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Dissertation présentée en vue de l'obtention du grade de Docteur en Sciences, en séance publique, le 10 décembre 2004, au Département de Mathématique de la Faculté des Sciences de l'Université catholique de Louvain, à Louvain-la-Neuve.

Promoteur: Enrico Maria Vitale

Jury: Francis Borceux (UCL)
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Enrico M. Vitale (UCL)
Richard J. Wood (Dalhousie University, Halifax, Canada)

AMS Subject Classification (2000): 18C10, 18E35, 18F20, 18C35, 18A25, 18A30, 18A35, 16D90.

Dépôt légal : D/2004/9964/39
ISBN : 2-930344-78-4

Diffusion: http://www.i6doc.com, l'édition universitaire en ligne.
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It is the task of history to explain how modern concepts arose and why they are interesting and relevant. It is the task of modern mathematics to explain the same thing independent of and irrespective of history.
– J. W. Gray. Fragments of the history of Sheaf Theory, *Applications of Sheaves*, Proceedings of the Research Symposium on Application of Sheaf Theory to Logic, Algebra, and Analysis, Durham, 1977 –

Contents

Introduction

Algebraic theories and algebraic categories were introduced by Lawvere in his pioneering Ph.D. thesis as invariant formulations of Birkhoff's universal algebra (see [30]). The innovative and revelatory approach consists of describing the syntax and the semantics in the following way. An *algebraic theory* is a concrete mathematical object – the concept – namely a set of variables together with formal symbols and equalities between these terms. Stated otherwise, an algebraic theory is a small category with finite products. An algebra or model of the theory is a set-theoretical interpretation – a possible meaning – or, more categorically, a finite product-preserving functor from the theory into the category of sets. We call the category of models of an algebraic theory an *algebraic category.* This description allows us to investigate "simultaneously" all those structures like groups, rings, modules, lattices, ... characterised by the existence of one or several operations defined everywhere and subjected to axioms expressed by equalities. The peculiar feature of the subject is the interplay between properties of the theory and properties of the models: how we can recollect aspects of the syntax knowing about the semantics and conversely. We mention that Lawvere's investigation concerned the one-sorted version; whereas, at the same period, many-sorted algebraic theories were studied by Bénabou (see [8]).

By generalising the theory we do generalise the models. This concept is the fascinating aspect of the subject and the reference point of our project. Replacing finite products by finite limits provides the notion of a locally finitely presentable category of models. A theory, in this case, is a small category $\mathcal{C}$ with finite limits; a model of the theory is a set-valued functor $F : \mathcal{C} \longrightarrow \mathbf{set}$ which preserves finite limits. If we consider the case of absence of limits, other generalisations are possible. A theory is a mere small category $\mathcal{C}$; a model of the theory is a flat functor $F : \mathcal{C} \longrightarrow \mathbf{set}$, namely a (small) filtered colimit of representable functors. This leads to the notion of accessible category of models. On the other hand, a model of the theory $\mathcal{C}$ could be just a set-valued functor $F : \mathcal{C} \longrightarrow \mathbf{set}$. In this case, we deal with presheaf categories.

Accessible categories admit an intrinsic axiomatic characterisation in terms of generators: they are those categories with a "good" set of generators. And locally presentable categories are those accessible categories which are cocomplete (or, equivalently, complete).

In this manuscript, we are interested in the study of categories of models. We pursue our task by means of considering models of different theories. We approach the investigation from two different – but related – points of view, as the structure of the thesis itself will denote.

The first idea is "unification": the theories are small categories with a specified class $\mathbb{D}$ of limits, the models of the theory are set-valued functors preserving the class of limits $\mathbb{D}$ under consideration. It is an attempt of describing "at the same time" similar situations, as the ones previously mentioned, where $\mathbb{D}$ can be the class of finite products, or the class of finite limits, or something else. This wants to be a generalisation of the notion of models of a theory, which encloses in one all the known cases of models, from presheaves to locally finitely presentable categories. The analysis leads to the definition of $\mathbb{D}$-theories and, consequently, of categories of $\mathbb{D}$-models. The theory of locally $\mathbb{D}$-presentable categories has been recently developed by Adámek, Borceux, Lack and Rosický in [1]. Starting from a doctrine – that is, an (essentially small) full subcategory of **cat**, the category of all small categories – $\mathbb{D}$, the notion of $\mathbb{D}$-limit – the limit of a functor with domain in $\mathbb{D}$ – the notion of $\mathbb{D}$-complete category – a category with all $\mathbb{D}$-limits – and the notion of $\mathbb{D}$-continuous functor – a functor, between $\mathbb{D}$-complete categories, which preserves all $\mathbb{D}$-limits – have been introduced. By generalising further these issues, one is led to define $\mathbb{D}$-filtered colimits: they are – as you might expect – those colimits commuting in **set** with $\mathbb{D}$-limits. Then, all notions of $\mathbb{D}$-presentable object, of $\mathbb{D}$-accessibility and of locally $\mathbb{D}$-presentability take shape quite naturally. We mention at once that this kind of generalisation is not possible for any class $\mathbb{D}$ of limits. In fact, a further assumption is required on the doctrine $\mathbb{D}$: the condition is called *soundness*. Roughly speaking, soundness is the hypothesis needed in order to achieve the analogous – to the locally finitely presentable case – implication that for a diagram $S : \mathcal{I} \longrightarrow \mathcal{C}$, for a small category $\mathcal{C}$, its category of cocones is connected implies that the category $\mathcal{C}$ is filtered. The definition might appear quite technical (and it is so) but it is the one needed to establish the expected correspondence between $\mathbb{D}$-flat functors and $\mathbb{D}$-continuous functors, precisely as in the case of finite limits. More explicitly, under the assumption on the doctrine $\mathbb{D}$ to be sound, a $\mathbb{D}$-flat functor $F : \mathcal{A} \longrightarrow \mathcal{B}$, defined on a $\mathbb{D}$-complete category $\mathcal{A}$, is, moreover, $\mathbb{D}$-continuous. We support the general theory of locally $\mathbb{D}$-presentable categories with examples. For our purposes, the illuminating examples are mainly three: the doctrine FIN of finite limits, the doctrine FINPR of finite products and the empty doctrine. The corresponding categories are, therefore, locally finitely presentable categories, (finitary multisorted) varieties and presheaf categories, respectively. We achieve a duality theorem, in terms of a contravariant biequivalence:

$$\mathbb{D}\text{-}\mathbf{Th}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{LP}$$

where $\mathbb{D}$-**Th** is the 2-category of essentially small Cauchy complete and $\mathbb{D}$-complete categories, $\mathbb{D}$-continuous functors and natural transformations, whereas $\mathbb{D}$-**LP** is the 2-category of locally $\mathbb{D}$-presentable categories, $\mathbb{D}$-accessible right adjoint functors and natural transformations (see [18]). This is the theorem which provides a unified proof of Gabriel-Ulmer duality for locally finitely presentable categories (see [23]), Adámek-Lawvere-Rosický duality for varieties (see [2]) and Morita duality for presheaf categories (see, for instance, [10]). Moreover, following the suggestion of the locally finitely presentable case, we investigate further these issues to finally propose a new characterisation of locally $\mathbb{D}$-presentable categories in terms of strong generators. Explicitly, we prove that a cocomplete category is locally $\mathbb{D}$-presentable if and only if it has a strong generator consisting of $\mathbb{D}$-presentable objects (see [17]). We mention that some work in this area has been undertaken independently by Diers ([20]).

The second point of view is "specialisation". We go back to a "case by case" analysis, in order to study *localizations* (namely, fully faithful right adjoint functors whose left adjoint preserves finite limits) of algebraic categories and localizations of presheaf categories. These are still categories of models of the corresponding theory.

Motivations to the study of localizations can be historically traced in homological algebra, for instance. The goal, which a wide class of mathematicians pursue, is to do homological algebra in a sheaf theoretical context. The fundamental, basic remark is that categories of presheaves of abelian groups or modules are abelian categories, meaning, roughly speaking, that techniques of homological algebra as "diagram lemmas" can be applied to categories of presheaves and performed quite easily. From where, the need of carrying this concept to categories of sheaves has arisen. This is the fruitful idea behind the formal definition of localization: homological algebra is inherited via localizations. More precisely, a localization of an abelian category is again abelian. The crucial reference point is that categories of sheaves are precisely the localizations of presheaf categories (see [25, 36]). The next, natural setting where to develop homological algebra and, therefore, where to study localizations are semi-abelian categories and, in particular, semi-abelian algebraic categories: the category of groups, the category of rings, the category Lie algebras, ... are all examples of such. As references for this more recent research developments, we mention to [12, 24]. Finally, to provide some context for localization, we mention the following example: given a ring A and a prime ideal I, by localising the ring A over the ideal I, one obtains the category of modules over the local ring A_I as a localization of the category of modules over the ring A.

This thesis sits inside a wider project, which is still in our minds. In the original classical spirit of Category Theory, our purpose is to produce a new notion which generalises localizations and allows us to treat them in the more general setting of the theory of locally $\mathbb{D}$-presentable categories. More explicitly, for a limit doctrine $\mathbb{D}$, we wish to define a proper notion of $\mathbb{D}$-localization, which should make "good sense" at least while specialising $\mathbb{D}$ in the three main examples we always keep in mind while dealing with $\mathbb{D}$-theories, namely, the doctrine FIN, the doctrine FINPR and the empty doctrine. A $\mathbb{D}$-*localization* $i : \mathcal{B} \longrightarrow \mathcal{A}$ should be a fully faithful functor i together with a left adjoint $L : \mathcal{A} \longrightarrow \mathcal{B}$ which preserves $\mathbb{D}$-limits (where $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{D}$-complete categories). Clearly, in the case of the doctrine $\mathbb{D}$ of finite limits, this definition returns the classical one of localization. The aim of some advanced investigation is to classify $\mathbb{D}$-localizations of categories of set-valued $\mathbb{D}$-continuous functors, that is, categories of the form $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, for some small $\mathbb{D}$-complete category $\mathcal{A}$, which are precisely all locally $\mathbb{D}$-presentable categories. In the case of the doctrine of finite limits, this project has already been undertaken by Day and Street (see [19]): they characterise localizations of locally presentable categories. Moreover, we mention that some work has been done, in a slightly different optic, when specifying $\mathbb{D}$ as the empty doctrine: in [39] (Theorem 17.3.3) they classify localizations of presheaf categories in terms of dense functors. Unfortunately, when we tried to adapt these kinds of results to the case of the doctrine FINPR, we did not succeed in describing the localizations of the categories of models. The nature of the problem is partially grounded on the condition on the left adjoint functor L to preserves finite products. These troubles in finding a general way of investigating $\mathbb{D}$-localizations, for a limit doctrine $\mathbb{D}$, determined the beginning of a deep investigation on the character of those functors $k : \mathcal{C} \longrightarrow \mathcal{E}$ defined on a category $\mathcal{C}$ with finite coproducts. It led us to the discovery of the characterisation theorems for localizations described in Chapter 3, for the presheaf case, and in Chapter 4, for the algebraic case.

