AXIOMATIZATION OF THE CORE OF POSITIVE GAMES

Pierre Dehez

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Voie du Roman Pays 34, L1.03.01

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Axiomatization of the core of positive games

Pierre DEHEZ [*]

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Abstract
We show that, when restricted to positive games, games whose Harsanyi dividends are non-negative, additivity together with efficiency, individual rationality and the null player property, characterizes the core as a maximal set-valued solution.

Keywords: core; convex games; positive games
JEL Classification: C71

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* CORE-LIDAM, UCLouvain, Voie du Roman Pays 34, 1348 Louvain-la-Neuve, Belgium.
Email: pierre.dehez@uclouvain.be
Orcid: 0000-0002-0752-7548

1. Introduction

A *solution* is a mapping that associates allocations to games. Convex games form a large class of transferable utility games on which solutions tend to converge: the core of a convex game contains the *Shapley value*, is the unique *stable set*, coincides with the *bargaining set* (relative to the grand coalition) and the *kernel* coincides with the *nucleolus*.[1] Moreover, Monderer et al. (1992) have shown that the core coincides with the set of *weighted Shapley values*.

The concept of dividend was introduced by Harsanyi (1959). Positive games are games whose Harsanyi dividends are non-negative and positive games are convex. The *Harsanyi set* (also known as the *selectope*) is the set of all distributions of dividends, a solution introduced independently by Hammer et al. (1977) and Vasil'ev (1975).[2]

Weber (1988) has shown that the core is a subset of the convex hull of the marginal contribution vectors, called the Weber set. Convex games were first investigated by Shapley (1971) who shows that the core of a convex game is the polyhedron whose vertices are the marginal contribution vectors. Ichiishi (1981) went further by showing that it is a *necessary and sufficient* condition for convexity. Hence the core and the Weber set coincide on the subspace of convex games. Vasil'ev and van der Laan (2002) have shown that the Weber set is a subset of the Harsanyi set. Hammer et al. (1977) and Vasil'ev (1975) have shown that the core is a subset of the Harsanyi set and that both coincide when applied to positive games. As a consequence, all three solution concepts coincide when applied to positive games.

The core has been axiomatized by Peleg (1986) using a reduced game property.[3] There he shows that the core is a superadditive solution. Dragan et al (1989) have later shown that the Weber set is a subadditive solution. These two results apply to all transferable utility games. The coincidence between the core and the Weber set of convex games then implies that the core is an *additive* solution on the subspace of convex games.

In terms of axiomatization, we limit ourselves to solutions that satisfy the axioms of *efficiency*, *additivity* and *null player*. Additional axioms are needed for characterizing particular solutions. Weber (1988) has axiomatized the Weber set by adding a monotonicity axiom requiring solutions to be non-negative for monotonic games. Vasil'ev (1981) has axiomatized the Harsanyi set by adding a positivity axiom requiring solutions to be non-negative when applied to positive games. Here, we show that adding *individual rationality* characterizes the core on the subspace of positive games.

The paper is organized as follows. Section 2 introduces transferable utility games. It is followed by a section where the core and the Weber set are defined. The axiomatization of the core on the subspace of positive games is the object of Section 3. Concluding remarks are offered in the last section and an appendix gathers some intermediary results.

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[1] Shapley (1971) and Maschler et al. (1971).

[2] See also Vasli'ev (1980 in Russian and 1981 in English) and Derks et al. (2000).

[3] Various axiomatizations are reviewed by Peleg (1992). Dietzenbacher and Sudhölter (2022) is a more recent reference that applies to convex games.

2. Transferable utility games: convexity and positivity

Following von Neumann and Morgenstern (1944), a cooperative game with transferable utility is defined by a set $N = \{1,...,n\}$ of players and a set function (called *characteristic function*) v that associates to each *coalition* $S \subset N$ a real number $v(S)$ that represents its (potential) worth, that is the minimum gain that it can realize without the participation of the others. In particular, $v(\{i\})$ is what player i can obtain alone and $v(N)$ is the maximum amount that the "grand coalition" is able to generate. By convention, $v(\varnothing) = 0$. A game (N, v) is *monotonic* if the characteristic function is non-decreasing with respect to set inclusion: $S \subset T \Rightarrow v(S) \le v(T)$.

