

Quasiparticle dynamics of symmetry-resolved entanglement after a quench: Examples of conformal field theories and free fermions

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The time evolution of the entanglement entropy is a key concept to understand the structure of a nonequilibrium quantum state. In a large class of models, such evolution can be understood in terms of a semiclassical picture of moving quasiparticles spreading the entanglement throughout the system. However, it is not yet known how the entanglement splits between the sectors of an internal local symmetry of a quantum many-body system. Here, guided by the examples of conformal field theories and free-fermion chains, we show that the quasiparticle picture can be adapted to this goal, leading to a general conjecture for the charged entropies whose Fourier transform gives the desired symmetry-resolved entanglement $S_n(q)$. We point out two physically relevant effects that should be easily observed in atomic experiments: a delay time for the onset of $S_n(q)$ which grows linearly with $|\Delta q|$ (the difference between the charge q and its mean value) and an effective equipartition when $|\Delta q|$ is much smaller than the subsystem size.

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I. INTRODUCTION

The time evolution of the entanglement in extended quantum systems starting from a nonequilibrium configuration became, in the last decade, a fundamental question with ramifications into many problems of contemporary physics, such as the equilibration and thermalization of isolated many-body systems [1–5], the emergence of thermodynamic entropy [6–9], and the effectiveness of classical computers to simulate the quantum dynamics [10–14].

These truly remarkable theoretical advances moved together with pioneering cold-atom and ion-trap experiments, with which it has been possible to directly measure the many-body entanglement of nonequilibrium quantum states [15–18]. In particular, in one of these experiments [16], it was recognized that a more refined understanding of the many-body dynamics comes from the knowledge of how entanglement splits in different symmetry sectors. Although the *symmetry resolution of the entanglement* has been the subject of intense research activity over the last few years [19–37], no results are available for the important case of a global quantum quench.

In this Letter, we start filling this gap by initiating the study of the symmetry-resolved entanglement after a global quantum quench in two paradigmatic instances of many-body systems which are free fermions and conformal field theories (CFTs). We will characterize these systems analytically and find interesting effects that are expected to hold in more general circumstances, as it can be argued based on the quasiparticle picture for the entanglement spreading [38–40]. Indeed, although we focus on rather simple models,

we expect our findings to be qualitatively the same for large classes of systems. CFT and free fermion models have been the settings in which the study of the time evolution of many entanglement-related quantities has been initiated (see, e.g., [38,41–43]) and, only after many years, generalized to more complex and realistic situations.

II. QUANTITIES OF INTEREST

We consider an extended quantum system with an internal $U(1)$ symmetry, with conserved local charge Q . We take a bipartition $A \cup \bar{A}$ such that the charge Q splits as $Q = Q_A + Q_{\bar{A}}$. Consequently, the reduced density matrix ρ_A satisfies $[\rho_A, Q_A] = 0$, implying a block diagonal form, in which each block corresponds to an eigenvalue q of Q_A , with normalized density matrix $\rho_A(q)$, i.e.,

$$\rho_A = \oplus_q \Pi_q \rho_A \Pi_q = \oplus_q [p(q) \rho_A(q)], \quad (1)$$

where Π_q is the projector on the subspace with eigenvalue q and $p(q) = \text{Tr}(\Pi_q \rho_A)$ is the probability of having q be the outcome of a measurement of Q_A . The *symmetry-resolved Rényi entropies* are the entropies of the given sector, i.e.,

$$S_n(q) = \frac{1}{1-n} \log \text{Tr}[\rho_A(q)^n], \quad (2)$$

where we use the natural base for the logarithm. For $n = 1$ they reduce to the von Neumann entropy $S_1(q) = -\text{Tr}[\rho_A(q) \log \rho_A(q)]$. The latter satisfies the remarkable sum

rule for the total entropy S_1 [16,44]:

$$S_1 = \sum_q p(q) S_1(q) - \sum_q p(q) \log[p(q)] \equiv S^c + S^n. \quad (3)$$

The two terms above have been dubbed configurational (S^c) [16,45,46] and number (S^n) entanglement entropy [16,47–49]. The former measures the average of the entanglement in the charge sectors, and the latter measures the entropy due to the fluctuations of the charge within subsystem A . In the experiment [16], the time evolution of these entropies was considered, and it was shown that, in the many-body localized phase, the number entropy grows quickly and soon saturates, irrespective of the strength of the interaction. Conversely, the configurational entropy moves up very slowly, logarithmically in time, but only after a time delay, whose precise value depends on the interaction strength and gets larger as the interaction decreases. These findings nicely explain former theoretical results for the total entanglement entropy [50,51].