Of course, localizations of presheaf categories and of algebraic categories are well-known and well-studied mathematical objects: to mention some references, consider Giraud's theorem for the former case ([25, 34]) and the theorems by Vitale for the latter one ([41, 42]). The innovation in this work consists of a different approach to the matter of localizations. We wish to characterise all functors $k : \mathcal{C} \longrightarrow \mathcal{E}$ from a small category $\mathcal{C}$ – with or without finite products, depending on the case treated – into a category $\mathcal{E}$ which satisfies certain properties, depending on the corresponding case, which realise the category $\mathcal{E}$ as a localization of the category $[\mathcal{C}^{op}, \mathbf{set}]$ of set-valued functors, or of the category $\mathbf{Finprod}[\mathcal{C}^{op}, \mathbf{set}]$ of set-valued finite product-preserving functors, which is precisely $\mathbf{Mod}(\mathcal{C})$. In other words, provided some category $\mathcal{E}$, we establish theorems which describe all its realisations as a localization of some appropriate category of set-valued functors. The suggestion comes from a Gabriel-Popescu theorem, which characterises Grothendieck categories (namely, cocomplete abelian categories with a generator and filtered colimits commuting with finite limits) as localizations of the additive functor category $\mathbf{Add}[\mathcal{C}^{op}, \mathbf{Ab}]$ of additive presheaves defined on a small pre-additive category $\mathcal{C}$, and its generalisation in the recent paper [32]. Following these lines, we achieve a classification of localizations, in both the presheaf and the algebraic context and, moreover, a classification of *geometric morphisms* (namely, functors together with a finite limit-preserving left adjoint). We are able, in fact, to state independent conditions on the functor $k : \mathcal{C} \longrightarrow \mathcal{E}$, which provide fully faithfulness of the embedding into the functor category on one side, and left exactness of the left adjoint functor on the other. It is precisely the nature of these properties which allows us to distinguish the classification of geometric morphisms from that of localizations. It will be redundant to list, in this introduction, the various conditions we came across for the functor k, but it is still worthwhile to mention that some of these might have – for those readers familiar with the subject – the same flavour of the properties characterising the *Lemme de Comparaison*, as in [28], whereas some others remind of the notion of *filtering functor*, as in [34]. In fact, we introduce the new notion of *special filtering* functor, which encapsulates the deep nature of the exact completion of the full subcategory of the category of models consisting of the free ones.

The classifying theorems contained in Chapter 3 and Chapter 4 might propose a new way of facing the issue of $\mathbb{D}$-localizations. Provided a category $\mathcal{E}$, we should hope to reach some results describing all its realisations as a $\mathbb{D}$-localization of some appropriate category of $\mathbb{D}$-continuous set-valued functors. The desirable achievement would be to establish some theorems giving necessary and sufficient conditions on a $\mathbb{D}$-continuous functor $k : \mathcal{C} \longrightarrow \mathcal{E}$, from a small $\mathbb{D}$-complete category $\mathcal{C}$ into a cocomplete, exact category $\mathcal{E}$ with $\mathbb{D}$-filtered colimits commuting with $\mathbb{D}$-limits, to realise $\mathcal{E}$ as a $\mathbb{D}$-localization of the category of $\mathbb{D}$-continuous set-valued functors defined on $\mathcal{C}$. This remains part of our outgoing project, still open to fruitful investigations.

In Chapter 1, we illustrate the theory of locally $\mathbb{D}$-presentable categories. We recall the basic definitions the reader will find useful to approach this thesis: starting from the notion of a doctrine $\mathbb{D}$, we present the whole series of $\mathbb{D}$-limit, $\mathbb{D}$-filtered colimit, $\mathbb{D}$-presentable object, $\mathbb{D}$-accessibility and locally $\mathbb{D}$-presentability definitions. The fundamental results concerning this material are listed and provided with (sketches of the) proofs. Classical examples constantly support this new-born theory.

In Chapter 2, we build our contravariant biequivalence between the 2-category $\mathbb{D}$-**Th**, of essentially small $\mathbb{D}$-complete and Cauchy complete categories, $\mathbb{D}$-continuous functors and natural transformations, and the 2-category $\mathbb{D}$-**LP**, of locally $\mathbb{D}$-presentable categories, $\mathbb{D}$-accessible right adjoint functors and natural transformations. This Duality theorem provides the desired, expected generalisation of Gabriel-Ulmer duality for locally finitely presentable categories, Adámek-Lawvere-Rosický duality for varieties and Morita duality for presheaf categories. Moreover, we state our characterisation of locally $\mathbb{D}$-presentable categories in terms of strong generators. As applications of the Duality theorem, we compare different doctrines and, consequently, different categories of models, providing a useful scheme to relate locally finitely presentable categories, (finitary multisorted) varieties and presheaves categories.

In Chapter 3, we briefly mention the fundamental definitions of geometric morphism and localization. Moreover, the notion of filtering functor is recalled. Given a small category $\mathcal{A}$ and a cocomplete, exact and extensive category $\mathcal{B}$, we establish the crucial equivalence between the category $\mathbf{GeoMor}[\mathcal{B}, [\mathcal{A}^{op}, \mathbf{set}]]$, of geometric morphisms between $\mathcal{B}$ and $[\mathcal{A}^{op}, \mathbf{set}]$ and geometric transformations between those, and the category $\mathbf{Filt}[\mathcal{A}, \mathcal{B}]$, of filtering functors from $\mathcal{A}$ to $\mathcal{B}$ and all natural transformations. We present some revelatory proprieties on functors $k : \mathcal{C} \longrightarrow \mathcal{E}$, which will enlighten our characterisation of localizations. Combining these conditions together, we state our theorem classifying localizations of presheaf categories. As a corollary, we present a classification of geometric morphisms into presheaf categories. Finally, an alternative proof of our classifying theorem based on sheaf theoretical techniques is offered.

In Chapter 4, we outline the basics for algebraic theories and algebraic categories, by listing the crucial results. We specify the coproduct completion and the exact completion in the case of an algebraic theory. This leads to the notion of *special filtering* functor and, further, to an equivalence between the category $\mathbf{GeoMor}[\mathcal{E}, \mathbf{Mod}(\mathcal{C})]$, of geometric morphisms between $\mathcal{E}$ and $\mathbf{Mod}(\mathcal{C})$ and geometric transformations between those, and the category $\mathbf{SFilt}[\mathcal{C}, \mathcal{E}]$, of special filtering functors which preserve finite coproducts from $\mathcal{C}$ to $\mathcal{E}$ and all natural transformations, provided $\mathcal{C}$ is an algebraic theory and $\mathcal{E}$ is a cocomplete and exact category in which filtered colimits commute with finite limits. Thanks to this equivalence, we are able to establish our theorem describing all realisations of a given category as localization of some (one sorted) algebraic category. As in the presheaf case, a direct corollary of this result is a classifying theorem for geometric morphisms into algebraic categories.

1

The theory of locally $\mathbb{D}$-presentable categories

Basic definitions are recalled. The idea of "$\mathbb{D}$-ification" is presented and justified by classical examples as algebraic categories, presheaf categories and locally finitely presentable categories of Gabriel and Ulmer. The theory of locally $\mathbb{D}$-presentable categories was quite recently introduced in an article by Adámek, Borceux, Lack and Rosický (see [1]). We present the main results concerning the subject always referring for details to the above-mentioned paper.

1.1 $\mathbb{D}$-filtered categories

All definitions and results contained in this section can be found in [1].

Definition 1 *A collection $\mathbb{D}$ of small categories is called a* limit doctrine*, or just* doctrine*, if $\mathbb{D}$, regarded as full subcategory of* **cat***, is essentially small. A $\mathbb{D}$-*limit *is the limit of a functor with domain in $\mathbb{D}$. A category which has all $\mathbb{D}$-limits is said to be $\mathbb{D}$-*complete*, and a functor between $\mathbb{D}$-complete categories is called $\mathbb{D}$-*continuous *if it preserves all $\mathbb{D}$-limits. Dually, there are the notions of $\mathbb{D}$-cocompleteness and $\mathbb{D}$-cocontinuity. $\mathbb{D}^{op}$ stands for the doctrine consisting of all categories $\mathcal{D}^{op}$, for $\mathcal{D} \in \mathbb{D}$.*

As general consideration, let us point out that a category $\mathcal{A}$ is $\mathbb{D}$-complete if and only if its dual category $\mathcal{A}^{op}$ is $\mathbb{D}^{op}$-cocomplete.

Definition 2 *We say that a small category $\mathcal{C}$ is $\mathbb{D}$-*filtered *if $\mathcal{C}$-colimits commute in* **set** *with $\mathbb{D}$-limits, namely for any $\mathcal{D} \in \mathbb{D}$ and any functor $F : \mathcal{C} \times \mathcal{D} \longrightarrow$* **set** *the canonical map*

$$\varphi : \operatorname*{colim}_{C\in\mathcal{C}} \lim_{D\in\mathcal{D}} F(C, D) \longrightarrow \lim_{D\in\mathcal{D}} \operatorname*{colim}_{C\in\mathcal{C}} F(C, D)$$

is an isomorphism. Explicitly, the morphism φ is defined by the universal property of the colimit, as in the following diagram:

$$\begin{tikzcd}
\operatorname*{colim}\limits_{C\in\mathcal{C}} \lim\limits_{D\in\mathcal{D}} F(C,D) \arrow[r, "\varphi"] & \lim\limits_{D\in\mathcal{D}} \operatorname*{colim}\limits_{C\in\mathcal{C}} F(C,D) \arrow[d, "\lambda_D"] \\
\lim\limits_{D\in\mathcal{D}} F(C,D) \arrow[u, "\rho_C"] \arrow[ur, dotted, "\varphi_C"] \arrow[d, "\mu_D"'] & \operatorname*{colim}\limits_{C\in\mathcal{C}} F(C,D) \\
F(C,D) \arrow[r, "1_{F(C,D)}"'] & F(C,D) \arrow[u, "\sigma_C"']
\end{tikzcd}$$

that is, φ is the unique arrow in **set** *satisfying $\varphi.\rho_C = \varphi_C$, for any object $C \in \mathcal{C}$, provided the morphism φ_C, defined via the universal property of the limit, is the unique arrow in* **set** *such that $\lambda_D.\varphi_C = \sigma_C.\mu_D$, for any $D \in \mathcal{D}$. (Here, we denote by λ_D and μ_D the canonical limit projections in* **set**, *whereas ρ_C and σ_C denote the canonical colimit injections in* **set**.*)*

By a $\mathbb{D}$-*filtered colimit* we mean the colimit of a functor defined on a $\mathbb{D}$-filtered category.

Example 3

1. The doctrine FIN of finite limits is the essentially small collection $\mathbb{D}$ of all finite categories. A small category $\mathcal{C}$ is said to be *filtered* if it is not empty; for any pair of objects $C_1, C_2 \in \mathcal{C}$, there exist an object $C_3 \in \mathcal{C}$ and a pair of arrows $f : C_1 \longrightarrow C_3$ and $g : C_2 \longrightarrow C_3$ of $\mathcal{C}$; and, finally, for any pair of objects $C_1, C_2 \in \mathcal{C}$ and for any pair of arrows $f, g : C_1 \longrightarrow C_2$ of $\mathcal{C}$, there exist an object $C_3 \in \mathcal{C}$ and a morphism $h : C_2 \longrightarrow C_3$ satisfying $h.f = h.g$. Filtered colimits have been defined as colimits of diagrams which have filtered categories as domain. It is a well-know fact that filtered colimits are those colimits commuting in **set** with finite limits (see [6]), that is, filtered colimits commute with FIN-limits. In other words,
$$\text{FIN} - \text{filtered} \equiv \text{filtered}$$

2. The doctrine FINPR of finite products is the essentially small collection of all finite discrete categories. A small category $\mathcal{C}$ is said to be *sifted* if $\mathcal{C}$-colimits commute in **set** with finite products, that is
$$\text{FINPR} - \text{filtered} \equiv \text{sifted}$$

Of course, any filtered category is sifted and reflexive coequalizers are sifted; moreover, any category with finite coproducts is so. Sifted categories were introduced in [2, 5]. Explicitly, sifted colimits are the colimits of diagrams which have sifted categories as domain; in other words, sifted colimits are those colimits commuting in **set** with finite products.

