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Torsion theories and Galois theories in preordered groups

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Abstract

The notion of torsion theory was first introduced in 1966 by S. Dickson in the study of abelian categories. Over the years, many non-abelian settings, such as the homological or, more generally, the normal context, have proven suitable for extending the definition of a torsion theory. In particular, the concept has been widely considered in several semi-abelian categories.

Another area of interest in category theory is categorical Galois theory, developed in the 1990s, mainly by G. Janelidze, with the aim of generalizing the classical Galois theory introduced in the 19th century by E. Galois. Since then, this subject has been studied in different contexts, in particular in the semi-abelian one in which many unexpected results have been obtained.

It turns out that there are interesting categories that are not semi-abelian but yet share some important properties with semi-abelian categories. The category of preordered groups, whose categorical behaviour was examined in 2019 by M. M. Clementino, N. Martins-Ferreira and A. Montoli, is one of them. In this thesis, we continue to explore the categorical properties of preordered groups through the study of torsion theories and categorical Galois theories.

More precisely, we show that there is a torsion theory in the category of preordered groups, which induces a reflection to the subcategory of partially ordered groups. This reflection, in turn, gives rise to a monotone-light factorization system as well as to an "absolute" Galois structure for which we give an explicit description of the coverings (also called central extensions). We prove that the coverings can be classified as internal actions of a Galois groupoid. Moreover, we notice that the category of preordered groups also contains a pretorsion theory. It turns out that all these results are more generally valid in any category of V-groups (under suitable assumptions on the quantale V) hence, in particular, in

the categories of Lawvere metric groups, Lawvere ultrametric groups, probabilistic metric groups, etc.

We then generalize the well-known Galois theory of groups induced by the abelianization functor. To do this, we consider the functor to the subcategory of abelian objects in the category of preordered groups. It turns out that this functor gives rise to an admissible Galois structure for which we provide an algebraic description of central and normal extensions. We prove that both notions do actually coincide in our context, as is the case in the above mentioned setting of groups.

Contents

Introduction

Category theory was introduced in the 1940s by S. Eilenberg and S. Mac Lane in their work on algebraic topology. In [28], they formalized the notion of *natural transformation* which was before then used informally by researchers in algebraic topology. In order to do so, they first had to define the concepts of *functor* and, before that, of *category.* This led to the birth of category theory which is nowadays used not only in algebraic topology but also in many other areas of mathematics. Category theory turns out to be in addition a powerful tool in theoretical computer science. One of the main features of category theory is to take a step back in order to get a broader view of mathematics so that the details no longer matter, while patterns, which in some cases would otherwise not be imagined, start to appear. Instead of comparing the elements of mathematical structures (such as sets, groups, Lie algebras or topological spaces), category theorists compare the structures themselves. More generally, one of their aims consists in understanding how different kinds of structures relate to each other. As an example, one could wonder which mathematical structures "behave" like the abelian groups, in the sense that they share similar properties with abelian groups. One could then ask the same question for other structures such as groups, for instance.

The notion of *abelian category*, introduced by D. Buchsbaum [16] and A. Grothendieck [48], appeared in the 1950s as a context where one can deal with mathematical structures sharing important properties with abelian groups. In other words, an abelian category is a setting in which many crucial properties of abelian groups can be stated and proven in an abstract way. The categories R-Mod of R-modules over any ring R are examples of such categories, as well as the category Ab of abelian groups of course. The context of abelian categories is particularly suitable for the development of homological algebra.

For many years, finding a (broader) setting which, this time, would be an abstraction of the category Grp of groups remained an open problem. Indeed, most classical homological lemmas are valid for groups, or even for Lie algebras, which however do not form abelian categories, making the setting of abelian categories somehow too restrictive. Many attempts were then made to weaken the axioms of the definition of an abelian category. Different lists of axioms were for instance suggested by P. J. Higgins [50], A. G. Kurosh [68], B. Eckmann and P. J. Hilton [25], E. G. Shul'geifer [79], O. Wyler [81], and by R. Fritsch and O. Wyler [36] (to cite only a few), in the study of isomorphism and decomposition theorems. Such lists were also proposed by S. A. Amitsur [1] and E. G. Shul'geifer [78] in radical theory, by A. Fröhlich [37] in (non-abelian) homological algebra, by S. A. Huq [54] in commutator theory, etc. *Semi-abelian categories* were eventually introduced in 2002 by G. Janelidze, L. Márki and W. Tholen in [60], with therefore the idea of treating axiomatically several concepts and properties of groups, Lie algebras and rings. The axioms of the definition of a semi-abelian category, namely the fact that the category has a *zero object* and *binary coproducts* and that it is both *Barr-exact* [2] and *protomodular* [6], indeed offer a suitable context for the study of exact sequences, actions, semi-direct products, commutators, homology, and so on. Some examples of semi-abelian categories are given by the categories Grp of groups, Lie_K of Lie algebras over a field K, Rng of rings, XMod of crossed modules, $\mathsf{Hopf}_{\mathsf{K,coc}}$ of cocommutative Hopf algebras, etc. This axiomatic approach has made it possible to prove new results and also consider new problems relating different areas of algebra.

Semi-abelian categories have now been studied for the last 20 years. In parallel, category theorists have realized that there are other interesting categories which share a lot of important properties with the category Grp but which are not semi-abelian. The category PreOrdGrp of *preordered groups*, for instance, is not semi-abelian because it is neither Barr-exact nor protomodular, but its categorical behaviour in some situations is similar to Grp [21]. A *preordered group* $(G, \leq)$ is a group $G = (G, +, 0)$ endowed with a preorder (i.e. a relation that is both reflexive and transitive) $\leq$ which is compatible with the addition $+$ of the group G in the sense that, if $a \leq c$ and $b \leq d$ for a, b, c and d in G, then $a+b \leq c+d$. This structure should not be confused with that of *internal group in the category of preordered sets* since the inversion of the group G need not be monotone (that is, we do not necessarily have that $-a \leq -b$ for $a \leq b$, with a and b in G).

The first categorical properties of preordered groups were studied in 2019 by M. M. Clementino, N. Martins-Ferreira and A. Montoli. In their article [21], they showed that $\mathsf{PreOrdGrp}$ is pointed (i.e. it has a zero object) and both complete and cocomplete, so that it has in particular binary coproducts. Unfortunately, it is not Barr-exact nor protomodular, hence not semi-abelian. However, the category of preordered groups is *regular* (as is the category Grp of groups) and has the powerful property that any *regular epimorphism* in it is an *effective descent morphism* (in the sense of [63] and [62]), this property being valid in any Barr-exact category. In addition, $\mathsf{PreOrdGrp}$ also shares a crucial property with protomodular categories, namely the fact that any regular epimorphism is a *normal epimorphism*, more precisely the *cokernel* of its *kernel*. The correspondence in $\mathsf{PreOrdGrp}$ between these three types of epimorphisms (which also occurs in Grp) is particularly interesting for the present thesis. With these observations, we then see that, even if the category $\mathsf{PreOrdGrp}$ is not semi-abelian, it is nevertheless “not too far” from the semi-abelian setting.

The general objective of this thesis is twofold:

(1) We first aim to study the preordered groups in themselves in order to better understand them. Indeed, preordered groups are mathematical structures used in different areas of mathematics, for example in K_0-*theory* [39], and in *directed algebraic topology* [45] with applications in fields where a privileged direction appears, such as *concurrent processes* [35], space-time models or biological systems.

(2) The second goal consists in studying, in the category $\mathsf{PreOrdGrp}$ of preordered groups, some notions rather well-known in the semi-abelian setting to see if certain results valid in the semi-abelian context extend or not to $\mathsf{PreOrdGrp}$. More precisely, in this thesis, we are mainly interested in *torsion* and *pretorsion theories*, and in *categorical Galois theory*, of which we give below a quick historical overview.

Torsion theories

The category Ab of abelian groups contains two particular subcategories: the subcategory $\mathsf{Ab}_{\mathsf{t.}}$ of *torsion abelian groups* (whose objects are abelian groups $A = (A, +, 0)$ such that, for any $a \in A$, there exists a non-zero natural n satisfying the identity $na = 0$) and the subcategory $\mathsf{Ab}_{\mathsf{t.f.}}$ of *torsion-free abelian groups* (whose objects are abelian groups $A = (A, +, 0)$ such that, if $na = 0$ for some element $a \in A$ and some non-zero natural n, then $a = 0$). These two subcategories are special in the sense that they are involved in the following two properties which have been fundamental in the study of the abelian groups.

(1) If $f\colon X \to Y$ is a group homomorphism such that $X \in \mathsf{Ab}_{\mathsf{t.}}$ and $Y \in \mathsf{Ab}_{\mathsf{t.f.}}$, then $f = 0$. Indeed, if $x \in X$, then there exists a non-zero natural n such that $nx = 0$, so that $nf(x) = f(nx) = f(0) = 0$, hence $f(x) = 0$.

(2) For any abelian group A, there exists a *short exact sequence*

$$0 \longrightarrow T \xrightarrow{\epsilon_A} A \xrightarrow{\eta_A} F \longrightarrow 0$$

with $T \in \mathsf{Ab}_{\mathsf{t.}}$ and $F \in \mathsf{Ab}_{\mathsf{t.f.}}$. It suffices to take $T = T(A)$, where

$$T(A) = \{a \in A \mid \exists n \in \mathbb{N}_* \text{ such that } na = 0\},$$

and $F = A/T(A)$. Then, the above sequence, with ϵ_A the inclusion and η_A the canonical quotient, is clearly exact. Moreover, the subgroup $T(A)$ is by definition a torsion abelian group and it is easy to check that $A/T(A)$ is torsion-free. Indeed, if we have $n\bar{a} = \bar{0}$ for $n \in \mathbb{N}_*$ and $\bar{a} \in A/T(A)$, then $\overline{na} = \bar{0}$, which means that $na \in T(A)$, that is, there exists a non-zero natural m such that $m(na) = 0$. In other words, $(mn)a = 0$ for $mn \in \mathbb{N}_*$, and this implies that $a \in T(A)$, hence $\bar{a} = \bar{0}$ in $A/T(A)$.

As already mentioned above, these two properties are fundamental. More precisely, they allow to prove many (other) interesting properties on abelian groups, involving torsion and torsion-free abelian groups. In 1966, S. Dickson published an important article [23] whose goal was to prove that all these nice properties still hold in any "suitable" abelian category. With this objective in mind, Dickson was therefore led to define the notion of *torsion theory* in the abelian setting: if $\mathscr{C}$ is an abelian category, a *torsion theory* is a pair $(\mathscr{T}, \mathscr{F})$ of *full* and *replete* subcategories of $\mathscr{C}$ satisfying two axioms which correspond to a generalization of the two properties explained above in the case $\mathscr{C} = \mathsf{Ab}$, $\mathscr{T} = \mathsf{Ab}_{\mathsf{t.}}$ and $\mathscr{F} = \mathsf{Ab}_{\mathsf{t.f.}}$.

The *homological* setting [6] (a non-abelian setting where most homological lemmas hold) was then found to be suitable for extending the definition of a torsion theory (see [14]). It soon turned out that this was further the case for the broader context of *normal categories* (in the sense of [66]). A different approach was also developed in [20]. Of course, when restricting to abelian categories, we find the classical definition of Dickson in both situations. Over the years, torsion theories have then been considered in several non-abelian pointed contexts (see [40, 44, 64], for instance, among the many existing references). Connections with other categorical concepts, such as *radicals*, *closure operators* and *factorization systems*, have also been established. Recently, the definition of a torsion theory has finally been extended to a non-necessarily pointed setting, to give the concept of *pretorsion theory* [32, 34]. The idea of this generalization, due to A. Facchini and C. Finocchiaro, consists in replacing, in the definition of a torsion theory (and in the subsequent properties), the zero object with a class $\mathscr{Z}$ of objects, called *$\mathscr{Z}$-trivial objects*, which somehow plays the role of the zero object. Further results in the context of pretorsion theories have recently been obtained in [7–10, 33, 82].

Categorical Galois theory

Classical Galois theory was introduced in the 19th century by E. Galois in order to answer the following question: is it possible to solve *by radicals* an equation of degree 5 or more? In other words, does there exist a formula for expressing the roots of a polynomial equation of degree 5 or more in terms of the coefficients of the polynomial, by using only algebraic operations (such as addition, subtraction, multiplication and division) and the square roots, cube roots, etc.? The answer to this question was already known since antiquity for the equations of degree 1 and 2. Formulas had also been discovered from the 16th century (by R. Bombelli, P. Ruffini, N. Abel, and other mathematicians) for equations of degree 3 and 4. The question of the existence of similar formulas for equations of degree 5 or more, though, remained an open problem until the beginning of the 19th century.

Galois proved the impossibility of solving by radicals a general equation of degree 5 or more and found some methods allowing to give, when possible, the solutions of such an equation. His work [38] remained unpublished several years after his death until 1846, thanks to the additional explanations provided by J. Liouville. As things progressed, mathematicians began to realize that the methods used by Galois turned out to be more interesting than the solution to the initial problem. Indeed, the new approach developed by Galois confirmed the importance of complex numbers and led to the beginning of group theory as a discipline. The main result of classical Galois theory is the *classical Galois theorem* whose statement is the following: the subgroups G of the *Galois group* $\mathsf{Gal}[L : K]$ of a given finite dimensional Galois extension $K \subseteq L$ classify the intermediate field extensions $K \subseteq M \subseteq L$.

In the 1960s, A. Grothendieck then generalized the classical Galois theorem to unitary commutative algebras [49]. The *Galois correspondence* involved in this generalization implicitly relies on the adjunction

$$\begin{tikzcd} \mathsf{K\text{-}Alg}^{\mathsf{op}} \arrow[r, bend left, "I"] \arrow[r, phantom, "\perp"] & \mathsf{Set}_{\mathsf{f}} \arrow[l, bend left, "H"] \end{tikzcd} \tag{1}$$

between the dual $\mathsf{K\text{-}Alg}^{\mathsf{op}}$ of the category of finite dimensional unitary commutative K-algebras, and the category $\mathsf{Set}_{\mathsf{f}}$ of finite sets. Indeed, the validity of the correspondence comes, among other things, from the *admissibility* of the *Galois structure* arising from the above adjunction. The *Galois theorem "of Grothendieck"* was later extended to arbitrary Galois extensions of fields (not necessarily of finite dimension), leading to the *infinitary Galois theorem "of Grothendieck"*.

In 1974, A. Magid wrote an article [71] in which he replaced the ground field K with a commutative ring R. In this paper, he developed a Galois theory for commutative rings, including its own version of the Galois theorem. This actually was the first step of a much more general theory.

Categorical Galois theory, developed in the 1990s by G. Janelidze [55] (and later in collaboration with other mathematicians), is stated in a categorical context suitable for the generalization of the classical Galois theory. By "suitable context", we mean a setting containing an *admissible Galois structure*, i.e. an adjunction satisfying some extra conditions (such as the adjunction (1)). In this situation, the *categorical Galois theorem* generalizes all the previously mentioned versions. Surprisingly, it turned out that categorical Galois theory also has deep connections with other parts of mathematics such as central extensions of groups and covering spaces for instance.

The book [11] by F. Borceux and G. Janelidze, considered as one of the main references in this field, explains in details the above listed steps, from classical to categorical Galois theory.

Thesis outline

In Chapter 1, we recall the basic notions needed for this thesis. This chapter is mainly intended for non-specialists so that the experienced category theorist could just skip this part and proceed immediately to the next chapter, unless unfamiliar with the notion of *effective descent morphism* discussed in Section 1.3.

Chapter 2 is devoted to the reminders of more advanced notions. More precisely, we explain the different concepts appearing in the title of this thesis and state several properties useful for the next sections. Among other things, we give the definition of a *preordered group* as well as some interesting properties of the category $\mathsf{PreOrdGrp}$ of preordered groups, we state the formal definition of a *torsion theory* in the normal setting together with that of a *pretorsion theory*, we explain everything one needs to know about *categorical Galois theory* for the purposes of this thesis, and finally we give an overview of *factorization systems* and their links with torsion theories and categorical Galois theory.

The third chapter is then dedicated to the study of torsion theories and *coverings* in the category $\mathsf{PreOrdGrp}$ of preordered groups. In particular, we show that there is a torsion theory in $\mathsf{PreOrdGrp}$ which gives rise to a *(normal epi)-reflection.* This reflection itself induces a *monotone-light* factorization system from which a precise characterization of the coverings (with respect to the induced reflection) is obtained. It is then possible to prove that these coverings can be classified in terms of *internal actions* of a *Galois groupoid.* Finally, we emphasize the fact that the category $\mathsf{PreOrdGrp}$ also contains a pretorsion theory whose torsion-free subcategory is the same as for the torsion theory.

In Chapter 4, we generalize the content of Chapter 3 to any category V-Grp of V*-groups* (up to some additional assumptions on the *quantale* V). We prove that all the results developed in the previous chapter can be extended to this broader context, with the exception of some results very specific to preordered groups.

The aim of Chapter 5 consists in extending to preordered groups the well-known Galois theory of groups, induced by the *abelianization functor* $ab\colon \mathsf{Grp} \to \mathsf{Ab}$. For this, we first consider the functor $C\colon \mathsf{PreOrdGrp} \to \mathsf{PreOrdAb}$ from $\mathsf{PreOrdGrp}$ to its full subcategory $\mathsf{PreOrdAb}$ of *preordered abelian groups*, whose objects are the preordered groups for which the underlying group is abelian. We prove that it is a reflector that gives rise to an *admissible Galois structure* Γ_C, and provide an explicit description of the Γ_C-normal extensions. We are next interested in the functor $F\colon \mathsf{PreOrdGrp} \to \mathsf{Mono(Ab)}$ from $\mathsf{PreOrdGrp}$ to its full subcategory

$\mathsf{Mono}(\mathsf{Ab})$ of preordered abelian groups such that the preorder is moreover an equivalence relation. This functor is indeed the generalization of the abelianization functor since the subcategory $\mathsf{Mono}(\mathsf{Ab})$ is the subcategory of *abelian objects* in $\mathsf{PreOrdGrp}$. It can be shown that the functor F is a reflector giving rise to an admissible Galois structure Γ. Eventually, we show that the Γ-*central* and Γ-*normal extensions* coincide (as is the case for the well-known Galois structure involving the abelianization functor) and give a purely algebraic description of them, as desired.

Personal contribution

The original content of this dissertation can be found in Chapters 3, 4 and 5, and is based on the three articles [41], [74] and [42] (respectively). The results in Appendix A, although unpublished, are also original.

1 | Background: basic notions

In this background chapter, we explain some essential notions that will be needed in order to understand the content of this dissertation. The material of the first two sections is basic and fundamental, and is intended for non-specialists. The third and last section is a bit more advanced, even if the concepts explained therein are often well-known to specialists.

1.1. Category theory: the basics

For more details on the basic notions that will be presented in this section, we recommend the celebrated book by S. Mac Lane [70] or the handbook by F. Borceux [3], or any introductory course in category theory.

1.1.1. Categories, functors and natural transformations

Let us start with the definition of a *category*:

Definition 1.1.1. A *category* $\mathscr{C}$ is given by

- a class $\mathscr{C}_0$ whose elements are called *objects* and are denoted with capital letters X, Y, etc;
- for any pair of objects $X, Y \in \mathscr{C}_0$, a class $\mathscr{C}(X, Y)$ whose elements are called *arrows* (or *morphisms*) and are denoted by $X \xrightarrow{f} Y$;
- for any object $X \in \mathscr{C}_0$, an arrow $1_X \in \mathscr{C}(X, X)$ called *identity* on X;

- for any triple of objects $X, Y, Z \in \mathscr{C}_0$, a function

  $$\circ\colon \mathscr{C}(X,Y) \times \mathscr{C}(Y,Z) \to \mathscr{C}(X,Z); (f,g) \mapsto g \circ f$$

  called *composition*;

such that the following two axioms hold:

(1) *Identity axiom*: for any arrow $f \in \mathscr{C}(X,Y)$, $f \circ 1_X = f = 1_Y \circ f$, i.e. the diagram below commutes.

$$\begin{tikzcd}
X \arrow[r, equal, "1_X"] \arrow[dr, "f"'] & X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[dl, equal, "1_Y"] \\
& Y &
\end{tikzcd}$$

(2) *Associativity axiom*: for any triple (f,g,h) of arrows in $\mathscr{C}$ such that $f \in \mathscr{C}(X,Y)$, $g \in \mathscr{C}(Y,Z)$ and $h \in \mathscr{C}(Z,W)$, we have that $(h \circ g) \circ f = h \circ (g \circ f)$, i.e. the diagram below commutes.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rrr, bend left, "(h\circ g)\circ f"] \arrow[rr, bend right, "g\circ f"'] \arrow[rrr, bend right=40, "h\circ(g\circ f)"'] & Y \arrow[r, "g"] \arrow[rr, bend left, "h\circ g"] & Z \arrow[r, "h"] & W
\end{tikzcd}$$

The most obvious examples are categories where the objects are sets endowed with a mathematical structure, and the arrows are *structure-preserving* functions:

- The category Set whose objects are sets and whose arrows are functions between sets. In this case, the composition is simply given by the usual composition of functions.
- The category Grp whose objects are groups and whose arrows are group homomorphisms. In this case again, the composition is just the usual composition of homomorphisms.
- The category Top whose objects are topological spaces and whose arrows are continuous maps, with the usual composition.

- The category $\mathsf{Vect}_{\mathsf{K}}$ whose objects are vector spaces over a field K and whose arrows are linear maps, with the usual composition.
- $\cdots$

Note that there are also some other examples of a different nature. For instance, any preordered set $(X, \leq)$ can be seen as a category where the objects are the elements $a, b, c \cdots$ of the set X and where there is an arrow from a to b whenever $a \leq b$ in X.

Notation 1.1.2. In literature, the composition is quite often written with the symbol $\cdot$ instead of $\circ$. This is the notation that we will use in this dissertation. When there is no risk of confusion, we will also write "$X \in \mathscr{C}$" instead of "$X \in \mathscr{C}_0$".

We can now consider the "morphisms between categories", which are called *functors*.

Definition 1.1.3. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A *functor* $F \colon \mathscr{C} \to \mathscr{D}$ is given by

- a function $F_0 \colon \mathscr{C}_0 \to \mathscr{D}_0$;
- for any pair of objects X and Y in $\mathscr{C}_0$, a function

$$F_{X,Y} \colon \mathscr{C}(X,Y) \to \mathscr{D}(F_0(X), F_0(Y)); f \mapsto F_{X,Y}(f)$$

such that

(1) for any $X \in \mathscr{C}_0$, $F_{X,X}(1_X) = 1_{F_0(X)}$;
(2) for any pair of morphisms $f \colon X \to Y$ and $g \colon Y \to Z$ in $\mathscr{C}$,

$$F_{X,Z}(g \cdot f) = F_{Y,Z}(g) \cdot F_{X,Y}(f).$$

Example 1.1.4.

(1) The *abelianization functor* $ab \colon \mathsf{Grp} \to \mathsf{Ab}$ is a functor between the categories Grp of groups and Ab of abelian groups. It sends any group G to the quotient $G/[G,G]$ of G by its derived subgroup $[G,G]$ (which is an abelian group by definition). The functor ab associates with any group morphism $f \colon G \to H$ the group morphism $ab(f) \colon G/[G,G] \to H/[H,H]$ defined, for any $[x]_G \in G/[G,G]$, by $ab(f)([x]_G) = [f(x)]_H$.

(2) We also have what is called the *forgetful functor* $u\colon \mathsf{Ab} \to \mathsf{Grp}$ which "forgets" the commutative structure of abelian groups.

(3) Of course, we can consider the *identity functor* $1_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ on any category $\mathscr{C}$.

Remark 1.1.5. It is possible to compose two functors when the domain-category of one of them coincides with the codomain-category of the other one. The identity for this composition is then given by the identity functor. We therefrom get the category Cat: its objects are all the (*small*) categories and its arrows are given by the functors between them.

Let us finally introduce the "morphisms between functors", namely the *natural transformations*:

Definition 1.1.6. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, and $F, G\colon \mathscr{C} \rightrightarrows \mathscr{D}$ two parallel functors between $\mathscr{C}$ and $\mathscr{D}$. A *natural transformation* $\alpha\colon F \to G$ is given by a family

$$\alpha = \{\alpha_X\colon F(X) \to G(X)\}_{X \in \mathscr{C}}$$

of arrows in $\mathscr{D}$ which is natural, i.e., for any arrow $f\colon X \to Y$ in $\mathscr{C}$, the following *naturality square* commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"'] & G(Y). \end{tikzcd}$$

For any $X \in \mathscr{C}$, the arrow $\alpha_X\colon F(X) \to G(X)$ is called the X*-component* of α.

Example 1.1.7. Consider the functors introduced in Example 1.1.4. Then, we have a natural transformation $\eta\colon 1_{\mathsf{Grp}} \to u \cdot ab$.

1.1.2. Limits and colimits

We now explain some constructions that we can do in categories. Let $\mathscr{C}$ be an arbitrary category.

Definition 1.1.8. The *binary product* of two objects X and Y in $\mathscr{C}$ is given by an object $X \times Y$ as well as two arrows $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ such that the *universal property of binary products* holds, that is, for any object $Z \in \mathscr{C}$ and any arrows $f \colon Z \to X$ and $g \colon Z \to Y$, there exists a unique arrow $\phi \colon Z \to X \times Y$, usually denoted by $\langle f, g \rangle$, making the following diagram commute:

$$\begin{tikzcd}
 & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & Y \\
 & Z. \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, dotted, "\phi=\langle f,g\rangle"] &
\end{tikzcd}$$

In the category Grp of groups, the (categorical) product of two groups is simply given by their cartesian product endowed with the induced group structure.

We also have the dual notion (in category theory, *dual* means that the direction of arrows is reversed):

Definition 1.1.9. The *binary coproduct* of two objects X and Y in $\mathscr{C}$ is given by an object $X + Y$ as well as two arrows $\iota_1 \colon X \to X + Y$ and $\iota_2 \colon Y \to X + Y$ such that the *universal property of binary coproducts* holds, that is, for any object $Z \in \mathscr{C}$ and any arrows $f \colon X \to Z$ and $g \colon Y \to Z$, there exists a unique arrow $\phi \colon X + Y \to Z$, usually denoted by $[f, g]$, making the following diagram commute:

$$\begin{tikzcd}
 & X + Y \arrow[dd, dotted, "\phi=[f,g]"] & \\
X \arrow[ur, "\iota_1"] \arrow[dr, "f"'] & & Y \arrow[ul, "\iota_2"'] \arrow[dl, "g"] \\
 & Z. &
\end{tikzcd}$$

In Grp, the coproduct corresponds to the "free product".

Remark 1.1.10. It is of course possible to define in a similar way n-ary (co)products and, more generally, infinite (co)products.

Let us now state the definition of an *equalizer* as well as of its dual, the *coequalizer*:

Definition 1.1.11. The *equalizer* of two parallel arrows $f, g\colon X \rightrightarrows Y$ in $\mathscr{C}$ is given by an object E as well as an arrow $e\colon E \to X$ such that

- $f \cdot e = g \cdot e$;
- the *universal property of equalizers* holds, that is, for any object A in $\mathscr{C}$ and any arrow $\alpha\colon A \to X$ such that $f \cdot \alpha = g \cdot \alpha$, there exists a unique arrow $\phi\colon A \to E$ making the following diagram commute:

$$\begin{tikzcd} E \arrow[rr, "e"] & & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ & A. \arrow[ul, dotted, "\phi"] \arrow[ur, "\alpha"'] & & \end{tikzcd}$$

In Grp, the object part of the equalizer of the group homomorphisms $f, g\colon X \rightrightarrows Y$ is given by the set $E = \{x \in X \mid f(x) = g(x)\}$ endowed with the group structure induced by X. The arrow part e of the equalizer is then just the inclusion $E \hookrightarrow X$.

Definition 1.1.12. The *coequalizer* of two parallel arrows $f, g\colon X \rightrightarrows Y$ in $\mathscr{C}$ is given by an object Q as well as an arrow $q\colon Y \to Q$ such that

- $q \cdot f = q \cdot g$;
- the *universal property of coequalizers* holds, that is, for any object A in $\mathscr{C}$ and any arrow $\alpha\colon Y \to A$ such that $\alpha \cdot f = \alpha \cdot g$, there exists a unique arrow $\phi\colon Q \to A$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[rr, "q"] \arrow[dr, "\alpha"'] & & Q \arrow[dl, dotted, "\phi"] \\ & & A. & \end{tikzcd}$$

We next have the notion of *pullback*:

Definition 1.1.13. Let $f\colon X \to Y$ and $g\colon Z \to Y$ be two arrows in $\mathscr{C}$. The *pullback* of f and g is given by an object $X \times_Y Z$ as well as two arrows $p_1\colon X \times_Y Z \to X$ and $p_2\colon X \times_Y Z \to Z$ such that

- $f \cdot p_1 = g \cdot p_2$;
- the *universal property of pullbacks* holds, that is, for any object A in $\mathscr{C}$ and any pair of arrows $\alpha\colon A \to X$ and $\beta\colon A \to Z$ such that $f \cdot \alpha = g \cdot \beta$, there exists a unique arrow $\phi\colon A \to X \times_Y Z$, usually denoted by $\langle \alpha, \beta \rangle$, making the following diagram commute:

$$\begin{tikzcd}
A \arrow[dr, dotted, "\phi=\langle\alpha,\beta\rangle"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& X \times_Y Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & Z \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd} \tag{1.1}$$

In the category Grp of groups, the object part of the pullback of two group homomorphisms $f\colon X \to Y$ and $g\colon Z \to Y$ is given by the group $X \times_Y Z = \{(x, z) \in X \times Z \mid f(x) = g(z)\}$ and the two morphisms p_1 and p_2 are, in this case, the two projections defined, for any $(x, z) \in X \times_Y Z$, by $p_1(x, z) = x$ and $p_2(x, z) = z$.

We can define in a similar way the concept of *pushout* which is the dual of the notion of pullback.

Remark 1.1.14. The pullback of an arrow $f\colon X \to Y$ with itself is called the *kernel pair* of f and is denoted by $\mathsf{Eq}(f)$. Note that any kernel pair is an *internal equivalence relation* (see Definition 1.2.4 and Example 1.2.5).

Let us now introduce two special kinds of objects (dual of each other):

Definition 1.1.15.

- An *initial object* in $\mathscr{C}$ is an object $I \in \mathscr{C}$ such that, for any object $X \in \mathscr{C}$, there is a unique arrow $I \to X$.
- A *terminal object* in $\mathscr{C}$ is an object $T \in \mathscr{C}$ such that, for any object $X \in \mathscr{C}$, there is a unique arrow $X \to T$.

In Grp, the initial and terminal objects are both isomorphic to the one-element group $\{e\}$, so that the two notions coincide. In the category

Set of sets though, they are distinct. The empty set $\emptyset$ is initial but not terminal while any singleton set $\{*\}$ is terminal but not initial. Note that the empty set $\emptyset$ is the unique initial object in Set.

Definition 1.1.16. A category $\mathscr{C}$ is *pointed* when there is an object 0, called the *zero object*, which is both initial and terminal.
In this situation, if an arrow $f\colon X \to Y$ in $\mathscr{C}$ factors through the zero object 0, f is then called the *zero morphism*, written $0\colon X \to Y$:

$$\begin{tikzcd} X \arrow[rr, "0"] \arrow[dr] & & Y \\ & 0. \arrow[ur] & \end{tikzcd}$$

Thanks to the observation above, one can easily deduce that the category Grp is pointed while the category Set is not.

In the context of a pointed category $\mathscr{C}$, we can then consider the notion of *kernel*:

Definition 1.1.17. The *kernel* of an arrow $f\colon X \to Y$ in a pointed category $\mathscr{C}$ is given by the equalizer of f and the zero morphism $0\colon X \to Y$.

Of course, there exists also the dual notion of *cokernel*. The object part of the kernel (respectively of the cokernel) of an arrow $f\colon X \to Y$ will be denoted by $\mathsf{Ker}(f)$ (respectively by $\mathsf{Coker}(f)$). For the arrow part of the kernel (respectively of the cokernel) of f, we will use the notation $\mathsf{ker}(f)$ (respectively $\mathsf{coker}(f)$).

The object part of the kernel of a morphism $f\colon X \to Y$ in the category Grp of groups is given by the set $\mathsf{Ker}(f) = \{x \in X \mid f(x) = 0\}$ endowed with the group structure induced by X. The arrow part $\mathsf{ker}(f)$ of the kernel of f is then just the inclusion morphism $\mathsf{Ker}(f) \hookrightarrow X$.

In the pointed setting, we therefore have a natural notion of *short exact sequence*:

Definition 1.1.18. A *short exact sequence* in a pointed category $\mathscr{C}$ is given by a pair $(k\colon K \to X, q\colon X \to Q)$ of composable arrows in $\mathscr{C}$ such that $k = \mathsf{ker}(q)$ and $q = \mathsf{coker}(k)$. This situation is usually depicted as follows:

$$0 \longrightarrow K \xrightarrow{k} X \xrightarrow{q} Q \longrightarrow 0.$$

Remark 1.1.19.

(1) The kernel (respectively the cokernel) of f can be seen as the pullback (respectively the pushout) of f and 0.
(2) Note also that the binary product of two objects X and Y in $\mathscr{C}$ can be constructed as a pullback: it is the pullback $X \times_T Y$ over a terminal object T of the (unique) arrows $X \to T$ and $Y \to T$. In particular, the binary product of X and Y in a pointed category is the pullback $X \times_0 Y$ over the zero object 0.

Products, equalizers, pullbacks, terminal objects and kernels are examples of the more abstract notion of *limit* (whose definition will not be needed for this thesis). Dually, coproducts, coequalizers, pushouts, initial objects and cokernels are examples of *colimits*.

Definition 1.1.20.

- A category $\mathscr{C}$ is said to be *complete* (respectively *cocomplete*) when it has all *small* limits (respectively all *small* colimits).
- A category $\mathscr{C}$ is said to be *finitely complete* (respectively *finitely cocomplete*) when it has all *finite* limits (respectively all *finite* colimits).

For example, the categories Grp and Set are both complete and cocomplete, while the category Fld of fields is neither finitely complete nor finitely cocomplete (it has neither products nor coproducts).

1.1.3. Some special arrows in categories

Let $\mathscr{C}$ be an arbitrary category. Let us define some special kinds of arrows in $\mathscr{C}$.

Definition 1.1.21.

- An arrow $f\colon X \to Y$ in $\mathscr{C}$ is a *split monomorphism* when there exists another arrow $g\colon Y \to X$ such that the left-hand diagram below commutes.
- An arrow $f\colon X \to Y$ in $\mathscr{C}$ is a *split epimorphism* when there exists another arrow $g\colon Y \to X$ such that the right-hand diagram below commutes.

$$
\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "f"'] & & X \\
& Y \arrow[ur, "g"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y \arrow[rr, equal] \arrow[dr, "g"'] & & Y \\
& X \arrow[ur, "f"'] &
\end{tikzcd}
$$

We now have the weaker notions of *monomorphism* and *epimorphism*:

Definition 1.1.22.

- The arrow $f\colon X \to Y$ in $\mathscr{C}$ is a *monomorphism* when, for any pair of arrows $h, k\colon Z \rightrightarrows X$ such that $f \cdot h = f \cdot k$, we have $h = k$.
- The arrow $f\colon X \to Y$ in $\mathscr{C}$ is an *epimorphism* when, for any pair of arrows $h, k\colon Y \rightrightarrows Z$ such that $h \cdot f = k \cdot f$, we have $h = k$.

Definition 1.1.23. An arrow $f\colon X \to Y$ in $\mathscr{C}$ is an *isomorphism* when it is both a split monomorphism and a split epimorphism.

Note that we can easily prove the following:

Proposition 1.1.24. *An arrow $f\colon X \to Y$ in $\mathscr{C}$ is an isomorphism if and only if it is both a split monomorphism and an epimorphism or, equivalently, if and only if it is both a split epimorphism and a monomorphism.*

In the category Set of sets, every monomorphism $X \to Y$ is split unless X is the empty set and Y is not, and split epimorphisms coincide with epimorphisms if and only if the *Axiom of Choice* holds. The isomorphisms in Set are then the bijections. In the category Grp of groups however, monomorphisms (i.e. injective group homomorphisms) and epimorphisms (i.e. surjective group homomorphisms) are in general not split! Nevertheless, an isomorphism in Grp is given by a bijective group homomorphism.

Proposition 1.1.25. *An arrow $f\colon X \to Y$ in $\mathscr{C}$ is an isomorphism if and only if it is both a regular epimorphism and a monomorphism.*

Note also the following important property about isomorphisms:

Proposition 1.1.26. *Isomorphisms are* pullback-stable*, that is, if in the pullback* (1.1) *the arrow f is an isomorphism, then so is p_2.*

It turns out that there is the same stability property for monomorphisms.

Proposition 1.1.27. *Monomorphisms are* pullback-stable*, that is, if in the pullback* (1.1) *the arrow f is a monomorphism, then so is p_2.*

Let us continue with some special types of monomorphisms and epimorphisms.

Definition 1.1.28.

- An arrow $f\colon X \to Y$ in $\mathscr{C}$ is a *normal monomorphism* (respectively a *normal epimorphism*) when it is the kernel (respectively the cokernel) of some arrow in $\mathscr{C}$.
- An arrow $f\colon X \to Y$ in $\mathscr{C}$ is a *regular monomorphism* (respectively a *regular epimorphism*) when it is the equalizer (respectively the coequalizer) of a pair of arrows in $\mathscr{C}$.

Definition 1.1.29. An arrow $f\colon X \to Y$ in $\mathscr{C}$ is a *strong epimorphism* when, given any commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[dl, dotted, "\phi"] \\ Z \arrow[r, tail, "m"'] & W \end{tikzcd}$$

where $m\colon Z \rightarrowtail W$ is a monomorphism, there exists a unique arrow $\phi\colon Y \to Z$ such that $\phi \cdot f = g$ and $m \cdot \phi = h$.

Definition 1.1.30.

- A monomorphism $f\colon X \to Y$ in $\mathscr{C}$ is an *extremal monomorphism* when the following property holds: if $f = \phi \cdot e$ for some arbitrary arrow ϕ and some epimorphism e in $\mathscr{C}$, then e is an isomorphism.

- An epimorphism $f\colon X \to Y$ in $\mathscr{C}$ is an *extremal epimorphism* when the following property holds: if $f = m \cdot \phi$ for some arbitrary arrow ϕ and some monomorphism m in $\mathscr{C}$, then m is an isomorphism.

Proposition 1.1.31. *In a category $\mathscr{C}$ with binary products, we have the following chains of inclusions:*

$$\{\textit{split epimorphisms}\} \subseteq \{\textit{regular epimorphisms}\} \subseteq \{\textit{strong epimorphisms}\} \subseteq \{\textit{extremal epimorphisms}\} \subseteq \{\textit{epimorphisms}\}$$

and

$$\{\textit{split monomorphisms}\} \subseteq \{\textit{regular monomorphisms}\} \subseteq \{\textit{monomorphisms}\}.$$

In the category Grp of groups, the four notions of regular epimorphism, strong epimorphism, extremal epimorphism and epimorphism coincide. These are the surjective group homomorphisms. More generally, regular, strong and extremal epimorphisms are the same in any *regular category* (see Definition 1.2.1).

1.1.4. Isomorphisms and equivalences of categories

We first present the definition of an *isomorphism of categories* which says that two categories are actually "the same":

Definition 1.1.32. An *isomorphism between two categories* $\mathscr{C}$ and $\mathscr{D}$ is given by two functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ such that $F \cdot G = 1_{\mathscr{D}}$ and $G \cdot F = 1_{\mathscr{C}}$.

It is possible to characterize the isomorphisms of categories. In order to do this, we start by defining different kinds of functors.

Definition 1.1.33. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor.

- F is *injective on objects* when, for any pair X, Y of objects in $\mathscr{C}$ such that $F(X) = F(Y)$, we have $X = Y$.
- F is *surjective on objects* when, for any object D in $\mathscr{D}$, there exists an object C in $\mathscr{C}$ such that $F(C) = D$.

- F is *bijective on objects* when it is both injective and surjective on objects.
- F is *essentially surjective on objects* when, for any object D in $\mathscr{D}$, there exists an object C in $\mathscr{C}$ such that $F(C)$ is isomorphic to D: $F(C) \cong D$.
- F is *faithful* when, for any pair $f, g \colon X \rightrightarrows Y$ of arrows in $\mathscr{C}$ such that $F(f) = F(g)$, we have $f = g$.
- F is *full* when, for any arrow $h \colon F(X) \to F(Y)$ in $\mathscr{D}$, there exists an arrow $f \colon X \to Y$ in $\mathscr{C}$ such that $F(f) = h$.
- F is *fully faithful* when it is both full and faithful.

Proposition 1.1.34. *An isomorphism between two categories $\mathscr{C}$ and $\mathscr{D}$ is given by a functor $F \colon \mathscr{C} \to \mathscr{D}$ which is bijective on objects and fully faithful.*

We will see in Chapter 2 an example of isomorphism of categories. More precisely, we will see that the category of *preordered groups* is isomorphic to another category (which will be specified in Proposition 2.1.5).

Nevertheless, the notion of isomorphism of categories is not so common in category theory; it is actually quite rare. A concept that appears more often is the weaker notion of *equivalence of categories* whose definition is presented below:

Definition 1.1.35. An *equivalence between two categories* $\mathscr{C}$ and $\mathscr{D}$ is given by a functor $F \colon \mathscr{C} \to \mathscr{D}$ which is essentially surjective on objects and fully faithful.

1.1.5. Adjunctions and reflections

We now introduce a notion that will be central in this thesis.

Definition 1.1.36. An *adjunction* between two categories $\mathscr{C}$ and $\mathscr{D}$ is given by

- a functor $F \colon \mathscr{C} \to \mathscr{D}$;
- a functor $G \colon \mathscr{D} \to \mathscr{C}$;

- a natural transformation $\eta\colon 1_{\mathscr{C}} \to G \cdot F$ (called the *unit* of the adjunction) which is *universal*, that is, for all objects $C \in \mathscr{C}$ and $D \in \mathscr{D}$, and any arrow $f\colon C \to G(D)$, there exists a unique arrow $\hat{f}\colon F(C) \to D$ making the following diagram commute:

$$\begin{tikzcd} C \arrow[rr, "\eta_C"] \arrow[dr, "f"'] & & GF(C) \arrow[dl, "G(\hat{f})"] \\ & G(D). & \end{tikzcd}$$

The functor F is then called a *left adjoint* and G a *right adjoint*.

Notation 1.1.37. When there is an adjunction between two categories $\mathscr{C}$ and $\mathscr{D}$ as in Definition 1.1.36, the following notation is generally used:

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D}. \arrow[l, bend left, "G"] \end{tikzcd} \tag{1.2}$$

Remark 1.1.38. There exist other equivalent definitions for the concept of adjunctions. One of them is the dual of Definition 1.1.36 and involves a natural transformation $\epsilon\colon F \cdot G \to 1_{\mathscr{D}}$ (called the *counit* of the adjunction) which is universal in a sense dual to what is stated in Definition 1.1.36. The two natural transformations η and ϵ are linked by the so-called *triangular identities*: for any $C \in \mathscr{C}$, $\epsilon_{F(C)} \cdot F(\eta_C) = 1_{F(C)}$ and, for any $D \in \mathscr{D}$, $G(\epsilon_D) \cdot \eta_{G(D)} = 1_{G(D)}$.

We have the following interesting result involving the counit of an adjunction:

Proposition 1.1.39. *Consider an adjunction as in* (1.2) *with unit* η *and counit* ϵ*. Then* ϵ *is an isomorphism (i.e. any of its components is an isomorphism) if and only if the right adjoint* G *is fully faithful.*

Let us also mention the fact that right adjoints preserve limits:

Proposition 1.1.40. *Consider an adjunction as in* (1.2)*. Then the right adjoint* G *preserves limits.*

For instance, if we consider the pullback square (1.1) in the category $\mathscr{D}$, then, when we apply the right adjoint G, we get a new pullback in the category $\mathscr{C}$. Left adjoints though do not preserve limits in general, but they preserve the colimits that exist in $\mathscr{C}$.

There is a particular type of adjunction, called *reflection*:

Definition 1.1.41. A *reflection* is an adjunction as in (1.2) where the right adjoint in an inclusion functor (that is, $\mathscr{D}$ is a subcategory of $\mathscr{C}$). In this situation, the subcategory $\mathscr{D}$ is said to be *reflective* and the functor F is called a *reflector*.

Recall that a *subcategory* $\mathscr{S}$ of a category $\mathscr{C}$ is a category whose class of objects is a subclass of the class $\mathscr{C}_0$ of objects of $\mathscr{C}$ and, for any $X, Y \in \mathscr{S}_0$, the class of arrows $\mathscr{S}(X,Y)$ from X to Y is a subclass of $\mathscr{C}(X,Y)$. The category $\mathscr{S}$ has the same composition law and the same identities as $\mathscr{C}$.

Remark 1.1.42. We also have the dual notion of *coreflection* (which requires the left adjoint to be an inclusion functor).

Example 1.1.43. There is a reflection between the categories Grp of groups and Ab of abelian groups

$$\begin{tikzcd} \mathsf{Grp} \arrow[r, bend left, "ab"] \arrow[r, phantom, "\perp"] & \mathsf{Ab} \arrow[l, bend left, "u"] \end{tikzcd} \tag{1.3}$$

where ab is the abelianization functor and u the forgetful functor. The unit of this adjunction is given by the natural transformation $\eta \colon 1_{\mathsf{Grp}} \to u \cdot ab$ of Example 1.1.7 whose X-component $\eta_X \colon X \twoheadrightarrow (u \cdot ab)(X) = X/[X,X]$ is the canonical quotient. Let us show that this is indeed an adjunction. Let $X \in \mathsf{Grp}$, $Y \in \mathsf{Ab}$ and $f \colon X \to u(Y) = Y$ be any group morphism.

$$\begin{tikzcd} X \arrow[rr, two heads, "\eta_X"] \arrow[dr, "f"'] & & X/[X,X] \arrow[dl, dotted, "\hat{f}"] \\ & Y & \end{tikzcd}$$

Let $z \in X/[X,X]$. By surjectivity of η_X, there exists an x in X such $\eta_X(x) = z$. Let us then define $\hat{f}$ as follows: $\hat{f}(z) = f(x)$. It is well-defined.

Indeed, if we have $x' \in X$ such that $\eta_X(x') = z$, then $\eta_X(x - x') = 0$ which means that $x - x' \in \mathsf{Ker}(\eta_X) = [X, X]$. This implies that

$$x - x' = x_1 + x_2 - x_1 - x_2 + x_3 + x_4 - x_3 - x_4 + \cdots + x_{n-1} + x_n - x_{n-1} - x_n,$$

for some $x_i \in X$, so that

$$\begin{aligned} f(x - x') = f(x_1) + f(x_2) - f(x_1) - f(x_2) + \cdots \\ + f(x_{n-1}) + f(x_n) - f(x_{n-1}) - f(x_n) = 0 \end{aligned}$$

since $f(x_i) \in Y$, for any $i \in \{1, 2, \cdots, n\}$, with Y an abelian group. Accordingly, $f(x') = f(x)$, which proves that $\hat{f}$ is well-defined. It is then clear that, defined in this way, it is a group morphism and that the above diagram commutes. The uniqueness of $\hat{f}$ such that $\hat{f} \cdot \eta_X = f$ comes from the fact that η_X is an epimorphism and u a faithful functor. This completes the proof that there is a reflection between the categories Grp and Ab.

1.1.6. Some useful properties in categories

Let us now see some well-known and often used properties in category theory. The proof of the first property is basic since it only uses the definition of a pullback.

Proposition 1.1.44. *Consider the following commutative diagram:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & C \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C'. \end{tikzcd}$$

(1) If the squares (1) *and* (2) *are pullbacks, then the external rectangle* $(1) + (2)$ *is also a pullback.*

(2) If the external rectangle $(1) + (2)$ *and the square* (2) *are pullbacks, then the square* (1) *is also a pullback.*

Thanks to Proposition 1.1.44 above and Proposition 1.1.26, it is therefore easy to show the following:

Proposition 1.1.45. *Let $\mathscr{C}$ be a category with kernels. If the square*

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & D \end{array}$$

is a pullback in $\mathscr{C}$, then there is an isomorphism between the kernel of f and the kernel of f' (and, symmetrically, between the kernel of g and the kernel of g'): $\mathsf{Ker}(f) \cong \mathsf{Ker}(f')$ (and, symmetrically, $\mathsf{Ker}(g) \cong \mathsf{Ker}(g')$).

Remark 1.1.46. The converse of Proposition 1.1.45 does not hold in general but is still valid in categories where the *Short Five Lemma* holds, such as the category Grp of groups.

Proposition 1.1.47. *The projections of a pullback are* jointly monomorphic*, that is, in a pullback diagram like* (1.1)*, for any pair $\alpha, \beta \colon W \rightrightarrows X \times_Y Z$ of parallel arrows such that $p_1 \cdot \alpha = p_1 \cdot \beta$ and $p_2 \cdot \alpha = p_2 \cdot \beta$, we have $\alpha = \beta$.*

This property follows directly from the universal property of pullbacks.

Proposition 1.1.48. *The projections of a pullback are* jointly extremally monomorphic*, that is, in a pullback diagram like* (1.1)*, if $p_1 = \alpha \cdot e$ and $p_2 = \beta \cdot e$ for arbitrary arrows α and β and some epimorphism e, then e is an isomorphism.*

This again is a consequence of the universal property of pullbacks, as well as of Propositions 1.1.47 and 1.1.24.

1.2. Some special kinds of categories

In this section, we define different types of categories.

1.2.1. Regular categories

Let us first introduce the *regular categories* (in the sense of M. Barr [2]). More results and details are available in [4], for example.

Definition 1.2.1. A category $\mathscr{C}$ is *regular* when

- it is finitely complete;
- the kernel pair $f_1, f_2 \colon \mathsf{Eq}(f) \rightrightarrows X$ of any arrow $f \colon X \to Y$ in $\mathscr{C}$ has a coequalizer;
- regular epimorphisms in $\mathscr{C}$ are *pullback-stable*: in the pullback diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

  if f is a regular epimorphism, then so is p_2.

The following easy criterion allows one to verify that a given category is regular:

Proposition 1.2.2. *Let $\mathscr{C}$ be a finitely complete category. Then $\mathscr{C}$ is regular if and only if*

- *any arrow in $\mathscr{C}$ has a* (regular epi, mono)-factorization *(unique up to isomorphism): for any arrow $f \colon X \to Y$ in $\mathscr{C}$, there exist a regular epimorphism $e \colon X \twoheadrightarrow W$ as well as a monomorphism $m \colon W \rightarrowtail Y$ such that $f = m \cdot e$;*
- *regular epimorphisms are pullback-stable.*

The categories Set of sets, Grp of groups, Mon of monoids, Ab of abelian groups and R-Mod of modules over a ring R are some examples of regular categories. The category Top of topological spaces is not regular (see Counterexample 2.4.5 in [4]).

Let us also mention the following important property valid in any regular category:

Proposition 1.2.3. *In any regular category, regular, strong and extremal epimorphisms coincide.*

1.2.2. (Barr-)exact categories

Let $\mathscr{C}$ be a category with pullbacks.

Definition 1.2.4. An *internal equivalence relation* is a diagram

$$\begin{tikzcd} R \times_X R \arrow[r, shift left=2, "p_1"] \arrow[r, "\tau" description] \arrow[r, shift right=2, "p_2"'] & R \arrow[loop above, "\sigma"] \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "\Delta" description] \end{tikzcd} \tag{1.4}$$

in $\mathscr{C}$, where $(R \times_X R, p_1, p_2)$ is defined by the following pullback

$$\begin{tikzcd} R \times_X R \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"'] & X, \end{tikzcd}$$

where the arrows r_1 and r_2 are jointly monomorphic, and where the following identities hold:

(R) $r_1 \cdot \Delta = 1_X = r_2 \cdot \Delta$;

(S) $r_1 \cdot \sigma = r_2$ and $r_2 \cdot \sigma = r_1$;

(T) $r_1 \cdot \tau = r_1 \cdot p_1$ and $r_2 \cdot \tau = r_2 \cdot p_2$.

Example 1.2.5.

(1) An internal equivalence relation in Set is nothing but an equivalence relation in the usual sense. The three identities (R), (S) and (T) are then a generalization of the axioms of reflexivity, symmetry and transitivity, respectively, for equivalence relations in Set. More generally, an internal equivalence relation in a *variety of universal algebras* is a congruence, i.e. an equivalence relation which is compatible with the operations of the algebraic theory of the variety.

