ORDERED SETS IN HOMOLOGICAL ALGEBRA

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1. Introduction

Historically, homological algebra grew out of algebraic topology, where one attempts to simplify and solve problems involving geometric structures by associating to them algebraic ones. Homological algebra then gives the calculus for extracting the required information from the algebraic objects. Today, the subject also has many applications in other areas of mathematics apart from algebraic topology, and so one may want to begin studying homological algebra without necessarily going through its geometric background. This is precisely what we will do in these notes, which attempt to give a quick introduction to basic concepts and constructions of homological algebra. Our approach here is inspired by a recent work of Marco Grandis (see the two monographs [1, 2] for a complete exposition of his theory), which bases the foundation of homological algebra on an interplay between Galois connections between lattices of "substructures", and "structure preserving maps" between the "structures". In fact, we slightly generalise the theory of Marco Grandis to cover further examples where homological algebra has been applied/developed. In this sense, our approach is new.[1]

2. Exact sequences

Homological algebra begins with the notion of an *exact sequence* of mathematical structures. Our approach to exact sequences will be "fully general" thanks to the fact that it makes use of ordered sets and Galois connections between them rather than ad-hoc definitions in terms of elements of certain mathematical structures.

We work in the context of a *category* equipped with its *subobject bifibration.* We shall see that certain simple conditions on the *subobjects* of the objects we consider suffice to sketch a general picture of basic homological algebra, including concepts such as exact sequence, chain complex, and homology of chain complexes, and results as for instance the basic diagram lemmas, including the 3×3 Lemma and the Snake Lemma.

2.1. **Categories.** With the exception of certain exercises, no prior knowledge of category theory is necessary to understand these lectures. It suffices to naively think of a category as just a collection of mathematical structures of a certain type, which we call **objects**, with the corresponding structure-preserving maps between them, called **morphisms**. There is a composition of morphisms (the "usual" composition of structure-preserving maps) which is associative and any object admits an identity morphism which acts as a left and right identity for the composition.

We shall be mostly interested in the following categories:

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[1]This aspect of these notes is part of an ongoing joint research project with Zurab Janelidze, and is along the lines of [3], although homological algebra is not explicitly treated there.

(a) modules over a ring R, and module morphisms between them;
(b) vector spaces over a field of scalars, e.g. the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers, and linear maps between them
(this is actually the same as (a) in the special case where the ring R there is a field, i.e. every nonzero element has a multiplicative inverse);
(c) abelian groups, and group homomorphisms between them
(this is (a) for $R = \mathbb{Z}$);
(d) non-abelian groups, and group homomorphisms between them.

In fact, the approach sketched here is general enough to include many other types of mathematical structure. To name a few: non-unitary rings, associative (non-unitary) algebras, Lie algebras, crossed modules, topological groups, loops, and sheaves of abelian groups. And even when the theory is not fully applicable, it is in any case worth finding out what the concepts mean for your favourite mathematical structures.

2.2. **The subobject bifibration.** Given an object X we shall consider the ordered set $\mathrm{Sub}(X)$ of its *subobjects*, ordered by *inclusion of subobjects*. Once again, instead of defining what subobjects (or inclusion of subobjects) are, we consider them abstractly, having in mind that for familiar mathematical structures we can take them to be the "substructures" (and "inclusion of substructures") in the *usual* sense.

A structure-preserving map between mathematical structures normally induces a Galois connection between the ordered sets of substructures of these structures. Similarly, in our abstract treatment we assume that a morphism $f\colon X \to Y$ between two objects X and Y, is given together with a Galois connection

$$\mathrm{Sub}(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Sub}(Y)$$

where f_* is called the **direct-image map**, and f^* is called the **inverse-image map**. We assume that the process of assigning f_* to f is functorial, so $(1_X)_* = 1_{\mathrm{Sub}(X)}$ and $(g \circ f)_* = g_* \circ f_*$, given any morphism $g\colon Y \to Z$. This determines the respective inverse-image maps uniquely. This structure on our category—the structure which assigns to an object its ordered set of subobjects, and to a morphism f the induced Galois connection (f_*, f^*)—will be called the **subobject bifibration**.

In this section we shall assume that the subobject bifibration satisfies two conditions. The first condition **(SB 1)** demands that for any object X, the ordered set $\mathrm{Sub}(X)$ is actually a bounded lattice, i.e. there is a smallest subobject $0 = \bot$ and a largest subobject $1 = \top$, and any two subobjects x and y admit a meet $x \wedge y$ and a join $x \vee y$. The second condition (SB 2) will be introduced in 2.10 below.

Exercise 2.3. *What are subobjects, their meets and joins, and their direct and inverse images for your favourite mathematical structures? Does* (SB 1) *hold?*

2.4. **Injections and surjections.** A morphism $g\colon Y \to Z$ is called an **injection** when g_* is injective. A morphism $f\colon X \to Y$ is a **surjection** when f_* is surjective.

Exercise 2.5. *What are the injections and surjections, in the above sense, in the setting of your favourite mathematical structures?*

Exercise 2.6. *What if above we ask that g^* is surjective? Or that f^* is injective? (Answering this question is needed to see that the concept of injection is formally dual to the concept of surjection.)*

Exercise 2.7. *Show that a composite of two injections is still an injection.*

Exercise 2.8. *A* **split monomorphism** *or* **section** *is a morphism $s\colon B \to E$ for which there exists a morphism $p\colon E \to B$ (a* **split epimorphism** *or* **retraction***) such that $p \circ s = 1_B$. Prove that*

(a) *any split monomorphism is an injection;*
(b) *any injection of vector spaces is a split monomorphism;*
(c) *there exist injections of abelian groups which need not be split monomorphisms.*

Verify the dual properties for split epimorphisms.

2.9. **Image and coimage.** The **image** of a morphism $f\colon X \to Y$ is the subobject $f_*(1)$ of Y. The **coimage** of f is the subobject $f^*(0)$ of X.

The coimage of f *contains* all subobjects x of X which are sent to 0 by f, since $x \leqslant f^*(0)$ is equivalent to $f_*(x) \leqslant 0$. Dually, the image of f is contained in all subobjects whose inverse image is all of X, because $f_*(1) \leqslant y$ if and only if $1 \leqslant f^*(y)$.

If a morphism $g\colon Y \to Z$ is an injection then it has a trivial coimage: $g^*(0) = 0$. Dually, a surjection $f\colon X \to Y$ has all of Y for its image: $f_*(1) = 1$. Indeed, if f is a surjection then $f_*(x) = 1$ for some subobject x of X. Since $x \leqslant 1$ this shows that $f_*(1) = 1$.

2.10. **The cartesianness condition.** It would be very nice to have a converse for these properties so that injections could be characterised as those morphisms with a trivial coimage, and likewise for the surjections. Unfortunately this is not true in general, as the example of pointed sets shows. The second condition (SB 2) mentioned above in 2.2 repairs this issue: any subobject which is contained in the image of a morphism should actually be the image of something and, dually, every subobject which contains the coimage should be the inverse image of something. More precisely, **(SB 2)** holds if and only if the subobject bifibration is **cartesian**:

$$
\begin{aligned}
y \leqslant f_*(1) &\quad\Rightarrow\quad y = f_*f^*(y)\\
f^*(0) \leqslant x &\quad\Rightarrow\quad f^*f_*(x) = x
\end{aligned}
$$

From now on we work in a cartesian setting.