3. For $\mathbb{D} = \emptyset$, the empty doctrine; any small category is $\emptyset$-filtered.

4. The doctrine FINCL of finite connected limits is the essentially small collection of all finite connected categories and

$$\text{FINCL} - \text{filtered} \equiv \text{small coproducts of filtered categories}$$

since coproducts commute in **set** with connected limits. For more details, we refer to [1].

5. The doctrine TERM of the terminal object is the doctrine consisting of the empty category and

$$\text{TERM} - \text{filtered} \equiv \text{connected}$$

Definition 4 *Let $\mathcal{K}$ and $\mathcal{K}'$ be categories with $\mathbb{D}$-filtered colimits. A functor $F : \mathcal{K} \longrightarrow \mathcal{K}'$ is* $\mathbb{D}$-accessible *if it preserves $\mathbb{D}$-filtered colimits. An object K of $\mathcal{K}$ is* $\mathbb{D}$-presentable *if the representable functor $\mathcal{K}(K, -) : \mathcal{K} \longrightarrow$ **set** is $\mathbb{D}$-accessible.*

We recall from [1] the following fundamental fact:

Lemma 5 *A $\mathbb{D}^{op}$-colimit of $\mathbb{D}$-presentable objects is $\mathbb{D}$-presentable itself.*

Proof. Consider a category $\mathcal{D} \in \mathbb{D}$ and a diagram $S : \mathcal{D}^{op} \longrightarrow \mathcal{K}$ for which $S(D)$ is a $\mathbb{D}$-presentable object of $\mathcal{K}$, for any $D \in \mathcal{D}$, and the colimit $\mathrm{colim}_{D \in \mathcal{D}^{op}} S(D)$ exists in $\mathcal{K}$. We wish to show that the representable functor $\mathcal{K}(\mathrm{colim}_{D \in \mathcal{D}^{op}} S(D), -) : \mathcal{K} \longrightarrow$ **set** preserves $\mathbb{D}$-filtered colimits. For it, take $\mathcal{J}$ a $\mathbb{D}$-filtered category and a functor $H : \mathcal{J} \longrightarrow \mathcal{K}$, we get:

$$\begin{array}{rcll} \mathcal{K}(\operatorname*{colim}_{D \in \mathcal{D}^{op}} S(D), \operatorname*{colim}_{J \in \mathcal{J}} H(J)) & \simeq & \lim_{D \in \mathcal{D}} \mathcal{K}(S(D), \operatorname*{colim}_{J \in \mathcal{J}} H(J)) & \\ & \simeq & \lim_{D \in \mathcal{D}} \operatorname*{colim}_{J \in \mathcal{J}} \mathcal{K}(S(D), H(J)) & (1) \\ & \simeq & \operatorname*{colim}_{J \in \mathcal{J}} \lim_{D \in \mathcal{D}} \mathcal{K}(S(D), H(J)) & (2) \\ & \simeq & \operatorname*{colim}_{J \in \mathcal{J}} \mathcal{K}(\operatorname*{colim}_{D \in \mathcal{D}^{op}} S(D), H(J)) & \end{array}$$

(where (1) follows from $S(D)$ being $\mathbb{D}$-presentable, for any $D \in \mathcal{D}$, and (2) is just the definition of $\mathbb{D}$-filtered colimit, namely a colimit commuting in **set** with $\mathbb{D}$-limits). ■

$\mathbb{D}$-flat functors and sound doctrines

For a given functor $F : \mathcal{A} \longrightarrow \mathbf{set}$, let us recall that the category $\mathbf{Elts}(F)$ of elements of F has for objects the pairs (A, a), where A is an object of $\mathcal{A}$ and $a \in FA$, and for morphisms $f : (A, a) \longrightarrow (B, b)$ those arrows $f : A \longrightarrow B$ of $\mathcal{A}$ such that $Ff(a) = b$. For a small category $\mathcal{A}$, let us also recall that a functor $F : \mathcal{A} \longrightarrow \mathbf{set}$ is said to be *flat* if it is a filtered colimit of representable functors. Equivalently, the functor F is flat if one of the following condition is satisfied:

- $\mathbf{Elts}(F)^{op}$ is filtered;
- the Kan extension $\mathbf{Lan}_{Y_{\mathcal{A}}}(F) : [\mathcal{A}^{op}, \mathbf{set}] \longrightarrow \mathbf{set}$ of F along the Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$ preserves finite limits.

For more details, see, for instance, [10].
The important fact about flat functors, that we want to underline, is:

Remark 6 If, moreover, $\mathcal{A}$ is finitely complete, a functor $F : \mathcal{A} \longrightarrow \mathbf{set}$ is flat if and only if F preserves finite limits.

The idea of generalising the definition of flat functor in terms of a limit doctrine $\mathbb{D}$ arises quite naturally and leads to the following:

Definition 7 *Let $\mathcal{A}$ be a small category. A functor $F : \mathcal{A} \longrightarrow \mathbf{set}$ is called* $\mathbb{D}$-flat *if the left Kan extension $\mathbf{Lan}_{Y_{\mathcal{A}}}(F) : [\mathcal{A}^{op}, \mathbf{set}] \longrightarrow \mathbf{set}$ of F along the Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$ preserves $\mathbb{D}$-limits.*

The aim, here, is to get a correspondence between $\mathbb{D}$-flat functors and $\mathbb{D}$-continuous functors in the same spirit as Remark 6: flat functors coincide with lex functors when the domain category is finitely complete. It is crucial to remark, though, that in order to achieve the expected equivalence when moving from finite limits to $\mathbb{D}$-limits, for a doctrine $\mathbb{D}$, some extra assumptions are needed on the doctrine $\mathbb{D}$. That is when the notion of soundness comes into play:

Definition 8 *A doctrine $\mathbb{D}$ is* sound *if for a functor $F : \mathcal{A} \longrightarrow \mathbf{set}$, with $\mathcal{A}$ a small category, the left Kan extension $\mathbf{Lan}_{Y_{\mathcal{A}}}(F) : [\mathcal{A}^{op}, \mathbf{set}] \longrightarrow \mathbf{set}$ of F along the Yoneda embedding preserves $\mathbb{D}$-limits if and only if it preserves $\mathbb{D}$-limits of representables.*

The previous definition can be proved to be equivalent to:

Definition 9 *A doctrine $\mathbb{D}$ is said to be* sound *if a small category $\mathcal{C}$ is $\mathbb{D}$-filtered whenever the category of cocones of any functor $S : \mathcal{D}^{op} \longrightarrow \mathcal{C}$, with $\mathcal{D} \in \mathbb{D}$, is connected.*

Example 10

1. The doctrine FIN of finite limits is sound.

2. The doctrine FINPR of finite products is sound.

3. The empty doctrine is sound.

4. The doctrine FINCL of finite connected limits is sound.

5. The doctrine TERM of the terminal object is sound.

6. The doctrine PB of pullbacks is not sound.

A fundamental result about sound doctrines is given by:

Proposition 11 *If the doctrine $\mathbb{D}$ is sound, then every small $\mathbb{D}^{op}$-co-complete category is $\mathbb{D}$-filtered.*

Proof. If $\mathcal{C}$ is a $\mathbb{D}^{op}$-cocomplete category, then for any functor $S : \mathcal{D}^{op} \longrightarrow \mathcal{C}$, where $\mathcal{D} \in \mathbb{D}$, its category of cocones has an initial object, namely the colimit of the diagram S, and, therefore, it is connected. Then, by soundness of $\mathbb{D}$, $\mathcal{C}$ is $\mathbb{D}$-filtered. ■

Therefore, thanks to the notion of soundness, the expected equivalence Theorem holds and we can characterise $\mathbb{D}$-flat functors in the following Theorem. We recall first that, for any object A of a category $\mathcal{A}$, the *comma category* $\mathcal{A}{\downarrow}A$ has for objects the pairs (B, b), where $b : B \longrightarrow A$ is an arrow of $\mathcal{A}$, and for morphisms $f : (B, b) \longrightarrow (C, c)$ those arrows $f : B \longrightarrow C$ of $\mathcal{A}$ such that $c.f = b$.

Theorem 12 *Let $\mathbb{D}$ be a sound doctrine and $\mathcal{A}$ be a small category. Then, the following conditions on a functor $F : \mathcal{A} \longrightarrow$ **set** are equivalent:*

1. $\mathbf{Lan}_{Y_\mathcal{A}}(F)$ *preserves $\mathbb{D}$-limits of representables;*

2. $\mathbf{Lan}_{Y_\mathcal{A}}(F)$ *is $\mathbb{D}$-continuous;*

3. F is a $\mathbb{D}$-filtered colimit of representables;

4. $\mathbf{Elts}(F)^{op}$ *is $\mathbb{D}$-filtered.*

If, moreover, $\mathcal{A}$ is $\mathbb{D}$-complete, these conditions are further equivalent to:

5. F is $\mathbb{D}$-continuous.

Sketch of the Proof. We just want to emphasise where the assumption on the soundness of the doctrine $\mathbb{D}$ is needed.

$1 \Leftrightarrow 2$. This is precisely the notion of a sound doctrine, as in Definition 8.

$3 \Rightarrow 2$. Consider a $\mathbb{D}$-filtered category $\mathcal{J}$ and a functor $H : \mathcal{J} \longrightarrow \mathcal{A}^{op}$ for which the colimit of the diagram $Y_{\mathcal{A}^{op}}.H : \mathcal{J} \longrightarrow [\mathcal{A}, \mathbf{set}]$ is precisely F. Provided that the functor $\mathbf{eval}_A : [\mathcal{A}^{op}, \mathbf{set}] \longrightarrow \mathbf{set}$ is given by evaluation at an object $A \in \mathcal{A}$, we have:

$$\begin{array}{rcll} \mathbf{Lan}_{Y_\mathcal{A}}(F) & \simeq & \mathbf{Lan}_{Y_\mathcal{A}}(\operatorname{colim}_{J\in\mathcal{J}} \mathcal{A}(H(J), -)) & \\ & \simeq & \operatorname{colim}_{J\in\mathcal{J}} \mathbf{Lan}_{Y_\mathcal{A}}(\mathcal{A}(H(J), -)) & (1) \\ & \simeq & \operatorname{colim}_{J\in\mathcal{J}} \mathbf{eval}_{H(J)} & (2) \end{array}$$

(where (1) holds since the left Kan extension $\mathbf{Lan}_{Y_\mathcal{A}}(-)$ is a left adjoint functor to $-.Y_\mathcal{A}$ and, therefore, preserves all existing colimits and (2) follows from the fact that, for any object $A \in \mathcal{A}$, the left Kan extension $\mathbf{Lan}_{Y_\mathcal{A}}(\mathcal{A}(A, -))$ of the representable functor $\mathcal{A}(A, -)$ along the Yoneda embedding $Y_\mathcal{A}$ is $\mathbf{eval}_A$; so, in particular, for $A = H(J)$, we get $\mathbf{Lan}_{Y_\mathcal{A}}(\mathcal{A}(H(J), -)) \simeq \mathbf{eval}_{H(J)}$). Knowing that such evaluation functors $\mathbf{eval}_{H(J)}$ preserve arbitrary limits since these are computed pointwise in $[\mathcal{A}^{op}, \mathbf{set}]$, we get that $\mathbf{Lan}_{Y_\mathcal{A}}(F)$ preserves all limits which happen to commute with $\mathcal{J}$-colimits, in particular, then, it preserves $\mathbb{D}$-limits.