(2) For any arrow $p\colon E \to B$ in a category with pullbacks,

$$\begin{tikzcd} \mathsf{Eq}(p) \times_E \mathsf{Eq}(p) \arrow[r, shift left=2, "p_1"] \arrow[r, "\langle \pi_1 \cdot p_1, \pi_2 \cdot p_2 \rangle" description] \arrow[r, shift right=2, "p_2"'] & \mathsf{Eq}(p) \arrow[loop above, "\langle \pi_2, \pi_1 \rangle"] \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & E \arrow[l, "\langle 1_E, 1_E \rangle" description] \end{tikzcd}$$

is an internal equivalence relation.

Notation 1.2.6. For convenience, we will use the notation

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$$

or (R, r_1, r_2) to refer to an internal equivalence relation (1.4).

Definition 1.2.7. An internal equivalence relation (R, r_1, r_2) in $\mathscr{C}$ is said to be *effective* when it is the kernel pair of some arrow in $\mathscr{C}$.

We are now ready for the definition of a *(Barr-)exact category* [2]:

Definition 1.2.8. A category $\mathscr{C}$ is *(Barr-)exact* when

- it is regular;
- any internal equivalence relation in $\mathscr{C}$ is effective.

The categories Set of sets, Grp of groups, Mon of monoids, Ab of abelian groups, R-Mod of modules over a ring R are all examples of (Barr-)exact categories. A counter-example is given by the category $\mathsf{Grp}(\mathsf{Top})$ of topological groups, which is regular but not (Barr-)exact.

1.2.3. Protomodular categories

The concept of *protomodularity* is due to D. Bourn [12] and is sometimes referred to as *Bourn-protomodularity.* The book of F. Borceux and D. Bourn [6] contains an in-depth study of that subject, among other things.

Definition 1.2.9. A pointed category $\mathscr{C}$ is *protomodular* when

- it admits all pullbacks;
- the *Split Short Five Lemma* holds in $\mathscr{C}$: given the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "k"] \arrow[d, "a"'] & B \arrow[r, shift left, "p"] \arrow[d, "b"'] & C \arrow[l, shift left, "s"] \arrow[d, "c"] \\
0 \arrow[r] & A' \arrow[r, tail, "k'"'] & B' \arrow[r, shift left, "p'"] & C' \arrow[l, shift left, "s'"]
\end{tikzcd}$$

where $k' \cdot a = b \cdot k$, $p' \cdot b = c \cdot p$, $s' \cdot c = b \cdot s$, $k = \mathsf{ker}(p)$, $k' = \mathsf{ker}(p')$, and p and p' are split epimorphisms with sections s and s' respectively (i.e. $p \cdot s = 1_C$ and $p' \cdot s' = 1_{C'}$), if the arrows a and c are isomorphisms, then so is b.

One can prove, by "*diagram chasing*", that the Split Short Five Lemma is valid in the category Grp of groups, so that Grp is protomodular. This is also the case for the categories Ab of abelian groups, Vect_K of vector spaces over a field K, Rng of (not necessarily unitary) rings, Lie_K of Lie algebras over a field K, R-Mod of modules over a ring R, etc. However, the Split Short Five Lemma does not hold, in general, in the categories Set of sets and Mon of monoids, so that these two categories are not protomodular.

Below is a very important property valid in any pointed protomodular category.

Proposition 1.2.10. [12] *In a pointed protomodular category, any regular epimorphism is the cokernel of its kernel.*

1.2.4. Semi-abelian categories

The "modern" definition of a *semi-abelian category* was first introduced in [60] by G. Janelidze, L. Márki and W. Tholen, after several years of research. Since then, many detailed surveys on semi-abelian categories have been written, such as [6] and [5] which are well-known references.

Definition 1.2.11. A category $\mathscr{C}$ is *semi-abelian* when

- it is pointed;
- it has binary coproducts;
- it is (Barr-)exact;
- it is protomodular.

The categories Grp, Ab, Vect_K, Rng, R-Mod, Lie_K are all examples of semi-abelian categories. It is also possible to show that the dual category $\mathsf{Set}_*^{\mathsf{op}}$ of the category of pointed sets is semi-abelian. However, the category Set of sets is not semi-abelian as is the case for the categories URng of unitary rings and Mon of monoids.

1.2.5. Normal categories

Definition 1.2.12. A category $\mathscr{C}$ is *normal* (in the sense of Z. Janelidze [66]) when

- it is pointed;
- it is regular;
- any regular epimorphism in $\mathscr{C}$ is the cokernel of its kernel.

A normal category is not necessarily protomodular but it keeps an interesting property valid in any pointed protomodular category (see Proposition 1.2.10). Of course, any semi-abelian category is normal. The notion of normal category is also weaker than that of *homological category* (which we will not define in this thesis) but some properties of homological categories, such as the following, are still valid in the normal setting:

Proposition 1.2.13. *Let $\mathscr{C}$ be a normal category. Consider a commutative diagram of short exact sequences in $\mathscr{C}$:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "k"] \arrow[d, "a"'] & B \arrow[r, two heads, "f"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, tail, "k'"'] & B' \arrow[r, two heads, "f'"'] & C' \arrow[r] & 0.
\end{tikzcd}$$

Then the left-hand square is a pullback if and only if the arrow c is a monomorphism.

1.2.6. Unital and strongly unital categories

Finally, we also mention *unital* and *strongly unital categories*, introduced by D. Bourn in [13]. The reader can also refer to [6] for additional information.

Definition 1.2.14. A category $\mathscr{C}$ is *unital* when

- it is pointed;

- is is finitely complete;
- for any pair X, Y of objects in $\mathscr{C}$, the arrows $l_X = \langle 1_X, 0\rangle$ and $r_Y = \langle 0, 1_Y\rangle$ are jointly strongly epimorphic:

$$X \xrightarrow{l_X} X \times Y \xleftarrow{r_Y} Y.$$

The categories Mag of magmas, Grp of groups, Ab of abelian groups, Mon of monoids, ComMon of commutative monoids, Rng of (not necessarily unitary) rings, R-Alg of algebras over a ring R, and even the dual $\mathsf{Set}_*^{\mathsf{op}}$ of the category of pointed sets, are examples of unital categories. The category Set of sets is not an example of such a category (it is not even pointed!).

Definition 1.2.15. A category $\mathscr{C}$ is *strongly unital* when

- it is pointed;
- is is finitely complete;
- for any object X in $\mathscr{C}$, the arrows $\Delta_X = \langle 1_X, 1_X\rangle$ and $r_X = \langle 0, 1_X\rangle$ are jointly strongly epimorphic:

$$X \xrightarrow{\Delta_X} X \times X \xleftarrow{r_X} X.$$

The categories Grp of groups and LCMag of left closed magmas are strongly unital while the category Mon of monoids is not.

It is possible to prove that any strongly unital category is unital. But the converse implication is not true in general; there exist some categories which are unital but not strongly unital as is the case for the category Mon of monoids, for instance. We will see that also the category of preordered groups is unital but not strongly unital.

1.3. Effective descent morphisms

Effective descent morphisms constitute a special class of arrows which is more difficult to define than the other classes already discussed in Subsection 1.1.3. This is why we devote an entire section to this subject.

The notion of effective descent morphism is the right notion of "good quotient" in a general category. It has been widely studied by different authors: the reader can refer to the work of G. Janelidze and W. Tholen in [63], and of G. Janelidze, M. Sobral and W. Tholen in [62]. This concept is fundamental in *Categorical Galois theory*, as we will see in Section 2.3.

Let $\mathscr{C}$ be a category with pullbacks.

In order to define the concept of effective descent morphism, we first need to understand the notion of *discrete fibration of internal equivalence relations*:

Definition 1.3.1. A *discrete fibration of (internal) equivalence relations* from an internal equivalence relation (R, r_1, r_2) on $X \in \mathscr{C}$ to another internal equivalence relation (R', r_1', r_2') on $X' \in \mathscr{C}$ is given by a pair (f_0, f_1) of arrows in $\mathscr{C}$ as in

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "f_1"'] & X \arrow[d, "f_0"] \\ R' \arrow[r, shift left, "r_1'"] \arrow[r, shift right, "r_2'"'] & X' \end{tikzcd}$$

such that $f_0 \cdot r_i = r_i' \cdot f_1$ for $i = 1, 2$, and such that the square

$$\begin{tikzcd} R \arrow[r, "r_2"] \arrow[d, "f_1"'] & X \arrow[d, "f_0"] \\ R' \arrow[r, "r_2'"'] & X' \end{tikzcd}$$

involving r_2 and r_2' is a pullback.

Remark 1.3.2. Note that, by symmetry of equivalence relations, the conditions of Definition 1.3.1 imply that also the square involving r_1 and r_1' is a pullback.

Let $p \colon E \to B$ be any arrow in $\mathscr{C}$. Then the discrete fibrations of equivalence relations with codomain the internal equivalence relation $\mathsf{Eq}(p)$ (see Example 1.2.5(2)) are the objects of a category:

Definition 1.3.3. Let $p\colon E \to B$ be any arrow in $\mathscr{C}$. The *category* $\mathsf{DiscFib}(\mathsf{Eq}(p))$ *of discrete fibrations over* $\mathsf{Eq}(p)$ is the category whose

- objects are discrete fibrations of equivalence relations with codomain $\mathsf{Eq}(p)$ as below:

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "f_1"'] & F \arrow[d, "f_0"] \\ \mathsf{Eq}(p) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & E; \end{tikzcd} \tag{1.5}$$

- arrows are pairs (ϕ_0, ϕ_1) of morphisms in $\mathscr{C}$ making the following diagram commute:

$$\begin{tikzcd} R \arrow[rr, shift left, "r_1"] \arrow[rr, shift right, "r_2"'] \arrow[dr, dotted, "\phi_1"] \arrow[dd, "f_1"'] & & F \arrow[dr, dotted, "\phi_0"] \arrow[dd, "f_0" near end] & \\ & R' \arrow[rr, shift left, "r'_1"] \arrow[rr, shift right, "r'_2"'] \arrow[dl, "f'_1"] & & F' \arrow[dl, "f'_0"] \\ \mathsf{Eq}(p) \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] & & E. & \end{tikzcd}$$

Let us also recall the definition of the *slice category over* $B \in \mathscr{C}$:

Definition 1.3.4. Let B be any object in $\mathscr{C}$. The *slice category* $\mathscr{C} \downarrow B$ *over* B is the category whose

- objects are arrows $A \xrightarrow{f} B$ in $\mathscr{C}$ with codomain B; we usually use the notation (A, f) to denote such an object;
- arrows between objects (A, f) and (A', f') are arrows $\phi\colon A \to A'$ in $\mathscr{C}$ making the triangle below commutative:

$$\begin{tikzcd} A \arrow[rr, dotted, "\phi"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B. & \end{tikzcd}$$

We now define a functor that compares the last two categories:

Definition 1.3.5. Let $p\colon E \to B$ be any arrow in $\mathscr{C}$. The *functor* $K^p\colon \mathscr{C} \downarrow B \to \mathsf{DiscFib}(\mathsf{Eq}(p))$ is the functor sending any object (A, f) of $\mathscr{C} \downarrow B$ to the left-hand double square in the diagram below:

$$\begin{tikzcd}
\mathsf{Eq}(\rho_2) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, dotted] & E \times_B A \arrow[r, "\rho_2"] \arrow[d, "\rho_1"] & A \arrow[d, "f"] \\
\mathsf{Eq}(p) \arrow[r, shift left] \arrow[r, shift right] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

where the right-hand square is a pullback and where the dotted arrow is induced by the universal property of the kernel pair $\mathsf{Eq}(p)$ of p.

Note that we can prove that the left-hand double square is indeed a discrete fibration over $\mathsf{Eq}(p)$, as desired. Remark also that the definition of the functor K^p on arrows is obvious (by using the universal properties of pullbacks and kernel pairs).

We are finally ready for the definition of an effective descent morphism:

Definition 1.3.6. A morphism $p\colon E \to B$ in $\mathscr{C}$ is an *effective descent morphism* when the functor $K^p\colon \mathscr{C} \downarrow B \to \mathsf{DiscFib}(\mathsf{Eq}(p))$ is an equivalence of categories.

Example 1.3.7.

(1) If $\mathscr{C}$ is a regular category, then a morphism $p\colon E \to B$ is an effective descent morphism if and only if p is a regular epimorphism and, moreover, for any discrete fibration over $\mathsf{Eq}(p)$ as in (1.5), the equivalence relation (R, r_1, r_2) is effective.

(2) If $\mathscr{C}$ is a (Barr-)exact category, then a morphism $p\colon E \to B$ is an effective descent morphism if and only if p is a regular epimorphism (as a consequence of (1)).

(3) If $\mathscr{C}$ is a semi-abelian category, then a morphism $p\colon E \to B$ is an effective descent morphism if and only if p is a normal epimorphism.

Remark 1.3.8. Note that there are other equivalent definitions for the notion of effective descent morphism. One of them is expressed in terms of *internal actions of equivalence relations*. An arrow $p\colon E \to B$ in a category $\mathscr{C}$ with pullbacks is then an effective descent morphism when the functor $\tilde{K}^p \colon \mathscr{C} \downarrow B \to \mathscr{C}^{\mathsf{Eq}(p)}$ from $\mathscr{C} \downarrow B$ to the *category of internal actions over* $\mathsf{Eq}(p)$ (that we do not define in this dissertation) is an equivalence of categories. Another (equivalent) definition states that p is an effective descent morphism when the *pullback functor* $p^* \colon \mathscr{C} \downarrow B \to \mathscr{C} \downarrow E$, which sends any object (A, f) in $\mathscr{C} \downarrow B$ to its pullback $(E \times_B A, \rho_1) \in \mathscr{C} \downarrow E$ along p, is *monadic* [70]. Now, it is well-known that any monadic functor is *conservative* (i.e. it reflects isomorphisms) so that, if $p \colon E \to B$ is an effective descent morphism, then the pullback functor $p^* \colon \mathscr{C} \downarrow B \to \mathscr{C} \downarrow E$ is conservative.

Let us also mention the weaker concept of *descent morphism*:

Definition 1.3.9. A morphism $p \colon E \to B$ in $\mathscr{C}$ is a *descent morphism* when the functor $K^p \colon \mathscr{C} \downarrow B \to \mathsf{DiscFib}(\mathsf{Eq}(p))$ is fully faithful.

It is possible to show the following assertion which follows from the above definition.

Proposition 1.3.10. *If $p \colon E \to B$ is a descent morphism, then the pullback functor $p^* \colon \mathscr{C} \downarrow B \to \mathscr{C} \downarrow E$ is conservative.*

Thanks to this statement, it is easy to prove the following useful Proposition.

Proposition 1.3.11. *Let $\mathscr{C}$ be a category with pullbacks. Consider the following commutative diagram in which p is a descent morphism:*

$$\begin{tikzcd}
C \arrow[r, "s"] \arrow[d, "g"'] \arrow[dr, phantom, "(1)"] & A \arrow[r, "t"] \arrow[d, "f"] \arrow[dr, phantom, "(2)"] & M \arrow[d, "h"] \\
E \arrow[r, "p"'] & B \arrow[r, "q"'] & N.
\end{tikzcd}$$

If the external rectangle $(1) + (2)$ *and the square* (1) *are pullbacks, then so is the square* (2).

Proof. Let us take the pullback (P, p_1, p_2) of q and h. By the universal property of pullbacks, there exists a unique morphism $\phi\colon A \to P$ such that $p_1 \cdot \phi = f$ and $p_2 \cdot \phi = t$. The rectangle

$$\begin{tikzcd} C \arrow[r, "\phi\cdot s"] \arrow[d, "g"'] & P \arrow[r, "p_2"] \arrow[d, "p_1"] & M \arrow[d, "h"] \\ E \arrow[r, "p"'] & B \arrow[r, "q"'] & N \end{tikzcd}$$

corresponds to the rectangle (1) + (2) and is therefore a pullback by assumption. Since the right-hand square is in addition a pullback (by construction), then the left-hand square is also a pullback thanks to Proposition 1.1.44(2). Now, the arrow ϕ can be seen as a morphism in the slice category $\mathscr{C} \downarrow B$, from (A, f) to (P, p_1), and we compute that

$$p^*(\phi) \cong 1_{(C,g)}\colon (C, g) \to (C, g).$$

Indeed, the pullback of f along p is g (since the square (1) is a pullback by assumption), while the pullback of p_1 along p is also g as explained above. But $1_{(C,g)}$ is an isomorphism in $\mathscr{C} \downarrow E$. Since p is a descent morphism, p^* is then conservative by Proposition 1.3.10, so that ϕ is an isomorphism in $\mathscr{C} \downarrow B$ and, as a consequence, an isomorphism in $\mathscr{C}$. This proves that the square (2) is a pullback. $\square$

2 | Background: advanced notions

In this chapter, we explain some more advanced notions that will be studied or used in the next chapters. Roughly speaking, we define the vocabulary needed to understand the title of this thesis.

2.1. The category $\mathsf{PreOrdGrp}$ of preordered groups

Definition 2.1.1. A *preordered group* $(G, \leq)$ is a group $G = (G, +, 0)$ endowed with a preorder relation (i.e. a relation that is both reflexive and transitive) $\leq$ which is compatible with the addition $+$ of the group G, that is, for $a, b, c, d \in G$, if $a \leq c$ and $b \leq d$, then $a + b \leq c + d$.

Remark 2.1.2.

- In this thesis, we use the additive notation for groups even if the groups we consider are not necessarily abelian.
- The compatibility of the preorder $\leq$ with the addition $+$ of the group G means that the operation $+ \colon G \times G \to G$ is monotone. But it is not required in Definition 2.1.1 that also the inversion map $- \colon G \to G$ of G is monotone. If so, then the preordered group $(G, \leq)$ is actually an *internal group in the category of preordered sets*.

Example 2.1.3. The prototypical example of a preordered group is given by $(\mathbb{Z}, \leq)$, where $\leq$ is the usual order on the group $\mathbb{Z}$ of integers.

This preordered group is actually special for two reasons. First of all, the underlying group $\mathbb{Z}$ is an abelian group and, secondly, the usual order $\leq$ on integers is in fact a *partial order*, i.e. it is an antisymmetric preorder.

Of course, there are many other non-trivial examples of preordered groups, some of which we will see later on.

Preordered groups give rise to a category that we describe below:

Definition 2.1.4. The *category* $\mathsf{PreOrdGrp}$ *of preordered groups* is the category whose

- objects are all preordered groups;
- arrows are *preorder preserving group morphisms*, i.e. an arrow between two preordered groups $(G, \leq_G)$ and $(H, \leq_H)$ is a group morphism $f\colon G \to H$ between the two underlying groups G and H such that, if $a \leq_G b$ in G, then $f(a) \leq_H f(b)$ in H.

It turns out that this category is isomorphic to another category:

Proposition 2.1.5. *The category* $\mathsf{PreOrdGrp}$ *of preordered groups is isomorphic to the category whose*

- *objects are pairs* (G, M)*, where* G *is a group and* M *a submonoid of* G *closed under conjugation in* G*; such an object is usually represented by the inclusion arrow* $M \rightarrowtail G$;
- *arrows between two given objects* (G, M) *and* (H, N) *are group morphisms* $f\colon G \to H$ *between the two underlying groups* G *and* H *such that* $f(M) \subseteq N$*; equivalently, such arrows can be seen as pairs* $(f, \bar{f})$*, where* f *is a group morphism and* $\bar{f}$ *a monoid morphism, making the following diagram commute:*

$$\begin{tikzcd} M \arrow[r, "\bar{f}"] \arrow[d, tail] & N \arrow[d, tail] \\ G \arrow[r, "f"'] & H. \end{tikzcd}$$

Proof. For the sake of clarity, let $\mathscr{P}$ denote the above category. The idea of the proof consists in checking that the functor $I\colon \mathsf{PreOrdGrp} \to \mathscr{P}$, sending any preordered group $(G, \leq)$ to the pair (G, M), where $M = \{x \in G \mid 0 \leq x\}$, is bijective on objects and fully faithful (see Proposition 1.1.34). This is an easy verification left to the reader. $\square$

Example 2.1.6. The preordered group $(\mathbb{Z}, \leq)$ of Example 2.1.3 corresponds via this isomorphism of categories to the pair $(\mathbb{Z}, \mathbb{N})$ (where $\mathbb{N}$ denotes the monoid of natural numbers).

By isomorphism of categories, we can now use this category $\mathscr{P}$ as the definition of the category of preordered groups. This is what we will do most of the time in this thesis since this latter formulation (Proposition 2.1.5) is much more "categorical" than the other (Definition 2.1.4).

Notation 2.1.7. The submonoid M in a given preordered group (G, M) is usually called the *positive cone* of G and is denoted by P_G accordingly.

Remark 2.1.8. If the underlying group G of a preordered group (G, P_G) is abelian, then all possible positive cones P_G are given by all submonoids of G (since P_G is then trivially closed under conjugation in G). More generally, there is a characterization of the positive cones of any preordered group (G, P_G), the proof of which can be found in [21, Corollary 3.3]: given a monoid M which is embeddable in a group, there exists a group G such that the pair (G, M) is a preordered group if and only if, for any $a, b \in M$, there exist x and y in M such that $a + b = b + x = y + a$. This characterization allows one to find examples of preordered groups.

The categorical behaviour of preordered groups was studied in 2019 by M. M. Clementino, N. Martins-Ferreira and A. Montoli. In what follows, we recall some of their results which will be useful later on. For more details, the reader can refer to their article [21].

2.1.1. Limits and colimits in PreOrdGrp

It was proved in [21] that the category PreOrdGrp of preordered groups is both complete and cocomplete. So let us describe some useful limits and colimits in this context:

- The binary product of two preordered groups (G, P_G) and (H, P_H) is given by the preordered group $(G \times H, P_G \times P_H)$, where $G \times H$ is the binary product of G and H in the category Grp of groups, and $P_G \times P_H$ the binary product of P_G and P_H in the category Mon of monoids.
- The equalizer of a pair of parallel arrows

$$\begin{tikzcd} P_G \arrow[r, shift left, "\bar{f}"] \arrow[r, shift right, "\bar{g}"'] \arrow[d, tail] & P_H \arrow[d, tail] \\ G \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & H \end{tikzcd} \tag{2.1}$$

in $\mathsf{PreOrdGrp}$ is given by $((E, P_E), (e, \bar{e}))$, where (E, e) is the equalizer of f and g in Grp, and the square

$$\begin{tikzcd} P_E \arrow[r, tail, "\bar{e}"] \arrow[d, tail] & P_G \arrow[d, tail] \\ E \arrow[r, tail, "e"'] & G \end{tikzcd}$$

is a pullback in Mon, i.e. $P_E = E \cap P_G$.
Equivalently, $((E, P_E), (e, \bar{e}))$ is the equalizer of $(f, \bar{f})$ and $(g, \bar{g})$ if and only if (E, e) is the equalizer of f and g in Grp and $(P_E, \bar{e})$ is the equalizer of $\bar{f}$ and $\bar{g}$ in Mon.
- The pullback of two morphisms $(f, \bar{f}) \colon (G, P_G) \to (H, P_H)$ and $(g, \bar{g}) \colon (X, P_X) \to (H, P_H)$ with same codomain in $\mathsf{PreOrdGrp}$ is $((P, P_P), (p_1, \bar{p}_1), (p_2, \bar{p}_2))$, with (P, p_1, p_2) the pullback of f and g in Grp and $(P_P, \bar{p}_1, \bar{p}_2)$ the pullback of $\bar{f}$ and $\bar{g}$ in Mon.

Let us remark that the limits are "well-behaved" in the category of preordered groups. Indeed, in order to compute them, it suffices to consider them "component-wise", i.e. at the level of groups and then at the level of monoids (at the level of positive cones). Colimits however are not as well-behaved since we cannot compute them component-wise. Some of them, such as coequalizers, are still not too difficult to determine:

- The coequalizer of two parallel arrows in $\mathsf{PreOrdGrp}$ as in diagram (2.1) is computed by first taking the coequalizer (Q, q) of f and g

in the category Grp. The positive cone P_Q of Q is then given by $P_Q = q(P_H)$, and the arrow $\bar{q}\colon P_H \twoheadrightarrow P_Q$ by the surjective monoid morphism induced by the group morphism q.

Let us also mention the fact that the initial and terminal objects coincide in PreOrdGrp with the trivial group $\{0\}$ endowed with the trivial positive cone: $(\{0\}, \{0\})$. Consequently:

Proposition 2.1.9. *The category* PreOrdGrp *of preordered groups is pointed.*

2.1.2. Some special arrows in PreOrdGrp

We continue our survey of the category of preordered groups with a description of some particular arrows, which will be needed later on.

An arrow

$$\begin{tikzcd} P_G \arrow[r, "\bar{f}"] \arrow[d, tail] & P_H \arrow[d, tail] \\ G \arrow[r, "f"'] & H \end{tikzcd} \tag{2.2}$$

in PreOrdGrp is

- a monomorphism if and only if f is injective (which implies that also $\bar{f}$ is injective by commutativity of the square (2.2));
- an epimorphism if and only if f is surjective;
- a regular epimorphism if and only if both f and $\bar{f}$ are surjections;
- a normal monomorphism if and only if f is a normal monomorphism in the category Grp of groups and the square (2.2) is a pullback in the category Mon of monoids;
- a normal epimorphism if and only if f is a normal epimorphism in the category Grp of groups and $\bar{f}$ is surjective.

As a consequence, we are able to describe the short exact sequences in PreOrdGrp:

Proposition 2.1.10. *Consider, in the category* PreOrdGrp *of preordered groups, a pair of composable arrows as in the following diagram:*

$$
\begin{tikzcd}
P_A \arrow[r, "\bar{k}"] \arrow[d, tail] & P_B \arrow[r, "\bar{f}"] \arrow[d, tail] & P_C \arrow[d, tail] \\
A \arrow[r, "k"'] & B \arrow[r, "f"'] & C.
\end{tikzcd}
\tag{2.3}
$$

Then, the sequence (2.3) *is a short exact sequence if and only if* $A \xrightarrow{k} B \xrightarrow{f} C$ *is a short exact sequence in* Grp*, the left-hand square is a pullback in* Mon *and* $\bar{f}$ *is surjective.*

Remark 2.1.11. Equivalently, the sequence (2.3) is a short exact sequence in PreOrdGrp if and only if $A \xrightarrow{k} B \xrightarrow{f} C$ is a short exact sequence in Grp, $\bar{k}$ is the kernel of $\bar{f}$ in Mon and $\bar{f}$ is surjective. But $\bar{f}$ is not necessarily the cokernel of $\bar{k}$ in Mon so that the sequence $P_A \xrightarrow{\bar{k}} P_B \xrightarrow{\bar{f}} P_C$ is not a short exact sequence in Mon in general. Accordingly, short exact sequences in PreOrdGrp are not computed component-wise. This is sometimes the case, but in some exceptional situations. We will see an example of such a situation later on (see Corollary 3.1.10(1)).

2.1.3. Some properties of the category PreOrdGrp

In this subsection, we prove that the category PreOrdGrp of preordered groups is regular, normal and unital, but neither (Barr-)exact nor protomodular (which, in particular, means that it is not semi-abelian). A description of effective descent morphisms in this setting is also given.

Proposition 2.1.12. [21] *The category* PreOrdGrp *of preordered groups is regular.*

Proof. As already stated, the category PreOrdGrp is first of all complete, hence finitely complete. We can then show that any arrow $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ in PreOrdGrp factors as

$$
\begin{tikzcd}
(G, P_G) \arrow[rr, "(f,\bar{f})"] \arrow[dr, two heads, "(e,\bar{e})"'] & & (H, P_H) \\
& (f(G), f(P_G)) \arrow[ur, tail, "(m,\bar{m})"'] &
\end{tikzcd}
$$

where $e\colon G \twoheadrightarrow f(G)$ is defined, for any $x \in G$, by $e(x) = f(x)$, and where $m\colon f(G) \hookrightarrow H$ is the inclusion morphism. It is easy to see that both $(e, \bar{e})$ and $(m, \bar{m})$ are morphisms in PreOrdGrp. Moreover, $(e, \bar{e})$ is a regular epimorphism since both e and $\bar{e}$ are surjections, and $(m, \bar{m})$ is a monomorphism since m is an injection. Now, the pullback-stability of regular epimorphisms follows from the pullback-stability of surjections in the categories Grp of groups and Mon of monoids, and from the fact that pullbacks are computed "component-wise" in PreOrdGrp (i.e. at the level of groups and then at the level of monoids). This completes the proof by Proposition 1.2.2. □

Proposition 2.1.13. [21] *In* PreOrdGrp*, any regular epimorphism is the cokernel of its kernel.*

Proof. Let $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ be a regular epimorphism, and consider $(k, \bar{k})\colon (K, P_K) \to (G, P_G)$ its kernel. By assumption, k is then the kernel of f in Grp with f surjective, so that f is the cokernel of k in Grp (since Grp is a normal category, thanks to the First Isomorphism Theorem). We conclude the proof by observing that $\bar{f}$ is also surjective by assumption. □

As already mentioned in Proposition 2.1.9, the category PreOrdGrp of preordered groups is pointed so that we get, together with Propositions 2.1.12 and 2.1.13, the following:

Corollary 2.1.14. [21] *The category* PreOrdGrp *of preordered groups is normal.*

Unfortunately, the category of preordered groups is not (Barr-)exact since, as explained in [21], it is possible to find an internal equivalence relation in PreOrdGrp which is not effective. Indeed, consider the following internal equivalence relation in PreOrdGrp

$$(\mathbb{Z}\times\mathbb{Z}, D) \underset{(\pi_2,\bar{\pi}_2)}{\overset{(\pi_1,\bar{\pi}_1)}{\rightrightarrows}} (\mathbb{Z}, \mathbb{N})$$

where $D = \{(z,z) \in \mathbb{Z}\times\mathbb{Z} \mid z \geq 0\}$, and where π_1 and π_2 are the two canonical projections of the product $\mathbb{Z}\times\mathbb{Z}$. Then, the universal arrow $\langle(\pi_1,\bar{\pi}_1),(\pi_2,\bar{\pi}_2)\rangle = (\langle\pi_1,\pi_2\rangle,\langle\bar{\pi}_1,\bar{\pi}_2\rangle)$ admits the following trivial factorization through $(\mathbb{Z}\times\mathbb{Z},\mathbb{N}\times\mathbb{N})$:

$$\begin{tikzcd}
(\mathbb{Z}\times\mathbb{Z}, D) \arrow[rr, "(\langle\pi_1,\pi_2\rangle,\langle\bar{\pi}_1,\bar{\pi}_2\rangle)"] \arrow[dr, two heads, "(\langle\pi_1,\pi_2\rangle,\langle\bar{\pi}_1,\bar{\pi}_2\rangle)"'] & & (\mathbb{Z}\times\mathbb{Z}, \mathbb{N}\times\mathbb{N}) \\
& (\mathbb{Z}\times\mathbb{Z}, \mathbb{N}\times\mathbb{N}). \arrow[ur, equal] &
\end{tikzcd}$$

It is clear that $(\langle\pi_1,\pi_2\rangle,\langle\bar{\pi}_1,\bar{\pi}_2\rangle)$ is an epimorphism which is not an isomorphism. Indeed, $\langle\pi_1,\pi_2\rangle = 1_{\mathbb{Z}\times\mathbb{Z}}$ is a surjective (even bijective) group morphism, but $\langle\bar{\pi}_1,\bar{\pi}_2\rangle$ is not surjective since $(1,2)$, for instance, belongs to $\mathbb{N}\times\mathbb{N}$ but not to D. This means that there exists a factorization of $\langle(\pi_1,\bar{\pi}_1),(\pi_2,\bar{\pi}_2)\rangle$ through an epimorphism which is not an isomorphism, i.e. $\langle(\pi_1,\bar{\pi}_1),(\pi_2,\bar{\pi}_2)\rangle$ is not an extremal monomorphism. In other words, the two projections $(\pi_1,\bar{\pi}_1)$ and $(\pi_2,\bar{\pi}_2)$ are not jointly extremally monomorphic which, by Proposition 1.1.48, implies that the above internal equivalence relation is not a kernel pair, that is, it is not effective.

Despite the fact that $\mathsf{PreOrdGrp}$ is not a (Barr-)exact category, it keeps one of the specific features of (Barr-)exact categories, namely the fact that the effective descent morphisms exactly coincide with the regular epimorphisms:

Proposition 2.1.15. [21] *In* $\mathsf{PreOrdGrp}$*, the effective descent morphisms exactly coincide with the regular epimorphisms, i.e. with the normal epimorphisms.*

Proof. Let $(p,\bar{p})\colon (E,P_E) \to (B,P_B)$ be any arrow in $\mathsf{PreOrdGrp}$. Consider the following discrete fibration over $\mathsf{Eq}(p,\bar{p}) = (\mathsf{Eq}(p),\mathsf{Eq}(\bar{p}))$:

$$\begin{tikzcd}
(R, P_R) \arrow[r, shift left, "(r_1,\bar{r}_1)"] \arrow[r, shift right, "(r_2,\bar{r}_2)"'] \arrow[d, "(f_1,\bar{f}_1)"'] & (F, P_F) \arrow[d, "(f_0,\bar{f}_0)"] \\
(\mathsf{Eq}(p), \mathsf{Eq}(\bar{p})) \arrow[r, shift left, "(\pi_1,\bar{\pi}_1)"] \arrow[r, shift right, "(\pi_2,\bar{\pi}_2)"'] & (E, P_E).
\end{tikzcd}$$

Consider also the kernel pair $\mathsf{Eq}(p \cdot f_0, \bar{p} \cdot \bar{f}_0) = (\mathsf{Eq}(p \cdot f_0), \mathsf{Eq}(\bar{p} \cdot \bar{f}_0))$ of $(p, \bar{p}) \cdot (f_0, \bar{f}_0)$, with its two projections $(\chi_1, \bar{\chi}_1)$ and $(\chi_2, \bar{\chi}_2)$. Then, by the universal property of kernel pairs, there exists a unique arrow $(\psi, \bar{\psi})\colon (\mathsf{Eq}(p \cdot f_0), \mathsf{Eq}(\bar{p} \cdot \bar{f}_0)) \to (\mathsf{Eq}(p), \mathsf{Eq}(\bar{p}))$ such that $\pi_1 \cdot \psi = f_0 \cdot \chi_1$ and $\pi_2 \cdot \psi = f_0 \cdot \chi_2$. It is now easy to see that the square

$$\begin{tikzcd}
(\mathsf{Eq}(p \cdot f_0), \mathsf{Eq}(\bar{p} \cdot \bar{f}_0)) \arrow[r, tail, "(\langle \chi_1,\chi_2 \rangle, \langle \bar{\chi}_1,\bar{\chi}_2 \rangle)"] \arrow[d, "(\psi,\bar{\psi})"'] & (F \times F, P_F \times P_F) \arrow[d, "(f_0 \times f_0, \bar{f}_0 \times \bar{f}_0)"] \\
(\mathsf{Eq}(p), \mathsf{Eq}(\bar{p})) \arrow[r, tail, "(\langle \pi_1,\pi_2 \rangle, \langle \bar{\pi}_1,\bar{\pi}_2 \rangle)"'] & (E \times E, P_E \times P_E)
\end{tikzcd}$$

is a pullback in PreOrdGrp. So, by the universal property of pullbacks, since

$$\langle \pi_1, \pi_2 \rangle \cdot f_1 = \langle \pi_1 \cdot f_1, \pi_2 \cdot f_1 \rangle = \langle f_0 \cdot r_1, f_0 \cdot r_2 \rangle = (f_0 \times f_0) \cdot \langle r_1, r_2 \rangle,$$

there exists a unique morphism $(\phi, \bar{\phi})\colon (R, P_R) \to (\mathsf{Eq}(p \cdot f_0), \mathsf{Eq}(\bar{p} \cdot \bar{f}_0))$ such that $\psi \cdot \phi = f_1$ and $\langle \chi_1, \chi_2 \rangle \cdot \phi = \langle r_1, r_2 \rangle$ (in particular, $\chi_1 \cdot \phi = r_1$ and $\chi_2 \cdot \phi = r_2$). From the last identity, we deduce that ϕ is injective since so is $\langle r_1, r_2 \rangle$. This implies that $(\phi, \bar{\phi})$ is a monomorphism. Let us show that it is actually a split monomorphism. By definition of a discrete fibration, the square

$$\begin{tikzcd}
(R, P_R) \arrow[r, "(r_2,\bar{r}_2)"] \arrow[d, "(f_1,\bar{f}_1)"'] & (F, P_F) \arrow[d, "(f_0,\bar{f}_0)"] \\
(\mathsf{Eq}(p), \mathsf{Eq}(\bar{p})) \arrow[r, "(\pi_2,\bar{\pi}_2)"'] & (E, P_E)
\end{tikzcd}$$

is a pullback. Recall also that $\pi_2 \cdot \psi = f_0 \cdot \chi_2$ whence, by the universal property of pullbacks, there is a unique arrow $(s, \bar{s})\colon (\mathsf{Eq}(p \cdot f_0), \mathsf{Eq}(\bar{p} \cdot \bar{f}_0)) \to (R, P_R)$ such that $f_1 \cdot s = \psi$ and $r_2 \cdot s = \chi_2$. We now compute that $f_1 \cdot s \cdot \phi = \psi \cdot \phi = f_1 = f_1 \cdot 1_R$ and $r_2 \cdot s \cdot \phi = \chi_2 \cdot \phi = r_2 = r_2 \cdot 1_R$. Thanks to Proposition 1.1.47, we get that $s \cdot \phi = 1_R$, and $(\phi, \bar{\phi})$ is then a split monomorphism in $\mathsf{PreOrdGrp}$. By Proposition 1.1.31, any split monomorphism is regular, so that $(\phi, \bar{\phi})$ is then an equalizer. In particular,

$$P_R = R \cap \mathsf{Eq}(\bar{p} \cdot \bar{f}_0) = R \cap \mathsf{Eq}(p \cdot f_0) \cap (P_F \times P_F).$$

Now, notice that the category Grp of groups is (Barr-)exact, which implies that R is effective, i.e. R is the kernel pair $\mathsf{Eq}(g)$ of some group homomorphism $g\colon F \to X$. Accordingly,

$$P_R = \mathsf{Eq}(g) \cap \mathsf{Eq}(p \cdot f_0) \cap (P_F \times P_F) = \mathsf{Eq}(g) \cap (P_F \times P_F)$$

since $\mathsf{Eq}(g) = R \subseteq \mathsf{Eq}(p \cdot f_0)$, and we then deduce that (R, P_R) is effective. Thanks to Example 1.3.7(1) and the fact that the category $\mathsf{PreOrdGrp}$ is regular (see Proposition 2.1.12), an arrow in $\mathsf{PreOrdGrp}$ is an effective descent morphism if and only if it is a regular epimorphism. $\square$

The category $\mathsf{PreOrdGrp}$ then has a fairly simple characterization for its effective descent morphisms:

Corollary 2.1.16. *A morphism $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ in $\mathsf{PreOrdGrp}$ is an effective descent morphism if and only if both f and $\bar{f}$ are surjective.*

Note also that, although any regular epimorphism in the category $\mathsf{PreOrdGrp}$ is normal, the category $\mathsf{PreOrdGrp}$ of preordered groups is not protomodular since the Split Short Five Lemma does not hold in it. Indeed, consider the following commutative diagram of split extensions

$$\begin{tikzcd}
(\mathbb{Z}, \mathbb{N}) \arrow[r, "(\langle 1,0\rangle, \langle 1,0\rangle)"] \arrow[d, equal] & (\mathbb{Z} \times \mathbb{Z}, \mathbb{N} \times \mathbb{N}) \arrow[r, shift left, "(\pi_2, \bar{\pi}_2)"] \arrow[d, "(\phi, \bar{\phi})"] & (\mathbb{Z}, \mathbb{N}) \arrow[l, shift left, "(\langle 0,1\rangle, \langle 0,1\rangle)"] \arrow[d, equal] \\
(\mathbb{Z}, \mathbb{N}) \arrow[r, "(\langle 1,0\rangle, \langle 1,0\rangle)"'] & (\mathbb{Z} \times \mathbb{Z}, P) \arrow[r, shift left, "(\pi_2, \bar{\pi}_2)"] & (\mathbb{Z}, \mathbb{N}) \arrow[l, shift left, "(\langle 0,1\rangle, \langle 0,1\rangle)"]
\end{tikzcd}$$

where $P = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid y > 0 \text{ or } (y = 0 \text{ and } x \geq 0)\}$ and $\phi\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ is the identity on the group $\mathbb{Z}\times\mathbb{Z}$. Remark that the morphism of preordered groups $(\phi,\bar{\phi})$ is well-defined since, for any $(x,y) \in \mathbb{N}\times\mathbb{N}$, we have that $\phi(x,y) = (x,y) \in P$. But $(\phi,\bar{\phi})$ is not an isomorphism in PreOrdGrp. Indeed, $(-1,2) \in P$ while $(-1,2) \notin \mathbb{N}\times\mathbb{N}$.

Below is a last property of PreOrdGrp that will be useful later:

Proposition 2.1.17. [21] *The category* PreOrdGrp *of preordered groups is unital.*

Proof. Let us first recall the fact that PreOrdGrp is pointed and finitely complete (see Subsection 2.1.1). It remains to check the last axiom of the definition of a unital category (i.e. Definition 1.2.14). Note that, since the category PreOrdGrp of preordered groups is regular, proving that two given arrows are jointly strongly epimorphic is the same as showing that they are jointly extremally epimorphic (thanks to Proposition 1.2.3). So consider two preordered groups (X,P_X) and (Y,P_Y) and assume that $(l_X,\bar{l}_X)$ and $(r_Y,\bar{r}_Y)$ (where $l_X = \langle 1_X, 0\rangle$ and $r_Y = \langle 0, 1_Y\rangle$) factor through some monomorphism $(m,\bar{m})\colon (Z,P_Z) \to (X\times Y, P_X\times P_Y)$, that is, $m\cdot f = l_X$ and $m\cdot g = r_Y$ for two morphisms $(f,\bar{f})$ and $(g,\bar{g})$ as below:

$$\begin{tikzcd}
(X,P_X) \arrow[r, "(l_X,\bar{l}_X)"] \arrow[dr, "(f,\bar{f})"'] & (X\times Y, P_X\times P_Y) \arrow[d, dotted, shift right, "(\phi,\bar{\phi})"'] & (Y,P_Y) \arrow[l, "(r_Y,\bar{r}_Y)"'] \arrow[dl, "(g,\bar{g})"] \\
& (Z,P_Z). \arrow[u, tail, shift right, "(m,\bar{m})"'] &
\end{tikzcd}$$

Define the function $\phi\colon X\times Y \to Z$ as follows: for any $(x,y) \in X\times Y$, $\phi(x,y) = f(x) + g(y)$. It is clearly a group morphism. Indeed, we compute that

$$\begin{aligned}
&m\left(\phi((x,y)+(x',y'))\right) = m\left(\phi(x+x',y+y')\right)\\
&\quad = m\left(f(x+x') + g(y+y')\right)\\
&\quad = (m\cdot f)(x) + (m\cdot f)(x') + (m\cdot g)(y) + (m\cdot g)(y')\\
&\quad = (x,0) + (x',0) + (0,y) + (0,y') = (x+x',y+y')
\end{aligned}$$

$$
\begin{aligned}
&= (x,y) + (x',y') = (x,0) + (0,y) + (x',0) + (0,y') \\
&= (m \cdot f)(x) + (m \cdot g)(y) + (m \cdot f)(x') + (m \cdot g)(y') \\
&= m\left((f(x) + g(y)) + (f(x') + g(y'))\right) \\
&= m\left(\phi(x,y) + \phi(x',y')\right)
\end{aligned}
$$

for any $(x,y),(x',y') \in X \times Y$, that is, $\phi((x,y)+(x',y')) = \phi(x,y) + \phi(x',y')$ since m is a monomorphism. In addition, the restriction of ϕ to $P_X \times P_Y$ takes its values in P_Z. Indeed, if $(x,y) \in P_X \times P_Y$, then $f(x) \in P_Z$ and $g(y) \in P_Z$ since $(f,\bar{f})$ and $(g,\bar{g})$ are two morphisms of preordered groups, so that $\phi(x,y) = f(x) + g(y) \in P_Z$ (because P_Z is a monoid). Finally, we show that $m \cdot \phi = 1_{X \times Y}$: for any $(x,y) \in X \times Y$,

$$
\begin{aligned}
(m \cdot \phi)(x,y) &= m(f(x) + g(y)) = (m \cdot f)(x) + (m \cdot g)(y) \\
&= (x,0) + (0,y) \\
&= (x,y).
\end{aligned}
$$

This proves that $(m,\bar{m})$ is a split epimorphism. Since it is also a monomorphism by assumption, it is then an isomorphism (by Proposition 1.1.24) and this concludes the proof. □

2.2. Torsion and pretorsion theories

One of the objectives of this thesis is to study *torsion* and *pretorsion theories* in the context of preordered groups. Let us therefore recall these two notions as well as their main properties.

2.2.1. Torsion theories in normal categories

We present here the definition of a *torsion theory* as well as some interesting properties that will be used later on. For more details on this subject, the reader can refer to [14] and [20], or also [64] for a slightly different approach.

Definition 2.2.1. A *torsion theory* in a normal category $\mathscr{C}$ is given by a pair $(\mathscr{T},\mathscr{F})$ of full and replete subcategories of $\mathscr{C}$ such that the following conditions hold:

(TT1) the only arrow from any $T \in \mathscr{T}$ to any $F \in \mathscr{F}$ is the zero arrow

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & F \\ & 0; \arrow[ur] & \end{tikzcd}$$

(TT2) for any object $C \in \mathscr{C}$, there exists a short exact sequence

$$0 \longrightarrow T \xrightarrow{\epsilon_C} C \xrightarrow{\eta_C} F \longrightarrow 0$$

with $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

By a *full* subcategory of $\mathscr{C}$, we mean a subcategory $\mathscr{S}$ such that the inclusion functor $\mathscr{S} \rightarrowtail \mathscr{C}$ is full. On the other hand, a subcategory $\mathscr{S}$ of $\mathscr{C}$ is said to be *replete* when, for any object $X \in \mathscr{S}$ and any isomorphism $f\colon X \to Y$ in $\mathscr{C}$, Y and f are in $\mathscr{S}$.

Example 2.2.2. The pair $(\mathsf{Ab}_{\mathsf{t.}}, \mathsf{Ab}_{\mathsf{t.f.}})$ is a torsion theory in the category Ab of abelian groups (see the Introduction).

The subcategory $\mathscr{T}$ of Definition 2.2.1 is then called a *torsion subcategory* of $\mathscr{C}$ and $\mathscr{F}$ a *torsion-free subcategory* by analogy with the terminology used in Example 2.2.2 above.

Proposition 2.2.3. *The short exact sequence of the axiom* $(TT2)$ *(from Definition 2.2.1) is unique up to isomorphism.*

Proof. Assume that, for an object $C \in \mathscr{C}$, we have two short exact sequences, with kernel in $\mathscr{T}$ and cokernel in $\mathscr{F}$, as below:

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r, "\epsilon_C"] \arrow[d, dotted, bend right, "t"'] & C \arrow[r, "\eta_C"] \arrow[d, equal] & F \arrow[r] \arrow[d, dotted, bend left, "f"] & 0 \\ 0 \arrow[r] & T' \arrow[r, "\epsilon'_C"'] \arrow[u, dotted, bend right, "s"'] & C \arrow[r, "\eta'_C"'] & F' \arrow[r] \arrow[u, dotted, bend left, "g"] & 0. \end{tikzcd}$$

Since η_C is the cokernel of ϵ_C and since $\eta'_C \cdot \epsilon_C = 0$ (by $(TT1)$), there exists a unique morphism $f\colon F \to F'$ such that $f \cdot \eta_C = \eta'_C$. We next observe that $\eta_C \cdot \epsilon'_C = 0$ (also by $(TT1)$) so that the universal property of cokernels ($\eta'_C = \mathsf{coker}(\epsilon'_C)$) again gives a unique arrow $g\colon F' \to F$ satisfying $g \cdot \eta'_C = \eta_C$. Knowing that any cokernel is an epimorphism (by Proposition 1.1.31), it is then easy to see that $f \cdot g = 1_{F'}$ and that $g \cdot f = 1_F$, i.e. that f is an isomorphism (whose inverse is g). Dually, the universal property of kernels applied twice gives a unique isomorphism $t\colon T \to T'$ with its inverse $s\colon T' \to T$ such that $\epsilon'_C \cdot t = \epsilon_C$ and $\epsilon_C \cdot s = \epsilon'_C$. □

Proposition 2.2.4. *Any torsion theory $(\mathscr{T}, \mathscr{F})$ in a normal category $\mathscr{C}$ induces two functors:*

(1) The functor $F\colon \mathscr{C} \to \mathscr{F}$ sends any object $C \in \mathscr{C}$ to the cokernel part $F \in \mathscr{F}$ of the (unique) short exact sequence in $(TT2)$*, and any arrow $\phi\colon C \to C'$ in $\mathscr{C}$ to the induced arrow $F(\phi)$ of the diagram* (2.4) *below.*

(2) The functor $T\colon \mathscr{C} \to \mathscr{T}$ sends any object $C \in \mathscr{C}$ to the kernel part $T \in \mathscr{T}$ of the (unique) short exact sequence in $(TT2)$*, and any arrow $\phi\colon C \to C'$ in $\mathscr{C}$ to the induced arrow $T(\phi)$ of the diagram* (2.4) *below.*

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "\epsilon_C"] \arrow[d, dotted, "T(\phi)"'] & C \arrow[r, "\eta_C"] \arrow[d, "\phi"] & F \arrow[r] \arrow[d, dotted, "F(\phi)"] & 0 \\
0 \arrow[r] & T' \arrow[r, "\epsilon_{C'}"'] & C' \arrow[r, "\eta_{C'}"'] & F' \arrow[r] & 0
\end{tikzcd} \tag{2.4}$$

The arrows $F(\phi)$ and $T(\phi)$ are induced by the universal properties of cokernels and kernels, respectively (as was done in the proof of Proposition 2.2.3), and they make the above diagram (2.4) commute.

Proposition 2.2.5. *Let $(\mathscr{T}, \mathscr{F})$ be a torsion theory in a normal category $\mathscr{C}$. Then*

(1) the induced functor $F\colon \mathscr{C} \to \mathscr{F}$ is a (normal epi)-reflector*, i.e. it is a reflector such that any component of the unit of the reflection is a normal epimorphism;*

(2) the induced functor $T\colon \mathscr{C} \to \mathscr{T}$ is a (normal mono)-coreflector*, i.e. it is a coreflector such that any component of the counit of the coreflection is a normal monomorphism.*

Proof. Let $C \in \mathscr{C}$. Consider the morphism $\eta_C \colon C \to F$ of $(TT2)$ and let us show that it is the C-component of the unit of the adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{F} \arrow[l, bend left, "U"] \end{tikzcd}.$$

In order to do this, consider an arrow $f \colon C \to F'$ with $F' \in \mathscr{F}$. Then $f \cdot \epsilon_C = 0$ (where ϵ_C is as in $(TT2)$) since $T \in \mathscr{T}$ and $F' \in \mathscr{F}$.

$$\begin{tikzcd} T \arrow[r, "\epsilon_C"] & C \arrow[rr, "\eta_C"] \arrow[dr, "f"'] & & F \arrow[dl, dotted, "\hat{f}"] \\ & & F' & \end{tikzcd}$$

By the universal property of cokernels ($\eta_C = \mathsf{coker}(\epsilon_C)$), there exists a unique morphism $\hat{f} \colon F \to F'$ such that $\hat{f} \cdot \eta_C = f$. This proves the first assertion since η_C is clearly a normal epimorphim for any $C \in \mathscr{C}$. The proof of the second assertion is dual. $\square$

2.2.2. Pretorsion theories in arbitrary categories

Now, suppose that our category $\mathscr{C}$ is arbitrary, in particular not necessarily pointed. In this context, the notions of kernel and cokernel, and then also of short exact sequence no longer have any meaning. This implies that Definition 2.2.1 does not make sense anymore. The concept of *pretorsion theory*, introduced by A. Facchini and C. Finocchiaro in [32] and investigated in [34], is an extension to not necessarily pointed categories of the notion of torsion theory. The idea is to replace in Definition 2.2.1 (and in what follows) the zero object with a special (non-empty) class of objects in $\mathscr{C}$. Let $\mathscr{Z}$ denote this class of objects, called *$\mathscr{Z}$-trivial objects*, and $\mathscr{N}$ the class of morphisms in $\mathscr{C}$, called *$\mathscr{Z}$-trivial morphisms*, that factor through a $\mathscr{Z}$-trivial object. Then the axiom $(TT1)$ of Definition 2.2.1 generalizes by saying that any arrow in $\mathscr{C}$ from any $T \in \mathscr{T}$ to any $F \in \mathscr{F}$ is a $\mathscr{Z}$-trivial morphism. It remains to extend the second axiom $(TT2)$. In order to do so, we first need to define the notions of *$\mathscr{Z}$-kernel*, *$\mathscr{Z}$-cokernel* and *short $\mathscr{Z}$-exact sequence.*

Definition 2.2.6. Let $f\colon A \to B$ be an arrow in $\mathscr{C}$.

