Appl Categor Struct
DOI 10.1007/s10485-013-9359-y

Approximate Hagemann–Mitschke Co-operations

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Received: 14 July 2012 / Accepted: 2 May 2013

Abstract We show that varietal techniques based on the existence of operations of a certain arity can be extended to n-permutable categories with binary coproducts. This is achieved via what we call *approximate Hagemann–Mitschke co-operations*, a generalisation of the notion of approximate Mal'tsev co-operation [2]. In particular, we extend characterisation theorems for n-permutable varieties due to J. Hagemann and A. Mitschke [8, 9] to regular categories with binary coproducts.

Keywords Mal'tsev category · n-permutable category · Binary coproduct · Approximate co-operation

Mathematics Subject Classifications (2010) 08C05 · 18C10 · 18B99 · 18E10

Dedicated to George Janelidze on the occasion of his sixtieth birthday

The first author was supported by Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT–Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2013 and grants number PTDC/MAT/120222/2010 and SFRH/BPD/69661/2010. The second author works as *chercheur qualifié* for Fonds de la Recherche Scientifique–FNRS and would like to thank CMUC for its kind hospitality during his stays in Coimbra.

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Published online: 24 December 2013

1 Introduction

A variety of universal algebras is called *n-permutable* when its congruence relations satisfy the *n-permutability condition*: for congruences R and S on an algebra X, the equality $(R, S)_n = (S, R)_n$ holds, where $(R, S)_n = RSRS\cdots$ denotes the composite of n alternating factors R and S. In a categorical context, this notion was first considered by A. Carboni, G. M. Kelly and M. C. Pedicchio in the article [6]. Here an *n-permutable category* is defined as a regular category [1] in which the (effective) equivalence relations satisfy the n-permutability condition.

For a variety $\mathbb{V}$ of universal algebras, it was shown by A. I. Mal'tsev in [12] that 2-permutability of congruences is equivalent to the condition that the theory of $\mathbb{V}$ admits a ternary operation p such that $p(x, y, y) = x$ and $p(x, x, y) = y$. Then $\mathbb{V}$ is called a *Mal'tsev variety* [16] or a 2*-permutable variety* and p a *Mal'tsev operation*. Similarly, for the strictly weaker 3-permutability condition [14], the theory admits ternary operations r and s such that $r(x, y, y) = x$, $r(x, x, y) = s(x, y, y)$ and $s(x, x, y) = y$. More generally, the n-permutability of congruences can be characterised by the existence of ternary operations satisfying suitable equations ([9], see Theorem 2 below) or, equivalently, by the existence of certain $(n + 1)$-ary operations ([7, 9, 15, 17], see Theorem 3 below).

The first aim of this work is to give a categorical version of such ternary (and $(n + 1)$-ary) operations for n-permutable categories. We do this in the context of regular categories with binary coproducts via *approximate Hagemann–Mitschke co-operations* with a certain *approximation* (see Definition 1 and Fig. 2). This method extends D. Bourn and Z. Janelidze's approach to Mal'tsev categories via *approximate Mal'tsev (co-)operations* [2], which makes it possible to lift varietal techniques to the categorical level and obtain general versions of the characterisation theorems for n-permutable varieties mentioned above (Theorems 4 and 5). We believe this aspect of our work gives a good illustration of the strength and generality of D. Bourn and Z. Janelidze's technique. See also [3–5, 10] where the authors further develop their theory of approximate operations.

The second aim of our paper—in fact, the problem which we originally set out to solve—is answering the following question. J. Hagemann discovered a purely categorical characterisation of n-permutable varieties [9]:

Theorem 1 *For any variety $\mathbb{V}$ of universal algebras, the following statements are equivalent:*

1. *the congruence relations of every algebra of $\mathbb{V}$ are n-permutable;*
2. *for $A \in \mathbb{V}$, every reflexive subalgebra R of $A \times A$ satisfies $R^{\mathrm{op}} \leqslant R^{n-1}$;*
3. *for $A \in \mathbb{V}$, every reflexive subalgebra R of $A \times A$ satisfies $R^n \leqslant R^{n-1}$.*

These three conditions make sense in arbitrary regular categories; nevertheless, they are not mentioned in the article [6]. For the proof the authors of [9] refer to the unpublished work [8]. It is indeed not difficult to find a proof which is valid in varieties of algebras (see [13] for part of it) but we failed to produce a categorical argument.

Assuming that binary coproducts exist—in fact, finite copowers suffice—we can mimic the varietal arguments in terms of ternary or $(n + 1)$-arity operations, using approximate Hagemann–Mitschke co-operations instead. This is what we do in the present paper. Thus we obtain a version of the above characterisation theorem, valid in any regular category with binary coproducts (Theorem 6). This, in turn, implies that in an n-permutable category with binary coproducts, any reflexive and transitive relation is symmetric (Corollary 1).

On the other hand, in recent work with Z. Janelidze [11] we prove that Theorem 1 is actually valid in regular categories, independently of the existence of binary coproducts—using a very different approach, since it does not (and can not) involve approximate (co-)operations.