The computation of the symmetry-resolved entanglement entropies from Eq. (2) requires knowledge of the spectrum of ρ_A and its resolution in Q_A , which is not easy because of the nonlocal nature of the projector Π_q . A more feasible path [20,21] is based on the computation of the *charged moments*

$$Z_n(\alpha) \equiv \text{Tr}[\rho_A^n e^{i\alpha Q_A}], \quad (4)$$

whose Fourier transform

$$\mathcal{Z}_n(q) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-iq\alpha} Z_n(\alpha) \equiv \text{Tr}[\Pi_q \rho_A^n] \quad (5)$$

readily provides the symmetry-resolved quantities (2) as

$$S_n(q) = \frac{1}{1-n} \log \left[\frac{\mathcal{Z}_n(q)}{\mathcal{Z}_1(q)^n} \right]. \quad (6)$$

The probability $p(q)$ is just $p(q) = \mathcal{Z}_1(q)$.

III. FREE FERMIONS

We study the evolution of the symmetry-resolved Rényi entropies in the tight-binding model with the Hamiltonian

$$H = \sum_{i=1}^L (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i), \quad (7)$$

where ladder operators satisfy anticommutation relations $\{c_i, c_j^\dagger\} = \delta_{i,j}$ and $\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$. The conserved charge is the fermion number $Q = \sum_j c_j^\dagger c_j$. The Jordan-Wigner transformation maps the model into the XX spin chain. We are interested in the entanglement of a block of ℓ consecutive sites in an infinite system. The reduced density matrix is obtained from the $\ell \times \ell$ matrix $C_A = \langle c_x^\dagger c_{x'} \rangle$ formed by the correlations with $x, x' \in A$ [52,53]. Using standard algebra of Gaussian operators, the charged moments $Z_n(\alpha)$ can be written as [20]

$$\log Z_n(\alpha) = \text{Tr} \log[(C_A)^n e^{i\alpha} + (1 - C_A)^n]. \quad (8)$$

We begin our study with the quench from the Néel state $|N\rangle = \prod_{j=1}^{L/2} c_{2j}^\dagger |0\rangle$ (in spin language $|N\rangle \equiv |\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\cdots\rangle$). We choose this initial state because it is the one engineered in most of the experiments [16,17] and it is simple enough to

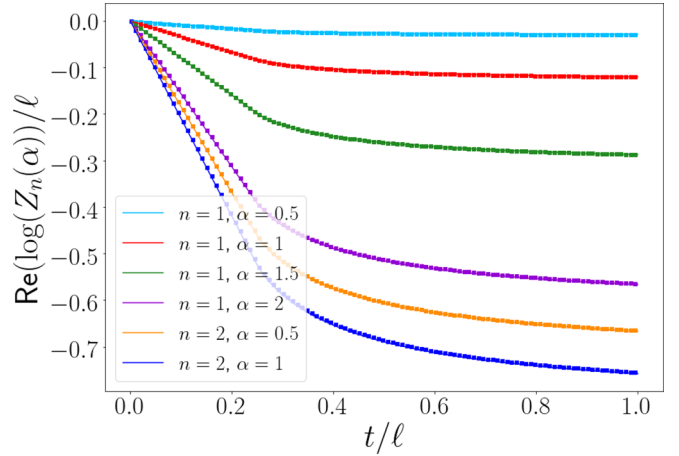


FIG. 1. The time evolution of the charged moments $Z_n(\alpha)$ after a quench from the Néel state in the free-fermion model (7). We fix $\ell = 120$ and plot the real part of $\log(Z_n(\alpha))$ as a function of t/ℓ for several values of n and α . Numerical data (symbols) perfectly match the analytic prediction (10) (solid lines).

allow full analytic computations, serving as guidance for the general case. The correlation function is (see, e.g., [54])

$$[C(t)]_{x,x'} = \frac{\delta_{x,x'}}{2} + \frac{(-1)^{x'}}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(x-x') + 4it \cos k}. \quad (9)$$