(1) An arrow $k\colon K \to A$ in $\mathscr{C}$ is said to be a *$\mathscr{Z}$-kernel* of f when
- $f \cdot k \in \mathscr{N}$;
- for any arrow $\alpha\colon X \to A$ such that $f \cdot \alpha \in \mathscr{N}$, there exists a unique arrow $\phi\colon X \to K$ such that $k \cdot \phi = \alpha$.

(2) An arrow $c\colon B \to C$ in $\mathscr{C}$ is said to be a *$\mathscr{Z}$-cokernel* of f when
- $c \cdot f \in \mathscr{N}$;
- for any arrow $\alpha\colon B \to X$ such that $\alpha \cdot f \in \mathscr{N}$, there exists a unique arrow $\phi\colon C \to X$ such that $\phi \cdot c = \alpha$.

Proposition 2.2.7. *Any $\mathscr{Z}$-kernel is a monomorphism and any $\mathscr{Z}$-cokernel is an epimorphism.*

Proof. This directly follows from the universal property of $\mathscr{Z}$-kernels (respectively of $\mathscr{Z}$-cokernels), that is, by using the second item of the definition of a $\mathscr{Z}$-kernel (respectively of a $\mathscr{Z}$-cokernel). $\square$

Definition 2.2.8. Let $f\colon A \to B$ and $g\colon B \to C$ be a pair of composable morphisms in $\mathscr{C}$. The sequence

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

is said to be a *short $\mathscr{Z}$-exact sequence* when f is a $\mathscr{Z}$-kernel of g and g a $\mathscr{Z}$-cokernel of f.

Remark 2.2.9. Of course, when the category $\mathscr{C}$ is pointed and the class $\mathscr{Z}$ of $\mathscr{Z}$-trivial objects is reduced to the zero object 0, the notions of $\mathscr{Z}$-kernel, $\mathscr{Z}$-cokernel and short $\mathscr{Z}$-exact sequence are exactly the classical concepts of kernel, cokernel and short exact sequence (respectively).

We are now ready to state the definition of a *$\mathscr{Z}$-pretorsion theory*:

Definition 2.2.10. A *$\mathscr{Z}$-pretorsion theory* in an arbitrary category $\mathscr{C}$ is given by a pair $(\mathscr{T}, \mathscr{F})$ of full and replete subcategories of $\mathscr{C}$, with $\mathscr{Z} = \mathscr{T} \cap \mathscr{F}$, such that the following conditions hold:

(PT1) any arrow in $\mathscr{C}$ from any $T \in \mathscr{T}$ to any $F \in \mathscr{F}$ belongs to $\mathscr{N}$

$$\begin{tikzcd}
T \arrow[rr] \arrow[dr] & & F \\
& Z \in \mathscr{Z}; \arrow[ur] &
\end{tikzcd}$$

(PT2) for any object $C \in \mathscr{C}$, there exists a short $\mathscr{Z}$-exact sequence

$$T \xrightarrow{\epsilon_C} C \xrightarrow{\eta_C} F$$

with $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

As before, the subcategory $\mathscr{T}$ is called a *torsion subcategory* of $\mathscr{C}$ and $\mathscr{F}$ a *torsion-free subcategory.* All the results previously exposed for torsion theories also hold for pretorsion theories up to some slight modifications. We state these results below, without proofs because they are very similar to what was done for torsion theories. For details or more results, the reader can refer to [32] and [34], and to [19, 46, 72] for closely related approaches.

Proposition 2.2.11. *The short $\mathscr{Z}$-exact sequence of the axiom* $(PT2)$ *(see Definition 2.2.10) is unique up to isomorphism.*

Proposition 2.2.12. *Any $\mathscr{Z}$-pretorsion theory $(\mathscr{T}, \mathscr{F})$ in an arbitrary category $\mathscr{C}$ induces two functors:*

(1) The functor $F\colon \mathscr{C} \to \mathscr{F}$ sends any object $C \in \mathscr{C}$ to the $\mathscr{Z}$-cokernel part $F \in \mathscr{F}$ of the (unique) short $\mathscr{Z}$-exact sequence in $(PT2)$, *and any arrow $\phi\colon C \to C'$ in $\mathscr{C}$ to the induced arrow $F(\phi)$ of the diagram* (2.5) *below.*

(2) The functor $T\colon \mathscr{C} \to \mathscr{T}$ sends any object $C \in \mathscr{C}$ to the $\mathscr{Z}$-kernel part $T \in \mathscr{T}$ of the (unique) short $\mathscr{Z}$-exact sequence in $(PT2)$, *and any arrow $\phi\colon C \to C'$ in $\mathscr{C}$ to the induced arrow $T(\phi)$ of the diagram* (2.5) *below.*

$$\begin{tikzcd}
T \arrow[r, "\epsilon_C"] \arrow[d, dotted, "T(\phi)"'] & C \arrow[r, "\eta_C"] \arrow[d, "\phi"] & F \arrow[d, dotted, "F(\phi)"] \\
T' \arrow[r, "\epsilon_{C'}"'] & C' \arrow[r, "\eta_{C'}"'] & F'
\end{tikzcd} \tag{2.5}$$

The arrows $F(\phi)$ and $T(\phi)$ are induced by the universal properties of $\mathscr{Z}$-cokernels and $\mathscr{Z}$-kernels, respectively, and therefore make the above diagram (2.5) commutative.

Proposition 2.2.13. *Let $(\mathscr{T}, \mathscr{F})$ be a pretorsion theory in an arbitrary category $\mathscr{C}$. Then*

(1) the induced functor $F\colon \mathscr{C} \to \mathscr{F}$ is an epi-reflector*, i.e. it is a reflector such that any component of the unit of the reflection is an epimorphism;*

(2) the induced functor $T\colon \mathscr{C} \to \mathscr{T}$ is a mono-coreflector*, i.e. it is a coreflector such that any component of the counit of the coreflection is a monomorphism.*

2.3. Categorical Galois theory

Another aim of this thesis is to study *categorical Galois theory* in the category PreOrdGrp of preordered groups. In this section, we recall the main definitions and results of categorical Galois theory that will be necessary for our work. We mainly follow the articles [55, 56] by G. Janelidze and [58] by G. Janelidze and G. M. Kelly.

2.3.1. Admissible Galois structures

Definition 2.3.1. A *Galois structure* is a system $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ in which

- $\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{F} \arrow[l, bend left, "U"] \end{tikzcd}$ is an adjunction, with unit η and counit ϵ;
- $\mathcal{E}$ and $\mathcal{Z}$ are classes of morphisms in $\mathscr{C}$ and $\mathscr{F}$, respectively,

such that

- $\mathscr{C}$ and $\mathscr{F}$ admit all pullbacks along morphisms from $\mathcal{E}$ and $\mathcal{Z}$, respectively;
- $\mathcal{E}$ and $\mathcal{Z}$ are closed under composition, contain all isomorphisms and are pullback-stable;
- $F(\mathcal{E}) \subseteq \mathcal{Z}$;
- $U(\mathcal{Z}) \subseteq \mathcal{E}$.

Let $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ be a Galois structure. For any object B in $\mathscr{C}$, let $\mathcal{E}(B)$ denote the full subcategory of the slice category $\mathscr{C} \downarrow B$ determined by the morphisms $A \xrightarrow{f} B$ in $\mathcal{E}$. The objects of this subcategory are called *extensions* of B and are denoted by (A, f). Let $p \colon E \to B$ be any arrow in $\mathscr{C}$. Then, for any object (A, f) in $\mathcal{E}(B)$, $p^*(A, f)$ belongs to $\mathcal{E}(E)$ thanks to the pulback-stability of $\mathcal{E}$. It follows that the restriction of the pullback functor $p^* \colon \mathscr{C} \downarrow B \to \mathscr{C} \downarrow E$ to $\mathcal{E}(B)$ corestricts to $\mathcal{E}(E)$.

It is well-known that a Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ induces, for any object $B \in \mathscr{C}$, an adjunction

$$\begin{tikzcd} \mathcal{E}(B) \arrow[r, bend left, "F^B"] \arrow[r, phantom, "\perp"] & \mathcal{Z}(F(B)) \arrow[l, bend left, "U^B"] \end{tikzcd} \tag{2.6}$$

with unit and counit denoted by η^B and ϵ^B, respectively. The left adjoint $F^B \colon \mathcal{E}(B) \to \mathcal{Z}(F(B))$ is defined, for any $(A, f) \in \mathcal{E}(B)$, by $F^B(A, f) = (F(A), F(f))(\in \mathcal{Z}(F(B)))$, whereas the right adjoint $U^B \colon \mathcal{Z}(F(B)) \to \mathcal{E}(B)$ sends any $(X, \phi) \in \mathcal{Z}(F(B))$ to $(B \times_{UF(B)} U(X), \eta_B^*(U(\phi)))$, where $\eta_B^*(U(\phi))$ is the pullback of $U(\phi)$ along η_B:

$$\begin{tikzcd} B \times_{UF(B)} U(X) \arrow[r, "\pi_2"] \arrow[d, "\pi_1 = \eta_B^*(U(\phi))"'] & U(X) \arrow[d, "U(\phi)"] \\ B \arrow[r, "\eta_B"'] & UF(B). \end{tikzcd} \tag{2.7}$$

For any $(A, f) \in \mathcal{E}(B)$, the (A, f)-component of the unit η^B is given by $\eta^B_{(A,f)} = \langle f, \eta_A \rangle$ where $\langle f, \eta_A \rangle$ is the (unique) dotted induced arrow in the pullback diagram

$$\begin{tikzcd} A \arrow[dr, dotted, "\langle f, \eta_A \rangle"] \arrow[drr, bend left, "\eta_A"] \arrow[ddr, bend right, "f"'] & & \\ & B \times_{UF(B)} UF(A) \arrow[r] \arrow[d] & UF(A) \arrow[d, "UF(f)"] \\ & B \arrow[r, "\eta_B"'] & UF(B) \end{tikzcd}$$

with η_A the A-component of the unit η of the adjunction $F \dashv U$. The (X, ϕ)-component of the counit ϵ^B (for $(X, \phi) \in \mathcal{Z}(F(B))$) is given by $\epsilon^B_{(X,\phi)} = \epsilon_X \cdot F(\pi_2)$ where ϵ_X is the X-component of the counit ϵ of the adjunction $F \dashv U$ and π_2 the second projection of the pullback (2.7).

Definition 2.3.2. A Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ is *admissible* when, for any $B \in \mathscr{C}$, the counit morphism ϵ^B of the adjunction (2.6) is an isomorphism.

There is now an equivalent way to define the admissibility of a Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$, which corresponds to the equivalence (1) $\Leftrightarrow$ (2) in the following Proposition 2.3.3. Under an additional condition on the Galois structure Γ, we also obtain the equivalence with (3). This extra condition is that the counit ϵ of the adjunction $F \dashv U$ is an isomorphism or, equivalently, that the functor $U : \mathscr{F} \to \mathscr{C}$ is fully faithful (by Proposition 1.1.39). In this thesis, we will always be in such a situation, so that it will be possible to use this last equivalent definition of the admissibility.

Proposition 2.3.3. *Let $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ be a Galois structure such that the counit ϵ of the adjunction $F \dashv U$ is an isomorphism. Then, the following conditions are equivalent:*

(1) Γ is admissible;

(2) for any $B \in \mathscr{C}$, the functor $U^B : \mathcal{Z}(F(B)) \to \mathcal{E}(B)$ is fully faithful;

(3) F preserves all pullbacks of the form (2.7) where $\phi \in \mathcal{Z}$.

Proof. The equivalence between (1) and (2) follows directly from the well-known general Proposition 1.1.39. It now remains to prove the equivalence between (1) and (3).
Assume that condition (1) holds and consider a pullback of the form (2.7) with $\phi \in \mathcal{Z}$. By assumption, the arrows ϵ_X and $\epsilon^B_{(X,\phi)}$ are then isomorphisms. Therefore, $F(\pi_2)$ is also an isomorphism (by definition of $\epsilon^B_{(X,\phi)}$). Now, from the fact that the arrow $\epsilon_{F(B)} : FUF(B) \to F(B)$ is an isomorphism, it follows that $F(\eta_B)$ is also an isomorphism since $\epsilon_{F(B)} \cdot F(\eta_B) = 1_{F(B)}$ (see Remark 1.1.38). The morphisms $F(\pi_2)$ and $F(\eta_B)$ being two isomorphisms implies that the square

$$\begin{tikzcd}
F(B \times_{UF(B)} U(X)) \arrow[r, "F(\pi_2)", "\cong"'] \arrow[d, "F(\pi_1)"'] & FU(X) \arrow[d, "FU(\phi)"] \\
F(B) \arrow[r, "\cong", "F(\eta_B)"'] & FUF(B)
\end{tikzcd}$$

is a pullback, i.e. we have proved condition (3).
Conversely, condition (3) implies that $F(\pi_2)$ is an isomorphism by pullback-stability of isomorphisms (as explained above, $F(\eta_B)$ is an isomorphism since so is $\epsilon_{F(B)}$). Then $\epsilon^B_{(X,\phi)}$ is an isomorphism because it is the composite of two isomorphisms (ϵ_X and $F(\pi_2)$). $\square$

2.3.2. Trivial, central and normal extensions

From now on, we assume that an admissible Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ as in Proposition 2.3.3 has been fixed, and we recall the notions of (Γ-)*trivial*, (Γ-)*central* and (Γ-)*normal extensions*.

Definition 2.3.4. An arrow $f\colon A \to B$ in $\mathcal{E}$ is a (Γ-)*trivial extension* (or a (Γ-)*trivial covering*) when the naturality square

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & UF(A) \arrow[d, "UF(f)"] \\
B \arrow[r, "\eta_B"'] & UF(B)
\end{tikzcd} \tag{2.8}$$

is a pullback.

There is also an equivalent definition for the above notion of (Γ-)trivial extension:

Proposition 2.3.5. *An arrow $f\colon A \to B$ in $\mathcal{E}$ is a* (Γ-)trivial extension *if and only if (A, f) lies in the image of the functor $U^B\colon \mathcal{Z}(F(B)) \to \mathcal{E}(B)$.*

Proof. Assume first that f is trivial (in the sense of Definition 2.3.4). This means that the square (2.8) is a pullback, i.e. $f = \eta_B^*(UF(f))$. In other words, $(A, f) = U^B(F(A), F(f))$.

Conversely, if (A, f) is in the image of U^B, then there exists an extension $(X, \phi) \in \mathscr{Z}(F(B))$ such that $f = \eta_B^*(U(\phi))$, which means that there is a pullback square of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & U(X) \arrow[d, "U(\phi)"] \\
B \arrow[r, "\eta_B"'] & UF(B).
\end{tikzcd} \tag{2.9}$$

By admissibility of Γ, F preserves the above pullback. And, since right adjoints always preserve pullbacks (see Proposition 1.1.40), we conclude that the composite $U \cdot F$ preserves the above pullback. Observe now that, for any $X \in \mathscr{F}$, the X-component $\epsilon_X \colon FU(X) \to X$ of the counit ϵ of the adjunction $F \dashv U$ is an isomorphism, so that $FU(X) \cong X$. As a consequence, if we apply the functor $U \cdot F$ to the pullback (2.9), we get the front square of the following commutative diagram:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "\eta_A"] \arrow[dd, "f"'] & & U(X) \arrow[dr, equal] \arrow[dd, "U(\phi)"' near start] & \\
& UF(A) \arrow[rr, crossing over] & & UFU(X) \cong U(X) \arrow[dd, "U(\phi)"] \\
B \arrow[rr, "\eta_B" near end] \arrow[dr, "\eta_B"'] & & UF(B) \arrow[dr, equal] & \\
& UF(B) \arrow[rr, equal] \arrow[from=uu, crossing over, "UF(f)" near end] & & UFUF(B) \cong UF(B).
\end{tikzcd}$$

The right-hand square is obviously a pullback. Since the back square is also a pullback by assumption, it follows, by commutativity of the cube and Proposition 1.1.44(1), that the rectangle

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & UF(A) \arrow[r] \arrow[d, "UF(f)"] & U(X) \arrow[d, "U(\phi)"] \\
B \arrow[r, "\eta_B"'] & UF(B) \arrow[r, equal] & UF(B)
\end{tikzcd}$$

is a pullback. Thanks to the observation made above (i.e. the fact that the right-hand square of the above rectangle is a pullback), the left-hand square is therefore a pullback according to Proposition 1.1.44(2), and this completes the proof. □

Definition 2.3.6. A morphism $p\colon E \to B$ in $\mathcal{E}$ is called a *monadic extension* when it is an effective descent morphism.

Definition 2.3.7. An arrow $f\colon A \to B$ in $\mathcal{E}$ is a $(\Gamma$-$)$*central extension* (or a $(\Gamma$-$)$*covering*) when there exists a monadic extension $p\colon E \to B$ such that $p^*(f)\colon E \times_B A \to E$ is a $(\Gamma$-$)$trivial extension, that is, the following square

$$\begin{tikzcd}
E \times_B A \arrow[r, "\eta_{E \times_B A}"] \arrow[d, "p^*(f)=\pi_1"'] & UF(E \times_B A) \arrow[d, "UF(\pi_1)"] \\
E \arrow[r, "\eta_E"'] & UF(E)
\end{tikzcd}$$

is a pullback, where π_1 is the first projection in the pullback

$$\begin{tikzcd}
E \times_B A \arrow[r, "\pi_2"] \arrow[d, "p^*(f)=\pi_1"'] & A \arrow[d, "f"] \\
E \arrow[r, "p"'] & B.
\end{tikzcd}$$

Definition 2.3.8. An arrow $f\colon A \to B$ in $\mathcal{E}$ is a $(\Gamma$-$)$*normal extension* (or a $(\Gamma$-$)$*normal covering*) when f is a monadic extension and $f^*(f)$ is a $(\Gamma$-$)$trivial extension.

Proposition 2.3.9 below now allows us to see the links between the different concepts of $(\Gamma$-$)$trivial, $(\Gamma$-$)$central and $(\Gamma$-$)$normal extensions.

Proposition 2.3.9.

(1) Any $(\Gamma$-$)$*trivial extension is* $(\Gamma$-$)$*central.*

(2) Any $(\Gamma$-$)$*normal extension is* $(\Gamma$-$)$*central.*

(3) Any $(\Gamma$-$)$*trivial extension is* $(\Gamma$-$)$*normal.*

Proof. The assertions (1) and (2) are clear by definition. Let us now prove the third statement. Let $f\colon A \to B$ be a $(\Gamma$-$)$trivial extension.

Then the rectangle

$$\begin{tikzcd}
\mathsf{Eq}(f) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[r, "\eta_A"] \arrow[d, "f"] & UF(A) \arrow[d, "UF(f)"] \\
A \arrow[r, "f"'] & B \arrow[r, "\eta_B"'] & UF(B)
\end{tikzcd} \tag{2.10}$$

is a pullback since it is the composite of two pullbacks (see Proposition 1.1.44(1)), which amounts to saying that the external rectangle

$$\begin{tikzcd}
\mathsf{Eq}(f) \arrow[r, "\eta_{\mathsf{Eq}(f)}"] \arrow[dd, "\pi_1"'] & UF(\mathsf{Eq}(f)) \arrow[rr, "UF(\pi_2)"] \arrow[dd, "UF(\pi_1)"'] \arrow[dr, dotted, "\phi"] & & UF(A) \arrow[dd, "UF(f)"] \\
& & \mathsf{Eq}(UF(f)) \arrow[ur, "p_2"'] \arrow[dl, "p_1"] & \\
A \arrow[r, "\eta_A"'] & UF(A) \arrow[rr, "UF(f)"'] & & UF(B)
\end{tikzcd} \tag{2.11}$$

is a pullback. Consider now the kernel pair $(\mathsf{Eq}(UF(f)), p_1, p_2)$ of $UF(f)$ as well as the induced arrow $\phi\colon UF(\mathsf{Eq}(f)) \to \mathsf{Eq}(UF(f))$. Knowing that U preserves limits (by Proposition 1.1.40), $\mathsf{Eq}(UF(f))$ is then isomorphic to $U(\mathsf{Eq}(F(f)))$, and $p_1 \cong U(\rho_1)$ and $p_2 \cong U(\rho_2)$ where ρ_1 and ρ_2 are the two projections of the kernel pair of $F(f)$. We next observe, thanks to Proposition 1.1.44(2) that the square

$$\begin{tikzcd}
\mathsf{Eq}(f) \arrow[r, "\phi \cdot \eta_{\mathsf{Eq}(f)}"] \arrow[d, "\pi_1"'] & U(\mathsf{Eq}(F(f))) \arrow[d, "U(\rho_1)"] \\
A \arrow[r, "\eta_A"'] & UF(A)
\end{tikzcd}$$

is a pullback. Now, we know that $f \in \mathcal{E}$, so $F(f) \in \mathcal{Z}$ and, by pullback-stability of $\mathcal{Z}$, ρ_1 is then in $\mathcal{Z}$. By admissibility of Γ, F then preserves the above pullback. The functor F also preserves the pullback

$$\begin{tikzcd}
U(\mathsf{Eq}(F(f))) \arrow[r, "U(\rho_2)"] \arrow[d, "U(\rho_1)"'] & UF(A) \arrow[d, "UF(f)"] \\
UF(A) \arrow[r, "UF(f)"'] & UF(B).
\end{tikzcd}$$

Indeed, since ϵ is an isomorphism, the functor F sends the above square to the following square

$$\begin{tikzcd} \mathsf{Eq}(F(f)) \arrow[r, "\rho_2"] \arrow[d, "\rho_1"'] & F(A) \arrow[d, "F(f)"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

which is obviously a pullback. According to Proposition 1.1.44(1), F therefore preserves the pullback rectangle (2.11), that is, F preserves the pullback rectangle (2.10). In other words, the square

$$\begin{tikzcd} F(\mathsf{Eq}(f)) \arrow[r, "F(\pi_2)"] \arrow[d, "F(\pi_1)"'] & F(A) \arrow[d, "F(f)"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

is a pullback (since ϵ is an isomorphism). Using again the fact that U preserves limits, we have then proved that the right-hand square of (2.11) is a pullback. We conclude by Proposition 1.1.44(2) that the left-hand square of (2.11) is a pullback, which means that π_1 is a (Γ-)trivial extension or, in other words, that f is a (Γ-)normal extension. $\square$

2.3.3. An important example

We now illustrate, in a particular situation, the different notions seen in the previous two subsections. The results proved below will be of major interest in our future developments, more particularly in Chapter 5.

Proposition 2.3.10. *Consider the reflection* (1.3) *of Example 1.1.43 between the categories* Grp *of groups and* Ab *of abelian groups. Let* $\mathcal{E}_{ab}$ *and* $\mathcal{Z}_{ab}$ *be the classes of surjective group homomorphisms in* Grp *and* Ab*, respectively. Then the system*

$$\Gamma_{ab} = (\mathsf{Grp}, \mathsf{Ab}, ab, u, \mathcal{E}_{ab}, \mathcal{Z}_{ab})$$

is a Galois structure.

Proof. It is clear that these two classes $\mathcal{E}_{ab}$ and $\mathcal{Z}_{ab}$ are closed under composition, contain all isomorphisms and are pullback-stable. Moreover, the categories Grp and Ab obviously admit all pullbacks (thus, in particular, all pullbacks along morphisms from $\mathcal{E}_{ab}$ and $\mathcal{Z}_{ab}$, respectively). Finally, we easily see that $U(\mathcal{Z}_{ab}) \subseteq \mathcal{E}_{ab}$ and $ab(\mathcal{E}_{ab}) \subseteq \mathcal{Z}_{ab}$. The first inclusion is clear. The second follows from the fact that the functor ab, as a left adjoint, preserves regular epimorphisms. $\square$

Proposition 2.3.11. *The Galois structure Γ_{ab} of Proposition 2.3.10 is admissible.*

Proof. Consider, in Grp, the following commutative diagram

$$\begin{tikzcd}
\mathsf{Ker}(p_2) \arrow[rr, tail, "\mathsf{ker}(p_2)"] & & P \arrow[rr, two heads, "p_2"] \arrow[dr, two heads, "\eta_P"'] \arrow[dd, two heads, "p_1"'] & & A \arrow[dd, two heads, "e"] \\
& \mathsf{Ker}(\eta_P) \arrow[ur, tail, "\mathsf{ker}(\eta_P)"'] \arrow[ul, dotted, tail, "\phi"] & & P/[P,P] \arrow[ur, dotted, two heads, "\alpha"'] & \\
& & G \arrow[rr, two heads, "\eta_G"'] & & G/[G,G]
\end{tikzcd}$$

where A is an abelian group, η_G the G-component of the adjunction (1.3), e a morphism in $\mathcal{Z}_{ab}$ (that is, a surjective group morphism), and (P, p_1, p_2) the pullback of η_G and e. Since $A \in \mathsf{Ab}$, there exists, by the universal property of adjunctions, a unique arrow $\alpha\colon P/[P,P] \to A$ such that $\alpha \cdot \eta_P = p_2$. We now compute that $p_2 \cdot \mathsf{ker}(\eta_P) = \alpha \cdot \eta_P \cdot \mathsf{ker}(\eta_P) = \alpha \cdot 0 = 0$ so that, by the universal property of kernels, there exists a unique morphism $\phi\colon \mathsf{Ker}(\eta_P) \to \mathsf{Ker}(p_2)$ satisfying the identity $\mathsf{ker}(p_2) \cdot \phi = \mathsf{ker}(\eta_P)$. Since $\mathsf{ker}(\eta_P)$ is a monomorphism, it follows that ϕ also is a monomorphism, that is, $\mathsf{Ker}(\eta_P) \subseteq \mathsf{Ker}(p_2)$. Let us now check the converse inclusion $\mathsf{Ker}(p_2) \subseteq \mathsf{Ker}(\eta_P)$. Take an element p in the kernel of p_2. Then, thanks to Proposition 1.1.45, it is of the form $(x, 0)$ for 0 the neutral element of A and x an element of G in the kernel of η_G, i.e. $x \in [G,G]$. As a consequence, we can find $g_1, g_2, \cdots, g_{n-1}, g_n \in G$ such that $x = g_1 + g_2 - g_1 - g_2 + \cdots + g_{n-1} + g_n - g_{n-1} - g_n$. Now, $\eta_G(g_i) \in G/[G,G]$ for any $i \in \{1, 2, \cdots, n\}$ and e is surjective. Accordingly, for any $i \in \{1, 2, \cdots, n\}$, there is an element a_i of A for which $e(a_i) = \eta_G(g_i)$.

This means in particular that $(g_i, a_i) \in P$ for any $i \in \{1, 2, \cdots, n\}$. We now observe that $a_{2i-1}+a_{2i}-a_{2i-1}-a_{2i} = a_{2i-1}-a_{2i-1}+a_{2i}-a_{2i} = 0$, for any $i \in \{1, 2, \cdots, n/2\}$, since the group A is abelian. As a consequence,

$$\begin{aligned}(x, 0) &= \left(\sum_{i=1}^{n/2} g_{2i-1} + g_{2i} - g_{2i-1} - g_{2i}, \sum_{i=1}^{n/2} a_{2i-1} + a_{2i} - a_{2i-1} - a_{2i}\right) \\ &= \sum_{i=1}^{n/2} (g_{2i-1}, a_{2i-1}) + (g_{2i}, a_{2i}) - (g_{2i-1}, a_{2i-1}) - (g_{2i}, a_{2i}) \\ &\in [P, P] = \mathsf{Ker}(\eta_P),\end{aligned}$$

which proves that $\mathsf{Ker}(p_2) \subseteq \mathsf{Ker}(\eta_P)$, and therefore that ϕ is an isomorphism. Note now that η_P is a regular epimorphism by construction and that p_2 is also regular thanks to the pullback-stability of regular epimorphisms in the regular category Grp of groups. The category Grp being normal, any regular epimorphism is the cokernel of its kernel. In particular,

$$\eta_P \cong \mathsf{coker}(\mathsf{ker}(\eta_P)) \cong \mathsf{coker}(\mathsf{ker}(p_2)) \cong p_2,$$

which proves that α is an isomorphism, i.e. $A \cong P/[P, P]$. As a consequence, the image under the functor ab of the pullback (P, p_1, p_2) is the square

$$\begin{tikzcd} P/[P,P] \arrow[r, equal] \arrow[d, two heads, "ab(p_1)\cong e"'] & P/[P,P] \arrow[d, two heads, "e"] \\ G/[G,G] \arrow[r, equal] & G/[G,G] \end{tikzcd}$$

which is obviously a pullback in Ab. This completes the proof of the admissibility of Γ_{ab}, thanks to Proposition 2.3.3. □

We have seen previously that any normal extension is central. While the converse of this statement does not hold in general, it does in our situation. In order to prove this (Proposition 2.3.16), it is first useful to make some preliminary observations.

Lemma 2.3.12. *Consider the admissible Galois structure Γ_{ab} of Proposition 2.3.11. Then a group homomorphism $p: E \to B$ is a monadic extension if and only if it is surjective.*

Proof. The proof is direct as soon as we remember that the category Grp of groups is (Barr-)exact so that the effective descent morphisms are given by the regular epimorphisms (see Example 1.3.7(2)). $\square$

Lemma 2.3.13. *Let $f\colon A \to B$ be any group homomorphism. If $K = \mathsf{Ker}(f)$ is in the center $Z(A) = \{x \in A \mid x + a = a + x \text{ for any } a \in A\}$ of A, then $\mathsf{Eq}(f) \cong K \times A$.*

Proof. It suffices to see that the function

$$\Phi\colon \mathsf{Eq}(f) \to K \times A; (x, y) \mapsto (x - y, y)$$

is a group homomorphism (since $K \subseteq Z(A)$), with inverse

$$\Psi\colon K \times A \to \mathsf{Eq}(f); (c, a) \mapsto (c + a, a).$$ $\square$

Lemma 2.3.14. *The abelianization functor $ab\colon \mathsf{Grp} \to \mathsf{Ab}$ preserves the binary products:*

$$ab(X \times Y) \cong ab(X) \times ab(Y) \qquad \text{for any } X, Y \in \mathsf{Grp}.$$

Proof. It is clear that any element of $[X \times Y, X \times Y]$ can be written as an element of $[X, X] \times [Y, Y]$ (and *vice versa*). The kernels of $\eta_{X\times Y}$ and $\eta_X \times \eta_Y$ are then isomorphic. Now, any regular epimorphism in a normal category is the cokernel of its kernel, so that

$$\eta_{X\times Y} \cong \mathsf{coker}(\mathsf{ker}(\eta_{X\times Y})) \cong \mathsf{coker}(\mathsf{ker}(\eta_X \times \eta_Y)) \cong \eta_X \times \eta_Y,$$

hence the announced isomorphism. $\square$

Lemma 2.3.15. *Consider the following pullback*

$$\begin{tikzcd} E \times_B A \arrow[r, two heads, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ E \arrow[r, two heads, "p"'] & B \end{tikzcd}$$

in the category Grp of groups, where p (and then also π_2) is surjective. Then $\mathsf{Ker}(f) \subseteq Z(A)$ if and only if $\mathsf{Ker}(\pi_1) \subseteq Z(E \times_B A)$.

Proof. Let $(0, a) \in \mathsf{Ker}(\pi_1)$ and $(e, a') \in E \times_B A$. In particular, we then have that $a \in \mathsf{Ker}(f)$ (by Proposition 1.1.45), so that

$$(0, a) + (e, a') = (e, a + a') = (e, a' + a) = (e, a') + (0, a)$$

since $\mathsf{Ker}(f) \subseteq Z(A)$.
Conversely, let $x \in \mathsf{Ker}(f)$ and $a \in A$. Then, we have that $(0, x) \in \mathsf{Ker}(\pi_1)$ (again by Proposition 1.1.45). In addition, since p is surjective, there exists an element $e \in E$ such that $p(e) = f(a)$, and then such that $(e, a) \in E \times_B A$. By assumption, $(0, x) + (e, a) = (e, a) + (0, x)$, which implies that $x + a = a + x$. $\square$

Proposition 2.3.16. [55] *Let $f \colon A \twoheadrightarrow B$ be a surjective group homomorphism. Then, the following conditions are equivalent:*
(1) $\mathsf{Ker}(f) \subseteq Z(A)$*;*
(2) f *is a* $(\Gamma$-$)$*normal extension;*
(3) f *is a* $(\Gamma$-$)$*central extension.*

Proof.
- (1) $\Rightarrow$ (2): If $K = \mathsf{Ker}(f) \subseteq Z(A)$, then $\mathsf{Eq}(f) \cong K \times A$ thanks to Lemma 2.3.13. By Lemma 2.3.14, we now observe that $ab(K \times A) \cong ab(K) \times ab(A) = K \times ab(A)$, since $K \in \mathsf{Ab}$, hence $ab(\mathsf{Eq}(f)) \cong K \times ab(A)$. Consider then the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Eq}(f) \cong K \times A \arrow[r, two heads, "\eta_{K\times A}"] \arrow[d, "\pi_2"'] & ab(\mathsf{Eq}(f)) \cong K \times ab(A) \arrow[r, "p_1"] \arrow[d, "ab(\pi_2)\cong p_2"] & K \arrow[d] \\
A \arrow[r, two heads, "\eta_A"'] & ab(A) \arrow[r] & 0
\end{tikzcd}$$

where π_2 is the second projection of the kernel pair $\mathsf{Eq}(f)$ of f, which, via the isomorphism $\mathsf{Eq}(f) \cong K \times A$, corresponds to the second projection of the product $K \times A$, and where p_1 and p_2 are the two projections of the product $K \times ab(A)$. Since products can be seen as pullbacks over the zero object 0 (see Remark 1.1.19(2)), the whole rectangle as well as the right-hand square are pullbacks. By Proposition 1.1.44(2), the left-hand square is then a pullback, which shows that π_2 is a $(\Gamma$-$)$trivial extension. In other words, f is a $(\Gamma$-$)$normal extension.

- (2) $\Rightarrow$ (3): Any (Γ-)normal extension is (Γ-)central by Proposition 2.3.9(2).
- (3) $\Rightarrow$ (1): By assumption, there exists a surjective group morphism $p\colon E \twoheadrightarrow B$ such that $p^*(f)$ is (Γ-)trivial. The two squares below (where we write P for $E \times_B A$) are then pullbacks:

$$\begin{tikzcd} P \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "p^*(f)=\pi_1"'] & A \arrow[d, two heads, "f"] \\ E \arrow[r, two heads, "p"'] & B \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, two heads, "\eta_P"] \arrow[d, two heads, "p^*(f)=\pi_1"'] & ab(P) \arrow[d, two heads, "ab(\pi_1)"] \\ E \arrow[r, two heads, "\eta_E"'] & ab(E). \end{tikzcd}$$

  Since the group $ab(P)$ is clearly abelian, it is obvious that $\mathsf{Ker}(ab(\pi_1)) \subseteq Z(ab(P))$. By Lemma 2.3.15 used twice, it follows first that $\mathsf{Ker}(\pi_1) \subseteq Z(P)$ and then that $\mathsf{Ker}(f) \subseteq Z(A)$. $\square$

2.4. Monotone-light factorization systems

Let us now introduce the notion of *monotone-light factorization system* and its links with torsion theories and categorical Galois theory. For this section, we mainly follow the articles [17], [18], [29] and [55] in which the reader will find more information.

Definition 2.4.1. [3] A *factorization system* on a category $\mathscr{C}$ is given by a pair $(\mathscr{E}, \mathscr{M})$ of classes of arrows in $\mathscr{C}$ such that

- any isomorphism of $\mathscr{C}$ is in both $\mathscr{E}$ and $\mathscr{M}$;
- $\mathscr{E}$ and $\mathscr{M}$ are closed under composition;
- for any morphism f in $\mathscr{C}$, there exist morphisms $e \in \mathscr{E}$ and $m \in \mathscr{M}$ such that $f = m \cdot e$;
- for any commutative square as below where $e \in \mathscr{E}$ and $m \in \mathscr{M}$, there exists a unique arrow ϕ making both triangles commute:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[dl, dotted, "\phi"] \\ C \arrow[r, "m"'] & D. \end{tikzcd}$$

Example 2.4.2. If the category $\mathscr{C}$ is regular, we directly obtain a factorization system $(\mathscr{E}, \mathscr{M})$ by taking the class of regular epimorphisms for $\mathscr{E}$ and that of monomorphisms for $\mathscr{M}$.

We will now see another (special) example of factorization system that arises when the category $\mathscr{C}$ has a full reflective subcategory $\mathscr{F}$ and when the reflector is *semi-left-exact* in the following sense:

Definition 2.4.3. [18] Let $\mathscr{C}$ be an arbitrary category and assume that $\mathscr{C}$ has a full reflective subcategory $\mathscr{F}$. Then the reflector $F\colon \mathscr{C} \to \mathscr{F}$ is said to be *semi-left-exact* when it preserves all pullbacks of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U(C) \arrow[d, "U(\phi)"] \\ B \arrow[r, "\eta_B"'] & UF(B) \end{tikzcd}$$

where $\eta_B\colon B \to UF(B)$ is the B-component of the unit of the reflection $F \dashv U$ and $\phi\colon C \to F(B)$ is any morphism in the subcategory $\mathscr{F}$.

Remark 2.4.4. If the categories $\mathscr{C}$ and $\mathscr{F}$ admit all pullbacks, the reflector $F\colon \mathscr{C} \to \mathscr{F}$ is then semi-left-exact if and only if the *absolute* Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U)$ is admissible (from the point of view of categorical Galois theory). Here, *absolute* means that the two classes of extensions are the classes of *all* morphisms in $\mathscr{C}$ and $\mathscr{F}$ respectively.

Let us also mention the following notion, stronger than that of semi-left-exactness:

Definition 2.4.5. Let $\mathscr{C}$ be an arbitrary category and assume that $\mathscr{C}$ has a full reflective subcategory $\mathscr{F}$. Then the reflector $F\colon \mathscr{C} \to \mathscr{F}$ is said to *have stable units* when it preserves all pullbacks of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d] & C \arrow[d, "\phi"] \\ B \arrow[r, "\eta_B"'] & UF(B) \end{tikzcd}$$

where $\eta_B\colon B \to UF(B)$ is the B-component of the unit of the reflection $F \dashv U$ and $\phi\colon C \to UF(B)$ is any morphism in the category $\mathscr{C}$.

From now on, assume that we have a full reflection

$$
\begin{tikzcd}
\mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{F} \arrow[l, bend left, "U"]
\end{tikzcd}, \tag{2.12}
$$

with unit η and counit ϵ, and define the following two classes of arrows:

- $\mathscr{E} = \{f \in \mathscr{C} \mid F(f) \text{ is an isomorphism}\}$;
- $\mathscr{M} = \{f \in \mathscr{C} \mid \text{ the naturality square (2.8) is a pullback}\}$.

It is then possible to show [18] that $(\mathscr{E}, \mathscr{M})$ is a factorization system if the reflector $F\colon \mathscr{C} \to \mathscr{F}$ is semi-left-exact. In this context, the morphisms in $\mathscr{M}$ are the (Γ-)trivial coverings (for Γ the absolute Galois structure mentioned in Remark 2.4.4).

Proposition 2.4.6. [30, Theorem 1.6], [17] *Any reflector $F\colon \mathscr{C} \to \mathscr{F}$ induced by a torsion theory $(\mathscr{T}, \mathscr{F})$ in a normal category $\mathscr{C}$ has stable units.*
As a consequence, the induced reflector F gives rise to a factorization system $(\mathscr{E}, \mathscr{M})$ as defined above.

Proof. Consider the following commutative diagram

$$
\begin{tikzcd}
 & P \arrow[r, two heads, "p_2"] \arrow[d, "p_1"] & C \arrow[d, "\phi"] \\
T(B) \arrow[ur, tail, "\ker(p_2)"] \arrow[r, tail, "\epsilon_B"'] & B \arrow[r, two heads, "\eta_B"'] & F(B)
\end{tikzcd} \tag{2.13}
$$

where η_B is the B-component of the unit of the reflection $F \dashv U$, $\phi\colon C \to F(B)$ any morphism in $\mathscr{C}$, and (P, p_1, p_2) the pullback of η_B and ϕ. Thanks to Proposition 1.1.45, the kernels of p_2 and η_B are isomorphic. Now, by pullback-stability of normal epimorphisms in the normal category $\mathscr{C}$, we have that p_2 is a normal epimorphism. More precisely, it is the cokernel of its kernel: $p_2 \cong \mathsf{coker}(\mathsf{ker}(p_2))$. Let us now apply the functor F to the diagram (2.13). We then get the following commutative diagram

$$
\begin{tikzcd}
 & F(P) \arrow[r, two heads, "F(p_2)"] \arrow[d, "F(p_1)"] & F(C) \arrow[d, "F(\phi)"] \\
0 \arrow[ur] \arrow[r] & F(B) \arrow[r, equal] & F(B)
\end{tikzcd} \tag{2.14}
$$

since, for any $Y \in \mathscr{T}$, $F(Y) = 0$ and, for any $X \in \mathscr{F}$, $F(X) = X$. Using the fact that F is a left adjoint, we know that F preserves colimits (by the dual of Proposition 1.1.40). In particular, $F(p_2) \cong \mathsf{coker}(F(\mathsf{ker}(p_2))) \cong \mathsf{coker}(0)$. It then follows that $F(p_2)$ is an isomorphism and therefore that the square in diagram (2.14) is a pullback. This proves that F preserves the pullback in diagram (2.13), i.e. F has stable units. $\square$

Given the reflection (2.12), consider now the following classes of arrows in $\mathscr{C}$:

- $\mathscr{E}'$ is the class of morphisms $f \in \mathscr{C}$ such that the pullback of f along any morphism in $\mathscr{C}$ is in $\mathscr{E}$;
- $\mathscr{M}^*$ is the class of morphisms $f \in \mathscr{C}$ for which there exists an effective descent morphism p such that $p^*(f) \in \mathscr{M}$.

The class $\mathscr{E}'$ (the so-called *stabilization* of $\mathscr{E}$) is in $\mathscr{E}$ while the class $\mathscr{M}^*$ (the so-called *localization* of $\mathscr{M}$) contains $\mathscr{M}$. The class $\mathscr{M}^*$ is the class of (Γ-)coverings (where, once again, Γ is the absolute Galois structure mentioned in Remark 2.4.4).
The pair $(\mathscr{E}', \mathscr{M}^*)$ is not a factorization system in general. In a few situations though, it is. In this case, we say that $(\mathscr{E}', \mathscr{M}^*)$ a *monotone-light factorization system*:

Definition 2.4.7. A factorization system is said to be *monotone-light* when it is of the form $(\mathscr{E}', \mathscr{M}^*)$ for some factorization system $(\mathscr{E}, \mathscr{M})$.

Example 2.4.8. The terminology "monotone-light" comes from the *Eilenberg/Whyburn monotone-light factorization system* $(\mathscr{E}', \mathscr{M}^*)$ in the category $\mathsf{CompHaus}$ of compact Hausdorff spaces [26, 80], where

- $\mathscr{E}'$ is the class of *monotone* maps, i.e. continuous maps between compact Hausdorff spaces whose fibers are connected;
- $\mathscr{M}^*$ is the class of *light* maps, i.e. continuous maps between compact Hausdorff spaces whose fibers are totally disconnected.

Let us now recall this really useful result which will turn out to be needed later on. For a proof of this, we refer the reader to [29, Theorem 1.6].

Theorem 2.4.9. *Let $\mathscr{C}$ be a normal category, $(\mathscr{T}, \mathscr{F})$ a torsion theory in $\mathscr{C}$, and $(\mathscr{E}, \mathscr{M})$ the associated factorization system. If*

- *(N) for any normal monomorphism $k\colon K \to A$, the monomorphism $k \cdot \epsilon_K\colon T(K) \to A$ is normal in $\mathscr{C}$, where $\epsilon_K\colon T(K) \to K$ is the K-component of the counit ϵ of the induced coreflection $T\colon \mathscr{C} \to \mathscr{T}$;*
- *(C) for any object $C \in \mathscr{C}$, there is an effective descent morphism $p\colon X \to C$ with $X \in \mathscr{F}$,*

then $(\mathscr{E}', \mathscr{M}^*)$ is a monotone-light factorization system, and moreover

- *$\mathscr{E}'$ is the class of normal epimorphisms in $\mathscr{C}$ whose kernel is in $\mathscr{T}$;*
- *$\mathscr{M}^*$ is the class of morphisms in $\mathscr{C}$ whose kernel is in $\mathscr{F}$.*

3 | Torsion theories and coverings of preordered groups

In the category $\mathsf{PreOrdGrp}$ of preordered groups, there is a torsion theory, whose torsion-free subcategory is the subcategory $\mathsf{ParOrdGrp}$ of *partially ordered groups* (i.e. preordered groups whose preorder is antisymmetric) and whose torsion subcategory is that of *indiscrete preordered groups* (i.e. preordered groups endowed with the indiscrete relation). From this torsion theory arises a (normal epi)-reflector $F\colon \mathsf{PreOrdGrp} \to \mathsf{ParOrdGrp}$ which has stable units. This, in turn, naturally induces a factorization system $(\mathscr{E}, \mathscr{M})$, where $\mathscr{E}$ is the class of arrows in $\mathsf{PreOrdGrp}$ inverted by the reflector F and $\mathscr{M}$ the class of trivial coverings (for the absolute Galois structure induced by F). A fairly simple characterization can also be obtained for this last class: a morphism $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ is a trivial covering (i.e. belongs to $\mathscr{M}$) if and only if the restriction $\phi\colon N_G \to N_H$ of f to $N_G = \{x \in G \mid x \in P_G \text{ and } -x \in P_G\}$ is a group isomorphism. We can further prove that, by a process of simultaneous stabilization and localization of these two classes $\mathscr{E}$ and $\mathscr{M}$, we again get a factorization system $(\mathscr{E}', \mathscr{M}^*)$, which is therefore monotone-light. Moreover, it is possible to describe precisely these two classes: the class $\mathscr{E}'$ is the class of normal epimorphisms in $\mathsf{PreOrdGrp}$ whose kernel is an indiscrete preordered group while the class $\mathscr{M}^*$ is the class of all morphisms in $\mathsf{PreOrdGrp}$ with a partially ordered kernel. This last class is the class of all coverings (with respect to the absolute Galois structure induced by the reflector F).

We then make two interesting observations. We first notice that these coverings can be classified in terms of *internal actions* over the *Galois* *groupoid* of some special arrow. This result can be obtained via the theory on *locally semisimple coverings* developed by G. Janelidze, L. Márki and W. Tholen in [59]. We next observe that the full subcategory ParOrdGrp of partially ordered groups is a torsion-free subcategory not only for the above mentioned torsion theory but also for a pretorsion theory. In this case, the torsion subcategory is the full subcategory of those preordered groups whose preorder is symmetric, which clearly coincides with the category Grp(PreOrd) of *internal groups in the category of preordered sets*. The category Grp(PreOrd) is then coreflective in PreOrdGrp. It turns out that it is in addition reflective.

This chapter is essentially based on the article [41] written in collaboration with M. Gran.

3.1. A torsion theory in the category of preordered groups

The main goal of this section is to prove that there is a torsion theory in the category PreOrdGrp of preordered groups. Its torsion-free and torsion subcategories are given, respectively, by the full subcategories of *partially ordered groups* and *indiscrete preordered groups*. Below is their description:

- The subcategory ParOrdGrp is the subcategory of *partially ordered groups*, i.e. the full subcategory whose objects are preordered groups (G, P_G) where the positive cone P_G is *reduced*. This means that, if $x + y = 0$ in P_G, then $x = y = 0$ or, in other words, that the only element of P_G having its inverse in P_G is the neutral element 0. Using the isomorphism of categories of Proposition 2.1.5, partially ordered groups can alternatively be seen as preordered groups $(G, \leq)$ where the preorder $\leq$ is a partial order (i.e. an antisymmetric preorder).
- The subcategory of *indiscrete preordered groups* is the full subcategory of preordered groups $(G, \leq)$ where the preorder $\leq$ is indiscrete, i.e. $a \leq b$ for any $a, b \in G$. Using the category isomorphism of Proposition 2.1.5, an object in this subcategory

corresponds to a pair (G, G), i.e. to a preordered group where the positive cone is the underlying group itself. This subcategory is therefore isomorphic to the category of groups and this is the reason why we will use the notation Grp for it.

Proposition 3.1.1. *The pair of full and replete subcategories* $(\mathsf{Grp}, \mathsf{ParOrdGrp})$ *of* $\mathsf{PreOrdGrp}$ *is a torsion theory in the normal category* $\mathsf{PreOrdGrp}$.

Proof. Let us first show that the only arrow in $\mathsf{PreOrdGrp}$ from an object of Grp to an object of $\mathsf{ParOrdGrp}$ is the zero morphism. Consider an arrow $(f, \bar{f})\colon (G, G) \to (H, P_H)$ in $\mathsf{PreOrdGrp}$, with (G, G) in Grp and (H, P_H) in $\mathsf{ParOrdGrp}$:

$$\begin{tikzcd} G \arrow[r, "\bar{f}"] \arrow[d, equal] & P_H \arrow[d, tail] \\ G \arrow[r, "f"'] & H. \end{tikzcd}$$

For any $x \in G$, its opposite $-x$ is also in G, and

$$0 = \bar{f}(x - x) = \bar{f}(x) + \bar{f}(-x),$$

where $\bar{f}(x), \bar{f}(-x) \in P_H$, with P_H a reduced monoid. This implies that $\bar{f}(x) = \bar{f}(-x) = 0$, $\bar{f} = 0$, and then $f = 0$.
Consider then an object (G, P_G) of $\mathsf{PreOrdGrp}$, and define

$$N_G = \{\, n \in G \mid n \in P_G \text{ and } -n \in P_G \,\}.$$

It is a normal subgroup of G: indeed, if $n \in N_G$ and $x \in G$, then we have that $x + n - x \in P_G$ and $-(x + n - x) = x - n - x \in P_G$, since the submonoid P_G is closed under conjugation in G. Accordingly, the sequence $N_G \overset{k_G}{\rightarrowtail} G \overset{\eta_G}{\twoheadrightarrow} G/N_G$ is a short exact sequence in the category Grp of groups. Consider next the direct image factorization of the morphism $\eta_G \cdot g$ in the category Mon of monoids, where $P_G \overset{g}{\rightarrowtail} G$ is the inclusion of P_G into G:

$$\begin{tikzcd} P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail, "g"'] & \eta_G(P_G) \arrow[d, tail, "\psi_G"] \\ G \arrow[r, two heads, "\eta_G"'] & G/N_G. \end{tikzcd}$$

Let us now prove that the sequence

$$\begin{tikzcd} N_G \arrow[r, tail, "\bar{k}_G"] \arrow[d, equal] & P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail, "g"'] & \eta_G(P_G) \arrow[d, tail, "\psi_G"] \\ N_G \arrow[r, tail, "k_G"'] & G \arrow[r, two heads, "\eta_G"'] & G/N_G \end{tikzcd} \tag{3.1}$$

is a short exact sequence in the category $\mathsf{PreOrdGrp}$ of preordered groups. This follows from Proposition 2.1.10, since the left-hand square in (3.1) is clearly a pullback in Mon, the lower sequence is exact, and the morphism $\bar{\eta}_G$ is surjective by construction.

It is obvious that $(N_G, N_G) \in \mathsf{Grp}$, so that the proof will be complete if we show that $(G/N_G, \eta_G(P_G))$ is in $\mathsf{ParOrdGrp}$. Now, if $y + z = 0$ for $y, z \in \eta_G(P_G)$, then there exist $x, x' \in P_G$ such that $\eta_G(x) = y$ and $\eta_G(x') = z$, so that $\eta_G(x + x') = y + z = 0$, that is $x + x' \in N_G$. Since N_G is a group and P_G is a monoid, it follows that

$$-x = x' - x' - x = x' - (x + x') \in P_G,$$

hence $x \in N_G$, which implies that $y = \eta_G(x) = 0$ and therefore $z = 0$. Accordingly, the submonoid $\eta_G(P_G)$ is reduced, and $(G/N_G, \eta_G(P_G))$ a partially ordered group. $\square$

Remark 3.1.2. Note that a similar result can be proved in the category of commutative monoids, as observed in [31]: the pair $(\mathsf{Ab}, \mathsf{RedComMon})$ is a torsion theory in the category ComMon of commutative monoids, where we write Ab for the category of abelian groups and $\mathsf{RedComMon}$ for the category of reduced commutative monoids.

