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CONDITIONAL MEAN RISK SHARING WITH SETTLEMENT DELAYS IN PEER-TO-PEER DISABILITY INSURANCE

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Abstract

This paper develops a dynamic conditional mean risk-sharing mechanism for peer-to-peer disability insurance when claim amounts are not observed at occurrence time. The model is motivated by cooperative arrangements such as Dutch bread funds, where participants mutually protect each other against the economic consequences of disability. We consider a two-state model in which participants move between active and disabled states, and where the ultimate cost of a disability claim is determined by the duration of the disability spell. Since this duration is only known at recovery, the conditional mean risk-sharing allocation cannot be implemented immediately when the claim is reported. The proposed mechanism requires active participants to contribute an initial reserve when a disability spell starts. This reserve is then dynamically updated as information about the claim development becomes available, and it converges at settlement to the terminal conditional mean risk-sharing allocation. We show that the dynamic reserve is a conditional expectation of the corresponding thinned individual loss under the information used by the pool. This representation yields martingale, conditional full-allocation and actuarial fairness properties. Most importantly, the cumulative contribution paid by each participant is smaller in convex order than the corresponding stand-alone incurred-and-reserved disability cost. Hence, every risk-averse participant is willing to join the pool. Numerical illustrations show how reserves and final adjustments depend on disability duration and heterogeneous recovery rates.

Keywords: conditional mean risk sharing; peer-to-peer insurance; disability insurance; multistate model; settlement delay; dynamic reserving; convex order.

1 Introduction and motivation

This paper is motivated by peer-to-peer arrangements designed to provide mutual protection against the economic consequences of disability. A prominent example is given by Dutch bread funds (*broodfondsen*), which have been developed by groups of self-employed workers in the Netherlands. Dutch self-employed workers are not automatically covered by social security against loss of income due to disability, and many of them do not purchase private disability insurance, often perceived as being too expensive. As an alternative, some of them have formed gift circles in which participants mutually support one another when disability occurs. See Oostveen (2018) for a presentation of bread funds.

When joining a bread fund, each participant transfers a monthly contribution to a personal account. If a participant becomes disabled, he or she receives monthly donations from the other members, usually for a limited period. These payments take the form of direct personal gifts from one member to another. There is no transfer of risk to an external insurer, and the arrangement does not qualify as a conventional insurance contract. This feature explains both the practical appeal of bread funds and their actuarial specificity: the system relies on mutual commitment, transparency and solidarity within a limited community.

Bread funds are typically organized with the support of a national organization, *De BroodfondsMakers* (see `broodfonds.nl`), which provides guidelines and operational rules. For instance, bread funds are usually kept small enough to preserve mutual trust, while a minimum size is required to ensure some diversification. Since a single fund may temporarily lack sufficient resources to pay all disability benefits, a “Bread Fund Alliance” (*Broodfonds Alliantie*) has been created to allow funds to support each other in case of exceptional pressure on liquidity. Similar initiatives have also been developed outside the Netherlands; see, for instance, `breadfunds.uk`. This mechanism reminds nineteen-century Irish mutual aid tontines and Scottish dividing friendly societies providing subscribers with sickness benefits (McDiarmid, 2025).

The actuarial question addressed in this paper is whether such a peer-to-peer disability arrangement can be supported by a rigorous risk-sharing rule satisfying a willingness-to-join property. In the terminology of Denuit et al. (2022a), willingness to join means that each risk-averse participant prefers joining the pool to remaining in the stand-alone situation where he or she bears his or her own disability losses. This preference is formalized by a convex-order comparison, which captures the common ranking of risks by all risk-averse agents.

To answer this question, we use a multistate framework. Multistate models provide a natural actuarial representation of life and health insurance contracts, including disability insurance and permanent health insurance policies. Each state represents a possible status of the insured individual, and benefits are associated with sojourns in states or transitions between states; see, for instance, Pitacco (2014). In this paper, we focus on a two-state model in which each participant is either active or disabled. Contrary to hierarchical multistate models used in long-term care or tontine applications, the disability model considered here allows participants to recover and subsequently experience several disability spells.

This distinguishes the present work from some existing multistate risk-sharing schemes. Chen et al. (2022) and Hieber and Lucas (2022), for instance, consider three-state models in which individuals are active, care-dependent or dead. Hieber and Lucas (2022) introduce life-care tontines, while Chen et al. (2022) study care-dependent tontines that combine retirement and long-term care benefits. In those models, the state process is hierarchical: once a state has been left, it cannot be re-entered. By contrast, the disability setting considered in this paper requires a recurrent multistate model, because participants may alternate between active and disabled states.

Kabuche et al. (2024) proposed a pooled annuity concept classifying individuals according to functional disability states and chronic illness status. In their multistate extension to group self-annuitization, these authors allow participants to share both longevity and health risks after retirement. To this end, annuity benefits are updated according to both realized health transitions and mortality experienced. Contrarily to previous contributions, recoveries are possible. The fund dynamics as well as the distribution of mortality credits and other transition credits and debits are carefully studied in discrete time. In a similar multistate setting, Zhou and Dhaene (2024) proposed and actuarially fair health-contingent mortality pooling system with health-state-dependent payments allowing for recovery from functional disability and for heterogeneity among members (for instance, with respect to age at entry). Compared to these papers, the contribution of the present work is to propose an appropriate way to share individual case estimates among participants, dynamically from reporting to final settlement. Here, mortality risk is not pooled because the mechanism aims to offer a substitute to disability insurance at working ages. Also, the present paper differs from Maggistro et al. (2026) who proposed a Markov aging model where states correspond to latent heterogeneity that cannot be observed.

The risk-sharing rule used in this paper is the conditional mean risk-sharing rule introduced by Denuit and Dhaene (2012). This mechanism is henceforth referred to as CMRS. Under this rule, the contribution assigned to a participant is the conditional expectation of the loss generated by this participant, given the aggregate loss experienced by the pool. Denuit and Robert (2023) adapted this rule to the compound Poisson risk model by allocating each reported claim at occurrence time, assuming that the claim severity is immediately observable. The present paper addresses a different and practically important situation: in disability insurance, the ultimate claim amount is not known at occurrence time, because it depends on the duration of the disability spell. The loss is progressively revealed and becomes known only at recovery.

This observation leads to the main contribution of the paper. We develop a dynamic CMRS mechanism with settlement delays. When a disability claim is reported, the active participants at risk of having generated the claim contribute an initial reserve. This reserve is then updated as information about the claim development becomes available. If the claim remains open, the reserve is revised by conditioning on the event that the disability spell has lasted beyond the current elapsed duration. At recovery, a final adjustment is made so that the total amount paid by each participant coincides with the terminal CMRS allocation.

The proposed mechanism has several important properties. First, the dynamic CMRS reserve is not an ad hoc provision. It is the conditional expectation of the corresponding thinned individual loss under the information used by the pool. This representation implies a martingale property for the reserve process and gives a natural actuarial fairness property. Second, before settlement, full allocation cannot mean equality with the realized ultimate claim amount, because this amount is not yet known. The relevant property is therefore conditional full allocation: the sum of the reserves is equal to the conditional expectation of the ultimate claim amount given the available claim-development information. Third, the dynamic CMRS contribution is less risky, in convex order, than the corresponding stand-alone incurred-and-reserved disability cost. Hence, every risk-averse agent is willing to join the peer-to-peer pool.

The paper is organized as follows. Section 2 introduces the multistate model, the disability cost process and the basic risk-sharing setting. Section 3 derives the conditional distribution of a reported disability claim, defines the terminal CMRS allocation, and constructs the dynamic reserve process used when the disability duration is not yet known. Section 4 establishes the main theoretical properties of the mechanism: martingale representations, conditional full allocation, actuarial fairness, claim-level and cumulative convex-order reductions, and the resulting dynamic willingness-to-join property. Section 5 provides numerical illustrations for a heterogeneous pool and shows how reserves and final adjustments depend on disability duration and recovery rates. Section 6 discusses extensions to more general multistate models, more flexible benefit structures and loss reserving problems in property and casualty insurance. Section 7 concludes and discusses the relevance of the proposed mechanism for cooperative and solidarity-based insurance arrangements.

2 Setting

2.1 Multistate model

Consider a pool of n participants, with initial ages $x_1, x_2, \ldots, x_n$. Since the paper focuses on short- and medium-term disability protection for relatively young participants, mortality is disregarded. We work in a two-state model with state space $\{e_0, e_1\}$, where e_0 denotes the active state and e_1 the disabled state. Time $t \geq 0$ measures the seniority of the pool, that is, the time elapsed since the pool started operating.

The trajectory of participant i across the two states is represented by a stochastic process $\mathcal{X}_i = \{X_{i,t},\, t \geq 0\}$, where $X_{i,t} \in \{e_0, e_1\}$ is the state occupied by participant i at time t, corresponding to age $x_i + t$. The processes $\mathcal{X}_1, \ldots, \mathcal{X}_n$ are assumed to be independent. All participants are assumed to be active at inception, so that $X_{i,0} = e_0$ for every $i = 1, \ldots, n$.

To obtain explicit formulas and to keep the presentation transparent, we assume that these processes are time-homogeneous Markov chains. For $s < t$, let

$$p_{lk,i}(s,t) = \mathrm{P}[X_{i,t} = e_k \mid X_{i,s} = e_l], \qquad l, k \in \{0, 1\},$$

be the transition probabilities of participant i. The transition intensities are assumed to be constant and are denoted by

$$\lambda_{01,i} = \lim_{\Delta t \downarrow 0} \frac{p_{01,i}(t, t+\Delta t)}{\Delta t}, \qquad \lambda_{10,i} = \lim_{\Delta t \downarrow 0} \frac{p_{10,i}(t, t+\Delta t)}{\Delta t}.$$

The intensity $\lambda_{01,i}$ is the disability incidence rate of participant i, while $\lambda_{10,i}$ is the recovery rate. These parameters may depend on age and on individual risk characteristics.

For $h \geq 0$, the transition probabilities are then given by

$$\begin{aligned} p_{00,i}(s, s+h) &= \frac{\lambda_{10,i}}{\lambda_{01,i} + \lambda_{10,i}} + \frac{\lambda_{01,i}}{\lambda_{01,i} + \lambda_{10,i}} \exp\{-(\lambda_{01,i} + \lambda_{10,i})h\}, \\ p_{01,i}(s, s+h) &= 1 - p_{00,i}(s, s+h), \end{aligned}$$

and

$$p_{11,i}(s, s+h) = \frac{\lambda_{01,i}}{\lambda_{01,i} + \lambda_{10,i}} + \frac{\lambda_{10,i}}{\lambda_{01,i} + \lambda_{10,i}} \exp\{-(\lambda_{01,i} + \lambda_{10,i})h\},$$
$$p_{10,i}(s, s+h) = 1 - p_{11,i}(s, s+h).$$

Because the model is time-homogeneous, these transition probabilities depend on s and $s+h$ only through the elapsed duration h.

Extensions to non-homogeneous Markov or semi-Markov models are discussed in Section 6. The risk-sharing mechanism remains essentially the same, but the explicit formulas derived below are replaced by conditional expectations computed under the corresponding transition structure. This is the reason why we first deal with the homogeneous Markov case where the mechanism is transparent, easing exposition.

2.2 Disability costs

Disability costs are attached to sojourns in state e_1. For $k \in \{0, 1\}$, define

$$\mathbf{1}_{\{X_{i,t}=e_k\}} = \begin{cases} 1, & \text{if } X_{i,t} = e_k, \\ 0, & \text{otherwise.} \end{cases}$$

We first assume that the benefit rate paid during disability is the same for all participants and is equal to one. This normalization is without loss of generality when benefits are proportional to time spent in disability. More general benefit structures, including individual benefit levels, deferred periods and maximum benefit durations, are discussed in Section 6.

The cumulative disability cost generated by participant i up to time t is therefore

$$L_{i,t} = \int_0^t \mathbf{1}_{\{X_{i,u}=e_1\}}\, du. \tag{2.1}$$

Thus, $L_{i,t}$ is the total time spent by participant i in disability over the interval $[0, t]$. In the present normalization, this time is also the total amount of disability benefits paid to participant i up to time t.