Notation. Finite sets are denoted by upper-case letters and lower-case letters are used to denote their cardinals: $t = |T|$, $s = |S|$, … Set inclusion (non-strict) is denoted by $\subset$. Braces will sometimes be omitted for coalitions, for instance, $v(i, j)$ replaces $v(\{i, j\})$. For a given vector x, $x(S)$ denotes the sum of its coordinates over the set S. By convention, a sum over an empty set is equal to 0.

The problem is to allocate the $v(N)$ among the players. The set of all allocations is given by $X(N,v) = \{x \in \mathbb{R}^n \mid x(N) = v(N)\}$. This is the requirement of *efficiency* in the sense of Pareto: allocating less than $v(N)$ is not efficient and allocating more than $v(N)$ is not feasible.

We denote by $G(N)$ the subspace of all characteristic functions defined on the finite set N. It is a *vector space* that can be identified with $\mathbb{R}^{2^n-1}$. The collection of *T-unanimity games* (N, u_T) defined for all nonempty subset $T \subset N$ by:

$$u_T(S) = 1 \text{ if } S \supset T,$$
$$= 0 \text{ otherwise,}$$

forms a basis of $G(N)$ i.e. for all $v \in G(N)$, there exists a unique $(2^n - 1)-$dimensional vector $(\alpha_T)_{\substack{T\subset N \\ T\neq\varnothing}}$ such that:[4]

$$v(S) = \sum_{T\subset N} \alpha_T u_T(S) = \sum_{T\subset S} \alpha_T, \tag{1}$$

assuming $\alpha_\varnothing = 0$. Harsanyi (1959) identifies α_T as the *dividend* accruing to coalition T once all sub-coalitions have got their dividends. By (1), the sum of all dividends is equal to $v(N)$. An allocation can then be obtained by distributing the dividends of each coalition among its members. The set of all possible distributions of the dividends defines the *Harsanyi set* $H(N, v)$, also called *selectope*, a solution concept introduced independently by Hammer at al. (1977) and by Vasil'ev (1975). The *Shapley value* is a particular element of the Harsanyi set that gives to each player the sum of the *per capita* dividend of the coalitions of which he is a member:

$$SV_i(N,v) = \sum_{T\subset N: i\in T} \frac{1}{t}\alpha_T(N,v) \ \ (i = 1,...,n).$$ [5]

[4] This basis was used by Shapley (1953) in his proof of the uniqueness of the value.

[5] This corresponds to Equation (10) in Shapley (1953).

The *marginal contribution* of player *i* to coalition *S* is defined by $v(S)-v(S\setminus i).$ Two players *i* and *j* are *substitutable* in a game (N,v) if they contribute equally to all coalitions to which they belong: $v(S)-v(S\setminus i)=v(S)-v(S\setminus j)$ for all $S\subset N$ such that $i,j\in S.$ A player *i* is *null* in a game (N,v) if he never contributes: $v(S)-v(S\setminus i)=0$ for all $S\subset N.$

A game (N,v) is *convex* (or *supermodular*) if $v(S)+v(T)\le v(S\cup T)+v(S\cap T)$ for all S and $T\subset N.$ It is a stronger property than superadditivity: a game (N,v) is *superadditive* if the above inequalities hold only for *disjoint* subsets: merging is never detrimental. Shapley (1971) shows that a game is convex *if and only if* players' marginal contributions do not decrease with coalition size:

$$i\in S\subset T\ \Rightarrow\ v(S)-v(S\setminus i)\le v(T)-v(T\setminus i).$$

As a consequence, the marginal contribution of a player is *maximal* at the grand coalition *N* and convexity means *increasing returns to size*. It is easily verified that *T*-unanimity games are convex. We denote by $CG(N)$ the subspace of convex games. It is a closed and convex cone in $G(N).$ Indeed, if *v* and *w* are convex on some common set of players *N*, so is the set function $\alpha v+w$ for all $\alpha>0.$