2 Preliminaries

We recall the main definitions and properties known for n-permutable varieties from [9] and for n-permutable categories we follow [6].

2.1 Relations

A category $\mathbb{C}$ with finite limits is called a **regular** category [1] when every kernel pair has a coequaliser and regular epimorphisms are stable under pulling back. In a regular category any morphism $f: A \to B$ can be decomposed into $f = mp$, where p is a regular epimorphism and m is a monomorphism. Regular categories give a suitable context for composing relations.

A **relation** R from A to B is a subobject $\langle r_1, r_2\rangle: R \rightarrowtail A \times B$. The opposite relation, denoted R^{op}, is the relation from B to A determined by the subobject $\langle r_2, r_1\rangle: R \rightarrowtail B \times A$. Given another relation S from B to C, the composite relation of R and S is a relation, denoted SR, from A to C.

Given morphisms $a: X \to A$ and $b: X \to B$, we say that $\langle a, b\rangle$ **belongs** to R when there exists a morphism $\chi: X \to R$ such that $r_1\chi = a$ and $r_2\chi = b$; we write $\langle a, b\rangle \in R$. For any morphism $c: X \to C$, we have $\langle a, c\rangle \in SR$ if and only if there exists a regular epimorphism $\zeta: Z \twoheadrightarrow X$ and a morphism $x: Z \to B$ such that $\langle a\zeta, x\rangle \in R$ and $\langle x, c\zeta\rangle \in S$ (see Proposition 2.1 in [6]). This observation trivially extends to the composite of more than two relations. Moreover, when R' is another relation from A to B, then $R \leqslant R'$ if and only if any pair of morphisms $\langle a, b\rangle$ that belongs to R also belongs to R'.

A relation R from an object A to A is simply called a **relation on** A. We say that R is **reflexive** when $1_A \leqslant R$, **symmetric** when $R^{\mathrm{op}} = R$ and **transitive** when $RR = R$. As usual, a relation R on A is an **equivalence relation** when it is reflexive, symmetric and transitive. In particular, the kernel relation $\langle f_1, f_2\rangle: A \times_B A \rightarrowtail A \times A$ of a morphism $f: A \to B$ is called an **effective** equivalence relation or **congruence**.

2.2 n-Permutable Varieties [9]

A **Mal'tsev** (or 2**-permutable**) variety of universal algebras is such that the composition of congruences is 2-permutable, i.e., $RS = SR$, for any pair of congruences R and S on the same object. 3**-permutable** varieties satisfy the strictly weaker 3-permutability condition: $RSR = SRS$. More generally, for $n \geqslant 2$, n**-permutable** varieties satisfy the n-permutability condition $(R, S)_n = (S, R)_n$, where $(R, S)_n = RSRS\cdots$ denotes the composite of n alternating factors R and S. We write $R^n = (R, R)_n$ for the n-th power of R.

It is well known that an n-permutable variety of universal algebras is characterised by the condition that its theory contains $n-1$ ternary or, equivalently, $n+1$ operations of arity $n+1$ satisfying appropriate identities:

Theorem 2 (Theorem 2 of [9]) *For any variety $\mathbb{V}$ of universal algebras, the following statements are equivalent:*

1. *the congruence relations of every algebra of $\mathbb{V}$ are n-permutable;*
2. *there exist ternary algebraic operations $w_1, \dots, w_{n-1}$ of $\mathbb{V}$ for which the identities*

$$\begin{cases} w_1(x,y,y) = x, \\ w_i(x,x,y) = w_{i+1}(x,y,y), & \textit{for } i \in \{1,\dots,n-2\}, \\ w_{n-1}(x,x,y) = y \end{cases}$$

hold.

Theorem 3 ([9, 15, 17]) *For any variety $\mathbb{V}$ of universal algebras, the following statements are equivalent:*

1. *the congruence relations of every algebra of $\mathbb{V}$ are n-permutable;*
2. *there exist $(n+1)$-ary algebraic operations $v_0, \dots, v_n$ of $\mathbb{V}$ for which the identities*

$$\begin{cases} v_0(x_0,\dots,x_n) = x_0, \\ v_{i-1}(x_0,x_0,x_2,x_2,\dots) = v_i(x_0,x_0,x_2,x_2,\dots), & i \textit{ even}, \\ v_{i-1}(x_0,x_1,x_1,x_3,x_3,\dots) = v_i(x_0,x_1,x_1,x_3,x_3,\dots), & i \textit{ odd}, \\ v_n(x_0,\dots,x_n) = x_n \end{cases}$$

hold.

In particular, a Mal'tsev variety has a **Mal'tsev operation** p which satisfies

$$\begin{cases} p(x,y,y) = x, \\ p(x,x,y) = y. \end{cases}$$

A 3-permutable variety can be characterised by the existence of two ternary operations, r and s, satisfying the left hand side identities

$$\begin{cases} r(x,y,y) = x, \\ r(x,x,y) = s(x,y,y), \\ s(x,x,y) = y \end{cases} \qquad\qquad \begin{cases} p(x,y,y,z) = x, \\ p(x,x,y,y) = q(x,x,y,y), \\ q(x,y,y,z) = z \end{cases}$$

or, equivalently, by the existence of two quaternary operations, p and q, (besides the first and forth projections), such that the identities on the right above hold.