$1 \Rightarrow 4$. Here the assumption on the soundness of the doctrine $\mathbb{D}$ appears again. In order to show that the category $\mathbf{Elts}(F)^{op}$ is $\mathbb{D}$-filtered, it suffices to prove that for any diagram $S : \mathcal{D}^{op} \longrightarrow \mathbf{Elts}(F)^{op}$, with $\mathcal{D} \in \mathbb{D}$, its category of cocones is connected, using the notion of a sound doctrine, as in Definition 9. For it, denoted by $U : \mathbf{Elts}(F) \longrightarrow \mathcal{A}$ the canonical forgetful functor, consider the composite $Y_\mathcal{A}.U.S^{op} : \mathcal{D} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$, we get:

$$\begin{array}{rcll} \lim_{D\in\mathcal{D}} F.U.S^{op} & \simeq & \lim_{D\in\mathcal{D}} \mathbf{Lan}_{Y_\mathcal{A}}(F).Y_\mathcal{A}.U.S^{op} & (1) \\ & \simeq & \mathbf{Lan}_{Y_\mathcal{A}}(F) \lim_{D\in\mathcal{D}} Y_\mathcal{A}.U.S^{op} & (2) \end{array}$$

(where (1) holds because, $Y_\mathcal{A}$ being fully faithful, the universal arrow for the left Kan extension is an isomorphism, that is, $F \simeq \mathbf{Lan}_{Y_\mathcal{A}}(F).Y_\mathcal{A}$ and (2) follows from the assumption on $\mathbf{Lan}_{Y_\mathcal{A}}(F)$ preserving $\mathbb{D}$-limits of representables). The limit in $[\mathcal{A}^{op}, \mathbf{set}]$ of the diagram $Y_\mathcal{A}.U.S^{op}$ can be thought of as a functor $H := \lim_{D\in\mathcal{D}} Y_\mathcal{A}.U.S^{op} : \mathcal{A}^{op} \longrightarrow \mathbf{set}$, which assigns to any object $A \in \mathcal{A}$ the set of all cones $\mathcal{A}(A, \lim_{D\in\mathcal{D}} U.S^{op}(D))$ of $U.S^{op}$ with domain A in $\mathcal{A}$. Knowing that the left Kan extension

$\mathbf{Lan}_{Y_{\mathcal{A}}}(F)$, evaluated at H, can be expressed as a colimit of the diagram

$$Y_{\mathcal{A}} \downarrow H \xrightarrow{\Gamma_H} \mathcal{A} \xrightarrow{F} \mathbf{set}\ ,\ (A, \alpha : \mathcal{A}(-, A) \longrightarrow H) \mapsto FA$$

it can be proved that $\mathbf{Lan}_{Y_{\mathcal{A}}}(F)(H)$ is, in fact, the colimit of the composite

$$\mathbf{Elts}(F)^{op} \xrightarrow{U^{op}} \mathcal{A}^{op} \xrightarrow{H} \mathbf{set}\ ,\ (A, a) \mapsto \lim_{D \in \mathcal{D}} \mathcal{A}(A, U.S^{op}(D))$$

Then, the above-mentioned canonical isomorphism $l : \mathbf{Lan}_{Y_{\mathcal{A}}}(F)(H) \longrightarrow \lim_{D \in \mathcal{D}} F.U.S^{op}$ (as in (2)) is defined, for any object $(A, a) \in \mathbf{Elts}(F)$, by $l_{(A,a)} : (f_D : A \longrightarrow U.S^{op}(D))_{D \in \mathcal{D}} \longrightarrow ((F f_D)a)_{D \in \mathcal{D}}$. Note that, provided the image of the functor S, for any $D \in \mathcal{D}$, is given by the pair (B_D, b_D), where $b_D \in FB_D$, then the collection $(b_D)_{D \in \mathcal{D}}$ is an element of $\lim_{D \in \mathcal{D}} F.U.S^{op}$. Therefore, thanks to the isomorphism l, there exists an object $(A, a) \in \mathbf{Elts}(F)$ and a cone $(f_D : A \longrightarrow U.S^{op}(D))_{D \in \mathcal{D}}$ such that $(F f_D)a = b_D$, for any $D \in \mathcal{D}$, and we can connect any other cocone of S in $\mathbf{Elts}(F)^{op}$ to it, via a zig-zag in the category of all cocones of S by the description of colimits in $\mathbf{set}$.

We omit the other implications, which are rather trivial, and refer to [1] for a more detailed proof. ■

Remark 13 A priori, as we have seen, there is no reason to define the notion of a $\mathbb{D}$-flat functor under the restricted hypothesis of a sound doctrine, but it is indeed just in this case that the notion becomes interesting for our purposes.

So, we assume for the sequel that the doctrine $\mathbb{D}$ is sound.

1.2 $\mathbb{D}$-accessible categories

The $\mathbb{D}$-ind completion

We recall the concept of a free completion $\mathbf{Ind}(\mathcal{A})$ of a category $\mathcal{A}$ under filtered colimits, introduced by Grothendieck (see [6]). The category $\mathbf{Ind}(\mathcal{A})$ can be explicitly described as the category of all functors in $[\mathcal{A}^{op}, \mathbf{set}]$ which are small, filtered colimits of representable functors. To be precise, note that the treatment of the $\mathbf{Ind}$ completion is done for any category $\mathcal{A}$, regardless of its size. Nevertheless, in the present work, we focus our attention on small categories and we consider the free completion $\mathbf{Ind}(\mathcal{A})$ only in the case of a small category $\mathcal{A}$. In light of Theorem 12, a functor $F : \mathcal{A}^{op} \longrightarrow \mathbf{set}$ is a small, filtered colimit of representables if and only if it is FIN-flat, that is, if and only if $\mathbf{Lan}_{Y_{\mathcal{A}}}(F)$

preserves finite limits. This can be generalised to an arbitrary sound doctrine $\mathbb{D}$ in the following terms: for any small category $\mathcal{A}$, we denote by $\mathbb{D}$-$\mathbf{ind}(\mathcal{A})$ the full subcategory of $[\mathcal{A}^{op}, \mathbf{set}]$ consisting of all $\mathbb{D}$-flat functors (as in Definition 7). Moreover:

Lemma 14 *The Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$ does restrict to $\mathbb{D}$-$\mathbf{ind}(\mathcal{A})$; we denote by $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbb{D}$-$\mathbf{ind}(\mathcal{A})$ such a restriction, as displayed in the following commutative diagram:*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "Y_{\mathcal{A}}"] \arrow[dr, dotted, "\eta_{\mathcal{A}}"'] & & {[\mathcal{A}^{op}, \mathbf{set}]} \\ & \mathbb{D}\text{-}\mathbf{ind}(\mathcal{A}) \arrow[ur] & \end{tikzcd}$$

Proof. We just need to show that, for any object $A \in \mathcal{A}$, the representable functor $\mathcal{A}(-, A)$ is $\mathbb{D}$-flat, which amounts to proving, by Theorem 12, that its left Kan extension along the Yoneda embedding preserves $\mathbb{D}$-limits, as in the following diagram:

$$\begin{tikzcd} \mathcal{A}^{op} \arrow[rr, "Y_{\mathcal{A}^{op}}"] \arrow[dr, "\mathcal{A}(-,A)"'] & & {[\mathcal{A}, \mathbf{set}]} \arrow[dl, "\mathbf{Lan}_{Y_{\mathcal{A}^{op}}}(\mathcal{A}(-,A))"] \\ & \mathbf{set} & \end{tikzcd}$$

We know that this left Kan extension is simply $\mathbf{eval}_A$, evaluation at A, which preserves all existing limits, since these are computed pointwise in $[\mathcal{A}, \mathbf{set}]$. ∎

By a free completion of $\mathcal{A}$ under $\mathbb{D}$-filtered colimits we mean a full embedding $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \tilde{\mathcal{A}}$ for which:

- $\tilde{\mathcal{A}}$ has $\mathbb{D}$-filtered colimits;
- for any functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ into a category $\mathcal{B}$ with $\mathbb{D}$-filtered colimits, there exists a unique (up to isomorphism) functor $\tilde{F} : \tilde{\mathcal{A}} \longrightarrow \mathcal{B}$ satisfying $F \simeq \tilde{F}.\eta_{\mathcal{A}}$ and preserving $\mathbb{D}$-filtered colimits.

Then, the following result holds:

Proposition 15 *Let $\mathbb{D}$ be a sound doctrine. For any small category $\mathcal{A}$, $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbb{D}$-$\mathbf{ind}(\mathcal{A})$ is a free completion of $\mathcal{A}$ under $\mathbb{D}$-filtered colimits.*

Proof. The free completion of $\mathcal{A}$ under $\mathbb{D}$-filtered colimits is the closure in $[\mathcal{A}^{op}, \mathbf{set}]$ of the representable functors under $\mathbb{D}$-filtered colimits. Since, by Theorem 12, any $\mathbb{D}$-flat functor is a $\mathbb{D}$-filtered colimit of representables, it only remains to prove that $\mathbb{D}$-flat functors are closed in $[\mathcal{A}^{op}, \mathbf{set}]$ under $\mathbb{D}$-filtered colimits. For it, consider a $\mathbb{D}$-filtered category $\mathcal{J}$ and a diagram $H : \mathcal{J} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$ of $\mathbb{D}$-flat functors, we have $\mathbf{Lan}_{Y_\mathcal{A}}(\mathrm{colim}_{J\in\mathcal{J}} H(J)) \simeq \mathrm{colim}_{J\in\mathcal{J}} \mathbf{Lan}_{Y_\mathcal{A}}(H(J))$, because of the left Kan extension $\mathbf{Lan}_{Y_\mathcal{A}}(-)$ being a left adjoint functor to $-.Y_\mathcal{A}$ and, therefore, preserving all existing colimits. Moreover, any functor $\mathbf{Lan}_{Y_\mathcal{A}}(H(J))$ preserves $\mathbb{D}$-limits, $H(J)$ being a $\mathbb{D}$-flat, and, since the category $\mathcal{J}$ is $\mathbb{D}$-filtered, $\mathcal{J}$-colimits commute with $\mathbb{D}$-limits. We can claim, then, that $\mathbf{Lan}_{Y_\mathcal{A}}(\mathrm{colim}_{J\in\mathcal{J}} H(J))$ preserves $\mathbb{D}$-limits, that is the functor $\mathrm{colim}_{J\in\mathcal{J}} H(J)$ is $\mathbb{D}$-flat. Concerning the universal property of the completion under $\mathbb{D}$-filtered colimits, let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor into a category $\mathcal{B}$ with $\mathbb{D}$-filtered colimits, and define a functor $\tilde{F} : \mathbb{D}\text{-}\mathbf{ind}(\mathcal{A}) \longrightarrow \mathcal{B}$, at any object $G \in \mathbb{D}\text{-}\mathbf{ind}(\mathcal{A})$, that is a canonical $\mathbb{D}$-filtered colimit of representables, as in the diagram