For simplicity, we work with even ℓ . The calculation of $Z_n(\alpha)$ in Eq. (8) with the correlation matrix above can be performed in the space-time scaling limit, i.e., $t, \ell \rightarrow \infty$ with a finite ratio, and it proceeds in full analogy to the case $\alpha = 0$ presented in Ref. [41]. We expand the logarithm in Eq. (8) in powers of C_A to rewrite $\log Z_n(\alpha)$ as a series in $\text{Tr}(C_A)^m$. These moments were already calculated by a multidimensional stationary phase technique [41]. The resulting power series can be summed up to obtain

$$Z_n(\alpha) = e^{i\ell \frac{\alpha}{2}} \left(\frac{\cos \frac{\alpha}{2}}{2^{n-1}} \right)^{\mathcal{J}}, \quad (10)$$

where

$$\mathcal{J} = \int \frac{dk}{2\pi} \min[2v_k t, \ell], \quad (11)$$

with $v_k = 2|\sin k|$. In Fig. 1 we compare this analytical prediction with *ab initio* computations, finding perfect agreement.

The symmetry-resolved moments for $q = \langle Q_A \rangle + \Delta q$, with $\langle Q_A \rangle = \ell/2$, are obtained by plugging Eq. (10) into (5) to get

$$\mathcal{Z}_n(q) = 2^{(1-n)\mathcal{J}} \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \left(\cos \frac{\alpha}{2} \right)^{\mathcal{J}} e^{-i\alpha \Delta q}. \quad (12)$$

This integral can be evaluated analytically [55], but for our aim it is convenient to use the saddle point method, which holds for large $\mathcal{J}$. We also assume $|\Delta q| \propto \ell$. Since the dependence on n in Eq. (12) is trivial, we focus on $\mathcal{Z}_1(q)$:

$$\mathcal{Z}_1(q) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{\ell h(\alpha)}, \quad h(\alpha) = -i\alpha \frac{\Delta q}{\ell} + \frac{\mathcal{J}}{\ell} \log \cos \frac{\alpha}{2}. \quad (13)$$

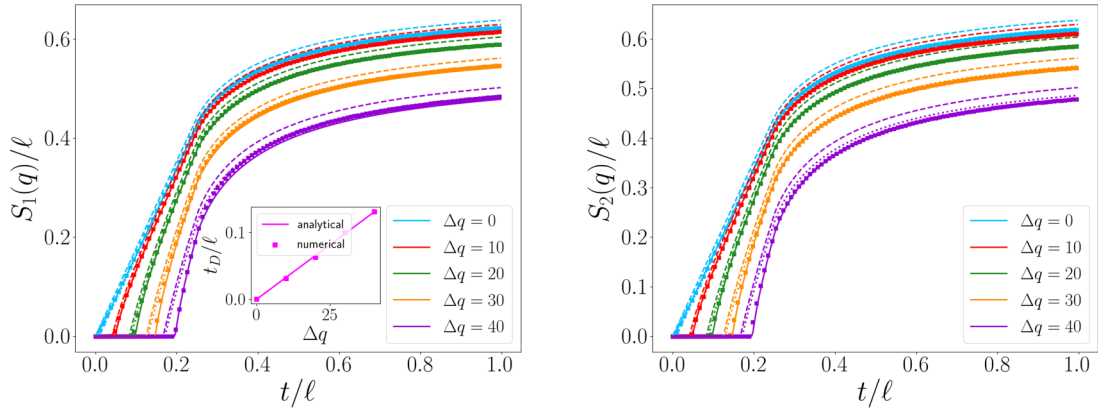


FIG. 2. Time evolution of the symmetry-resolved entanglement entropies $S_n(q)$ after a quench from the Néel state in the free-fermion model (7). The symbols are the exact numerical results for $\ell = 160$ and various Δq . The solid line is our prediction (18) with (16) that perfectly matches the numerical data. Notice the delay $\propto \Delta q$ that is well captured by our prediction. The dashed line is the expansion for large ℓ and small Δq , while the dotted one is the same with logarithmic correction. In the inset, we report the numerically calculated delays for $\ell = 240$, obtained as the time when $S_1(q)/\ell = 0.007$ and the analytic prediction $t_D = \pi|\Delta q|/4$.