Proposition 2.11. For any morphism $g\colon Y \to Z$, the following are equivalent:

(i) g is an injection;
(ii) $g^*(0) = 0$;
(iii) $g^* \circ g_* = 1_{\mathrm{Sub}(Y)}$;
(iv) (g_*, g^*) is an *embedding-projection pair.*

Dually, for any $f\colon X \to Y$, the following are equivalent:

(v) f is a surjection;
(vi) $f_*(1) = 1$;
(vii) $f_* \circ f^* = 1_{\mathrm{Sub}(Y)}$;
(viii) (f^*, f_*) is an *embedding-closure pair.*

Proof. We already explained that (i) $\Rightarrow$ (ii). If $g^*(0) = 0$ then $g^* \circ g_* = 1_{\mathrm{Sub}(Y)}$, because any y in Y is larger than $g^*(0)$, so $g^*g_*(y) = y$. It follows that (ii) implies (iii). Conditions (iii) and (iv) are equivalent by definition. (iii) $\Rightarrow$ (i) since

$$
g_*(y) = g_*(y') \quad\Rightarrow\quad g^*g_*(y) = g^*g_*(y') \quad\Rightarrow\quad y = y'. \qquad \square
$$

Exercise 2.12. *Complete the proof of Proposition 2.11.*

Exercise 2.13. *Verify that all settings* (a) *to* (d) *above are cartesian. What happens in other settings you know?*

Exercise 2.14. *We consider the category of sets with a chosen subset: an object is a pair (X, A) where $A \subseteq X$, and a morphism $f\colon (X, A) \to (Y, B)$ between such is a function $f\colon X \to Y$ for which $f(A) \subseteq B$. In this context a subobject of an* *object* (X, A) *is* defined *to be a pair* (S, A) *such that* $A \subseteq S \subseteq X$*. Does this setting satisfy* (SB 1) *and* (SB 2)*?*

Exercise 2.15. *Extend the previous exercise to topological spaces with a chosen subspace.*

Exercise 2.16. *What about the category of bounded lattices and Galois connections itself?*

2.17. **Null morphisms and zero objects.** We say that $f\colon X \to Y$ is **null** when the image of f is zero, so $f_*(1) = 0$.

An object O is called a **zero object** when the identity morphism $1_O\colon O \to O$ is null, so $0 = (1_O)_*(1) = 1$ in O. Hence a zero object only has one subobject. We usually write 0 for such an O.

Any morphism that factors through a zero object is null: given a composite

$$X \xrightarrow{f} 0 \xrightarrow{g} Z$$

we have $(g \circ f)_*(1) = g_* f_*(1) = g_*(0) = 0$.

Exercise 2.18. *Show that in the settings* (a) *to* (d) *considered above there is always a unique zero object, and a morphism is null if and only if it factors through* 0*.*

2.19. **Exact sequences.** Consider a composable pair of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Note that the composite $g \circ f$ is null (so $g_* f_*(1) \leqslant 0$) if and only if $f_*(1) \leqslant g^*(0)$: the image of f is contained in the coimage of g. By asking that this inequality is an equality we obtain the concept of **exact sequence**, any sequence of morphisms as above such that the image of the first coincides with the coimage of the second.

In this situation, any subobject y of Y which is sent to 0 by g is the image of a subobject x of X through f (we may take $x = f^*(y)$); dually, if y contains the image of X, then it is the inverse image of some subobject z of Z (namely of $g_*(y)$).

Before giving any concrete examples we first treat two important special cases.

2.20. **Injections and surjections revisited.** Injectivity and surjectivity can be expressed in terms of exact sequences as follows.

$$0 \xrightarrow{f} Y \xrightarrow{g} Z \qquad\qquad X \xrightarrow{f} Y \xrightarrow{g} 0$$

If the sequence on the left is exact, this means that g has trivial coimage: $g^*(0) = f_*(1) = f_*(0) = 0$. So g is an injection by Proposition 2.11. Conversely, the equality $f_*(1) = g^*(0)$ follows form injectivity of g applied to

$$g_* f_*(1) = g_* f_*(0) = 0 = g_* g^*(0).$$

Exercise 2.21. *Check that the sequence on the right is exact if and only if f is a surjection.*

A slight inconvenience here is that it depends on the given context whether or not a given morphism $g\colon Y \to Z$ can be written as in the above sequence on the left: the morphism $f\colon 0 \to Y$ may not exist. Unless it always does, we cannot characterise injections as those g which may appear in an exact sequence as above. Such issues involving the construction of new arrows will be dealt with in Section 3.

Exercise 2.22. *Show that in the settings* (a) *to* (d) *considered above there the unique zero object is such that it admits a unique morphism to and from any object, so that any morphism may be seen as one of the morphisms in the above sequences.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K' \arrow[r, "l"] \arrow[d, "c"'] & X' \arrow[r, "g"] \arrow[d, "a"'] & Y' \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & K \arrow[r, "k"'] \arrow[d, "z"'] & X \arrow[r, "f"'] \arrow[d, "x"'] & Y \arrow[r] \arrow[d, "y"] & 0 \\
0 \arrow[r] & K'' \arrow[r, "m"'] \arrow[d] & X'' \arrow[r, "h"'] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

FIGURE 1. Diagram of the 3×3 Lemma

Definition 2.23. A sequence of morphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots \qquad (\star)$$

is called **exact at** C_n for some n when the sequence

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

is exact. The sequence $(\star)$ is called **exact** when any two successive morphisms in it form an exact sequence.

A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 0.$$

Note that the given sequence is a short exact sequence precisely when k is an injection, f is a surjection, and the image of k is the coimage of f.

Exercise 2.24. *What if K or Y is zero? What if X is zero?*

Exercise 2.25. *What are the zero objects and the short exact sequences in Exercise 2.14?*

2.26. **Diagram chasing: the 3×3 Lemma.** Given a diagram containing several sequences, in many cases the exactness of some of them implies that other sequences in the diagram will also be exact. To deal with such problems, one uses a standard technique called **diagram chasing**. As an illustration we prove the 3×3 **Lemma**:

Theorem 2.27. Given a diagram such as in Figure 1 in which the composite of any two successive arrows is null, if the horizontal rows are exact as well as two out of three of the vertical columns, then also the third column is exact.

Proof. We shall assume that the first and the second column are exact and prove exactness of the third. The proofs of the other cases are either similar or dual. We shall also only prove exactness at Y' and leave the other cases as an exercise. This goes as follows:

$$\begin{aligned}
b^*(0) &\overset{\text{(a)}}{=} g_*g^*b^*(0) = g_*a^*f^*(0) \overset{\text{(b)}}{=} g_*a^*k_*(1) \overset{\text{(c)}}{=} g_*a^*(k_*(1) \wedge a_*(1)) \\
&\overset{\text{(d)}}{=} g_*a^*(k_*(1) \wedge x^*(0)) \overset{\text{(e)}}{=} g_*a^*k_*k^*(k_*(1) \wedge x^*(0)) \\
&\overset{\text{(f)}}{=} g_*a^*k_*k^*x^*(0) = g_*a^*k_*z^*m^*(0) \overset{\text{(g)}}{=} g_*a^*k_*z^*(0) \overset{\text{(h)}}{=} g_*a^*k_*c_*(1) \\
&= g_*a^*a_*l_*(1) \overset{\text{(i)}}{=} g_*l_*(1) \overset{\text{(j)}}{=} g_*g^*(0) \overset{\text{(k)}}{=} 0,
\end{aligned}$$

because

(a) since g is a surjection;
(b) by (horizontal) exactness at X;
(c) by Lemma 2.28;
(d) by (vertical) exactness at X;
(e) by cartesianness;
(f) by Lemma 2.28;
(g) since m is an injection;
(h) by (vertical) exactness at K;
(i) since a is an injection;
(j) by (horizontal) exactness at X';
(k) since g is a surjection. □

Lemma 2.28. For any injection $g\colon Y \to Z$ we have $g^*(z) = g^*(z \wedge g_*(1))$.