$$\eta_\mathcal{A}\!\downarrow\! G \xrightarrow{\Gamma_G} \mathcal{A} \xrightarrow{\eta_\mathcal{A}} \mathbb{D}\text{-}\mathbf{ind}(\mathcal{A})\ ,\ (A, \alpha : \mathcal{A}(-,A) \longrightarrow G) \mapsto \mathcal{A}(-,A)$$

to be $\tilde{F}(\mathrm{colim}\, \eta_\mathcal{A}.\Gamma_G) = \mathrm{colim}\, F.\Gamma_G$. To check that the functor $\tilde{F}$ does preserve $\mathbb{D}$-filtered colimits, consider a $\mathbb{D}$-filtered category $\mathcal{J}$ and a diagram $S : \mathcal{J} \longrightarrow \mathbb{D}\text{-}\mathbf{ind}(\mathcal{A})$ and compute its colimit $(\mathrm{colim}_{J\in\mathcal{J}} S(J), \delta_J : S(J) \longrightarrow \mathrm{colim}_{J\in\mathcal{J}} S(J))$ in $\mathbb{D}\text{-}\mathbf{ind}(\mathcal{A})$. As object of $\mathbb{D}\text{-}\mathbf{ind}(\mathcal{A})$, it can be expressed itself as the canonical $\mathbb{D}$-filtered colimit of the diagram

$$\eta_\mathcal{A}\!\downarrow\! \mathop{\mathrm{colim}}_{J\in\mathcal{J}} S(J) \xrightarrow{\Gamma_{\mathrm{colim}_{J\in\mathcal{J}} S(J)}} \mathcal{A} \xrightarrow{\eta_\mathcal{A}} \mathbb{D}\text{-}\mathbf{ind}(\mathcal{A})$$

$$\left(A, \alpha : \mathcal{A}(-,A) \longrightarrow \mathop{\mathrm{colim}}_{J\in\mathcal{J}} S(J)\right) \mapsto \mathcal{A}(-,A)$$

Since the representable functors are absolutely presentable, taking the full subcategory of $\eta_\mathcal{A}\!\downarrow\! \mathrm{colim}_{J\in\mathcal{J}} S(J)$ consisting of those natural transformations $\alpha : \mathcal{A}(-,A) \longrightarrow \mathrm{colim}_{J\in\mathcal{J}} S(J)$ which factor through some colimit injection δ_J, we get a cofinal subcategory of $\eta_\mathcal{A}\!\downarrow\! \mathrm{colim}_{J\in\mathcal{J}} S(J)$. Therefore, we have $\tilde{F}(\mathrm{colim}_{J\in\mathcal{J}} S(J)) \simeq \mathrm{colim}_{J\in\mathcal{J}} \tilde{F}(S(J))$, as desired. Moreover, the functor $\tilde{F}$ clearly makes the triangle:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\eta_\mathcal{A}"] \arrow[dr, "F"'] & & \mathbb{D}\text{-}\mathbf{ind}(\mathcal{A}) \arrow[dl, "\tilde{F}"] \\ & \mathcal{B} & \end{tikzcd}$$

commute, and, in fact, it is essentially unique with this property, since, by definition, $\tilde{F}$ is entirely determined by its value on representables

$\mathcal{A}(-, A)$, via its property of preserving $\mathbb{D}$-filtered colimits. ∎

Recall that a category $\mathcal{K}$ is said to be *finitely accessible* if it has filtered colimits and a small set $\mathcal{A}$ of *finitely presentable* objects (that is, those objects A such that the corresponding representable functor $\mathcal{K}(A, -) : \mathcal{K} \longrightarrow \mathbf{set}$ preserves filtered colimits) with the property that every object of $\mathcal{K}$ is a filtered colimit of objects of $\mathcal{A}$. Following the suggestion of the finitely accessible case and extending this idea to the case of an arbitrary sound doctrine $\mathbb{D}$ leads to:

Definition 16 *Let $\mathbb{D}$ be a sound doctrine. To say that a locally small category $\mathcal{K}$ is* $\mathbb{D}$-accessible *is to say that it has $\mathbb{D}$-filtered colimits and admits a small set $\mathcal{A}$ of $\mathbb{D}$-presentable objects such that any object of $\mathcal{K}$ is a $\mathbb{D}$-filtered colimit of objects of $\mathcal{A}$.*

In fact, we can characterise $\mathbb{D}$-accessible categories in terms of $\mathbb{D}$-**ind** completion:

Theorem 17 *For a locally small category $\mathcal{K}$, the following conditions are equivalent:*

1. *$\mathcal{K}$ is $\mathbb{D}$-accessible;*
2. *$\mathcal{K}$ is equivalent to $\mathbb{D}$-$\mathbf{ind}(\mathcal{A})$, for a small category $\mathcal{A}$;*
3. *$\mathcal{K}$ is a free completion of a small category $\mathcal{A}$ under $\mathbb{D}$-filtered colimits.*

Sketch of the Proof. 2 ⇔ 3. This equivalence has been proved in Proposition 15.

1 ⇒ 3. We refer to [27] Proposition 5.62.

3 ⇒ 1. Once again by Proposition 5.62 in [27], we know that a free completion $\mathcal{K}$ of a small category $\mathcal{A}$ under $\mathbb{D}$-filtered colimits has $\mathbb{D}$-filtered colimits itself and, moreover, there is a set $\mathcal{A}$ of $\mathbb{D}$-presentable colimits whose closure in $\mathcal{K}$ under $\mathbb{D}$-filtered colimits is $\mathcal{K}$ itself. Then, it only remains to show that this closure can be formed in one step, namely, that any object of $\mathcal{K}$ is a $\mathbb{D}$-filtered colimit of objects of $\mathcal{A}$. For it, note that $\mathcal{K} \simeq \mathbb{D}$-$\mathbf{ind}(\mathcal{A})$ and, by Theorem 12, any $\mathbb{D}$-flat functor is a $\mathbb{D}$-filtered colimit of representable functors. ∎

The full subcategory of $\mathbb{D}$-presentable objects

For any locally small category $\mathcal{K}$, we denote by $\mathcal{K}_{\mathbb{D}}$ the full subcategory of the $\mathbb{D}$-presentable objects of $\mathcal{K}$.

We recall that an object B of a category $\mathcal{A}$ is said to be a *retract* of

$A \in \mathcal{A}$ if there exist arrows $a : B \longrightarrow A$ and $b : A \longrightarrow B$ of $\mathcal{A}$ satisfying $b.a = 1_B$.

Proposition 18 *If $\mathcal{K}$ is a $\mathbb{D}$-accessible category, then the following properties hold:*

1. *$\mathcal{K}_{\mathbb{D}}$ is dense in $\mathcal{K}$, that is, for any object K in $\mathcal{K}$, the colimit of the canonical diagram $\mathcal{K}_{\mathbb{D}} \downarrow K \xrightarrow{\Gamma_K} \mathcal{K}_{\mathbb{D}} \hookrightarrow \mathcal{K}$, which associates with any pair (X, x), where the arrow $x : X \longrightarrow K$ is in $\mathcal{K}$ and $X \in \mathcal{K}_{\mathbb{D}}$, its image X through the forgetful functor Γ_K, is precisely K;*

2. *$\mathcal{K}_{\mathbb{D}}$ is essentially small and consists of the retracts of objects of $\mathcal{A}$ (as in Definition 16);*

3. *for any object $K \in \mathcal{K}$, the comma category $\mathcal{K}_{\mathbb{D}} \downarrow K$ is $\mathbb{D}$-filtered.*

Sketch of the Proof. Condition 1. By definition of $\mathbb{D}$-accessible category, any object $K \in \mathcal{K}$ is a $\mathbb{D}$-filtered colimit of objects of $\mathcal{A}$, which are, in particular, objects of $\mathcal{K}_{\mathbb{D}}$; namely, $K \simeq \mathrm{colim}_{J \in \mathcal{J}} H(J)$, where $H : \mathcal{J} \longrightarrow \mathcal{A}$ and $\mathcal{J}$ is a $\mathbb{D}$-filtered category. We wish to prove that the colimit of the canonical diagram $\Gamma_K : \mathcal{K}_{\mathbb{D}} \downarrow K \longrightarrow \mathcal{K}_{\mathbb{D}} \hookrightarrow \mathcal{K}$ coincides with K, that is $\mathrm{colim}_{(X,x) \in \mathcal{K}_{\mathbb{D}} \downarrow K} X \simeq \mathrm{colim}_{J \in \mathcal{J}} H(J) \simeq K$. In order to do so, we just point out that, for any object $(X, x) \in \mathcal{K}_{\mathbb{D}} \downarrow K$, we have $x \in \mathcal{K}(X, K) \simeq \mathcal{K}(X, \mathrm{colim}_{J \in \mathcal{J}} H(J)) \simeq \mathrm{colim}_{J \in \mathcal{J}} \mathcal{K}(X, H(J))$, X being a $\mathbb{D}$-presentable object of $\mathcal{K}$. It is easy to show, then, that the canonical comparison morphisms between the two colimits give an isomorphism, as desired.

Condition 2. We want to show that any $X \in \mathcal{K}_{\mathbb{D}}$ is a retract of an object of $\mathcal{A}$. For it, we express X, which, in particular, is an object of $\mathcal{K}$ as a $\mathbb{D}$-filtered colimit of objects of $\mathcal{A}$; namely $X \simeq \mathrm{colim}_{J \in \mathcal{J}} H(J)$, where $H : \mathcal{J} \longrightarrow \mathcal{A}$ and $\mathcal{J}$ is a $\mathbb{D}$-filtered category. Since X is a $\mathbb{D}$-presentable object of $\mathcal{K}$, we have $\mathcal{K}(X, X) \simeq \mathcal{K}(X, \mathrm{colim}_{J \in \mathcal{J}} H(J)) \simeq \mathrm{colim}_{J \in \mathcal{J}} \mathcal{K}(X, H(J))$. Therefore, the identity arrow 1_X on X has to factor through one of the colimit injection σ_J, for some $J \in \mathcal{J}$, as follows:

$$
\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[dr, dotted, "f_J"'] & X \simeq \underset{J \in \mathcal{J}}{\mathrm{colim}}\, H(J) \\
& H(J) \arrow[u, "\sigma_J"']
\end{tikzcd}
$$

Namely, we get $\sigma_J.f_J = 1_X$, for some morphism $f_J : X \longrightarrow H(J)$ in $\mathcal{K}$. Therefore, X can be regarded as a retract of the object $H(J) \in \mathcal{A}$.

Moreover, since $\mathcal{A}$ is a small set, one can prove that, for any object $A \in \mathcal{A}$, the collection of all its retracts is a small set. Therefore, $\mathcal{K}_{\mathbb{D}}$, which is nothing but the collection of all retracts of objects of $\mathcal{A}$, is an essentially small category.