2.3 Risk-sharing setting

The participants form a pool in order to mutually protect against the economic consequences of disability. The pool compensates disabled participants for their disability costs, and participants contribute to the pool according to a risk-sharing rule. We denote by $C_{i,t}$ the total contribution paid by participant i over the time interval $[0, t]$.

Let

$$M_k(t) = \sum_{j=1}^{n} \mathbf{1}_{\{X_{j,t}=e_k\}}, \qquad k \in \{0, 1\},$$

be the number of participants in state e_k at time t. Hence, $M_0(t)$ is the number of active participants and $M_1(t)$ is the number of disabled participants. Under the normalized benefit rate used in this paper, $M_1(t)$ is the instantaneous aggregate benefit rate paid by the pool at time t, and

$$\int_0^t M_1(u)\,du = \sum_{i=1}^{n} L_{i,t}$$

is the total amount of benefits paid by the pool up to time t.

One possible approach would be to share the instantaneous amount $M_1(t)$ among all participants according to the conditional probabilities that each participant is disabled at time t. Such a rule would be close to a continuous-time version of the risk-sharing mechanisms studied in Denuit et al. (2022b). This paper follows a different approach. Instead of allocating the instantaneous aggregate benefit rate, we allocate each disability spell as a claim reported when a participant enters state e_1 and settled when the participant recovers.

This distinction is central. At the reporting time of a disability claim, the ultimate cost of the claim is not known, because it is determined by the future duration of the disability spell. The risk-sharing rule must therefore account for the development of the claim from occurrence to settlement. The next section introduces this claim-level representation and constructs the corresponding dynamic CMRS mechanism.

3 CMRS rule with settlement delays

3.1 Individual disability spells

We first describe the individual claim process generated by the two-state Markov model introduced in Section 2. For participant i, let $A_{0,i}, A_{1,i}, A_{2,i}, \ldots$ denote the successive sojourn times in the active state e_0. In the homogeneous Markov setting considered here, these random variables are independent and exponentially distributed with parameter $\lambda_{01,i}$. Thus,

$$\mathrm{P}[A_{\ell,i} > t] = \exp(-\lambda_{01,i} t), \qquad t \geq 0.$$

Similarly, let $D_{1,i}, D_{2,i}, D_{3,i}, \ldots$ denote the successive sojourn times in the disabled state e_1. These random variables are independent and exponentially distributed with parameter $\lambda_{10,i}$, so that

$$\mathrm{P}[D_{\ell,i} > t] = \exp(-\lambda_{10,i} t), \qquad t \geq 0.$$

All active and disabled sojourn times are independent. Under the normalized benefit rate, the duration $D_{\ell,i}$ is also the amount of benefits paid during the ℓ-th disability spell of participant i.

Participant i first remains active during $A_{0,i}$, then enters disability for duration $D_{1,i}$, then becomes active again during $A_{1,i}$, and so on. The reporting time of the k-th disability claim of participant i is therefore

$$T_{k,i} = A_{0,i} + \sum_{\ell=1}^{k-1} \big(D_{\ell,i} + A_{\ell,i}\big), \qquad k \geq 1,$$

with the convention that the empty sum is equal to zero. The corresponding individual claim-counting process is

$$N_{i,t}^{01} = \sum_{k \geq 1} \mathbf{1}_{\{T_{k,i} \leq t\}}, \qquad t \geq 0. \tag{3.1}$$

The severity of the k-th disability claim of participant i is $Y_{k,i} = D_{k,i}$. It is noteworthy that $Y_{k,i}$ is not observed at the reporting time $T_{k,i}$. It is only known at the recovery time $T_{k,i} + D_{k,i}$. Thus, in the present disability setting, claim occurrence and claim settlement are separated by a random delay. This is one of the main differences with the occurrence-time setting of Denuit and Robert (2023), where the claim severity is assumed to be observable when the claim is reported.

Under more general benefit structures, discussed in Section 6, the claim amount $Y_{k,i}$ may differ from the disability duration $D_{k,i}$. For instance, it may include individual benefit levels, deferred periods or maximum payment durations. In the present section, however, $Y_{k,i} = D_{k,i}$.

3.2 Pooled claim process

The pool is impacted by the disability claims reported by all participants. The aggregate claim-counting process is

$$N_t^{01} = \sum_{i=1}^{n} N_{i,t}^{01}, \qquad t \geq 0.$$

Let

$$T_k = \inf\{t \geq 0 : N_t^{01} \geq k\}$$

be the reporting time of the k-th claim at the pool level. Since the individual transition times are continuous random variables, simultaneous claim reports occur with probability zero.

Let I_k denote the identity of the participant who reports the k-th claim:

$$I_k = \sum_{i=1}^{n} i \, \mathbf{1}_{\{N_{i,T_k}^{01} - N_{i,T_k-}^{01} = 1\}}.$$

The corresponding claim amount is

$$Y_k = \sum_{i=1}^{n} \mathbf{1}_{\{N_{i,T_k}^{01} - N_{i,T_k-}^{01} = 1\}} Y_{N_{i,T_k}^{01}, i}. \tag{3.2}$$

Equivalently, $Y_k = Y_{N_{I_k,T_k}^{01}, I_k}$.

Only participants who are active immediately before T_k can report a claim at time T_k. We denote the corresponding risk set by

$$\mathcal{R}_k = \{i : X_{i,T_k-} = e_0\},$$

and set

$$\Lambda_k = \sum_{j \in \mathcal{R}_k} \lambda_{01,j}.$$

Conditionally on the information available immediately before T_k, the probability that participant i reports the k-th claim is

$$\mathrm{P}[I_k = i \mid T_k, \mathcal{R}_k] = \begin{cases} \dfrac{\lambda_{01,i}}{\Lambda_k}, & i \in \mathcal{R}_k, \\ 0, & i \notin \mathcal{R}_k. \end{cases} \tag{3.3}$$

Given $I_k = i$, the claim duration Y_k is exponentially distributed with parameter $\lambda_{10,i}$.

The following result gives the conditional distribution of the k-th claim amount.

Property 3.1. *Given T_k and $\mathcal{R}_k$, the conditional distribution of the claim amount Y_k is described by:*

(i) the probability density function

$$f_k(y) = \frac{1}{\Lambda_k} \sum_{j \in \mathcal{R}_k} \lambda_{01,j} \lambda_{10,j} \exp(-\lambda_{10,j} y), \qquad y > 0;$$

(ii) the survival function

$$\overline{F}_k(y) = \frac{1}{\Lambda_k} \sum_{j \in \mathcal{R}_k} \lambda_{01,j} \exp(-\lambda_{10,j} y), \qquad y > 0;$$

(iii) the hazard function

$$\gamma_k(y) = \frac{\sum_{j \in \mathcal{R}_k} \lambda_{01,j} \lambda_{10,j} \exp(-\lambda_{10,j} y)}{\sum_{j \in \mathcal{R}_k} \lambda_{01,j} \exp(-\lambda_{10,j} y)}, \qquad y > 0.$$

Proof. For $y > 0$, the conditional survival function is obtained from

$$\begin{aligned}
\mathrm{P}[Y_k > y \mid T_k, \mathcal{R}_k] &= \sum_{j=1}^{n} \mathrm{P}[Y_k > y \mid T_k, \mathcal{R}_k, I_k = j] \, \mathrm{P}[I_k = j \mid T_k, \mathcal{R}_k] \\
&= \sum_{j \in \mathcal{R}_k} \mathrm{P}[D_j > y] \, \frac{\lambda_{01,j}}{\Lambda_k} \\
&= \frac{1}{\Lambda_k} \sum_{j \in \mathcal{R}_k} \lambda_{01,j} \exp(-\lambda_{10,j} y).
\end{aligned}$$

The density follows by differentiation, and the hazard function is obtained from $\gamma_k(y) = \frac{f_k(y)}{\overline{F}_k(y)}$. □

We also define the cumulative hazard function

$$\Gamma_k(y) = \int_0^y \gamma_k(u) \, \mathrm{d}u = -\ln \overline{F}_k(y), \qquad y > 0.$$

Hence,

$$f_k(y) = \overline{F}_k(y) \gamma_k(y) = \exp\{-\Gamma_k(y)\} \gamma_k(y).$$

3.3 Terminal CMRS allocation

At settlement time $T_k + Y_k$, the ultimate amount Y_k is known. The CMRS rule allocates this amount by considering, for each participant i, the individual loss component

$$Z_{i,k} = Y_k \mathbf{1}_{\{I_k=i\}}. \tag{3.4}$$

Thus $Z_{i,k} = Y_k$ if participant i reported the k-th claim, and $Z_{i,k} = 0$ otherwise. By construction,

$$\sum_{i=1}^{n} Z_{i,k} = Y_k.$$

The CMRS allocation of the k-th claim to participant i is defined as

$$h_{i,k}^{\mathrm{cmrs}}(Y_k) = \mathrm{E}[Z_{i,k} \mid T_k, \mathcal{R}_k, Y_k].$$

The next result gives its explicit expression.

Proposition 3.2. *At settlement time $T_k + Y_k$, the CMRS allocation of the k-th disability claim to participant i is*

$$h_{i,k}^{\mathrm{cmrs}}(Y_k) = \begin{cases} Y_k \dfrac{\lambda_{01,i}\lambda_{10,i}\exp(-\lambda_{10,i}Y_k)}{\sum_{j\in\mathcal{R}_k}\lambda_{01,j}\lambda_{10,j}\exp(-\lambda_{10,j}Y_k)}, & i \in \mathcal{R}_k, \\ 0, & i \notin \mathcal{R}_k. \end{cases}$$

Proof. If $i \notin \mathcal{R}_k$, participant i cannot have reported the claim at time T_k, and the allocation is zero. For $i \in \mathcal{R}_k$, Bayes' formula gives, for $y > 0$,

$$\begin{aligned} \mathrm{P}[I_k = i \mid T_k, \mathcal{R}_k, Y_k = y] &= \frac{f_{D_i}(y)\mathrm{P}[I_k = i \mid T_k, \mathcal{R}_k]}{f_k(y)} \\ &= \frac{\lambda_{01,i}\lambda_{10,i}\exp(-\lambda_{10,i}y)}{\sum_{j\in\mathcal{R}_k}\lambda_{01,j}\lambda_{10,j}\exp(-\lambda_{10,j}y)}. \end{aligned}$$

Since $Z_{i,k} = Y_k\mathbf{1}_{\{I_k=i\}}$, we obtain

$$\mathrm{E}[Z_{i,k} \mid T_k, \mathcal{R}_k, Y_k = y] = y\mathrm{P}[I_k = i \mid T_k, \mathcal{R}_k, Y_k = y],$$

which gives the stated expression. □

The allocation is fully allocated at settlement since we obviously have $\sum_{i=1}^{n} h_{i,k}^{\mathrm{cmrs}}(Y_k) = Y_k$. Moreover, for $i \in \mathcal{R}_k$ and $y > 0$, $h_{i,k}^{\mathrm{cmrs}}(\cdot)$ in Proposition 3.2 can be rewritten as

$$\begin{aligned} h_{i,k}^{\mathrm{cmrs}}(y) &= y\frac{\lambda_{01,i}f_{D_i}(y)}{\sum_{j\in\mathcal{R}_k}\lambda_{01,j}f_{D_j}(y)} \\ &= y\frac{\lambda_{01,i}\lambda_{10,i}}{\sum_{j\in\mathcal{R}_k}\lambda_{01,j}\lambda_{10,j}\exp\{-(\lambda_{10,j}-\lambda_{10,i})y\}}. \end{aligned}$$

This expression shows that long-lasting claims are increasingly allocated to participants with smaller recovery rates. This feature is illustrated numerically in Section 5.