Remark 1 The equivalence between the existence of ternary operations and the existence of quaternary operations is given by the identities

$$p(x,y,z,w) = r(x,y,z) \qquad \text{and} \qquad q(x,y,z,w) = s(y,z,w)$$

on the one hand,

$$r(x,y,z) = p(x,y,z,z) \qquad \text{and} \qquad s(x,y,z) = q(x,x,y,z)$$

on the other. More generally [9], the equivalence between the existence of ternary and the existence of $(n+1)$-ary operations for n-permutable varieties is given by the identities

$$\begin{cases} v_0(x_0,\dots,x_n) = x_0, \\ v_i(x_0,\dots,x_n) = w_i(x_{i-1},x_i,x_{i+1}), & \text{for } i \in \{1,\dots,n-1\}, \\ v_n(x_0,\dots,x_n) = x_n \end{cases}$$

and $w_i(x, y, z) = v_i(\underbrace{x, \ldots, x}_{i}, y, \underbrace{z, \ldots, z}_{n-i})$ for $i \in \{1, \ldots, n-1\}$.

As mentioned in the introduction (Theorem 1), J. Hagemann also obtained alternative characterisations which involve conditions on reflexive relations.

2.3 n-Permutable Categories [6]

A regular category is an n**-permutable category** when the composition of (effective) equivalence relations on a given object is n-permutable: for two (effective) equivalence relations R and S on the same object, we have $(R, S)_n = (S, R)_n$. In fact, it suffices to have one of the inequalities, say $(R, S)_n \leqslant (S, R)_n$. These categories can be characterised by the condition that, for every reflexive relation E, the relation $(E, E^{\mathrm{op}})_{n-1}$ is an equivalence relation or, equivalently, a transitive relation.

3 Main Result

In this section we prove a categorical version of Theorem 2, valid in regular categories with binary coproducts. We shall repeatedly use the techniques from Subsection 2.1 without further mention.

Definition 1 Let $\mathbb{C}$ be a category with binary products. We say that morphisms w_1, $\ldots$, $w_{n-1}\colon X^3 \to A$ are **approximate Hagemann–Mitschke operations** (on X) **with approximation** $\alpha\colon X \to A$ if the diagram in Fig. 1 commutes.

If $(A, \alpha, w_1, \ldots, w_{n-1})$ is the colimit of the outer solid diagram, then $w_1, \ldots, w_{n-1}$ are called **universal** approximate Hagemann–Mitschke operations with approximation α.

The approximation α is uniquely determined by any of its operations w_i since we have $\alpha = w_i\langle 1_X, 1_X, 1_X\rangle$. To say that $w_1, \ldots, w_{n-1}$ are approximate Hagemann–Mitschke operations with approximation α is equivalent to having, for every object W and all morphisms $x_0, \ldots, x_n\colon W \to X$, identities which correspond to those given in Theorem 2.

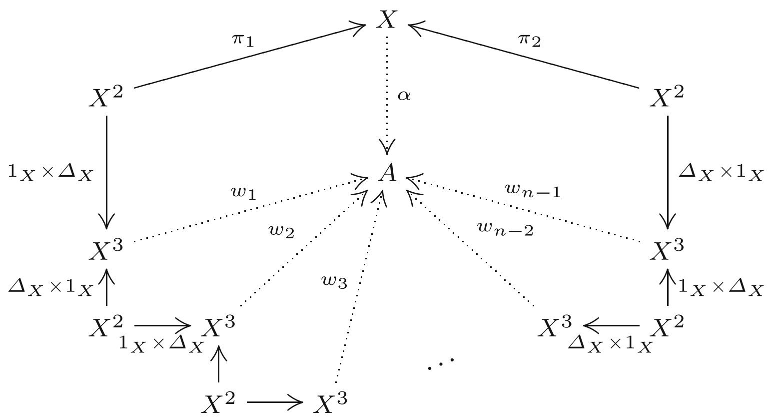

Fig. 1 Approximate Hagemann–Mitschke operations

When $\mathbb{C}$ is a category with binary products and finite colimits, then there always exist universal approximate Hagemann–Mitschke operations given by the colimit of the outer diagram of Fig. 1.

If $A = X$ and $\alpha = 1_X$ then the $w_1, \ldots, w_{n-1}$ are "real" operations on X which provide X with an internal structure. For instance, in Set this means that $(X, w_1, \ldots, w_n)$ is a kind of generalised Mal'tsev algebra.