Condition 3. Consider an object $K \in \mathcal{K}$ and the comma category $\mathcal{K}_{\mathbb{D}} \downarrow K$. By Lemma 5, we can already claim that the category $\mathcal{K}_{\mathbb{D}}$ is a $\mathbb{D}^{op}$-cocomplete. It is easy to prove, then, that the comma category $\mathcal{K}_{\mathbb{D}} \downarrow K$ is $\mathbb{D}^{op}$-cocomplete itself. Therefore, the doctrine $\mathbb{D}$ being sound, by Proposition 11 we can derive that $\mathcal{K}_{\mathbb{D}} \downarrow K$ is $\mathbb{D}$-filtered, provided it is small. This is the case, since we know by Condition 2 that $\mathcal{K}_{\mathbb{D}}$ is essentially small and the category $\mathcal{K}$ is, by assumption, locally small. ∎

As a consequence of Lemma 5, we also know that, for any category $\mathcal{K}$ having $\mathbb{D}^{op}$-colimits, its full subcategory $\mathcal{K}_{\mathbb{D}}$ is closed with respect to $\mathbb{D}^{op}$-colimits in $\mathcal{K}$ and, hence, is $\mathbb{D}^{op}$-cocomplete. Stated otherwise, for future purposes:

Remark 19 Let $\mathcal{K}$ be a $\mathbb{D}^{op}$-cocomplete category. The category ${\mathcal{K}_{\mathbb{D}}}^{op}$ is $\mathbb{D}$-complete.

We present here a crucial result which will be applied to any situation concerning retracts:

Proposition 20 *Consider two functors $F, G : \mathcal{A} \longrightarrow$ **set**, with $\mathcal{A}$ a small category, and a diagram $S : \mathcal{I} \longrightarrow \mathcal{A}$, for some category $\mathcal{I}$, such that a colimit of S exists in $\mathcal{A}$. If the functor F preserves such a colimit of S and G is a retract of F, then the functor G itself preserves the colimit of S under consideration.*

Proof. Let the functor G be a retract of F, as in the diagram:

$$\begin{tikzcd} G \arrow[r, shift right, tail, "\alpha"'] & F \arrow[l, shift right, two heads, "\beta"'] \end{tikzcd}$$

where $\beta \cdot \alpha = 1_G$. Suppose the functor F preserves a colimit of the diagram $S : \mathcal{I} \longrightarrow \mathcal{A}$. We wish to show that $G : \mathcal{A} \longrightarrow$ **set** preserves such a colimit. The condition on G being a retract of F can be expressed in terms of a coequalizer diagram; explicitly, G is the coequalizer in $[\mathcal{A}, \mathbf{set}]$ of the pair $(\alpha.\beta, 1_F)$, as displayed:

$$\begin{tikzcd} F \arrow[r, shift left, "\alpha.\beta"] \arrow[r, shift right, "1_F"'] & F \arrow[r, two heads, "\beta"] & G \end{tikzcd}$$

Therefore, G, being a coequalizer, is, in particular, a colimit of functors, namely F, which preserve a colimit of the diagram S. Since colimits obviously commute with colimits, it follows immediately that G itself does preserve such a colimit of S. ∎

Recall that a category $\mathcal{C}$ is *Cauchy complete* if all idempotents of $\mathcal{C}$ split (see [10]). We can achieve Cauchy completeness for $\mathcal{K}_{\mathbb{D}}$ as a consequence of the previous Proposition 20:

Corollary 21 *Let $\mathcal{K}$ be a locally small category with $\mathbb{D}$-filtered colimits. A retract of a $\mathbb{D}$-presentable object is itself $\mathbb{D}$-presentable.*

Proof. Let $A \in \mathcal{K}$ be a $\mathbb{D}$-presentable object and consider B a retract of A, as in the diagram:

$$\begin{tikzcd} B \arrow[r, shift right, tail, "a"'] & A \arrow[l, shift right, two heads, "b"'] \end{tikzcd}$$

where $b.a = 1_B$. The object A being $\mathbb{D}$-presentable amounts to the corresponding representable functor $\mathcal{K}(A,-) : \mathcal{K} \longrightarrow \mathbf{set}$ preserving $\mathbb{D}$-filtered colimits. Moreover, the morphisms a and b provide the functor $\mathcal{K}(B,-)$ as retract of $\mathcal{K}(A,-)$, since $\mathcal{K}(a,-).\mathcal{K}(b,-) = \mathcal{K}(b.a,-) = 1_{\mathcal{K}(B,-)}$, as in the following diagram:

$$\begin{tikzcd} \mathcal{K}(B,-) \arrow[r, shift right, tail, "\mathcal{K}(b,-)"'] & \mathcal{K}(A,-) \arrow[l, shift right, two heads, "\mathcal{K}(a,-)"'] \end{tikzcd}$$

Therefore, by Proposition 20, the representable functor $\mathcal{K}(B,-) : \mathcal{K} \longrightarrow \mathbf{set}$ is $\mathbb{D}$-accessible too, that is, the object B is $\mathbb{D}$-presentable in $\mathcal{K}$. ∎

As first consequence of the previous Corollary, the full subcategory $\mathcal{K}_{\mathbb{D}}$ of $\mathcal{K}$ consisting of all $\mathbb{D}$-presentable objects is closed under retracts, and, therefore:

Corollary 22 *Let $\mathcal{K}$ be a locally small category with $\mathbb{D}$-filtered colimits and denote by $\mathcal{K}_{\mathbb{D}}$ its full subcategory given by all $\mathbb{D}$-presentable objects. The category $\mathcal{K}_{\mathbb{D}}$, being closed under retracts, is Cauchy complete.*

Proof. Let $e : A \longrightarrow A$ be an idempotent on an object A of $\mathcal{K}_{\mathbb{D}}$, that is, A is $\mathbb{D}$-presentable of $\mathcal{K}$ and $e.e = e$. Consider the pair of arrows $(1_A, e)$ of $\mathcal{K}_{\mathbb{D}}$:

$$\begin{tikzcd} A \arrow[r, shift left, "1_A"] \arrow[r, shift right, "e"'] & A \end{tikzcd}$$

and take its coequalizer $b : A \longrightarrow B$ in $\mathcal{K}$. Since e is idempotent, it factors through the coequalizer b, namely there exists a morphism $a : B \longrightarrow A$ in $\mathcal{K}$ such that $a.b = e$ and, in fact, a satisfies the extra property $b.a = 1_B$, b being an epimorphism. This is precisely the condition for an idempotent e to split in $\mathcal{K}$. But since B is a retract of a $\mathbb{D}$-presentable object A, it is $\mathbb{D}$-presentable itself by Corollary 21. Therefore, the idempotent e actually splits in $\mathcal{K}_{\mathbb{D}}$. Note that the splitting of an idempotent is a colimit (as well as a limit); moreover, it is an absolute colimit, namely it is preserved by any functor. As a consequence such a colimit commutes with any existing limit. Therefore, in particular, $\mathbb{D}$-filtered colimits always contain as a special case the splitting of idempotents. This is the reason why the only hypothesis of having $\mathbb{D}$-filtered colimits is required on the category $\mathcal{K}$. ■

Remark 23 Equivalently, in Corollary 22, we could have just asked for the category $\mathcal{K}$ being Cauchy complete, in order to get the Cauchy completion of its full subcategory $\mathcal{K}_{\mathbb{D}}$. In fact, not even Cauchy completion is required as an extra assumption, since for the category $\mathcal{K}$ having $\mathbb{D}$-filtered colimits directly implies being Cauchy complete, as we will point out in Section 1.4.

1.3 Locally $\mathbb{D}$-presentable categories

Keeping in mind the definition of a locally finitely presentable category, that is a cocomplete, finitely accessible category, and moving to the case of an arbitrary sound doctrine $\mathbb{D}$, we are led to the following generalisation:

Definition 24 *Let $\mathbb{D}$ be a sound doctrine. We say that a locally small category $\mathcal{K}$ is* locally $\mathbb{D}$-presentable *if it is cocomplete and $\mathbb{D}$-accessible.*

It turns out that a $\mathbb{D}$-accessible category is complete if and only if it is cocomplete (see [1]). So, a locally $\mathbb{D}$-presentable category is not only cocomplete but also complete, and, in particular, $\mathbb{D}$-complete.
For any essentially small and $\mathbb{D}$-complete category $\mathcal{A}$, we denote by $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{set}]$ the full subcategory of the functor category $[\mathcal{A}, \mathbf{set}]$ given by $\mathbb{D}$-continuous functors. By general reasons (see, for example, [27]), the full inclusion

$$i_{\mathcal{A}} : \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}] \longrightarrow [\mathcal{A}, \mathbf{set}]$$

has a left adjoint, which we denote by $r_{\mathcal{A}}$. Namely, $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{set}]$ is a full reflective subcategory of $[\mathcal{A}, \mathbf{set}]$.

A characterisation for locally $\mathbb{D}$-presentable categories can be fully stated as in the next Theorem. We recall first from [29, 35] that a *sketch* is a quadruple $\mathcal{S} = (\mathcal{A}, \mathbb{L}, \mathbb{C}, \sigma)$ consisting of a small category $\mathcal{A}$, sets $\mathbb{L}$ and $\mathbb{C}$ of small diagrams in $\mathcal{A}$ and a map σ assigning a cone to every diagram in $\mathbb{L}$ and a cocone to every diagram in $\mathbb{C}$. *Models* of $\mathcal{S}$ are functors in $[\mathcal{A}, \mathbf{set}]$ mapping $\sigma(D)$ to a limit cone if $D \in \mathbb{L}$ and to a colimit cocone if $D \in \mathbb{C}$. We write $\mathbf{Mod}(\mathcal{S})$ for the full subcategory of $[\mathcal{A}, \mathbf{set}]$ given by all models. We say that a category is *sketchable* by $\mathcal{S}$ if it is equivalent to $\mathbf{Mod}(\mathcal{S})$. Then, in terms of a $\mathbb{D}$ doctrine, we get from [1] the following:

Definition 25 *A sketch is called a* $\mathbb{D}$-sketch *if every diagram in* $\mathbb{L}$ *has scheme in* $\mathbb{D}$ *(no restriction is made on the diagrams in* $\mathbb{C}$*).*

Theorem 26 *Let* $\mathbb{D}$ *be a sound doctrine. For a locally small category* $\mathcal{K}$*, the following condition are equivalent:*

1. *$\mathcal{K}$ is locally $\mathbb{D}$-presentable;*
2. *$\mathcal{K}$ is a free completion of a small $\mathbb{D}^{op}$-cocomplete category under $\mathbb{D}$-filtered colimits;*
3. *$\mathcal{K}$ is sketchable by a limit $\mathbb{D}$-sketch;*
4. *$\mathcal{K}$ is equivalent to* $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{set}]$ *for a small $\mathbb{D}$-complete category $\mathcal{A}$.*

Sketch of the Proof. $1 \Rightarrow 2$. Given a locally $\mathbb{D}$-presentable category $\mathcal{K}$, we know that $\mathcal{K}$ is a free completion of its full subcategory $\mathcal{K}_{\mathbb{D}}$, consisting of $\mathbb{D}$-presentable objects, under $\mathbb{D}$-filtered colimits. The fact that $\mathcal{K}_{\mathbb{D}}$ is a $\mathbb{D}^{op}$-cocomplete category is an immediate consequence of Lemma 5.

$2 \Rightarrow 3$. Let $\mathcal{K}$ be a free completion of a small $\mathbb{D}^{op}$-cocomplete category $\mathcal{C}$ under $\mathbb{D}$-filtered colimits. Take the limit $\mathbb{D}$-sketch $\mathcal{S}$ on $\mathcal{C}^{op}$ given by the set $\mathbb{L}$ of all $\mathbb{D}$-diagrams and the map σ which assigns limit cones. Then, we get $\mathcal{K} \simeq \mathbf{Mod}(\mathcal{S})$.