Proof. $g^*(z \wedge g_*(1)) = g^*(z) \wedge g^* g_*(1) = g^*(z) \wedge 1 = g^*(z)$ since inverse images preserve meets and g is an injection. □

Exercise 2.29. *What is the corresponding lemma for surjections?*

Exercise 2.30. *Complete the proof of the* 3×3 *Lemma.*

Exercise 2.31. *Prove the* **Short Five Lemma**: *given a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, "k'"] \arrow[d, "z"'] & X' \arrow[r, "f'"] \arrow[d, "x"] & Y' \arrow[r] \arrow[d, "y"] & 0 \\
0 \arrow[r] & K \arrow[r, "k"'] & X \arrow[r, "f"'] & Y \arrow[r] & 0
\end{tikzcd}$$

in which the rows are short exact sequences, if z and y are bijections then also x is a bijection.

Exercise 2.32. *Prove the* **Five Lemma**: *given a diagram with exact rows*

$$\begin{tikzcd}
A' \arrow[r, "f"] \arrow[d, "a"'] & B' \arrow[r, "g"] \arrow[d, "b"'] & C' \arrow[r, "h"] \arrow[d, "c"] & D' \arrow[r, "k"] \arrow[d, "d"] & E' \arrow[d, "e"] \\
A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r, "w"'] & D \arrow[r, "x"'] & E
\end{tikzcd}$$

in which the rows are exact sequences,

(a) *if a is a surjection, while b and d are injections, then c is an injection;*
(b) *if e is a injection, while b and d are surjections, then c is a surjection;*
(c) *if a is a surjection, e is a injection, and b and d are bijections, then c is a bijection.*

Exercise 2.33. *To see that bijections are isomorphisms we need to assume an additional condition* (SB 3) *which allows us to construct new maps from given ones. Can you find a category where not all bijections are isomorphisms?*

3. Active aspects of the subobject bifibration

The first section treated only *passive* aspects of the subobject bifibration, in the sense that we only considered given diagrams and never attempted to construct any new arrows. The aim of the present section is to study some more powerful *active* techniques and constructions. Next to the conditions (SB 1) and (SB 2) introduced in 2.2 and 2.10 we shall assume that the subobject bifibration satisfies a third and a fourth condition, (SB 3) and (SB 4).

3.1. **The surjection-injection factorisation.** Condition **(SB 3)** asks that any morphism $f\colon X \to Y$ admits a factorisation $f = m \circ e$

$$\begin{tikzcd}
& I \arrow[dr, "m"] & \\
X \arrow[ur, "e"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

into a surjection e followed by an injection m, and that these factorisations are unique up to isomorphism, in the following precise sense. Given another such factorisation $m' \circ e'$ of f,

$$\begin{tikzcd}
& I \arrow[dr, "m"] \arrow[dd, dotted, "\exists i"'] & \\
X \arrow[ur, "e"] \arrow[dr, "e'"'] & & Y \\
& I' \arrow[ur, "m'"'] &
\end{tikzcd}$$

there exists a (necessarily unique) morphism $i\colon I \to I'$ such that $i \circ e = e'$ and $m' \circ i = m$. Note that this i is automatically a bijection. In fact it is easily seen to be an isomorphism. More interestingly, as soon as (SB 3) holds, *any* bijection $f\colon X \to Y$ will be an isomorphism: it suffices to notice that $f \circ 1_X$ and $1_Y \circ f$ are two surjection-injection factorisations,

$$\begin{tikzcd}
& Y \arrow[dr, equal] \arrow[dd, dotted, "\exists i"'] & \\
X \arrow[ur, "f"] \arrow[dr, equal] & & Y \\
& X \arrow[ur, "f"'] &
\end{tikzcd}$$

so that the essential uniqueness of such factorisations gives the needed inverse i for f.

Exercise 3.2. *How is this factorisation obtained in the context of abelian groups, modules, or groups?*

3.3. **Induced injection and surjection.** Any injection $g\colon Y \to Z$ "represents" a subobject of its codomain, namely its image $g_*(1)$. Conversely, any surjection $f\colon X \to Y$ "represents" a subobject of its domain, namely its coimage $f^*(0)$.

We shall ask that this process is reversible. (As we shall see in 3.12, full reversibility can in general only work for *certain* subobjects, but there is a procedure which gets sufficiently close for our purposes.) We assume condition **(SB 4)** which demands that for any subobject y of Y, there exists

(a) a morphism $\lfloor y \rfloor\colon L(y) \to Y$, universal amongst $f\colon X \to Y$ with $f_*(1) \leqslant y$;
(b) a morphism $\lceil y \rceil\colon Y \to R(y)$, universal amongst $g\colon Y \to Z$ with $y \leqslant g^*(0)$.

The universal properties mean that

(a) $\lfloor y \rfloor_*(1) \leqslant y$, and if any other f is such that $f_*(1) \leqslant y$, then there exists a unique dotted arrow $\underline{f}$ making the triangle on the left below commute;
(b) $y \leqslant \lceil y \rceil^*(0)$, and if any other g is such that $y \leqslant g^*(0)$, then there exists a unique dotted arrow $\overline{g}$ making the triangle on the right below commute.

$$\begin{tikzcd}
X \arrow[d, dotted, "\exists!\underline{f}"'] \arrow[dr, "\forall f"] & \\
L(y) \arrow[r, "\lfloor y \rfloor"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& Z \\
Y \arrow[ur, "\forall g"] \arrow[r, "\lceil y \rceil"'] & R(y) \arrow[u, dotted, "\exists!\overline{g}"']
\end{tikzcd}$$

It follows immediately from the universal properties that the morphisms $\lfloor y \rfloor$ and $\lceil y \rceil$ are necessarily unique up to isomorphism. We call them the **induced injection** and the **induced surjection**. This makes sense because for any subobject y of Y, by (SB 3) we may factorise $\lfloor y \rfloor$ into a surjection e followed by an injection m:

$$\begin{tikzcd}
& I \arrow[dr, "m"] \arrow[dl, dotted, shift left, "\underline{m}"] & \\
L(y) \arrow[ur, "e"] \arrow[rr, "\lfloor y \rfloor"'] & & Y
\end{tikzcd}$$

Now $m_*(1) = m_* e_*(1) = \lfloor y \rfloor_*(1) \leqslant y$, so that the existence in the universal property of $\lfloor y \rfloor$ induces a unique $\underline{m}$ for which $\lfloor y \rfloor \circ \underline{m} = m$. We see that $\underline{m} \circ e = 1_{L(y)}$ by the uniqueness in the universal property of $\lfloor y \rfloor$, since $\lfloor y \rfloor \circ \underline{m} \circ e = \lfloor y \rfloor \circ 1_{L(y)}$. This implies that also e is an injection. It follows that, as a composite of two injections, the morphism $\lfloor y \rfloor$ is indeed an injection.

Exercise 3.4. *Show that the* induced surjection *is indeed a surjection.*

Exercise 3.5. *In the present context, a morphism g is an injection if and only if it fits into an exact sequence as on the left*

$$0 \xrightarrow{f} Y \xrightarrow{g} Z \qquad\qquad X \xrightarrow{f} Y \xrightarrow{g} 0$$

and morphism f is a surjection if and only if it fits into an exact sequence as on the right.

Exercise 3.6. *Show that* (SB 4) *holds for abelian groups, vector spaces, modules, groups.*

Exercise 3.7. *Does the context of sets with a chosen subset introduced in Exercise 2.14 satisfy the conditions* (SB 3) *and* (SB 4)*? What about topological spaces with a chosen subspace?*

Lemma 3.8. If a subobject y of Y is the image $f_*(1)$ of a morphism $f\colon X \to Y$, then it is also the image $\lfloor y \rfloor_*(1)$ of its induced injection. Furthermore, the comparison $\underline{f}\colon X \to L(y)$ is a surjection.