Positive games, games whose dividends are non-negative, were introduced as a class of games by Vasil'ev (1975). *T*-unanimity games are convex and, given (1), positive games are positive linear combinations of *T*-unanimity games. As a consequence, positive games are convex. For any given set of player *N*, the subspace of positive games *PG*(*N*) is a closed and convex cone contained in $CG(N).$

Imputations are *individually rational* allocations: an individual player cannot object to an imputation. This is a minimal requirement to be imposed on allocations. The set of imputations $I(N,v)=\{x\in\mathbb{R}^n\mid x(N)=v(N),\ x(i)\ge v(i)\ \text{for all}\ i\in N\}$ is a regular simplex and superadditivity ensures that it is nonempty. We denote by $I_0(N,v)$ the modified imputation set that imposes a zero payoff to null players.

We denote by Π_N the set of all players' orderings and we define the *marginal contributions vector* $\mu^\pi(N,v)$ associated to the players' ordering $\pi=(i_1,...,i_n)\in\Pi_N$ by:

$$\begin{aligned}&\mu^\pi_{i_1}(N,v)=v(i_1)-v(\varnothing)=v(i_1),\\&\mu^\pi_{i_k}(\pi)=v(i_1,...,i_k)-v(i_1,...,i_{k-1})\quad(k=2,...,n).\end{aligned}\tag{2}$$

There are *n*! marginal contribution vectors they are, not necessarily distinct. Under the assumption of superadditivity, marginal contributions vectors belong to the modified imputations set: $\mu^\pi(N,v)\in I_0(N,v)$ for all $\pi\in\Pi_N.$

3. The core and the Weber set

A *set-valued solution* is a mapping φ that associates a set of allocations $\varphi(N,v)\subset X(N,v)$ to a game $(N,v).$ Notice that *efficiency* is built in the definition of a solution. The core is a set-valued solution. It is the set of allocations that no coalition can improve upon:[6]

[6] The term "core" was introduced by Gillies (1953) in connection with the notion of stable set. It was later introduced as an independent solution concept by Shapley (1955). See Zhao (2018) for a clarification of the respective contributions of Gillies and Shapley as to the core.

$$C(N,v)=\left\{x\in\mathbb{R}^n \mid x(N)=v(N),\ x(S)\geq v(S) \text{ for all } S\subset N\right\}. \tag{3}$$

It may be empty. Shapley (1971) shows that the core of a convex game is the convex polyhedron whose vertices are the marginal contribution vectors.[7] Ichiishi (1981) shows that this is actually a necessary and sufficient condition for convexity. In particular, the core of a convex game contains the Shapley value defined as the average marginal of the contribution vectors (Shapley, 1971). The object that is obtained by considering all convex combinations of the marginal contribution vectors is called the *Weber set*. In other words, the Weber set is the convex hull of the marginal contribution vectors:[8]

$$W(N,v)=co\left\{x\in\mathbb{R}^n \mid x=\mu^{\pi}(N,v) \text{ for some } \pi\in\Pi_N\right\}.$$

Weber (1988) shows that it contains the core and therefore, under convexity, the two solutions coincide: $W(N,v)=C(N,v)$ if the game (N,v) is convex. The core is an *additive* solution on the subspace of convex games. This is a corollary of the following lemmas that apply to the space of *all* games. The proofs are reproduced in Appendix.