If Y_k were known at the reporting time T_k, the amount $h_{i,k}^{\mathrm{cmrs}}(Y_k)$ could be allocated immediately. In the present disability setting, however, Y_k is only known at recovery. The risk-sharing rule must therefore be implemented dynamically through a reserve process.

3.4 Dynamic CMRS reserve

Let $u = t - T_k$ denote the elapsed duration since the reporting time of the k-th claim. Before settlement, the information available about the claim development is that the disability spell is still open, that is $Y_k > u$. After settlement, the realized value Y_k is known.

For participant i, define the dynamic CMRS reserve attached to the k-th claim by

$$R_{i,k}^{\text{cmrs}}(u) = \mathrm{E}\left[h_{i,k}^{\text{cmrs}}(Y_k) \mid T_k, \mathcal{R}_k, \text{claim-development information up to elapsed time } u\right].$$

Equivalently, before settlement,

$$R_{i,k}^{\text{cmrs}}(u) = \mathrm{E}\left[h_{i,k}^{\text{cmrs}}(Y_k) \mid T_k, \mathcal{R}_k, Y_k > u\right], \qquad 0 \le u < Y_k,$$

whereas after settlement,

$$R_{i,k}^{\text{cmrs}}(u) = h_{i,k}^{\text{cmrs}}(Y_k), \qquad u \ge Y_k. \tag{3.5}$$

For $i \notin \mathcal{R}_k$, the reserve is identically equal to zero.

Let us now derive the evolution of the CMRS reserve over time for $i \in \mathcal{R}_k$. At reporting time, the initial reserve is

$$\begin{aligned} R_{i,k}^{\text{cmrs}}(0) &= \mathrm{E}[h_{i,k}^{\text{cmrs}}(Y_k) \mid T_k, \mathcal{R}_k] \\ &= \int_0^\infty h_{i,k}^{\text{cmrs}}(y) f_k(y)\,\mathrm{d}y \\ &= \frac{\lambda_{01,i}}{\Lambda_k} \int_0^\infty y\lambda_{10,i} \exp(-\lambda_{10,i} y)\,\mathrm{d}y \\ &= \frac{\lambda_{01,i}}{\Lambda_k} \frac{1}{\lambda_{10,i}}, \qquad i \in \mathcal{R}_k. \end{aligned} \tag{3.6}$$

Equivalently,

$$R_{i,k}^{\text{cmrs}}(0) = \mathrm{P}[I_k = i \mid T_k, \mathcal{R}_k]\mathrm{E}[D_i], \qquad i \in \mathcal{R}_k.$$

For $0 < u < Y_k$, we obtain

$$\begin{aligned} R_{i,k}^{\text{cmrs}}(u) &= \int_u^\infty h_{i,k}^{\text{cmrs}}(y) \frac{f_k(y)}{\overline{F}_k(u)}\,\mathrm{d}y \\ &= \frac{\lambda_{01,i}}{\sum_{j \in \mathcal{R}_k} \lambda_{01,j} \exp(-\lambda_{10,j} u)} \int_u^\infty y\lambda_{10,i} \exp(-\lambda_{10,i} y)\,\mathrm{d}y \\ &= \frac{\lambda_{01,i} \exp(-\lambda_{10,i} u)}{\sum_{j \in \mathcal{R}_k} \lambda_{01,j} \exp(-\lambda_{10,j} u)} \left(u + \frac{1}{\lambda_{10,i}}\right). \end{aligned} \tag{3.7}$$

The last equality follows from integration by parts. Equivalently,

$$R_{i,k}^{\text{cmrs}}(u) = \mathrm{P}[I_k = i \mid T_k, \mathcal{R}_k, Y_k > u]\,\mathrm{E}[D_i \mid D_i > u].$$

This formula has a natural interpretation. While the claim is still open after elapsed duration u, the reserve is the product of two terms: the updated conditional probability that participant i generated the claim, and the conditional expected ultimate duration of a disability spell of participant i given survival beyond u.

In calendar time, for $T_k \le t < T_k + Y_k$, this reserve can be written as

$$\widehat{h}_{i,k|t}^{\mathrm{cmrs}} = R_{i,k}^{\mathrm{cmrs}}(t - T_k).$$

Thus,

$$\widehat{h}_{i,k|t}^{\mathrm{cmrs}} = \frac{\lambda_{01,i} \exp\{-\lambda_{10,i}(t - T_k)\}}{\sum_{j \in \mathcal{R}_k} \lambda_{01,j} \exp\{-\lambda_{10,j}(t - T_k)\}} \left(t - T_k + \frac{1}{\lambda_{10,i}} \right).$$

The reserve is continuously updated while the claim remains open. Its adjustment rate, as a function of elapsed duration u, is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u} R_{i,k}^{\mathrm{cmrs}}(u) &= \frac{\mathrm{d}}{\mathrm{d}u} \left(\int_u^\infty h_{i,k}^{\mathrm{cmrs}}(y) \frac{f_k(y)}{\overline{F}_k(u)} \,\mathrm{d}y \right) \\ &= \gamma_k(u) \left(R_{i,k}^{\mathrm{cmrs}}(u) - h_{i,k}^{\mathrm{cmrs}}(u) \right).\end{aligned}$$

The adjustment rate may be negative for some participants, leading to a partial reimbursement of previous contributions.

At settlement time $T_k + Y_k$, the ultimate claim amount becomes known. A final adjustment is then made so that the reserve reaches the terminal CMRS allocation. The final adjustment paid by participant i is

$$\begin{aligned}h_{i,k}^{\mathrm{cmrs}}(Y_k) - R_{i,k}^{\mathrm{cmrs}}(Y_k-) = Y_k &\frac{\lambda_{01,i}\lambda_{10,i} \exp(-\lambda_{10,i} Y_k)}{\sum_{j \in \mathcal{R}_k} \lambda_{01,j}\lambda_{10,j} \exp(-\lambda_{10,j} Y_k)} \\ &- \frac{\lambda_{01,i} \exp(-\lambda_{10,i} Y_k)}{\sum_{j \in \mathcal{R}_k} \lambda_{01,j} \exp(-\lambda_{10,j} Y_k)} \left(Y_k + \frac{1}{\lambda_{10,i}} \right).\end{aligned}$$

3.5 Cumulative individual contributions

Each participant contributes to claims that occur while he or she belongs to the risk set, that is, while he or she is active and therefore could have reported the claim. Participants do not contribute to disability claims starting while they are themselves disabled. They nevertheless continue to adjust reserves for claims that started while they were active.

At calendar time t, the cumulative incurred-and-reserved CMRS contribution of participant i is

$$C_{i,t}^{\mathrm{cmrs}} = \sum_{k : T_k \le t} R_{i,k}^{\mathrm{cmrs}}(t - T_k). \tag{3.8}$$

Equivalently,

$$C_{i,t}^{\mathrm{cmrs}} = \sum_{k : T_k + Y_k \le t} h_{i,k}^{\mathrm{cmrs}}(Y_k) + \sum_{k : T_k \le t < T_k + Y_k} \widehat{h}_{i,k|t}^{\mathrm{cmrs}}.$$

The first sum corresponds to claims already settled by time t. The second sum corresponds to claims reported before time t but still open at time t; for these claims, the participant holds a dynamically updated reserve.

This expression makes clear that $C_{i,t}^{\mathrm{cmrs}}$ is not merely the amount already paid for closed claims. It is an incurred-and-reserved quantity: it combines final CMRS allocations for settled claims and dynamic CMRS reserves for open claims. This is the relevant object for the fairness and willingness-to-join properties established in the next section.

4 Properties of the dynamic risk-sharing mechanism

This section establishes the main properties of the dynamic risk-sharing mechanism introduced in Section 3. The key difference with the occurrence-time compound Poisson framework of Denuit and Robert (2023) is that the ultimate claim amount is not known when the claim is reported. It is progressively revealed through the duration of the disability spell and becomes observable only at recovery. Therefore, the relevant objects are not only terminal claim allocations, but incurred-and-reserved contributions attached to reported claims.

Let $\mathcal{F}_{T_k-}$ denote the information available immediately before the k-th claim is reported. Conditionally on this pre-claim information, participant $i \in \mathcal{R}_k$ reports the claim with probability given in (3.3) whereas participants outside $\mathcal{R}_k$ cannot report the claim. Given $I_k = i$, the duration Y_k is exponentially distributed with parameter $\lambda_{10,i}$.

For a fixed participant i, the thinned individual loss associated with the k-th reported claim is $Z_{i,k}$ defined in (3.4). Thus, $Z_{i,k}$ is the ultimate loss generated by participant i at the k-th pool claim time. It is equal to zero if the claim was reported by another participant. Conditionally on $\mathcal{F}_{T_k-}$,

$$\mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}] = \mathbf{1}_{\{i\in\mathcal{R}_k\}}\frac{\lambda_{01,i}}{\Lambda_k}\frac{1}{\lambda_{10,i}}.$$

The dynamic CMRS mechanism and the stand-alone benchmark considered below are two conditional projections of this same terminal random variable under two different information structures. This allows us to establish comparison in the convex order, whose definition is recalled next.

Given two random variables V and W, V is said to be smaller than W in the convex order, henceforth denoted by $V \preceq_{\text{cx}} W$ if the inequality $\mathrm{E}[g(V)] \leq \mathrm{E}[g(W)]$ holds for all convex functions g for which the expectations exist. Since the functions $x \mapsto \pm x$ are both convex, $\preceq_{\text{cx}}$ only applies to random variables with the same expected value. The stochastic inequality $V \preceq_{\text{cx}} W$ intuitively means that V and W have the same "size" (as $\mathrm{E}[V] = \mathrm{E}[W]$ holds) but that W is "more variable" than V. For instance, the variance of W is larger than the variance of V. In the expected utility setting, $V \preceq_{\text{cx}} W$ means that all risk-averse agents prefer to be exposed to the loss V than to the loss W. A general treatment of this order relation can be found in Shaked and Shanthikumar (2007), for instance.

4.1 Terminal CMRS allocation

At settlement time $T_k + Y_k$, the ultimate amount Y_k is known. The CMRS allocation of the k-th claim to participant i is

$$h^{\text{cmrs}}_{i,k}(Y_k) = \mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}, Y_k]$$

where $h^{\text{cmrs}}_{i,k}(\cdot)$ is given in Proposition 3.2.

Because $h^{\text{cmrs}}_{i,k}(Y_k)$ is a conditional expectation of $Z_{i,k}$, Jensen's inequality gives

$$h^{\text{cmrs}}_{i,k}(Y_k) \preceq_{\text{cx}} Z_{i,k}, \qquad \text{conditionally on } \mathcal{F}_{T_k-}.$$

This is the terminal convex-order comparison. In the present disability setting, however, it cannot be implemented at the reporting time T_k, because Y_k is not yet known.

4.2 Thinned individual reserving benchmark

We now define the stand-alone incurred-and-reserved benchmark at pool claim times. Let

$$B_{i,k} = \mathbf{1}_{\{I_k=i\}} \text{ so that } Z_{i,k} = B_{i,k} Y_k.$$

Conditionally on $\mathcal{F}_{T_k-}$,

$$\mathrm{P}[B_{i,k} = 1 \mid \mathcal{F}_{T_k-}] = \mathbf{1}_{\{i\in\mathcal{R}_k\}} \frac{\lambda_{01,i}}{\Lambda_k}.$$

Let $\mathcal{G}_{k,u}$ denote the information generated by the development of the k-th claim up to elapsed duration u, without revealing the identity of the participant who reported the claim. Thus, on the event $\{Y_k > u\}$, the claim is known to be still open after elapsed duration u; on the event $\{Y_k \leq u\}$, the claim is settled and the realized duration Y_k is known.