Proof. The image $\underline{f}_*(1) \leqslant 1$ of the universally induced morphism $\underline{f}$ is such that

$$y = f_*(1) = \lfloor y \rfloor_* \underline{f}_*(1) \leqslant \lfloor y \rfloor_*(1).$$

The other inequality $\lfloor y \rfloor_*(1) \leqslant y$ holds by definition of $\lfloor y \rfloor$. Since $\lfloor y \rfloor$ is an injection, the equality $\lfloor y \rfloor_* \underline{f}_*(1) = \lfloor y \rfloor_*(1)$ implies $\underline{f}_*(1) = 1$, so $\underline{f}$ is a surjection. □

Lemma 3.9. If $g\colon Y \to Z$ is an injection, then it is isomorphic to $\lfloor g_*(1) \rfloor$, so we may choose $\lfloor g_*(1) \rfloor = g$. If $f\colon X \to Y$ is a surjection, then it is isomorphic to $\lceil f^*(0) \rceil$, so we may choose $\lceil f^*(0) \rceil = f$.

Proof. By Lemma 3.8, the injection g has two surjection-injection factorisations:

$$\begin{tikzcd}
& I \arrow[dr, "\lfloor g_*(1) \rfloor"] \arrow[dd, dotted, "\exists i"] & \\
Y \arrow[ur, "\underline{g}"] \arrow[dr, equal] \arrow[r, phantom, "\cong"] & {} & Z \\
& Y \arrow[ur, "g"'] &
\end{tikzcd}$$

The induced i is the needed isomorphism. □

Proposition 3.10. For any morphism $f\colon X \to Y$, an induced injection and surjection may be chosen such that the diagram

$$\begin{tikzcd} & R(f^*(0)) \arrow[r, equal] & L(f_*(1)) \arrow[dr, "\lfloor f_*(1)\rfloor"] & \\ X \arrow[ur, "\lceil f^*(0)\rceil"] \arrow[rrr, "f"'] & & & Y \end{tikzcd}$$

is a surjection-injection factorisation of f.

Proof. This follows immediately from Lemma 3.9. □

Exercise 3.11. *Formulate the dual of Lemma 3.8 and complete the proofs of Lemma 3.9 and Proposition 3.10.*

3.12. **Normal subobjects.** In the context of non-abelian groups, not all subgroups of a given group X may appear as a coimage $f^*(0)$ of a group homomorphism $f\colon X \to Y$: only the *normal* subgroups do. This means that only normal subgroups can be represented as a coimage of their induced surjection, so that for those, and only for those, the process described above is reversible.

In general, we say that a subobject of an object is **normal** when there exists a morphism of which it is the image, and a morphism of which it is the coimage.

Exercise 3.13. *Show that in a category of modules, all subobjects are normal.*

Exercise 3.14. *What are the normal subobjects in the categories of groups, non-unitary rings, associative non-unitary algebras?*

Proposition 3.15. A normal subobject is always the image of its induced injection and the coimage of its induced surjection.

Proof. This follows immediately from Lemma 3.8 and its dual. □

Corollary 3.16. For any normal subobject y of an object Y, the sequence

$$0 \longrightarrow L(y) \xrightarrow{\lfloor y\rfloor} Y \xrightarrow{\lceil y\rceil} R(y) \longrightarrow 0$$

is exact.

Proof. First of all, the arrows $0 \to L(y)$ and $R(y) \to 0$ are induced by 0 and 1, respectively. Exactness at $L(y)$ and $R(y)$ holds because $\lfloor y\rfloor$ and $\lceil y\rceil$ are injective and surjective, respectively. Finally, $\lceil y\rceil^*(0) = y = \lfloor y\rfloor_*(1)$ by Proposition 3.15. □

Exercise 3.17. *Prove that up to isomorphism, all short exact sequences are such.*

3.18. **Kernels and cokernels.** Given a morphism $f\colon X \to Y$, its **kernel** $\ker(f)$ is the induced injection $\lfloor f^*(0)\rfloor$ of its coimage $f^*(0)$, and its **cokernel** is the induced surjection $\lceil f_*(1)\rceil$ of its image $f_*(1)$. These satisfy the following universal properties:

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "\ker(f)"] & X \arrow[r, "f"] & Y \arrow[r, "\mathrm{coker}(f)"] \arrow[dr, "\forall y"'] & \mathrm{Coker}(f) \arrow[d, dotted, "\exists!\overline{y}"] \\ X' \arrow[u, dotted, "\exists!\underline{x}"] \arrow[ur, "\forall x"'] & & & Y' \end{tikzcd}$$

For all x as in the diagram, $f \circ x$ is null iff $f_*x_*(1) = 0$ iff $x_*(1) \leqslant f^*(0)$, so by the universal property of the induced injection $\lfloor f^*(0)\rfloor$ that there is a unique $\underline{x}$ making the diagram commute. Likewise, any y as above induces a unique $\overline{y}$ which makes the diagram commute.

Kernels and cokernels are unique up to isomorphism.

Proposition 3.19. A morphism k is a kernel of a morphism with a normal coimage f if and only if it appears in an exact sequence such as on the left, and f is a cokernel of a morphism with a normal image k if and only if it appears in an exact sequence such as on the right.

$$0 \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \qquad\qquad K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 0$$

Proof. First suppose that we are given the exact sequence on the left. Being an injection, k is isomorphic to $\lfloor k_*(1) \rfloor$ by Lemma 3.9. Exactness at X implies that $k_*(1) = f^*(0)$. Hence k is a kernel of f.

Conversely, if k is a kernel then it is an (induced) injection $\lfloor f^*(0) \rfloor$, which proves exactness at K. Furthermore, $k_*(1) = \lfloor f^*(0) \rfloor_*(1) = f^*(0)$ which proves exactness at X. □

Briefly, a kernel is precisely an injection with a normal image, while a cokernel is nothing but a surjection with a normal coimage.

Corollary 3.20. For any normal subobject y of an object Y,
(a) $\lfloor y \rfloor$ is a kernel of $\lceil y \rceil$;
(b) $\lceil y \rceil$ is a cokernel of $\lfloor y \rfloor$.

Proof. This is a consequence of Proposition 3.19 and Corollary 3.16. □

Exercise 3.21. *Let $f\colon X \to Y$ be a morphism. If $m\colon Y \to Z$ is an injection, then* $\ker(f) = \ker(m \circ f)$*. If $e\colon W \to X$ is a surjection, then* $\operatorname{coker}(f) = \operatorname{coker}(f \circ e)$*.*

It follows that in a short exact sequence

$$0 \longrightarrow K \rightarrowtail^{k} X \xrightarrow{f} Y \longrightarrow 0$$

the morphism f is a cokernel of k, k is a kernel of f, the image of k is normal, and the coimage of f is normal. This characterises the exactness of the sequence completely. We sometimes change the shape of the arrows to indicate that a certain arrow is a kernel or a cokernel. In particular, in a short exact sequence, any of the "outer" objects is completely determined by the other two.

Example 3.22. Given an (abelian) group X, any presentation F/R of X in terms of generators and relations forms a short exact sequence

$$0 \longrightarrow R \longrightarrow F \longrightarrow X \longrightarrow 0$$

where F is a free group.

3.23. **Extracting information from exact sequences.** Let us now illustrate with a simple example how the diagram lemmas of homological algebra are used. We sometimes want to use the information in the outer objects to reconstruct the "middle object" of a short exact sequence. It turns out that is not always possible. But if we are given a diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "k"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r, "1_K"'] & K \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

in which the rows are short exact sequences of modules, then we always have an isomorphism $X \cong K \times Y$. (It suffices to check that the morphism $\langle s, f \rangle\colon X \to K \times Y$ is a bijection by hand, or to use the Short Five Lemma, Exercise 2.31.) So the mere fact of being part of some bigger diagram gives information about the objects in a short exact sequence.