Lemma 1 (Peleg 1986) The core is a *superadditive* solution:

$$C(N,v_1)+C(N,v_2)\subset C(N,v_1+v_2) \text{ for all } v_1,v_2\in G(N).$$

Lemma 2 (Dragan et al. 1989) The Weber set is a *subadditive* solution:

$$W(N,v_1+v_2)\subset W(N,v_1)+W(N,v_2) \text{ for all } v_1,v_2\in G(N).$$

An allocation in the Weber set of a game (N,v) may *equivalently* be regarded as an average marginal contribution computed with respect to a probability distribution $p\in\Delta(\Pi_N)$ defined over players' orderings. It defines the *random order value* associated to p:

$$RV_i(N,v,\lambda)=\sum_{\pi\in\Pi_N} p(\pi)\mu_i^{\pi}(N,v) \quad (i=1,...,n).$$

Weighted Shapley values are random order values. Monderer et al. (1992) have shown that the core is a subset of the set $WS(N,v)$ of all weighted values.[9] Furthermore, the Weber set is a subset of the Harsanyi set as shown by Vasil'ev and van der Laan (2002). Hence, the following sequence of inclusions holds:

$$C(N,v)\subset WS(N,v)\subset W(N,v)\subset H(N,v) \text{ for all } v\in G(N).$$

The first three solutions coincide when applied to a convex game. Furthermore, the core and the Harsanyi set coincide if (and actually only if) they apply to positive games, as shown by Hammer et al. (1977) and Vasil'ev (1981). As a consequence, all four solutions coincide on the subspace of positive games.[10]

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[7] By Caratheodory's Theorem, core allocations can actually be written as a convex combination of at most $n+1$ marginal contribution vectors. See Theorem 17.1 in Rockafellar (1970).

[8] The convex hull of a set A, denoted $co(A)$, is the smallest convex set containing A. See Theorem 2.3 in Rockafellar (1970).

[9] The concept of weighted values was introduced by Shapley (1953) and later axiomatized by Shapley (1981) and by Kalai and Samet (1987). See Dehez (2011) for a characterization of the weighted Shapley value in a cost sharing context, along the lines suggested by Shapley (1981).

[10] See Dehez (2017) for a detailed analysis of the relations between these four solution concepts.

4. Axiomatization of the core of positive games

Given a player set N, consider the following four axioms applied to a set-valued solution $\varphi : G(N) \to \mathbb{R}^n$:

Efficiency (EFF): $x \in \varphi(N,v) \Rightarrow \sum_{i \in N} x_i = v(N)$.

Individual rationality (IR): $x \in \varphi(N,v) \Rightarrow x(i) \geq v(i)$ for all $i \in N$,

Null player (NULL): $x \in \varphi(N,v) \Rightarrow x_i = 0$ if i is a null player,

Additivity (ADD): $\varphi(N, v_1 + v_2) = \varphi(N, v_1) + \varphi(N, v_2)$.[11]

We have the following characterization of the core as a set-valued solution $\varphi : PG(N) \to \mathbb{R}^n$.

Proposition For a given a positive game (N,v), the core is the *maximal* set-valued solution $\varphi(N,v)$ *if and only if* φ satisfies EFF, IR, NULL and ADD.[12]

Proof Clearly, as a set-valued solution defined on the space of all games, the core satisfies EFF and IR by definition. It satisfies NULL as a consequence of the inequalities $v(i) \leq x_i \leq v(N) - v(N \setminus i)$ for all $i \in N$. It satisfies ADD on the subspace of convex games, and thereby also on the subspace of positive games.

Consider additive set-valued solutions φ applied to positive games. We will prove that, for any given positive game (N,v), the set $\varphi(N,v)$ of imputations that allocate 0 to null players coincides with the core.

Consider the game $(N, \lambda u_T)$ where $\lambda > 0$ and $T \subset N$. The set of imputations that allocate 0 to null players is given by:

$$\varphi(N, \lambda u_T) = \left\{ x \in \mathbb{R}^n_+ \mid x(N) = \lambda \text{ and } x_i = 0 \text{ for all } i \notin T \right\}.$$

Indeed, players outside T are null players and we have $u_{\{i\}}(i) = 1$ and $u_T(i) = 0$ if $T \neq \{i\}$, for all $i \in N$. Clearly, by definition of the core given in (3), we have $\varphi(N, \lambda u_T) = C(N, \lambda u_T)$.