For participant i, define the finer information

$$\mathcal{H}^i_{k,u} = \mathcal{G}_{k,u} \vee \sigma(B_{i,k}),$$

which additionally reveals whether the k-th claim belongs to participant i. The thinned individual reserve benchmark is

$$R^{\text{thin}}_{i,k}(u) = \mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}, \mathcal{H}^i_{k,u}]. \tag{4.1}$$

It is explicitly given by

$$R^{\text{thin}}_{i,k}(u) = \begin{cases} 0, & \text{if } B_{i,k} = 0, \\ u + \dfrac{1}{\lambda_{10,i}}, & \text{if } B_{i,k} = 1,\ Y_k > u, \\ Y_k, & \text{if } B_{i,k} = 1,\ Y_k \leq u. \end{cases}$$

Thus, $R^{\text{thin}}_{i,k}(u)$ is the incurred-and-reserved cost of participant i attached to the k-th reported claim, under the stand-alone information revealing whether this claim is his or hers. At occurrence time,

$$R^{\text{thin}}_{i,k}(0) = \frac{B_{i,k}}{\lambda_{10,i}},$$

and after settlement,

$$R^{\text{thin}}_{i,k}(u) = Z_{i,k}, \qquad u \geq Y_k.$$

4.3 Dynamic CMRS reserve as a coarser projection

The dynamic CMRS mechanism uses the coarser information $\mathcal{G}_{k,u}$, which follows the development of the claim but does not use the identity of the participant who generated it. The CMRS reserve attached to the k-th claim is therefore

$$R^{\text{cmrs}}_{i,k}(u) = \mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}]. \tag{4.2}$$

Equivalently,

$$R^{\text{cmrs}}_{i,k}(u) = \mathrm{E}[h^{\text{cmrs}}_{i,k}(Y_k) \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}].$$

On the event $\{Y_k > u\}$, $R^{\mathrm{cmrs}}_{i,k}(u)$ is given in (3.7) for $i \in \mathcal{R}_k$. For $i \notin \mathcal{R}_k$, the reserve is equal to zero. Once the claim is settled, we know from (3.5) that $R^{\mathrm{cmrs}}_{i,k}(u) = h^{\mathrm{cmrs}}_{i,k}(Y_k)$, $u \geq Y_k$. In particular, at reporting time, $R^{\mathrm{cmrs}}_{i,k}(0)$ is given in (3.6).

Since $\mathcal{G}_{k,u} \subset \mathcal{H}^i_{k,u}$, the tower property gives the fundamental projection identity

$$R^{\mathrm{cmrs}}_{i,k}(u) = \mathrm{E}[R^{\mathrm{thin}}_{i,k}(u) \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}].$$

This identity is the main structural property of the dynamic CMRS mechanism. It shows that the CMRS reserve is obtained by smoothing the individual incurred-and-reserved benchmark over the coarser information used by the pool.

4.4 Martingale and fairness properties

The conditional-projection representation directly yields martingale and fairness properties.

Proposition 4.1 (Martingale properties)**.** *For every participant i and every claim k, the process $\{R^{\mathrm{thin}}_{i,k}(u),\, u \geq 0\}$ is a martingale with respect to $\{\mathcal{H}^i_{k,u}, u \geq 0\}$, conditionally on $\mathcal{F}_{T_k-}$. Similarly, the process $\{R^{\mathrm{cmrs}}_{i,k}(u),\, u \geq 0\}$ is a martingale with respect to $\{\mathcal{G}_{k,u}, u \geq 0\}$, conditionally on $\mathcal{F}_{T_k-}$.*

Proof. Considering (4.1) and (4.2), both processes are conditional expectations of the same terminal random variable $Z_{i,k}$ with respect to increasing claim-development filtrations. The announced result then follows from the tower property of conditional expectations. □

The same representation gives the following claim-by-claim fairness property.

Proposition 4.2 (Claim-by-claim fairness)**.** *For every participant i, every claim k, and every elapsed duration $u \geq 0$,*

$$\mathrm{E}[R^{\mathrm{cmrs}}_{i,k}(u) \mid \mathcal{F}_{T_k-}] = \mathrm{E}[R^{\mathrm{thin}}_{i,k}(u) \mid \mathcal{F}_{T_k-}] = \mathbf{1}_{\{i\in\mathcal{R}_k\}} \frac{\lambda_{01,i}}{\Lambda_k} \frac{1}{\lambda_{10,i}}.$$

Proof. Taking conditional expectations in (4.1) and (4.2) gives

$$\mathrm{E}[R^{\mathrm{cmrs}}_{i,k}(u) \mid \mathcal{F}_{T_k-}] = \mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}],$$

and

$$\mathrm{E}[R^{\mathrm{thin}}_{i,k}(u) \mid \mathcal{F}_{T_k-}] = \mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}].$$

Since

$$\mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}] = R^{\mathrm{cmrs}}_{i,k}(0) = \mathbf{1}_{\{i\in\mathcal{R}_k\}} \frac{\lambda_{01,i}}{\Lambda_k} \frac{1}{\lambda_{10,i}},$$

the announced result follows. □

The cumulative CMRS contribution $C^{\mathrm{cmrs}}_{i,t}$ of participant i at calendar time t is defined in (3.8). Similarly, define the cumulative thinned individual reserve benchmark by

$$C^{\mathrm{thin}}_{i,t} = \sum_{k:T_k\leq t} R^{\mathrm{thin}}_{i,k}(t - T_k).$$

Summing the claim-by-claim fairness identities yields $\mathrm{E}[C_{i,t}^{\mathrm{cmrs}}] = \mathrm{E}[C_{i,t}^{\mathrm{thin}}]$, $t \geq 0$. The process $C_{i,t}^{\mathrm{thin}}$ is the thinned representation of the stand-alone incurred-and-reserved disability cost of participant i. Hence, the dynamic risk-sharing mechanism is fair in the actuarial sense that the expected contribution of participant i is equal to the expected amount of his or her own incurred and reserved losses.

This fairness statement is more appropriate than an equality with $L_{i,t}$ in (2.1) which only records benefits already paid up to time t. At a finite time horizon, $C_{i,t}^{\mathrm{cmrs}}$ also contains reserves for claims that have started before t but are still open at t.

4.5 Conditional full allocation

For a settled claim, the CMRS allocation is fully allocated. Before settlement, full allocation cannot hold in the usual sense because the ultimate claim amount is not known. It is therefore replaced with conditional full allocation of the ultimate claim amount.

Proposition 4.3 (Conditional full allocation)**.** *For every claim k and every elapsed duration $u \geq 0$,*

$$\sum_{i=1}^{n} R_{i,k}^{\mathrm{cmrs}}(u) = \mathrm{E}[Y_k \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}].$$

In particular, on the event $\{Y_k > u\}$,

$$\sum_{i=1}^{n} R_{i,k}^{\mathrm{cmrs}}(u) = \frac{\sum_{j \in \mathcal{R}_k} \lambda_{01,j} \exp(-\lambda_{10,j} u) \left(u + \frac{1}{\lambda_{10,j}}\right)}{\sum_{j \in \mathcal{R}_k} \lambda_{01,j} \exp(-\lambda_{10,j} u)},$$

whereas on the event $\{Y_k \leq u\}$, $\sum_{i=1}^{n} R_{i,k}^{\mathrm{cmrs}}(u) = Y_k$.

Proof. Since $\sum_{i=1}^{n} Z_{i,k} = Y_k$, we have

$$\begin{aligned} \sum_{i=1}^{n} R_{i,k}^{\mathrm{cmrs}}(u) &= \sum_{i=1}^{n} \mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}] \\ &= \mathrm{E}[Y_k \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}]. \end{aligned}$$

The explicit expression on $\{Y_k > u\}$ follows from the conditional mixture distribution of Y_k given $\mathcal{F}_{T_k-}$ and survival beyond u. On $\{Y_k \leq u\}$, the realized duration is known, hence the conditional expectation is Y_k. □

Thus, before settlement, the sum of the reserves is not equal to the amount of benefits already paid. It is equal to the conditional expectation of the ultimate claim amount. This is the appropriate full-allocation property when the ultimate claim amount is still unknown.

4.6 Claim-level convex-order reduction

We now compare the dynamic CMRS reserve with the thinned individual reserve. The projection identity

$$R_{i,k}^{\mathrm{cmrs}}(u) = \mathrm{E}[R_{i,k}^{\mathrm{thin}}(u) \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}]$$

implies the following convex-order comparison.

Proposition 4.4 (Claim-level convex-order reduction)**.** *For every participant i, every claim k, and every elapsed duration $u \geq 0$,*

$$R_{i,k}^{\mathrm{cmrs}}(u) \preceq_{\mathrm{cx}} R_{i,k}^{\mathrm{thin}}(u), \qquad \textit{conditionally on } \mathcal{F}_{T_k-}.$$

Proof. Let φ be any convex function for which the expectations exist. Since

$$R_{i,k}^{\mathrm{cmrs}}(u) = \mathrm{E}[R_{i,k}^{\mathrm{thin}}(u) \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}],$$

Jensen's inequality gives

$$\varphi(R_{i,k}^{\mathrm{cmrs}}(u)) \leq \mathrm{E}[\varphi(R_{i,k}^{\mathrm{thin}}(u)) \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}].$$

Taking conditional expectations given $\mathcal{F}_{T_k-}$ yields

$$\mathrm{E}[\varphi(R_{i,k}^{\mathrm{cmrs}}(u)) \mid \mathcal{F}_{T_k-}] \leq \mathrm{E}[\varphi(R_{i,k}^{\mathrm{thin}}(u)) \mid \mathcal{F}_{T_k-}].$$

Moreover, the two random variables have the same conditional mean by Proposition 4.2. This proves the conditional convex-order comparison. □

At occurrence time $u = 0$, the comparison reduces to

$$\mathbf{1}_{\{i \in \mathcal{R}_k\}} \frac{\lambda_{01,i}}{\Lambda_k} \frac{1}{\lambda_{10,i}} \preceq_{\mathrm{cx}} \frac{B_{i,k}}{\lambda_{10,i}}.$$

Thus the CMRS initial reserve is deterministic conditionally on the pre-claim information, whereas the thinned individual reserve is a Bernoulli mixture. At settlement, the comparison becomes

$$h_{i,k}^{\mathrm{cmrs}}(Y_k) \preceq_{\mathrm{cx}} Z_{i,k},$$

which is the terminal analogue of the convex-order result obtained by Denuit and Robert (2023) in the compound Poisson framework.

4.7 Cumulative convex-order reduction

We now aggregate the claim-level comparison. Some care is needed because convex order is not, in general, preserved by summation when the summands are dependent. We therefore do not rely on a direct summation argument. Instead, we show that the cumulative CMRS contribution is itself a conditional expectation of the cumulative thinned individual benchmark with respect to a coarser information structure.

Fix a calendar time $t \geq 0$ and set

$$K_t = N_t^{01}, \qquad u_k(t) = t - T_k, \qquad k = 1, \ldots, K_t.$$

The quantity $u_k(t)$ is the elapsed duration since the reporting time of the k-th claim.

Let $\mathcal{A}_t$ denote the claim-arrival skeleton up to time t, that is, the sigma-field generated by the number of reported claims K_t, the reporting times $T_1, \ldots, T_{K_t}$, and the corresponding risk sets $\mathcal{R}_1, \ldots, \mathcal{R}_{K_t}$. Conditionally on $\mathcal{A}_t$, each reported claim has a well-defined risk set and a mixture distribution over the participants in this risk set.

For each claim $k \le K_t$, let $\mathcal{G}_{k,u_k(t)}$ be the claim-development information used by the risk-sharing mechanism up to time t. This information tells whether the claim is still open at time t, and, if it is already closed, its realized duration. It does not reveal which participant generated the claim for allocation purposes.

For a fixed participant i, define the global coarse and fine sigma-fields

$$\mathcal{G}_t = \mathcal{A}_t \vee \bigvee_{k=1}^{K_t} \mathcal{G}_{k,u_k(t)} \text{ and } \mathcal{H}_t^i = \mathcal{G}_t \vee \bigvee_{k=1}^{K_t} \sigma(B_{i,k}).$$

Clearly, $\mathcal{G}_t \subset \mathcal{H}_t^i$.