We work in the dual category $\mathbb{C}^{\mathrm{op}} = \mathbb{X}^{\mathbb{X}}$, where $\mathbb{X}$ is a category with finite limits and binary coproducts. So, (universal) approximate Hagemann–Mitschke operations $w_1, \ldots, w_{n-1} \colon X^3 \to A$ with approximation $\alpha \colon X \to A$ in $\mathbb{C}$ are, in fact, **(universal) approximate Hagemann–Mitschke co-operations** $w_1, \ldots, w_{n-1} \colon A \to 3X$ with approximation $\alpha \colon A \to X$ in $\mathbb{C}^{\mathrm{op}}$. We consider the particular case when $X = 1_{\mathbb{X}}$. Again, universal approximate co-operations always exist: $w_1, \ldots, w_{n-1}$ and α are natural transformations defined, for each object X in $\mathbb{X}$, by the limit of one of the outer solid diagram in Fig. 2.

Now we obtain the claimed categorical version of Theorem 2:

Theorem 4 *Let $\mathbb{X}$ be a regular category with binary coproducts. The following statements are equivalent:*

1. *the approximation $\alpha \colon A \Rightarrow 1_{\mathbb{X}}$ of the universal approximate Hagemann–Mitschke co-operations on $1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*
2. *there exist approximate Hagemann–Mitschke co-operations on $1_{\mathbb{X}}$ such that the approximation $\alpha \colon A \Rightarrow 1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*
3. *$\mathbb{X}$ is an n-permutable category.*

Proof 1. $\Rightarrow$ 2. is obvious.

2. $\Rightarrow$ 3. Let R and S be equivalence relations on an object Y. We must prove that $(R, S)_n \leqslant (S, R)_n$. Let $a, b \colon X \to Y$ be morphisms such that $\langle a, b\rangle \in (R, S)_n$. Then there exists a regular epimorphism $\zeta \colon Z \twoheadrightarrow X$ together with morphisms $x_1, \ldots, x_{n-1} \colon Z \to Y$ such that

$$\langle a\zeta, x_1\rangle \in R,\ \langle x_1, x_2\rangle \in S,\ \ldots,\ \langle x_{n-2}, x_{n-1}\rangle \in S \text{ and } \langle x_{n-1}, b\zeta\rangle \in R,$$

if n is odd. Then the relation S contains the couples $\langle a\zeta, a\zeta\rangle$, $\langle x_1, x_1\rangle$, $\langle x_1, x_2\rangle$, $\langle x_3, x_3\rangle$, $\langle x_3, x_4\rangle$ $\ldots$, $\langle x_{n-4}, x_{n-4}\rangle$, $\langle x_{n-4}, x_{n-3}\rangle$, $\langle x_{n-2}, x_{n-2}\rangle$, $\langle x_{n-2}, x_{n-1}\rangle$ and $\langle b\zeta, b\zeta\rangle$. If we

$$\begin{tikzcd}
 & & & X \arrow[dlll, "\iota_1"'] \arrow[drr, "\iota_2"] & & \\
2X & & & & & 2X \\
 & & & A(X) \arrow[u, dotted, "\alpha_X"'] \arrow[dlll, dotted, "(w_1)_X"'] \arrow[ddll, dotted, "(w_2)_X"] \arrow[dddl, dotted, "(w_3)_X"] \arrow[ddr, dotted, "(w_{n-2})_X"'] \arrow[drr, dotted, "(w_{n-1})_X"] & & \\
3X \arrow[uu, "1_X+\nabla_X"] \arrow[d, "\nabla_X+1_X"'] & & & & & 3X \arrow[uu, "\nabla_X+1_X"'] \arrow[d, "1_X+\nabla_X"] \\
2X & 3X \arrow[l, "1_X+\nabla_X"] \arrow[d] & & & 3X \arrow[r, "\nabla_X+1_X"'] & 2X \\
 & 2X & 3X \arrow[l] & \cdots & &
\end{tikzcd}$$

Fig. 2 Approximate Hagemann–Mitschke co-operations compose the first triple $\langle a\zeta, a\zeta\rangle$, $\langle x_1, x_1\rangle$, $\langle x_1, x_2\rangle$ with $(w_1)_Z$, the second triple $\langle x_1, x_1\rangle$, $\langle x_1, x_2\rangle$, $\langle x_3, x_3\rangle$ with $(w_2)_Z$, and so on, we get

$$\begin{cases} \left\langle a\zeta\alpha_Z, \begin{bmatrix} a\zeta \\ x_1 \\ x_2 \end{bmatrix}(w_1)_Z \right\rangle \in S \\ \left\langle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}(w_2)_Z, \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}(w_3)_Z \right\rangle \in SS^{\mathrm{op}} = S \\ \qquad \vdots \\ \left\langle \begin{bmatrix} x_{n-4} \\ x_{n-3} \\ x_{n-2} \end{bmatrix}(w_{n-3})_Z, \begin{bmatrix} x_{n-3} \\ x_{n-2} \\ x_{n-1} \end{bmatrix}(w_{n-2})_Z \right\rangle \in SS^{\mathrm{op}} = S \\ \left\langle b\zeta\alpha_Z, \begin{bmatrix} x_{n-2} \\ x_{n-1} \\ b\zeta \end{bmatrix}(w_{n-1})_Z \right\rangle \in S \end{cases}$$