In the case of vector spaces this technique works very well:

$$\begin{tikzcd}
& \mathrm{Ker}(z) \arrow[r, dashed] \arrow[d, dotted] & \mathrm{Ker}(x) \arrow[r, dashed] \arrow[d, dotted] & \mathrm{Ker}(y) \arrow[d, dotted] \arrow[dddll, dashed] & \\
& K' \arrow[r, "k'"] \arrow[d, "z"'] & X' \arrow[r, "f'"'] \arrow[d] & Y' \arrow[r] \arrow[d, "y"] & 0 \\
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, dotted] & X \arrow[r, "f"'] \arrow[d, dotted] & Y \arrow[d, dotted] & \\
& \mathrm{Coker}(z) \arrow[r, dashed] & \mathrm{Coker}(x) \arrow[r, dashed] & \mathrm{Coker}(y) &
\end{tikzcd}$$

FIGURE 2. The Snake Lemma

Proposition 3.24. In a short exact sequence of vector spaces

$$0 \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 0,$$

always $X \cong K \times Y$.

Proof. Choose a basis $(z_i)_{i \in I}$ of K and extend it to a basis $(x_j)_{j \in J}$ of X, so that $I \subseteq J$ and $k(z_i) = x_i$ for $i \in I$. Define $s\colon X \to K$ by

$$s(x_j) = \begin{cases} z_j & j \in I \\ 0 & j \in J \backslash I, \end{cases}$$

then apply 3.23. □

This is far from true outside the context of vector spaces, as indicated for instance in the following exercise.

Exercise 3.25. *For any $n \geqslant 2$,*

$$0 \longrightarrow n\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

is a short exact sequence of abelian groups. Show that here, the middle object is not a product of the outer objects.

This means that in order to extract information about the middle object in a short exact sequence, we need more subtle techniques. One such technique, called the **Snake Lemma**, produces new exact sequences from given ones.

Theorem 3.26. Given a commutative diagram

$$\begin{tikzcd}
& K' \arrow[r, "k'"] \arrow[d, "z"'] \arrow[dr] & X' \arrow[r, "f'"] \arrow[d, "x"] \arrow[dr] & Y' \arrow[r] \arrow[d, "y"] & 0 \\
0 \arrow[r] & K \arrow[r, "k"'] & X \arrow[r, "f"'] & Y &
\end{tikzcd}$$

with exact rows and in which all arrows have normal images and coimages, take kernels and cokernels as in Figure 2. Then there exists a **connecting morphism** $\partial\colon \mathrm{Ker}(y) \to \mathrm{Coker}(z)$ such that the sequence

$$\mathrm{Ker}(z) \to \mathrm{Ker}(x) \to \mathrm{Ker}(y) \xrightarrow{\partial} \mathrm{Coker}(z) \to \mathrm{Coker}(x) \to \mathrm{Coker}(y)$$

is exact.

Proof. We explain how the connecting morphism ∂ is constructed and leave the exactness of the sequence as an (elaborate) exercise.

We first show that we have an exact sequence

$$K' \xrightarrow{k''} \mathrm{Ker}(y \circ f') \xrightarrow{f''} \mathrm{Ker}(y) \longrightarrow 0$$

of which the cokernel part is induced by f'. Certainly k' factors over the kernel of $y \circ f'$ to a morphism k'', so it suffices to prove that the arrow on the right is a surjection and the sequence is exact in $\mathrm{Ker}(y \circ f')$. We first prove surjectivity:

$$\begin{aligned} f''_*(1) &= \ker(y)^* \ker(y)_* f''_*(1) = \ker(y)^* f'_* \ker(y \circ f')_*(1) \\ &= \ker(y)^* f'_*(y \circ f')^*(0) = \ker(y)^* f'_*(f')^* y^*(0) = \ker(y)^* y^*(0) = 1, \end{aligned}$$

where the first equality on the second line holds because the coimage of $y \circ f'$ is normal. Next we prove that f'' is the cokernel of k''. Here we need to show that $(f'')^*(0) = (k'')_*(1)$:

$$\begin{aligned} (f'')^*(0) &= (f'')^* \ker(y)^*(0) = \ker(y \circ f')^*(f')^*(0) \\ &= \ker(y \circ f')^*(k')_*(1) = (k'')_*(1), \end{aligned}$$

where to see that the last equality holds we may take its direct image along the injection $\ker(y \circ f')$.

The needed morphism ∂ is now constructed in two steps as follows. First the morphism x restricts to the kernels of f and of $f \circ x = y \circ f'$ to a morphism $x' \colon \mathrm{Ker}(y \circ f') \to K$ which makes the diagram

$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r, "k''"] \arrow[d, equal] & \mathrm{Ker}(y \circ f') \arrow[r] \arrow[d, "x'"] & \mathrm{Ker}(y) \arrow[r] \arrow[d, dashed, "\partial"] & 0 \\ & K' \arrow[r, "z"'] & K \arrow[r] & \mathrm{Coker}(z) \arrow[r] & 0 \end{tikzcd}$$

commute. Taking cokernels now, we find ∂ as the induced arrow. $\square$

Exercise 3.27. *Given an epimorphism of abelian groups $f \colon X \to Y$, we present X and Y by generators and relations as follows:*

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & L \arrow[r] & F \arrow[r] & Y \arrow[r] & 0 \end{tikzcd}$$

Use the Snake Lemma to obtain a presentation of the kernel K of f.

Exercise 3.28. *Complete the proof of the Snake Lemma.*

4. Homology of chain complexes

We know that exactness of a sequence allows us to extract information about its objects. The concept of *homology* allows us to approach the following question: *How to extract information from a non-exact sequence?*

Definition 4.1. A **chain complex** (C, d) is a sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

such that

(a) $d_n \circ d_{n+1}$ is null, so $(d_{n+1})_*(1) \leqslant d_n^*(0)$, and
(b) $d_n^*(0)$ and $(d_{n+1})_*(1)$ are normal subobjects of C_n,

for all $n \in \mathbb{Z}$. The maps d_n are called **differentials**. We sometimes omit the d and write C for the chain complex.

Exercise 4.2. *Given a linear map $f\colon V \to V$ where V is the real vector space $\mathbb{R}^2$, find examples where (C, d) given by*

$$\cdots \longrightarrow V \xrightarrow{f} V \xrightarrow{f} V \longrightarrow \cdots,$$

so $C_n = V$, $d_n = f$ for all $n \in \mathbb{Z}$ is, or isn't, a chain complex. Give a precise characterisation (for instance, a matrix with respect to the standard basis). What about $V = \mathbb{R}$ and $V = \mathbb{R}^3$?

Definition 4.3. We say that a chain complex C is **exact at** C_n when the sequence

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

is exact. C is called **exact** or **acyclic** when it is exact at in all of its objects C_n, so $(d_{n+1})_*(1) = d_n^*(0)$, for all $n \in \mathbb{Z}$.

Example 4.4. Any (short) exact sequence can be considered as part of an exact chain complex (by adding zeroes).

A sequence of modules, whether exact or not, in which the composite of any two successive arrows is null, is always a chain complex. Outside the context of modules, however, condition (b) need not be automatically satisfied: differentials need not have normal images and coimages—see Exercise 4.16.

Example 4.5. For any module M, the sequence

$$\cdots \xrightarrow{0} M = M \xrightarrow{0} M = M \xrightarrow{0} \cdots$$

given by $C_n = M$ and

$$d_n = \begin{cases} 1_M & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

for $n \in \mathbb{Z}$ is an exact chain complex.

Example 4.6. Given a sequence of modules $(M_n)_{n \in \mathbb{Z}}$, we can consider modules $C_{n+1} = M_{n+1} \times M_n$ and differentials

$$d_{n+1}\colon M_{n+1} \times M_n \to M_n \times M_{n-1}\colon (x, y) \mapsto (y, 0)$$

for all $n \in \mathbb{Z}$. This defines an exact chain complex (C, d).