Remark 3.1.3. In view of the proof of Proposition 3.1.1, it would be legitimate to ask the following question: are there examples of preordered groups (G, P_G) for which the normal subgroup N_G is not trivial (i.e. not the trivial group $\{0\}$ nor the group G itself)? Indeed, in our prototypical example of preordered group $(\mathbb{Z}, \mathbb{N})$ for instance, we have $N_G = \{0\}$. Here are now two non-trivial examples:

(1) Consider the group $G = \mathbb{R}^{2\times 2}$ of real square matrices of order 2. In this case, G is an abelian group (for the addition) and so, by

Remark 2.1.8, all submonoids of G can be considered as positive cones of G. Take the submonoid

$$P_G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in \mathbb{R}^{2\times 2} \mid a \geq 0 \right\}$$

in particular. Then, (G, P_G) is a preordered abelian group: $(G, P_G) \in \mathsf{PreOrdAb}$. In this situation, the normal subgroup N_G is given by

$$N_G = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2} \right\},$$

which is non-trivial.

(2) As another example, we can take

$$G = \left\{ \begin{pmatrix} q & r \\ 0 & q \end{pmatrix} \in \mathbb{R}^{2\times 2} \mid q, r \in \mathbb{Q}, q \neq 0 \right\}$$

endowed with the matrix multiplication. It is an abelian group with inverse

$$\frac{1}{q^2} \cdot \begin{pmatrix} q & -r \\ 0 & q \end{pmatrix}$$

for any element

$$\begin{pmatrix} q & r \\ 0 & q \end{pmatrix} \in G.$$

Thanks again to Remark 2.1.8, all submonoids of G can be considered as positive cones of G, so that $(G, P_G) \in \mathsf{PreOrdAb}$ for

$$P_G = \left\{ \begin{pmatrix} q & r \\ 0 & q \end{pmatrix} \in \mathbb{R}^{2\times 2} \mid q, r \in \mathbb{Q}, q > 0 \right\}.$$

In this specific case, N_G is then the positive cone itself: $N_G = P_G$.

As a consequence of Proposition 3.1.1, we get the following result:

Corollary 3.1.4.

- *The category* $\mathsf{ParOrdGrp}$ *is reflective in* $\mathsf{PreOrdGrp}$

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{ParOrdGrp}, \arrow[l, shift left, hook, "U"] \end{tikzcd} \tag{3.2}$$

and each component of the unit η of the adjunction (as in (3.1)*) is a normal epimorphism.*

- *The category* Grp *is coreflective in* $\mathsf{PreOrdGrp}$ *and each component of the counit κ of the adjunction (as in* (3.1)*) is a normal monomorphism.*

Proof. This follows directly from Propositions 3.1.1 and 2.2.5. □

Now, we would like to make some useful comments on the short exact sequence (3.1) constructed in the proof of Proposition 3.1.1. To do this, we need to make a digression on *Schreier points* and *special Schreier morphisms in monoids.*

3.1.1. A short reminder on Schreier points and special Schreier morphisms in monoids

As we saw in Section 2.1, the category of preordered groups is isomorphic to the one whose objects are pairs (G, P_G) where G is a group and P_G a submonoid of G closed under conjugation. While the category of groups is protomodular, which means that the Split Short Five Lemma holds in Grp, this is not the case for the category of monoids [6]. Moreover, actions in monoids are not equivalent to split extensions of monoids (while this is the case for groups). Nevertheless, we can restrict our attention to a class of points (a "*point*" (A, B, p, s) being a split epimorphism $p\colon A \to B$ with fixed section $s\colon B \to A$) in Mon, called *Schreier points* (or, equivalently, *Schreier split epimorphisms*) [15], which have a behavior which is quite similar to the ones in the category of groups. The class of Schreier points corresponds to monoid actions, and it was shown in [15] that (a restricted version of) the Split Short Five Lemma does hold for such points.

Definition 3.1.5. A *Schreier point* in the category Mon of monoids is a point (A, B, p, s) such that, for any element a in A, there exists a unique element x in the kernel $\mathsf{Ker}(p)$ of p such that

$$a = x + (s \cdot p)(a).$$

This kind of points is useful to "locally" extend some classical properties of split extensions of groups to the context of monoids. We shall not develop these interesting aspects here, but we refer the reader to [15] for a thorough introduction to this subject. What will be of interest for the purpose of this thesis is to briefly recall the properties of *special Schreier morphisms* in the category of monoids.

Definition 3.1.6.

- An internal reflexive relation in the category Mon of monoids

$$R \underset{r_2}{\overset{r_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;s\;}}}}} A$$

  is said to be a *Schreier reflexive relation* when the point (R, A, r_1, s) is a Schreier one.

- A morphism $f\colon A \to B$ in the category Mon of monoids is said to be a *special Schreier morphism* when its kernel pair

$$\mathsf{Eq}(f) \underset{f_2}{\overset{f_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;\langle 1_A, 1_A\rangle\;}}}}} A$$

  is a Schreier reflexive relation, where $\langle 1_A, 1_A\rangle\colon A \to \mathsf{Eq}(f)$ is such that $f_1 \cdot \langle 1_A, 1_A\rangle = 1_A = f_2 \cdot \langle 1_A, 1_A\rangle$.

It is then possible to prove [15] that any surjective special Schreier morphism $f\colon A \to B$ is the cokernel of its kernel. Accordingly, we get an extension of monoids:

$$0 \longrightarrow \mathsf{Ker}(f) \xrightarrow{\;k\;} A \xrightarrow{\;f\;} B \longrightarrow 0.$$

These *special Schreier extensions* satisfy some remarkable properties. The following proposition states two of them, which will be needed for our future investigations:

Proposition 3.1.7. [15]

(1) Special Schreier extensions are pullback-stable in Mon.

(2) The Short Five Lemma *holds for special Schreier extensions. This means that given any commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "k"] \arrow[d, "a"'] & B \arrow[r, "f"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "k'"'] & B' \arrow[r, "f'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

of short exact sequences in Mon*, where* f *and* f' *are special Schreier morphisms, and* a *and* c *are isomorphisms, then* b *is also an isomorphism.*

3.1.2. On the canonical short exact sequence of $(\mathsf{Grp}, \mathsf{ParOrdGrp})$

We now make some useful comments on the short exact sequence (3.1) constructed in the proof of Proposition 3.1.1:

Lemma 3.1.8. *Consider the following commutative diagram in the category* Mon *of monoids, where the two rows are special Schreier extensions:*

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Ker}(f) \arrow[r, "k"] \arrow[d, "a"'] & A \arrow[r, "f"] \arrow[d, "b"'] & B \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & \mathsf{Ker}(f') \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B' \arrow[r] & 0. \end{tikzcd} \tag{3.3}$$

If the morphism a *is an isomorphism, then the right-hand square of diagram* (3.3) *is a pullback.*

Proof. Let us consider the pullback $(P, p_{A'}, p_B)$ of f' and c, and then the kernel $(\mathsf{Ker}(p_B), k'')$ of p_B.

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Ker}(f) \arrow[rr, "k"] \arrow[dr, dotted, "\gamma"] \arrow[dd, "a"'] & & A \arrow[rr, "f"] \arrow[dr, dotted, "\phi"] \arrow[dd, "b"'] & & B \arrow[r] \arrow[dr, equal] \arrow[dd, "c"] & 0 \\ & & \mathsf{Ker}(p_B) \arrow[rr, "k''"] \arrow[dl, dotted, "\psi"] & & P \arrow[rr, "p_B"] \arrow[dl, "p_{A'}"] & & B \arrow[dl, "c"] \\ 0 \arrow[r] & \mathsf{Ker}(f') \arrow[rr, "k'"'] & & A' \arrow[rr, "f'"'] & & B' \arrow[r] & 0 \end{tikzcd}$$

We are going to show that the arrow ϕ induced by the universal property of the pullback $(P, p_{A'}, p_B)$ is an isomorphism. By the universal property of the kernel $k' = \mathsf{ker}(f')$, we first get the morphism ψ such that $k' \cdot \psi = p_{A'} \cdot k''$. In the same way, the universal property of the kernel $k'' = \mathsf{ker}(p_B)$ gives a unique arrow γ such that $k'' \cdot \gamma = \phi \cdot k$. It is easily seen that $\psi \cdot \gamma = a$, with a an isomorphism by assumption. In addition, thanks to Proposition 1.1.45, ψ is also an isomorphism since $(P, p_{A'}, p_B)$ is a pullback, so that γ is itself an isomorphism. We have that the bottom row is a special Schreier extension, hence, by the pullback-stability of special Schreier extensions (Proposition 3.1.7(1)), we get that the middle row of the above diagram is a special Schreier extension. By Proposition 3.1.7(2), we now apply the Short Five Lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Ker}(f) \arrow[r, "k"] \arrow[d, "\gamma"'] & A \arrow[r, "f"] \arrow[d, "\phi"'] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathsf{Ker}(p_B) \arrow[r, "k''"'] & P \arrow[r, "p_B"'] & B \arrow[r] & 0
\end{tikzcd}$$

to conclude that ϕ is an isomorphism, that is, the right-hand square in the diagram (3.3) is a pullback. □

Corollary 3.1.9. *Consider a short exact sequence in* $\mathsf{PreOrdGrp}$ *of the following form:*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\bar{k}"] \arrow[d, equal] & P_G \arrow[r, "\bar{f}"] \arrow[d, tail] & P_H \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & K \arrow[r, "k"'] & G \arrow[r, "f"'] & H \arrow[r] & 0.
\end{tikzcd} \tag{3.4}$$

Then the upper sequence is a special Schreier extension, and the right-hand square in (3.4) *is a pullback in the category* Mon *of monoids.*

Proof. By Lemma 3.1.8, since any short exact sequence in the category Grp of groups is a special Schreier extension, it suffices to show that the sequence $K \xrightarrow{\bar{k}} P_G \xrightarrow{\bar{f}} P_H$ is a special Schreier extension in Mon. Consider the kernel pair $(\mathsf{Eq}(\bar{f}), r_1, r_2)$ of $\bar{f}$ where the projections r_1 and r_2 are split by the diagonal morphism $s\colon P_G \to \mathsf{Eq}(\bar{f})$. Note that there is no restriction in assuming that $f\colon G \to H \cong G/K$ is the canonical quotient of G by its normal subgroup K, and we write $f(g) = \bar{g}^K$. For any $(a,b) \in \mathsf{Eq}(\bar{f})$, one has the equalities $f(a) = \bar{f}(a) = \bar{f}(b) = f(b)$, since $\bar{f}$ is the restriction of f to the positive cone P_G of G. It follows that $\bar{a}^K = \bar{b}^K$, and there exists $x \in K$ such that $b = x + a$. As a consequence $(0,x) \in \mathsf{Ker}(r_1)$ satisfies the equalities

$$(0,x) + (s \cdot r_1)(a,b) = (0,x) + (a,a) = (a, x+a) = (a,b),$$

and $(0,x)$ is the only element of $\mathsf{Ker}(r_1)$ with this property. This shows that (r_1, s) is a Schreier point, and the arrow $\bar{f}\colon P_G \to P_H$ is a special Schreier morphism. Since any surjective special Schreier morphism is the cokernel of its kernel and since $\bar{f}$ is surjective, this shows that $K \xrightarrow{\bar{k}} P_G \xrightarrow{\bar{f}} P_H$ is a special Schreier extension. $\square$

In particular the previous result implies the following

Corollary 3.1.10. *Consider the short exact sequence* (3.1) *in* $\mathsf{PreOrdGrp}$. *Then:*

(1) the upper sequence $N_G \rightarrowtail^{\bar{k}_G} P_G \xrightarrow{\bar{\eta}_G} \eta_G(P_G)$ *is a short exact sequence in the category* Mon *of monoids.*

(2) the square

$$\begin{tikzcd} P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail, "g"'] & \eta_G(P_G) \arrow[d, tail, "\psi_G"] \\ G \arrow[r, two heads, "\eta_G"'] & G/N_G \end{tikzcd}$$

is a pullback in the category Mon *of monoids.*

Remark 3.1.11. As a consequence of Corollary 3.1.10(1), from now on, we will write P_G/N_G instead of $\eta_G(P_G)$ for the codomain of $\bar{\eta}_G$ in the short exact sequence (3.1). This short exact sequence therefore provides an example of a short exact sequence in the category $\mathsf{PreOrdGrp}$ of preordered groups, where we have short exact sequences both at the group and at the monoid level, as announced in Remark 2.1.11.

3.2. Monotone-light factorization system and coverings in PreOrdGrp

Proposition 3.2.1. *The reflector* $F\colon \mathsf{PreOrdGrp} \to \mathsf{ParOrdGrp}$ *in the adjunction* (3.2) *has stable units.*

Proof. This follows directly from Proposition 2.4.6. ☐

The reflector $F\colon \mathsf{PreOrdGrp} \to \mathsf{ParOrdGrp}$ then induces a factorization system $(\mathscr{E}, \mathscr{M})$ whose description is quite simple in our situation:

Proposition 3.2.2. *Given the adjunction* (3.2)*, we have a factorization system* $(\mathscr{E}, \mathscr{M})$ *in* $\mathsf{PreOrdGrp}$ *where:*

- $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ *is in the class* $\mathscr{E}$ *if and only if the following conditions hold:*
  (a) $f^{-1}(N_H) = N_G$*;*
  (b) for any $y \in H$*, there exists* $x \in G$ *such that* $\overline{f(x)}^{N_H} = \overline{y}^{N_H}$*;*
  (c) for any $y \in P_H$*, there exists* $x \in P_G$ *such that* $\overline{\bar{f}(x)}^{N_H} = \overline{y}^{N_H}$*.*
- $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ *is in the class* $\mathscr{M}$ *if and only if the restriction* $\phi\colon N_G \to N_H$ *of* $f\colon G \to H$ *to* N_G *is a group isomorphism.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
 & & & P_H \arrow[rrr, two heads, "\bar{\eta}_H"] \arrow[dd, tail] & & & P_H/N_H \arrow[dd, tail] \\
 & & P_G \arrow[ur, "\bar{f}"] \arrow[rrr, two heads, "\bar{\eta}_G"] \arrow[dd, tail] & & & P_G/N_G \arrow[ur, "\bar{\alpha}"'] \arrow[dd, tail] & \\
 & N_H \arrow[rr, tail, "\ker(\eta_H)"] \arrow[uurr, tail, bend left, "\ker(\bar{\eta}_H)"] & & H \arrow[rrr, two heads, "\eta_H"] & & & H/N_H \\
N_G \arrow[ur, "\phi"] \arrow[rr, tail, "\ker(\eta_G)"'] \arrow[uurr, tail, bend left, "\ker(\bar{\eta}_G)"] & & G \arrow[ur, "f"] \arrow[rrr, two heads, "\eta_G"'] & & & G/N_G \arrow[ur, "\alpha"'] &
\end{tikzcd} \tag{3.5}$$

where $(\alpha, \bar{\alpha})$ stands for $F(f, \bar{f})$, and where the front and the back squares of the cube are the (G, P_G)-component and the (H, P_H)-component of the unit of the adjunction (3.2), respectively.

- Assume that $(f, \bar{f}) \colon (G, P_G) \to (H, P_H)$ is in the class $\mathscr{E}$, so that $(\alpha, \bar{\alpha})$ is an isomorphism in $\mathsf{PreOrdGrp}$. The fact that α is a monomorphism in the normal category Grp of groups implies that the square

$$\begin{tikzcd} N_G \arrow[r, tail, "\ker(\eta_G)"] \arrow[d, "\phi"'] & G \arrow[d, "f"] \\ N_H \arrow[r, tail, "\ker(\eta_H)"'] & H \end{tikzcd} \tag{3.6}$$

  is a pullback, i.e. $f^{-1}(N_H) = N_G$ (by Lemma 1.2.13). Furthermore, knowing that α is surjective, for any y in H, there exists an x in G such that $\alpha(\bar{x}^{N_G}) = \bar{y}^{N_H}$, that is, $\overline{f(x)}^{N_H} = \bar{y}^{N_H}$. We can show the analogue assertion for $\bar{f}$ in a similar way, since $\bar{\alpha}$ is surjective. Conversely, if $f^{-1}(N_H) = N_G$, the square (3.6) is a pullback, and this implies that α is a monomorphism (by Lemma 1.2.13 in the category Grp). Now, the assumption (b) guarantees that $\eta_H \cdot f = \alpha \cdot \eta_G$ is surjective, hence α is surjective. Similarly, $\bar{\alpha}$ is surjective because the assumption (c) says that $\bar{\eta}_H \cdot \bar{f} = \bar{\alpha} \cdot \bar{\eta}_G$ is surjective. It follows that $(\alpha, \bar{\alpha})$ is both a monomorphism and a regular epimorphism, and then an isomorphism in the regular category $\mathsf{PreOrdGrp}$ of preordered groups (see Proposition 1.1.25), i.e. that $(f, \bar{f})$ belongs to the class $\mathscr{E}$.

- To prove the second point, we observe that the arrow $(f, \bar{f})$ belongs to the class $\mathscr{M}$ if and only if the cube in the diagram (3.5) is a pullback in $\mathsf{PreOrdGrp}$, and this is equivalent to the bottom and the top squares of this cube being pullbacks in the categories Grp and Mon, respectively.
  If these two squares are pullbacks, then the induced arrow $\phi \colon N_G \to N_H$, the restriction of the morphism $f \colon G \to H$ to N_G, is obviously an isomorphism by Proposition 1.1.45.
  Conversely, if ϕ is an isomorphism, then the bottom square of the cube is a pullback (since the Short Five Lemma holds in the

category Grp of groups, so that the converse of Proposition 1.1.45 is valid in this context). The fact that the top square of the same cube is a pullback (in the category Mon of monoids) is a consequence of Corollary 3.1.10(2). Indeed, this latter states that the front and the back squares in the cube of diagram (3.5) are pullbacks, hence, thanks to Proposition 1.1.44, the top square is a pullback since the bottom one is a pullback. $\square$

Now that we have a description of the trivial coverings (relative to the absolute Galois structure induced by the reflection (3.2)), i.e. the morphisms in the class $\mathscr{M}$, we would like to have a description of the class $\mathscr{M}^*$ of coverings. We shall actually prove that there is a monotone-light factorization system $(\mathscr{E}', \mathscr{M}^*)$, by applying Theorem 2.4.9. In the following two propositions, we verify the two fundamental conditions (N) and (C) needed to apply that theorem.

Let us start with the *normality condition* (N):

Proposition 3.2.3. *For any normal monomorphism* $(i, \bar{i})\colon (K, P_K) \rightarrowtail (G, P_G)$ *in* $\mathsf{PreOrdGrp}$*, the monomorphism* $(i, \bar{i}) \cdot (k, \bar{k})\colon (N_K, N_K) \rightarrowtail (G, P_G)$ *is normal, where* $(k, \bar{k})\colon (N_K, N_K) \rightarrowtail (K, P_K)$ *is the* (K, P_K)*-component of the counit of the coreflection* $\mathsf{PreOrdGrp} \to \mathsf{Grp}$*.*

Proof. Since $(i, \bar{i})$ is a normal monomorphism, there exists an arrow $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ in $\mathsf{PreOrdGrp}$ such that $(i, \bar{i}) = \mathsf{ker}(f, \bar{f})$. With that notation, we then have that $K = \mathsf{Ker}(f)$ and that $P_K = K \cap P_G$. Let us first show that N_K is normal in G, where

$$\begin{aligned} N_K &= \{\, x \in K \mid x \in P_K \text{ and } -x \in P_K \,\} \\ &= \{\, x \in K \mid x \in K \cap P_G \text{ and } -x \in K \cap P_G \,\} \\ &= K \cap N_G. \end{aligned}$$

Since K and N_G are two normal subgroups of G, N_K is normal in G. In order to prove that the inclusion $(i, \bar{i}) \cdot (k, \bar{k})\colon (N_K, N_K) \rightarrowtail (G, P_G)$ is a normal monomorphism in $\mathsf{PreOrdGrp}$, one observes that the rectangle

$$\begin{tikzcd} N_K \arrow[r, tail, "\bar{k}"] \arrow[d, equal] & P_K \arrow[r, tail, "\bar{i}"] \arrow[d, tail] & P_G \arrow[d, tail] \\ N_K \arrow[r, tail, "k"'] & K \arrow[r, tail, "i"'] & G, \end{tikzcd}$$

is a pullback since it is made of two pullbacks, and the result then follows from the definition of a normal monomorphism in PreOrdGrp (see Subsection 2.1.2). □

Let us now check the *covering condition* (C):

Proposition 3.2.4. *For any object (G, P_G) in the category* PreOrdGrp *of preordered groups, there exists an object (H, P_H) in the subcategory* ParOrdGrp *of partially ordered groups and an effective descent morphism*

$$(f, \bar{f})\colon (H, P_H) \to (G, P_G)$$

from (H, P_H) to (G, P_G).

Proof. Let $(G, P_G) \in$ PreOrdGrp. Define (H, P_H) in the following way:

- $H = \mathbb{Z} \times G$;
- $P_H = (\mathbb{N} \times P_G) \setminus \{ (0, g) \mid g \neq 0 \}$.

It is easy to check that P_H is a submonoid of the group H (endowed with the natural group structure). To show that P_H is closed in H under conjugation, consider $(z, g) \in H$ and $(n, h) \in P_H$. Then

$$(z, g) + (n, h) - (z, g) = (z + n - z, g + h - g) \in \mathbb{N} \times P_G$$

since P_G is closed under conjugation in G, and if $z+n-z = 0$, i.e. if $n = 0$, then $(n, h) \in P_H$ implies that $h = 0$, that is, $g + h - g = g - g = 0$. This means that $(z, g)+(n, h)-(z, g) \in P_H$, and (H, P_H) is a preordered group, which actually lies in ParOrdGrp since, by definition, the submonoid P_H is reduced (the only element of P_H having its inverse in P_H is $(0, 0)$).
Let us next consider the function $f\colon H \to G$ defined, for any $(z, g) \in H$, by $f(z, g) = g$. It is a morphism in PreOrdGrp, since it is a group morphism and $f(z, g) = g \in P_G$, for any $(z, g) \in P_H$. In other words,

the restriction $\bar{f}$ of f to P_H takes its values in P_G. The morphism $(f, \bar{f}) : (H, P_H) \to (G, P_G)$ is also a normal epimorphism in $\mathsf{PreOrdGrp}$, since both f and $\bar{f}$ are easily seen to be surjective. Since effective descent morphisms coincide with normal epimorphisms in $\mathsf{PreOrdGrp}$ (see Proposition 2.1.15), the proof is complete. $\square$

We are now ready to state the final results of this section:

Theorem 3.2.5. *The pair $(\mathscr{E}', \mathscr{M}^*)$ is a monotone-light factorization system and, moreover,*

- $\mathscr{E}' = \{\, (f, \bar{f}) \in \mathsf{PreOrdGrp} \mid (f, \bar{f})$ *is a normal epimorphism such that* $\mathsf{Ker}(f, \bar{f}) \in \mathsf{Grp} \,\}$*;*
- $\mathscr{M}^* = \{\, (f, \bar{f}) \in \mathsf{PreOrdGrp} \mid \mathsf{Ker}(f, \bar{f}) \in \mathsf{ParOrdGrp} \,\}$*.*

Proof. This result follows from Theorem 2.4.9, which can be applied to the reflection (3.2) thanks to the two previous propositions. $\square$

Corollary 3.2.6. *A morphism $(f, \bar{f}) : (G, P_G) \to (H, P_H)$ in the category* $\mathsf{PreOrdGrp}$ *of preordered groups is a covering (with respect to the absolute Galois structure induced by the reflection* (3.2)*) if and only if its kernel* $\mathsf{Ker}(f, \bar{f})$ *belongs to the subcategory* $\mathsf{ParOrdGrp}$ *of partially ordered groups.*

This description is then similar to the one of the *locally semisimple coverings* relative to a *generalized semisimple class*, given by G. Janelidze, L. Márki and W. Tholen in [59]. We explain the link with this latter approach in the next section.

3.3. Classification of the coverings of preordered groups

3.3.1. Locally semisimple coverings

Let us first recall the approach to *locally semisimple coverings* based on Galois theory developed in [59]. Here below we shall adapt the context in order to include the example of the category $\mathsf{PreOrdGrp}$ of preordered groups.

Let $\mathscr{C}$ be any normal category in which normal epimorphisms and effective descent morphisms coincide. Let us consider a fixed class $\mathcal{X}$ of objects in $\mathscr{C}$, called a *generalized semisimple class*, such that the following two properties hold for any pullback

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B, \end{tikzcd}$$

where p is a normal epimorphism in $\mathscr{C}$:

(1) $E \in \mathcal{X}$ and $A \in \mathcal{X}$ implies that $E \times_B A \in \mathcal{X}$;

(2) $B \in \mathcal{X}$, $E \in \mathcal{X}$ and $E \times_B A \in \mathcal{X}$ implies that $A \in \mathcal{X}$.

The notion of *locally semisimple covering* is then defined relatively to a generalized semisimple class $\mathcal{X}$ in a category $\mathscr{C}$: a morphism $\alpha \colon A \to B$ is a locally semisimple covering in $\mathscr{C}$ if there is a normal epimorphism $p \colon E \to B$ such that the pullback $p^*(\alpha)$ of α along p lies in the corresponding full subcategory $\mathcal{X}$ of $\mathscr{C}$.

For a fixed $B \in \mathscr{C}$, let $\mathsf{LocSSimple}_{\mathcal{X}}(B)$ be the full subcategory of the slice category $\mathscr{C} \downarrow B$ over B whose objects are pairs (A, α), where $\alpha \colon A \to B$ is a locally semisimple covering (relatively to the generalized semisimple class $\mathcal{X}$). We then have the following equivalence of categories:

Theorem 3.3.1. [59] *Consider a normal category $\mathscr{C}$ where normal epimorphisms are effective descent morphisms, and $\mathcal{X}$ a generalized semisimple class in $\mathscr{C}$. If $p \colon E \to B$ is a normal epimorphism in $\mathscr{C}$ such that $E \in \mathcal{X}$, there is the equivalence of categories*

$$\mathsf{LocSSimple}_{\mathcal{X}}(B) \cong \mathsf{DiscFib}_{\mathcal{X}}(\mathsf{Eq}(p)) \tag{3.7}$$

where $\mathsf{DiscFib}_{\mathcal{X}}(\mathsf{Eq}(p))$ is the full subcategory of $\mathsf{DiscFib}(\mathsf{Eq}(p))$ whose objects are the discrete fibrations over $\mathsf{Eq}(p)$ (as in (1.5)*) with $F \in \mathcal{X}$.*

Proof. [59] Since $p\colon E \to B$ is an effective descent morphism in $\mathscr{C}$, the comparison functor $K^p\colon \mathscr{C} \downarrow B \to \mathsf{DiscFib}(\mathsf{Eq}(p))$ is therefore an equivalence of categories (see Definition 1.3.6). Let us prove that its restriction to $\mathsf{LocSSimple}_{\mathcal{X}}(B)$ gives the desired category equivalence (3.7). So, we need to show that (A, α) is a locally semisimple covering if and only if $K^p(A, \alpha) \in \mathsf{DiscFib}_{\mathcal{X}}(\mathsf{Eq}(p))$. Now, we observe that $K^p(A, \alpha) \in \mathsf{DiscFib}_{\mathcal{X}}(\mathsf{Eq}(p))$ if and only if $E \times_B A \in \mathcal{X}$, so that it suffices to prove that (A, α) is a locally semisimple covering if and only if $E \times_B A \in \mathcal{X}$.
If $E \times_B A \in \mathcal{X}$, then obviously $p^*(\alpha)\colon E \times_B A \to E$ lies in the full subcategory $\mathcal{X}$ since $E \in \mathcal{X}$ by assumption. This means that (A, α) is a locally semisimple covering.
Conversely, if (A, α) is a locally semisimple covering, there exists a normal epimorphism $p'\colon E' \to B$ such that $p'^*(\alpha)$ lies in $\mathcal{X}$. Let $D = E \times_B E'$ and consider the following diagram in which all squares are pullbacks:

$$
\begin{tikzcd}
 & D \times_B A \arrow[rr] \arrow[dl] \arrow[dd] & & E' \times_B A \arrow[dl] \arrow[dd, "p'^*(\alpha)"] \\
E \times_B A \arrow[rr] \arrow[dd, "p^*(\alpha)"'] & & A \arrow[dd, "\alpha"] & \\
 & D \arrow[rr] \arrow[dl] & & E' \arrow[dl, "p'"] \\
E \arrow[rr, "p"'] & & B. &
\end{tikzcd}
$$

Then, by the first property of the definition of a generalized semisimple class, $D \in \mathcal{X}$ since $E \in \mathcal{X}$ and $E' \in \mathcal{X}$. It follows from the same property that $D \times_B A \in \mathcal{X}$ since $D \in \mathcal{X}$ and $E' \times_B A \in \mathcal{X}$. We finally conclude that $E \times_B A \in \mathcal{X}$ by the second property of the definition of a generalized semisimple class, since $E \in \mathcal{X}$, $D \in \mathcal{X}$ and $D \times_B A \in \mathcal{X}$. $\square$

3.3.2. Coverings of preordered groups classified as internal actions

In particular, we can consider $\mathscr{C} = \mathsf{PreOrdGrp}$, and $\mathcal{X}$ the class of objects of $\mathsf{ParOrdGrp}$, which is easily seen (by using an extension, from the homological to the normal setting, of Lemma 2.7 in [44], for instance) to be a generalized semisimple class. We are therefore in a situation where we can apply Theorem 3.3.1. We first of all prove the following lemma:

Lemma 3.3.2. *A morphism* $(h, \bar{h})\colon (H, P_H) \to (G, P_G)$ *in* $\mathsf{PreOrdGrp}$ *is a locally semisimple covering (relatively to the subcategory* $\mathsf{ParOrdGrp}$*) if and only if its kernel is a partially ordered group.*

Proof. If $(h, \bar{h})$ is a locally semisimple covering, there exists a normal epimorphism $(p, \bar{p})\colon (E, P_E) \to (G, P_G)$ such that $(p, \bar{p})^*(h, \bar{h}) \in \mathsf{ParOrdGrp}$. It follows that $\mathsf{Ker}\left((p, \bar{p})^*(h, \bar{h})\right) \in \mathsf{ParOrdGrp}$. Now, since the diagram

$$\begin{tikzcd}
(E, P_E) \times_{(G,P_G)} (H, P_H) \arrow[r] \arrow[d, "(p,\bar{p})^*(h,\bar{h})"'] & (H, P_H) \arrow[d, "(h,\bar{h})"] \\
(E, P_E) \arrow[r, "(p,\bar{p})"'] & (G, P_G)
\end{tikzcd} \tag{3.8}$$

is a pullback, we have that $\mathsf{Ker}(h, \bar{h}) \cong \mathsf{Ker}\left((p, \bar{p})^*(h, \bar{h})\right)$ (thanks to Proposition 1.1.45), so that $\mathsf{Ker}(h, \bar{h}) \in \mathsf{ParOrdGrp}$.
Conversely, by Proposition 3.2.4, we know that there exists an effective descent morphism (i.e. a normal epimorphism) $(p, \bar{p})\colon (E, P_E) \to (G, P_G)$ with $(E, P_E) \in \mathsf{ParOrdGrp}$. Since the diagram (3.8) is a pullback, we have (again by Proposition 1.1.45) that

$$\mathsf{Ker}\left((p, \bar{p})^*(h, \bar{h})\right) \cong \mathsf{Ker}(h, \bar{h}),$$

with $\mathsf{Ker}(h, \bar{h}) \in \mathsf{ParOrdGrp}$ by assumption. Knowing that any torsion-free subcategory is stable by extensions (see [64] for instance) and that $\mathsf{ParOrdGrp}$ is a torsion-free subcategory of $\mathsf{PreOrdGrp}$ (by Proposition 3.1.1), it follows that $(p, \bar{p})^*(h, \bar{h})$ lies in $\mathsf{ParOrdGrp}$. □

Remark 3.3.3. The previous lemma is a particular case of a more general fact observed in [59] (Proposition 2.3) where, more generally, the role of the kernel of an arrow was played by the "*fibers*" (as defined in [59]).

Theorem 3.3.4. *Let* $(G, P_G) \in \mathsf{PreOrdGrp}$. *Consider the effective descent morphism*

$$(f, \bar{f})\colon\ (\mathbb{Z} \times G, (\mathbb{N} \times P_G) \backslash \{\, (0, g) \mid g \neq 0 \,\}) \to (G, P_G)$$

from Proposition 3.2.4. Then there exists an equivalence of categories

$$\mathscr{M}^* \downarrow (G, P_G) \cong \mathsf{DiscFib}_{\mathsf{ParOrdGrp}}(\mathsf{Eq}(f, \bar{f}))$$

where $\mathscr{M}^* \downarrow (G, P_G)$ *is the category of coverings over* (G, P_G)*.*

Proof. Since the morphism

$$(f, \bar{f})\colon\ (\mathbb{Z} \times G, (\mathbb{N} \times P_G)\backslash\{\, (0, g) \mid g \neq 0 \,\}) \to (G, P_G)$$

from Proposition 3.2.4 is an effective descent morphism in $\mathsf{PreOrdGrp}$ such that

$$(\mathbb{Z} \times G, (\mathbb{N} \times P_G)\backslash\{\, (0, g) \mid g \neq 0 \,\}) \in \mathsf{ParOrdGrp},$$

we are allowed to apply Theorem 3.3.1: there exists then an equivalence of categories

$$\mathsf{LocSSimple}_{\mathsf{ParOrdGrp}}(G, P_G) \cong \mathsf{DiscFib}_{\mathsf{ParOrdGrp}}(\mathsf{Eq}(f, \bar{f})).$$

Thanks to Lemma 3.3.2 and Corollary 3.2.6, the proof is complete since both the coverings and the locally semisimple coverings (over (G, P_G)) are described as the arrows $(h, \overline{h})\colon (H, P_H) \to (G, P_G)$ with $\mathsf{Ker}(h, \overline{h}) \in \mathsf{ParOrdGrp}$. □

Note that the internal equivalence relation $\mathsf{Eq}(f, \bar{f})$ from Theorem 3.3.4 is in fact the *Galois groupoid* $\mathsf{Gal}(f, \bar{f})$ associated with the effective descent morphism $(f, \bar{f})$ [55]. By definition, the *Galois groupoid* associated with $(f, \bar{f})$ is indeed the image of $\mathsf{Eq}(f, \bar{f})$ by the reflector $F\colon \mathsf{PreOrdGrp} \to \mathsf{ParOrdGrp}$. But since the diagram

$$(\mathsf{Eq}(f), \mathsf{Eq}(\bar{f})) \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; (E, P_E)$$

lies in $\mathsf{ParOrdGrp}$ (where we write (E, P_E) for

$$(\mathbb{Z} \times G, (\mathbb{N} \times P_G)\backslash\{\, (0, g) \mid g \neq 0 \,\})),$$

the image of $\mathsf{Eq}(f, \bar{f})$ by the reflector F is $\mathsf{Eq}(f, \bar{f})$ itself. In other words, $\mathsf{Eq}(f, \bar{f})$ is $\mathsf{Gal}(f, \bar{f})$, and

$$\mathscr{M}^* \downarrow (G, P_G) \cong \mathsf{DiscFib}_{\mathsf{ParOrdGrp}}(\mathsf{Gal}(f, \bar{f})). \tag{3.9}$$

This equivalence is the *classification* of the coverings as internal $\mathsf{Gal}(f, \bar{f})$-actions.

Remark 3.3.5. Besides its interest for the classification of coverings in the category of preordered groups, the above result also provides an example of application of Theorem 3.1 in [59] in a non-exact setting (see Remark 3.2 (e) in [59]).

3.4. A pretorsion theory in the category of preordered groups

In this last section, we show that the reflection (3.2) gives also rise to a pretorsion theory in the category $\mathsf{PreOrdGrp}$ of preordered groups. The torsion-free subcategory is, as in Proposition 3.1.1, the subcategory $\mathsf{ParOrdGrp}$ of partially ordered groups while the torsion subcategory is, this time, the full subcategory $\mathsf{Grp}(\mathsf{PreOrd})$ of *internal groups in the category of preordered sets* (already mentioned before). An object in this category is a preordered group $(G, \leq)$ for which the inversion map $-\colon G \to G$ is monotone, that is, the preorder $\leq$ is symmetric (i.e. an equivalence relation). Via the category isomorphism of Proposition 2.1.5, such a preordered group corresponds to a preordered group (G, P_G) with the property that the positive cone P_G is a group.

Proposition 3.4.1. *The pair* $(\mathsf{Grp}(\mathsf{PreOrd}), \mathsf{ParOrdGrp})$ *of full replete subcategories of* $\mathsf{PreOrdGrp}$ *is a* $\mathscr{Z}$*-pretorsion theory in* $\mathsf{PreOrdGrp}$*, where* $\mathscr{Z} = \mathsf{Grp}(\mathsf{PreOrd}) \cap \mathsf{ParOrdGrp}$ *is given by*

$$\mathscr{Z} = \{ (G, P_G) \mid P_G = 0 \},$$

i.e. by the class of preordered groups endowed with the discrete order.

Proof. To prove that any arrow $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ in $\mathsf{PreOrdGrp}$, with $(G, P_G) \in \mathsf{Grp}(\mathsf{PreOrd})$ and $(H, P_H) \in \mathsf{ParOrdGrp}$, factorizes through an object in $\mathscr{Z}$, first observe that the following diagram commutes

$$
\begin{tikzcd}
P_G \arrow[rr, "\bar{f}"] \arrow[dddd, tail] \arrow[dr, dotted, tail] & & P_H \arrow[dddd, tail] \\
& 0 \arrow[ur, dotted, tail] \arrow[dd, tail] & \\
& & \\
& f(G) \arrow[dr, dotted, tail] & \\
G \arrow[rr, "f"'] \arrow[ur, dotted, two heads] & & H,
\end{tikzcd}
\tag{3.10}
$$

where $f(G)$ is the image of the group morphism $f\colon G \to H$. Indeed, as explained in the first part of the proof of Proposition 3.1.1, any monoid morphism from a group to a reduced monoid is the zero morphism. Since $f(G)$ is a group and since any part of the diagram (3.10) commutes, we conclude that any arrow in PreOrdGrp from an object of Grp(PreOrd) to an object of ParOrdGrp factorizes through an object of $\mathscr{Z}$, i.e. it belongs to $\mathscr{N}$.

Consider now any preordered group (G, P_G). We will work as before with the normal subgroup N_G of G. We then consider the (regular epi, mono)-factorization of $\eta_G \cdot g$ in the category Mon of monoids, where $G \overset{\eta_G}{\twoheadrightarrow} G/N_G$ is the quotient morphism and $P_G \overset{g}{\rightarrowtail} G$ is the inclusion arrow:

$$
\begin{tikzcd}
P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail, "g"'] & P_G/N_G \arrow[d, tail, "\psi_G"] \\
G \arrow[r, two heads, "\eta_G"'] & G/N_G.
\end{tikzcd}
$$

Let us prove that the sequence

$$
\begin{tikzcd}
N_G \arrow[r, tail, "i"] \arrow[d, tail, "\phi_G"'] & P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail, "g"'] & P_G/N_G \arrow[d, tail, "\psi_G"] \\
G \arrow[r, equal] & G \arrow[r, two heads, "\eta_G"'] & G/N_G,
\end{tikzcd}
$$

in which $(G, N_G) \in$ Grp(PreOrd) and $(G/N_G, P_G/N_G) \in$ ParOrdGrp, is a short $\mathscr{Z}$-exact sequence in PreOrdGrp.

We begin by showing that $(\eta_G, \bar{\eta}_G)$ is the $\mathscr{Z}$-cokernel of the arrow $(1_G, i)$. Let us consider a morphism $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ in PreOrdGrp such that $(f, \bar{f}) \cdot (1_G, i) \in \mathscr{N}$, i.e. such that $(f, \bar{f}) \cdot (1_G, i)$ factorizes through an object $(A, 0)$ of $\mathscr{Z}$: $(f, \bar{f}) \cdot (1_G, i) = (b, \bar{b}) \cdot (a, \bar{a})$.

$$\begin{tikzcd}
N_G \arrow[rr, tail, "i"] \arrow[dr, "\bar{a}"] \arrow[ddd, tail, "\phi_G"'] & & P_G \arrow[rr, two heads, "\bar{\eta}_G"] \arrow[dr, "\bar{f}"] \arrow[ddd, tail, "g"' near start] & & P_G/N_G \arrow[dl, dotted, "\bar{\alpha}"'] \arrow[ddd, tail, "\psi_G"] \\
& 0 \arrow[rr, "\bar{b}"] \arrow[d, tail] & & P_H \arrow[d, tail, "h"] & \\
& A \arrow[rr, "b"'] & & H & \\
G \arrow[ur, "a"'] \arrow[rr, equal] & & G \arrow[ur, "f"'] \arrow[rr, two heads, "\eta_G"'] & & G/N_G \arrow[ul, dotted, "\alpha"]
\end{tikzcd}$$

In particular, $f \cdot \phi_G = b \cdot a \cdot \phi_G = 0$. Since η_G is the cokernel of ϕ_G in the category Grp of groups, by the universal property of cokernels, there exists a unique arrow $\alpha\colon G/N_G \to H$ in Grp such that $\alpha \cdot \eta_G = f$. Now, seeing that $\bar{\eta}_G$ is a regular epimorphism, and therefore (by Proposition 1.2.3) a strong epimorphism, in the regular category Mon of monoids, and that h is a monomorphism, the universal property of strong epimorphisms yields a unique arrow $\bar{\alpha}\colon P_G/N_G \to P_H$ such that $\bar{\alpha} \cdot \bar{\eta}_G = \bar{f}$ and $h \cdot \bar{\alpha} = \alpha \cdot \psi_G$. In other words, there exists a unique arrow $(\alpha, \bar{\alpha})\colon (G/N_G, P_G/N_G) \to (H, P_H)$ in $\mathsf{PreOrdGrp}$ such that $(\alpha, \bar{\alpha}) \cdot (\eta_G, \bar{\eta}_G) = (f, \bar{f})$, i.e. $(\eta_G, \bar{\eta}_G)$ is the $\mathscr{Z}$-cokernel of $(1_G, i)$.
Next, we show that $(1_G, i)$ is the $\mathscr{Z}$-kernel of $(\eta_G, \bar{\eta}_G)$. Consider $(f, \bar{f})\colon (H, P_H) \to (G, P_G)$ in $\mathsf{PreOrdGrp}$ such that $(\eta_G, \bar{\eta}_G) \cdot (f, \bar{f}) \in \mathscr{N}$, i.e. $(\eta_G, \bar{\eta}_G) \cdot (f, \bar{f})$ factorizes through an object $(A, 0)$ of $\mathscr{Z}$: $(\eta_G, \bar{\eta}_G) \cdot (f, \bar{f}) = (b, \bar{b}) \cdot (a, \bar{a})$.

$$\begin{tikzcd}
N_G \arrow[rr, tail, "i"] \arrow[ddd, tail, "\phi_G"'] & & P_G \arrow[rr, two heads, "\bar{\eta}_G"] \arrow[ddd, tail, "g"' near start] & & P_G/N_G \arrow[ddd, tail, "\psi_G"] \\
& P_H \arrow[ul, dotted, "\bar{\alpha}"'] \arrow[ur, "\bar{f}"] \arrow[rr, "\bar{a}"] \arrow[d, tail, "h"'] & & 0 \arrow[ur, "\bar{b}"] \arrow[d, tail] & \\
& H \arrow[rr, "a"'] \arrow[dl, dotted, "\alpha"] \arrow[dr, "f"'] & & A \arrow[dr, "b"] & \\
G \arrow[rr, equal] & & G \arrow[rr, two heads, "\eta_G"'] & & G/N_G
\end{tikzcd}$$

Let us take $\alpha = f$, since this is the only possible arrow such that $1_G \cdot \alpha = f$. Now we have that $\bar{\eta}_G \cdot \bar{f} = 0$, hence $\eta_G \cdot g \cdot \bar{f} = 0$. Since ϕ_G is the kernel of η_G in Mon (and in Grp), there is a unique arrow $\bar{\alpha}\colon P_H \to N_G$ such that $\phi_G \cdot \bar{\alpha} = g \cdot \bar{f}$. Then

$$g \cdot i \cdot \bar{\alpha} = \phi_G \cdot \bar{\alpha} = g \cdot \bar{f}$$

and, since g is a monomorphism, it follows that $i \cdot \bar{\alpha} = \bar{f}$. The morphism $\bar{\alpha}$ is moreover unique with this property, because i is a monomorphism. As a conclusion, $(\alpha, \bar{\alpha})\colon (H, P_H) \to (G, N_G)$ is the unique arrow such that $(1_G, i) \cdot (\alpha, \bar{\alpha}) = (f, \bar{f})$, and $(1_G, i)$ is the $\mathscr{Z}$-kernel of $(\eta_G, \bar{\eta}_G)$. □

From Propositions 2.2.13 and 3.4.1, we deduce in particular that the subcategory $\mathsf{Grp}(\mathsf{PreOrd})$ is mono-coreflective in $\mathsf{PreOrdGrp}$:

Corollary 3.4.2. *The subcategory* $\mathsf{Grp}(\mathsf{PreOrd})$ *is mono-coreflective in the category* $\mathsf{PreOrdGrp}$ *of preordered groups:*

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, shift right, "T"'] \arrow[r, phantom, "\perp"] & \mathsf{Grp}(\mathsf{PreOrd}). \arrow[l, hook', shift right, "V"'] \end{tikzcd} \tag{3.11}$$

It turns out that the inclusion functor $V\colon \mathsf{Grp}(\mathsf{PreOrd}) \hookrightarrow \mathsf{PreOrdGrp}$ is not only a left adjoint but also a right adjoint:

Proposition 3.4.3. *The functor* $V\colon \mathsf{Grp}(\mathsf{PreOrd}) \hookrightarrow \mathsf{PreOrdGrp}$ *has a left adjoint functor* $E\colon \mathsf{PreOrdGrp} \to \mathsf{Grp}(\mathsf{PreOrd})$*:*

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, shift left, "E"] \arrow[r, phantom, "\perp"] & \mathsf{Grp}(\mathsf{PreOrd}). \arrow[l, hook', shift left, "V"] \end{tikzcd} \tag{3.12}$$

Proof. We begin with the construction of the functor E. Let (G, P_G) be any preordered group. Consider the subgroup M_G of G generated by all elements in $P_G \cup (-P_G)$, where we write $-P_G$ for the submonoid

$$-P_G = \{\, x \in G \mid \exists\; g \in P_G \text{ such that } x = -g \,\}.$$

Any element m of M_G is of the form $m = g_1 - g_2 + g_3 - \cdots + g_{n-1} - g_n$ for $g_1, \ldots, g_n \in P_G$. Since both P_G and $-P_G$ are submonoids of G, it is clear that M_G is a subgroup of G. And this subgroup is in addition normal in G. Indeed, let $g \in G$ and let $m = g_1 - g_2 + \cdots + g_{n-1} - g_n$ be an element in M_G (where $g_1, \ldots, g_n \in P_G$). Then

$$\begin{aligned} g + m - g &= g + g_1 - g_2 + \cdots + g_{n-1} - g_n - g \\ &= (g + g_1 - g) + (g - g_2 - g) + \cdots + (g + g_{n-1} - g) + (g - g_n - g) \\ &= (g + g_1 - g) - (g + g_2 - g) + \cdots + (g + g_{n-1} - g) - (g + g_n - g) \end{aligned}$$

with $g + g_i - g \in P_G$ for any $i \in \{1, \ldots, n\}$ since P_G is closed under conjugation in G, and $g+m-g \in M_G$. Accordingly, (G, M_G) in an object of the subcategory $\mathsf{Grp}(\mathsf{PreOrd})$. This construction is obviously functorial, and we write $E \colon \mathsf{PreOrdGrp} \to \mathsf{Grp}(\mathsf{PreOrd})$ for the functor defined on objects by $E(G, P_G) = (G, M_G)$, for any $(G, P_G) \in \mathsf{PreOrdGrp}$.
Let us now prove that this functor E is a left adjoint of the functor $V \colon \mathsf{Grp}(\mathsf{PreOrd}) \to \mathsf{PreOrdGrp}$. Let $(G, P_G) \in \mathsf{PreOrdGrp}$, and let us check that the (G, P_G)-component of the unit of the adjunction is given by the arrow $(1_G, j)$

$$\begin{tikzcd} P_G \arrow[r, tail, "j"] \arrow[d, tail, "g"'] & M_G \arrow[d, tail, "n"] \\ G \arrow[r, equal] & G \end{tikzcd}$$

where $P_G \overset{j}{\rightarrowtail} M_G$ is the inclusion morphism. Let (H, P_H) be an object in $\mathsf{Grp}(\mathsf{PreOrd})$ and consider any morphism $(f, \bar{f}) \colon (G, P_G) \to V(H, P_H) = (H, P_H)$.

$$\begin{tikzcd} P_G \arrow[rr, tail, "j"] \arrow[dr, "\bar{f}"'] \arrow[ddd, tail, "g"'] & & M_G \arrow[dl, dotted, "\bar{\phi}"] \arrow[ddd, tail, "n"] \\ & P_H \arrow[d, tail, "h"] & \\ & H & \\ G \arrow[ur, "f"] \arrow[rr, equal] & & G \arrow[ul, dotted, "\phi"'] \end{tikzcd}$$

There exists a unique morphism $\phi = f \colon G \to H$ with the property that $\phi \cdot 1_G = f$. We then observe that, for any $m = g_1 - g_2 + \cdots + g_{n-1} - g_n \in$

M_G (with $g_1, \dots, g_n \in P_G$),

$$\begin{aligned} f(m) &= f(g_1 - g_2 + \cdots + g_{n-1} - g_n) \\ &= f(g_1) - f(g_2) + \cdots + f(g_{n-1}) - f(g_n) \end{aligned}$$

with $f(g_i) \in P_H$ since $g_i \in P_G$ for any $i \in \{1, \dots, n\}$. Hence $f(m) \in P_H$ since P_H is a group by assumption. This means that the restriction $f_{|M_G}$ of f to M_G takes its values in P_H. Let us then define $\bar{\phi} = f_{|M_G} \colon M_G \to P_H$. We can then check that $\bar{\phi} \cdot j = \bar{f}$, and we observe that, for any $m \in M_G$,

$$(\phi \cdot n)(m) = (f \cdot n)(m) = f(m) = \bar{\phi}(m) = (h \cdot \bar{\phi})(m),$$

so that $\phi \cdot n = h \cdot \bar{\phi}$. Accordingly, there exists a unique morphism $(\phi, \bar{\phi}) = (f, f_{|M_G}) \colon (G, M_G) \to (H, P_H)$ such that $(\phi, \bar{\phi}) \cdot (1_G, j) = (f, \bar{f})$, and the proof is complete. $\square$

Remark 3.4.4. We end this chapter by mentioning that it is possible to associate, with the above pretorsion theory, a torsion theory in a pointed category called the *stable category* [32]. This torsion theory is *universal*; this means that it is "the best" one that can be associated with the pretorsion theory ($\mathsf{Grp}(\mathsf{PreOrd})$, $\mathsf{ParOrdGrp}$) in the category $\mathsf{PreOrdGrp}$ of preordered groups. The description of the stable category and the proof of its universal property are given in Appendix A. These results have not been published yet.

4 | Torsion theories and coverings of V-groups

In Chapter 3, torsion theories and coverings were studied in the category PreOrdGrp of preordered groups. As a reminder, a preordered group is given by a group and a relation satisfying some additional conditions (the reflexivity and the transitivity of the given relation as well as its compatibility with the addition law $+$). Now, a relation R on a set X can be seen as a function $r\colon X \times X \to 2 = \{\top, \perp\}$ where, for any $(x, x') \in X \times X$, $r(x, x') = \top$ if $(x, x') \in R$ and $r(x, x') = \perp$ if $(x, x') \notin R$. This suggests to consider a generalization of the notion of relation: for a commutative and unital *quantale* V (that is, V is a complete lattice $(V, \wedge, \vee, \top, \perp)$ equipped with a binary operation $\otimes$ which is associative, commutative, distributes over arbitrary joins, and has a unit k), we could consider *V-relations*, a V-relation $r\colon X \nrightarrow X$ on a set X being a function $r\colon X \times X \to V$.

Based on this notion of V-relation, a generalization of the category PreOrdGrp of preordered groups was considered in the article [22] by M. M. Clementino and A. Montoli. A *V-group* is a group $(X, +, 0)$ endowed with a V-relation $a\colon X \nrightarrow X$ satisfying two properties (called the reflexivity and the transitivity axioms), and which is invariant by shifting: for any $x, x', x'' \in X$, $a(x', x'') = a(x' + x, x'' + x)$. In this sense, a preordered group is then a particular case of V-group: it is a V-group for the quantale $V = 2 = \{\top, \perp\}$. There are many other very interesting examples of V-groups, such as the *Lawvere metric groups*, the *Lawvere ultrametric groups*, the *probabilistic metric groups*, etc [69].