because $(\nabla_Z + 1_Z)\,(w_j)_Z = (1_Z + \nabla_Z)\,(w_{j+1})_Z$, for j even, $j \in \{2, \ldots, n-3\}$. Similarly, since $\langle a\zeta, a\zeta\rangle$, $\langle x_1, a\zeta\rangle$, $\langle x_2, x_3\rangle$, $\langle x_3, x_3\rangle$, $\ldots$, $\langle x_{n-3}, x_{n-2}\rangle$, $\langle x_{n-2}, x_{n-2}\rangle$, $\langle x_{n-1}, b\zeta\rangle$ and $\langle b\zeta, b\zeta\rangle$ are all in R, we get

$$\begin{cases} \left\langle \begin{bmatrix} a\zeta \\ x_1 \\ x_2 \end{bmatrix}(w_1)_Z, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}(w_2)_Z \right\rangle \in R^{\mathrm{op}}R = R \\ \qquad \vdots \\ \left\langle \begin{bmatrix} x_{n-3} \\ x_{n-2} \\ x_{n-1} \end{bmatrix}(w_{n-2})_Z, \begin{bmatrix} x_{n-2} \\ x_{n-1} \\ b\zeta \end{bmatrix}(w_{n-1})_Z \right\rangle \in R^{\mathrm{op}}R = R. \end{cases}$$

We can now conclude that $\langle a\zeta\alpha_Z, b\zeta\alpha_Z\rangle \in (S, R)_n$, which implies that $\langle a, b\rangle \in (S, R)_n$, since ζ and α_Z are regular epimorphisms.

For n even the proof is similar. Now we have

$$\langle a\zeta, x_1\rangle \in S, \quad \langle x_1, x_2\rangle \in R, \quad \ldots,$$

$\langle x_{n-2}, x_{n-1}\rangle \in S$ and $\langle x_{n-1}, b\zeta\rangle \in R$ and should consider $\langle a\zeta, a\zeta\rangle$, $\langle x_1, x_1\rangle$, $\langle x_1, x_2\rangle$, $\langle x_3, x_3\rangle$, $\ldots$, $\langle x_{n-3}, x_{n-3}\rangle$, $\langle x_{n-3}, x_{n-2}\rangle$, $\langle x_{n-1}, x_{n-1}\rangle$, $\langle x_{n-1}, b\zeta\rangle \in R$ and $\langle a\zeta, a\zeta\rangle$, $\langle x_1, a\zeta\rangle$, $\langle x_2, x_3\rangle$, $\langle x_3, x_3\rangle$, $\ldots$, $\langle x_{n-3}, x_{n-3}\rangle$, $\langle x_{n-2}, x_{n-1}\rangle$, $\langle x_{n-1}, x_{n-1}\rangle$, $\langle b\zeta, b\zeta\rangle \in S$ and proceed as above.

3. $\Rightarrow$ 1. We must prove that α_X in the limit diagram of Fig. 2 is a regular epimorphism for every object X in $\mathbb{X}$. If n is odd, we take $k = \frac{n+1}{2}$ and let R be the kernel relation of $k\nabla_X$ and S the kernel relation of $1_X + (k-1)\nabla_X + 1_X$, and if n is even, we take R and S to be the respective kernel relations of $1_X + k\nabla_X$ and $k\nabla_X + 1_X$ where $k = \frac{n}{2}$. For the coproduct inclusions $\iota_1, \ldots, \iota_{n+1}\colon X \to (n+1)X$ we have

$$\langle \iota_1, \iota_{n+1}\rangle \in (R, S)_n = (S, R)_n.$$

So there exist a regular epimorphism $\zeta\colon Z \twoheadrightarrow X$ as well as morphisms

$$x_1, \ldots, x_{n-1}\colon Z \to (n+1)X$$

such that

$$\begin{cases} \langle \iota_1\zeta, x_1\rangle \in S,\ \langle x_1, x_2\rangle \in R,\ \langle x_2, x_3\rangle \in S,\ \ldots, & \\ \quad \langle x_{n-2}, x_{n-1}\rangle \in R,\ \langle x_{n-1}, \iota_{n+1}\zeta\rangle \in S, & n \text{ odd} \\ & \\ \langle \iota_1\zeta, x_1\rangle \in R,\ \langle x_1, x_2\rangle \in S,\ \langle x_2, x_3\rangle \in R,\ \ldots, & \\ \quad \langle x_{n-2}, x_{n-1}\rangle \in R,\ \langle x_{n-1}, \iota_{n+1}\zeta\rangle \in S, & n \text{ even.} \end{cases}$$

The regular epimorphism $\zeta\colon Z \twoheadrightarrow X$ and the morphisms

$$((\nabla_i)_X + 1_X + (\nabla_{n-i})_X)\, x_i\colon Z \to 3X, \quad i \in \{1, \ldots, n-1\},$$

give a cone of the outer diagram in Fig. 2: see Remark 1. Here

$$(\nabla_i)_X = [1_X, \ldots, 1_X]^{\mathrm{T}}\colon iX \to X;$$

in particular, $(\nabla_1)_X = 1_X$ and $(\nabla_2)_X = \nabla_X$. This guarantees the existence of a unique morphism $\lambda\colon Z \to A(X)$ such that, in particular, $\zeta = \alpha_X \lambda$. Hence α_X is a regular epimorphism. $\square$

4 Applications

We now extend Theorem 3 to the context of n-permutable categories. As for n-permutable varieties (Remark 1), we also have a correspondence between approximate Hagemann–Mitschke co-operations and certain $(n+1)$-ary co-operations. Similarly, (universal) approximate $(n+1)$-ary co-operations $v_1, \ldots, v_{n-1}$ with approximation β are natural transformations defined, for each object X in $\mathbb{X}$, by the (limit of the outer solid) commutative diagrams in Fig. 3, when n is odd, and in Fig. 4, when n is even.

