$
3	$v^{\frac{\omega_x\mu_1-\mu_2}{\omega_x-\mu_1}} \frac{(\omega_x-\mu_1)^2}{(\omega_x-\mu_1)^2+\sigma_x{}^2} + v^{\omega_x}\frac{\sigma_x{}^2}{(\omega_x-\mu_1)^2+\sigma_x{}^2}$	$\frac{\sigma_x{}^2}{\mu_2} + v^{\frac{\mu_2}{\mu_1}}\frac{(\mu_1)^2}{\mu_2}$
4	$\left(1-\frac{\mu_1-r_-}{r_+-r_-}\right)v^{r_-} + \frac{\mu_1-r_-}{r_+-r_-}v^{r_+}$	$(1-p_1-p_2) + p_1 v^{\frac{\mu_3-\omega_x\mu_2}{\mu_2-\omega_x\mu_1}} + p_2 v^{\omega_x}$
5	$p_+v^{z_+} + p_-v^{z_-} + (1-p_+-p_-)v^{\omega_x}$	$1 - q_+(1-v^{t_+}) - q_-(1-v^{t_-})$

Table 7.1: Bounds on $\overline{A}_x$ when $X \in \mathcal{B}_s([0,\omega_x];\boldsymbol{\mu})$ for $s = 2$ to 5.

As an example, we plotted the graph of Figure 1 for $i = 3.5\%$. The dotted line represent the extremal approximations $\overline{A}_{x;\min}^{(3)}$ and $\overline{A}_{x;\max}^{(3)}$ and the solid line the canonical approximation $Ev^{\widetilde{X}_4}$; the computations have been made on the basis of a life table relating to Belgian men published by the Belgian National Institute of Statistics in 1992. Details of the numerical values can be found in Table 7.2.

Now, assume you are given a price list and you are interested in the effective interest rate of a whole life insurance policy for an insured person. Assume that the future lifetime L of this individual is described by a mortality table established after a census. The aim is to determine i_* such that $Ev_*^L = \pi$, where π is the amount of premium charged by the insurer.

The idea is to determine the first three moments of L and then to derive the extremal approximations $L_{\min}^{(4)}$ and $L_{\max}^{(4)}$, as well as its canonical approximation $\widetilde{L}_4$. Then, a good approximation for i_* is furnished by the root of the equation

$$E(1+\xi)^{-\widetilde{L}_4} = \pi.$$

A lower bound (resp. upper bound) for i_* is given by the root of the equation

$$E(1+\xi)^{-L_{\min}^{(4)}} = \pi \text{ (resp. } E(1+\xi)^{-L_{\max}^{(4)}} = \pi).$$