$3 \Rightarrow 4$. If $\mathcal{K} \simeq \mathbf{Mod}(\mathcal{S})$, for a limit $\mathbb{D}$-sketch $\mathcal{S}$ on $\mathcal{A}$, define a fully faithful functor $\xi : \mathcal{A} \longrightarrow \mathbf{Mod}(\mathcal{S})^{op}$ by assigning to any object $A \in \mathcal{A}$ the representable functor $\mathcal{A}(A, -)$; denote by $\tilde{\mathcal{A}}$ the closure of its image under $\mathbb{D}$-limits. Then, we have $\mathbf{Mod}(\mathcal{S}) \simeq \mathbb{D}\text{-}\mathbf{cont}[\tilde{\mathcal{A}}, \mathbf{set}]$.

$4 \Rightarrow 1$. We know that, by definition, $\mathbb{D}\text{-}\mathbf{ind}(\mathcal{A}) = \mathbb{D}\text{-}\mathbf{flat}[\mathcal{A}^{op}, \mathbf{set}]$ and we also know that, while working with small $\mathbb{D}$-complete categories, the notions of $\mathbb{D}$-flat functor and $\mathbb{D}$-continuous functor coincide, by Theorem 12. Therefore, for any small $\mathbb{D}$-complete category $\mathcal{A}$, we have:

$$\mathbb{D}\text{-}\mathbf{ind}(\mathcal{A}^{op}) = \mathbb{D}\text{-}\mathbf{flat}[\mathcal{A}, \mathbf{set}] \simeq \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$$

It only remains to remark, then, that $\mathbb{D}\text{-}\mathbf{ind}(\mathcal{A}^{op})$ is $\mathbb{D}$-accessible, by Theorem 17, and $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, being reflective in $[\mathcal{A}, \mathbf{set}]$, is cocomplete. ■

Remark 27 The two main facts of Theorem 26 can be rephrased in the following terms:

1. Let $\mathcal{A}$ be a small and $\mathbb{D}$-complete category. Then, the category $\mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{set}]$ is locally $\mathbb{D}$-presentable.

2. Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category. Then, $\mathcal{K}$ is equivalent to
$\mathbb{D}$-$\mathbf{cont}[\mathcal{K}_{\mathbb{D}}{}^{op}, \mathbf{set}]$, where $\mathcal{K}_{\mathbb{D}} \hookrightarrow \mathcal{K}$ is the full subcategory of $\mathcal{K}$ consisting of its $\mathbb{D}$-presentable objects.

Going back to our leading examples, we have:

Example 28

1. Let $\mathbb{D}$ be the doctrine FIN of finite limits categories. Then, locally $\mathbb{D}$-presentable categories are precisely the locally finitely presentable ones.

2. Let $\mathbb{D}$ be the doctrine FINPR of finite discrete categories. Then, locally $\mathbb{D}$-presentable categories are multisorted finitary varieties (see [5]).

3. Let $\mathbb{D} = \emptyset$ be the empty doctrine. Then, locally $\mathbb{D}$-presentable categories are just presheaf categories.

The first important property of the inclusion $i_{\mathcal{A}}$ we want to mention is:

Lemma 29 *Let $\mathcal{A}$ be an essentially small and $\mathbb{D}$-complete category. The full embedding $i_{\mathcal{A}} : \mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{set}] \longrightarrow [\mathcal{A}, \mathbf{set}]$ does preserve $\mathbb{D}$-filtered colimits. In fact, $i_{\mathcal{A}}$ reflects them too, being full and faithful.*

Proof. For any $\mathbb{D}$-filtered category $\mathcal{J}$ and for any functor $H : \mathcal{J} \longrightarrow \mathbb{D}$-$\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, consider the colimit of such a diagram in $[\mathcal{A}, \mathbf{set}]$, say it $M = \mathrm{colim}_{J \in \mathcal{J}} H(J) : \mathcal{A} \longrightarrow \mathbf{set}$. We shall prove that M is a $\mathbb{D}$-continuous functor. For it, given $\mathcal{D} \in \mathbb{D}$ and a functor $S : \mathcal{D} \longrightarrow \mathcal{A}$ we obtain:

$$
\begin{aligned}
(\operatorname*{colim}_{J\in\mathcal{J}} H(J))(\lim_{D\in\mathcal{D}} S(D)) &\simeq (\operatorname*{colim}_{J\in\mathcal{J}})(H(J)(\lim_{D\in\mathcal{D}} S(D))) && (1)\\
&\simeq (\operatorname*{colim}_{J\in\mathcal{J}})(\lim_{D\in\mathcal{D}} H(J)(S(D))) && (2)\\
&\simeq (\operatorname*{colim}_{J\in\mathcal{J}})(\lim_{D\in\mathcal{D}}) H(J)(S(D)) && (3)\\
&\simeq (\lim_{D\in\mathcal{D}})(\operatorname*{colim}_{J\in\mathcal{J}}) H(J)(S(D)) && (4)\\
&\simeq (\lim_{D\in\mathcal{D}})(\operatorname*{colim}_{J\in\mathcal{J}} H(J))(S(D)) && (5)
\end{aligned}
$$

(where (1) and (5) follow from the fact that colimits are computed pointwise in $[\mathcal{A}, \mathbf{set}]$, (2) holds because $H(J)$ is a$\mathbb{D}$-continuous functor, for any $J \in \mathcal{J}$, (3) follows from the fact that limits as well are computed pointwise in $[\mathcal{A}, \mathbf{set}]$ and (4) is just the definition of $\mathbb{D}$-filtered colimit). Therefore, $M \in \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, or, stated otherwise, $i_{\mathcal{A}}$ preserves $\mathbb{D}$-filtered colimits. ■

Another way of looking at this result it is as a consequence of Theorem 26, which states that $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$ is a free completion of $\mathcal{A}^{op}$ under $\mathbb{D}$-filtered colimits. In fact, we have just proved that:

Corollary 30 *Let $\mathcal{A}$ be an essentially small and $\mathbb{D}$-complete category. In the category $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, $\mathbb{D}$-filtered colimits are computed pointwise.*

Restricting Yoneda

Lemma 31 *Let $\mathcal{A}$ be an essentially small and $\mathbb{D}$-complete category. Any representable functor $\mathcal{A}(A, -)$ is $\mathbb{D}$-continuous; in other words, the Yoneda embedding $Y_{\mathcal{A}^{op}} : \mathcal{A}^{op} \longrightarrow [\mathcal{A}, \mathbf{set}]$ restricts to $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, as displayed in the following commutative diagram:*

$$
\begin{tikzcd}
\mathcal{A}^{op} \arrow[rr, "Y_{\mathcal{A}^{op}}"] \arrow[dr, dotted, "Y_{\mathcal{A}^{op}}"'] & & {[\mathcal{A}, \mathbf{set}]} \\
& \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}] \arrow[ur, "i_{\mathcal{A}}"'] &
\end{tikzcd}
$$

Proof. We want to prove that for any object $A \in \mathcal{A}$ the functor $\mathcal{A}(A, -)$ is $\mathbb{D}$-continuous, that is, it preserves all $\mathbb{D}$-limits. This follows from the fact that the representable functor $\mathcal{A}(A, -) : \mathcal{A} \longrightarrow \mathbf{set}$ preserves all (existing) limits, then, in particular, the $\mathbb{D}$-ones. ■

With abuse of notation, we denote the restricted Yoneda embedding also by $Y_{\mathcal{A}^{op}}$.

Proposition 32 *Let $\mathcal{A}$ be an essentially small and $\mathbb{D}$-complete category. The restricted Yoneda embedding $Y_{\mathcal{A}^{op}} : \mathcal{A}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$ transforms $\mathbb{D}$-limits in $\mathcal{A}$ into $\mathbb{D}^{op}$-colimits.*

Proof. For any $\mathcal{D} \in \mathbb{D}$ and any functor $S : \mathcal{D} \longrightarrow \mathcal{A}$, consider its $\mathbb{D}$-limit cone $(\lim_{D \in \mathcal{D}} S(D); \lambda_D : \lim_{D \in \mathcal{D}} S(D) \longrightarrow S(D))$. We wish to show that applying Yoneda to this diagram produces a $\mathbb{D}^{op}$-colimit cone in $\mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$, namely that:

$$(\mathcal{A}(\lim_{D\in\mathcal{D}} S(D),-); \mathcal{A}(\lambda_D,-) : \mathcal{A}(S(D),-) \longrightarrow \mathcal{A}(\lim_{D\in\mathcal{D}} S(D),-))$$

is the $\mathbb{D}^{op}$-colimit of the diagram $Y_{\mathcal{A}^{op}}.S$. This is obviously a cocone in $\mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$. To show that it coincides with $\mathrm{colim}_{D\in\mathcal{D}^{op}} \mathcal{A}(S(D),-)$ (which exists, $\mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$ being cocomplete), take any object $E \in \mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$ and just use the Yoneda Lemma, once remarked that the functor E is in fact $\mathbb{D}$-continuous:

$$\begin{aligned}
\mathbf{Nat}(\mathcal{A}(\lim_{D\in\mathcal{D}} S(D),-),E) &\simeq E(\lim_{D\in\mathcal{D}} S(D)) && (1)\\
&\simeq \lim_{D\in\mathcal{D}} E(S(D)) && (2)\\
&\simeq \lim_{D\in\mathcal{D}} \mathbf{Nat}(\mathcal{A}(S(D),-),E) && (3)\\
&\simeq \mathbf{Nat}(\mathop{\mathrm{colim}}_{D\in\mathcal{D}^{op}} \mathcal{A}(S(D),-),E)
\end{aligned}$$

(where (1) and (3) are just instances of the Yoneda Lemma and (2) follows from E being a $\mathbb{D}$-continuous functor). Since this is valid for any $E \in \mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$, again from the Yoneda Lemma it follows that $\mathcal{A}(\lim_{D\in\mathcal{D}} S(D),-) \simeq \mathrm{colim}_{D\in\mathcal{D}^{op}} \mathcal{A}(S(D),-)$, as desired. ∎

It might be more convenient to rephrase the previous Proposition in the following terms: *the Yoneda embedding $Y_{\mathcal{A}^{op}} : \mathcal{A}^{op} \longrightarrow \mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$ does preserve $\mathbb{D}^{op}$-colimits. So, it is clear that $Y_{\mathcal{A}^{op}}$, being full and faithful, reflects them too.*

Remark 33 Note that if $\mathcal{A}$ is a Cauchy complete category, then the previous Proposition is just a reformulation of Lemma 5. In fact, in the previous Proposition we have shown that a $\mathbb{D}^{op}$-colimit of representables is representable itself. Of course, in the case of a Cauchy complete category, this coincides with a special instance of the statement "a $\mathbb{D}^{op}$-colimit of $\mathbb{D}$-presentables is $\mathbb{D}$-presentable itself", provided that the $\mathbb{D}$-presentables objects of $\mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$ are retracts of representable functors. (This will be proved in Proposition 38.)

1.4 $\mathbb{D}$-completeness and Cauchy completeness

Let $\mathcal{E}$ be the category consisting of:

- a single object $*$;
- two morphisms: 1_* and e, with the property $e = e.e$.