Consider a positive game (N,v). Referring to the decomposition given in (1) and using additivity, we have:

$$\varphi(N,v) = \sum_{T \subset N} \varphi(N, \alpha_T u_T) = \sum_{T \subset N} C(N, \alpha_T u_T).$$

Using additivity again, we have:

$$\sum_{T \subset N} C(N, \alpha_T u_T) = C(N,v). \ \blacklozenge$$

Remark 1 As a set-valued solution, the imputation set satisfies individual rationality (by definition) and additivity but not the null player property. However, the modified imputation set $I_0(N,v)$ results is a solution that fails to be additive. The following two games $v_1 = (0,1,2 \mid 1,2,4 \mid 4)$ and $v_2 = (2,0,3 \mid 2,8,3 \mid 8)$ are positive. Player 1 is null in the first game and player 2 is null in the second while no player is null in the game sum $v = v_1 + v_2$ that is given by $v = (2,1,5 \mid 3,10,7 \mid 12)$. We do have $I_0(N,v_1) + I_0(N,v_2) \subset I_0(N,v)$: as an allocation

[11] Sums and differences of sets are defined vector wise.

[12] Maximal is to be understood *inclusion-wise*.

rule, the modified imputation set is indeed *superadditive*. It is however *not subadditive*: the allocation $x = (6,1,5)$ belongs to $I_0(N,v)$ and cannot be obtained as a sum of the allocations $x_1 \in I_0(N,v_1)$ and $x_2 \in I_0(N,v_2)$: the maximum player 1 can obtain in $I_0(N,v_2)$ is 5.

One could be tempted to follow the steps in Shapley's proof of the uniqueness proof of his value but it does not work: the core does not result from EFF, IR, NULL and ADD when applied to a non-positive convex game. The game $v = (0,0,0|1,1,2|3)$ is convex, its dividends are given by $\alpha = (0,0,0|1,1,2|-1)$ and its core is given by:

$$C(N,v) = \{x \in \mathbb{R}^3_+ \,|\, x_1 + x_2 + x_3 = 3, x_1 \le 1, x_2 \le 2, x_3 \le 2\}.$$

The characteristic function v can be decomposed as a difference between two positive games $v = v^+ - v^-$ where:

$$v^+(S) = \sum_{T \subset S:\, \alpha_T > 0} \alpha_T \quad \to \; v^+ = (0,0,0|1,1,2|4),$$

$$v^-(S) = \sum_{T \subset S:\, \alpha_T < 0} (-\alpha_T) \; \to \; v^- = (0,0,0|0,0,0|1).$$

By ADD, we have successively:

$$\varphi(N,v^+) \; = \sum_{T \subset N:\, \alpha_T > 0} C(N, \alpha_T u_T) = C(N,v^+)$$

$$= \{x \in \mathbb{R}^3_+ \,|\, x_1 + x_2 + x_3 = 4, x_1 \le 2, x_2 \le 3, x_3 \le 3\},$$

and

$$\varphi(N,v^-) \; = \sum_{T \subset N:\, \alpha_T < 0} C(N, -\alpha_T u_T) = C(N,v^-)$$

$$= \{x \in \mathbb{R}^3_+ \,|\, x_1 + x_2 + x_3 = 1\}.$$

The convex game v^+ is a sum of two convex games: $v^+ = v + v^-$. Then, by ADD, we have $\varphi(N,v^+) = \varphi(N,v) + \varphi(N,v^-)$ or, alternatively, $\varphi(N,v) = C(N,v^+) - C(N,v^-)$. Considering the allocations $(2,2,0) \in C(N,v^+)$ and $(0,1,0) \in C(N,v^-)$, we must have $(2,1,0) \in \varphi(N,v)$, while $(2,1,0) \notin C(N,v)$. As a consequence, $\varphi(N,v) \ne C(N,v)$.