The aim of this chapter consists in generalizing to any category of V-groups (up to an additional assumption on the quantale V) the results on torsion theories and coverings developed in [41] in the particular setting of preordered groups (see Chapter 3). We start with some prerequisites needed to understand its content. Among other things, we present a summary of the categorical properties of V-groups (as described in [22]).

In the next section, we prove our first result: when V is an *integral* quantale (i.e. when the unit k of $\otimes$ coincides with the top element $\top$), there is a torsion theory in the category $V\text{-}\mathsf{Grp}$ of V-groups. The torsion subcategory $V\text{-}\mathsf{Grp}_{\mathsf{ind}}$ is the full subcategory of $V\text{-}\mathsf{Grp}$ whose objects are V-groups (X, a) such that $a(x, x') = \top$ for any $x, x' \in X$, while the torsion-free subcategory, denoted by $V\text{-}\mathsf{Grp}_{\mathsf{sep}}$, contains all V-groups (X, a) satisfying the separation axiom: for any $x, x' \in X$,

$$a(x, x') \geq k \qquad \text{and} \qquad a(x', x) \geq k \qquad \Longrightarrow \qquad x = x'.$$

Of course, whenever V is integral, this axiom reduces to

$$a(x, x') = \top = a(x', x) \qquad \Longrightarrow \qquad x = x'.$$

The objects in $V\text{-}\mathsf{Grp}_{\mathsf{ind}}$ and in $V\text{-}\mathsf{Grp}_{\mathsf{sep}}$ are called *indiscrete* and *separated* V-groups, respectively. As a direct consequence, we get that $V\text{-}\mathsf{Grp}_{\mathsf{sep}}$ is a *(normal epi)-reflective* subcategory and that $V\text{-}\mathsf{Grp}_{\mathsf{ind}}$ is *(normal mono)-coreflective* in $V\text{-}\mathsf{Grp}$.

The above mentioned torsion theory also induces a factorization system $(\mathscr{E}, \mathscr{M})$ for which we give a fairly simple description. The class $\mathscr{M}$ of morphisms in $V\text{-}\mathsf{Grp}$ corresponds to the class of *trivial coverings* (in the sense of categorical Galois theory). We then prove two intermediate propositions. In one of them, we build, for any $(X, a) \in V\text{-}\mathsf{Grp}$, a separated V-group (Y, b) as well as an effective descent morphism

$$f : (Y, b) \to (X, a) \tag{4.1}$$

in the category of V-groups. Thanks to these two propositions and Theorem 2.4.9, it is possible to conclude that $(\mathscr{E}', \mathscr{M}^*)$ is a monotone-light factorization system, where $\mathscr{E}'$ is the stabilization of $\mathscr{E}$ and $\mathscr{M}^*$ the localization of $\mathscr{M}$. The class $\mathscr{M}^*$ is then actually the class of coverings in V-Grp. We are moreover able to characterize the morphisms in both classes. In particular, when V is an integral quantale, the coverings in V-Grp are given by the morphisms whose kernel is a separated V-group.

In Sections 4.4 and 4.5, we then make two observations related to the results of the two previous sections. We first notice that it is possible to classify the coverings in V-Grp in terms of $\mathsf{Gal}(f)$-actions, where f is the special effective descent morphism as in (4.1) previously constructed and $\mathsf{Gal}(f)$ the *Galois groupoid* associated with f. This can be done by using the notion of *locally semisimple covering* introduced by G. Janelidze, L. Márki and W. Tholen (see Theorem 3.1 in [59]). We then observe that V-$\mathsf{Grp}_{\mathsf{sep}}$ is a torsion-free subcategory not only for a torsion theory but also for a pretorsion theory. In this case, the torsion subcategory is given by the full subcategory V-$\mathsf{Grp}_{\mathsf{sym}}$ of *symmetric* V-groups whose objects are V-groups (X, a) satisfying the symmetry axiom: for any $x, x' \in X$, $a(x, x') = a(x', x)$. Note that, contrary to the above, for this last result, we do not need to assume that the quantale V is integral. From this pretorsion theory, we deduce that V-$\mathsf{Grp}_{\mathsf{sym}}$ is coreflective in V-Grp. This was already mentioned in [22], where it is also proven that V-$\mathsf{Grp}_{\mathsf{sym}}$ is reflective.

We finally remark, in the last section, that a pretorsion theory can also be obtained in the more general context of *V-categories*.

The results of this chapter mainly come from the paper [74].

4.1. The category V-Grp of V-groups

We devote this first section to the notion of *V-group*: among other things, we define such a notion through the concepts of *V-relation* and *V-category*, we give some examples of V-groups (including the example of preordered groups), and we describe some limits and colimits as well as the different kinds of epimorphisms and monomorphisms, and the short exact sequences in this special setting. We then state some properties about the category of V-groups which will be useful for our work. For this part, we follow the recent paper [22] where the reader will find more details.

Consider V a commutative and unital *quantale*, i.e. V is a complete lattice (with bottom element $\perp$ and top element $\top$) endowed with an associative and commutative tensor product $\otimes$ which has a unit k and which preserves arbitrary joins: for any $v \in V$ and any family $\{u_i\}_{i\in I}$ in V,

$$v \otimes \left(\bigvee_{i\in I} u_i \right) = \bigvee_{i\in I} (v \otimes u_i) \qquad \text{and} \qquad v \otimes \perp = \perp.$$

For the purpose of this chapter, we add two assumptions on the quantale V. We first suppose that arbitrary joins distribute over finite meets so that, as a lattice, V is a frame. Indeed, following [22, Remark 4.2(6)], this extra condition allows one to check the pullback-stability of regular epimorphisms in V-Grp and then the regularity of this category (see Subsection 4.1.3). The second additional assumption that we consider is that V is non-trivial, i.e. $\perp \neq \top$. This is in particular used in the proof of Proposition 4.3.4.

Remark 4.1.1. For part of our results (which will be specified later), we will assume that V is in addition an *integral* quantale, i.e. that $k = \top$ in V.

In order to understand the notion of V*-group*, we first need to define the concepts of V*-relation* and of V*-category*.

A V*-relation* $r : X \nrightarrow Y$ from a set X to a set Y is a function $r\colon X \times Y \to V$. As for ordinary relations, a V-relation $r\colon X \nrightarrow Y$ can be composed with a V-relation $s\colon Y \nrightarrow Z$, via a "matrix multiplication": for any $x \in X$ and any $z \in Z$,

$$(s \cdot r)(x, z) = \bigvee_{y\in Y} r(x, y) \otimes s(y, z),$$

and this defines a V-relation $s \cdot r\colon X \nrightarrow Z$. The identity for the composition is given by the V-relation $1_X\colon X \nrightarrow X$ defined, for any $x, x' \in X$, by

$$1_X(x, x') = \begin{cases} k & \text{if} \quad x = x' \\ \perp & \text{if} \quad x \neq x'. \end{cases}$$

The behaviour of V-relations under composition explains the terminology of *V-matrices* we sometimes use instead of that of V-relations (see for instance [77] for the use of this terminology). Indeed, the above definition of the composition as well as its properties (associativity and identity) suggest an analogy with the matrix product.

We then get the category V-Rel of V-relations whose objects are sets and whose arrows are V-relations. Now, there exists a non-full, bijective on objects, embedding functor $\mathsf{Set} \to V$-Rel which associates with any function $f\colon X \to Y$ the V-relation $f\colon X \nrightarrow Y$ defined, for any $x \in X$ and any $y \in Y$, by

$$f(x,y) = \begin{cases} k & \text{if } f(x) = y \\ \perp & \text{otherwise.} \end{cases}$$

In addition, for any pair (X,Y) of sets, we have an order on the set V-$\mathsf{Rel}(X,Y)$ which is induced by the one on the quantale V: for $r, r'\colon X \nrightarrow Y$ two V-relations from X to Y, we say that $r \leq r'$ if and only if, in V, $r(x,y) \leq r'(x,y)$ for any $x \in X$ and $y \in Y$.

We are now ready for the definition of a *V-category*:

Definition 4.1.2. A *V-category* is a pair (X,a) where X is a set and $a\colon X \nrightarrow X$ a V-relation such that

$$1_X \leq a \qquad \text{and} \qquad a \cdot a \leq a.$$

In pointwise notation, these last two inequalities give the *reflexivity* and *transitivity* axioms:

(R) $k \leq a(x,x)$ for any $x \in X$;

(T) $a(x,x') \otimes a(x',x'') \leq a(x,x'')$ for any $x, x', x'' \in X$.

Definition 4.1.3. Given two V-categories (X,a) and (Y,b), a *V-functor* $f\colon (X,a) \to (Y,b)$ is a function $f\colon X \to Y$ such that $f \cdot a \leq b \cdot f$, that is, such that

$$a(x,x') \leq b(f(x), f(x')) \qquad \text{for any } x, x' \in X.$$

This gives rise to the category V-Cat of V-categories and V-functors.

Remark 4.1.4. Pairs (X,a) with X a set and $a\colon X \nrightarrow X$ a V-relation satisfying the reflexivity axiom (R) only are said to be *V-graphs*.

Remark 4.1.5. The notions of V-category and V-functor we are considering are particular cases of the broader concepts of *$\mathscr{V}$-category* and *$\mathscr{V}$-functor*, where $\mathscr{V}$ is more generally a *symmetric monoidal closed category* [27].

We can now state the definition of a *V-group*:

Definition 4.1.6. A *V-group* $(X, a, +)$ is a V-category (X, a) endowed with a group structure

$$(X, +\colon X \times X \to X, -\colon X \to X, 0\colon 0 \to X)$$

such that $+\colon (X, a) \otimes (X, a) \to (X, a)$ is a V-functor, where we define $(X, a) \otimes (X, a)$ by $(X \times X, a \otimes a)$ with

$$(a \otimes a)((x_1, x_1'), (x_2, x_2')) = a(x_1, x_2) \otimes a(x_1', x_2').$$

Remark 4.1.7. Even though we are using the additive notation, the group structure involved in Definition 4.1.6 is not necessarily abelian.

Definition 4.1.8. A *V-homomorphism* is a V-functor which is also a group homomorphism.

V-groups and V-homomorphisms define the category V-Grp of V-groups.

When we want to check that a given pair (X, a), with $(X, +)$ a group, is a V-group, it is sometimes easier to use the following proposition:

Proposition 4.1.9. [22, Proposition 3.1] *Let (X, a) be a V-graph and $(X, +)$ be a group. Then, the following conditions are equivalent:*

(1) $+\colon (X, a) \otimes (X, a) \to (X, a)$ *is a V-functor;*

(2) (X, a) *is a V-category and a is* invariant by shifting*, i.e., for any* $x, x', x'' \in X$,
$$a(x', x'') = a(x' + x, x'' + x).$$

Example 4.1.10. If we consider the quantale $V = 2 = (\{\perp, \top\}, \wedge, \top)$ (i.e. $\otimes = \wedge$ and $k = \top$), then any V-group is in fact a preordered group, and the converse also holds. Indeed, given a V-group (X, a) with $V = 2$, we can get a preordered group given by $(X, \leq)$, where $x \leq x'$ in X if and only if $a(x,x') = \top$. Conversely, any preordered group $(X,\leq)$ gives rise to a V-group (X,a) with $V = 2$ by defining $a\colon X \nrightarrow X$, for any $x,x' \in X$, by

$$a(x,x') = \begin{cases} \top & \text{if } x \leq x' \\ \bot & \text{otherwise.} \end{cases}$$

As a consequence, the category of V-groups (and V-homomorphisms), with $V = 2$, is isomorphic to the category $\mathsf{PreOrdGrp}$ of preordered groups (and preorder preserving group homomorphisms), and this latter category is then an example of a category of V-groups. Other interesting examples of categories of V-groups are given by the categories MetGrp of *Lawvere (generalized) metric groups* and non-expansive group homomorphisms (with $V = ([0,\infty],\geq)$, $\otimes = +$ and $k = 0$), $\mathsf{UMetGrp}$ of *Lawvere (generalized) ultrametric groups* and non-expansive group homomorphisms (with $V = ([0,\infty],\geq)$, $\otimes =$ the max for the usual order on real numbers and $k = 0$), $\mathsf{ProbMetGrp}$ of *probabilistic metric groups* (with

$$V = \left\{ \phi : [0,\infty] \to [0,1] \mid \phi \text{ is monotone and } \phi(x) = \bigvee_{y<x} \phi(y) \right\},$$

with the pointwise order, and with $\otimes$ defined by

$$(\phi \otimes \psi)(u) = \bigvee_{x+y\leq u} \phi(x) \times \psi(y)),$$

etc [69].

4.1.1. Limits and colimits in V-Grp

The category V-Grp of V-groups is complete and cocomplete. Before describing the limits and some colimits in this particular setting, we first remark that the initial object in V-Grp is given by the V-group $(\{*\},\kappa)$ where $\kappa(*,*) = k$ while the terminal object is $(\{*\},c)$ where $c(*,*) = \top$, so that the category V-Grp of V-groups is pointed if and only if $k = \top$, i.e. if and only if V is integral.

In V-Grp, the binary product of two V-groups (X,a) and (Y,b) is given by $(X \times Y, a \wedge b)$, where

$$(a \wedge b)((x_1,y_1),(x_2,y_2)) = a(x_1,x_2) \wedge b(y_1,y_2),$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$. The equalizer of a pair

$$f, g\colon (X, a) \rightrightarrows (Y, b) \tag{4.2}$$

of arrows in V-Grp is given by $(E, \tilde{a}) \xrightarrow{e} (X, a)$ where (E, e) is the equalizer of f and g in the category Grp of groups and $\tilde{a}$ is the V-category structure induced by the one of X: $\tilde{a} = e^\circ \cdot a \cdot e$ or, equivalently, $\tilde{a}(x, x') = a(e(x), e(x'))$ for any $x, x' \in E$.

Remark 4.1.11. For any V-relation $a\colon X \nrightarrow Y$, we can define the *opposite (or dual) V-relation* $a^\circ\colon Y \nrightarrow X$: $a^\circ(y, x) = a(x, y)$ for $x \in X$ and $y \in Y$.

The pullback of two arrows $f\colon (X, a) \to (Z, c)$ and $g\colon (Y, b) \to (Z, c)$ is $((X \times_Z Y, a \wedge b), \pi_1, \pi_2)$, where $(X \times_Z Y, \pi_1, \pi_2)$ is the pullback of f and g in the category Grp of groups. In particular, the kernel of f is given by $(K, \tilde{a}) \xrightarrow{\kappa} (X, a)$ where (K, κ) is the kernel of f in Grp and $\tilde{a}$ is the V-category structure induced by the one of X.

It is also possible to describe the colimits in V-Grp. The coequalizer of two parallel arrows f and g as in (4.2) is obtained by computing the coequalizer (Q, q) of f and g in Grp and then by equipping the group Q with the final V-category structure $\bar{b} = q \cdot b \cdot q^\circ$, that is

$$\bar{b}(z_1, z_2) = \bigvee_{q(y_i) = z_i} b(y_1, y_2)$$

for any $z_1, z_2 \in Q$. In particular, the cokernel of f is given by $(Y, b) \xrightarrow{q} (Q, \bar{b})$ where (Q, q) is the cokernel of f in Grp and $\bar{b}$ is the final V-category structure. The description of coproducts in V-Grp is more complex but it will not be needed for this thesis so that we do not give details about it here and refer the interested reader to [22].

4.1.2. Some special arrows in V-Grp

Let us now study the different kinds of epimorphisms and monomorphisms which will be of interest for our work. An arrow $f\colon (X, a) \to (Y, b)$ in V-Grp is an epimorphism if and only if it is surjective. It is a regular epimorphism whenever it is also final in the sense that $b = f \cdot a \cdot f^\circ$. Moreover, the category of V-groups has the property that any regular epimorphism is a normal epimorphism (whenever V is integral). The morphism f is a monomorphism if and only if it injective, and it is a normal monomorphism whenever it is a normal monomorphism in Grp and $a(x, x') = b(f(x), f(x'))$ for any $x, x' \in X$.

In the next proposition, we gather the information from [22] which is useful to describe short exact sequences in the category V-Grp of V-groups:

Proposition 4.1.12. *A pair of composable arrows*

$$(X, a) \xrightarrow{\kappa} (Y, b) \xrightarrow{f} (Z, c) \tag{4.3}$$

is a short exact sequence in V-Grp if and only if

$$0 \longrightarrow X \xrightarrow{\kappa} Y \xrightarrow{f} Z \longrightarrow 0$$

is a short exact sequence in Grp, $a(x, x') = b(\kappa(x), \kappa(x'))$ for any $x, x' \in X$ (i.e. $a = \kappa^\circ \cdot b \cdot \kappa$) and $c(z_1, z_2) = \bigvee_{f(y_i)=z_i} b(y_1, y_2)$ for any $z_1, z_2 \in Z$ (i.e. $c = f \cdot b \cdot f^\circ$).

4.1.3. Some properties of the category V-Grp

We end this section by stating some properties which will turn out to be useful later on. The category V-Grp is first of all regular, but not (Barr-)exact, as observed in [22, Proposition 4.3]. In particular, every V-homomorphism $f: (X, a) \to (Y, b)$ factors as

$$\begin{tikzcd}
(X,a) \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & (Y,b) \\
& (f(X), c) \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

with e a regular epimorphism in V-Grp, m a monomorphism in V-Grp and $c = e \cdot a \cdot e^\circ$. It is moreover *normal* whenever the quantale V is integral, since any regular epimorphism is then a normal epimorphism, and the effective descent morphisms exactly coincide with the regular epimorphisms. The detailed proofs of these assertions are due to M. M. Clementino and A. Montoli [22].

4.2. A torsion theory in the category of V-groups

In this section, we prove that the category V-Grp of V-groups admits a torsion theory whenever the quantale V is integral (so that V-Grp is then normal). Its torsion-free subcategory is the category V-$\mathsf{Grp}_{\mathsf{sep}}$ of *separated V-groups* while its torsion subcategory is the category V-$\mathsf{Grp}_{\mathsf{ind}}$ of *indiscrete V-groups*. Let us recall the definition of these two subcategories:

- The subcategory V-$\mathsf{Grp}_{\mathsf{sep}}$ of separated V-groups is the subcategory of V-groups $(X, a, +)$ satisfying the following axiom: for $x, x' \in X$,

$$a(x, x') \geq k \quad \text{and} \quad a(x', x) \geq k \qquad \Longrightarrow \qquad x = x'.$$

  Note that, since a is invariant by shifting, the above property is equivalent to the following : for $x \in X$,

$$a(x, 0) \geq k \quad \text{and} \quad a(0, x) \geq k \qquad \Longrightarrow \qquad x = 0.$$

  We borrow this terminology (which, to our knowledge, does not exist in the literature yet) from the more general notion of *separated V-category* (mentioned, for instance, in the Appendix of [51] and in [52]). Note that, for $V = ([0, \infty], \geq)$ (with $\otimes = +$), separated V-groups correspond to the usual separated Lawvere metric groups (i.e. those Lawvere metric groups (X, d) satisfying the separation axiom: $d(x, x') = 0 = d(x', x) \implies x = x'$).
- We can next consider V-groups $(X, a, +)$ with the *indiscrete V-category structure*: $a(x, x') = \top$ for any $x, x' \in X$. This structure gives rise to the category V-$\mathsf{Grp}_{\mathsf{ind}}$ of indiscrete V-groups which is a full subcategory of V-Grp.

We first of all state a lemma which will turn out to be useful not only in the present section but also in Section 4.5.

Lemma 4.2.1. *Let (X, a) be a V-group. Then:*

(1) the subset

$$N_X = \{x \in X | a(0, x) \geq k \text{ and } a(x, 0) \geq k\}$$

of X is a normal subgroup of X;

(2) the pair $(X/N_X, \bar{a})$ is a separated V-group, where $\bar{a} = \eta_X \cdot a \cdot \eta_X^\circ$ with $X \xrightarrow{\eta_X} X/N_X$ the quotient group morphism.

Proof.

(1) We first remark that the subset N_X is a subgroup of X since

- $0 \in N_X$: $a(0,0) \geq k$;
- $x \in N_X \implies -x \in N_X$: $a(0,-x) = a(x,0) \geq k$ and $a(-x,0) = a(0,x) \geq k$;
- $x, y \in N_X \implies x+y \in N_X$:

$$a(0, x+y) \geq a(0,x) \otimes a(0,y) \geq k \otimes k = k$$

and

$$a(x+y, 0) \geq a(x,0) \otimes a(y,0) \geq k \otimes k = k$$

since $+\colon (X,a) \otimes (X,a) \to (X,a)$ is a V-functor.

This subgroup N_X is actually normal in X. Indeed, by invariance of a by shifting, for $x \in X$ and $n \in N_X$, we have that

$$a(0, x+n-x) = a(-x+x, n) = a(0,n) \geq k$$

and

$$a(x+n-x, 0) = a(n, -x+x) = a(n,0) \geq k,$$

which implies that $x+n-x \in N_X$, as desired.

(2) Assume that $\bar{a}(y,0) \geq k$ and that $\bar{a}(0,y) \geq k$ for $y \in X/N_X$. Then,

$$k \leq \bar{a}(y,0) = \bigvee_{\substack{\eta_X(x_1)=y \\ \eta_X(x_2)=0}} a(x_1, x_2) \tag{4.4}$$

and

$$k \leq \bar{a}(0,y) = \bigvee_{\substack{\eta_X(x_1')=0 \\ \eta_X(x_2')=y}} a(x_1', x_2'). \tag{4.5}$$

Now, we observe that, for any $x_1, x_2, x_1', x_2' \in X$ such that $\eta_X(x_1) = \eta_X(x_1')$ and $\eta_X(x_2) = \eta_X(x_2')$, we have

$$\begin{aligned} a(x_1, x_2) &= a(x_1 - x_1' + x_1', x_2 - x_2' + x_2') \\ &\geq a(x_1 - x_1', 0) \otimes a(0, x_2 - x_2') \otimes a(x_1', x_2') \\ &\geq k \otimes k \otimes a(x_1', x_2') = a(x_1', x_2') \end{aligned}$$

since $+\colon (X,a)\otimes(X,a)\to(X,a)$ is a V-functor and since $x_1 - x_1' \in N_X$ and $x_2 - x_2' \in N_X$. Symmetrically, we also show that $a(x_1', x_2') \geq a(x_1, x_2)$ for such x_1, x_2, x_1', x_2', and this proves that $a(x_1, x_2) = a(x_1', x_2')$ for any $x_1, x_2, x_1', x_2' \in X$ such that $\eta_X(x_1) = \eta_X(x_1')$ and $\eta_X(x_2) = \eta_X(x_2')$. Since the morphism $\eta_X\colon X \to X/N_X$ is surjective, there exists an $x \in X$ such that $\eta_X(x) = y$. Equations (4.4) and (4.5) then imply, by idempotence of the operation $\vee$, that $a(x,0) \geq k$ and that $a(0,x) \geq k$. In other words, $x \in N_X$. This shows that $(X/N_X, \bar{a}) \in V\text{-}\mathsf{Grp}_{\mathsf{sep}}$ since

$$y = \eta_X(x) = 0. \qquad \square$$

As previously announced, from now on, the commutative and unital quantale V will be assumed to be also integral. Under this additional assumption, the reflexivity axiom from the definition of a V-category (Definition 4.1.2) now becomes:

$$a(x,x) = k = \top \qquad \text{for any } x \in X.$$

Proposition 4.2.2. *The pair of full and replete subcategories* $(V\text{-}\mathsf{Grp}_{\mathsf{ind}}, V\text{-}\mathsf{Grp}_{\mathsf{sep}})$ *of* V-Grp *is a torsion theory in the normal category* V-Grp.

Proof. Let us first show that the only arrow in V-Grp from an object of $V\text{-}\mathsf{Grp}_{\mathsf{ind}}$ to an object of $V\text{-}\mathsf{Grp}_{\mathsf{sep}}$ is the zero morphism. For this, consider an arrow $f\colon (X,a)\to(Y,b)$ in V-Grp, with $(X,a) \in V\text{-}\mathsf{Grp}_{\mathsf{ind}}$ and $(Y,b) \in V\text{-}\mathsf{Grp}_{\mathsf{sep}}$. Then, since f is a V-homomorphism, for any $x \in X$,

$$b(f(x),0) \geq a(x,0) = \top \qquad \Longrightarrow \qquad b(f(x),0) \ = \top = k$$

and, similarly,

$$b(0,f(x)) \geq a(0,x) = \top \qquad \Longrightarrow \qquad b(0,f(x)) \ = \top = k,$$

which implies that $f(x) = 0$ since $f(x) \in Y$ with (Y,b) a separated V-group, and then $f = 0$.
Let now (X,a) be an object of V-Grp, and consider the normal subgroup

$$N_X = \{x \in X | a(0,x) \geq k \text{ and } a(x,0) \geq k\}$$

of X (see Lemma 4.2.1(1)). So the sequence $N_X \xrightarrow{\kappa_X} X \xrightarrow{\eta_X} X/N_X$, where κ_X is the inclusion of N_X in X and η_X the quotient morphism, is a short exact sequence in Grp. Now, we endow N_X with the V-category structure $\tilde{a}$ induced by the one of X, i.e. $\tilde{a} = \kappa_X^\circ \cdot a \cdot \kappa_X$, and we equip the quotient X/N_X with the final structure $\bar{a}$, that is, $\bar{a} = \eta_X \cdot a \cdot \eta_X^\circ$. By Proposition 4.1.12, the sequence

$$(N_X, \tilde{a}) \xrightarrow{\kappa_X} (X, a) \xrightarrow{\eta_X} (X/N_X, \bar{a}) \tag{4.6}$$

is a short exact sequence in the category V-Grp of V-groups. It remains to show that $(N_X, \tilde{a}) \in V\text{-}\mathsf{Grp}_{\mathsf{ind}}$ and that $(X/N_X, \bar{a}) \in V\text{-}\mathsf{Grp}_{\mathsf{sep}}$. Thanks to Lemma 4.2.1(2), it is first of all clear that $(X/N_X, \bar{a}) \in V\text{-}\mathsf{Grp}_{\mathsf{sep}}$. We then compute that, for any $n, n' \in N_X$,

$$\tilde{a}(n, n') = a(\kappa_X(n), \kappa_X(n')) = a(n, n') = a(0, n' - n) \geq k = \top,$$

since $n' - n \in N_X$ and V is integral, which implies that $\tilde{a}(n, n') = \top$, and this shows that $(N_X, \tilde{a}) \in V\text{-}\mathsf{Grp}_{\mathsf{ind}}$. $\square$

Example 4.2.3. In all the examples of V-groups discussed in Example 4.1.10, the quantale V is integral. Let us then take a look at what Proposition 4.2.2 means in these different cases:

(1) If we consider the quantale $2 = (\{\perp, \top\}, \wedge, \top)$ (i.e. the context of preordered groups), then Proposition 4.2.2 states that $(\mathsf{Grp}, \mathsf{ParOrdGrp})$ forms a torsion theory in the category $\mathsf{PreOrdGrp}$ of preordered groups, where Grp is the full subcategory of $\mathsf{PreOrdGrp}$ whose objects are given by preordered groups endowed with the indiscrete relation and $\mathsf{ParOrdGrp}$ the subcategory of partially ordered groups. Proposition 4.2.2 is then a generalization of Proposition 3.1.1.

(2) Another consequence of Proposition 4.2.2 is that $(\mathsf{MetGrp}_{\mathsf{ind}}, \mathsf{MetGrp}_{\mathsf{sep}})$ is a torsion theory in the category MetGrp of Lawvere (generalized) metric groups (i.e. in V-Grp, where $V = ([0, \infty], \geq)$, $\otimes = +$ and $k = 0$). The objects of its torsion subcategory $\mathsf{MetGrp}_{\mathsf{ind}}$ are the Lawvere metric groups (X, d) whose distance d is always 0, while the objects of the torsion-free subcategory $\mathsf{MetGrp}_{\mathsf{sep}}$ correspond to the usual separated Lawvere metric groups, i.e. those Lawvere metric groups (X, d) satisfying the separation axiom: $d(x, x') = 0 = d(x', x) \implies x = x'$.

(3) Similarly to (2), by Proposition 4.2.2, there exists a torsion theory in the category $\mathsf{UMetGrp}$ of Lawvere (generalized) ultrametric groups (case where $V = ([0,\infty], \geq)$, $\otimes =$ the max for the usual order on real numbers and $k = 0$) which is given by $(\mathsf{UMetGrp}_{\mathsf{ind}}, \mathsf{UMetGrp}_{\mathsf{sep}})$.

(4) By Proposition 4.2.2, $(\mathsf{ProbMetGrp}_{\mathsf{ind}}, \mathsf{ProbMetGrp}_{\mathsf{sep}})$ is a torsion theory in the category $\mathsf{ProbMetGrp}$ of probabilistic metric groups.

As a direct consequence of Proposition 4.2.2, we get the following result:

Corollary 4.2.4.

- *The category $V\text{-}\mathsf{Grp}_{\mathsf{sep}}$ is reflective in the category $V\text{-}\mathsf{Grp}$*

$$\begin{tikzcd} V\text{-}\mathsf{Grp} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & V\text{-}\mathsf{Grp}_{\mathsf{sep}} \arrow[l, hook', shift left=1.5, "U"] \end{tikzcd} \tag{4.7}$$

  and each component of the unit η of the adjunction (as in (4.6)*) is a normal epimorphism.*

- *The category $V\text{-}\mathsf{Grp}_{\mathsf{ind}}$ is coreflective in $V\text{-}\mathsf{Grp}$ and each component of the counit κ of the adjunction (as in* (4.6)*) is a normal monomorphism.*

Proof. This follows from Propositions 4.2.2 and 2.2.5. □

4.3. Monotone-light factorization system and coverings in V-Grp

Proposition 4.3.1. *The reflector $F\colon V\text{-}\mathsf{Grp} \to V\text{-}\mathsf{Grp}_{\mathsf{sep}}$ in the adjunction* (4.7) *has stable units.*

Proof. This follows directly from Proposition 2.4.6. □

The reflector $F\colon V\text{-}\mathsf{Grp} \to V\text{-}\mathsf{Grp}_{\mathsf{sep}}$ then induces a factorization system $(\mathscr{E}, \mathscr{M})$ whose description is as below:

Proposition 4.3.2. *Consider, in $V\text{-}\mathsf{Grp}$, the commutative diagram*

$$\begin{tikzcd}
(N_X, \tilde{a}) \arrow[r, "\kappa_X"] \arrow[d, "\phi"'] \arrow[dr, phantom, "(1)"] & (X, a) \arrow[r, two heads, "\eta_X"] \arrow[d, "f"] \arrow[dr, phantom, "(2)"] & (X/N_X, \bar{a}) \arrow[d, "\alpha"] \\
(N_Y, \tilde{b}) \arrow[r, "\kappa_Y"'] & (Y, b) \arrow[r, two heads, "\eta_Y"'] & (Y/N_Y, \bar{b})
\end{tikzcd}$$

where, as before, $\tilde{a} = \kappa_X^\circ \cdot a \cdot \kappa_X$, $\tilde{b} = \kappa_Y^\circ \cdot b \cdot \kappa_Y$, $\bar{a} = \eta_X \cdot a \cdot \eta_X^\circ$, *and* $\bar{b} = \eta_Y \cdot b \cdot \eta_Y^\circ$, *where* η_X *(respectively* η_Y*) is the* (X, a)*-component (respectively the* (Y, b)*-component) of the unit of the adjunction* (4.7), ϕ *is the restriction of* f *to* $(N_X, \tilde{a})$ *and where we write* α *for* $F(f)$.
The adjunction (4.7) *induces a factorization system* $(\mathscr{E}, \mathscr{M})$ *in* V-Grp *where:*

- $f\colon (X, a) \to (Y, b)$ *is in the class* $\mathscr{E}$ *if and only if the following conditions hold:*
  (a) the square (1) *is a pullback;*
  (b) $\eta_Y \cdot f$ *is a regular epimorphism;*
- $f\colon (X, a) \to (Y, b)$ *is in the class* $\mathscr{M}$ *if and only if the restriction* $\phi\colon N_X \to N_Y$ *of* f *to* N_X *is a group isomorphism and* $a = b \wedge \bar{a}$.

Proof.

- Assume that $f\colon (X, a) \to (Y, b)$ is in the class $\mathscr{E}$, so that α is an isomorphism in V-Grp. Since V-Grp is a normal category, by Proposition 1.2.13, the fact that α is a monomorphism implies that the square (1) in the above diagram is a pullback. Now, knowing that α is a regular epimorphism, we have that the composite $\alpha \cdot \eta_X$ is also a regular epimorphism (since η_X is itself a regular epimorphism and the category V-Grp is regular), that is, $\eta_Y \cdot f$ is a regular epimorphism by commutativity of the square (2).
  Conversely, if the square (1) in the above diagram is a pullback, then this implies, by Proposition 1.2.13, that α is a monomorphism since V-Grp is a normal category. The assumption (b) then implies that $\alpha \cdot \eta_X$ is a regular epimorphism (by commutativity of the square (2)). Knowing that the category V-Grp is regular, it follows that α

is a regular epimorphism. As a conclusion, α is an isomorphism (by Proposition 1.1.25), that is, f is in the class $\mathscr{E}$.

- Assume that $f\colon (X,a) \to (Y,b)$ is in the class $\mathscr{M}$. This means that the square (2) in the above diagram is a pullback. It is then easily seen that $a = b \wedge \bar{a}$ and that the arrow ϕ is an isomorphism in the category of V-groups (see Proposition 1.1.45), so in particular in the category of groups.
  Conversely, if we suppose that ϕ is a group isomorphism, then the square
  $$\begin{tikzcd} X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"'] & X/N_X \arrow[d, "\alpha"] \\ Y \arrow[r, two heads, "\eta_Y"'] & Y/N_Y \end{tikzcd}$$
  is a pullback in the category Grp of groups since this category is indeed protomodular (see Remark 1.1.46). By assumption, we also know that $a = b \wedge \bar{a}$ so that the square (2) in the above diagram is a pullback in V-Grp. It follows that f is in the class $\mathscr{M}$. ☐

In particular, the previous proposition provides us with a description of the trivial coverings (i.e. the V-homomorphisms in the class $\mathscr{M}$). In order to find *all* the coverings, we use Theorem 2.4.9 ([29, Theorem 1.6], see also [17]), which not only gives a very simple characterization of the morphisms in the class $\mathscr{M}^*$ but also states that the pair $(\mathscr{E}', \mathscr{M}^*)$ is a monotone-light factorization system.

As was the case for preordered groups in Chapter 3, it now remains to check two assumptions (conditions (N) and (C)) in order to apply Theorem 2.4.9 to our context. This is the aim of the following two propositions.

Proposition 4.3.3. *For any normal monomorphism $i\colon (K,b) \rightarrowtail (X,a)$ in V-Grp, the monomorphism $i \cdot \kappa\colon (N_K, \tilde{b}) \rightarrowtail (X,a)$ is normal, where $\kappa\colon (N_K, \tilde{b}) \rightarrowtail (K,b)$ is the (K,b)-component of the counit of the coreflection V-$\mathsf{Grp} \to V$-$\mathsf{Grp}_{\mathsf{ind}}$.*

Proof. Since i is a normal monomorphism, there exists an arrow $f\colon (X,a) \to (Z,c)$ in V-Grp such that $i = \mathsf{ker}(f)$. It means that K is a normal subgroup of X and that, for any $x, x' \in K$,

$$b(x, x') = a(i(x), i(x')).$$

Let us show that N_K is a normal subgroup of X:

$$\begin{aligned} N_K &= \{x \in K \mid b(0, x) \geq k \text{ and } b(x, 0) \geq k\} \\ &= K \cap \{x \in X \mid a(0, x) \geq k \text{ and } a(x, 0) \geq k\} \\ &= K \cap N_X \end{aligned}$$

with K and N_X two normal subgroups of X. Hence, N_K is normal in X. One then observes that, for any $n, n' \in N_K$,

$$a((i \cdot \kappa)(n), (i \cdot \kappa)(n')) = a(i(\kappa(n)), i(\kappa(n'))) = b(\kappa(n), \kappa(n')) = \tilde{b}(n, n').$$

We conclude that the monomorphism $i \cdot \kappa$ is normal. □

Proposition 4.3.4. *For any object (X, a) in the category V-Grp of V-groups, there exist an object (Y, b) in the subcategory V-$\mathsf{Grp}_{\mathsf{sep}}$ of separated V-groups and an effective descent morphism*

$$f \colon (Y, b) \to (X, a)$$

from (Y, b) to (X, a).

Proof. Let $(X, a) \in V$-Grp. Define (Y, b) in the following way:

- $Y = \mathbb{Z} \times X$ as a group;
- for any $z, z' \in \mathbb{Z}$ and any $x, x' \in X$,

$$b((z, x), (z', x')) = \begin{cases} a(x, x') & \text{if } z < z' \\ \top & \text{if } z = z' \text{ and } x = x' \\ \bot & \text{otherwise.} \end{cases}$$

It is then clear that (Y, b) is a V-graph: for any $(z, x) \in Y$,

$$b((z, x), (z, x)) = \top = k.$$

Let us now check that $+ \colon (Y, b) \otimes (Y, b) \to (Y, b)$ is a V-functor. We recall that we have to prove the following: for any $(z_1, x_1), (z_1', x_1'), (z_2, x_2), (z_2', x_2') \in Y$,

$$\begin{aligned} b((z_1, x_1), (z_1', x_1')) \otimes b((z_2, x_2), (z_2', x_2')) \\ \leq b((z_1 + z_2, x_1 + x_2), (z_1' + z_2', x_1' + x_2')). \quad (4.8) \end{aligned}$$

For this, we consider the different possible cases.

(1) If either $b((z_1,x_1),(z_1',x_1')) = \bot$ or $b((z_2,x_2),(z_2',x_2')) = \bot$, then the identity (4.8) is obviously satisfied.

(2) If both $b((z_1,x_1),(z_1',x_1')) = \top$ and $b((z_2,x_2),(z_2',x_2')) = \top$, then $z_1+z_2 = z_1'+z_2'$ (since $z_1 = z_1'$ and $z_2 = z_2'$) and $x_1+x_2 = x_1'+x_2'$ (since $x_1 = x_1'$ and $x_2 = x_2'$), which implies that

$$b((z_1+z_2, x_1+x_2),(z_1'+z_2', x_1'+x_2')) = \top.$$

In this case, the identity (4.8) is then also verified.

(3) If $b((z_1,x_1),(z_1',x_1')) = \top$ and $b((z_2,x_2),(z_2',x_2')) = a(x_2,x_2')$, then

$$\begin{aligned} b((z_1,x_1),(z_1',x_1')) \otimes b((z_2,x_2),(z_2',x_2')) &= \top \otimes a(x_2,x_2') \\ &= a(x_1,x_1) \otimes a(x_2,x_2') \\ &\leq a(x_1+x_2, x_1+x_2') = a(x_1+x_2, x_1'+x_2') \end{aligned}$$

since (X,a) is a V-group and $x_1 = x_1'$. This leads to (4.8) because, in this case,

$$b((z_1+z_2, x_1+x_2),(z_1'+z_2', x_1'+x_2')) = a(x_1+x_2, x_1'+x_2')$$

since $z_1+z_2 = z_1'+z_2 < z_1'+z_2'$. The symmetric situation ($b((z_1,x_1),(z_1',x_1')) = a(x_1,x_1')$ and $b((z_2,x_2),(z_2',x_2')) = \top$) is treated in a similar way.

(4) If $b((z_1,x_1),(z_1',x_1')) = a(x_1,x_1')$ (i.e. $z_1 < z_1'$) and $b((z_2,x_2),(z_2',x_2')) = a(x_2,x_2')$ (i.e. $z_2 < z_2'$), then

$$\begin{aligned} b((z_1,x_1),(z_1',x_1')) \ \otimes\ & b((z_2,x_2),(z_2',x_2')) = a(x_1,x_1') \otimes a(x_2,x_2') \\ &\leq a(x_1+x_2, x_1'+x_2') = b((z_1+z_2, x_1+x_2),(z_1'+z_2', x_1'+x_2')) \end{aligned}$$

since $+\colon (X,a)\otimes(X,a) \to (X,a)$ is a V-functor and $z_1+z_2 < z_1'+z_2'$.

As a conclusion, thanks to Proposition 4.1.9, we deduce that (Y,b) is a V-group. In particular, $(Y,b) \in V\text{-}\mathsf{Grp}_{\mathsf{sep}}$. Indeed, for $(z,x) \in Y$, if

$$b((z,x),(0,0)) = \top = b((0,0),(z,x)),$$

then $(z,x) = (0,0)$ since

- if $z < 0$, then
$$b((0,0),(z,x)) = \bot \neq \top;$$

- if $z > 0$, then
$$b((z,x),(0,0)) = \bot \neq \top;$$
- if $z = 0$ with $x \neq 0$, then
$$b((z,x),(0,0)) = b((0,0),(z,x)) = \bot \neq \top.$$

Let us now consider the function $f\colon Y \to X$ defined, for any $(z,x) \in Y$, by
$$f(z,x) = x.$$

It is clear that f is a group homomorphism. We now prove that, in addition, $f\colon (Y,b) \to (X,a)$ is a V-functor, i.e., for any $(z,x),(z',x') \in Y$,
$$b((z,x),(z',x')) \leq a(f(z,x), f(z',x')),$$
i.e.
$$b((z,x),(z',x')) \leq a(x,x').$$

Again, we consider the different possible cases:

- if $z < z'$, then
$$b((z,x),(z',x')) = a(x,x');$$
- if $z = z'$ and $x = x'$, then
$$b((z,x),(z',x')) = \top = a(x,x) = a(x,x');$$
- in the other cases,
$$b((z,x),(z',x)) = \bot \leq a(x,x').$$

We can then conclude that f is a V-homomorphism. It just remains to show that f is an effective descent morphism in V-Grp, which is the same as showing that it is a regular epimorphism. It is first of all clear that f is surjective. Let us next prove that f is also final, i.e. that, for any $x_1, x_2 \in X$,
$$a(x_1,x_2) = \bigvee_{f(y_i)=x_i} b(y_1,y_2).$$

We compute, for any $x_1, x_2 \in X$, that
$$\bigvee_{f(y_i)=x_i} b(y_1,y_2) = \bigvee_{z_1,z_2 \in \mathbb{Z}} b((z_1,x_1),(z_2,x_2)).$$

Accordingly,

- if $x_1 = x_2$, then
$$\bigvee_{f(y_i)=x_i} b(y_1, y_2) = \top = a(x_1, x_2)$$
since $b((z, x_1), (z, x_2)) = \top$ for any $z \in \mathbb{Z}$;
- if $x_1 \neq x_2$, then
$$\begin{aligned}\bigvee_{f(y_i)=x_i} b(y_1, y_2) &= \left(\bigvee_{z_1<z_2} b((z_1, x_1), (z_2, x_2))\right) \\ &\qquad\qquad \vee \left(\bigvee_{z_1 \geq z_2} b((z_1, x_1), (z_2, x_2))\right) \\ &= a(x_1, x_2) \vee \bot \\ &= a(x_1, x_2).\end{aligned}$$

This completes the proof. □

It is now possible to apply Theorem 2.4.9:

Theorem 4.3.5. *The pair $(\mathscr{E}', \mathscr{M}^*)$ is a monotone-light factorization system and, moreover,*

- $\mathscr{E}' = \{f \in V\text{-}\mathsf{Grp} \mid f$ *is a normal epimorphism such that* $\mathsf{Ker}(f) \in V\text{-}\mathsf{Grp}_{\mathsf{ind}}\}$*;*
- $\mathscr{M}^* = \{f \in V\text{-}\mathsf{Grp} \mid \mathsf{Ker}(f) \in V\text{-}\mathsf{Grp}_{\mathsf{sep}}\}$*.*

Proof. This result follows from Theorem 2.4.9, which can be applied to the reflection (4.7) thanks to the two previous propositions. □

Corollary 4.3.6. *A V-homomorphism $f\colon (X, a) \to (Y, b)$ in the category V-Grp of V-groups is a covering (with respect to the absolute Galois structure induced by the reflection* (4.7)*) if and only if its kernel $\mathsf{Ker}(f)$ belongs to the subcategory V-$\mathsf{Grp}_{\mathsf{sep}}$ of separated V-groups.*

Example 4.3.7.

(1) The coverings in $\mathsf{PreOrdGrp}$ with respect to the adjunction (4.7) (where $V = (\{\bot, \top\}, \wedge, \top)$) are the preorder preserving group morphisms whose kernel is a partially ordered group. As for the

morphisms in the class $\mathscr{E}'$, they are the normal epimorphisms in $\mathsf{PreOrdGrp}$ whose kernel is in the subcategory Grp. We therefore recover, from Theorem 4.3.5, the result obtained in Theorem 3.2.5 for preordered groups.

(2) When $V = ([0,\infty], \geq)$, $\otimes = +$ and $k = 0$, the coverings are given by those non-expansive group homomorphisms with kernel in $\mathsf{MetGrp}_{\mathsf{sep}}$, while the elements of the class $\mathscr{E}'$ are the normal epimorphisms in MetGrp whose kernel is in $\mathsf{MetGrp}_{\mathsf{ind}}$.

(3) In the setting of Lawvere (generalized) ultrametric groups, we get a result analogous to (2).

(4) By Theorem 4.3.5, the coverings with respect to the adjunction (4.7) (for V the suitable quantale) in the category $\mathsf{ProbMetGrp}$ of probabilistic metric groups are the morphisms whose kernel is in $\mathsf{ProbMetGrp}_{\mathsf{sep}}$, and the morphisms in this context which are in the class $\mathscr{E}'$ coincide with the normal epimorphisms whose kernel is in $\mathsf{ProbMetGrp}_{\mathsf{ind}}$.

4.4. Coverings of V-groups classified as internal actions

As it was the case for preordered groups, it is possible to classify the coverings in the category V-Grp of V-groups in terms of internal actions (provided, as before, that the quantale V is integral). This can be done by means of a result of [59], namely Theorem 3.3.1. We invite the reader to have a look at Subsection 3.3.1, if necessary.

We consider here the particular case where $\mathscr{C} = V$-Grp (with V an integral quantale so that V-Grp is a normal category) and $\mathcal{X} = V$-$\mathsf{Grp}_{\mathsf{sep}}$ which is easily seen (by using for instance an extension, from the homological to the normal setting, of Lemma 2.7 in [44]) to be a generalized semisimple class. By means of Theorem 3.3.1 and also of the following lemma, we will see that it is then possible to characterize the coverings of V-groups as internal actions.

Lemma 4.4.1. *A morphism $h\colon (Z,c) \to (X,a)$ in V-Grp is a locally semisimple covering (relatively to the subcategory V-$\mathsf{Grp}_{\mathsf{sep}}$) if and only if its kernel is a separated V-group.*

We refer to the proof of Lemma 3.3.2 (in the particular case of preordered groups) for a general proof of the above lemma since it is exactly the same idea for an arbitrary integral quantale V.

Remark 4.4.2. The previous lemma is a particular case of a more general fact observed in [59, Proposition 2.3] where the role of the kernel of an arrow was played by the "fibers" (as defined in [59]).

Theorem 4.4.3. *Let* $(X, a) \in V\text{-}\mathsf{Grp}$. *Consider the effective descent morphism*

$$f\colon (Y, b) \to (X, a)$$

from Proposition 4.3.4. Then there exists an equivalence of categories

$$\mathscr{M}^* \downarrow (X, a) \cong \mathsf{DiscFib}_{V\text{-}\mathsf{Grp}_{\mathsf{sep}}}(\mathsf{Eq}(f))$$

where $\mathscr{M}^* \downarrow (X, a)$ *is the category of coverings over* (X, a).

Proof. Since the V-homomorphism $f\colon (Y, b) \to (X, a)$ from Proposition 4.3.4 is an effective descent morphism such that $(Y, b) \in V\text{-}\mathsf{Grp}_{\mathsf{sep}}$, by Theorem 3.3.1, we have the following equivalence of categories:

$$\mathsf{LocSSimple}_{V\text{-}\mathsf{Grp}_{\mathsf{sep}}}(X, a) \cong \mathsf{DiscFib}_{V\text{-}\mathsf{Grp}_{\mathsf{sep}}}(\mathsf{Eq}(f)).$$

The proof is now complete thanks to Lemma 4.4.1 and Corollary 4.3.6 since both the locally semisimple coverings and the coverings in V-Grp are given by the V-homomorphisms having their kernel in the subcategory of separated V-groups. $\square$

As a reminder, the *Galois groupoid* $\mathsf{Gal}(f)$ associated with $f\colon (Y, b) \to (X, a)$ (where, as before, f denotes the effective descent morphism from Proposition 4.3.4) is the image of $\mathsf{Eq}(f)$ by the reflector $F\colon V\text{-}\mathsf{Grp} \to V\text{-}\mathsf{Grp}_{\mathsf{sep}}$. But since the diagram

$$(\mathsf{Eq}(f), b \wedge b) \rightrightarrows (Y, b)$$

lies in V-$\mathsf{Grp}_{\mathsf{sep}}$, the image of $\mathsf{Eq}(f)$ by F is $\mathsf{Eq}(f)$ itself. Accordingly, $\mathsf{Eq}(f)$ is the Galois groupoid associated with f, and

$$\mathscr{M}^* \downarrow (X, a) \cong \mathsf{DiscFib}_{V\text{-}\mathsf{Grp}_{\mathsf{sep}}}(\mathsf{Gal}(f)). \tag{4.9}$$

This equivalence *classifies* the coverings in V-Grp as internal $\mathsf{Gal}(f)$-actions.

Example 4.4.4.

(1) When $V = (\{\bot, \top\}, \wedge, \top)$, Equation (4.9) results in

$$\mathscr{M}^* \downarrow (G, P_G) \cong \mathsf{DiscFib}_{\mathsf{ParOrdGrp}}(\mathsf{Gal}(f, \bar{f})),$$

for any preordered group (G, P_G) and for $(f, \bar{f})$ the morphism of preordered groups of Proposition 3.2.4. This is precisely the equivalence (3.9) obtained in Chapter 3.

(2) When $V = ([0, \infty], \geq)$, $\otimes = +$ and $k = 0$, Equation (4.9) results in

$$\mathscr{M}^* \downarrow (X, d) \cong \mathsf{DiscFib}_{\mathsf{MetGrp_{sep}}}(\mathsf{Gal}(f)),$$

for any Lawvere (generalized) metric group (X, d).

(3) For Lawvere (generalized) ultrametric groups, we have a result which is similar to (2).

(4) In the context of probabilistic metric groups, we get from Equation (4.9) that

$$\mathscr{M}^* \downarrow (X, d) \cong \mathsf{DiscFib}_{\mathsf{ProbMetGrp_{sep}}}(\mathsf{Gal}(f)),$$

for any probabilistic metric group (X, d).

4.5. A pretorsion theory in the category of V-groups

Next, we show that the reflection (4.7) also gives rise to a pretorsion theory in the category V-Grp of V-groups. The torsion-free subcategory is the same as for the torsion theory of Proposition 4.2.2 (i.e. the subcategory V-$\mathsf{Grp_{sep}}$ of separated V-groups), while the torsion subcategory is this time given by the subcategory V-$\mathsf{Grp_{sym}}$ of *symmetric V-groups*, that is, V-groups $(X, a, +)$ which coincide with their dual $(X, a^\circ, +)$ (i.e. such that $a(x, x') = a(x', x)$ for any $x, x' \in X$). Note that, for $V = ([0, \infty], \geq)$ (with $\otimes = +$), the symmetric V-groups coincide with the usual symmetric Lawvere metric groups (which are Lawvere metric groups (X, d) with $d(x, x') = d(x', x)$ for any $x, x' \in X$).

Remark 4.5.1. For this section, we do not need to assume that the quantale V is integral.

Lemma 4.5.2. *The subcategories V-$\mathsf{Grp}_{\mathsf{sym}}$ and V-$\mathsf{Grp}_{\mathsf{sep}}$ are closed in V-Grp under regular epimorphisms and monomorphisms, respectively.*

Lemma 4.5.3. *For any V-group (X,a), the pair $(X,\hat{a})$, where $\hat{a}\colon X \nrightarrow X$ is the V-relation on X defined, for any $x,x' \in X$, by*

$$\hat{a}(x,x') = a(x,x') \wedge a(x',x),$$

is a symmetric V-group.