Exercise 4.7. *Can Example 4.5 be seen as a special case of 4.6?*

Exercise 4.8. *Which of the chain complexes found in Exercise 4.2 are exact?*

4.9. **Measuring how far a complex is from being exact.** Given an object C_n in a chain complex (C, d), we can always take the kernel of d_n and obtain a factorisation $\underline{d_{n+1}}$ of d_{n+1} over $\ker(d_n)$:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, dotted, "\underline{d_{n+1}}"'] & & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots \\ & & \operatorname{Ker}(d_n) \arrow[ur, tail, "\ker(d_n)"'] & & & \end{tikzcd}$$

The complex C is exact at C_n when $(d_{n+1})_*(1) = d_n^*(0)$, which happens if and only if $\underline{d_{n+1}}$ is a surjection. Indeed

$$\underline{d_{n+1}}_*(1) = 1 \quad \Leftrightarrow \quad \ker(d_n)_* \underline{d_{n+1}}_*(1) = \ker(d_n)_*(1) \quad \Leftrightarrow \quad (d_{n+1})_*(1) = d_n^*(0).$$

Now in general $\underline{d_{n+1}}$ need not be surjective. However, the image $(\underline{d_{n+1}})_*(1)$ of $\underline{d_{n+1}}$ is normal, since so is the image of d_{n+1}. We saw above (Exercise 3.5 combined with Proposition 3.19) that a morphism with a normal image is a surjection if and only if its cokernel is zero. This immediately leads to the following definition, as well as the subsequent theorem:

Definition 4.10. The **n-th homology** H_nC of a chain complex (C, d) is the cokernel of the induced morphism $\underline{d_{n+1}}\colon C_{n+1} \to \mathrm{Ker}(d_n)$.

Theorem 4.11. For a chain complex (C, d) and an integer n, the following conditions are equivalent:

(i) C is exact at C_n;
(ii) the morphism $\underline{d_{n+1}}\colon C_{n+1} \to \mathrm{Ker}(d_n)$ is a surjection;
(iii) $H_nC = 0$.

Thus exactness of chain complexes is captured completely by the concept of homology. It turns out that when a chain complex is not exact, its homology modules may still contain interesting information about the complex. Our aim today is to give some examples of this phenomenon.

Example 4.12. Let A be an $i \times j$ matrix and B a $j \times l$ matrix of real numbers such that the product $B \cdot A$ is zero. Then there may exist a column vector V (a $j \times 1$ matrix) such that $A \cdot V = 0$ while not being of the form $V = B \cdot W$ for any $l \times 1$ matrix W. This failure is measured by the **defect**

$$d = j - \mathrm{rank}(A) - \mathrm{rank}(B).$$

This defect turns out to be precisely the dimension of the homology H_1C of the chain complex C defined to be zero everywhere except in C_2, C_1 and C_0 where it is

$$\mathbb{R}^l \xrightarrow{g} \mathbb{R}^j \xrightarrow{f} \mathbb{R}^i.$$

Here g and f have matrix B and A with respect to the standard bases. Writing $\mathrm{Im}(g) = \lfloor g_*(1) \rfloor$ we see that $H_1C = \mathrm{Ker}(f)/\mathrm{Im}(g)$ and $\dim(\mathrm{Ker}(f)) = j - \mathrm{rank}(A)$ while $\dim(\mathrm{Im}(g)) = \mathrm{rank}(B)$.

Exercise 4.13. *We take $R = \mathbb{Z}$ so that R-modules are abelian groups. Consider*

$$C_n = \begin{cases} \mathbb{Z}_8 & \text{if } n \geqslant 0 \\ 0 & \text{if } n < 0 \end{cases}$$

and $d_{n+1}\colon C_{n+1} \to C_n$ defined by $d_n(m) = 4m$. Show that C is a chain complex and compute its homology groups.

Exercise 4.14. *Same question for*

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 4} \mathbb{Z}_8 \xrightarrow{\cdot 8} \mathbb{Z}_{16} \longrightarrow \cdots$$

Exercise 4.15. *Find a chain complex of abelian groups C of which the homology alternates between 0 and $\mathbb{Z}_2$, so*

$$H_nC = \begin{cases} 0 & n \text{ even} \\ \mathbb{Z}_2 & n \text{ odd.} \end{cases}$$

Exercise 4.16. *Give an example of a sequence of non-abelian groups which is not exact but still has trivial homology groups. (The sequence may fail to satisfy condition* (b) *in Definition 4.1.)*

Exercise 4.17. *Show that for chain complexes of modules, products commute with homology: given complexes (C, d^C) and (D, d^D), the pointwise product $C \times D$ defined by*

$$(C \times D, d^C \times d^D) = (C_n \times D_n, d_n^C \times d_n^D)_{n \in \mathbb{Z}}$$

satisfies

$$H_n(C \times D) \cong H_nC \times H_nD$$

for all $n \in \mathbb{Z}$.

Exercise 4.18. *Let Γ be a finite graph with V vertices $(v_1, \ldots, v_V)$ and E edges $(e_1, \ldots, e_E)$. Loops are not allowed. If we orient the edges, we can form the incidence matrix of the graph. This is a $V \times E$ matrix whose (i, j) entry is*

$$\begin{cases} +1 & \text{if the edge } e_j \text{ starts at } v_i, \\ -1 & \text{if } e_j \text{ ends at } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let C_0 be $\mathbb{R}^V$, C_1 the real vector space $\mathbb{R}^E$, $C_n = 0$ if $n \notin \{0, 1\}$, and $d_1 \colon C_1 \to C_0$ the linear map determined by the incidence matrix.

If Γ is connected (which means that we can get from the vertex v_1 to every other vertex by tracing a path with edges), show that H_0C and H_1C have dimensions 1 and $V - E + 1$ respectively. (The number $V - E + 1$ is the number of circuits of the graph.)

4.19. **Morphisms of chain complexes; homology as a functor.** Morphisms of chain complexes are "commutative ladders"

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

This picture can be formalised as follows.

Definition 4.20. Given chain complexes (C, d^C) and (D, d^D), a **morphism of chain complexes** or **chain morphism** $f \colon C \to D$ is a sequence of morphisms $(f_n \colon C_n \to D_n)_{n \in \mathbb{Z}}$ such that for any $n \in \mathbb{Z}$ the equality $f_n \circ d^C_{n+1} = d^D_{n+1} \circ f_{n+1}$ holds.

Chain complexes and chain morphisms again form a category, in which composites of chain morphisms are defined degreewise, and the identity on a chain complex C is just $(1_{C_n})_{n \in \mathbb{Z}}$. Also the subobject bifibration lifts to the level of chain complexes.

Exercise 4.21. *Show that a morphism of chain complexes induces morphisms between the homology objects. This process preserves composition and identities, so that for each n we obtain a* **homology functor** *H_n sending a chain complex C to its n-th homology H_nC and a morphism of chain complexes $f \colon C \to D$ to a morphism $H_nf \colon H_nD \to H_nC$.*

Exercise 4.22. *Use Exercise 4.17 to show that in the modules case, the functors H_n are* **additive***: for any two chain morphisms f, $g \colon C \to D$ we have*

$$H_n(f + g) = H_nf + H_ng.$$

Exercise 4.23. *When R is a field we may extend Example 4.6 to the case of non-exact complexes. Given sequences of vector spaces $(V_n)_{n\in\mathbb{Z}}$ and $(H_n)_{n\in\mathbb{Z}}$, we consider the vector spaces $C_{n+1} = V_{n+1} \times H_{n+1} \times V_n$ and differentials*

$$d_{n+1} \colon C_{n+1} \to C_n \colon (x, y, z) \mapsto (z, 0, 0)$$

for all $n \in \mathbb{Z}$.