The saddle point is

$$\alpha^* = -2i \operatorname{arctanh}\left(\frac{2\Delta q}{\mathcal{J}}\right) = i \log\left(\frac{\mathcal{J} - 2\Delta q}{\mathcal{J} + 2\Delta q}\right), \quad (14)$$

which is purely imaginary only for $\mathcal{J} > 2|\Delta q|$, where we can deform the contour of integration to pass through α^* while staying in the region of analyticity of the integrand. Conversely, for $\mathcal{J} < 2|\Delta q|$, α^* acquires a real part that leads to a nonzero imaginary part of $h(\alpha^*)$, making $\mathcal{Z}_1(q)$ quickly oscillating in ℓ around zero, a value to which it averages for all the relevant physics. This is one of the main results of our Letter: the symmetry-resolved entanglement entropies start only after a delay time t_D which grows linearly with $|\Delta q|$. In fact, the equation $\mathcal{J}(t_D) = 2|\Delta q|$ reads (as long as $2v_M t_D < \ell$ self-consistently and $v_M \equiv \max v_k = 2$)

$$4t_D \int_{-\pi}^{\pi} \frac{dk}{2\pi} |\sin k| = 2|\Delta q| \quad \Rightarrow \quad t_D = \pi \frac{|\Delta q|}{4}. \quad (15)$$

Instead, for $t > t_D$ (i.e., $\mathcal{J} > 2|\Delta q|$), we have

$$\mathcal{Z}_1(q) \approx e^{\ell h(\alpha^*)} \sqrt{\frac{1}{2\pi \ell |h''(\alpha^*)|}}, \quad (16)$$

which for large ℓ (i.e., large $\mathcal{J}$) is

$$\begin{aligned} \log \mathcal{Z}_1(q) = & -\left(\frac{\mathcal{J}}{2} + |\Delta q|\right) \log\left(1 + \frac{2|\Delta q|}{\mathcal{J}}\right) \\ & -\left(\frac{\mathcal{J}}{2} - |\Delta q|\right) \log\left(1 - \frac{2|\Delta q|}{\mathcal{J}}\right). \end{aligned} \quad (17)$$

From Eq. (6), we can finally get the symmetry-resolved entropies as

$$S_n(q) = \mathcal{J} \log 2 + \log \mathcal{Z}_1(q). \quad (18)$$

All these results are also found taking the large- ℓ limit of the exact integral (10) [55], but the saddle point approach remains valid when $Z_n(\alpha)$ is not as simple as Eq. (10). Interestingly, $S_n(q)$ does not depend on n . Plugging Eq. (16) into the above, one obtains the curves reported as solid lines in Fig. 2 that perfectly match the numerical data for $\ell = 160$ and $n = 1, 2$.

For $|\Delta q| \ll \mathcal{J}$ we have

$$S_n(q) = \mathcal{J} \left[\log 2 - 2 \left(\frac{|\Delta q|}{\mathcal{J}} \right)^2 \right], \quad (19)$$

reported as dashed lines in Fig. 2. The deviations observed for small $|\Delta q|$ are due to the square-root factor in Eq. (16) that leads to a logarithmic correction to $S_n(q)$, negligible in the limit of large ℓ . This correction has been included as the dotted lines, which match the data well as one moves away from t_D to make $\mathcal{J}$ significantly larger than $|\Delta q|$. Equation (19) is another main result: for small $|\Delta q|$ there is an effective equipartition of entanglement [21] with violations of order $(\Delta q)^2/\ell$.

To understand what happens for a more general initial state, we move to the dimer state, i.e., a collection of neighbor singlets $|D\rangle = \bigotimes_j [|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]_{2j-1, 2j}$. Following the same logic as before, with the correlation function, e.g., in Ref. [56], one obtains after long, but simple, algebra

$$\log Z_n(\alpha) = i\ell \frac{\alpha}{2} + \int \frac{dk}{2\pi} h_{n,\alpha}(n_k) \min[2v_k t, \ell], \quad (20)$$

with $n_k = \frac{1+\cos k}{2}$, i.e., the mode occupation of the stationary state, and

$$h_{n,\alpha}(z) = \operatorname{Re}\left\{ \log \left[z^n e^{i\frac{\alpha}{2}} + (1-z)^n e^{-i\frac{\alpha}{2}} \right] \right\}. \quad (21)$$