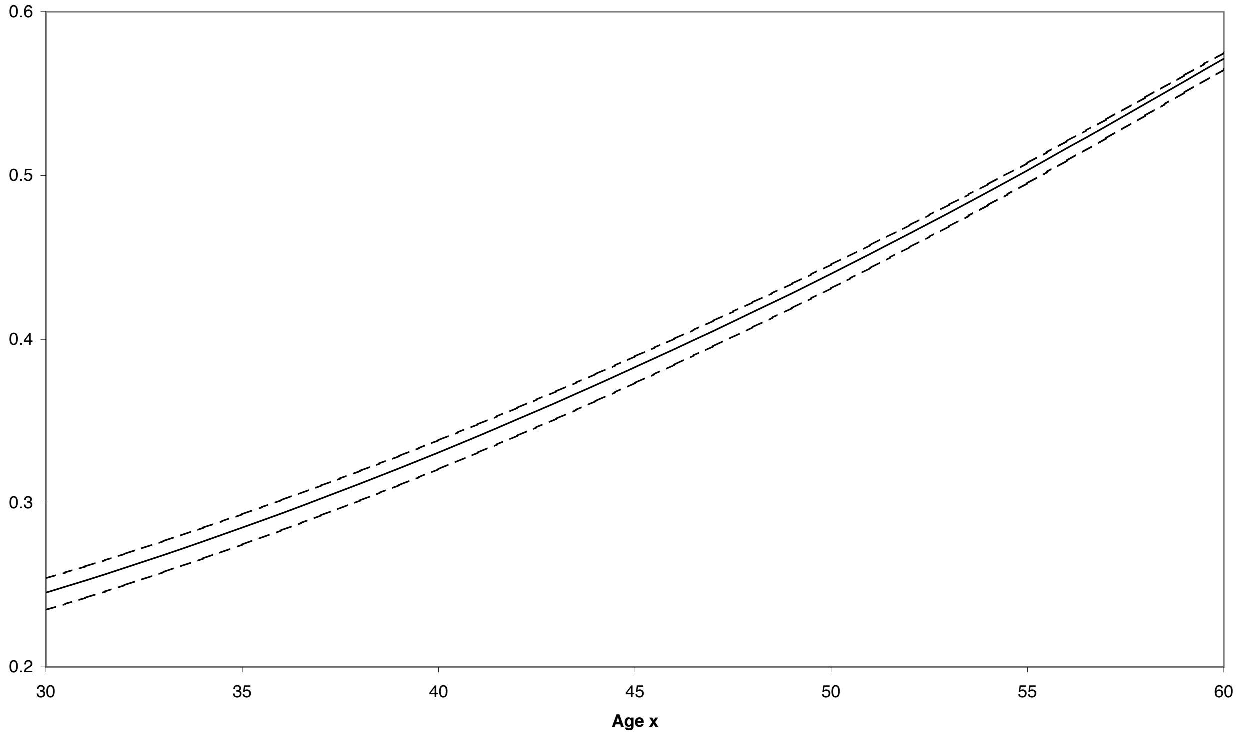

Figure 1: *Extremal approximations* $\overline{A}^{(3)}_{x;\min}$ *and* $\overline{A}^{(3)}_{x;\max}$ *and canonical approximation* $Ev^{\widetilde{X}_4}$.

To end with, let us mention that extremal approximations have been applied by Denuit, Lefèvre and Shaked (1999b) to find bounds for Lundberg's coefficient or certainty equivalences. Canonical bounds can also be used in this context to get accurate approximations for these quantities.

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Age x	$\overline{A}^{(3)}_{x;\min}$	$\overline{A}^{(4)}_{x;\min}$	$Ev^{\widetilde{X}_4}$	$\overline{A}^{(4)}_{x;\max}$	$\overline{A}^{(3)}_{x;\max}$
30	0.23477	0.242526	0.245327	0.24567	0.254149
31	0.242166	0.249917	0.252649	0.252987	0.261357
32	0.249942	0.257736	0.260405	0.26074	0.268994
33	0.257936	0.265765	0.26837	0.268701	0.276834
34	0.266174	0.274038	0.276578	0.276905	0.284912
35	0.274685	0.28259	0.285066	0.285389	0.293264
36	0.2834	0.29133	0.293739	0.294058	0.301797
37	0.292384	0.30034	0.302682	0.302997	0.310593
38	0.301614	0.309589	0.311863	0.312173	0.319622
39	0.311043	0.319017	0.321222	0.321526	0.328825
40	0.320792	0.328775	0.33091	0.33121	0.338351
41	0.330778	0.338759	0.340824	0.341118	0.348097
42	0.34108	0.349062	0.351057	0.351346	0.358157
43	0.351484	0.359426	0.361348	0.36163	0.368271
44	0.362267	0.370186	0.372036	0.372312	0.378777
45	0.373327	0.381217	0.382995	0.383265	0.389549
46	0.384373	0.392175	0.393878	0.39414	0.400243
47	0.395761	0.403481	0.405111	0.405364	0.411282
48	0.407397	0.415026	0.416582	0.416828	0.422557
49	0.419107	0.426608	0.42809	0.428326	0.433867
50	0.431165	0.438545	0.439954	0.440182	0.445531
51	0.443563	0.450827	0.452165	0.452384	0.457539
52	0.456194	0.463331	0.464599	0.464809	0.469767
53	0.468975	0.475965	0.477164	0.477365	0.482126
54	0.482051	0.488894	0.490024	0.490217	0.494779
55	0.495372	0.502063	0.503127	0.503311	0.507671
56	0.509113	0.515666	0.516665	0.516841	0.520997
57	0.522771	0.529151	0.530085	0.530252	0.534207
58	0.536673	0.542879	0.54375	0.543908	0.547661
59	0.550685	0.556704	0.557515	0.557664	0.561215
60	0.564747	0.570566	0.571318	0.571458	0.57481

Table 7.2: *Extremal approximations and canonical approximation of* Ev^X.