(a) *Prove that C is a chain complex and compute its homology objects.*
(b) *Show that every chain complex of vector spaces is isomorphic to a complex of this form.*
(c) *Find an example (necessarily outside the context of vector spaces) of a chain complex which is not of this form.*

4.24. **A "dual" definition of homology.** Instead of computing homology as a quotient of a kernel, we can equally well obtain it through a subobject of a cokernel, as in the following picture.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "\underline{d_{n+1}}"'] & & C_n \arrow[rr, "d_n"] \arrow[dr, "\mathrm{coker}(d_{n+1})"] & & C_{n-1} \arrow[r] & \cdots \\
& & \mathrm{Ker}(d_n) \arrow[ur, tail, "\ker(d_n)"] \arrow[d, two heads, "\mathrm{coker}(\underline{d_{n+1}})"'] \arrow[drr, dotted] & & \mathrm{Coker}(d_{n+1}) \arrow[ur, "\overline{d_n}"'] & & \\
& & H_nC \arrow[rr, dotted, "\lambda_n"'] \arrow[urr, dotted] & & K_nC \arrow[u, tail, "\ker(\overline{d_n})"'] & &
\end{tikzcd}$$

Proposition 4.25. For any chain complex C and any $n \in \mathbb{Z}$, we have an isomorphism $H_nC \cong K_nC$.

Proof. The universal properties of kernels and cokernels give us the diagonal dotted arrows, then the horizontal dotted arrow such that the entire diagram commutes. We do a diagram chase to prove that the horizontal arrow is an isomorphism. By duality it suffices to show that it is an injection, so we chase 0 around the diagram:

$$\begin{aligned}
&\mathrm{coker}(\underline{d_{n+1}})_* \ker(d_n)^* \mathrm{coker}(d_{n+1})^* \ker(\overline{d_n})_*(0) \\
&= \mathrm{coker}(\underline{d_{n+1}})_* \ker(d_n)^* \mathrm{coker}(d_{n+1})^*(0) \\
&= \mathrm{coker}(\underline{d_{n+1}})_* \ker(d_n)^* (d_{n+1})_*(1) \\
&= \mathrm{coker}(\underline{d_{n+1}})_* \ker(d_n)^* \ker(d_n)_* \underline{d_{n+1}}_*(1) \\
&= \mathrm{coker}(\underline{d_{n+1}})_* \underline{d_{n+1}}_*(1) = 0.
\end{aligned}$$

Note that $\mathrm{coker}(d_{n+1})^*(0) = (d_{n+1})_*(1)$ if and only if $(d_{n+1})_*(1)$ has a normal image (by the dual of Lemma 3.8).

We still have to explain why this diagram chase does the job! It does, because by commutativity of the diagram,

$$\ker(\overline{d_n}) \circ \lambda_n \circ \mathrm{coker}(\underline{d_{n+1}}) = \mathrm{coker}(d_{n+1}) \circ \ker(d_n)$$

so $\mathrm{coker}(\underline{d_{n+1}})^* \circ \lambda_n^* \circ \ker(\overline{d_n})^* = \ker(d_n)^* \circ \mathrm{coker}(d_{n+1})^*$, which implies

$$\lambda_n^* = \mathrm{coker}(\underline{d_{n+1}})_* \circ \ker(d_n)^* \circ \mathrm{coker}(d_{n+1})^* \circ \ker(\overline{d_n})_*$$

since $\mathrm{coker}(\underline{d_{n+1}})$ is a surjection and $\ker(\overline{d_n})$ is an injection. □

We shall need this later on when constructing the long exact homology sequence induced by a short exact sequence of chain complexes.

Exercise 4.26. *Why are the (co)images of the $\widehat{d_{n+1}}$ normal?*

$$\begin{tikzcd}
0 \arrow[r] & A_{n+2} \arrow[r] \arrow[d, "d^A_{n+2}"'] & B_{n+2} \arrow[r] \arrow[d, "d^B_{n+2}"] & C_{n+2} \arrow[r] \arrow[d, "d^B_{n+2}"] & 0 \\
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d, dotted] & B_{n+1} \arrow[r] \arrow[d, dotted] & C_{n+1} \arrow[r] \arrow[d, dotted] & 0 \\
& \mathrm{Coker}(d^A_{n+1}) \arrow[r] \arrow[d, "\widehat{d^A_{n+1}}"'] & \mathrm{Coker}(d^B_{n+1}) \arrow[r] \arrow[d, "\widehat{d^B_{n+1}}"] & \mathrm{Coker}(d^C_{n+1}) \arrow[r] \arrow[d, "\widehat{d^C_{n+1}}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(d^A_n) \arrow[r] \arrow[d, dotted] & \mathrm{Ker}(d^B_n) \arrow[r] \arrow[d, dotted] & \mathrm{Ker}(d^C_n) \arrow[d, dotted] & \\
0 \arrow[r] & A_n \arrow[r] \arrow[d, "d^A_n"'] & B_n \arrow[r] \arrow[d, "d^B_n"] & C_n \arrow[r] \arrow[d, "d^C_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

FIGURE 3. Applying the Snake Lemma to find the long exact homology sequence

Exercise 4.27. *Give an example of a sequence of non-abelian groups for which the two definitions of homology do not coincide. (Same explanation as Exercise 4.16: strictly speaking, homology is only defined for chain complexes, not for all sequences.)*

4.28. **The long exact homology sequence.** Even though chain complexes themselves need not be exact sequences, they can be used to produce new exact sequences, which may then provide additional information about the situation at hand. One very important result of this kind is the *long exact homology sequence* induced by a short exact sequence of chain complexes.

Theorem 4.29. Let

$$0 \longrightarrow A \xrightarrow{k} B \xrightarrow{f} C \longrightarrow 0$$

be a short exact sequence of chain complexes such that the composites $d^B_n \circ k_n$ and $f_n \circ d^B_n$ have a normal images and coimages. Then there exist morphisms

$$\partial_{n+1} \colon H_{n+1}C \to H_n A$$

called **connecting homomorphisms** such that the sequence

$$\cdots \longrightarrow H_{n+1}C \xrightarrow{\partial_{n+1}} H_n A \xrightarrow{H_n k} H_n B \xrightarrow{H_n f} H_n C \xrightarrow{\partial_n} H_{n-1}A \longrightarrow \cdots$$

is exact.

Proof. The Snake Lemma 3.26 applied to the middle part of Figure 3 gives us an exact sequence

$$K_{n+1}A \longrightarrow K_{n+1}B \longrightarrow K_{n+1}C \xrightarrow{\partial_{n+1}} H_n A \longrightarrow H_n B \longrightarrow H_n C.$$

The result follows by gluing such sequences together using Proposition 4.25. □

5. Homotopy of chain complexes

In a topological setting, *homotopy* is about continuous deformation of one function into another. Here we shall study homotopies between morphisms of chain complexes of modules and prove that homotopic morphisms have the same homology. We obtain a concept of *homotopy equivalence* between chain complexes, which turns out to be stronger than the concept of *weak equivalence* induced by homology.

Definition 5.1. A morphism of chain complexes $f\colon C \to D$ is called a **weak equivalence** when for all $n \in \mathbb{Z}$, the induced morphism $H_n f\colon H_n C \to H_n D$ is an isomorphism. Two chain complexes are said to be **weakly equivalent** when there exist a weak equivalence between them.

Clearly, any isomorphism is a weak equivalence. The converse need not hold, as shows the following example.

Example 5.2. For any module M we can consider the morphism of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & M \arrow[r, equal] \arrow[d, equal] & M \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & M \arrow[r, equal] & M \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Since both complexes are exact, this morphism is a weak equivalence. Unless M is zero, it is not an isomorphism.

Exercise 5.3. *Prove that a chain complex C is exact precisely when the unique morphism $C \to 0$ is a weak equivalence.*

Exercise 5.4. *Let $f\colon C \to D$ be a morphism of chain complexes with normal image and coimage. Show that if f is a weak equivalence, then* $\mathrm{Ker}(f)$ *and* $\mathrm{Coker}(f)$ *are exact. Is the converse true?*

5.5. **Split (exact) complexes.** A chain complex C is called **split** if for all $n \in \mathbb{Z}$ there exists a morphism $s_n\colon C_n \to C_{n+1}$ such that $d_{n+1} = d_{n+1} \circ s_n \circ d_{n+1}$. It is called **split exact** when moreover it is exact.