The main difference compared to the Néel quench is the function $h_{n,\alpha}(n_k)$, which is not constant. This modification makes the computation of $Z_n(q)$ not feasible analytically. However, we can infer the validity of the two main physical results, namely, the presence of a time delay $t_D \propto |\Delta q|$ and the effective equipartition for small $|\Delta q|$. Indeed, writing $Z_n(q) \propto \int d\alpha e^{\ell g_n(\alpha)}$, the saddle point is given by $g'_n(\alpha^*) = 0 = -i\Delta q + it \int \frac{dk}{2\pi} (\partial_{i\alpha} h_{n,\alpha}) 2v_k$. Now, for purely imaginary α and arbitrary n , this last integral is a monotonic decreasing function of $i\alpha$ going from $4/\pi$ at $\alpha = -i\infty$ to $-4/\pi$ at $i\infty$. Hence, the saddle point equation admits a purely imaginary solution only for $t > t_D = |\Delta q|\pi/4$, which is the same delay time as for the Néel state. It is straightforward to realize that

the delay time is the same for any n_k in Eq. (20) as long as $n_k \neq 0, 1$ except at isolated points.

Finally, proving equipartition for small $|\Delta q|$ is easy. It is enough to approximate $Z_n(\alpha)$ at the Gaussian level as $Z_n(\alpha) \simeq Z_n(0)e^{-\mathcal{J}_n\alpha^2}$ to get immediately

$$S_n(q) = S_n - \frac{\Delta q^2}{4(1-n)} \left\{ \frac{1}{\mathcal{J}_n} - \frac{n}{\mathcal{J}_1} \right\}, \quad (22)$$

with S_n being the total entropy. It is slightly more complicated than Eq. (19) but physically equivalent.

IV. CONFORMAL FIELD THEORY

The CFT approach to global quantum quenches was developed in Refs. [38,57,58]. The main idea is that the correlation functions at time t , following a quantum quench from a low-entangled state $|\psi_0\rangle$, can be mapped to the path integral in a strip in Euclidean time of width $2\tau_0$ in which the operators are inserted at τ , which must be analytically continued to $\tau \rightarrow \tau_0 + it$. The variable τ_0 is an appropriate extrapolation length which, in some sense [57], measures the distance of the initial state from an ideal conformally invariant boundary state, and hence, the results are valid only for times t and separations ℓ much larger than τ_0 . The total Rényi entanglement entropies within this approach were obtained [38] by exploiting the fact that they are related to two-point correlators of properly defined twist fields. The very same ideas apply to the charged moments of a compact boson φ (which includes free fermions), with conserved charge $Q_A = \int_A dx \partial_x \varphi$. Indeed, the moments $Z_n(\alpha)$ are correlations of composite twist fields $\mathcal{T}_{n,\alpha}$ [20] of dimension $\Delta_{n,\alpha} = \frac{1}{12}(n - \frac{1}{n}) + \frac{K}{n} \frac{\alpha^2}{4\pi^2}$, with K being related to the compactification radius. Hence, the expectation value in the strip is the same as the one for the total moments $Z_n(\alpha = 0)$ [38] with the change in the conformal dimension. Thus, after the analytic continuation, we have

$$Z_n(\alpha) = c_{n,\alpha} \left[\frac{\pi^2}{4\tau_0^2} \frac{\cosh\left(\frac{\pi\ell}{2\tau_0}\right) + \cosh\left(\frac{\pi vt}{\tau_0}\right)}{2 \sinh^2\left(\frac{\pi\ell}{4\tau_0}\right) \cosh^2\left(\frac{\pi vt}{2\tau_0}\right)} \right]^{2\Delta_{n,\alpha}}, \quad (23)$$

with $c_{n,\alpha}$ being a normalization constant. For large ℓ/τ_0 and t/τ_0 , this simplifies to

$$\log Z_n(\alpha) = \log Z_n(0) - \frac{K\alpha^2}{4\pi n} \frac{\min[2vt, \ell]}{\tau_0}. \quad (24)$$

This result resembles free-fermion ones (10) and (20) with $\langle Q_A \rangle = 0$ and with the difference that there is a single velocity v . As a consequence of this single velocity and of the too simple dependence on α and n in Eq. (24), the symmetry-resolved moments and entropies do not show the time delay t_D in Eq. (15), and the functional form in t, ℓ is different. The conclusion is that while CFT captures *universal aspects* of the charged entropy, it expectedly fails to reproduce nonuniversal ones for the symmetry-resolved quantities such as the time delay t_D . Instead, the entanglement equipartition trivially follows from the Gaussian form of Eq. (24).