The cumulative thinned ultimate loss generated by participant i among the claims reported before time t is

$$Z_{i,t} = \sum_{k=1}^{K_t} Z_{i,k}.$$

The thinned individual incurred-and-reserved benchmark is

$$C_{i,t}^{\text{thin}} = \mathrm{E}[Z_{i,t} \mid \mathcal{H}_t^i] = \sum_{k=1}^{K_t} R_{i,k}^{\text{thin}}(u_k(t)).$$

Similarly, the cumulative CMRS contribution is the conditional expectation of the same cumulative thinned ultimate loss under the coarser information used by the risk-sharing mechanism:

$$C_{i,t}^{\text{cmrs}} = \mathrm{E}[Z_{i,t} \mid \mathcal{G}_t] = \sum_{k=1}^{K_t} R_{i,k}^{\text{cmrs}}(u_k(t)).$$

Proposition 4.5 (Cumulative convex-order reduction)**.** *For every participant i and every time horizon $t \ge 0$, $C_{i,t}^{\text{cmrs}} \preceq_{\text{cx}} C_{i,t}^{\text{thin}}$. More precisely, conditionally on the claim-arrival skeleton $\mathcal{A}_t$, $C_{i,t}^{\text{cmrs}} \preceq_{\text{cx}} C_{i,t}^{\text{thin}}$.*

Proof. Since $\mathcal{G}_t \subset \mathcal{H}_t^i$, the tower property gives

$$\mathrm{E}[C_{i,t}^{\text{thin}} \mid \mathcal{G}_t] = \mathrm{E}\big[\mathrm{E}[Z_{i,t} \mid \mathcal{H}_t^i]\big|\mathcal{G}_t\big] = \mathrm{E}[Z_{i,t} \mid \mathcal{G}_t] = C_{i,t}^{\text{cmrs}}.$$

, Thus, $C_{i,t}^{\text{cmrs}} = \mathrm{E}[C_{i,t}^{\text{thin}} \mid \mathcal{G}_t]$. Let φ be any convex function for which the expectations exist. By Jensen's inequality,

$$\varphi(C_{i,t}^{\text{cmrs}}) = \varphi\left(\mathrm{E}[C_{i,t}^{\text{thin}} \mid \mathcal{G}_t]\right) \le \mathrm{E}[\varphi(C_{i,t}^{\text{thin}}) \mid \mathcal{G}_t].$$

Taking expectations yields

$$\mathrm{E}[\varphi(C_{i,t}^{\text{cmrs}})] \le \mathrm{E}[\varphi(C_{i,t}^{\text{thin}})].$$

Moreover, $\mathrm{E}[C_{i,t}^{\text{cmrs}}] = \mathrm{E}[C_{i,t}^{\text{thin}}]$. Therefore, $C_{i,t}^{\text{cmrs}} \preceq_{\text{cx}} C_{i,t}^{\text{thin}}$.

The same argument applies conditionally on $\mathcal{A}_t$. Conditionally on the claim-arrival skeleton, the sigma-fields $\mathcal{G}_t$ and $\mathcal{H}_t^i$ remain ordered, and the identity

$$C_{i,t}^{\text{cmrs}} = \mathrm{E}[C_{i,t}^{\text{thin}} \mid \mathcal{G}_t]$$

still holds. Hence, for every convex φ,

$$\mathrm{E}[\varphi(C_{i,t}^{\text{cmrs}}) \mid \mathcal{A}_t] \leq \mathrm{E}[\varphi(C_{i,t}^{\text{thin}}) \mid \mathcal{A}_t],$$

with equality of conditional means. This proves the conditional convex-order comparison. $\square$

We can now state the willingness-to-join property induced by the dynamic risk-sharing mechanism.

Corollary 4.6 (Dynamic willingness to join)**.** *For every participant i, every time horizon $t \geq 0$, and every initial wealth w,*

$$\mathrm{E}[u(w - C_{i,t}^{\text{cmrs}})] \geq \mathrm{E}[u(w - C_{i,t}^{\text{thin}})]$$

for every increasing concave utility function u, whenever the expectations exist. Thus every risk-averse participant prefers the dynamic CMRS mechanism to the stand-alone incurred-and-reserved benchmark.

Proof. Let u be an increasing concave utility function and define

$$\varphi(x) = -u(w - x).$$

Then, φ is convex. Hence,

$$\mathrm{E}[\varphi(C_{i,t}^{\text{cmrs}})] \leq \mathrm{E}[\varphi(C_{i,t}^{\text{thin}})],$$

which is equivalent to

$$\mathrm{E}[u(w - C_{i,t}^{\text{cmrs}})] \geq \mathrm{E}[u(w - C_{i,t}^{\text{thin}})].$$

The convex-order comparison from Proposition 4.5 proves the willingness-to-join property. $\square$

5 Numerical illustration

This section illustrates the dynamic CMRS mechanism derived in the previous sections. The purpose is not to calibrate a realistic disability portfolio, but to show how the proposed allocation reacts to heterogeneity in disability incidence and recovery rates. We focus on the first claim reported to the pool and compare the dynamic reserves of representative participants belonging to different risk groups.

The illustration highlights three features of the mechanism. First, the terminal CMRS allocation depends on the duration of the disability spell: short claims are mostly allocated to participants with high recovery rates, whereas long claims are increasingly allocated to participants with low recovery rates. Second, before settlement, the amounts held by participants are dynamic reserves, not final contributions. They are updated as the claim remains open. Third, at the pool level, the sum of the reserves is equal to the conditional expectation of the ultimate claim amount, and therefore exceeds the benefits already paid while the claim is still open.

5.1 A heterogeneous pool

Consider a pool of $n = 100$ participants divided into three homogeneous groups. Recovery rates are given by

$$\lambda_{10,i} = \begin{cases} 4, & i = 1, 2, \ldots, 25, \\ 3, & i = 26, 27, \ldots, 75, \\ 2, & i = 76, 77, \ldots, 100. \end{cases}$$

Thus, the average disability duration is 1/4 year, or three months, in the first group; 1/3 year, or four months, in the second group; and 1/2 year, or six months, in the third group.

Disability incidence rates are chosen by fixing the expected number of disability claims over the first year. For participant i,

$$\begin{aligned} \mathrm{E}[N_{i,1}^{01}] &= \int_0^1 p_{00,i}(0,t)\lambda_{01,i}\,\mathrm{d}t \\ &= \frac{\lambda_{01,i}\lambda_{10,i}}{\lambda_{01,i}+\lambda_{10,i}}\left(1 + \frac{1-\exp\{-(\lambda_{01,i}+\lambda_{10,i})\}}{\lambda_{01,i}+\lambda_{10,i}}\frac{\lambda_{01,i}}{\lambda_{10,i}}\right). \end{aligned}$$

Here, we impose

$$\mathrm{E}[N_{i,1}^{01}] = \begin{cases} 0.01, & i = 1, 2, \ldots, 25, \\ 0.025, & i = 26, 27, \ldots, 75, \\ 0.05, & i = 76, 77, \ldots, 100. \end{cases}$$

Solving for the incidence rates gives

$$\lambda_{01,i} = \begin{cases} 0.01002, & i = 1, 2, \ldots, 25, \\ 0.02514, & i = 26, 27, \ldots, 75, \\ 0.05072, & i = 76, 77, \ldots, 100. \end{cases}$$

The three groups are therefore ordered by increasing disability frequency and increasing disability severity. Participants in the third group have both the highest probability of entering disability and the longest expected disability duration.

5.2 Terminal CMRS allocation of the first claim

We consider the first claim reported to the pool and set $T_1 = 0.5$. Since this is the first claim, all participants are active immediately before the reporting time, so that

$$\mathcal{R}_1 = \{1, 2, \ldots, n\}.$$

The conditional density f_1 of the claim duration Y_1, given T_1 and $\mathcal{R}_1$, is a mixture of exponential densities with weights proportional to the disability incidence rates. It is displayed in Figure 1.

Figure 2 displays the terminal CMRS allocation $h_{i,1}^{\mathrm{cmrs}}(y)$ as a function of the realized claim duration y, for a representative participant in each group. For short disability spells, the posterior probability that the claim was generated by a participant with a high recovery

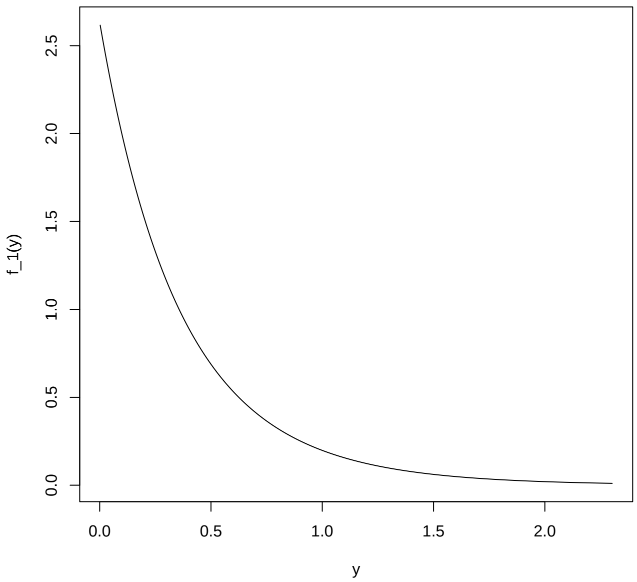

Figure 1: Conditional probability density function f_1 of the first claim reported to the pool.

rate is relatively large. Participants in the first group therefore receive a larger share of short claims. As the claim duration increases, the posterior probability shifts towards participants with lower recovery rates. Consequently, participants in the third group progressively bear a larger share of long-lasting claims.

5.3 Dynamics of the reserves

We now study the dynamic reserves associated with four possible realized durations of the first claim. The first three durations correspond to the medians of the exponential disability durations in groups 1, 2 and 3. The fourth duration corresponds to the 99-th percentile of the disability duration in group 3 and represents a long-lasting disability spell.

5.3.1 Dynamic reserves for short and medium disability spells

We first assume that the duration of the first claim is equal to the median disability duration of a participant in the first group:

$$Y_1 = F_{D_1}^{-1}(0.5) = 0.1733.$$

Since $T_1 = 0.5$, settlement occurs at time

$$T_1 + Y_1 = 0.6733.$$

Figure 3 shows the dynamic CMRS reserve $\widehat{h}_{i,1|t}^{\text{cmrs}}$ for a representative participant in each group. The dots at $T_1 = 0.5$ represent the initial reserves paid when the claim is reported.

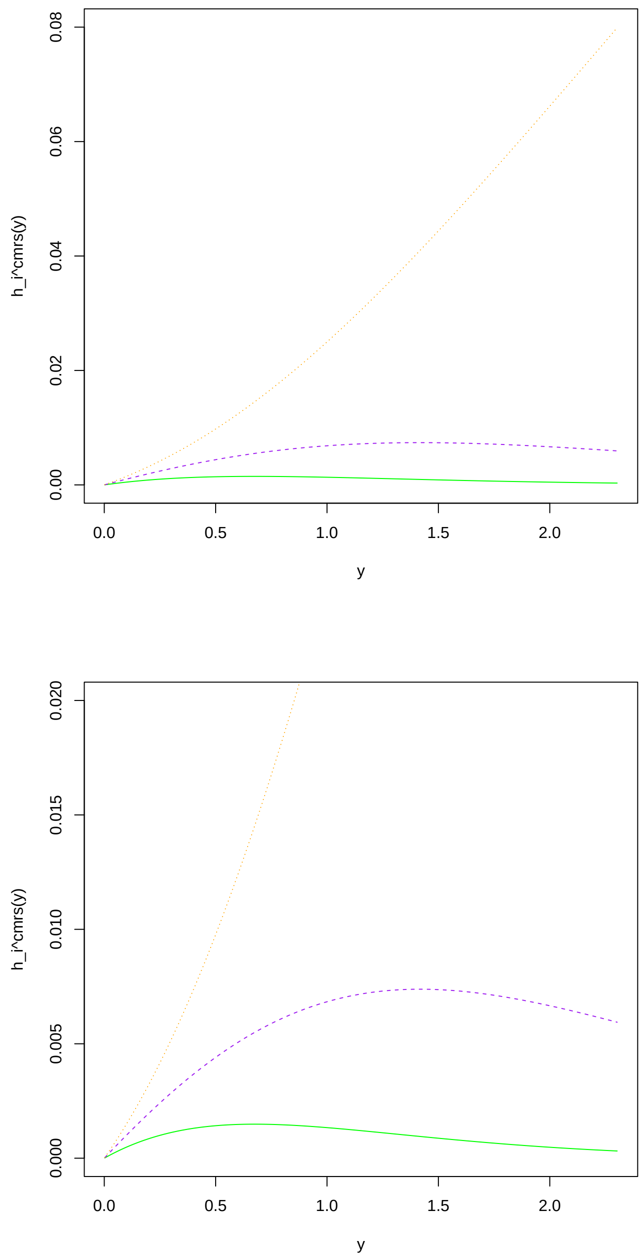

Figure 2: Terminal individual CMRS allocations $h_{i,1}^{\mathrm{cmrs}}(y)$ for a representative participant in the first group (solid line, in green), in the second group (dashed line, in purple), and in the third group (dotted line, in orange). The bottom panel zooms on amounts less than 0.02.