Proposition 4.5.4. *The pair of full and replete subcategories $(V\text{-}\mathsf{Grp}_{\mathsf{sym}}, V\text{-}\mathsf{Grp}_{\mathsf{sep}})$ of V-Grp is a $\mathscr{Z}$-pretorsion theory in V-Grp, where $\mathscr{Z} = V\text{-}\mathsf{Grp}_{\mathsf{sym}} \cap V\text{-}\mathsf{Grp}_{\mathsf{sep}}$ is given by*

$$\mathscr{Z} = \{(X,a) \in V\text{-}\mathsf{Grp} \mid a(x,x') = a(x',x)\ \forall x,x' \in X \\ \text{and } a(x,x') \geq k \Longrightarrow x = x'\}.$$

Proof. Let us first prove that any arrow

$$f\colon (X,a) \to (Y,b),$$

with $(X,a) \in V$-$\mathsf{Grp}_{\mathsf{sym}}$ and $(Y,b) \in V$-$\mathsf{Grp}_{\mathsf{sep}}$, belongs to $\mathscr{N}$. Consider then, in V-Grp, the (regular epi, mono)-factorization of f:

$$\begin{tikzcd}
(X,a) \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & (Y,b) \\
& (f(X),\bar{a}). \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

By Lemma 4.5.2, we have that $(f(X),\bar{a})$ is in both V-$\mathsf{Grp}_{\mathsf{sym}}$ and V-$\mathsf{Grp}_{\mathsf{sep}}$. Accordingly, f factors in V-Grp through an object of $\mathscr{Z}$, and we have proved the first point (PT1) of Definition 2.2.10.

Consider now any V-group (X,a) and let us show that the sequence

$$(X,\hat{a}) \xrightarrow{1_X} (X,a) \xrightarrow{\eta_X} (X/N_X,\bar{a})$$

is a short $\mathscr{Z}$-exact sequence where, as before, we write $X \xrightarrow{\eta_X} X/N_X$ for the quotient morphism, and where $(X,\hat{a})$ is a symmetric V-group (by Lemma 4.5.3) and $(X/N_X,\bar{a})$ a separated V-group (by Lemma 4.2.1(2)). We begin by showing that η_X is the $\mathscr{Z}$-cokernel of the arrow 1_X. Let us consider a morphism $f\colon (X,a) \to (Y,b)$ in V-Grp such that $f \cdot 1_X \in \mathscr{N}$, i.e. such that $f \cdot 1_X$ factors through an object (Z,c) of $\mathscr{Z}$: $f \cdot 1_X = \alpha \cdot \beta$.

$$\begin{tikzcd}
(X,\hat{a}) \arrow[rr, "1_X"] \arrow[dr, "\beta"'] & & (X,a) \arrow[rr, two heads, "\eta_X"] \arrow[dr, "f"] & & (X/N_X,\bar{a}) \arrow[dl, dotted, "\phi"] \\
& (Z,c) \arrow[rr, "\alpha"'] & & (Y,b) &
\end{tikzcd}$$

Let $x \in N_X$. Then,

$$a(0,x) \geq k \qquad \text{and} \qquad a(x,0) \geq k$$

which implies that

$$k \leq a(0,x) \wedge a(x,0) = \hat{a}(0,x) \leq c(0,\beta(x)),$$

since β is a V-homomorphism, so that

$$c(0,\beta(x)) \geq k.$$

It follows that $\beta(x) = 0$ since $(Z,c) \in \mathscr{Z}$ and then that

$$f(x) = (\alpha \cdot \beta)(x) = 0$$

for any $x \in N_X$. Accordingly, by the universal property of the cokernel η_X, there exists a unique arrow $\phi\colon X/N_X \to Y$ in the category Grp of groups such that $\phi \cdot \eta_X = f$:

$$\begin{tikzcd}
N_X \arrow[r, tail, "\kappa_X"] & X \arrow[rr, two heads, "\eta_X"] \arrow[dr, "f"'] & & X/N_X \arrow[dl, dotted, "\phi"] \\
& & Y. &
\end{tikzcd}$$

Let us show that this group morphism ϕ is in fact a V-functor. Consider $w_1, w_2 \in X/N_X$ and $x_1, x_2 \in X$ such that $\eta_X(x_i) = w_i$ for any $i = 1,2$. Then, we have that

$$a(x_1,x_2) \leq b(f(x_1),f(x_2)) = b((\phi \cdot \eta_X)(x_1),(\phi \cdot \eta_X)(x_2)) = b(\phi(w_1),\phi(w_2))$$

since f is a V-homomorphism, which implies that

$$\bar{a}(w_1,w_2) = \bigvee_{\eta_X(x_i)=w_i} a(x_1,x_2) \leq b(\phi(w_1),\phi(w_2))$$

and this shows that the group morphism ϕ is a V-functor and so a V-homomorphism. As a conclusion, there exists a unique morphism $\phi\colon (X/N_X,\bar{a}) \to (Y,b)$ in V-Grp such that $\phi \cdot \eta_X = f$.

Next, we show that 1_X is the $\mathscr{Z}$-kernel of η_X. Consider $f\colon (Y,b) \to (X,a)$ in V-Grp such that $\eta_X \cdot f \in \mathscr{N}$, i.e. $\eta_X \cdot f$ factors through an object (Z,c) of $\mathscr{Z}$: $\eta_X \cdot f = \alpha \cdot \beta$.

$$\begin{tikzcd}
(X,\hat{a}) \arrow[rr, "1_X"] & & (X,a) \arrow[rr, two heads, "\eta_X"] & & (X/N_X,\bar{a}) \\
& (Y,b) \arrow[ul, dotted, "\phi"] \arrow[ur, "f"'] \arrow[rr, "\beta"'] & & (Z,c) \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

Let us take $\phi = f$, since this is the only possible arrow in Grp such that $1_X \cdot \phi = f$. It remains to show that ϕ is a V-functor, i.e. that, for any $y, y' \in Y$,

$$b(y,y') \leq \hat{a}(\phi(y),\phi(y')),$$

with

$$\hat{a}(\phi(y),\phi(y')) = a(f(y),f(y')) \wedge a(f(y'),f(y)).$$

It is clear that

$$b(y,y') \leq a(f(y),f(y'))$$

for any $y, y' \in Y$ since f is a V-homomorphism. Let us then prove that we also have the following:

$$b(y,y') \leq a(f(y'),f(y)).$$

Let $y, y' \in Y$. We compute that

$$\begin{aligned}
b(y,y') &\leq c(\beta(y),\beta(y')) = c(\beta(y'),\beta(y)) \\
&\leq \bar{a}((\alpha\cdot\beta)(y'),(\alpha\cdot\beta)(y)) \\
&= \bar{a}((\eta_X\cdot f)(y'),(\eta_X\cdot f)(y)) = \bigvee_{\substack{\eta_X(x')=(\eta_X\cdot f)(y') \\ \eta_X(x)=(\eta_X\cdot f)(y)}} a(x',x)
\end{aligned}$$

since $(Z,c) \in \mathscr{Z}$. But, for any $x, x' \in X$ such that $\eta_X(x') = (\eta_X \cdot f)(y')$ and $\eta_X(x) = (\eta_X \cdot f)(y)$, we have that

$$\begin{aligned}
a(f(y'),f(y)) &= a(f(y')-x'+x', f(y)-x+x) \\
&\geq a(f(y')-x',0) \otimes a(0,f(y)-x) \otimes a(x',x) \\
&\geq k \otimes k \otimes a(x',x) = a(x',x)
\end{aligned}$$

since $+\colon (X,a)\otimes(X,a) \to (X,a)$ is a V-functor and since $f(y')-x' \in N_X$ and $f(y)-x \in N_X$. As a consequence, for any $y, y' \in Y$,

$$b(y,y') \leq \bigvee_{\substack{\eta_X(x')=(\eta_X\cdot f)(y') \\ \eta_X(x)=(\eta_X\cdot f)(y)}} a(x',x) \leq a(f(y'),f(y)),$$

and this completes the proof: there exists a unique morphism $\phi\colon (Y, b) \to (X, \hat{a})$ in V-Grp such that $1_X \cdot \phi = f$. $\square$

Example 4.5.5.

(1) If we consider the quantale $2 = (\{\bot, \top\}, \wedge, \top)$, then Proposition 4.5.4 states that ($\mathsf{Grp}(\mathsf{PreOrd})$, $\mathsf{ParOrdGrp}$) forms a pretorsion theory in $\mathsf{PreOrdGrp}$ where the objects of the torsion subcategory are the preordered groups whose preorder is an equivalence relation and the objects of the torsion-free subcategory are the partially ordered groups. This shows that Proposition 4.5.4 is a generalization of Proposition 3.4.1.

(2) Another application of Proposition 4.5.4 is that ($\mathsf{MetGrp}_{\mathsf{sym}}$, $\mathsf{MetGrp}_{\mathsf{sep}}$) is a pretorsion theory in MetGrp. The objects of its torsion subcategory $\mathsf{MetGrp}_{\mathsf{sym}}$ coincide with the usual symmetric Lawvere metric groups, which are Lawvere metric groups (X, d) with $d(x, x') = d(x', x)$ for any $x, x' \in X$.

(3) Similarly to (2), by Proposition 4.5.4, there exists a pretorsion theory in $\mathsf{UMetGrp}$, which is given by ($\mathsf{UMetGrp}_{\mathsf{sym}}, \mathsf{UMetGrp}_{\mathsf{sep}}$).

(4) According to Proposition 4.5.4, ($\mathsf{ProbMetGrp}_{\mathsf{sym}}, \mathsf{ProbMetGrp}_{\mathsf{sep}}$) is a pretorsion theory in $\mathsf{ProbMetGrp}$.

From Propositions 2.2.13 and 4.5.4, we deduce in particular that the subcategory V-$\mathsf{Grp}_{\mathsf{sym}}$ of symmetric V-groups is mono-coreflective in V-Grp:

Corollary 4.5.6. *The subcategory V-$\mathsf{Grp}_{\mathsf{sym}}$ of symmetric V-groups is mono-coreflective in the category V-Grp of V-groups*

$$\begin{tikzcd} V\text{-}\mathsf{Grp} \arrow[r, shift right, "T"'] \arrow[r, phantom, "\perp"] & V\text{-}\mathsf{Grp}_{\mathsf{sym}}, \arrow[l, shift right, hook', "V"'] \end{tikzcd}$$

with its coreflection defined, for any $(X, a) \in V$-Grp, by $T(X, a) = (X, \hat{a})$.

Remark 4.5.7. The coreflectivity of the subcategory V-$\mathsf{Grp}_{\mathsf{sym}}$ of symmetric V-groups has already been observed and proved in [22, Proposition 3.5]. We mention this result here because we view it as a consequence of our pretorsion theory (which has not, to our knowledge, been discovered before) and also because we want to give the generalization of Corollary 3.4.2. Note that Proposition 3.5 in [22] states that V-$\mathsf{Grp}_{\mathsf{sym}}$ is reflective in V-Grp as well.

4.6. A pretorsion theory in the category of V-categories

As a matter of fact, there is also a pretorsion theory in the category V-Cat of V-categories. The torsion-free subcategory is given by the category V-$\mathsf{Cat}_{\mathsf{sep}}$ of separated V-categories, namely of V-categories (X, a) satisfying the separation axiom ($a(x, x') \geq k$ and $a(x', x) \geq k \implies x = x'$) while the torsion subcategory is the category V-$\mathsf{Cat}_{\mathsf{sym}}$ of symmetric V-categories, i.e. of V-categories (X, a) such that $a(x, x') = a(x', x)$ for any $x, x' \in X$.

Note that the content of this section is not part of the article [74] and has not been published.

Proposition 4.6.1. *The pair of full and replete subcategories* (V-$\mathsf{Cat}_{\mathsf{sym}}$,$V$-$\mathsf{Cat}_{\mathsf{sep}}$) *of* V-Cat *is a* $\mathscr{Z}$*-pretorsion theory in* V-Cat*, where* $\mathscr{Z} = V\text{-}\mathsf{Cat}_{\mathsf{sym}} \cap V\text{-}\mathsf{Cat}_{\mathsf{sep}}$ *is given by*

$$\mathscr{Z} = \{(X, a) \in V\text{-}\mathsf{Cat} \mid a(x, x') = a(x', x)\ \forall x, x' \in X \\ \text{and } a(x, x') \geq k \Longrightarrow x = x'\}.$$

Proof. Let $f\colon (X, a) \to (Y, b)$ be a V-functor with $(X, a) \in V$-$\mathsf{Cat}_{\mathsf{sym}}$ and $(Y, b) \in V$-$\mathsf{Cat}_{\mathsf{sep}}$. We consider the V-relation $\hat{b}\colon f(X) \nrightarrow f(X)$ defined, for any $y, y' \in f(X)$, by

$$\hat{b}(y, y') = b(y, y') \wedge b(y', y).$$

Then $(f(X), \hat{b})$ is a V-category. Indeed, for any $y \in f(X)$,

$$\hat{b}(y, y) = b(y, y) \geq k,$$

and, for any $y, y', y'' \in f(X)$, we have that

$$(b(y, y') \wedge b(y', y)) \otimes (b(y', y'') \wedge b(y'', y')) \leq b(y, y') \otimes b(y', y'') \leq b(y, y'')$$

and that

$$\begin{aligned}(b(y,y')\wedge b(y',y))\otimes(b(y',y'')\wedge b(y'',y'))&\leq b(y',y)\otimes b(y'',y')\\&=b(y'',y')\otimes b(y',y)\leq b(y'',y),\end{aligned}$$

so that

$$\begin{aligned}\hat{b}(y,y')\otimes\hat{b}(y',y'')&=(b(y,y')\wedge b(y',y))\otimes(b(y',y'')\wedge b(y'',y'))\\&\leq b(y,y'')\wedge b(y'',y)=\hat{b}(y,y'')\end{aligned}$$

(since $b(y,y'')\wedge b(y'',y)$ is the biggest lower bound of $b(y,y'')$ and $b(y'',y)$). To be more precise, $(f(X),\hat{b})\in\mathscr{Z}$ since $\hat{b}$ is obviously symmetric and because, if $\hat{b}(y,y')\geq k$ for $y,y'\in f(X)$, then

$$b(y,y')\geq b(y,y')\wedge b(y',y)=\hat{b}(y,y')\geq k$$

and

$$b(y',y)\geq b(y,y')\wedge b(y',y)=\hat{b}(y,y')\geq k,$$

which implies that $y=y'$ because $y,y'\in Y$ and $(Y,b)\in V\text{-}\mathsf{Cat}_{\mathsf{sep}}$. We then factor f as follows:

$$\begin{tikzcd}(X,a) \arrow[rr, "f"] \arrow[dr, "e"'] & & (Y,b) \\ & (f(X),\hat{b}) \arrow[ur, "m"'] & \end{tikzcd}$$

where $e\colon X\to f(X)$ is defined, for any $x\in X$, by $e(x)=f(x)$, and $m\colon f(X)\to Y$ is the inclusion of $f(X)$ in Y. Let us check that $e\colon(X,a)\to(f(X),\hat{b})$ and $m\colon(f(X),\hat{b})\to(Y,b)$ are V-functors:

- for any $x,x'\in X$, by symmetry of $a\colon X\nrightarrow X$ and since f is a V-functor, we have that

$$\begin{aligned}a(x,x')&=a(x,x')\wedge a(x',x)\\&\leq b(f(x),f(x'))\wedge b(f(x'),f(x))=\hat{b}(f(x),f(x'))\\&=\hat{b}(e(x),e(x'));\end{aligned}$$

- for any $y, y' \in f(X)$,

$$\hat{b}(y, y') = b(y, y') \wedge b(y', y) \leq b(y, y') = b(m(y), m(y')).$$

Accordingly, f belongs to $\mathscr{N}$.
Consider now $(X, a) \in V\text{-}\mathsf{Cat}$. We endow the set X with the symmetric V-category structure $\hat{a}\colon X \nrightarrow X$ defined, for any $x, x' \in X$, by

$$\hat{a}(x, x') = a(x, x') \wedge a(x', x),$$

so that $(X, \hat{a}) \in V\text{-}\mathsf{Cat}_{\mathsf{sym}}$. Let us next denote $\sim$ for the following equivalence relation defined on the set X: for $x, x' \in X$, $x \sim x'$ if and only if $a(x, x') \geq k$ and $a(x', x) \geq k$. We then consider the quotient set $X/\sim$ as well as the quotient function $\eta_X\colon X \twoheadrightarrow X/\sim$. We equip the set $X/\sim$ with the V-relation $\bar{a} = \eta_X \cdot a \cdot \eta_X^\circ$, i.e., for any $y_1, y_2 \in X/\sim$,

$$\bar{a}(y_1, y_2) = \bigvee_{\eta_X(x_i) = y_i} a(x_1, x_2).$$

We then notice that $a(x_1, x_2) = a(x_1', x_2')$ for any $x_i, x_i' \in X$ such that $x_1 \sim x_1'$ and $x_2 \sim x_2'$. Indeed, by the transitivity axiom for V-categories,

$$\begin{aligned} a(x_1, x_2) &\geq a(x_1, x_1') \otimes a(x_1', x_2') \otimes a(x_2', x_2) \\ &\geq k \otimes a(x_1', x_2') \otimes k = a(x_1', x_2') \end{aligned}$$

and, similarly, $a(x_1', x_2') \geq a(x_1, x_2)$. By idempotence of the operation $\vee$, it follows that

$$\bar{a}(y_1, y_2) = a(x_1, x_2)$$

for some $x_1, x_2 \in X$ such that $\eta_X(x_i) = y_i$ $(i = 1, 2)$. It is then straightforward to see that $(X/\sim, \bar{a}) \in V\text{-}\mathsf{Cat}$. Let us now show that it is a separated V-category: let $y, y' \in X/\sim$ be such that $\bar{a}(y, y') \geq k$ and $\bar{a}(y', y) \geq k$. Then, $a(x, x') \geq k$ and $a(x', x) \geq k$ for some $x, x' \in X$ such that $\eta_X(x) = y$ and $\eta_X(x') = y'$. It follows that $x \sim x'$ and therefore that

$$y = \eta_X(x) = \eta_X(x') = y',$$

which means that $(X/\sim, \bar{a}) \in V\text{-}\mathsf{Cat}_{\mathsf{sep}}$. Now, consider the sequence

$$(X, \hat{a}) \xrightarrow{1_X} (X, a) \xrightarrow{\eta_X} (X/\sim, \bar{a})$$

and let us show that it is a short $\mathscr{Z}$-exact sequence.
We first prove that η_X is the $\mathscr{Z}$-cokernel of the arrow 1_X. Let $f\colon (X,a) \to (Y,b)$ in V-Cat be such that $f \cdot 1_X \in \mathscr{N}$, i.e. such that $f \cdot 1_X$ factors through an object (Z,c) of $\mathscr{Z}$: $f \cdot 1_X = \alpha \cdot \beta$.

$$\begin{tikzcd}
(X,\hat{a}) \arrow[rr, "1_X"] \arrow[dr, "\beta"'] & & (X,a) \arrow[rr, "\eta_X"] \arrow[dr, "f"] & & (X/\sim,\bar{a}) \arrow[dl, dotted, "\phi"] \\
& (Z,c) \arrow[rr, "\alpha"'] & & (Y,b) &
\end{tikzcd}$$

Let $(x,x') \in \sim$. Then $a(x,x') \geq k$ and $a(x',x) \geq k$ which implies that

$$k \leq a(x,x') \wedge a(x',x) = \hat{a}(x,x') \leq c(\beta(x),\beta(x'))$$

so that $\beta(x) = \beta(x')$ since $(Z,c) \in \mathscr{Z}$. We deduce from this equality that $f(x) = f(x')$, meaning that $(x,x') \in \mathsf{Eq}(f)$. Accordingly, $\sim \subset \mathsf{Eq}(f)$, and, by the universal property of quotients in Set, there exists a unique function $\phi\colon X/\sim \to Y$ such that $\phi \cdot \eta_X = f$. Let us check that $\phi\colon (X/\sim,\bar{a}) \to (Y,b)$ is a V-functor: for any $w_1, w_2 \in X/\sim$, there exist $x_1, x_2 \in X$ such that $\eta_X(x_i) = w_i$ $(i = 1,2)$ and such that

$$\begin{aligned}
&\bar{a}(w_1,w_2) = a(x_1,x_2) \\
&\leq b(f(x_1),f(x_2)) = b((\phi \cdot \eta_X)(x_1),(\phi \cdot \eta_X)(x_2)) = b(\phi(w_1),\phi(w_2))
\end{aligned}$$

since f is a V-functor. As a consequence, there exists a unique morphism $\phi\colon (X/\sim,\bar{a}) \to (Y,b)$ in V-Cat such that $\phi \cdot \eta_X = f$.
Next, we show that 1_X is the $\mathscr{Z}$-kernel of η_X. Let $f\colon (Y,b) \to (X,a)$ in V-Cat be such that $\eta_X \cdot f \in \mathscr{N}$, i.e. such that $\eta_X \cdot f$ factors through an object (Z,c) of $\mathscr{Z}$: $\eta_X \cdot f = \alpha \cdot \beta$.

$$\begin{tikzcd}
(X,\hat{a}) \arrow[rr, "1_X"] & & (X,a) \arrow[rr, "\eta_X"] & & (X/\sim,\bar{a}) \\
& (Y,b) \arrow[ul, dotted, "\phi"] \arrow[ur, "f"'] \arrow[rr, "\beta"'] & & (Z,c) \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

Let us take $\phi = f$. It is indeed the only arrow in Set such that $1_X \cdot \phi = f$. We now have to show that $\phi\colon (Y,b) \to (X,\hat{a})$ is a V-functor. It is first of all clear that, for any $y, y' \in Y$,

$$b(y, y') \leq a(f(y), f(y'))$$

since f is a V-functor. Next, as $(Z, c) \in \mathscr{Z}$,

$$\begin{aligned} b(y, y') &\leq c(\beta(y), \beta(y')) = c(\beta(y'), \beta(y)) \\ &\leq \bar{a}((\alpha \cdot \beta)(y'), (\alpha \cdot \beta)(y)) = \bar{a}((\eta_X \cdot f)(y'), (\eta_X \cdot f)(y)) = a(x', x) \end{aligned}$$

for some $x, x' \in X$ such that $\eta_X(x) = (\eta_X \cdot f)(y)$ and $\eta_X(x') = (\eta_X \cdot f)(y')$. These last two equalities imply that $x \sim f(y)$ and that $x' \sim f(y')$. As a consequence, by the transitivity axiom for V-categories,

$$\begin{aligned} a(x', x) &= k \otimes a(x', x) \otimes k \\ &\leq a(f(y'), x') \otimes a(x', x) \otimes a(x, f(y)) \\ &\leq a(f(y'), f(y)) \end{aligned}$$

and then

$$b(y, y') \leq a(x', x) \leq a(f(y'), f(y)).$$

We conclude that, for any $y, y' \in Y$,

$$b(y, y') \leq a(f(y), f(y')) \wedge a(f(y'), f(y)) = \hat{a}(\phi(y), \phi(y')).$$

Therefore, there is a unique arrow $\phi \colon (Y, b) \to (X, \hat{a})$ in V-Cat such that $1_X \cdot \phi = f$. □

According to Propositions 2.2.13 and 4.6.1, we get the following:

Corollary 4.6.2.

- *The subcategory* V-$\mathsf{Cat}_{\mathsf{sep}}$ *of separated* V*-categories is epi-reflective in the category* V-Cat.
- *The subcategory* V-$\mathsf{Cat}_{\mathsf{sym}}$ *of symmetric* V*-categories is mono-coreflective in the category* V-Cat.

Remark 4.6.3. The above corollary is an already known result (see, for instance, the Appendix of [51], or [22] for the second statement). The fact that (V-$\mathsf{Cat}_{\mathsf{sym}}$, V-$\mathsf{Cat}_{\mathsf{sep}}$) is a pretorsion theory in V-Cat is nevertheless a new result, as far as we know.

5 | Central extensions of preordered groups

The aim of this chapter is to generalize the well-known Galois theory existing in the category Grp of groups to the context of preordered groups. Indeed, as explained in Subsection 2.3.3, the abelianization functor $ab\colon \mathsf{Grp} \to \mathsf{Ab}$ to the category of abelian groups induces an admissible Galois structure Γ_{ab}. In this situation, it then turns out that the Γ_{ab}-central and Γ_{ab}-normal extensions both coincide with the *algebraically central extensions*, i.e. with the surjective group homomorphisms $f\colon G \twoheadrightarrow H$ whose kernel $\mathsf{Ker}(f)$ is in the center $Z(G)$ of G [55].

With this in mind, we first consider the functor $C\colon \mathsf{PreOrdGrp} \to \mathsf{PreOrdAb}$ from $\mathsf{PreOrdGrp}$ to its full subcategory $\mathsf{PreOrdAb}$ of *preordered abelian groups*, whose objects are the preordered groups (G, P_G) such that G is an abelian group. For any preordered group (G, P_G), one defines $C(G, P_G) = (ab(G), \eta_G(P_G))$ where, as before, $ab\colon \mathsf{Grp} \to \mathsf{Ab}$ denotes the abelianization functor and where $\eta_G\colon G \twoheadrightarrow G/[G,G]$ is the quotient of G by its derived subgroup $[G,G]$. We prove that C is a reflector which induces an admissible Galois structure Γ_C. By doing so, we observe that the admissibility of Γ_C essentially follows from the *relative modularity* of $\mathsf{PreOrdGrp}$. It is then possible to give a purely algebraic description of the (Γ_C-)normal extensions: a regular epimorphism $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ in $\mathsf{PreOrdGrp}$ is a (Γ_C-)normal extension if and only if $f\colon G \twoheadrightarrow H$ is an algebraically central extension and, for any $a, b, c \in P_G$ such that $\eta_G(a) = \eta_G(b)$ and $f(b) = f(c)$, we have that $a - b + c \in P_G$.

We then observe that the abelianization functor $ab\colon \mathsf{Grp} \to \mathsf{Ab}$ is the functor from Grp to its subcategory of *abelian objects* [6] while $C\colon \mathsf{PreOrdGrp} \to \mathsf{PreOrdAb}$ is the functor from $\mathsf{PreOrdGrp}$ to its subcategory $\mathsf{PreOrdAb}$ of *commutative objects* [6]. It turns out that, in the setting of preordered groups, the notions of abelian and of commutative objects do not coincide (contrary to what happens in Grp). In $\mathsf{PreOrdGrp}$, abelian objects are given by those preordered groups (G, P_G) such that not only G but also P_G are abelian groups. The subcategory $\mathsf{Ab}(\mathsf{PreOrdGrp})$ of abelian objects in $\mathsf{PreOrdGrp}$ is then (isomorphic to) the category $\mathsf{Mono}(\mathsf{Ab})$ of monomorphisms in the category Ab of abelian groups. The fact that abelian and commutative objects do not coincide in $\mathsf{PreOrdGrp}$ allows us to conclude that the category $\mathsf{PreOrdGrp}$ of preordered groups is not strongly unital (which was first observed in [21]), and therefore not subtractive in the sense of [65].

This observation also suggests to consider the functor $F\colon \mathsf{PreOrdGrp} \to \mathsf{Mono}(\mathsf{Ab})$ defined, for any preordered group (G, P_G), by $F(G, P_G) = (ab(G), grp(\eta_G(P_G)))$, where $grp(\eta_G(P_G))$ is the *group completion* (or *Grothendieck group*) of the commutative monoid $\eta_G(P_G)$. Thanks to the results of A. Montoli, D. Rodelo and T. Van der Linden [75] about the group completion functor $grp\colon \mathsf{ComMon} \to \mathsf{Ab}$ (where ComMon denotes the category of commutative monoids), it is then possible to prove that F gives rise to an admissible Galois structure Γ. We show that, in this situation, the (Γ-)central and (Γ-)normal extensions both coincide with the regular epimorphisms $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ in $\mathsf{PreOrdGrp}$ such that $f\colon G \twoheadrightarrow H$ is an algebraically central extension and such that $\bar{f}\colon P_G \twoheadrightarrow P_H$ is a *special homogeneous surjection* [15] of monoids (or, equivalently, a *special Schreier surjection* of monoids).

The content of this chapter comes from the preprint [42], written jointly with M. Gran.

5.1. The reflector $C\colon \mathsf{PreOrdGrp} \to \mathsf{PreOrdAb}$ and its induced admissible Galois structure

Let us denote by $\mathsf{PreOrdAb}$ the full subcategory of $\mathsf{PreOrdGrp}$ whose objects (G, P_G) are *preordered abelian groups*, i.e. such that the group G is abelian.

Proposition 5.1.1. *There is an adjunction*

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathsf{PreOrdAb} \arrow[l, bend left, hook, "V"] \end{tikzcd} \tag{5.1}$$

between the category $\mathsf{PreOrdGrp}$ *of preordered groups and its full subcategory* $\mathsf{PreOrdAb}$ *of preordered abelian groups, where the right adjoint V is the inclusion functor and the left adjoint C is defined, for any* $(G, P_G) \in \mathsf{PreOrdGrp}$*, by* $C(G, P_G) = (G/[G, G], \eta_G(P_G))$

$$\begin{tikzcd} P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail] & \eta_G(P_G) \arrow[d, tail] \\ G \arrow[r, two heads, "\eta_G"'] & G/[G,G] = ab(G), \end{tikzcd}$$

where $\eta_G(P_G)$ is the direct image of P_G along the quotient η_G of G by its derived subgroup $[G, G]$.

Proof. Let (G, P_G) be any preordered group. Then, the (G, P_G)-component of the unit of the adjunction (5.1) is given by the above morphism $(\eta_G, \bar{\eta}_G)$. Indeed, consider any preordered abelian group (A, P_A) and any morphism $(f, \bar{f})\colon (G, P_G) \to (A, P_A)$ in $\mathsf{PreOrdGrp}$:

$$\begin{tikzcd} P_G \arrow[rr, two heads, "\bar{\eta}_G"] \arrow[dr, "\bar{f}"'] \arrow[dd, tail] & & \eta_G(P_G) \arrow[dl, dotted, "\bar{g}"] \arrow[dd, tail, "i_{ab(G)}"] \\ & P_A \arrow[dd, tail, "i_A"] & \\ G \arrow[rr, two heads, "\eta_G"] \arrow[dr, "f"'] & & ab(G). \arrow[dl, dotted, "g"] \\ & A & \end{tikzcd}$$

Then, the universal property of the abelianization $ab(G)$ of G yields a unique arrow $g\colon ab(G) \to A$ such that $g \cdot \eta_G = f$ in the category Grp (see Example 1.1.43). Knowing that $\bar{\eta}_G$ is a strong epimorphism in the category Mon of monoids, there is also a unique monoid morphism $\bar{g}\colon \eta_G(P_G) \to P_A$ making the following diagram commute:

$$\begin{tikzcd} P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, "\bar{f}"'] & \eta_G(P_G) \arrow[d, "g\cdot i_{ab(G)}"] \arrow[dl, dotted, "\bar{g}"] \\ P_A \arrow[r, tail, "i_A"'] & A. \end{tikzcd}$$

Accordingly, there exists a unique morphism $(g, \bar{g})\colon (ab(G), \eta_G(P_G)) \to (A, P_A)$ in PreOrdGrp such that $(g, \bar{g}) \cdot (\eta_G, \bar{\eta}_G) = (f, \bar{f})$. □

Corollary 5.1.2. *If $\mathcal{E}_C$ denotes the class of normal epimorphisms in* PreOrdGrp *and $\mathcal{Z}_C$ the class of normal epimorphisms in* PreOrdAb*, then*

$$\Gamma_C = (\mathsf{PreOrdGrp}, \mathsf{PreOrdAb}, C, V, \mathcal{E}_C, \mathcal{Z}_C) \tag{5.2}$$

is a Galois structure.

We will now prove that this Galois structure is actually admissible. In order to do this, we will use the following result which is a reformulation of a result due to G. Janelidze and G. M. Kelly [58]:

Proposition 5.1.3. *Let $\mathscr{C}$ be a normal category. Let $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ be a Galois structure, where $\mathcal{E}$ is the class of normal epimorphisms in $\mathscr{C}$ and $\mathscr{F}$ a full reflective subcategory of $\mathscr{C}$ closed under subobjects and quotients. If the lattice* Norm$(\mathscr{C})$ *of normal subobjects in $\mathscr{C}$ is modular, then the Galois structure Γ is admissible.*

Proof. Consider any pullback in $\mathscr{C}$ of the following form

$$\begin{tikzcd} P \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"'] & A \arrow[d, two heads, "\phi"] \\ B \arrow[r, two heads, "\eta_B"'] & F(B) \end{tikzcd} \tag{5.3}$$

where η_B is the B-component of the unit of the adjunction $F \dashv U$, A an object of the subcategory $\mathscr{F}$ and ϕ any morphism in the class $\mathcal{E}$. We write $s\colon S \rightarrowtail P$ and $t\colon T \rightarrowtail P$ for the kernels of π_1 and π_2, respectively. We need to prove that this pullback is preserved by the reflector F. In order to do this, we consider the next commutative diagram where $r\colon R \rightarrowtail P$ is the kernel of the P-component η_P of the unit of the adjunction $F \dashv U$, $R \vee S$ is the supremum of R and S in $\mathsf{Norm}(\mathscr{C})$, q is the composite $\eta_B \cdot \pi_1 = F(\pi_1) \cdot \eta_P$, and ψ is the unique morphism induced by the universal property of η_P such that $\psi \cdot \eta_P = \pi_2$:

$$\begin{tikzcd}
R \vee S \arrow[dr, tail] & S \arrow[l, tail] \arrow[d, tail, "s"] & & \\
R \arrow[u, tail] \arrow[r, tail, "r"] \arrow[d, dotted, tail, "m"] & P \arrow[r, two heads, "\eta_P"] \arrow[rr, bend left, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"'] \arrow[dr, two heads, "q"] & F(P) \arrow[r, two heads, "\psi"] \arrow[d, two heads, "F(\pi_1)"] & A \arrow[d, two heads, "\phi"] \\
T \arrow[ur, tail, "t"'] \arrow[uu, dotted, tail, bend left, "n"] & B \arrow[r, two heads, "\eta_B"'] & F(B) \arrow[r, equal] & F(B).
\end{tikzcd} \tag{5.4}$$

We compute that $\pi_2 \cdot r = \psi \cdot \eta_P \cdot r = 0$ so that, by the universal property of kernels, there exists a unique arrow $m\colon R \to T$ such that $t \cdot m = r$, hence $R \leq T$. The assumption that $\mathscr{F}$ is a full reflective subcategory of $\mathscr{C}$ closed under subobjects and quotients implies that the middle square of the above diagram (5.4) is a pushout. As a consequence, the supremum $R \vee S$ exists, and it is the kernel of the morphism q. Accordingly, since $q \cdot t = F(\pi_1) \cdot \eta_P \cdot t = \phi \cdot \psi \cdot \eta_P \cdot t = \phi \cdot \pi_2 \cdot t = 0$, there exists a unique morphism $n\colon T \to R \vee S$ such that $\ker q \cdot n = t$, so that $T \leq R \vee S$ in $\mathsf{Norm}(\mathscr{C})$. Observe then that

$$R = R \vee \{0\} = R \vee (S \wedge T) = (R \vee S) \wedge T = T,$$

where the second equality follows from the fact that π_1 and π_2 are jointly monomorphic (by Proposition 1.1.47), the third equality by modularity of $\mathsf{Norm}(\mathscr{C})$ (since $R \leq T$), and the last one from the fact that $T \leq R \vee S$. This means that $\mathsf{ker}(\eta_P) = \mathsf{ker}(\pi_2)$, and the induced morphism ψ is then an isomorphism. If we apply the reflector F to the pullback (5.3), we then get a pullback in $\mathscr{F}$, and the Galois structure Γ is therefore admissible, as desired. $\square$

We will now apply this Proposition to the adjunction we are interested in. It just remains to prove that the lattice $\mathsf{Norm}(\mathsf{PreOrdGrp})$ of normal subobjects in $\mathsf{PreOrdGrp}$ is modular. In order to do this, we first need to describe the supremum of two normal subobjects in $\mathsf{PreOrdGrp}$.

Lemma 5.1.4. *Let (A, P_A) and (B, P_B) be two normal subobjects in $\mathsf{PreOrdGrp}$ of a preordered group (G, P_G). The supremum of (A, P_A) and (B, P_B) in $\mathsf{Norm}(\mathsf{PreOrdGrp})$ is given by*

$$(A, P_A) \vee (B, P_B) = (A \cdot B, (A \cdot B) \cap P_G)$$

where $A \cdot B = \{a + b \mid a \in A \text{ and } b \in B\}$ is the supremum of the normal subgroups A and B in the category Grp of groups.

Proof. We first note that $(A \cdot B, (A \cdot B) \cap P_G)$ is a normal subobject of (G, P_G). Indeed, $A \cdot B$ is a normal subgroup of G and the square

$$\begin{tikzcd}
(A \cdot B) \cap P_G \arrow[r, tail] \arrow[d, tail] & P_G \arrow[d, tail] \\
A \cdot B \arrow[r, tail] & G
\end{tikzcd}$$

is a pullback in the category Mon of monoids. It is also clear that $(A, P_A) \leq (A \cdot B, (A \cdot B) \cap P_G)$ and that $(B, P_B) \leq (A \cdot B, (A \cdot B) \cap P_G)$ (since $P_A = A \cap P_G$ and $P_B = B \cap P_G$). Consider next that we have another normal subobject (N, P_N) of (G, P_G) such that $(A, P_A) \leq (N, P_N)$ and $(B, P_B) \leq (N, P_N)$. Then of course $A \cdot B \leq N$ since $A \cdot B = A \vee B$ in Grp. As a consequence, $(A \cdot B, (A \cdot B) \cap P_G) \leq (N, P_N)$. $\square$

Proposition 5.1.5. *The lattice $\mathsf{Norm}(\mathsf{PreOrdGrp})$ of normal subobjects in the category $\mathsf{PreOrdGrp}$ of preordered groups is modular: for any triple (A, P_A), (B, P_B) and (C, P_C) of normal subobjects in $\mathsf{PreOrdGrp}$ of a given preordered group (G, P_G), such that $(C, P_C) \leq (A, P_A)$, we have that*

$$(A, P_A) \wedge ((B, P_B) \vee (C, P_C)) = ((A, P_A) \wedge (B, P_B)) \vee (C, P_C).$$

Proof. We already know that $A \wedge (B \vee C) = (A \wedge B) \vee C$ since the lattice $\mathsf{Norm}(\mathsf{Grp})$ of normal subgroups of G is modular. We therefore compute that

$$
\begin{aligned}
(A,P_A)\wedge((B,P_B)\vee(C,P_C)) &= (A,P_A)\wedge(B\cdot C,(B\cdot C)\cap P_G)\\
&= (A\cap(B\cdot C),P_A\cap(B\cdot C)\cap P_G)\\
&= (A\cap(B\cdot C),A\cap P_G\cap(B\cdot C)\cap P_G)\\
&= (A\cap(B\cdot C),(A\cap(B\cdot C))\cap P_G)\\
&= (A\wedge(B\vee C),(A\wedge(B\vee C))\cap P_G)\\
&= ((A\wedge B)\vee C,((A\wedge B)\vee C)\cap P_G)\\
&= (A\cap B)\cdot C,((A\cap B)\cdot C)\cap P_G)\\
&= (A\cap B,P_A\cap P_B)\vee(C,P_C)\\
&= ((A,P_A)\wedge(B,P_B))\vee(C,P_C),
\end{aligned}
$$

where we have used the fact that the infimum of two normal subobjects in $\mathsf{Norm}(\mathsf{PreOrdGrp})$ is given by their pullback. □

Corollary 5.1.6. *The Galois structure* (5.2) *is admissible.*

Proof. This is a direct consequence of Propositions 5.1.3 and 5.1.5, by taking into account the fact that $\mathsf{PreOrdAb}$ is closed under subobjects and quotients in $\mathsf{PreOrdGrp}$. □

We will prove in Theorem 5.1.9 that the Γ_C-normal extensions are given by the normal epimorphisms $(f,\bar{f})\colon (G,P_G) \longrightarrow\!\!\!\!\!\rightarrow (H,P_H)$ such that

(i) $f\colon G \longrightarrow\!\!\!\!\!\rightarrow H$ is an *algebraically central extension*, i.e. $\mathsf{Ker}(f) \subseteq Z(G)$;

(ii) the following Condition ($\star$) holds for $(f,\bar{f})$:

($\star$) for any $(a,b,c) \in \mathsf{Eq}(\bar{\eta}_G) \times_{P_G} \mathsf{Eq}(\bar{f})$, $a-b+c \in P_G$,

where $\bar{\eta}_G\colon P_G \twoheadrightarrow \eta_G(P_G)$ is the restriction of $\eta_G\colon G \twoheadrightarrow G/[G,G]$ to the positive cones.

The following two lemmas will be helpful to establish this characterization.

Lemma 5.1.7. *Consider the following pullback*

$$\begin{tikzcd}
(P, P_P) \arrow[r, two heads, "{(p_2,\bar{p}_2)}"] \arrow[d, two heads, "{(p_1,\bar{p}_1)}"'] & (G, P_G) \arrow[d, two heads, "{(f,\bar{f})}"] \\
(E, P_E) \arrow[r, two heads, "{(p,\bar{p})}"'] & (H, P_H)
\end{tikzcd}$$

in $\mathsf{PreOrdGrp}$*, where all the arrows are regular epimorphisms.*

(1) If Condition $(\star)$ *holds for* $(f, \bar{f})$*, then it holds for* $(p_1, \bar{p}_1)$*.*

(2) If $(p, \bar{p}) = (f, \bar{f})$*, then Condition* $(\star)$ *holds for* $(p_1, \bar{p}_1)$ *if and only if it holds for* $(f, \bar{f})$*.*

Proof.

(1) Let $((e_1, x_1), (e_2, x_2), (e_3, x_3)) \in \mathsf{Eq}(\bar{\eta}_P) \times_{P_P} \mathsf{Eq}(\bar{p}_1)$. Then, this means that

- $e_i \in P_E$ and $x_i \in P_G$ for any $i = 1, 2, 3$;
- $p(e_i) = f(x_i)$ for any $i = 1, 2, 3$;
- $\eta_P(e_1, x_1) = \eta_P(e_2, x_2)$, which implies that

$$(e_1 - e_2, x_1 - x_2) \in \mathsf{Ker}(\eta_P) = [P, P],$$

hence, in particular, $x_1 - x_2 \in [G, G]$;
- $p_1(e_2, x_2) = p_1(e_3, x_3)$, i.e. $e_2 = e_3$.

As a consequence,

- $\eta_G(x_1) = \eta_G(x_2)$, i.e. $(x_1, x_2) \in \mathsf{Eq}(\bar{\eta}_G)$;
- $f(x_2) = p(e_2) = p(e_3) = f(x_3)$, i.e. $(x_2, x_3) \in \mathsf{Eq}(\bar{f})$.

In other words, $(x_1, x_2, x_3) \in \mathsf{Eq}(\bar{\eta}_G) \times_{P_G} \mathsf{Eq}(\bar{f})$. By assumption, the element $x_1 - x_2 + x_3$ then belongs to P_G, and one has that

$$\begin{aligned}(e_1, x_1) - (e_2, x_2) + (e_3, x_3) &= (e_1 - e_2 + e_3, x_1 - x_2 + x_3)\\ &= (e_1, x_1 - x_2 + x_3) \in P_P,\end{aligned}$$

as desired.

(2) Thanks to (1), it suffices to check that, if Condition $(\star)$ holds for $(p_1, \bar{p}_1)$ (with $(p_1, \bar{p}_1)$ the first projection of the kernel pair of $(f, \bar{f})$), then it also holds for $(f, \bar{f})$.

Let $(a,b,c) \in \mathsf{Eq}(\bar{\eta}_G) \times_{P_G} \mathsf{Eq}(\bar{f})$. In particular, this means that $a,b,c \in P_G$, $a-b \in \mathsf{Ker}(\eta_G) = [G,G]$, and $f(b) = f(c)$. The fact that $a-b \in [G,G]$ implies that

$$(a,a)-(b,b) = (a-b,a-b) = ([x_1,x_2]+[x_3,x_4]+\cdots+[x_{n-1},x_n],\\ [x_1,x_2]+[x_3,x_4]+\cdots+[x_{n-1},x_n]),$$

for some suitable elements $x_i \in G$ (with $i \in \{1,\cdots,n\}$). It follows that

$$(a,a)-(b,b) = (x_1,x_1)+(x_2,x_2)-(x_1,x_1)-(x_2,x_2)+\cdots\\ +(x_{n-1},x_{n-1})+(x_n,x_n)-(x_{n-1},x_{n-1})-(x_n,x_n),$$

so that $(a,a)-(b,b) \in [\mathsf{Eq}(f),\mathsf{Eq}(f)] = \mathsf{Ker}(\eta_{\mathsf{Eq}(f)})$. This means that $((a,a),(b,b)) \in \mathsf{Eq}(\bar{\eta}_{\mathsf{Eq}(f)})$. On the other hand, $p_1(b,b) = p_1(b,c)$, and this implies that $((b,b),(b,c)) \in \mathsf{Eq}(\bar{p}_1)$, and then

$$((a,a),(b,b),(b,c)) \in \mathsf{Eq}(\bar{\eta}_{\mathsf{Eq}(f)}) \times_{\mathsf{Eq}(\bar{f})} \mathsf{Eq}(\bar{p}_1).$$

So, by assumption,

$$(a,a-b+c) = (a,a)-(b,b)+(b,c) \in \mathsf{Eq}(\bar{f}),$$

hence, in particular, $a-b+c \in P_G$. □

Lemma 5.1.8. *Condition* $(\star)$ *holds for any morphism* $(f,\bar{f})\colon (G,P_G) \to (H,P_H)$ *in* $\mathsf{PreOrdGrp}$ *where* $(G,P_G) \in \mathsf{PreOrdAb}$.

Proof. Let $(a,b,c) \in \mathsf{Eq}(\bar{\eta}_G) \times_{P_G} \mathsf{Eq}(\bar{f})$. Since $G \in \mathsf{Ab}$, we have that $\eta_G = 1_G$, and then that $a = \eta_G(a) = \eta_G(b) = b$. It follows that $a-b+c = c \in P_G$. □

Theorem 5.1.9. *Let* $(f,\bar{f})\colon (G,P_G) \twoheadrightarrow (H,P_H)$ *be a regular epimorphism in* $\mathsf{PreOrdGrp}$. *The following conditions are equivalent:*

(1) *(i)* $\mathsf{Ker}(f) \subseteq Z(G)$;

(ii) *for any* $(a,b,c) \in \mathsf{Eq}(\bar{\eta}_G) \times_{P_G} \mathsf{Eq}(\bar{f})$, *we have* $a-b+c \in P_G$.

(2) $(f,\bar{f})$ *is a* $(\Gamma_C$-$)$*normal extension.*

Proof.

- (1) $\Rightarrow$ (2): We need to prove that the first projection $(\pi_1, \bar{\pi}_1)$ of the kernel pair of $(f, \bar{f})$ is a $(\Gamma_C$-)trivial extension, i.e. that the square below

$$\begin{tikzcd}
\mathsf{Eq}(f,\bar{f}) \arrow[r, two heads, "(\eta_{\mathsf{Eq}(f)},\bar{\eta}_{\mathsf{Eq}(f)})"] \arrow[d, two heads, "(\pi_1,\bar{\pi}_1)"'] & C(\mathsf{Eq}(f,\bar{f})) \arrow[d, two heads, "C(\pi_1,\bar{\pi}_1)=(C(\pi_1),C(\bar{\pi}_1))"] \\
(G,P_G) \arrow[r, two heads, "(\eta_G,\bar{\eta}_G)"'] & C(G,P_G)
\end{tikzcd} \tag{5.5}$$

is a pullback in the category $\mathsf{PreOrdGrp}$ of preordered groups. First of all, it is well-known that (i) implies that its restriction to the category Grp of groups is a pullback (in Grp) [55]. It remains to show that the external square in the diagram

$$\begin{tikzcd}
\mathsf{Eq}(\bar{f}) \arrow[rr, two heads, "\bar{\eta}_{\mathsf{Eq}(f)}"] \arrow[dd, two heads, "\bar{\pi}_1"'] \arrow[dr, tail, dotted, "\bar{\phi}"'] & & \eta_{\mathsf{Eq}(f)}(\mathsf{Eq}(\bar{f})) \arrow[dd, two heads, "C(\bar{\pi}_1)"] \\
& P_P \arrow[ur, two heads, "\bar{q}_2"'] \arrow[dl, two heads, "\bar{q}_1"] & \\
P_G \arrow[rr, two heads, "\bar{\eta}_G"'] & & \eta_G(P_G)
\end{tikzcd}$$

is a pullback in Mon. Consider then the pullback (P, P_P) of $(\eta_G, \bar{\eta}_G)$ and $C(\pi_1, \bar{\pi}_1)$ (denoted with a slight abuse of notation by $(C(\pi_1), C(\bar{\pi}_1))$ in $\mathsf{PreOrdGrp}$ (with the two projections $(q_1, \bar{q}_1)$ and $(q_2, \bar{q}_2)$), as well as the induced morphism $(\phi, \bar{\phi})\colon (\mathsf{Eq}(f), \mathsf{Eq}(\bar{f})) \to (P, P_P)$ to the pullback (with ϕ an isomorphism, as already observed). Since ϕ is an isomorphism, obviously $\bar{\phi}$ is a monomorphism by commutativity of the following square:

$$\begin{tikzcd}
\mathsf{Eq}(\bar{f}) \arrow[r, tail, "\bar{\phi}"] \arrow[d, tail] & P_P \arrow[d, tail] \\
\mathsf{Eq}(f) \arrow[r, "\cong", "\phi"'] & P.
\end{tikzcd}$$

So it suffices to show that $\bar{\phi}$ is surjective. Let $(a, \eta_{\mathsf{Eq}(f)}(b,c)) \in P_P$. In other words, $a, b, c \in P_G$ are such that $f(b) = f(c)$ and $\eta_G(a) = C(\bar{\pi}_1)(\eta_{\mathsf{Eq}(f)}(b,c)) = (\eta_G \cdot \pi_1)(b,c) = \eta_G(b)$, that is, $(a,b,c) \in \mathsf{Eq}(\bar{\eta}_G) \times_{P_G} \mathsf{Eq}(\bar{f})$. By (ii), we then have that $(a, a-b+c) \in \mathsf{Eq}(\bar{f})$. We are going to show that this element is sent by ϕ to $(a, \eta_{\mathsf{Eq}(f)}(b,c))$. We first observe that, since $(a,b) \in \mathsf{Eq}(\bar{\eta}_G)$, there exist $x_i \in G$ (for $i \in \{1, \cdots, n\}$) such that

$$a - b = [x_1, x_2] + [x_3, x_4] + \cdots + [x_{n-1}, x_n].$$

This implies that

$$\begin{aligned}(a-b, a-b) = {} & (x_1,x_1) + (x_2,x_2) - (x_1,x_1) - (x_2,x_2) + \cdots \\ & + (x_{n-1},x_{n-1}) + (x_n,x_n) - (x_{n-1},x_{n-1}) - (x_n,x_n),\end{aligned}$$

i.e. $(a-b, a-b) \in [\mathsf{Eq}(f), \mathsf{Eq}(f)] = \mathsf{Ker}(\eta_{\mathsf{Eq}(f)})$. As a consequence, we can compute

$$\begin{aligned}\phi(a, a-b+c) &= (a, \eta_{\mathsf{Eq}(f)}(a, a-b+c)) \\ &= (a, \eta_{\mathsf{Eq}(f)}(a-b, a-b) + \eta_{\mathsf{Eq}(f)}(b,c)) \\ &= (a, \eta_{\mathsf{Eq}(f)}(b,c)),\end{aligned}$$

and deduce that the homomorphism $\bar{\phi}$ is then surjective, hence an isomorphism. This proves that the square (5.5) is a pullback in $\mathsf{PreOrdGrp}$, i.e. that $(f, \bar{f})$ is a (Γ_C-)normal extension.