Example 5.6. Consider sequences of modules $(M_n)_{n\in\mathbb{Z}}$ and $(H_n)_{n\in\mathbb{Z}}$. Like in Exercise 4.23, we may consider $C_{n+1} = M_{n+1} \times H_{n+1} \times M_n$ and differentials

$$d_{n+1}\colon C_{n+1} \to C_n\colon (x, y, z) \mapsto (z, 0, 0)$$

for all $n \in \mathbb{Z}$. Such a complex is always split, with $s_n\colon C_n \to C_{n+1}$ defined by $s_n(x, y, z) = (0, 0, x)$. In particular, part (b) of Exercise 4.23 shows that all complexes of vector spaces are split.

Exercise 5.7. *Prove that the homology of the above complex C is $H_n C = H_n$, so that C is split exact if and only if all given H_n vanish.*

Exercise 5.8. *Prove that a chain complex is split if and only if it is isomorphic to a chain complex as in Example 5.6, so that it is split exact precisely when it takes the shape of Example 4.6.*

Example 5.9. The complex of abelian groups

$$\cdots \longrightarrow \mathbb{Z}_4 \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 2} \mathbb{Z}_4 \longrightarrow \cdots$$

is exact but not split exact. To see this, it suffices to check for all possible morphisms $\mathbb{Z}_4 \to \mathbb{Z}_4$ that they do not have the property s_n should satisfy.

5.10. **Restricting the context to modules.** From now on, unless mentioned otherwise, we work in a category of modules over a ring R. This allows us to take sums of morphisms, which are crucial in the definition of chain homotopy.

In what follows, for the sake of convenience we write $\mathrm{Im}(f) = L(f_*(1))$.

Proposition 5.11. For any chain complex of modules C, the following conditions are equivalent:

(i) C is split exact;
(ii) C is exact and for all n we have $C_n \cong \mathrm{Im}(d_{n+1}) \times \mathrm{Ker}(d_{n-1})$, with d_n corresponding to the morphism that sends (b, z) to $(z, 0)$;
(iii) there exist morphisms $s_n \colon C_n \to C_{n+1}$ such that for all $n \in \mathbb{Z}$

$$1_{C_{n+1}} = d_{n+2} \circ s_{n+1} + s_n \circ d_{n+1}.$$

Proof. If (iii) holds then already $d_{n+1} = d_{n+1} \circ s_n \circ d_{n+1}$ for all n. If now $x \in \mathrm{Ker}(d_{n+1})$, then

$$x = 1_{C_{n+1}}(x) = d_{n+2}(s_{n+1}(x)) + s_n(d_{n+1}(x)) = d_{n+2}(s_{n+1}(x))$$

so that $x \in \mathrm{Im}(d_{n+2})$. This shows that C is exact at C_{n+1}, so (iii) implies (i).

(i) $\Rightarrow$ (ii) is the content of Exercise 5.8. To see that (ii) implies (iii), for the sake of simplicity, given any n we identify C_n with the product to which it is isomorphic and define

$$s_n \colon C_n \to C_{n+1} \colon (b, z) \mapsto (0, b).$$

This works because C is exact, so that $\mathrm{Im}(d_{n+1}) = \mathrm{Ker}(d_n)$. We get

$$\begin{aligned}(d_{n+2} \circ s_{n+1} + s_n \circ d_{n+1})(b, z) &= d_{n+2}(s_{n+1}(b, z)) + s_n(d_{n+1}(b, z)) \\ &= d_{n+2}(0, b) + s_n(z, 0) \\ &= (b, 0) + (0, z) = (b, z) = 1_{C_{n+1}}(b, z)\end{aligned}$$

for all $(b, z) \in C_{n+1}$, which finishes the proof. □

Exercise 5.12. *Show that a short exact sequence of modules, considered as a chain complex, is split exact if and only if the middle object decomposes as a product of the outer objects.*

5.13. **Chain homotopy.** We just saw that an exact chain complex of modules is split if and only if its components decompose into a product. This difference between exactness and split exactness will turn out to be an instance of the difference between weak equivalences and homotopy equivalences, which we are now about to introduce.

Definition 5.14. Given chain morphisms $f, g \colon C \to D$, we say that f **is (chain) homotopic to** g and write $f \simeq g$ when there exist morphisms $h_n \colon C_n \to D_{n+1}$ such that

$$f_{n+1} - g_{n+1} = d^D_{n+2} \circ h_{n+1} + h_n \circ d^C_{n+1}$$

for all $n \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, dotted, "h_n"'] & C_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, dotted, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

The collection $h = (h_n)_{n \in \mathbb{Z}}$ is called a **chain homotopy from f to g**.

Exercise 5.15. *Show that if f is homotopic to g, then g is homotopic to f.*

Exercise 5.16. *Show that $f \simeq f$ for any chain morphism f.*

Example 5.17. A morphism of chain complexes $f\colon C \to D$ is said to be **null homotopic** when it is chain homotopic to the zero morphism. Proposition 5.11 tells us that a chain complex C is split exact if and only if 1_C is null homotopic. In general, $f \simeq g$ if and only if $f - g$ is null homotopic.

Exercise 5.18. *Given chain complexes C and D, show that for* any *collection of morphisms $(h_n\colon C_n \to C_{n+1})_{n\in\mathbb{Z}}$ the expression*

$$f_n = d^D_{n+1} \circ h_n + h_{n-1} \circ d^C_n$$

defines a null homotopic chain morphism $f\colon C \to D$.

Exercise 5.19. *Show that* any *two chain morphisms of vector spaces $f\colon C \to D$ are homotopic as soon as D is exact.*

Definition 5.20. A **homotopy equivalence** is a chain morphism $f\colon C \to D$ that admits a **homotopy inverse**, which is a chain morphism $g\colon D \to C$ such that $g \circ f \simeq 1_D$ and $f \circ g \simeq 1_C$.

Exercise 5.21. *Show that two chain complexes can be weakly equivalent without being homotopy equivalent.*

Theorem 5.22. If f, $g\colon C \to D$ are chain homotopic, then they induce the same morphisms H_nf and $H_ng\colon H_nC \to H_nD$. In particular,

(a) if f is null homotopic, then H_nf is null for all n;
(b) if f is a homotopy equivalence, then it is a weak equivalence;
(c) if C and D are homotopically equivalent, then they are weakly equivalent.

Proof. Thanks to Exercise 4.22 and Example 5.17 it suffices to prove (a). For the sake of simplicity we omit all indexes. Suppose that $f = dh + hd$ and consider the diagram

$$\begin{tikzcd}
C \arrow[rrr, "dh+hd"] \arrow[dr, "\underline{d}"] \arrow[dd, "d"'] & & & D \arrow[dl, "\underline{d}"'] \arrow[dd, "d"] \\
& \mathrm{Ker}(d) \arrow[r, dotted] \arrow[dl, "\ker(d)"] \arrow[r, phantom, shift right=5ex, "(*)"] & \mathrm{Ker}(d) \arrow[dr, "\ker(d)"'] & \\
C \arrow[rrr, "dh+hd"'] \arrow[d, "d"'] & & & D \arrow[d, "d"] \\
C \arrow[rrr, "dh+hd"'] & & & D
\end{tikzcd}$$

explaining how Hf is obtained (namely, as the arrow between the cokernels of the $\underline{d}$ which is induced by the dotted arrow). The dotted arrow is nothing but $\underline{d}h\ker(d)$, since it is the (necessarily unique) arrow that makes the square $(*)$ commute: indeed,

$$(dh + hd)\ker(d) = dh\ker(d) = \ker(d)\underline{d}h\ker(d).$$

Since $\underline{d}h\ker(d)$ factors over $\underline{d}$, the morphism Hf is null. □

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