Theorem 5 *Let $\mathbb{X}$ be a regular category with binary coproducts. The following statements are equivalent:*

1. *the approximation $\beta\colon B \Rightarrow 1_{\mathbb{X}}$ of the universal approximate $(n+1)$-ary co-operations in* Figs. 3 and 4 *is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*
2. *there exist approximate $(n+1)$-ary co-operations as in* Figs. 3 and 4 *such that the approximation $\beta\colon B \Rightarrow 1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*
3. *$\mathbb{X}$ is an n-permutable category.*

Proof It suffices to show that condition 1. is equivalent to condition 1. from Theorem 4. First, we suppose that Fig. 3, for n odd, or Fig. 4, for n even, represents a limit where β_X is a regular epimorphism. For the limit of the outer diagram in Fig. 2, we want to prove that α_X is a regular epimorphism. As in Remark 1, the regular epimorphism $\beta_X\colon B(X) \twoheadrightarrow X$

$$\begin{tikzcd}
 & & & X \arrow[dlll, "\iota_1"'] \arrow[drrr, "\iota_{k+1}"] & & & \\
(k+1)X & & & & & & (k+1)X \\
 & & & B(X) \arrow[uu, dotted, "\beta_X"'] \arrow[dlll, dotted, "(v_1)_X"'] \arrow[ddll, dotted, "(v_2)_X"] \arrow[dddl, dotted, "(v_3)_X"] \arrow[ddrr, dotted, "(v_{n-2})_X"] \arrow[drrr, dotted, "(v_{n-1})_X"] & & & \\
(n+1=2k)X \arrow[uu, "1_X+(k-1)\nabla_X+1_X"] \arrow[d, "k\nabla_X"'] & & & & & & (n+1=2k)X \arrow[uu, "1_X+(k-1)\nabla_X+1_X"'] \arrow[d, "k\nabla_X"] \\
kX & 2kX \arrow[l, "k\nabla_X"] \arrow[d] & & & & 2kX \arrow[r, "k\nabla_X"'] & kX \\
 & (k+1)X & 2kX \arrow[l] & & \cdots & &
\end{tikzcd}$$

Fig. 3 Approximate $(n+1)$-ary co-operation, odd case ($n = 2k-1$ for $k \geqslant 2$)

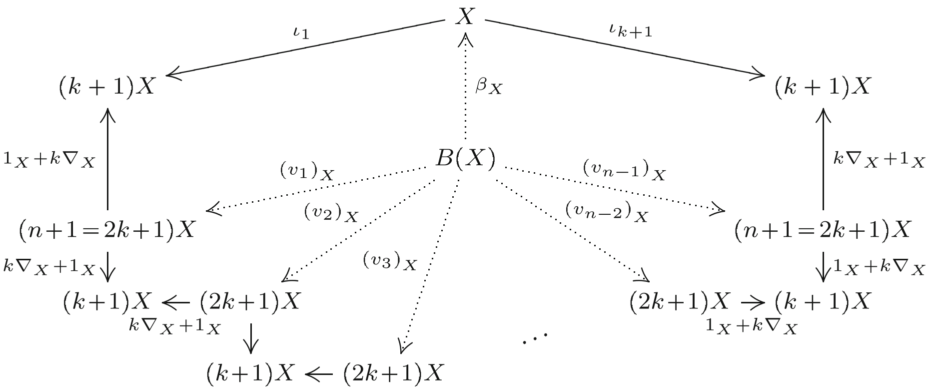

Fig. 4 Approximate $(n+1)$-ary co-operation, even case ($n = 2k$ for $k \geqslant 2$)

and the morphisms

$$((\nabla_i)_X + 1_X + (\nabla_{n-i})_X)\,(v_i)_X \colon B(X) \to 3X, \quad i \in \{1, \dots, n-1\},$$

give another cone on the outer diagram of Fig. 2. Here, once again,

$$(\nabla_i)_X = [1_X, \dots, 1_X]^{\mathrm{T}} \colon iX \to X.$$

This guarantees the existence of a unique morphism $\lambda \colon B(X) \to A(X)$ such that, in particular, $\beta_X = \alpha_X \lambda$. Hence α_X is a regular epimorphism.