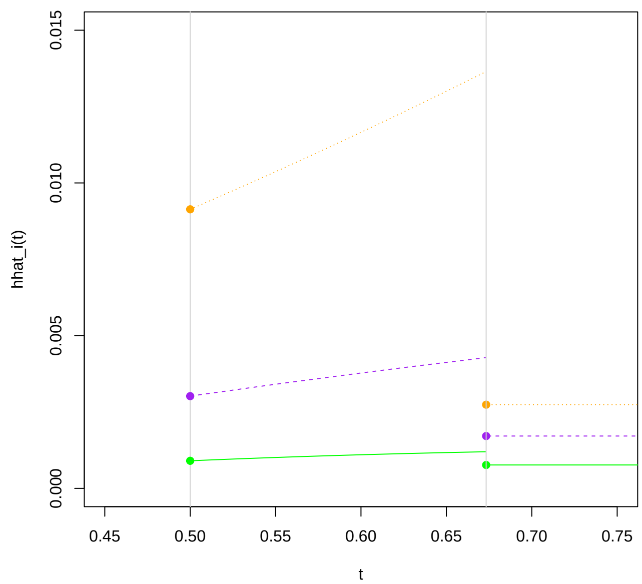

Figure 3: Dynamic CMRS reserve $\widehat{h}^{\text{cmrs}}_{i,1|t}$ for a representative participant in the first group (solid line, in green), in the second group (dashed line, in purple), and in the third group (dotted line, in orange), when the claim duration is the median disability duration of a participant in the first group.

These initial reserves differ across groups because they depend on both the disability incidence rate and the expected disability duration. While the claim remains open, the reserves are updated by conditioning on the event that the disability duration exceeds $t-T_1$. For this short claim, the reserves increase only during a short time interval and the final adjustment at recovery is negative for all three groups.

Figure 4 displays the sum of the reserves over all participants, together with the benefits already paid to the claimant. For $T_1 < t < T_1 + Y_1$, the benefits already paid are equal to $t - T_1$, whereas the sum of the dynamic reserves is

$$\begin{aligned}\sum_{i=1}^{n} C^{\text{cmrs}}_{i,t} &= \sum_{i=1}^{n} \widehat{h}^{\text{cmrs}}_{i,1|t} \\ &= t - T_1 + \sum_{i=1}^{n} \frac{1}{\lambda_{10,i}} \frac{\lambda_{01,i} \exp\{-\lambda_{10,i}(t-T_1)\}}{\sum_{j=1}^{n} \lambda_{01,j} \exp\{-\lambda_{10,j}(t-T_1)\}}.\end{aligned}$$

Hence,

$$\sum_{i=1}^{n} C^{\text{cmrs}}_{i,t} > t - T_1, \qquad T_1 < t < T_1 + Y_1.$$

The difference between the sum of the reserves and the benefits already paid is the conditional expected remaining cost of the claim. It is a weighted average of the mean disability durations $1/\lambda_{10,i}$, where the weights are the posterior probabilities $\mathrm{P}[I_1 = i \mid T_1, \mathcal{R}_1, Y_1 > t - T_1]$. As

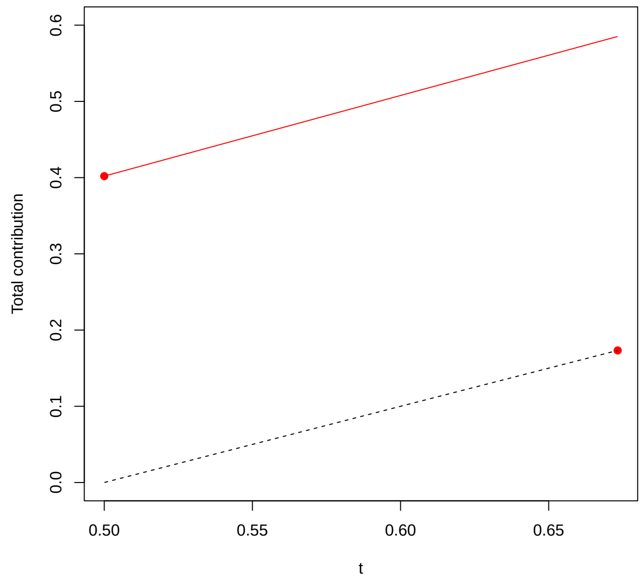

Figure 4: Sum of dynamic CMRS reserves over the entire pool (solid line, in red) and benefits already paid to the claimant (dashed line, in black), when the claim duration is the median disability duration of a participant in the first group.

the claim remains open, more weight is assigned to groups with smaller recovery rates. At settlement, the ultimate duration is known and the total allocation becomes equal to Y_1.

We next consider a duration equal to the median disability duration of a participant in the second group:

$$Y_1 = F_{D_2}^{-1}(0.5) = 0.2311.$$

Settlement then occurs at time $T_1 + Y_1 = 0.7311$.

Figure 5 shows the same reserve dynamics as before, but over a slightly longer disability spell. The initial reserves are unchanged because they are determined at the reporting time and do not depend on the future realized duration. However, since the claim remains open for longer, the posterior probability shifts more strongly towards groups with lower recovery rates. The reserves of participants in groups 2 and 3 are therefore higher than in the previous case before settlement. The final adjustment is still a reimbursement for all groups.

Figure 6 again illustrates conditional full allocation before settlement. The sum of the reserves exceeds the benefits already paid while the claim is open and coincides with the realized claim amount at settlement.

We then consider a duration equal to the median disability duration of a participant in the third group:

$$Y_1 = F_{D_3}^{-1}(0.5) = 0.3466.$$

Settlement occurs at time $T_1 + Y_1 = 0.8466$.

Figure 7 shows that the qualitative behavior remains the same, but the longer duration increases the reserves before settlement. The effect is particularly visible for the representa-

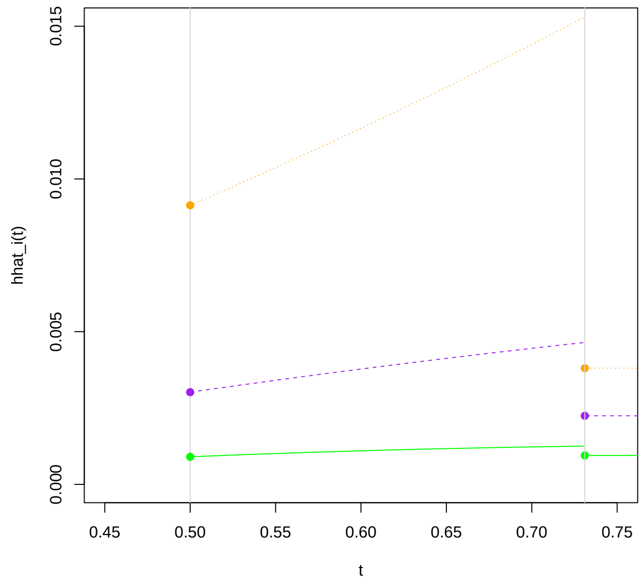

Figure 5: Dynamic CMRS reserve $\widehat{h}^{\text{cmrs}}_{i,1|t}$ for a representative participant in the first group (solid line, in green), in the second group (dashed line, in purple), and in the third group (dotted line, in orange), when the claim duration is the median disability duration of a participant in the second group.

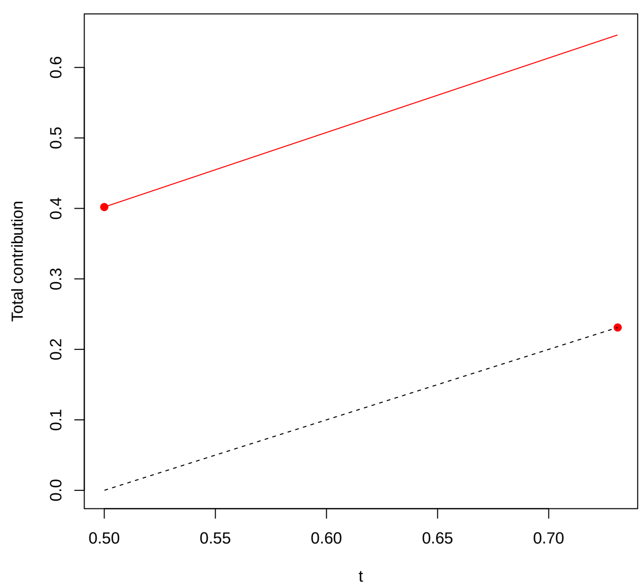

Figure 6: Sum of dynamic CMRS reserves over the entire pool (solid line, in red) and benefits already paid to the claimant (dashed line, in black), when the claim duration is the median disability duration of a participant in the second group.

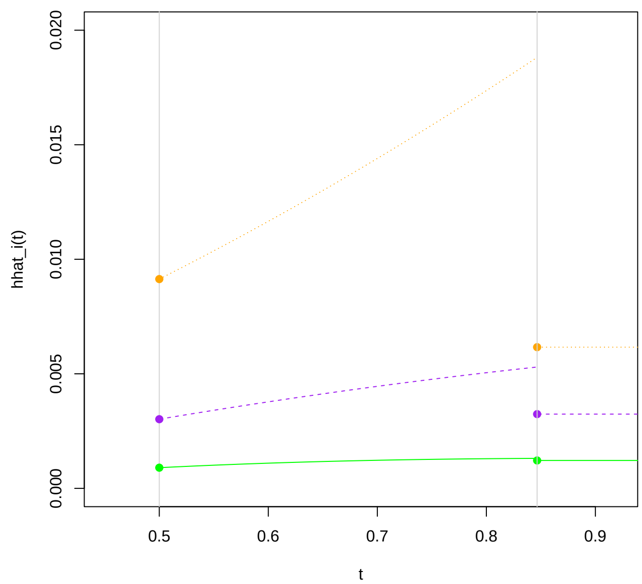

Figure 7: Dynamic CMRS reserve $\widehat{h}^{\text{cmrs}}_{i,1|t}$ for a representative participant in the first group (solid line, in green), in the second group (dashed line, in purple), and in the third group (dotted line, in orange), when the claim duration is the median disability duration of a participant in the third group.

tive participant in the third group, whose lower recovery rate makes this group increasingly likely to have generated a claim that remains open for several months.

Figure 8 shows the corresponding pool-level reserves. As in the previous cases, the total reserve is larger than the amount already paid while the claim is open. At recovery, the final adjustment brings the total allocation down to the realized claim duration Y_1.