- (2) $\Rightarrow$ (1): Since $(f, \bar{f})$ is a (Γ_C-)normal extension, the two squares below are then pullbacks in $\mathsf{PreOrdGrp}$:

$$\begin{tikzcd}(\mathsf{Eq}(f), \mathsf{Eq}(\bar{f})) \arrow[r, two heads, "(\pi_2, \bar{\pi}_2)"] \arrow[d, two heads, "(\pi_1, \bar{\pi}_1)"'] & (G, P_G) \arrow[d, two heads, "(f, \bar{f})"] \\ (G, P_G) \arrow[r, two heads, "(f, \bar{f})"'] & (H, P_H)\end{tikzcd}$$

$$\begin{tikzcd}(\mathsf{Eq}(f), \mathsf{Eq}(\bar{f})) \arrow[r, two heads, "(\eta_{\mathsf{Eq}(f)}, \bar{\eta}_{\mathsf{Eq}(f)})"] \arrow[d, two heads, "(\pi_1, \bar{\pi}_1)"'] & (ab(\mathsf{Eq}(f)), \eta_{\mathsf{Eq}(f)}(\mathsf{Eq}(\bar{f}))) \arrow[d, two heads, "C(\pi_1, \bar{\pi}_1)"] \\ (G, P_G) \arrow[r, two heads, "(\eta_G, \bar{\eta}_G)"'] & (ab(G), \eta_G(P_G)).\end{tikzcd}$$

In this second diagram, $C(\pi_1, \bar{\pi}_1) \in \mathsf{PreOrdAb}$. So, thanks to Lemma 5.1.8, we know that $C(\pi_1, \bar{\pi}_1)$ satisfies Condition ($\star$). Now, using Lemma 5.1.7(1) with the second pullback, we get that Condition ($\star$) holds for $(\pi_1, \bar{\pi}_1)$. The application of Lemma 5.1.7(2) to the first pullback next gives the validity of Condition ($\star$) for $(f, \bar{f})$, which corresponds to (ii). Condition (i) follows directly from the fact that $f: G \longrightarrow\!\!\!\!\!\rightarrow H$ is a normal extension with respect to the admissible Galois structure induced by the abelianization functor $ab: \mathsf{Grp} \to \mathsf{Ab}$ [55, 58] (see Proposition 2.3.16). $\square$

5.2. Commutative and abelian objects in the category of preordered groups

This section is devoted to the characterization of the commutative and the abelian objects in the category of preordered groups. For the reader's convenience, we first recall some definitions and results (we refer to [6] for more details).

Definition 5.2.1. Let $\mathscr{C}$ be a unital category. An object $X \in \mathscr{C}$ is said to be *commutative* when there exists a morphism $\phi: X \times X \to X$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "l_X"] \arrow[dr, equal] & X \times X \arrow[d, dotted, "\phi"] & X \arrow[l, "r_X"'] \arrow[dl, equal] \\ & X. & \end{tikzcd}$$

Note that the morphism ϕ above is necessarily unique by unitality of the category $\mathscr{C}$. It turns out that any commutative object in a unital category is an (internal) commutative monoid.

Proposition 5.2.2. [6] *Let $\mathscr{C}$ be a unital category. The category* $\mathsf{ComMon}(\mathscr{C})$ *of* internal commutative monoids *in $\mathscr{C}$ is the full subcategory of commutative objects in $\mathscr{C}$.*

Definition 5.2.3. Let $\mathscr{C}$ be a unital category. An object $X \in \mathscr{C}$ is said to be *abelian* when it is commutative and the corresponding internal commutative monoid is an internal abelian group.

The following Proposition gives a useful characterization of abelian objects in a unital category:

Proposition 5.2.4. [6] *Let $\mathscr{C}$ be a unital category. Then, an object X of $\mathscr{C}$ is abelian if and only if there exists a morphism $\phi\colon X \times X \to X$ making the diagram*

$$\begin{tikzcd} X \arrow[r, "r_X"] \arrow[dr, equal] & X \times X \arrow[d, dotted, "\phi"] & X \arrow[l, "\Delta_X"'] \arrow[dl, "0"] \\ & X & \end{tikzcd}$$

commute, where $\Delta_X = \langle 1_X, 1_X \rangle$ is the diagonal of X.

Any abelian object is by definition commutative, but the converse is not true in general. In the strongly unital context though, the converse holds:

Proposition 5.2.5. [6] *In a strongly unital category, any commutative object is an abelian object.*

As proved in [21] (see proof of Proposition 2.1.17), the category $\mathsf{PreOrdGrp}$ of preordered groups is a unital category, so that the notions of commutative and abelian objects both make sense in this setting. It turns out that the commutative objects in $\mathsf{PreOrdGrp}$ are precisely the preordered abelian groups:

Proposition 5.2.6. *The category $\mathsf{PreOrdAb}$ of preordered abelian groups coincides with the category $\mathsf{ComMon}(\mathsf{PreOrdGrp})$ of internal commutative monoids in $\mathsf{PreOrdGrp}$:*

$$\mathsf{PreOrdAb} = \mathsf{ComMon}(\mathsf{PreOrdGrp}).$$

Proof. Let (G, P_G) be a commutative object in $\mathsf{PreOrdGrp}$. Then, this means that there exists a morphism $(\phi, \bar{\phi})\colon (G, P_G) \times (G, P_G) \to (G, P_G)$ in $\mathsf{PreOrdGrp}$ such that the diagram

$$\begin{tikzcd} (G, P_G) \arrow[r, "(l_G, \bar{l}_G)"] \arrow[dr, equal] & (G \times G, P_G \times P_G) \arrow[d, dotted, "(\phi, \bar{\phi})"] & (G, P_G) \arrow[l, "(r_G, \bar{r}_G)"'] \arrow[dl, equal] \\ & (G, P_G) & \end{tikzcd}$$

commutes. In particular, for any $x \in G$ and any $y \in G$,

$$\begin{aligned} x + y &= (\phi \cdot l_G)(x) + (\phi \cdot r_G)(y) = \phi(x, 0) + \phi(0, y) \\ &= \phi((x, 0) + (0, y)) \\ &= \phi(x, y) \\ &= \phi((0, y) + (x, 0)) \\ &= \phi(0, y) + \phi(x, 0) = (\phi \cdot r_G)(y) + (\phi \cdot l_G)(x) \\ &= y + x, \end{aligned}$$

which means that G is an abelian group and then that $(G, P_G) \in \mathsf{PreOrdAb}$.
Conversely, consider $(G, P_G) \in \mathsf{PreOrdAb}$ and define, for any $(x, y) \in G \times G$, the requested morphism $\phi\colon G \times G \to G$ as follows: $\phi(x, y) = x + y$. Then, it is easy to check that ϕ is a group morphism and that $\phi \cdot l_G = 1_G$ and $\phi \cdot r_G = 1_G$. Moreover, it is clear that the restriction $\bar{\phi}$ of ϕ to $P_G \times P_G$ takes its values in P_G, since P_G is a submonoid of G. As a consequence, $(\phi, \bar{\phi})\colon (G, P_G) \times (G, P_G) \to (G, P_G)$ is a morphism in $\mathsf{PreOrdGrp}$ making the above diagram commute in $\mathsf{PreOrdGrp}$. This means that (G, P_G) is a commutative object in $\mathsf{PreOrdGrp}$. The result then follows from Proposition 5.2.2. □

Proposition 5.2.7. *The abelian objects in* $\mathsf{PreOrdGrp}$ *are the preordered groups* (G, P_G) *where* G *is an abelian group and* P_G *a (normal) subgroup of* G*.*

Proof. Let (G, P_G) be an abelian object in $\mathsf{PreOrdGrp}$. In particular, it is a commutative object, so that $G \in \mathsf{Ab}$ (by Proposition 5.2.6). Thanks to Proposition 5.2.4, we also have that there exists a morphism $(\phi, \bar{\phi})\colon (G, P_G) \times (G, P_G) \to (G, P_G)$ making the following diagram commute:

$$\begin{tikzcd} (G, P_G) \arrow[r, "(r_G, \bar{r}_G)"] \arrow[dr, equal] & (G \times G, P_G \times P_G) \arrow[d, dotted, "(\phi, \bar{\phi})"] & (G, P_G) \arrow[l, "(\Delta_G, \bar{\Delta}_G)"'] \arrow[dl, "(0,0)"] \\ & (G, P_G). & \end{tikzcd}$$

The restriction $\bar{\phi}$ of ϕ to $P_G \times P_G$ takes its values in P_G. Accordingly, for any $x \in P_G$, $\phi(x, 0) \in P_G$. Now we compute that

$$\begin{aligned}\phi(x,0) &= \phi((0,-x)+(x,x)) = \phi(0,-x)+\phi(x,x)\\ &= (\phi\cdot r_G)(-x)+(\phi\cdot\Delta_G)(x) = -x+0=-x.\end{aligned}$$

As a consequence, $-x \in P_G$ for any $x \in P_G$, which proves that P_G is a subgroup of G, as desired.
Conversely, consider any preordered abelian group (G, P_G) where the positive cone P_G is a subgroup of G. Define the morphism $\phi \colon G \times G \to G$ by $\phi(x, y) = -x + y$, for any $(x, y) \in G \times G$. It is easily seen that ϕ is a group morphism since the group G is abelian. Moreover, for any $x \in G$,

- $(\phi \cdot r_G)(x) = \phi(0, x) = -0 + x = x$, so that $\phi \cdot r_G = 1_G$;
- $(\phi \cdot \Delta_G)(x) = \phi(x, x) = -x + x = 0$, so that $\phi \cdot \Delta_G = 0$.

Let us now prove that the restriction $\bar{\phi}$ of ϕ to $P_G \times P_G$ takes its values in P_G. Let $(x, y) \in P_G \times P_G$. Since P_G is a subgroup of G, then $-x \in P_G$ and then $-x + y = \phi(x, y) \in P_G$. Accordingly, the pair $(\phi, \bar{\phi}) \colon (G, P_G) \times (G, P_G) \to (G, P_G)$ is a morphism in $\mathsf{PreOrdGrp}$ making the diagram above commute. By Proposition 5.2.4, we conclude that (G, P_G) is an abelian object in $\mathsf{PreOrdGrp}$. □

Corollary 5.2.8. *The category* $\mathsf{Ab}(\mathsf{PreOrdGrp})$ *of internal abelian groups in* $\mathsf{PreOrdGrp}$ *is isomorphic to the category* $\mathsf{Mono}(\mathsf{Ab})$ *of monomorphisms in the category* Ab *of abelian groups:*

$$\mathsf{Ab}(\mathsf{PreOrdGrp}) = \mathsf{Mono}(\mathsf{Ab}).$$

Proof. This is a direct consequence of Proposition 5.2.7 and Definition 5.2.3. □

Another consequence of Proposition 5.2.7 is the following remark, that was first observed in [21]:

Remark 5.2.9. The category $\mathsf{PreOrdGrp}$ of preordered groups is not strongly unital.

Proof. This follows from Propositions 5.2.6, 5.2.7, and 5.2.5. For instance, the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ of the monoid $\mathbb{N}$ of non-negative integers in the abelian group $\mathbb{Z}$ of integers is an example of a commutative object in $\mathsf{PreOrdGrp}$ that is not an abelian object. $\square$

Corollary 5.2.10. *The category* $\mathsf{PreOrdGrp}$ *is not* subtractive *(in the sense of* [65]*).*

Proof. This follows from the Remark 5.2.9 and the well-known fact that

$$\text{strongly unital} = \text{unital} + \text{subtractive}$$

(see [43], for instance). $\square$

We will now consider the functor F from the category $\mathsf{PreOrdGrp}$ of preordered groups to its subcategory $\mathsf{Mono(Ab)}$ of abelian objects. We will see it as the composite of the following two functors

$$\mathsf{PreOrdGrp} \xrightarrow{C} \mathsf{PreOrdAb} \xrightarrow{A} \mathsf{Mono(Ab)}$$

where C is defined as in Section 5.1 and A is defined, for any preordered abelian group (G, P_G), by

$$A(G, P_G) = (G, grp(P_G))$$

with $grp(P_G)$ the *group completion* of the monoid P_G. As a consequence, for any $(G, P_G) \in \mathsf{PreOrdGrp}$,

$$F(G, P_G) = (ab(G), grp(\eta_G(P_G))).$$

$$\begin{tikzcd}
P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail] \arrow[rr, bend left, "\hat{\eta}_G"] & \eta_G(P_G) \arrow[r, dotted, "j_G"] \arrow[d, tail] & grp(\eta_G(P_G)) \arrow[d, tail, dotted, "i_{ab(G)}"] \\
G \arrow[r, two heads, "\eta_G"'] \arrow[rr, bend right, two heads, "\eta_G"'] & ab(G) \arrow[r, equal] & ab(G)
\end{tikzcd} \tag{5.6}$$

Let us recall the general construction of the *group completion* (also called *Grothendieck group*) of an additive commutative monoid M. On the cartesian product $M \times M$, one defines an equivalence relation $\sim$ in the following way: $(m_1, m_2) \sim (n_1, n_2)$ if and only if there exists an element k in M such that $m_1 + n_2 + k = m_2 + n_1 + k$. The Grothendieck group $grp(M)$ of M is then given by the quotient $(M \times M)/\sim$, which turns out to be an abelian group. Note that there is a monoid homomorphism $j \colon M \to grp(M)$ sending any element m of M to the equivalence class $[(m, 0)]$ (with respect to $\sim$). This homomorphism satisfies a universal property: for any monoid homomorphism $\phi \colon M \to X$ from M to an abelian group X, there is a unique group homomorphism $\psi \colon grp(M) \to X$ such that $\phi = \psi \cdot j$.

This universal property yields the existence of the unique morphism $i_{ab(G)}$ making the diagram (5.6) commute. The following lemma implies that $grp(\eta_G(P_G))$ is in addition a submonoid of $ab(G)$, so that $(ab(G), grp(\eta_G(P_G)))$ is then indeed a preordered group.

Lemma 5.2.11. *Let M be a submonoid of an abelian group X and*

$$Y = \{x \in X \mid x = a - b, \text{ for } a, b \in M\},$$

which is a (normal) subgroup of X. Then, there is a group isomorphism

$$grp(M) \cong Y$$

which implies that $grp(M)$ is a (normal) subgroup of X.

Proof. By definition, $grp(M) = (M \times M)/\sim$. Now, we observe that, for $(m_1, m_2), (n_1, n_2) \in M \times M$, $(m_1, m_2) \sim (n_1, n_2)$ if and only if there exists a k in M such that $m_1 + n_2 + k = m_2 + n_1 + k$. We can see this equality in the group X so that, by the cancellation property, $(m_1, m_2) \sim (n_1, n_2)$ if and only if $m_1 + n_2 = m_2 + n_1$, which is equivalent to $m_1 - m_2 = n_1 - n_2$. Accordingly, the group homomorphism $\Phi \colon grp(M) \to Y$ defined, for any $[(m_1, m_2)]$ in $grp(M)$, by $\Phi([(m_1, m_2)]) = m_1 - m_2$, is an isomorphism. $\square$

Remark 5.2.12. The above construction is valid for commutative monoids but is different in the general case.

5.3. The Galois theory corresponding to the group completion functor

If we take a look at the restriction of the functor A to the positive cones (i.e. to the category ComMon of commutative monoids), we then get the *group completion functor grp* studied in [75]. In this section, we recall the results of this article that will be useful for our work.

In their paper [75], A. Montoli, D. Rodelo and T. Van der Linden study the adjunction

$$\begin{tikzcd} \mathsf{Mon} \arrow[r, bend left, "grp"] \arrow[r, phantom, "\perp"] & \mathsf{Grp} \arrow[l, bend left, "mon"] \end{tikzcd}$$

between the categories Mon of monoids and Grp of groups, where the right adjoint mon is the forgetful functor while the left adjoint grp is the *group completion functor*.

A significant part of the article [75] is devoted to the proof that the Galois structure

$$\Gamma_{grp} = (\mathsf{Mon}, \mathsf{Grp}, grp, mon, \mathcal{E}_{grp}, \mathcal{Z}_{grp})$$

is admissible when $\mathcal{E}_{grp}$ and $\mathcal{Z}_{grp}$ are the classes of surjective homomorphisms in Mon and Grp, respectively. At the end of the paper, a characterization of (Γ_{grp}-)normal and (Γ_{grp}-)central extensions is given: both notions coincide with the one of *special homogeneous surjection* that we are now going to recall.

Definition 5.3.1. [15] Let $f\colon X \to Y$ be a split epimorphism in Mon, with section s and kernel k:

$$K \overset{k}{\rightarrowtail} X \underset{s}{\overset{f}{\rightleftarrows}} Y. \tag{5.7}$$

The split epimorphism (f, s) is *homogeneous* when, for any $y \in Y$, the functions $\mu_y\colon K \to f^{-1}(y)$ and $\nu_y\colon K \to f^{-1}(y)$, defined, for any $x \in K$, by $\mu_y(x) = x + s(y)$ and $\nu_y(x) = s(y) + x$, are bijective.

Definition 5.3.2. [15] Let $f\colon X \twoheadrightarrow Y$ be a surjective homomorphism in the category Mon of monoids. Consider its kernel pair $(\mathsf{Eq}(f), \pi_1, \pi_2)$ with the diagonal Δ:

$$\begin{tikzcd} \mathsf{Eq}(f) \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & X \arrow[l, "\Delta" description] \arrow[r, two heads, "f"] & Y. \end{tikzcd}$$

The morphism f is said to be a *special homogeneous surjection* when (π_1, Δ) is a homogeneous split epimorphism.

Special homogeneous surjections are pullback-stable and, moreover, one has the following property:

Proposition 5.3.3. [15] *Consider in* Mon *the following pullback where both f and f' are surjective homomorphisms:*

$$\begin{tikzcd} P \arrow[r, two heads, "f'"] \arrow[d, two heads, "g'"'] & Z \arrow[d, two heads, "g"] \\ X \arrow[r, two heads, "f"'] & Y. \end{tikzcd}$$

If g' is a special homogeneous surjection, then so is g.

We now recall in Theorem 5.3.5 the characterization of (Γ_{grp}-)normal and (Γ_{grp}-)central extensions considered in [75].

Proposition 5.3.4. [75] *Consider an arbitrary split epimorphism (f, s) as in (5.7). Then the following conditions are equivalent:*

(1) f is a (Γ_{grp}-)trivial extension.

(2) f is a special homogeneous surjection.

Theorem 5.3.5. [75] *Let $f\colon X \twoheadrightarrow Y$ be a surjective homomorphism of monoids. Then the following conditions are equivalent:*

(1) f is a special homogeneous surjection.

(2) f is a (Γ_{grp}-)normal extension.

(3) f is a (Γ_{grp}-)central extension.

5.4. The functor $F\colon \mathsf{PreOrdGrp} \to \mathsf{Mono}(\mathsf{Ab})$ and its induced admissible Galois structure

Proposition 5.4.1. *There is an adjunction*

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Mono}(\mathsf{Ab}) \arrow[l, bend left, "U"] \end{tikzcd} \tag{5.8}$$

between the category $\mathsf{PreOrdGrp}$ *of preordered groups and its full subcategory* $\mathsf{Mono}(\mathsf{Ab})$ *of abelian objects, where the right adjoint U is the inclusion functor and the left adjoint F is defined as in Section 5.2: for any* $(G, P_G) \in \mathsf{PreOrdGrp}$*,*

$$F(G, P_G) = (ab(G) = G/[G, G], grp(\eta_G(P_G))).$$

Proof. The adjunction (5.8) can be seen as the composite of the two adjunctions

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathsf{PreOrdAb} \arrow[l, bend left, "V"] \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathsf{Mono}(\mathsf{Ab}) \arrow[l, bend left, "W"] \end{tikzcd}$$

where the left-hand one has been studied in Section 5.1 and (the restriction to the positive cones of) the right-hand one has been considered in Section 5.3. Note that the (G, P_G)-component of the unit of the composite adjunction is given by the morphism $(\eta_G, \hat{\eta}_G)$ (in the notations of diagram (5.6)). $\square$

Remark 5.4.2. For a given preordered group (G, P_G), the (G, P_G)-component $(\eta_G, \hat{\eta}_G)$ of the unit of the adjunction (5.8) is not a regular epimorphism, in general. The morphism $(\eta_G, \hat{\eta}_G)$ is just an epimorphism, since η_G is surjective, while $\hat{\eta}_G$ is not necessarily surjective. Note also that the morphism $j_G\colon \eta_G(P_G) \to grp(\eta_G(P_G))$ is a monomorphism because $\eta_G(P_G)$ is a commutative monoid with cancellation.

Proposition 5.4.3. *If $\mathcal{E}$ denotes the class of regular epimorphisms in* $\mathsf{PreOrdGrp}$ *and $\mathcal{Z}$ the class of regular epimorphisms in* $\mathsf{Mono}(\mathsf{Ab})$*, then*

$$\Gamma = (\mathsf{PreOrdGrp}, \mathsf{Mono}(\mathsf{Ab}), F, U, \mathcal{E}, \mathcal{Z}) \tag{5.9}$$

is a Galois structure.

Since the Galois structure (5.9) is a composite of two "compatible" Galois structures that are admissible, we may use the following known result in order to prove its admissibility (the argument given to prove Lemma 6.2 in [24], for instance, still holds in our situation).

Proposition 5.4.4. *Consider the following chain of adjunctions*

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "J"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "L"] \arrow[r, bend left, "I"] \arrow[r, phantom, "\perp"] & \mathscr{A} \arrow[l, bend left, "H"] \end{tikzcd} \tag{5.10}$$

where $\mathscr{A}$ is a full subcategory of $\mathscr{B}$ and $\mathscr{B}$ a full subcategory of $\mathscr{C}$. Assume moreover that

- $\Gamma_J = (\mathscr{C}, \mathscr{B}, J, L, \mathcal{E}_J, \mathcal{Z}_J)$;
- $\Gamma_I = (\mathscr{B}, \mathscr{A}, I, H, \mathcal{E}_I, \mathcal{Z}_I)$

are admissible Galois structures that are "compatible" in the sense that $J(\mathcal{E}_J) \subset \mathcal{E}_I$ and $H(\mathcal{Z}_I) \subset \mathcal{Z}_J$. Then the composite of these two admissible Galois structures

$$\Gamma = (\mathscr{C}, \mathscr{A}, I \cdot J, L \cdot H, \mathcal{E}, \mathcal{Z}),$$

where $\mathcal{E} = \mathcal{E}_J$ and $\mathcal{Z} = \mathcal{Z}_I$, is an admissible Galois structure.

Corollary 5.4.5. *The Galois structure* (5.9) *is admissible.*

5.5. Characterization of Γ-normal and Γ-central extensions

This section is devoted to Γ-normal and Γ-central extensions which will be characterized in Theorem 5.5.3. The following two lemmas will be needed for the proof of this result.

Lemma 5.5.1. *Consider a chain of adjunctions as in* (5.10) *and assume that we have admissible Galois structures Γ_J and Γ_I as in Proposition 5.4.4. Assume, moreover, that any component of the unit of the adjunction $J \dashv L$ is a descent morphism. Then, for an extension $f\colon X \to Y$ in $\mathscr{C}$, the following conditions are equivalent:*

(1) f is a Γ-trivial extension for the admissible Galois structure Γ described in Proposition 5.4.4.

(2) f is a Γ_J-trivial extension and $J(f)$ is a Γ_I-trivial extension.

Proof.

- (1) $\Rightarrow$ (2): By assumption, the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & (I \cdot J)(X) \arrow[d, "(I \cdot J)(f)"] \\ Y \arrow[r, "\eta_Y"'] & (I \cdot J)(Y) \end{tikzcd}$$

is a pullback in $\mathscr{C}$. Note that this pullback decomposes into the following two squares:

$$\begin{tikzcd} X \arrow[r, "\eta^J_X"] \arrow[d, "f"'] & J(X) \arrow[r, "\eta^I_{J(X)}"] \arrow[d, "J(f)"] & (I \cdot J)(X) \arrow[d, "(I \cdot J)(f)"] \\ Y \arrow[r, "\eta^J_Y"'] & J(Y) \arrow[r, "\eta^I_{J(Y)}"'] & (I \cdot J)(Y) \end{tikzcd} \tag{5.11}$$

(where η^J and η^I are the units of the adjunctions $J \dashv L$ and $I \dashv H$, respectively). Equivalently, the fact that f is a Γ-trivial extension can also be formulated by saying that

$$f = (L \cdot H)^Y(g)$$

for some extension $g\colon A \to (I \cdot J)(Y)$ in $\mathscr{A}$. Now,

$$(L \cdot H)^Y(g) = L^Y(H^{J(Y)}(g)),$$

so that $f = L^Y(\bar{g})$, for an extension $\bar{g} = H^{J(Y)}(g)$ in $\mathscr{B}$ with codomain $J(Y)$. This precisely means that f is a Γ_J-trivial extension, hence in the diagram (5.11) both the left-hand square and the external rectangle are pullbacks. Since η^J_Y is a descent morphism, it then follows, thanks to Proposition 1.3.11, that also the right-hand square is a pullback, meaning that $J(f)$ is a Γ_I-trivial extension.

- (2) $\Rightarrow$ (1): Statement (2) means that both squares in diagram (5.11) are pullbacks, so that the external rectangle is also a pullback (by Proposition 1.1.44(1)), and f is a Γ-trivial extension. $\square$

Lemma 5.5.2. *Let* $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ *be a regular epimorphism in* $\mathsf{PreOrdGrp}$. *Then, the following conditions are equivalent:*

(1) for any $(x, y) \in \mathsf{Eq}(\bar{f})$, $y - x \in P_G$ *and* $-x + y \in P_G$*;*

(2) $\bar{f}$ *is a special homogeneous surjection in* Mon*;*

(3) for any $(x, y) \in \mathsf{Eq}(\bar{f})$, $y - x \in P_G$*;*

(4) $\bar{f}$ *is a special Schreier surjection in* Mon.

Proof. The surjective homomorphism $\bar{f}$ is special homogeneous precisely when $\mathsf{Eq}(\bar{f}) \underset{\bar{\Delta}}{\overset{\bar{\pi}_1}{\rightleftarrows}} P_G$ is a homogeneous split epimorphism (see Definition 5.3.2). This happens if and only if, for any $x \in P_G$, the functions

$$\mu_x\colon \mathsf{Ker}(\bar{\pi}_1) \cong \mathsf{Ker}(\bar{f}) \to \bar{\pi}_1^{-1}(x); (0, a) \mapsto (0, a) + \bar{\Delta}(x) = (x, a + x)$$

and

$$\nu_x\colon \mathsf{Ker}(\bar{\pi}_1) \cong \mathsf{Ker}(\bar{f}) \to \bar{\pi}_1^{-1}(x); (0, b) \mapsto \bar{\Delta}(x) + (0, b) = (x, x + b)$$

are bijective. This is equivalent to the fact that, for any $(x, y) \in \mathsf{Eq}(\bar{f})$, there exist a unique $a \in \mathsf{Ker}(\bar{f})$ and a unique $b \in \mathsf{Ker}(\bar{f})$ such that $(x, y) = (x, a + x)$ and $(x, y) = (x, x + b)$. This condition also amounts to asking the existence of a unique $a \in \mathsf{Ker}(f) \cap P_G$ and a unique $b \in \mathsf{Ker}(f) \cap P_G$ such that $y = a + x$ and $y = x + b$, so that a and b are elements of G of the form $a = y - x$ and $b = -x + y$. As a consequence, the surjective homomorphism $\bar{f}$ is special homogeneous if and only if, for any $(x, y) \in \mathsf{Eq}(\bar{f})$, $y - x \in P_G$ and $-x + y \in P_G$. This proves the equivalence between (1) and (2).
The equivalence between (3) and (4) is proved similarly. Indeed, the fact that $\bar{f}$ is a special Schreier surjection is equivalent to the fact that the function μ_x defined above is bijective for any $x \in P_G$.
It remains to check that condition (3) implies condition (1). Let $(x, y) \in \mathsf{Eq}(\bar{f})$. Then, by assumption, we already have that $y - x \in P_G$. Now, we compute that $-x + y = -x + y + (-x + x) = -x + (y - x) + x$. This allows one to conclude that $-x + y \in P_G$ since P_G is closed in G under conjugation. $\square$

Theorem 5.5.3. *Let* $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ *be a regular epimorphism in* $\mathsf{PreOrdGrp}$. *Then, the following conditions are equivalent:*

(1) *(a)* $\mathsf{Ker}(f) \subseteq Z(G)$*;*
(b) $\bar{f}$ *is a special homogeneous surjection in* Mon*.*

(2) *(a)* $\mathsf{Ker}(f) \subseteq Z(G)$*;*
(b) $\bar{f}$ *is a special Schreier surjection in* Mon*.*

(3) $(f, \bar{f})$ *is a* $(\Gamma$-$)$*normal extension.*

(4) $(f, \bar{f})$ *is a* $(\Gamma$-$)$*central extension.*

Proof.

- (1) $\Leftrightarrow$ (2): This follows directly from Lemma 5.5.2.
- (1) $\Rightarrow$ (3): First remark that condition (b) implies that $a - b + c \in P_G$ for any $(a, b, c) \in \mathsf{Eq}(\bar{\eta}_G) \times_{P_G} \mathsf{Eq}(\bar{f})$. Indeed, thanks to Lemma 5.5.2, we know that $-b + c \in P_G$ since $(b, c) \in \mathsf{Eq}(\bar{f})$, and then $a - b + c \in P_G$ since $a \in P_G$, and P_G is a submonoid of G. So condition (b) implies condition (ii) of Theorem 5.1.9 and it then follows, thanks to the validity of condition (a), that $(f, \bar{f})$ is a Γ_C-normal extension. In other words, the first projection $(\pi_1, \bar{\pi}_1)\colon (\mathsf{Eq}(f), \mathsf{Eq}(\bar{f})) \twoheadrightarrow (G, P_G)$, of the kernel pair of $(f, \bar{f})$ is a Γ_C-trivial extension. According to Lemma 5.5.1, it now remains to prove that $C(\pi_1, \bar{\pi}_1)$ is a Γ_A-trivial extension (where Γ_A denotes the admissible Galois structure associated with the reflection $A \dashv W$). By condition (b), $\bar{f}$ is a special homogeneous surjection, which implies that also $\bar{\pi}_1$ is a special homogeneous surjection by pullback-stability. This in turn implies that the restriction $C(\bar{\pi}_1)$ of $C(\pi_1, \bar{\pi}_1)$ to the positive cones is a special homogeneous surjection thanks to Proposition 5.3.3. Indeed, the square

$$\begin{tikzcd}
\mathsf{Eq}(\bar{f}) \arrow[r, two heads, "\bar{\eta}_{\mathsf{Eq}(f)}"] \arrow[d, two heads, "\bar{\pi}_1"'] & \eta_{\mathsf{Eq}(f)}(\mathsf{Eq}(\bar{f})) \arrow[d, two heads, "C(\bar{\pi}_1)"] \\
P_G \arrow[r, two heads, "\bar{\eta}_G"'] & \eta_G(P_G)
\end{tikzcd}$$

is a pullback of regular epimorphisms in Mon, since $(\pi_1, \bar{\pi}_1)$ is a Γ_C-trivial extension. Accordingly, $C(\bar{\pi}_1)$ is a split epimorphism which is a special homogeneous surjection, and is then a Γ_{grp}-trivial extension by Proposition 5.3.4. This means that the square

$$\begin{tikzcd}
\eta_{\mathsf{Eq}(f)}(\mathsf{Eq}(\bar{f})) \arrow[r, tail, "j_{\mathsf{Eq}(f)}"] \arrow[d, two heads, "C(\bar{\pi}_1)"'] & grp(\eta_{\mathsf{Eq}(f)}(\mathsf{Eq}(\bar{f}))) \arrow[d, two heads, "grp(C(\bar{\pi}_1))"] \\
\eta_G(P_G) \arrow[r, tail, "j_G"'] & grp(\eta_G(P_G))
\end{tikzcd}$$

is a pullback in the category ComMon of commutative monoids. As a consequence, the square

$$\begin{tikzcd}
(ab(\mathsf{Eq}(f)), \eta_{\mathsf{Eq}(f)}(\mathsf{Eq}(\bar{f}))) \arrow[r, tail, "(1, j_{\mathsf{Eq}(f)})"] \arrow[d, two heads, "C(\pi_1, \bar{\pi}_1)"'] & (ab(\mathsf{Eq}(f)), grp(\eta_{\mathsf{Eq}(f)}(\mathsf{Eq}(\bar{f})))) \arrow[d, two heads, "(A\cdot C)(\pi_1, \bar{\pi}_1) = F(\pi_1, \bar{\pi}_1)"] \\
(ab(G), \eta_G(P_G)) \arrow[r, tail, "(1, j_G)"'] & (ab(G), grp(\eta_G(P_G)))
\end{tikzcd}$$

is a pullback in the category $\mathsf{PreOrdAb}$ of preordered abelian groups, and this proves that $C(\pi_1, \bar{\pi}_1)$ is a Γ_A-trivial extension. By Lemma 5.5.1, we conclude that $(\pi_1, \bar{\pi}_1)$ is a Γ-trivial extension, which is equivalent to saying that $(f, \bar{f})$ is a Γ-normal extension.

- (3) ⇒ (4): Any normal extension is central by definition.
- (4) ⇒ (1): By definition of a (Γ-)central extension, there exists an effective descent morphism (i.e. a regular epimorphism) $(p, \bar{p})\colon (E, P_E) \twoheadrightarrow (H, P_H)$ in $\mathsf{PreOrdGrp}$ such that the pullback $(p, \bar{p})^*(f, \bar{f})$ (which we will denote by $(\pi_1, \bar{\pi}_1)$) of $(f, \bar{f})$ along $(p, \bar{p})$ is a Γ-trivial extension. Using Lemma 5.5.1, this means that

  (α) $(\pi_1, \bar{\pi}_1)$ is a Γ_C-trivial extension, and

  (β) $C(\pi_1, \bar{\pi}_1)$ is a Γ_A-trivial extension.

  The first statement (α) means of course that $(f, \bar{f})$ is a Γ_C-central extension. In particular, f is algebraically central, that is, $f\colon G \twoheadrightarrow H$ satisfies condition (a). The second statement (β) now implies that the square

$$\begin{tikzcd}
(ab(P), \eta_P(P_P)) \arrow[r, tail, "{(1, j_P)}"] \arrow[d, two heads, "{(C(\pi_1), C(\bar{\pi}_1)) = C(\pi_1, \bar{\pi}_1)}"'] & (ab(P), grp(\eta_P(P_P))) \arrow[d, two heads, "{(A \cdot C)(\pi_1, \bar{\pi}_1) = F(\pi_1, \bar{\pi}_1)}"] \\
(ab(E), \eta_E(P_E)) \arrow[r, tail, "{(1, j_E)}"'] & (ab(E), grp(\eta_E(P_E)))
\end{tikzcd}$$

(where $P = E \times_H G$ and $P_P = P_E \times_{P_H} P_G$) is a pullback in $\mathsf{PreOrdAb}$. In particular, its restriction to ComMon

$$\begin{tikzcd}
\eta_P(P_P) \arrow[r, tail, "j_P"] \arrow[d, two heads, "C(\bar{\pi}_1)"'] & grp(\eta_P(P_P)) \arrow[d, two heads, "grp(C(\bar{\pi}_1))"] \\
\eta_E(P_E) \arrow[r, tail, "j_E"'] & grp(\eta_E(P_E))
\end{tikzcd}$$

is a pullback. This means that $C(\bar{\pi}_1)$ is a Γ_{grp}-trivial extension, and then a Γ_{grp}-normal extension. Indeed, since Γ_{grp} is admissible, any Γ_{grp}-trivial extension is Γ_{grp}-normal (see Proposition 2.3.9(3)). Thanks to Theorem 5.3.5, we can conclude that $C(\bar{\pi}_1)$ is a special homogeneous surjection. Since special homogeneous surjections are pullback-stable, it then follows that $\bar{\pi}_1$ has this same property. Indeed, the statement (α) implies (among other things) that the square

$$\begin{tikzcd}
P_P \arrow[r, two heads, "\bar{\eta}_P"] \arrow[d, two heads, "\bar{\pi}_1"'] & \eta_P(P_P) \arrow[d, two heads, "C(\bar{\pi}_1)"] \\
P_E \arrow[r, two heads, "\bar{\eta}_E"'] & \eta_E(P_E)
\end{tikzcd}$$

is a pullback in Mon. Applying now Proposition 5.3.3 to the following pullback of regular epimorphisms

$$\begin{tikzcd}
P_P \arrow[r, two heads, "\bar{\pi}_2"] \arrow[d, two heads, "\bar{\pi}_1"'] & P_G \arrow[d, two heads, "\bar{f}"] \\
P_E \arrow[r, two heads, "\bar{p}"'] & P_H,
\end{tikzcd}$$

we obtain that $\bar{f}$ is a special homogeneous surjection, i.e. condition (b) is satisfied. This concludes the proof. $\square$

Conclusion and outlook

Conclusion

At the end of this thesis, we now have a better understanding of the categorical behaviour of preordered groups. This exploration of the exactness properties of the category $\mathsf{PreOrdGrp}$ of preordered groups will also have shed light on the study of torsion theories and Galois theories in a non-exact (and therefore non-semi-abelian) context.

In Chapter 1, we have recalled the basic definitions and properties of category theory needed for this dissertation. Next, we have explained in Chapter 2 more advanced notions, directly related to the topics treated in this thesis, allowing one to understand the title and then the contents of the following chapters, which contain our new results.

Chapter 3 was first of all devoted to the study of torsion theories in $\mathsf{PreOrdGrp}$. We have proved that the pair $(\mathsf{Grp}, \mathsf{ParOrdGrp})$ forms a torsion theory in the normal category $\mathsf{PreOrdGrp}$, where Grp denotes the full subcategory of preordered groups whose preorder is indiscrete (or, equivalently, whose positive cone is the underlying group itself) and where $\mathsf{ParOrdGrp}$ is the full subcategory of partially ordered groups, i.e. of preordered groups whose preorder is antisymmetric (or, equivalently, whose positive cone is a reduced monoid). We have then seen that, like any torsion theory in a normal category, it induces a (normal epi)-reflection (3.2) which, in turn, gives rise to an absolute Galois structure. The latter is admissible or, in other words, the reflector is semi-left-exact, so that we naturally obtain a factorization system $(\mathscr{E}, \mathscr{M})$ where the class $\mathscr{M}$ of trivial coverings has the following description: a morphism $(f,\bar{f})\colon (G,P_G)\to(H,P_H)$ in $\mathsf{PreOrdGrp}$ is a trivial covering if and only if the restriction $\phi\colon N_G\to N_H$ of f to $N_G=\{x\in G\mid x\in P_G \text{ and } -x\in P_G\}$ is a group isomorphism. In order to prove this result, we first had to make some useful remarks on the short exact sequence (3.1) constructed in the proof of the torsion theory mentioned above. By doing so, we have realized that this short exact sequence is special in the sense that not only the sequence at the group level is exact but also the sequence at the monoid (positive cone) level, which is not the case in full generality for an arbitrary short exact sequence in $\mathsf{PreOrdGrp}$.

We have then focused on the characterization of all coverings relative to the above mentioned admissible Galois structure. For this, we have used Theorem 2.4.9 and therefore had to check the validity of the two involved axioms (N) and (C) in the context of preordered groups. In particular, we have seen that it is possible to build, for any preordered group (G,P_G), a partially ordered group (H,P_H) as well as an effective descent morphism $(f,\bar{f})\colon (H,P_H)\to(G,P_G)$. This showed the axiom (C) of "covering", which basically means that any preordered group can be obtained as a "good quotient" of a partially ordered group, a result that has proven to be quite useful for several purposes. Thanks to Theorem 2.4.9 and the validity of the two axioms (N) and (C) in the setting of preordered groups, it was finally possible to describe all the coverings in $\mathsf{PreOrdGrp}$ (again with respect to the above Galois structure): a morphism $(f,\bar{f})\colon (G,P_G)\to(H,P_H)$ in $\mathsf{PreOrdGrp}$ is a covering if and only if its kernel $\mathsf{Ker}(f,\bar{f})$ is a partially ordered group. Moreover, we have also obtained that $(\mathscr{E}',\mathscr{M}^*)$ is a monotone-light factorization system, where $\mathscr{M}^*$ is the class of coverings described above, and $\mathscr{E}'$ the class of normal epimorphisms in $\mathsf{PreOrdGrp}$ whose kernel belongs to the torsion subcategory Grp. This result is all the more interesting since this is a rare phenomenon, starting from a general reflective factorization system $(\mathscr{E},\mathscr{M})$.

Next, we have classified the coverings in terms of internal actions of the Galois groupoid of the morphism $(f,\bar{f})\colon (H,P_H)\to(G,P_G)$ of the axiom (C). For this, we have used the article [59] by G. Janelidze, L. Márki and W. Tholen on locally semisimple coverings. In our analysis, we have realized that our result provided an example of application of

their theorem (Theorem 3.1 in [59]) in a non-exact setting, and therefore gave a positive answer to the last remark of their article.
In the last part of Chapter 3, we have noticed that the subcategory $\mathsf{ParOrdGrp}$ of partially ordered groups is a torsion-free subcategory not only for the previously mentioned torsion theory but also for a pretorsion theory. The torsion part of this pretorsion theory is given by the category $\mathsf{Grp}(\mathsf{PreOrd})$ of internal groups in the category of preordered sets (i.e. of preordered groups whose preorder is an equivalence relation or, equivalently, whose positive cone is a group). From the pretorsion theory $(\mathsf{Grp}(\mathsf{PreOrd}), \mathsf{ParOrdGrp})$, we have deduced in particular that $\mathsf{Grp}(\mathsf{PreOrd})$ is coreflective in $\mathsf{PreOrdGrp}$. It actually turns out to be reflective as well, as shown in Proposition 3.4.3.

In Chapter 4, we have then generalized the results of Chapter 3 to any category V-Grp of V-groups, for V a commutative, unital and integral quantale. More precisely, in V-Grp, there is a torsion theory (V-$\mathsf{Grp}_{\mathsf{ind}}$, V-$\mathsf{Grp}_{\mathsf{sep}}$) whose torsion subcategory V-$\mathsf{Grp}_{\mathsf{ind}}$ is the category of indiscrete V-groups and whose torsion-free subcategory is the category of separated V-groups. In particular, the category MetGrp of Lawvere metric groups then admits a torsion theory whose torsion part is the subcategory of Lawvere metric groups such that the distance is always 0 and whose torsion-free part is the subcategory of separated Lawvere metric groups, that is, of Lawvere metric groups (X, d) satisfying the separation axiom: $d(x, x') = 0 = d(x', x) \Rightarrow x = x'$. The torsion theory ($V$-$\mathsf{Grp}_{\mathsf{ind}}$, V-$\mathsf{Grp}_{\mathsf{sep}}$) naturally induces a (normal epi)-reflection (4.7) which gives rise to an admissible absolute Galois structure. In other words, the reflector $F\colon V\text{-}\mathsf{Grp} \to V\text{-}\mathsf{Grp}_{\mathsf{sep}}$ is semi-left-exact. This implies that there is a factorization system $(\mathscr{E}, \mathscr{M})$ of which we have provided a precise description. The class $\mathscr{M}$ of trivial coverings with respect to the above mentioned absolute Galois structure is given by the class of morphisms $f\colon (X, a) \to (Y, b)$ in V-Grp such that the restriction $\phi\colon N_X \to N_Y$ of f to $N_X = \{x \in X \mid a(0, x) \geq k \text{ and } a(x, 0) \geq k\}$ is a group isomorphism and such that $a = b \wedge \bar{a}$ where $\bar{a} = \eta_X \cdot a \cdot \eta_X^\circ$ for $\eta_X : X \twoheadrightarrow X/N_X$ the quotient morphism.
As was already the case for the category $\mathsf{PreOrdGrp}$ of preordered groups, the two axioms (N) and (C) of Theorem 2.4.9 are valid in V-Grp, so

that the pair $(\mathscr{E}', \mathscr{M}^*)$ is a monotone-light factorization system. It is furthermore possible to provide an explicit description of the two classes $\mathscr{E}'$ and $\mathscr{M}^*$: the class $\mathscr{E}'$ is the class of normal epimorphisms in V-Grp whose kernel is in the torsion subcategory V-$\mathsf{Grp}_{\mathsf{ind}}$ while $\mathscr{M}^*$ is the class of morphisms in V-Grp whose kernel is in the torsion-free subcategory V-$\mathsf{Grp}_{\mathsf{sep}}$. This in particular gave us a description of all the coverings with respect to the above mentioned Galois structure.
We have then observed that these coverings can be classified in terms of internal actions. Again, as in the case of preordered groups, we were led to this conclusion thanks to the paper [59] by G. Janelidze, L. Márki and W. Tholen, and this has provided a lot of other examples of application of their theorem in the non-exact setting.
For the last part of this chapter, we have then noticed that there is a pretorsion theory in any category V-Grp of V-groups. The torsion-free subcategory is the same as for the torsion theory (i.e. V-$\mathsf{Grp}_{\mathsf{sep}}$), but the torsion subcategory is, this time, given by the category V-$\mathsf{Grp}_{\mathsf{sym}}$ of symmetric V-groups, i.e. of those V-groups (X, a) satisfying the symmetry axiom: $a(x, x') = a(x', x)$ for any $x, x' \in X$. By proving this statement, we have realized that the quantale V need not be integral. This has then given us many examples of pretorsion theories in non-pointed categories (one for any category of V-groups where V is not integral). Actually, this result turns out to be somewhat more interesting than its restriction to the context of preordered groups discussed in Chapter 3. Indeed, in the particular case of preordered groups, the class of $\mathscr{Z}$-trivial objects is simply given by the class of discrete preordered groups, which is not so interesting, whereas the class of $\mathscr{Z}$-trivial objects in the general case is much richer. For instance, in the category MetGrp of Lawvere metric groups, it corresponds to the class of both symmetric and separated Lawvere metric groups, which is more classical.
Finally, we have also proved that this pretorsion theory is valid, more generally, in any category V-Cat of V-categories.

Motivated and inspired by the admissible Galois structure that exists in the setting of groups as well as by the description of the related central and normal extensions (see Subsection 2.3.3), we have developed in Chapter 5 a Galois theory for preordered groups. With this generalization in view, the first idea was to consider the functor $C\colon \mathsf{PreOrdGrp} \to \mathsf{PreOrdAb}$ from $\mathsf{PreOrdGrp}$ to its subcategory $\mathsf{PreOrdAb}$ of preordered abelian groups, i.e. of those preordered groups (G, P_G) for which G is an abelian group. This was indeed the most natural first step, leading to an extension of the abelianization functor $ab\colon \mathsf{Grp} \to \mathsf{Ab}$.

We have proved that C is a left adjoint for the inclusion functor $V\colon \mathsf{PreOrdAb} \hookrightarrow \mathsf{PreOrdGrp}$, and this then gives rise to an admissible Galois structure Γ_C. The proof of the admissibility of Γ_C is based on a reformulation of a remark by G. Janelidze and G. M. Kelly [58], and involves the lattice of normal subobjects in $\mathsf{PreOrdGrp}$ which turns out to be modular. As desired, we were able to give an explicit description of the Γ_C-normal extensions: a regular epimorphism $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ in $\mathsf{PreOrdGrp}$ is a Γ_C-normal extension if and only if f is algebraically central, i.e. the kernel $\mathsf{Ker}(f)$ of f is in the center $Z(G)$ of G, and, for any $(a, b, c) \in \mathsf{Eq}(\bar{\eta}_G) \times_{P_G} \mathsf{Eq}(\bar{f})$, we have that $a - b + c \in P_G$, where $(\eta_G, \bar{\eta}_G)$ is the (G, P_G)-component of the unit of the adjunction $C \dashv V$.

As explained in Chapter 5, the subcategory $\mathsf{PreOrdAb}$ is not the subcategory of abelian objects in $\mathsf{PreOrdGrp}$, but only the subcategory of commutative objects. The subcategory of abelian objects in $\mathsf{PreOrdGrp}$ is given by the full subcategory $\mathsf{Mono}(\mathsf{Ab})$ of preordered abelian groups (G, P_G) for which the positive cone P_G is a group. In the light of this observation, we have then considered the composite $F = A \cdot C$ of C with the functor $A\colon \mathsf{PreOrdAb} \to \mathsf{Mono}(\mathsf{Ab})$ defined, for any $(G, P_G) \in \mathsf{PreOrdAb}$, by $A(G, P_G) = (G, grp(P_G))$, where $grp(P_G)$ is the group completion of the commutative monoid P_G.

We have shown that F is a reflector which induces an admissible Galois structure Γ. A regular epimorphism $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ in $\mathsf{PreOrdGrp}$ is then a Γ-normal extension if and only if f is an algebraically central extension ($\mathsf{Ker}(f) \subseteq Z(G)$) and $\bar{f}$ is a special homogeneous surjection in the category Mon of monoids, that is, for any $(x, y) \in \mathsf{Eq}(\bar{f})$, we have $y - x \in P_G$ and $-x + y \in P_G$. Equivalently, $(f, \bar{f})$ is a Γ-normal extension if and only if f is an algebraically central extension and $\bar{f}$ is a special Schreier surjection in Mon, i.e., for any $(x, y) \in \mathsf{Eq}(\bar{f})$, $y - x \in P_G$. It turns out that the class of Γ-central extensions coincides with that of Γ-normal extensions, as desired. The above algebraic description of these classes was obtained using the results of the article [75] by A. Montoli, D. Rodelo and T. Van der Linden in which a Galois theory for monoids was studied.

Outlook

This thesis raises several new questions. In this section, we present some of them as well as a few suggestions for future research.

Torsion theories and coverings of preordered groups with operations

In Chapter 4, we have considered a generalization of Chapter 3 to the context of V-groups, for V a commutative, unital and integral quantale. Instead of studying torsion theories and coverings in a category whose objects are pairs (X, R) with X a group and R a relation on X satisfying additional assumptions, we were more generally interested in categories whose objects are pairs (X, a) with X a group and a a V-relation satisfying extra conditions. In other words, we have replaced the relation with the more general notion of V-relation. Now, why couldn't we replace the group structure with another suitable algebraic structure?

One that seems appropriate is the structure of *group with operations* [76] (see also [73]). Roughly speaking, a group with operations is an algebraic structure which has all the operations as well as all the identities of the groups, and which can also admit some additional operations, of arity 1 and 2, satisfying particular axioms (which will not be specified in this thesis). Rings without units, vector spaces, Lie algebras and, of course, groups are the most typical examples of groups with operations. It is then possible to endow some of these groups with operations with a preorder "compatible with the operations". This gives rise to what we have called "*preordered groups with operations*", which include preordered groups.

It seems possible to generalize to this specific context the results of Chapter 3 concerning the torsion theory. The generalization of the part about coverings however turns out to be much more difficult. More precisely, there is a real obstruction in the extension (to preordered groups with operations) of the proof of Proposition 3.2.4. Indeed, the proof in the particular case of preordered groups is really specific and makes deep use of the group structure, so that a proof for the extension of Proposition 3.2.4, if valid, should be made on a case-by-case basis. We are actually quite confident that it should be possible to adapt the proof for "preordered rings" and for "preordered vector spaces". Unfortunately, this does not seem to work the same way for "preordered Lie algebras".

(Non-abelian) homology in $\mathsf{PreOrdGrp}$

The goal of Chapter 5 was to generalize the well-known Galois theory induced by the abelianization functor $ab\colon \mathsf{Grp} \to \mathsf{Ab}$ between the category Grp of groups and its subcategory Ab of abelian groups. We were finally able to find an extension of the abelianization functor $ab\colon \mathsf{Grp} \to \mathsf{Ab}$ for preordered groups (i.e. from $\mathsf{PreOrdGrp}$ to its subcategory of abelian objects) and provide a characterization of the Γ-central (and Γ-normal) extensions with respect to the induced Galois structure Γ. Now, we could continue the generalization further. Indeed, in the case of groups, it is possible to obtain, thanks to the characterization of the central (and normal) extensions in this context, a formula for the *fundamental group* [57] of any group G: the fundamental group $\pi_1(G)$ of any group G is given by

$$\pi_1(G) \cong \frac{K \wedge [P,P]}{[K,P]}$$

where K and P are the groups involved in any free presentation

$$0 \longrightarrow K \overset{k}{\rightarrowtail} P \overset{p}{\twoheadrightarrow} G \longrightarrow 0$$

of G. This is the well-known *Hopf formula* [53], which corresponds to the second integral homology group $H_2(G,\mathbb{Z})$ of G.

The idea consists in generalizing this result to preordered groups in order to get a "Hopf formula for the second homology object in $\mathsf{PreOrdGrp}$". The main difficulty in reaching our goal is to construct, for any preordered group (G,P_G), a *weakly universal central extension* of (G,P_G). Of course, studying higher homology in PreOrdGrp, as was done in Grp, might be an interesting and natural next step.

A commutator theory for relatively modular quasivarieties

Since the category PreOrdGrp is a *relatively modular quasivariety*, it could also be relevant to see if the results of Chapter 5 extend to any relatively modular quasivariety and then to study, if possible, homology in this context. But, in order to do this, we would first need to develop a categorical approach to *commutator theory* for relatively modular quasivarieties, as was already done in [61] for *congruence modular varieties*. A good starting point would consist in developing a categorical interpretation of the results in the paper [67] by K. Kearnes and R. McKenzie in which the authors present, from a purely algebraic point of view, a commutator theory for relatively modular quasivarieties.