Conversely, suppose that Fig. 2 represents a limit where α_X is a regular epimorphism. The regular epimorphism $\alpha_X \colon A(X) \twoheadrightarrow X$ and the morphisms

$$\begin{bmatrix} \iota_i \\ \iota_{i+1} \\ \iota_{i+2} \end{bmatrix} (w_i)_X \colon A(X) \to (n+1)X, \quad i \in \{1, \dots, n-1\},$$

for the coproduct inclusions $\iota_1, \dots, \iota_{n+1} \colon X \to (n+1)X$, give another cone on the outer diagram of Fig. 3, for n odd, or Fig. 4, for n even. Consequently, β_X is a regular epimorphism. □

We end with a categorical version of Theorem 1.

Lemma 1 *Let $\mathbb{X}$ be a regular category such that, for some natural number $n \geqslant 2$, we have $E^n \leqslant E^{n-1}$ for every reflexive relation E. Then $\mathbb{X}$ is $(2n-2)$-permutable.*

Proof Let R and S be equivalence relations on a given object Y. For $E = SR$, we have $(S, R)_{2n} \leqslant (S, R)_{2n-2}$ by assumption. But

$$(R, S)_{2n-2} \leqslant S(R, S)_{2n-2}R = (S, R)_{2n} \leqslant (S, R)_{2n-2},$$

which proves our claim. □

Theorem 6 *Let $\mathbb{X}$ be a regular category with binary coproducts. The following statements are equivalent:*

1. *$\mathbb{X}$ is an n-permutable category;*
2. *for every reflexive relation R, we have $R^{\mathrm{op}} \leqslant R^{n-1}$;*
3. *for every reflexive relation R, we have $R^n \leqslant R^{n-1}$.*

Proof 1. $\Rightarrow$ 2. Let R be a reflexive relation on Y and consider morphisms $x, y \colon X \to Y$ such that $\langle x, y\rangle \in R^{\mathrm{op}}$; hence $\langle y, x\rangle \in R$. Since $\mathbb{X}$ is an n-permutable category, there exist approximate Hagemann–Mitschke co-operations $w_1, \ldots, w_{n-1}$ with approximation α. These are defined, for each object X in $\mathbb{X}$, as in Fig. 2, where α_X is a regular epimorphism. Since R is a reflexive relation, we have $\langle x, x\rangle$, $\langle y, x\rangle$, $\langle y, y\rangle \in R$, so that also

$$\left\langle \begin{bmatrix} x \\ y \\ y \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \end{bmatrix} \right\rangle \in R.$$

Precomposing with each approximate co-operation, we get

$$\begin{cases} \left\langle x\alpha_X, \begin{bmatrix} x \\ x \\ y \end{bmatrix}(w_1)_X \right\rangle \in R \\ \left\langle \begin{bmatrix} x \\ y \\ y \end{bmatrix}(w_2)_X, \begin{bmatrix} x \\ x \\ y \end{bmatrix}(w_2)_X \right\rangle \in R \\ \qquad \vdots \\ \left\langle \begin{bmatrix} x \\ y \\ y \end{bmatrix}(w_{n-2})_X, \begin{bmatrix} x \\ x \\ y \end{bmatrix}(w_{n-2})_X \right\rangle \in R \\ \left\langle \begin{bmatrix} x \\ y \\ y \end{bmatrix}(w_{n-1})_X, y\alpha_X \right\rangle \in R. \end{cases}$$

From $(\nabla_X + 1_X)(w_j)_X = (1_X + \nabla_X)(w_{j+1})_X$, for $j \in \{1, \ldots, n-2\}$, we can conclude that

$$\langle x\alpha_X, y\alpha_X\rangle = \langle x, y\rangle\alpha_X \in R^{n-1}.$$

So $\langle x, y\rangle \in R^{n-1}$, since α_X is a regular epimorphism.

2. $\Rightarrow$ 1. For any object X in $\mathbb{X}$, consider the following reflexive graph and the reflexive relation R on $2X$ which results by taking the (regular epi, mono) factorisation in

$$\begin{tikzcd} 3X \arrow[rr, shift left=2, "\nabla_X + 1_X"] \arrow[rr, shift right=2, "1_X + \nabla_X"'] \arrow[dr, two heads, "\pi"'] & & 2X. \arrow[ll] \arrow[dl] \\ & R \arrow[ur, shift left, "r_1"] \arrow[ur, shift right=2, "r_2"'] & \end{tikzcd}$$

From $\langle \iota_1, \iota_2\rangle \in R$ we get $\langle \iota_2, \iota_1\rangle \in R^{\mathrm{op}} \leqslant R^{n-1}$. So, there exists a regular epimorphism $\zeta \colon Z \twoheadrightarrow X$ together with morphisms $x_1, \ldots, x_{n-2} \colon Z \to 2X$ such that

$$\langle \iota_2\zeta, x_1\rangle,\ \langle x_1, x_2\rangle\ , \ldots ,\ \langle x_{n-3}, x_{n-2}\rangle,\ \langle x_{n-2}, \iota_1\zeta\rangle \in R.$$