5.3.2 Dynamic reserves for a long-lasting disability spell

We finally consider a long-lasting disability claim. The duration is set equal to the 99-th percentile of the disability duration of a participant in the third group:

$$Y_1 = F^{-1}_{D_3}(0.99) = 2.3026.$$

Since $T_1 = 0.5$, settlement takes place at time

$$T_1 + Y_1 = 2.8026.$$

Figure 9 shows a substantially different reserve pattern. During the first part of the disability spell, reserves evolve as in the previous cases. As the claim remains open for an unusually long time, however, the posterior probability that the claim was generated by a participant with a high recovery rate becomes very small. Consequently, the reserves of participants in groups 1 and 2 start decreasing. By contrast, the reserve of a participant in

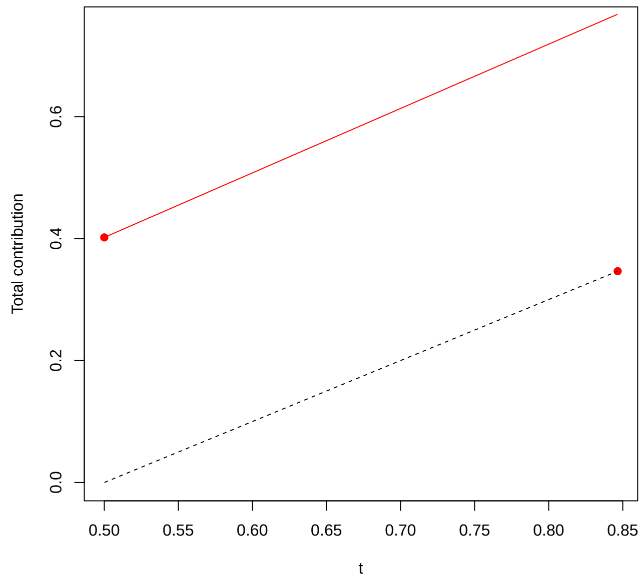

Figure 8: Sum of dynamic CMRS reserves over the entire pool (solid line, in red) and benefits already paid to the claimant (dashed line, in black), when the claim duration is the median disability duration of a participant in the third group.

group 3 keeps increasing for a longer period and becomes the dominant component of the allocation.

At settlement, the terminal allocation $h_{i,1}^{\mathrm{cmrs}}(Y_1)$ differs markedly from the reserve immediately before recovery. For participants in groups 1 and 2, the final adjustment is positive: although it has become unlikely that they generated such a long claim, the terminal CMRS allocation still assigns them a small positive share of the realized loss. For participants in group 3, the reserve immediately before settlement remains larger than the terminal allocation, so that the final adjustment is a reimbursement.

Figure 10 displays the pool-level effect. The sum of the dynamic reserves remains above the benefits already paid throughout the open period, because it is the conditional expectation of the ultimate claim amount given the information that the claim is still open. For a long-lasting claim, this conditional expectation evolves almost in parallel with the benefits already paid once the posterior distribution is dominated by the slowest recovery group. At settlement, the ultimate duration is observed and the total CMRS allocation becomes exactly equal to Y_1.

Overall, the numerical illustration confirms the interpretation of the dynamic CMRS mechanism as a conditional reserving procedure. Before settlement, participants do not pay the final allocation, because the final claim amount is not yet known. They hold dynamically updated reserves based on the information that the claim is still open. At settlement, these reserves are adjusted so that the terminal allocation is fully allocated and coincides with the CMRS rule applied to the realized claim duration.

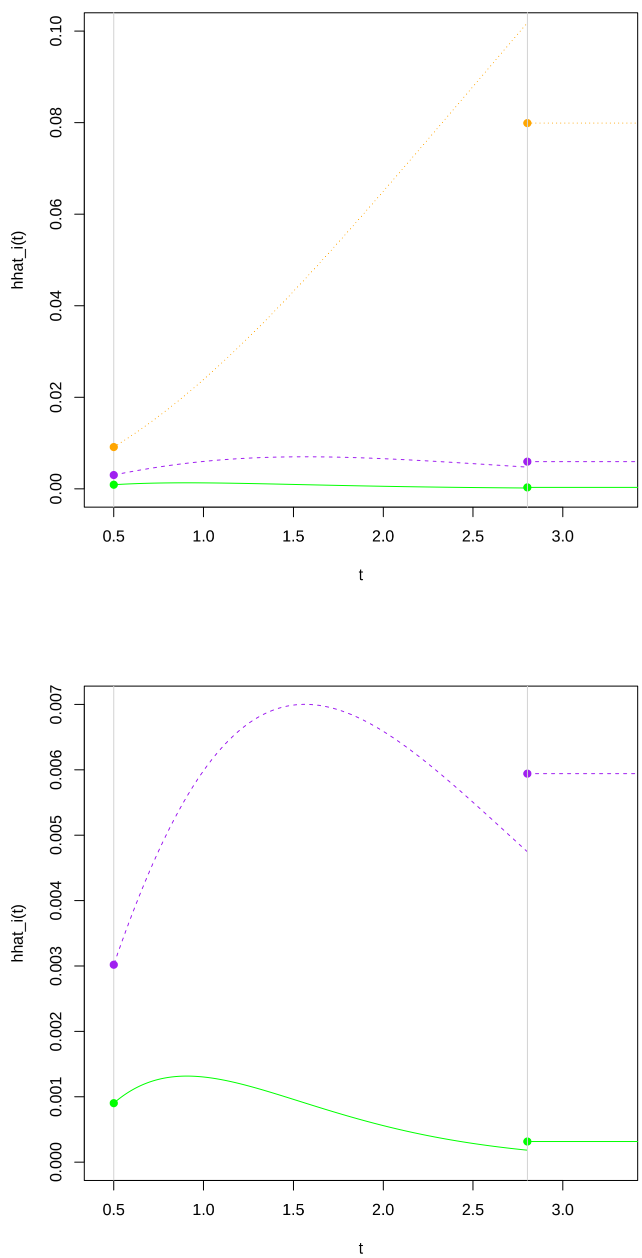

Figure 9: Dynamic CMRS reserve $\widehat{h}^{\mathrm{cmrs}}_{i,1|t}$ for a representative participant in the first group (solid line, in green), in the second group (dashed line, in purple), and in the third group (dotted line, in orange), when the claim duration is the 99-th percentile of the disability duration of a participant in the third group. The bottom panel zooms on the reserves of participants in the first and second groups.

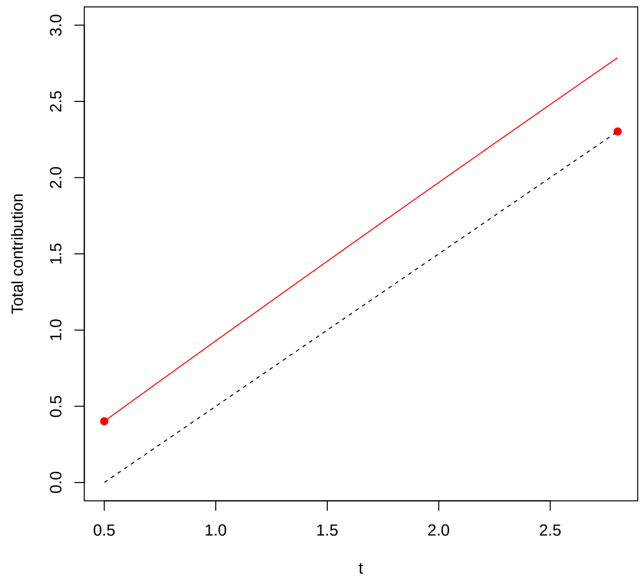

Figure 10: Sum of dynamic CMRS reserves over the entire pool (solid line, in red) and benefits already paid to the claimant (dashed line, in black), when the claim duration is the 99-th percentile of the disability duration of a participant in the third group.

6 Extensions

The previous sections were written in a homogeneous two-state Markov setting with a common unit benefit rate. This framework has the advantage of yielding closed-form expressions for the terminal CMRS allocation and for the dynamic reserve process. The construction, however, is not specific to exponential disability durations. The essential ingredients are the conditional distribution of the ultimate claim amount at reporting time and the claim-development information progressively revealed before settlement.

This section discusses three extensions. First, we explain how the mechanism extends to non-homogeneous Markov and semi-Markov multistate models. Second, we discuss more general benefit structures. Third, we indicate how the same conditional-reserving idea can be used in property and casualty insurance when claim severities develop after notification.

6.1 General multistate models

The dynamic CMRS mechanism can be extended to more general multistate models, including non-homogeneous Markov and semi-Markov models. In such models, disability incidence rates may depend on calendar time, age or duration in the active state, and recovery rates may depend on the duration already spent in disability. Closed-form formulas are then usually no longer available, but the risk-sharing mechanism remains based on the same conditional expectation principle.

Consider a claim reported at time T_k. Let $\mathcal{F}_{T_k-}$ still denote the information available immediately before T_k, and let $\mathcal{R}_k = \{i : X_{i,T_k-} = e_0\}$ be the corresponding risk set. For $i \in \mathcal{R}_k$, denote by $\alpha_{i,k}$ the instantaneous disability incidence rate of participant i at time T_k, conditionally on $\mathcal{F}_{T_k-}$. In the homogeneous Markov model studied above, $\alpha_{i,k} = \lambda_{01,i}$. In a non-homogeneous or semi-Markov model, $\alpha_{i,k}$ may depend on age, calendar time and past duration in the active state.

Assuming that simultaneous transitions occur with probability zero, the conditional probability that participant i reports the k-th claim is

$$\pi_{i,k} = \mathrm{P}[I_k = i \mid \mathcal{F}_{T_k-}] = \begin{cases} \dfrac{\alpha_{i,k}}{\sum_{j\in\mathcal{R}_k} \alpha_{j,k}}, & i \in \mathcal{R}_k, \\ 0, & i \notin \mathcal{R}_k. \end{cases}$$

Given $I_k = i$, let $F_{i,k}$, $\overline{F}_{i,k}$ and $f_{i,k}$ denote the conditional distribution function, survival function and density of the ultimate claim amount Y_k. These quantities are conditional on the pre-claim information and on the fact that participant i has entered disability at time T_k. The conditional mixture distribution of Y_k is then

$$\overline{F}_k(y) = \sum_{j\in\mathcal{R}_k} \pi_{j,k}\overline{F}_{j,k}(y),$$

and, when densities exist,

$$f_k(y) = \sum_{j\in\mathcal{R}_k} \pi_{j,k} f_{j,k}(y).$$

At settlement, the terminal CMRS allocation is still obtained by conditioning the thinned individual loss $Z_{i,k} = Y_k \mathbf{1}_{\{I_k=i\}}$ on the realized claim amount. Thus,

$$h^{\mathrm{cmrs}}_{i,k}(Y_k) = \mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}, Y_k].$$

When Y_k admits a conditional density, this gives

$$h^{\mathrm{cmrs}}_{i,k}(y) = y\frac{\pi_{i,k} f_{i,k}(y)}{\sum_{j\in\mathcal{R}_k} \pi_{j,k} f_{j,k}(y)}, \qquad y > 0.$$

This formula reduces to the expression obtained in Section 3 when $f_{i,k}$ is the exponential density with parameter $\lambda_{10,i}$ and $\pi_{i,k}$ is proportional to $\lambda_{01,i}$.

Let $\mathcal{G}_{k,u}$ denote the claim-development information available after elapsed duration u since reporting. Before settlement, this information contains the event $\{Y_k > u\}$; after settlement, it contains the realized value of Y_k. The dynamic CMRS reserve is defined by

$$R^{\mathrm{cmrs}}_{i,k}(u) = \mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}].$$

Equivalently,

$$R^{\mathrm{cmrs}}_{i,k}(u) = \mathrm{E}[h^{\mathrm{cmrs}}_{i,k}(Y_k) \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}].$$

On the event $\{Y_k > u\}$, this becomes

$$R^{\mathrm{cmrs}}_{i,k}(u) = \frac{\pi_{i,k} \int_u^\infty y f_{i,k}(y)\,\mathrm{d}y}{\sum_{j\in\mathcal{R}_k} \pi_{j,k}\overline{F}_{j,k}(u)}.$$

Equivalently,

$$R_{i,k}^{\mathrm{cmrs}}(u) = \mathrm{P}[I_k = i \mid \mathcal{F}_{T_k-}, Y_k > u]\, \mathrm{E}[Y_k \mid \mathcal{F}_{T_k-}, I_k = i, Y_k > u].$$

After settlement,

$$R_{i,k}^{\mathrm{cmrs}}(u) = h_{i,k}^{\mathrm{cmrs}}(Y_k), \qquad u \geq Y_k.$$

This formulation shows that the dynamic CMRS mechanism does not rely on the memoryless property of the exponential distribution. In the exponential case, the conditional expectation of the ultimate duration given survival beyond u is $u+1/\lambda_{10,i}$. In a semi-Markov model, this conditional expectation is replaced by the corresponding residual-duration formula under the conditional distribution $F_{i,k}$.