Remark 2 Actually, $C(N,v^+) - C(N,v^-) = H(N,v)$ as shown by Derks et al. (2000) and Vasil'ev (2006), and we have seen that coincidence with the core occurs if (and only if) the game (N,v) is positive.

Our proposition is to be compared to the axiomatization of the Harsanyi set on the space of all games proposed by Vasil'ev (1982) and Derks et al. (2000). They introduce the following axiom:

Weak positivity (WPOS): if (N,v) is a positive game, then $\varphi(N,v) \subset \mathbb{R}^n_+$.

and prove that EFF, NULL, WPOS and ADD characterizes the Harsanyi set on the space $G(N)$ of all games.[13]

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[13] See Proposition 7 in Dehez (2011) for a simple proof. Derks et al. (2000) show that strengthening the positivity axiom characterizes the Weber set on the space *G*(*N*) of all games.

5. Concluding remarks

Adding the axiom of symmetry requiring that substitutable players be treated equally, characterizes the Shapley value. Indeed, the set $\varphi(N, \lambda u_T)$ then reduces to the single allocation x given by:

$$x_i = \frac{\lambda}{t} \quad \text{for all } i \in T,$$
$$= 0 \quad \text{for all } i \notin T.$$

Last but not least, one question: what is the proportion of positive games within the subspace of convex games? For 3-player games, Dehez and Pacini (2024) show that 30% of the balanced superadditive games are convex and 1 out of 5 of them are positive. For a larger number of players, the question is open.

Appendix

Proof of Lemma 1

Consider two set functions v_1, v_2 in $G(N)$ and an allocation $x \in C(N, v_1) + C(N, v_2)$. Then using (3), there exist $x^1 \in C(N, v_1)$ and $x^2 \in C(N, v_2)$ such that $x = x^1 + x^2$ i.e.

$$x^1(S) \geq v_1(S) \text{ and } x^2(S) \geq v_2(S) \text{ for all } S \subset N.$$

Hence $x(S) = x^1(S) + x^2(S) \geq (v_1 + v_2)(S)$ for all $S \subset N$ i.e. $x \in C(N, v_1 + v_2)$.

Proof of Lemma 2

Consider two games v_1 and v_2 in $G(N)$, and their marginal contribution vectors $\mu^\pi(N, v_1)$ and $\mu^\pi(N, v_2)$ as defined by (2). By definition of convex hull, we have:

$$\begin{aligned} W(N, v_1) + W(N, v_2) &= co\{\mu^\pi(N, v_1) \mid \pi \in \Pi_N\} + co\{\mu^\pi(N, v_2) \mid \pi \in \Pi_N\} \\ &= co\left(\{\mu^\pi(N, v_1) \mid \pi \in \Pi_N\} + \{\mu^\pi(N, v_2) \mid \pi \in \Pi_N\}\right). \end{aligned}$$

The marginal contribution vectors associated to the game $(N, v^1 + v^2)$ are the sum of the marginal contribution vectors. Hence, $W(N, v^1 + v^2) = co\{\mu^\pi(N, v_1) + \mu^\pi(N, v_2) \mid \pi \in \Pi_N\}$ and $\{\mu^\pi(N, v_1) + \mu^\pi(N, v_2) \mid \pi \in \Pi_N\} \subset \{\mu^\pi(N, v_1) \mid \pi \in \Pi_N\} + \{\mu^\pi(N, v_2) \mid \pi \in \Pi_N\}$. Consequently, by the definition of the convex hull, we have:

$$co\{\mu^\pi(N, v_1) + \mu^\pi(N, v_2) \mid \pi \in \Pi_N\} \subset co\left(\{\mu^\pi(N, v_1) \mid \pi \in \Pi_N\} + \{\mu^\pi(N, v_2) \mid \pi \in \Pi_N\}\right). \ \blacklozenge$$

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