Let $k_1, \ldots, k_{n-1} \colon Z \to R$ be the morphisms such that $\langle r_1, r_2\rangle k_1 = \langle \iota_2\zeta, x_1\rangle$, $\langle r_1, r_2\rangle k_i = \langle x_{i-1}, x_i\rangle$, $i \in \{2, \ldots, n-2\}$, and $\langle r_1, r_2\rangle k_{n-1} = \langle x_{n-2}, \iota_1\zeta\rangle$. From the pullback

$$\begin{tikzcd} A(X) \arrow[r, "\langle (w_{n-1})_X, \ldots, (w_1)_X\rangle"] \arrow[d, two heads, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & (3X)^{n-1} \arrow[d, two heads, "\pi^{n-1}"] \\ Z \arrow[r, "\langle k_1, \ldots, k_{n-1}\rangle"'] & R^{n-1} \end{tikzcd}$$

we get morphisms $(w_1)_X, \ldots, (w_{n-1})_X$ and a regular epimorphism defined by $\alpha_X = \zeta\pi'$ such that the diagram in Fig. 2 commutes. Then $\mathbb{X}$ is an n-permutable category by Theorem 4.

1. $\Rightarrow$ 3. Let R be a reflexive relation on Y and consider morphisms $a, b\colon X \to Y$ such that $\langle a, b\rangle \in R^n$. Then there exists a regular epimorphism $\zeta\colon Z \twoheadrightarrow X$ together with morphisms $x_1, \ldots, x_{n-1}\colon Z \to Y$ such that $\langle a\zeta, x_1\rangle$, $\langle x_1, x_2\rangle$, $\ldots$, $\langle x_{n-2}, x_{n-1}\rangle$, $\langle x_{n-1}, b\zeta\rangle \in R$. Since $\mathbb{X}$ is an n-permutable category, there are approximate Hagemann–Mitschke co-operations $w_1, \ldots, w_{n-1}$ with approximation α defined, for each object X in $\mathbb{X}$, as in Fig. 2, where α_X is a regular epimorphism. Since R is a reflexive relation, we have

$$\langle a\zeta, x_1\rangle, \langle x_1, x_1\rangle, \langle x_1, x_2\rangle \in R \Rightarrow \left\langle a\zeta\alpha_X, \left[\begin{smallmatrix} x_1 \\ x_1 \\ x_2 \end{smallmatrix}\right](w_1)_X \right\rangle \in R,$$

$$\langle x_1, x_2\rangle, \langle x_2, x_2\rangle, \langle x_2, x_3\rangle \in R \Rightarrow \left\langle \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_2 \end{smallmatrix}\right](w_2)_X, \left[\begin{smallmatrix} x_2 \\ x_2 \\ x_3 \end{smallmatrix}\right](w_2)_X \right\rangle \in R,$$

$$\vdots$$

$$\langle x_{n-3}, x_{n-2}\rangle, \langle x_{n-2}, x_{n-2}\rangle, \langle x_{n-2}, x_{n-1}\rangle \in R \Rightarrow \left\langle \left[\begin{smallmatrix} x_{n-3} \\ x_{n-2} \\ x_{n-2} \end{smallmatrix}\right](w_{n-2})_X, \left[\begin{smallmatrix} x_{n-2} \\ x_{n-2} \\ x_{n-1} \end{smallmatrix}\right](w_{n-2})_X \right\rangle \in R,$$

$$\langle x_{n-2}, x_{n-1}\rangle, \langle x_{n-1}, x_{n-1}\rangle, \langle x_{n-1}, b\zeta\rangle \in R \Rightarrow \left\langle \left[\begin{smallmatrix} x_{n-2} \\ x_{n-1} \\ x_{n-1} \end{smallmatrix}\right](w_{n-1})_X, b\zeta\alpha_X \right\rangle \in R.$$

From $(\nabla_X + 1_X)(w_j)_X = (1_X + \nabla_X)(w_{j+1})_X$ for $j \in \{1, \ldots, n-2\}$ we conclude that

$$\langle a\zeta\alpha_X, b\zeta\alpha_X\rangle = \langle a, b\rangle\zeta\alpha_X \in R^{n-1},$$

so $\langle a, b\rangle \in R^{n-1}$ since ζ and α_X are regular epimorphisms.

3. $\Rightarrow$ 2. By Lemma 1, we know that $\mathbb{X}$ is $(2n-2)$-permutable. Let R be a reflexive relation. Using the equivalence 1. $\Leftrightarrow$ 2. for $(2n-2)$-permutability, we have $R^{\mathrm{op}} \leqslant R^{2n-3}$. Using our assumption $R^n \leqslant R^{n-1}$ several (in fact, $n-2$) times we obtain

$$R^{\mathrm{op}} \leqslant R^{2n-3} \leqslant R^{2n-2} \leqslant \cdots \leqslant R^n \leqslant R^{n-1}.$$

This finishes the proof. $\square$

Corollary 1 *In an n-permutable category with binary coproducts, any reflexive and transitive relation is symmetric.*

Proof It suffices to combine $R^{\mathrm{op}} \leqslant R^{n-1}$ for R reflexive with $RR \leqslant R$ for R transitive to see that $R^{\mathrm{op}} \leqslant R$. $\square$

Acknowledgments Thanks to the referee for many detailed comments and important suggestions.

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