If f_k is the density of the conditional mixture distribution and $\gamma_k(u) = \frac{f_k(u)}{\overline{F}_k(u)}$ is its hazard rate, then the reserve satisfies, on the open-claim event,

$$\frac{\mathrm{d}}{\mathrm{d}u} R_{i,k}^{\mathrm{cmrs}}(u) = \gamma_k(u) \left(R_{i,k}^{\mathrm{cmrs}}(u) - h_{i,k}^{\mathrm{cmrs}}(u) \right),$$

whenever the derivative exists. This is the same adjustment equation as in the homogeneous Markov case.

The martingale and convex-order arguments of Section 4 remain valid in this general setting. Indeed, they only use conditional expectation and the inclusion of information structures. If the thinned individual reserve is defined by

$$R_{i,k}^{\mathrm{thin}}(u) = \mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}, \mathcal{H}_{k,u}^i], \qquad \mathcal{H}_{k,u}^i = \mathcal{G}_{k,u} \vee \sigma(\mathbf{1}_{\{I_k=i\}}),$$

then

$$R_{i,k}^{\mathrm{cmrs}}(u) = \mathrm{E}[R_{i,k}^{\mathrm{thin}}(u) \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}].$$

Consequently, $R_{i,k}^{\mathrm{cmrs}}(u) \preceq_{\mathrm{cx}} R_{i,k}^{\mathrm{thin}}(u)$ holds conditionally on $\mathcal{F}_{T_k-}$. The cumulative willingness-to-join property is obtained by the same projection argument as in Proposition 4.5.

6.2 General benefit structures

Throughout Sections 2–4, the benefit rate paid in state e_1 is normalized to one and is the same for all participants. In that case, the claim amount is simply the disability duration: $Y_{k,i} = D_{k,i}$. In applications, however, benefit levels may depend on the participant, and payments may include deferred periods or maximum benefit durations.

A simple extension is obtained by assigning a benefit rate $b_i > 0$ to participant i, so that $Y_{k,i} = b_i D_{k,i}$. One may also introduce a deferred period $\delta_i \geq 0$ and a maximum payment duration $\omega_i \in (0, \infty]$, leading to

$$Y_{k,i} = b_i \min\{(D_{k,i} - \delta_i)_+, \omega_i\}.$$

More generally, the claim amount may be written as

$$Y_{k,i} = g_i(D_{k,i}),$$

where g_i is a non-negative benefit function describing the contractual payment rule.

The CMRS mechanism is unchanged in principle. The only modification concerns the conditional distribution of the claim amount Y_k. Given that the claim was reported by participant i, the law of Y_k is now the image of the disability-duration distribution through the benefit function g_i. The terminal CMRS allocation is still

$$h_{i,k}^{\mathrm{cmrs}}(Y_k) = \mathrm{E}[Y_k \mathbf{1}_{\{I_k=i\}} \mid \mathcal{F}_{T_k-}, Y_k],$$

and the dynamic reserve remains

$$R_{i,k}^{\mathrm{cmrs}}(u) = \mathrm{E}[h_{i,k}^{\mathrm{cmrs}}(Y_k) \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}].$$

When the distribution of Y_k has atoms, as it may happen with deferred periods or maximum benefit durations, density formulas must be replaced by their measure-theoretic counterparts. The conditional expectation representation remains valid without change. Thus the mechanism is better viewed as a conditional projection rule rather than as a formula depending on a particular density.

This extension is particularly relevant for disability schemes in which participants choose different income-replacement levels. Higher benefit levels affect both the distribution of the ultimate claim amount and the corresponding CMRS reserve, while the willingness-to-join argument continues to follow from the same convex-order comparison between the pooled dynamic reserve and the thinned stand-alone incurred-and-reserved benchmark.

6.3 Loss reserving in Property and Casualty insurance

The construction developed in this paper is also relevant for property and casualty insurance. Denuit and Robert (2023) considered CMRS of losses at occurrence time in a compound Poisson risk model. In that setting, the claim amount is assumed to be known when the claim is reported. This assumption is convenient mathematically, but it is often unrealistic in practice.

In many property and casualty lines, the ultimate claim severity is revealed only progressively. After notification, a case reserve may be opened by a claim handler. Payments can then occur in several installments, recoveries or refunds may be recorded, and the claim is closed only when the ultimate cost is known. Continuous-time models for the development of individual claims were studied, among others, by Arjas (1989), Norberg (1993, 1999) and Haastrup and Arjas (1996).

The dynamic CMRS mechanism provides a natural way to incorporate this development delay into peer-to-peer or mutual risk-sharing schemes. Suppose that a claim is reported at time T_k, but that its ultimate severity Y_k is not known at that time. Let $\mathcal{G}_{k,u}$ denote the information generated by the development of the claim during the elapsed time u: case estimates, payments, recoveries, medical or legal information, or any other observable development variables. The contribution of participant i can then be defined as

$$R_{i,k}^{\mathrm{cmrs}}(u) = \mathrm{E}[Z_{i,k} \mid \mathcal{F}_{T_k-}, \mathcal{G}_{k,u}],$$

where $Z_{i,k}$ is the thinned ultimate loss generated by participant i for the k-th reported claim.

At notification, participants contribute an initial reserve based on the information available at reporting. As the claim develops, this reserve is updated by conditioning on the enriched claim-development information. At closure, when the ultimate severity becomes known, a final adjustment is made so that the amount paid coincides with the terminal CMRS allocation. The same projection identity as in Section 4 then yields a martingale property, actuarial fairness and a convex-order reduction relative to the corresponding thinned stand-alone incurred-and-reserved benchmark.

Thus, the approach proposed here can be interpreted as a dynamic reserving version of occurrence-time CMRS in Denuit and Robert (2023). It extends the compound Poisson framework by replacing immediate observation of severities with progressive learning about ultimate claim amounts.

7 Discussion and conclusion

This paper develops a dynamic CMRS mechanism for peer-to-peer disability insurance when claim amounts are not known at occurrence time. The motivating example is provided by bread funds operating in the Netherlands, where participants mutually protect one another against the economic consequences of disability. From an actuarial viewpoint, the central difficulty is that a disability claim is not a one-shot loss observed at reporting time. Its ultimate amount depends on the duration of the disability spell and is only known at recovery. The risk-sharing rule must therefore be formulated not only at settlement, but also during the development period of the claim.

The first contribution of the paper is to adapt the CMRS rule to this settlement-delay setting. When a participant enters disability, the active members of the pool constitute an initial reserve. This reserve is then updated as information on the claim development becomes available. If the claim remains open, the reserve is recalculated by conditioning on the fact that the disability spell has already lasted beyond the current elapsed duration. At recovery, a final adjustment is made so that the total amount allocated to each participant coincides with the terminal CMRS allocation. The mechanism therefore provides a dynamic implementation of CMRS when the ultimate claim amount is progressively revealed.

The second contribution is to clarify the appropriate benchmark for assessing such a mechanism. In a model with settlement delays, the relevant comparison is not between contributions and benefits already paid up to time t. Benefits already paid ignore the outstanding part of open claims. The proper benchmark is an incurred-and-reserved stand-alone cost, obtained by combining settled claims and reserves for claims that are still open. This distinction is important because it changes the meaning of fairness and full allocation at finite horizons.

The third contribution is the conditional-projection representation of the dynamic CMRS reserve. For each claim, the CMRS reserve of a participant is the conditional expectation of the corresponding thinned individual loss under the information used by the pool. The stand-alone incurred-and-reserved benchmark is the conditional expectation of the same thinned loss under a finer information structure, which reveals whether the claim belongs to the participant. The dynamic CMRS reserve is therefore a coarser projection of the individual benchmark. This representation is the key mathematical mechanism behind the main results of the paper.

Several properties follow from this projection structure. First, the reserve process is a martingale with respect to the claim-development information. Second, the mechanism satisfies a natural actuarial fairness property: the expected dynamic contribution of each participant is equal to the expected incurred-and-reserved cost generated by that participant. Third, before settlement, full allocation must be understood conditionally. The sum of the reserves is not equal to the benefits already paid; it is equal to the conditional expectation of the ultimate claim amount given the available claim-development information. At settlement, when the ultimate amount is known, this conditional full allocation reduces to the usual full allocation of the realized claim.

The main welfare result is a dynamic willingness-to-join property. For each participant and each time horizon t, the cumulative contribution under the dynamic CMRS mechanism is smaller in convex order than the corresponding stand-alone incurred-and-reserved benchmark: the stochastic inequality $C_{i,t}^{\mathrm{cmrs}} \preceq_{\mathrm{cx}} C_{i,t}^{\mathrm{thin}}$ holds for all t. Consequently, every risk-averse participant prefers joining the pool to remaining exposed to his or her own incurred-and-reserved disability cost. The result is stronger than a comparison at settlement only, because it holds for the cumulative dynamic reserve process observed at any time horizon.

The numerical illustration shows how the mechanism behaves in a heterogeneous pool. Participants with high recovery rates tend to receive larger shares of short disability claims, whereas long-lasting disability claims are increasingly allocated to participants with lower recovery rates. The examples also illustrate the difference between benefits already paid and dynamic reserves: while a claim is open, the total reserve exceeds the benefits already paid because it includes the conditional expected future cost of the claim. At settlement, the final adjustment reconciles the reserve process with the terminal CMRS allocation.

The results extend to richer multistate models. The homogeneous two-state Markov model gives explicit formulas and makes the mechanism transparent, but practical disability models may require age-dependent, calendar-time-dependent or duration-dependent transition rates. Semi-Markov models are particularly relevant when recovery probabilities depend on the elapsed duration in disability. The conditional-expectation structure of the mechanism remains valid in such settings, even though numerical methods would generally replace closed-form formulas.

The paper also suggests several directions for future research. A first one concerns benefit design. Bread funds and similar peer-to-peer schemes often allow participants to choose different levels of income replacement, may impose deferred periods, and may limit the maximum benefit duration. Such contractual features change the conditional distribution of the ultimate claim amount and therefore the dynamic reserve. Studying how benefit design affects fairness, willingness to join and liquidity needs would be important for practical implementation.

A second direction concerns solvency and liquidity. The present paper focuses on individual contributions and risk-sharing properties. In practice, a pool must also be able to finance benefits when several claims are open simultaneously or when long-lasting claims occur. This raises questions about buffer accounts, liquidity rules, inter-pool support mechanisms and reinsurance-type arrangements. The dynamic reserve process derived in this paper could provide a basis for defining solvency indicators adapted to peer-to-peer disability pools.

A third direction is behavioral and informational. Peer-to-peer insurance schemes rely on trust, transparency and participation incentives. The willingness-to-join result shows that risk-averse participants benefit from the mechanism under the model assumptions, but further work could study adverse selection, moral hazard, participation constraints and the effect of heterogeneous information about disability risks. These issues are especially relevant for small pools, where mutual monitoring and social ties may play an important role.

Finally, the proposed approach is not limited to bread funds or disability insurance. It can be applied more broadly to cooperative and mutual insurance arrangements in which claims are reported before their ultimate amounts are known. This includes property and casualty insurance with claim development, health insurance with treatment episodes, long-term care benefits, and other solidarity-based schemes. Takaful (Billah, 2019) provides one possible example, since it is based on risk sharing rather than pure risk transfer, but it should be viewed as one application among several. The broader message of the paper is that CMRS can be combined with actuarial reserving to handle settlement delays while preserving fairness and willingness to join.

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