V. THE GENERAL QUASIPARTICLE INTERPRETATION

In integrable systems, the time evolution of the entanglement entropy can be understood in terms of the quasiparticle

picture [38,39], in which quasiparticle excitations of opposite momenta are produced in pairs at time $t = 0$ and then move ballistically through the system, spreading entanglement and correlations. Assuming that the contributions of pairs of quasiparticles of momentum $\pm k$ to the charged entropy can be encoded in a single factor $f_{n,\alpha}(k)$, it naturally follows that

$$\log Z_n(\alpha) = i\langle Q_A \rangle \alpha + \int \frac{dk}{2\pi} f_{n,\alpha}(k) \min[2v_k t, \ell], \quad (25)$$

where v_k is the velocity of the entangling quasiparticles in the stationary state [39,59], $\langle Q_A \rangle$ is the conserved mean charge within the subsystem, and the factor $\min[2v_k t, \ell]$ just comes from counting which pairs are shared between A and $\bar{A}$ [38]. This form agrees and generalizes previous results for free fermions (20) and CFTs (24). Equation (25) is written for a single species of quasiparticles, but the generalization to multiple ones is straightforward since it just requires summing over all of them, like for the total entanglement [39].

Independent of the precise form of $f_{n,\alpha}(k)$, we can infer the main general features of the symmetry-resolved entanglement from Eq. (25). The first one is the existence of the delay time t_D : in the saddle point calculation, t_D is nonzero and proportional to $|\Delta q|$ if the domain of the derivative with respect to $i\alpha$ of the integral on the right-hand side of Eq. (25) is finite for purely imaginary α . The precise value of t_D depends on the details of the function $f_{n,\alpha}(k)$ and thus on the specific quench. In the quasiparticle picture, this delay can be physically understood as the time needed to change the charge of an amount $|\Delta q|$ within subsystem A ; for example, having in mind a spin chain, this is the time to turn $|\Delta q|$ spins by local spin flips. The other general result is the effective equipartition for small $|\Delta q|$ that follows from expanding Eq. (25) at quadratic order, leading immediately to Eq. (22).

Having understood the main qualitative features of the symmetry-resolved entanglement, all the quantitative parts are encoded in the function $f_{n,\alpha}(k)$. For free systems, either we can exactly solve the quench (as we did above), or we can infer it from knowledge of the stationary state, conjecturing $f_{n,\alpha}(k) = h_{n,\alpha}(n_k)$, with n_k being the mode distribution in the stationary state [which is $n_k = 1/2$ for the Néel initial state and $n_k = (1 + \cos k)/2$ for the dimer state, matching all previous results]. Anyhow, this last conjecture, although very reasonable, is not obvious at all; for example, it is known [60] that the full counting statistics of the nonconserved order parameter in the Ising model has contributions that are not captured by an equivalent of Eq. (25), and similar considerations have been drawn for the work statistics [61]. For genuinely interacting integrable models, for the total Rényi entropies [i.e., $Z_n(0)$] there are well-known problems to reconstruct the time evolution [62], which can be circumvented close to $n = 1$ [39]. The generalization to the charged entropy is in progress.

VI. CONCLUSIONS

In this Letter, we initiated the study of symmetry-resolved entanglement after a quantum quench. Based on results from free fermions and CFTs, we conjectured a general formula (25) for the charged entropy which is expected to hold, within

the quasiparticle picture, for arbitrary integrable models. From this general form two main physical results follow: (i) the symmetry-resolved entanglement with charge $|\Delta q|$ starts evolving only after a (calculable) time delay proportional to $|\Delta q|$; (ii) for small $|\Delta q|$ there is an effective equipartition of entanglement broken at order $\Delta q^2/\ell$.

These findings, however, are the tip of an iceberg with ramifications into many branches of the quench dynamics. One important aspect is that our results allow us to tackle (for free fermions quantitatively) more complicated entanglement measures such as mutual information and negativity [54,63]. Within the quasiparticle picture, the same remains true for the charged and symmetry-resolved quantities. Also, it is natural to wonder how to adapt the results to inhomogeneous initial states [64–66]. Finally, it is a must to understand whether

some of the techniques developed for nonintegrable models [67–78] apply to the symmetry-resolved quantities and whether our main physical findings, time delay and equipartition, are robust.

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