A | The stable category of preordered groups

A.1. The universal torsion theory associated with the pretorsion theory $(\mathsf{Grp}(\mathsf{PreOrd}), \mathsf{ParOrdGrp})$

Consider the class $\mathscr{Z}$ of $\mathscr{Z}$-trivial objects of Proposition 3.4.1, i.e.

$$\mathscr{Z} = \{(G, P_G) \in \mathsf{PreOrdGrp} \mid P_G = 0\}\,.$$

The $\mathscr{Z}$-trivial morphisms are then characterized as follows:

Proposition A.1.1. *A morphism* $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ *in* $\mathsf{PreOrdGrp}$ *is* $\mathscr{Z}$*-trivial if and only if* $\bar{f} = 0$*.*

Proof. If $(f, \bar{f})$ is $\mathscr{Z}$-trivial then, by definition, $(f, \bar{f})$ factors through a $\mathscr{Z}$-trivial object $(A, 0)$:

$$\begin{tikzcd}
P_G \arrow[rr, "\bar{f}"] \arrow[dr] \arrow[ddd, tail] & & P_H \arrow[ddd, tail] \\
& 0 \arrow[ur] \arrow[d, tail] & \\
& A \arrow[dr] & \\
G \arrow[ur] \arrow[rr, "f"'] & & H.
\end{tikzcd}$$

In particular $\bar{f} = 0$.
Conversely, consider in the category Grp of groups the (regular epi,

mono)-factorization of f. Then, by assumption, the following diagram commutes:

$$\begin{tikzcd}
P_G \arrow[rr, "\bar{f}"] \arrow[dr] \arrow[ddd, tail] & & P_H \arrow[ddd, tail] \\
& 0 \arrow[ur] \arrow[d, tail] & \\
& f(G) \arrow[dr, tail] & \\
G \arrow[ur, two heads] \arrow[rr, "f"'] & & H.
\end{tikzcd}$$

Accordingly, $(f, \bar{f})$ factors through $(f(G), 0) \in \mathscr{Z}$, i.e. it is a $\mathscr{Z}$-trivial morphism. $\square$

The question we may ask now is: what is the "universal" torsion theory associated with the above pretorsion theory? More precisely, we need to find a pointed category $\mathscr{S}$, which will be called the *stable category* of $\mathsf{PreOrdGrp}$, containing a torsion theory $(\mathscr{T}, \mathscr{F})$, as well as a functor Φ from $\mathsf{PreOrdGrp}$ to $\mathscr{S}$ which is a *torsion theory functor*, i.e.

(1) $\Phi(G, P_G) \in \mathscr{T}$ for any $(G, P_G) \in \mathsf{Grp}(\mathsf{PreOrd})$ and $\Phi(G, P_G) \in \mathscr{F}$ for any $(G, P_G) \in \mathsf{ParOrdGrp}$;

(2) for any preordered group (G, P_G), Φ sends the *canonical short $\mathscr{Z}$-exact sequence* associated with (G, P_G) in the pretorsion theory $(\mathsf{Grp}(\mathsf{PreOrd}), \mathsf{ParOrdGrp})$ to a short exact sequence in $\mathscr{S}$.

In particular, this functor Φ must send any $\mathscr{Z}$-trivial object to the zero object in the stable category $\mathscr{S}$ and then any $\mathscr{Z}$-trivial morphism to the zero morphism.

Remark A.1.2. Such a construction was initially realized by A. Facchini and C. Finocchiaro in [32], in the context of preordered sets. F. Borceux, F. Campanini and M. Gran then extended this construction to the setting of internal preorders in a *coherent category* (see articles [7] and [9]). In the more recent article [8], the results were then widened to the context of *lextensive categories.*

A.2. Definition of the stable category of PreOrdGrp

In order to define the stable category of PreOrdGrp, we first recall two well-known results about monoids, already discussed in [21].

Lemma A.2.1. *If a monoid M is embeddable in a group G, then it is embeddable in its* group completion *(also called* Grothendieck group*)* $grp(M)$*.*

Proof. Consider the canonical morphism $j\colon M \to grp(M)$ as well as the injection $i\colon M \rightarrowtail G$ to the group G:

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, tail, "i"'] & & grp(M) \arrow[dl, dotted, "\phi"] \\ & G. & \end{tikzcd}$$

By the universal property of the Grothendieck group of M, there exists a unique group homomorphism $\phi\colon grp(M) \to G$ such that $\phi \cdot j = i$ in the category Mon of monoids. Since i is a monomorphism, it then easily follows that also j is a monomorphism, hence M is embeddable in $grp(M)$. $\square$

Lemma A.2.2. *Let M be a monoid which is embeddable in a group. Then, M is the positive cone of a group G if and only if, for any $a, b \in M$, there exist $x, y \in M$ such that $a + b = b + x = y + a$.*

Proof. If M is the positive cone of a group G, then, for any $a, b \in M$, $-b + a + b \in M$ and $a + b - a \in M$ (since M is closed under conjugation in G). By taking $x = -b + a + b$ and $y = a + b - a$, we can then check that $a + b = b + x = y + a$.
Assuming now that this last condition holds, let us prove that M is the positive cone of its group completion $grp(M)$. Let $m \in M$ and let $x \in grp(M)$, that is, $x = x_1 - x_2 + \cdots + x_{n-1} - x_n$ for $x_1, \cdots, x_n \in M$. Then,

$$x + m - x = x_1 - x_2 + \cdots + x_{n-1} - x_n + m + x_n - x_{n-1} + \cdots + x_2 - x_1.$$

We observe that $-x_n + m + x_n \in M$. Indeed, by assumption, there exist $a_1, b_1 \in M$ such that $m + x_n = x_n + a_1 = b_1 + m$ so that, in particular, $-x_n+m+x_n = a_1 \in M$. We next notice that $x_{n-1}+a_1-x_{n-1} \in M$ since, by hypothesis, $x_{n-1}+a_1 = a_1+b_2 = a_2+x_{n-1}$ for some $a_2, b_2 \in M$, which implies that $x_{n-1} + a_1 - x_{n-1} = a_2 \in M$, that is, $x_{n-1} - x_n + m + x_n - x_{n-1} \in M$. We then proceed by induction and show that $x+m-x \in M$. This proves that M is closed under conjugation in $grp(M)$. Since M is embeddable in a group, it is therefore embeddable in $grp(M)$ thanks to Lemma A.2.1. As a conclusion, $(grp(M), M) \in \mathsf{PreOrdGrp}$. □

Consider then the full subcategory $\mathsf{Mon}_{\geq 0}$ of the category Mon of monoids, whose objects are all monoids M which are embeddable in a group and such that, for any $a, b \in M$, there exist $x, y \in M$ for which the identity $a + b = b + x = y + a$ holds.

Consider also the functor $P\colon \mathsf{PreOrdGrp} \to \mathsf{Mon}_{\geq 0}$ defined, for any preordered group (G, P_G), by $P(G, P_G) = P_G$ and, for any morphism $(f, \bar{f})$ of preordered groups, by $P(f, \bar{f}) = \bar{f}$. This is the *positive cone functor*. Note that, thanks to Lemmas A.2.1 and A.2.2, this functor is actually well-defined.

We will prove in the subsequent sections that the category $\mathsf{Mon}_{\geq 0}$ is the stable category of $\mathsf{PreOrdGrp}$.

A.3. $\mathscr{Z}$-kernels and $\mathscr{Z}$-cokernels in PreOrdGrp

In this section, we present an explicit description of $\mathscr{Z}$-kernels and $\mathscr{Z}$-cokernels in the category $\mathsf{PreOrdGrp}$ of preordered groups.

A.3.1. Description of $\mathscr{Z}$-kernels and $\mathscr{Z}$-cokernels in PreOrdGrp

Proposition A.3.1. *Let $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ be a morphism in* $\mathsf{PreOrdGrp}$*. Then, a $\mathscr{Z}$-kernel of $(f, \bar{f})$ is given by the morphism*

$$(k, \bar{k})\colon (G, P_G \cap \mathsf{Ker}(f)) \to (G, P_G)$$

where k is the identity morphism 1_G. Note that $P_G \cap \mathsf{Ker}(f) = \mathsf{Ker}(\bar{f})$ in the category Mon *of monoids.*

Proof. We first observe that $\bar{f} \cdot \bar{k} = 0$ since $\bar{k}$ is the kernel of $\bar{f}$ in Mon. It then follows that $(f, \bar{f}) \cdot (k, \bar{k})$ is a $\mathscr{Z}$-trivial morphism thanks to Proposition A.1.1.
We next consider a morphism $(\alpha, \bar{\alpha})\colon (A, P_A) \to (G, P_G)$ in PreOrdGrp such that $(f, \bar{f}) \cdot (\alpha, \bar{\alpha})$ is $\mathscr{Z}$-trivial, i.e. such that $\bar{f} \cdot \bar{\alpha} = 0$ thanks to Proposition A.1.1.

$$\begin{tikzcd}
\mathsf{Ker}(\bar{f}) \arrow[rr, tail, "\bar{k}"] \arrow[ddd, tail, "\psi"'] & & P_G \arrow[r, "\bar{f}"] \arrow[ddd, tail, "g"] & P_H \arrow[ddd, tail] \\
& P_A \arrow[ul, dotted, "\bar{\phi}"] \arrow[ur, "\bar{\alpha}"'] \arrow[d, tail, "a"] & & \\
& A \arrow[dl, dotted, "\phi"'] \arrow[dr, "\alpha"] & & \\
G \arrow[rr, equal, "k=1_G"'] & & G \arrow[r, "f"'] & H
\end{tikzcd}$$

By the universal property of kernels in Mon, there exists a unique morphism $\bar{\phi}\colon P_A \to \mathsf{Ker}(\bar{f})$ such that $\bar{k} \cdot \bar{\phi} = \bar{\alpha}$. Now, we put $\phi = \alpha$ since this is the only possible group morphism $\phi\colon A \to G$ with $k \cdot \phi = \alpha$. We moreover compute that

$$\phi \cdot a = k \cdot \phi \cdot a = \alpha \cdot a = g \cdot \bar{\alpha} = g \cdot \bar{k} \cdot \bar{\phi} = k \cdot \psi \cdot \bar{\phi} = \psi \cdot \bar{\phi},$$

which means that $(\phi, \bar{\phi})\colon (A, P_A) \to (G, \mathsf{Ker}(\bar{f}))$ is a morphism in PreOrdGrp. It is obviously unique with the property $(k, \bar{k}) \cdot (\phi, \bar{\phi}) = (\alpha, \bar{\alpha})$. $\square$

Proposition A.3.2. *Let $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ be a morphism in* PreOrdGrp *and denote by S the normal closure of $\bar{f}(P_G)$ in H, i.e.*

$$S = \{s_1 - s_2 + \cdots - s_n \mid s_i = h_i + f(g_i) - h_i, \text{ with } h_i \in H \text{ and } g_i \in P_G, \text{ for any } i \in \{1, \cdots, n\}\}.$$

If we write $q\colon H \twoheadrightarrow H/S$ for the quotient morphism, then the arrow

$$(q,\bar{q})\colon (H,P_H) \twoheadrightarrow (H/S, q(P_H))$$

is a $\mathscr{Z}$-cokernel of $(f,\bar{f})$.

Proof. For any $x \in P_G$, $f(x) \in \bar{f}(P_G) \subseteq S$, so that $q(f(x)) = 0$ since S is the kernel of q in the category Grp of groups. This implies that $\bar{q} \cdot \bar{f} = 0$, which means that $(q,\bar{q}) \cdot (f,\bar{f})$ is a $\mathscr{Z}$-trivial morphism.
Consider then a morphism $(\alpha,\bar{\alpha})\colon (H,P_H) \to (A,P_A)$ in $\mathsf{PreOrdGrp}$ such that $(\alpha,\bar{\alpha}) \cdot (f,\bar{f})$ is $\mathscr{Z}$-trivial, that is, such that $\bar{\alpha} \cdot \bar{f} = 0$.

$$\begin{tikzcd}
P_G \arrow[rrr, "\bar{f}"] \arrow[dr, two heads] \arrow[ddd, tail] & & & P_H \arrow[rr, two heads, "\bar{q}"] \arrow[dr, "\bar{\alpha}"] \arrow[ddd, tail] & & q(P_H) \arrow[dl, dotted, "\bar{\phi}"'] \arrow[ddd, tail, "\psi"] \\
& \bar{f}(P_G) \arrow[urr, tail] \arrow[dr, tail] & & & P_A \arrow[d, tail, "a"] & \\
& & S \arrow[dr, tail, "t"] & & A & \\
G \arrow[rrr, "f"'] & & & H \arrow[ur, "\alpha"] \arrow[rr, two heads, "q"'] & & H/S \arrow[ul, dotted, "\phi"']
\end{tikzcd}$$

Let $s \in S$. Then, there exist $h_i \in H$ and $g_i \in P_G$ such that

$$s = s_1 - s_2 + \cdots - s_n$$

where $s_i = h_i + f(g_i) - h_i$ for any $i \in \{1, \cdots, n\}$. One then computes that, for any $i \in \{1, \cdots, n\}$,

$$\alpha(s_i) = \alpha(h_i) + (\alpha \cdot f)(g_i) - \alpha(h_i) = \alpha(h_i) + 0 - \alpha(h_i) = 0$$

since $\bar{\alpha} \cdot \bar{f} = 0$, so that $\alpha(s) = \alpha(s_1) - \alpha(s_2) + \cdots - \alpha(s_n) = 0$. Since q is the cokernel of $t\colon S \to H$ in the category Grp of groups, there is a unique group morphism $\phi\colon H/S \to A$ such that $\phi \cdot q = \alpha$. Now, seeing that $\bar{q}$ is a regular epimorphism in the category Mon of monoids and that $a\colon P_A \to A$ is a monomorphism, the universal property of strong epimorphisms yields a unique arrow $\bar{\phi}\colon q(P_H) \to P_A$ in Mon such that $\bar{\phi} \cdot \bar{q} = \bar{\alpha}$ and $a \cdot \bar{\phi} = \phi \cdot \psi$. Accordingly, there exists a unique morphism $(\phi, \bar{\phi})\colon (H/S, q(P_H)) \to (A, P_A)$ in PreOrdGrp such that $(\phi, \bar{\phi}) \cdot (q, \bar{q}) = (\alpha, \bar{\alpha})$. The uniqueness of such an arrow comes from the fact that $(q, \bar{q})$ is an epimorphism in PreOrdGrp. $\square$

It is then easy to prove the following:

Corollary A.3.3. *For any preordered group (G, P_G), the sequence*

$$\begin{tikzcd} N_G \arrow[r, tail] \arrow[d, tail] & P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail] & \eta_G(P_G) = P_G/N_G \arrow[d, tail] \\ G \arrow[r, equal] & G \arrow[r, two heads, "\eta_G"'] & G/N_G \end{tikzcd}$$

is a short $\mathscr{Z}$-exact sequence in PreOrdGrp. *It is the* canonical short $\mathscr{Z}$-exact sequence *associated with (G, P_G) in the pretorsion theory* (Grp(PreOrd), ParOrdGrp).

Remark A.3.4. As observed in Corollary 3.1.10(1), the arrow $\bar{\eta}_G\colon P_G \twoheadrightarrow \eta_G(P_G)$ is actually the cokernel of $N_G \rightarrowtail P_G$, that is, $\eta_G(P_G) = P_G/N_G$.

A.3.2. $\mathscr{Z}$-kernels and $\mathscr{Z}$-cokernels as special pullbacks and pushouts in PreOrdGrp

We then notice that these $\mathscr{Z}$-kernels and $\mathscr{Z}$-cokernels can be seen as some particular pullbacks and pushouts, respectively, as is the case for the classical notions of kernels and cokernels. To do this, we first recall some results from the second section of the article [47] by M. Grandis, G. Janelidze and L. Márki.

In their paper, M. Grandis, G. Janelidze and L. Márki consider a full subcategory $\mathscr{C}_0$ of a category $\mathscr{C}_1$, which is simultaneously reflective and coreflective, i.e. a diagram

$$\begin{tikzcd} \mathscr{C}_1 \arrow[r, bend left, "D"] \arrow[r, bend right, "C"'] & \mathscr{C}_0 \arrow[l, tail, "E"'] \end{tikzcd}$$

where C, D and E are three functors as follows:

- E is fully faithful;
- D is a right adjoint left inverse of E;
- C is a left adjoint left inverse of E.

Denote by ι the counit of the adjunction $E \dashv D$ and by π the unit of the adjunction $C \dashv E$, and assume, for simplicity, that $\mathscr{C}_0$ is a full replete subcategory of $\mathscr{C}_1$ and that E is the inclusion functor.

A category $\mathscr{C}_1$ with the above data can be seen as a category equipped with an *ideal of null morphisms* (which is generated by $\mathscr{C}_0$, in the sense that an arrow is in the ideal if and only if it factors through an object of $\mathscr{C}_0$). A *semiexact category* is then a category equipped with a closed ideal, where any morphism has both a kernel and a cokernel relatively to this ideal.

The link between these "relative" kernels and cokernels on the one hand, and the usual limits and colimits on the other hand is as follows:

Proposition A.3.5. [47] *Let $f\colon A \to B$ be a morphism in $\mathscr{C}_1$ for which ι_B is a monomorphism. Then, the following conditions are equivalent:*

(1) an arrow $k\colon K \to A$ is a "relative" kernel of $f\colon A \to B$;

(2) there exists a unique morphism $g\colon K \to D(B)$ making the diagram

$$\begin{tikzcd} K \arrow[r, dotted, "g"] \arrow[d, "k"'] & D(B) \arrow[d, tail, "\iota_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

a pullback.

Proposition A.3.6. [47] *Let $f\colon A \to B$ be a morphism in $\mathscr{C}_1$ for which π_A is an epimorphism. Then, the following conditions are equivalent:*

(1) an arrow $q\colon B \to Q$ is a "relative" cokernel of $f\colon A \to B$;

(2) there exists a unique morphism $g\colon C(A) \to Q$ making the diagram

$$\begin{tikzcd} A \arrow[r, two heads, "\pi_A"] \arrow[d, "f"'] & C(A) \arrow[d, dotted, "g"] \\ B \arrow[r, "q"'] & Q \end{tikzcd}$$

a pushout.

Let $\mathscr{Z}$ denote the full subcategory of $\mathsf{PreOrdGrp}$ whose objects are all $\mathscr{Z}$-trivial objects. In this case, a kernel (respectively a cokernel) with respect to the ideal generated by $\mathscr{Z}$ is nothing but a $\mathscr{Z}$-kernel (respectively a $\mathscr{Z}$-cokernel). So, let us take $\mathscr{C}_0 = \mathscr{Z}$ and $\mathscr{C}_1 = \mathsf{PreOrdGrp}$ in the theory developed by Grandis, Janelidze and Márki. The functor E is then the inclusion functor and we define the two functors C and D in the following way: for any preordered group (G, P_G),

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "D"] \arrow[r, bend right, "C"'] & \mathscr{Z} \arrow[l, hook', "E"'] \end{tikzcd}$$

- $D(G, P_G) = (G, 0)$;
- $C(G, P_G) = (G/M_G, 0)$ where $M_G = \{x_1 - x_2 + \cdots + x_{n-1} - x_n \mid x_i \in P_G \ \forall 1 \leq i \leq n\}$ is the normal closure of the submonoid P_G in G.

Let us show that we then have the following chain of adjunctions: $C \dashv E \dashv D$.

Proposition A.3.7. *The functor $D \colon \mathsf{PreOrdGrp} \to \mathscr{Z}$ is a right adjoint left inverse of the inclusion functor $E \colon \mathscr{Z} \hookrightarrow \mathsf{PreOrdGrp}$.*

Proof. It is first of all clear that D is a left inverse of E. Indeed, for any $\mathscr{Z}$-trivial object $(G, 0)$, $(D \cdot E)(G, 0) = D(G, 0) = (G, 0)$.
We are now going to prove that, for any preordered group (G, P_G), the arrow $(\iota_G, \bar{\iota}_G) = (1_G, 0) \colon ED(G, P_G) = (G, 0) \to (G, P_G)$ is the (G, P_G)-component of the counit ι of the adjunction $E \dashv D$.

$$\begin{tikzcd} 0 \arrow[rr, "\bar{\iota}_G = 0"] \arrow[dd, tail] & & P_G \arrow[dd, tail] \\ & 0 \arrow[ul, dotted, "\bar{\psi} = 0"] \arrow[ur, "\bar{\phi} = 0"'] \arrow[dd, tail] & \\ G \arrow[rr, "\iota_G = 1_G"] & & G \\ & A \arrow[ul, dotted, "\psi = \phi"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

Let $(A, 0) \in \mathscr{Z}$ and consider an arrow $(\phi, \bar{\phi}): (A, 0) \to (G, P_G)$. It suffices to take $\psi = \phi$ (since ϕ is the unique morphism such that $1_G \cdot \phi = \phi$) and $\bar{\psi} = 0$ (since of course $\bar{\iota}_G \cdot 0 = 0 = \bar{\phi}$). The pair $(\psi, \bar{\psi})$ is then clearly a morphism of preordered groups, unique with the property $(\iota_G, \bar{\iota}_G) \cdot (\psi, \bar{\psi}) = (\phi, \bar{\phi})$. We conclude that D is a right adjoint for E. $\square$

Proposition A.3.8. *The functor C: $\mathsf{PreOrdGrp} \to \mathscr{Z}$ is a left adjoint left inverse of the inclusion functor E: $\mathscr{Z} \hookrightarrow \mathsf{PreOrdGrp}$.*

Proof. We first check that C is a left inverse of E. For any $(G, 0) \in \mathscr{Z}$, $M_G = 0$, so that $(C \cdot E)(G, 0) = C(G, 0) = (G/0, 0) = (G, 0)$, as desired. Let us now show that, for any preordered group (G, P_G), the (G, P_G)-component of the unit π of the adjunction $C \dashv E$ is given by the morphism $(\pi_G, \bar{\pi}_G): (G, P_G) \to (G/M_G, 0)$, where $\pi_G: G \twoheadrightarrow G/M_G$ is the canonical quotient.

$$\begin{tikzcd}
 & P_G \arrow[rr, "\bar{\pi}_G=0"] \arrow[ddl, tail] \arrow[dd, tail] \arrow[dr, "\bar{\phi}=0"'] & & 0 \arrow[dl, dotted, "\bar{\psi}=0"'] \arrow[dd, tail] \\
 & & 0 \arrow[dd, tail] & \\
M_G \arrow[r, tail, "i"'] & G \arrow[rr, two heads, "\pi_G"'] \arrow[dr, "\phi"'] & & G/M_G \arrow[dl, dotted, "\psi"] \\
 & & A &
\end{tikzcd}$$

Consider a morphism $(\phi, \bar{\phi}): (G, P_G) \to (A, 0)$ (where $(A, 0)$ is then a $\mathscr{Z}$-trivial object). For any $m \in M_G$, we can compute that $\phi(m) = 0$. Indeed, $m = x_1 - x_2 + \cdots + x_{n-1} - x_n$ for some $x_i \in P_G$ ($i \in \{1, \cdots, n\}$), which implies that

$$\begin{aligned}
\phi(m) &= \phi(x_1) - \phi(x_2) + \cdots + \phi(x_{n-1}) - \phi(x_n) \\
&= \bar{\phi}(x_1) - \bar{\phi}(x_2) + \cdots + \bar{\phi}(x_{n-1}) - \bar{\phi}(x_n) \\
&= 0
\end{aligned}$$

since $\bar{\phi} = 0$. This means that $\phi \cdot i = 0$ where $i: M_G \rightarrowtail G$ is the inclusion of M_G in G. By the universal property of cokernels, there exists a unique morphism $\psi\colon G/M_G \to A$ such that $\psi \cdot \pi_G = \phi$. By taking $\bar{\psi} = 0$, we then get a morphism of preordered groups $(\psi, \bar{\psi})\colon (G/M_G, 0) \to (A, 0)$, unique with the property $(\psi, \bar{\psi}) \cdot (\pi_G, \bar{\pi}_G) = (\phi, \bar{\phi})$. This proves that C is a left adjoint for E. □

According to Propositions A.3.5 and A.3.6, our $\mathscr{Z}$-kernels and $\mathscr{Z}$-cokernels are then actually some special pullbacks and pushouts.

A.4. Properties of the stable category $\mathsf{Mon}_{\geq 0}$ and of the functor P

Let us start with two obvious observations:

Proposition A.4.1. *The category* $\mathsf{Mon}_{\geq 0}$ *is pointed with zero object the trivial monoid* 0.

Proposition A.4.2.

(1) For any $\mathscr{Z}$-trivial object (G, P_G) in $\mathsf{PreOrdGrp}$*, $P(G, P_G) = 0$ in* $\mathsf{Mon}_{\geq 0}$.

(2) For any $\mathscr{Z}$-trivial morphism $(f, \bar{f})$ in $\mathsf{PreOrdGrp}$*, $P(f, \bar{f})$ is the zero morphism in* $\mathsf{Mon}_{\geq 0}$.

It is now useful to note that there is a torsion theory in the category $\mathsf{Mon}_{\geq 0}$. The torsion-free subcategory is the full subcategory $\mathsf{RedMon}_{\geq 0}$ of $\mathsf{Mon}_{\geq 0}$ whose objects are in addition reduced monoids, while the torsion subcategory is the one generated by the objects in $\mathsf{Mon}_{\geq 0}$ which are also groups. Notice that this latter subcategory then clearly corresponds to the category Grp of groups.

Proposition A.4.3. *The pair* $(\mathsf{Grp}, \mathsf{RedMon}_{\geq 0})$ *of full replete subcategories of* $\mathsf{Mon}_{\geq 0}$ *is a torsion theory in* $\mathsf{Mon}_{\geq 0}$.

Proof. Let us first prove that any morphism $f\colon A \to B$ in $\mathsf{Mon}_{\geq 0}$, where $A \in \mathsf{Grp}$ and $B \in \mathsf{RedMon}_{\geq 0}$, is the zero morphism. For any $x \in A$, we have that $-x \in A$ (since A is a group), so that both $f(x)$ and

$f(-x) = -f(x)$ are in B. This implies that $f(x) = 0$ since B is reduced. Let now $M \in \mathsf{Mon}_{\geq 0}$, and consider the subgroup

$$U(M) = \{x \in M \mid -x \in M\} \in \mathsf{Grp}$$

of invertible elements of M. Define also, for any $a, b \in M$, the following equivalence relation:

$$\begin{aligned} a \sim b \quad &\Leftrightarrow \quad \exists u \in U(M) \text{ such that } a = b + u \\ &\Leftrightarrow \quad \exists v \in U(M) \text{ such that } b = a + v. \end{aligned}$$

It is actually a congruence. Indeed, let $a, b, c, d \in M$ be such that $a \sim b$ and $c \sim d$. Then, there exists $u \in U(M)$ such that $a = b + u$ and there exists $u' \in U(M)$ such that $c = d + u'$. Accordingly,

$$a + c = b + u + d + u' = (b + d) + (-d + u + d + u').$$

Let us prove that $-d+u+d \in U(M)$. Since $M \in \mathsf{Mon}_{\geq 0}$, there exist $x, y \in M$ such that $u + d = d + x = y + u$. In particular, $-d + u + d = x \in M$. On the other hand, there exist $z, w \in M$ such that $-u+d = d+z = w-u$, so that $-d - u + d = z \in M$, i.e. $-(-d + u + d) \in M$. This shows that $-d + u + d \in U(M)$, and then that $-d + u + d + u' \in U(M)$, as desired. As a consequence, $a + c \sim b + d$ and $\sim$ is a congruence on M. Let us then consider the monoid $M/U(M) := M/\sim$ and let us check that it is reduced. Let $y \in M/U(M)$ such that also $-y \in M/U(M)$. Then, there exist x and z in M such that $\bar{x} = y$ and $\bar{z} = -y$ (where $\bar{x}$ denotes the equivalence class of x with respect to $\sim$). We compute that $\overline{x + z} = \bar{x} + \bar{z} = y - y = 0$, which implies that $x + z \in U(M)$. In particular, $-(x + z) \in M$. From this, we deduce that

$$-x = z - z - x = z - (x + z) \in M,$$

i.e. that $x \in U(M)$. Accordingly, $y = \bar{x} = 0$, hence $M/U(M) \in \mathsf{RedMon}_{\geq 0}$. As a conclusion, we have built a short exact sequence

$$U(M) \xrightarrow{\kappa_M} M \xrightarrow{\eta_M} M/U(M)$$

in $\mathsf{Mon}_{\geq 0}$ with $U(M) \in \mathsf{Grp}$ and $M/U(M) \in \mathsf{RedMon}_{\geq 0}$. $\square$

Proposition A.4.4. *The functor $P \colon \mathsf{PreOrdGrp} \to \mathsf{Mon}_{\geq 0}$ is a torsion theory functor.*

Proof. It is first of all clear that $P(G, P_G) \in \mathsf{Grp}$ (respectively $\in \mathsf{RedMon}_{\geq 0}$) for any $(G, P_G) \in \mathsf{Grp}(\mathsf{PreOrd})$ (respectively $\in \mathsf{ParOrdGrp}$). We next easily observe that, for any preordered group (G, P_G), if we apply the functor P to the canonical short $\mathscr{Z}$-exact sequence

$$\begin{tikzcd} N_G \arrow[r, tail] \arrow[d, tail] & P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail] & P_G/N_G \arrow[d, tail] \\ G \arrow[r, equal] & G \arrow[r, two heads, "\eta_G"'] & G/N_G \end{tikzcd}$$

associated with (G, P_G) in the pretorsion theory $(\mathsf{Grp}(\mathsf{PreOrd}), \mathsf{ParOrdGrp})$ (see Corollary A.3.3), we then get

$$N_G = U(P_G) \rightarrowtail P_G \twoheadrightarrow P_G/N_G$$

which is a short exact sequence in $\mathsf{Mon}_{\geq 0}$. $\square$

Moreover, it is easy to see that the positive cone functor $P\colon \mathsf{PreOrdGrp} \to \mathsf{Mon}_{\geq 0}$ preserves a specific form of short exact sequences.

Proposition A.4.5. *The functor $P\colon \mathsf{PreOrdGrp} \to \mathsf{Mon}_{\geq 0}$ preserves short exact sequences of the following form:*

$$\begin{tikzcd} P_G \arrow[r, equal] \arrow[d, tail] & P_G \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\ H \arrow[r, tail] & G \arrow[r, two heads] & G/H \end{tikzcd} \tag{A.1}$$

for any $(G, P_G) \in \mathsf{PreOrdGrp}$ such that H is a normal subgroup of G containing the positive cone P_G of G.

A.5. The universal property of the stable category

In this section, we prove that the torsion theory $(\mathsf{Grp}, \mathsf{RedMon}_{\geq 0})$ in $\mathsf{Mon}_{\geq 0}$ is the "best" one can associate with the pretorsion theory $(\mathsf{Grp}(\mathsf{PreOrd}), \mathsf{ParOrdGrp})$ in $\mathsf{PreOrdGrp}$, i.e. we show that it is the *universal* torsion theory associated with $(\mathsf{Grp}(\mathsf{PreOrd}), \mathsf{ParOrdGrp})$.

Theorem A.5.1. *Let $\mathscr{C}$ be a pointed category containing a torsion theory $(\mathscr{T}, \mathscr{F})$. Let $F\colon \mathsf{PreOrdGrp} \to \mathscr{C}$ be a functor satisfying the following properties:*

(1) F is a torsion theory functor;

(2) F preserves short exact sequences of the form (A.1).

Then, there exists a unique functor $\hat{F}\colon \mathsf{Mon}_{\geq 0} \to \mathscr{C}$ such that the triangle

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[rr, "P"] \arrow[dr, "F"'] & & \mathsf{Mon}_{\geq 0} \arrow[dl, dotted, "\hat{F}"] \\ & \mathscr{C} & \end{tikzcd}$$

commutes (up to isomorphism).

Proof. Let $M \in \mathsf{Mon}_{\geq 0}$. Then, $(grp(M), M) \in \mathsf{PreOrdGrp}$. Let us therefore define $\hat{F}$ on objects as follows: $\hat{F}(M) = F(grp(M), M)$ for any $M \in \mathsf{Mon}_{\geq 0}$. It now remains to prove that $F(grp(M), M) \cong F(G, M)$ for any group G such that $(G, M) \in \mathsf{PreOrdGrp}$. This will then prove that $\hat{F} \cdot P \cong F$ on objects. Suppose we have a preordered group (G, M). By the universal property of the Grothendieck group $grp(M)$, there exists a unique group morphism $\alpha\colon grp(M) \to G$ making the square

$$\begin{tikzcd} M \arrow[r, equal, "\bar{\alpha}=1_M"] \arrow[d, tail] & M \arrow[d, tail] \\ grp(M) \arrow[r, dotted, "\alpha"'] & G \end{tikzcd}$$

commute, which entails that $(\alpha, \bar{\alpha})\colon (grp(M), M) \to (G, M)$ is a morphism of preordered groups. We now observe that

$$\begin{tikzcd} M \arrow[r, equal, "\bar{\alpha}=1_M"] \arrow[d, tail] & M \arrow[r, two heads, "\bar{q}"] \arrow[d, tail] & 0 \arrow[d, tail] \\ grp(M) \arrow[r, tail, "\alpha"'] & G \arrow[r, two heads, "q"'] & G/grp(M) \end{tikzcd}$$

is a short exact sequence in $\mathsf{PreOrdGrp}$. Indeed, since the monoid M is closed under conjugation in G, its group completion $grp(M)$ is then a normal subgroup of G. By the assumption (2) on F, if we apply the

functor F to this sequence, we then obtain a short exact sequence in $\mathscr{C}$. In particular, $F(\alpha, \bar{\alpha}) = \mathsf{ker}(F(q, \bar{q}))$. Notice now that $F(q, \bar{q})$ is the zero morphism in $\mathscr{C}$. Indeed, according to the assumption (1) on the functor F, $(q, \bar{q})$ being a $\mathscr{Z}$-trivial morphism in $\mathsf{PreOrdGrp}$ is sent by F to the zero morphism. As a conclusion, $F(\alpha, \bar{\alpha})$ is the kernel of its cokernel which is 0, that is, it is an isomorphism. We deduce that $F(grp(M), M) \cong F(G, M)$.
The definition of $\hat{F} \colon \mathsf{Mon}_{\geq 0} \to \mathscr{C}$ on arrows is now easy. Let $\bar{f} \colon M \to N$ be a morphism in $\mathsf{Mon}_{\geq 0}$. Thanks to the universal property of the Grothendieck group $grp(M)$, there exists a unique group morphism $g \colon grp(M) \to grp(N)$ such that the square

$$\begin{tikzcd} M \arrow[r, "\bar{f}"] \arrow[d, tail] & N \arrow[d, tail] \\ grp(M) \arrow[r, dotted, "g"'] & grp(N) \end{tikzcd}$$

commutes, which means that $(g, \bar{f}) \colon (grp(M), M) \to (grp(N), N)$ is a morphism of preordered groups. Let us then define $\hat{F}(\bar{f})$ as follows: $\hat{F}(\bar{f}) = F(g, \bar{f})$ where g is the above induced group morphism. It remains to check that $F(g, \bar{f}) \cong F(f, \bar{f})$ for any group morphism $f \colon G \to H$ such that $(f, \bar{f})$ is a morphism of preordered groups. By the universal property of the Grothendieck groups $grp(M)$ and $grp(N)$, there exist a unique group morphism $\sigma \colon grp(M) \to G$ and a unique group morphism $\tau \colon grp(N) \to H$ such that $\sigma \cdot i_M = i_G$ and $\tau \cdot i_N = i_H$:

$$\begin{tikzcd} & M \arrow[rr, "\bar{f}"] \arrow[dl, equal] \arrow[dd, tail, "i_G"] & & N \arrow[dl, equal] \arrow[dd, tail, "i_H"] \\ M \arrow[rr, "\bar{f}"'] \arrow[dd, tail, "i_M"'] & & N \arrow[dd, tail, "i_N"] & \\ & G \arrow[rr, "f"] & & H \\ grp(M) \arrow[ur, dotted, "\sigma"] \arrow[rr, "g"'] & & grp(N). \arrow[ur, dotted, "\tau"'] & \end{tikzcd}$$

We observe that, for any element $x_1 - x_2 + \cdots + x_{n-1} - x_n$ of $grp(M)$,

$$\begin{aligned}
(f \cdot \sigma)&(x_1 - x_2 + \cdots + x_{n-1} - x_n)\\
&= f(x_1 - x_2 + \cdots + x_{n-1} - x_n)\\
&= f(x_1) - f(x_2) + \cdots + f(x_{n-1}) - f(x_n)\\
&= g(x_1) - g(x_2) + \cdots + g(x_{n-1}) - g(x_n)\\
&= g(x_1 - x_2 + \cdots + x_{n-1} - x_n)\\
&= (\tau \cdot g)(x_1 - x_2 + \cdots + x_{n-1} - x_n)
\end{aligned}$$

since f and g coincide on M, which means that $f \cdot \sigma = \tau \cdot g$. In other words, there exist two isomorphisms $F(\sigma, 1_M)$ and $F(\tau, 1_N)$ in $\mathscr{C}$ (see the first part of the proof) such that $F(f, \bar{f}) \cdot F(\sigma, 1_M) = F(\tau, 1_N) \cdot F(g, \bar{f})$, which proves that $F(g, \bar{f}) \cong F(f, \bar{f})$.
Finally, the uniqueness of $\hat{F}$ such that $\hat{F} \cdot P \cong F$ follows from the surjectivity on objects of the positive cone functor P. □

Note that the induced functor $\hat{F} \colon \mathsf{Mon}_{\geq 0} \to \mathscr{C}$ of Theorem A.5.1 is itself a torsion theory functor (as is the case for F and P):

Proposition A.5.2. *The induced functor $\hat{F} \colon \mathsf{Mon}_{\geq 0} \to \mathscr{C}$ of Theorem A.5.1 is a torsion theory functor.*

Proof. We first easily check that

- for any $M \in \mathsf{Grp}$, $\hat{F}(M) = F(grp(M), M) \in \mathscr{T}$ since $(grp(M), M) \in \mathsf{Grp}(\mathsf{PreOrd})$ and F is a torsion theory functor;
- for any $M \in \mathsf{RedMon}_{\geq 0}$, $\hat{F}(M) = F(grp(M), M) \in \mathscr{F}$ since $(grp(M), M) \in \mathsf{ParOrdGrp}$ and F is a torsion theory functor.

Consider now the canonical short exact sequence associated with any $M \in \mathsf{Mon}_{\geq 0}$ in the torsion theory $(\mathsf{Grp}, \mathsf{RedMon}_{\geq 0})$ (see Proposition A.4.3):

$$U(M) \xrightarrow{\kappa_M} M \xrightarrow{\eta_M} M/U(M). \tag{A.2}$$

Then,

$$\begin{tikzcd}
U(M) \arrow[r, tail, "\kappa_M"] \arrow[d, tail] & M \arrow[r, two heads, "\eta_M"] \arrow[d, tail] & M/U(M) \arrow[d, tail] \\
grp(M) \arrow[r, equal] & grp(M) \arrow[r, two heads, "\eta_{grp(M)}"'] & grp(M)/U(M)
\end{tikzcd} \tag{A.3}$$

is the canonical short $\mathscr{Z}$-exact sequence associated with the preordered group $(grp(M), M)$ in the pretorsion theory $(\mathsf{Grp}(\mathsf{PreOrd}), \mathsf{ParOrdGrp})$ (see Corollary A.3.3). Since F is a torsion theory functor, it sends this short $\mathscr{Z}$-exact sequence in $\mathsf{PreOrdGrp}$ to a short exact sequence in $\mathscr{C}$. Now, applying the functor $\hat{F}$ to the short exact sequence (A.2) amounts to applying the functor F to the short $\mathscr{Z}$-exact sequence (A.3), so that the short exact sequence (A.2) is sent by $\hat{F}$ to a short exact sequence in $\mathscr{C}$. $\square$

Bibliography

[1] S. A. Amitsur. *A general theory of radicals, II. Radicals in rings and bicategories.* Amer. J. Math., 76:100–125, 1954.

[2] M. Barr. *Exact categories, in: Exact categories and categories of sheaves.* Lecture Notes in Math., vol. 236, Springer-Verlag, 1971, pp. 1-120.

[3] F. Borceux. *Handbook of categorical algebra, Volume I: Basic category theory.* Encyclopedia Math. Appl., vol. 50, Cambridge Univ. Press, 1994.

[4] F. Borceux. *Handbook of categorical algebra, Volume II: Categories and structures.* Encyclopedia Math. Appl., vol. 51, Cambridge Univ. Press, 1994.

[5] F. Borceux. *A survey of semi-abelian categories.* Fields Inst. Commun., 43:27–60, 2004.

[6] F. Borceux and D. Bourn. *Mal'cev, protomodular, homological and semi-abelian categories.* Math. Appl., vol. 566, Kluwer Acad. Publ., 2004.

[7] F. Borceux, F. Campanini, and M. Gran. *The stable category of preorders in a pretopos I: General theory.* J. Pure Appl. Algebra, 226:106997, 2021.

[8] F. Borceux, F. Campanini, and M. Gran. *Pretorsion theories in lextensive categories.* Accepted in Israel Journal of Mathematics, 2022.

[9] F. Borceux, F. Campanini, and M. Gran. *The stable category of preorders in a pretopos II: the universal property.* Ann. Mat. Pura Appl., 201:2847–2869, 2022.

[10] F. Borceux, F. Campanini, M. Gran, and W. Tholen. *Groupoids and skeletal categories form a pretorsion theory in Cat.* Advances in Mathematics, 426:109110, 2023.

[11] F. Borceux and G. Janelidze. *Galois theories.* Cambridge Stud. Adv. Math., vol. 72, Cambridge Univ. Press, 2001.

[12] D. Bourn. *Normalization equivalence, kernel equivalence and affine categories.* Lecture Notes Math., vol. 1488, Springer-Verlag, 1991, pp. 43-62.

[13] D. Bourn. *Mal'cev categories and fibrations of pointed objects.* Appl. Categ. Struct., 4:307–327, 1996.

[14] D. Bourn and M. Gran. *Torsion theories in homological categories.* J. Algebra, 305:18–47, 2006.

[15] D. Bourn, N. Martins-Ferreira, A. Montoli, and M. Sobral. *Schreier split epimorphisms in monoids and in semirings.* Textos de Matemática (Série B), vol. 45, Departamento de Matemática da Universidade de Coimbra, 2013.

[16] D. Buchsbaum. *Exact categories and duality.* Trans. Amer. Math. Soc., 80:1–34, 1955.

[17] A. Carboni, G. Janelidze, G. M. Kelly, and R. Paré. *On localization and stabilization for factorization systems.* Appl. Categ. Struct., 5:1–58, 1997.

[18] C. Cassidy, M. Hébert, and G. M. Kelly. *Reflective subcategories, localizations and factorization systems.* J. Austral. Math. Soc., 38:287–329, 1985.

[19] M. M. Clementino. *Constant morphisms and constant subcategories.* Appl. Categ. Struct., 3:119–137, 1995.

[20] M. M. Clementino, D. Dikranjan, and W. Tholen. *Torsion theories and radicals in normal categories.* J. Algebra, 305:98–129, 2006.

[21] M. M. Clementino, N. Martins-Ferreira, and A. Montoli. *On the categorical behaviour of preordered groups.* J. Pure Appl. Algebra, 223:4226–4245, 2019.

[22] M. M. Clementino and A. Montoli. *On the categorical behaviour of V-groups.* J. Pure Appl. Algebra, 225:106550, 2021.

[23] S. Dickson. *A torsion theory for abelian categories.* Trans. Amer. Math. Soc., 121:223–235, 1966.

[24] M. Duckerts, T. Everaert, and M. Gran. *A description of the fundamental group in terms of commutators and closure operators.* J. Pure Appl. Algebra, 216:1837–1851, 2012.

[25] B. Eckmann and P. J. Hilton. *Group-like structures in general categories I.* Math. Ann., 145:227–255, 1962.

[26] S. Eilenberg. *Sur les transformations continues d'espaces métriques compacts.* Fund. Math., 22:292–296, 1934.

[27] S. Eilenberg and G. M. Kelly. *Closed categories.* Proc. Conf. on Categorical Algebra (La Jolla, 1965), Springer-Verlag, Berlin-Heidelberg-New York, 1966, pp. 421-562.

[28] S. Eilenberg and S. Mac Lane. *General theory of natural equivalences.* Trans. Amer. Math. Soc., vol. 58, 1945, pp. 231-294.

[29] T. Everaert and M. Gran. *Monotone-light factorisation systems and torsion theories.* Bull. Sci. Math., 137:996–1006, 2013.

[30] T. Everaert and M. Gran. *Protoadditive functors, derived torsion theories and homology.* J. Pure Appl. Algebra, 219:3629–3676, 2015.

[31] A. Facchini. *Commutative monoids, noncommutative rings and modules.* In: M. M. Clementino, A. Facchini and M. Gran (eds.) New Perspectives in Algebra, Topology and Categories, Coimbra Mathematical Texts, 2021.

[32] A. Facchini and C. Finocchiaro. *Pretorsion theories, stable category and preordered sets.* Ann. Mat. Pura Appl., 199:1073–1089, 2020.

[33] A. Facchini, C. Finocchiaro, and M. Gran. *A new Galois structure in the category of internal preorders.* Theory Appl. Categ., 35 (11):326–349, 2020.

[34] A. Facchini, C. Finocchiaro, and M. Gran. *Pretorsion theories in general categories.* J. Pure Appl. Algebra, 225:106503, 2021.

[35] L. Fajstrup, E. Goubault, and M. Raussen. *Algebraic topology and concurrency.* Theor. Comput. Sci., 357:241–278, 2006.

[36] R. Fritsch and O. Wyler. *The Schreier Refinement Theorem for categories.* Arch. Math., 22:570–572, 1971.

[37] A. Fröhlich. *Non-abelian homological algebra I, Derived functors and satelites.* Proc. London Math. Soc., (3) 11:239–275, 1961.

[38] E. Galois. *Mémoire sur les conditions de résolubilité des équations par radicaux.* Journal de mathématiques pures et appliquées, 417-433, 1830.

[39] K. Goodearl. *Von Neumann regular rings.* Monographs and studies in mathematics, no. 4, Pitman, London-San Francisco-Melbourne, 1979.

[40] M. Gran, G. Kadjo, and J. Vercruysse. *A torsion theory in the category of cocommutative Hopf algebras.* Appl. Categ. Struct., 24:269–282, 2016.

[41] M. Gran and A. Michel. *Torsion theories and coverings of preordered groups.* Algebra Univers., 82 (22), 2021.

[42] M. Gran and A. Michel. *Central extensions of preordered groups.* Preprint, 2023.

[43] M. Gran and D. Rodelo. *On the characterization of Jonsson-Tarski and of subtractive varieties.* Diagrammes, Tome S67-68:101–115, 2012.

[44] M. Gran and V. Rossi. *Torsion theories and Galois coverings of topological groups.* J. Pure Appl. Algebra, 208:135–151, 2007.

[45] M. Grandis. *Directed algebraic topology: Models of non-reversible worlds.* Cambridge Univ. Press, 2009.

[46] M. Grandis and G. Janelidze. *From torsion theories to closure operators and factorization systems.* Categ. Gen. Algebr. Struct. Appl., 12 (1):89–121, 2020.

[47] M. Grandis, G. Janelidze, and L. Márki. *Non-pointed exactness, radicals, closure operators.* J. Austral. Math. Soc., 94:348–361, 2013.

[48] A. Grothendieck. *Sur quelques points d'algèbre homologique.* Tohoku Math. J. (2), 9 (2):119–221, 1957.

[49] A. Grothendieck. *Revêtements étales et groupe fondamental (SGA1).* Lecture Notes in Math., vol. 224, Springer-Verlag, 1971.

[50] P. J. Higgins. *Groups with multiple operators.* Proc. London Math. Soc., (3) 6:366–416, 1956.

[51] D. Hofmann and P. Nora. *Hausdorff coalgebras.* Appl. Categ. Struct., 28:773–806, 2020.

[52] D. Hofmann and C. D. Reis. *Probabilistic metric spaces as enriched categories.* Fuzzy Sets Syst., 210:1–21, 2013.

[53] H. Hopf. *Fundamentalgruppe und zweite Bettische Gruppe.* Comment. Math. Helv., 14:257–309, 1942.

[54] S. A. Huq. *Commutator, nilpotency and solvability in categories.* Quart. J. Math. Oxford, (2) 19:363–389, 1968.

[55] G. Janelidze. *Pure Galois theory in categories.* J. Algebra, 132 (2):270–286, 1990.

[56] G. Janelidze. *Categorical Galois theory: revision and some recent developments.* Galois connections and applications, Math. Appl., vol. 565, Kluwer Acad. Publ., 2004, pp. 139-171.

[57] G. Janelidze. *Galois groups, abstract commutators and Hopf formula.* Appl. Categ. Struct., 16:653–668, 2008.

[58] G. Janelidze and G. M. Kelly. *Galois theory and a general notion of central extension.* J. Pure Appl. Algebra, 97:135–161, 1994.

[59] G. Janelidze, L. Márki, and W. Tholen. *Locally semisimple coverings.* J. Pure Appl. Alg., 128:281–289, 1998.

[60] G. Janelidze, L. Márki, and W. Tholen. *Semi-abelian categories.* J. Pure Appl. Alg., 168 (2-3):367–386, 2002.

[61] G. Janelidze and M. C. Pedicchio. *Pseudogroupoids and commutators.* Theory Appl. Categ., 8 (15):408–456, 2001.

[62] G. Janelidze, M. Sobral, and W. Tholen. *Beyond Barr exactness: effective descent morphisms.* In: M.C. Pedicchio, W. Tholen (eds.), Categorical Foundations, Encyclopedia Math. Appl., vol. 97, Cambridge Univ. Press, 2004, pp. 359-405.

[63] G. Janelidze and W. Tholen. *Facets of Descent, II.* Appl. Categ. Struct., 5:229–248, 1997.

[64] G. Janelidze and W. Tholen. *Characterization of torsion theories in general categories.* Contemp. Math., 431:249–256, 2007.

[65] Z. Janelidze. *Subtractive categories.* Appl. Categor. Struct., 13:343–350, 2005.

[66] Z. Janelidze. *The pointed subobject functor,* 3×3 *lemmas, and subtractivity of spans.* Theory Appl. Categ., 23 (11):221–242, 2010.

[67] K. Kearnes and R. McKenzie. *Commutator theory for relatively modular quasivarieties.* Trans. Amer. Math. Soc., 331 (2):465–502, 1992.

[68] A. G. Kurosh. *Direct decompositions in algebraic categories.* Trudy Mosk. Obshch., 8:391–412, 1959.

[69] F. W. Lawvere. *Metric spaces, generalized logic, and closed categories.* Rend. Semin. Mat. Fis. Milano 43 (1973), 135-166. Republished in:

Reprints in Theory and Applications of Categories, no. 1, 2002, pp. 1-37.

[70] S. Mac Lane. *Categories for the working mathematician.* Second ed., Grad. Texts in Math., vol. 5, Springer-Verlag, 1998.

[71] A. Magid. *The separable Galois theory of commutative rings.* Marcel Dekker, New York, 1974.

[72] S. Mantovani. *Torsion theories for crossed modules.* Invited talk at the "Workshop in category theory and algebraic topology", September 2015, Université catholique de Louvain.

[73] N. Martins-Ferreira, A. Montoli, and M. Sobral. *Semidirect products and crossed modules in monoids with operations.* J. Pure Appl. Algebra, 217:334–347, 2013.

[74] A. Michel. *Torsion theories and coverings of V-groups.* Appl. Categ. Struct., 30:659–684, 2022.

[75] A. Montoli, D. Rodelo, and T. Van der Linden. *A Galois theory for monoids.* Theory Appl. Categ., 29 (7):198–214, 2014.

[76] T. Porter. *Extensions, crossed modules and internal categories in categories of groups with operations.* Proc. Edinburgh Math. Soc., 30:373–381, 1987.

[77] G. J. Seal. *Canonical and op-canonical lax algebras.* Theory Appl. Categ., 14 (10):221–243, 2005.

[78] E. G. Shul'geifer. *On the general theory of radicals in categories.* Math. Sb. (N.S.), 51:487–500, 1960.

[79] E. G. Shul'geifer. *The Schreier Theorem in normal categories.* Sib. Mat. Z., 13:688–697, 1972.

[80] G. T. Whyburn. *Open mappings on locally compact spaces.* Mem. Amer. Math. Soc. 1, 1950.

[81] O. Wyler. *The Zassenhaus Lemma for categories.* Arch. Math., 22:561–569, 1971.

[82] J. J. Xarez. *A pretorsion theory for the category of all categories.* Cahiers Top. Géom. Diff. Catég., LXIII-1:25–34, 2022.