For any category $\mathcal{A}$ and for any diagram $F : \mathcal{E} \longrightarrow \mathcal{A}$ admitting a limit or a colimit, such a limit or such a colimit is absolute, namely it is preserved by any functor, and it is a well-known fact that it leads, in both cases, to the splitting of the idempotent $F(e)$ in $\mathcal{A}$. As a consequence, such a limit commutes with any existing colimit, and, analogously such a colimit commutes with any existing limit. Therefore, in terms of a limit doctrine $\mathbb{D}$, we can claim that $\mathbb{D}$-filtered colimits always contain, as a special instance, the splitting of idempotents. In other words, for a category having $\mathbb{D}$-filtered colimits implies being Cauchy complete.

These simple facts have a useful impact on enlightening the interplay between $\mathbb{D}$-completeness and Cauchy completeness. For a given doctrine $\mathbb{D}$, we denote by $\mathbb{D}^+$ the doctrine given by the union $\mathbb{D} \cup \{\mathcal{E}\}$. Since $\mathcal{E}$-limits are preserved by any functor and, therefore, commute with any colimit, we get the following results:

Lemma 34 *The following properties hold:*

1. *$\mathbb{D}^+$-completeness coincides with $\mathbb{D}$-completeness plus Cauchy completeness;*
2. *$\mathbb{D}^+$-continuity coincides with $\mathbb{D}$-continuity;*
3. *$\mathbb{D}^+$-filtered colimits coincide with $\mathbb{D}$-filtered colimits;*
4. *$\mathbb{D}^+$-presentable objects coincide with $\mathbb{D}$-presentable objects;*
5. *$\mathbb{D}^+$-accessible categories are precisely $\mathbb{D}$-accessible categories;*
6. *locally $\mathbb{D}^+$-presentable categories are precisely locally $\mathbb{D}$-presentable categories.*

It is a crucial point to realise that $\mathbb{D}$-filtered colimit and $\mathbb{D}^+$-filtered colimits are precisely the same. This means, in fact, that any proof we have achieved relative to categories having $\mathbb{D}$-filtered colimits could be reformulated in terms of categories having $\mathbb{D}^+$-filtered colimits. The immediate advantage clearly is that of gaining Cauchy completeness. Moreover, while replacing $\mathbb{D}$ by $\mathbb{D}^+$, the condition on the doctrine of being sound is retained, since, for a given doctrine $\mathbb{D}$, soundness is a property which can be merely expressed in terms of $\mathbb{D}$-filtered colimits:

Lemma 35 *If the doctrine $\mathbb{D}$ is sound, then the doctrine $\mathbb{D}^+$ is itself sound.*

The whole of these observations is illuminating while dealing with the splitting of the idempotents, as in the next Chapter, for instance, and categories which are at once $\mathbb{D}$-complete and Cauchy complete.

2

The duality theorem for a limit doctrine

The duality Theorem concerning locally $\mathbb{D}$-presentable categories is presented. This is the theorem that puts the definition of a locally $\mathbb{D}$-presentable category in its natural context by taking into account the notion of morphisms between such categories. In other words, the functoriality of the process of "$\mathbb{D}$-ification" is analysed. Basic definitions concerning generators of a category are recalled in order to present a further characterisation of locally $\mathbb{D}$-presentable categories, this time in terms of strong generators. Applications of the duality Theorem go in the direction of comparing different doctrines. As an example of this kind of comparison, we mention [37].

2.1 The duality Theorem

The 2-functor ε

Let $\mathbb{D}$ be a sound doctrine, which we suppose fixed throughout the whole section. We denote by $\mathbb{D}$-**LP** the 2-category of locally $\mathbb{D}$-presentable categories, $\mathbb{D}$-accessible right adjoint functors and natural transformations. Whereas, we write $\mathbb{D}$-**Th** for the 2-category of essentially small $\mathbb{D}$-complete and Cauchy complete categories, $\mathbb{D}$-continuous functors and natural transformations.

Our purpose is to build a biequivalence of 2-categories between $\mathbb{D}$-**Th** and $\mathbb{D}$-**LP** which will provide the natural generalisation of Gabriel-Ulmer duality for locally finitely presentable categories, Adámek - Lawvere - Rosický duality for varieties and Morita duality for presheaf categories

we were seeking.
We start by defining a 2-functor

$$\varepsilon : \mathbb{D}\text{-}\mathbf{Th}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{LP}$$

which assigns, to every object $\mathcal{A}$ in $\mathbb{D}$-**Th**, the category of $\mathbb{D}$-continuous set-valued functors $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, which is locally $\mathbb{D}$-presentable by Remark 27 (Condition 1). On morphisms $F : \mathcal{A} \longrightarrow \mathcal{B}$ of $\mathbb{D}$-**Th**, ε is defined by composing with F, as shown in the following diagram:

$$\begin{tikzcd} \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{set}] \arrow[r, tail, "i_{\mathcal{B}}"] \arrow[d, dotted, "\varepsilon(F)"'] & {[\mathcal{B}, \mathbf{set}]} \arrow[d, "- \cdot F"] \\ \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}] \arrow[r, tail, "i_{\mathcal{A}}"'] & {[\mathcal{A}, \mathbf{set}]} \end{tikzcd}$$ Diagram 2.1

where $i_{\mathcal{A}}$ displays $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$ as the full reflective subcategory of $[\mathcal{A}, \mathbf{set}]$ whose left adjoint is denoted by $r_{\mathcal{A}}$, and analogously for $i_{\mathcal{B}}$ (and $r_{\mathcal{B}} \dashv i_{\mathcal{B}}$). Since F is a $\mathbb{D}$-continuous functor, composing with F on the right actually restricts to a functor $\varepsilon(F) : \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{set}] \longrightarrow \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, as desired. Moreover, if we consider the two Yoneda embeddings $Y_{\mathcal{A}^{op}} : \mathcal{A}^{op} \longrightarrow [\mathcal{A}, \mathbf{set}]$ and $Y_{\mathcal{B}^{op}} : \mathcal{B}^{op} \longrightarrow [\mathcal{B}, \mathbf{set}]$, taking the left Kan extension of $Y_{\mathcal{B}^{op}}.F^{op}$ along $Y_{\mathcal{A}^{op}}$ (which always exists, the category $[\mathcal{B}, \mathbf{set}]$ being cocomplete) gives rise to a left adjoint functor to $- \cdot F : [\mathcal{B}, \mathbf{set}] \longrightarrow [\mathcal{A}, \mathbf{set}]$, call it $T = \mathbf{Lan}_{Y_{\mathcal{A}^{op}}}(Y_{\mathcal{B}^{op}}.F^{op}) : [\mathcal{A}, \mathbf{set}] \longrightarrow [\mathcal{B}, \mathbf{set}]$ satisfying $T \dashv - \cdot F$:

$$\begin{tikzcd} \mathcal{A}^{op} \arrow[rr, "Y_{\mathcal{A}^{op}}"] \arrow[d, "F^{op}"'] & & {[\mathcal{A}, \mathbf{set}]} \arrow[dl, dotted, "T"', shift right] \\ \mathcal{B}^{op} \arrow[r, "Y_{\mathcal{B}^{op}}"'] & {[\mathcal{B}, \mathbf{set}]} \arrow[ur, "- \cdot F"', shift right] & \end{tikzcd}$$

Our present aim is to show that $\varepsilon(F)$ is, in fact, a morphism of the 2-category $\mathbb{D}$-**LP**, namely a right adjoint, $\mathbb{D}$-filtered colimit-preserving functor.

The following is helpful to that end.

Lemma 36 *Consider the following diagram of categories and functors:*

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "h"'] \arrow[r, shift right, "i"'] & \mathcal{B} \arrow[l, shift right, "r"'] \arrow[d, shift left, "g"] \\
\mathcal{C} \arrow[r, tail, "j"'] & \mathcal{D} \arrow[u, shift left, "f"]
\end{tikzcd}$$

Suppose that j is full and faithful, $g \cdot i \simeq j \cdot h$, $r \dashv i$ and $f \dashv g$. Then, the composite $r \cdot f \cdot j$ satisfies the condition $r \cdot f \cdot j \dashv h$.

Proof. In order to establish the adjunction $r \cdot f \cdot j \dashv h$, we have to show that there is an isomorphism $\mathcal{A}(r \cdot f \cdot j(C), A) \simeq \mathcal{C}(C, h(A))$ natural in both the variables $C \in \mathcal{C}$ and $A \in \mathcal{A}$:

$$\begin{array}{rcll}
\mathcal{A}(r \cdot f \cdot j(C), A) & \simeq & \mathcal{B}(f \cdot j(C), i(A)) & (1) \\
& \simeq & \mathcal{D}(j(C), g \cdot i(A)) & (2) \\
& \simeq & \mathcal{D}(j(C), j \cdot h(A)) & (3) \\
& \simeq & \mathcal{C}(C, h(A)) & (4)
\end{array}$$

(where (1) holds by $r \dashv i$, (2) holds by $f \dashv g$, (3) follows from $g \cdot i \simeq j \cdot h$ and (4) comes from j being fully faithful). Clearly all isomorphisms are natural by definition of the adjunctions involved. Therefore, we have proved that $r \cdot f \cdot j$ is left adjoint to h, as desired. ■

So, going back to the Diagram 2.1 and applying Lemma 36 to the case of $\varepsilon(F) : \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{set}] \longrightarrow \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$ will provide a left adjoint functor to $\varepsilon(F)$, namely the composite $r_{\mathcal{B}}.T.i_{\mathcal{A}}$, as displayed:

$$\begin{tikzcd}
\mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{set}] \arrow[d, dotted, "\varepsilon(F)"'] \arrow[r, tail, shift right, "i_{\mathcal{B}}"'] & {[\mathcal{B}, \mathbf{set}]} \arrow[l, shift right, "r_{\mathcal{B}}"'] \arrow[d, shift left, "- \cdot F"] \\
\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}] \arrow[r, tail, "i_{\mathcal{A}}"'] & {[\mathcal{A}, \mathbf{set}]} \arrow[u, shift left, "T"]
\end{tikzcd}$$

It only remains to prove that $\varepsilon(F)$ does preserve $\mathbb{D}$-filtered colimits. In order to do that, we remark that the functor $- \cdot F : [\mathcal{B}, \mathbf{set}] \longrightarrow [\mathcal{A}, \mathbf{set}]$ preserves all colimits (in particular, the $\mathbb{D}$-filtered ones). This follows immediately from the property that colimits in $[\mathcal{B}, \mathbf{set}]$ are computed pointwise (that is, in $\mathbf{set}$). Namely, for any small category $\mathcal{I}$, given a functor $S : \mathcal{I} \longrightarrow [\mathcal{B}, \mathbf{set}]$ we get:

$$((\operatorname*{colim}_{I \in \mathcal{I}} S(I)) \cdot F)(B) \simeq \operatorname*{colim}_{I \in \mathcal{I}}(S(I)(F(B))) \simeq (\operatorname*{colim}_{I \in \mathcal{I}}(S(I) \cdot F))(B)$$

for any object $B \in \mathcal{B}$, that is $(\mathrm{colim}_{I \in \mathcal{I}} S(I)) \cdot F = \mathrm{colim}_{I \in \mathcal{I}}(S(I) \cdot F)$, as required. From Lemma 29 we can derive that the composite $(- \cdot F).i_{\mathcal{B}}$ preserves $\mathbb{D}$-filtered colimits, but this is precisely the functor $i_{\mathcal{A}}.\varepsilon(F)$. Moreover, again by Lemma 29, $i_{\mathcal{A}}$ reflects $\mathbb{D}$-filtered colimits, therefore, the functor $\varepsilon(F)$ does preserve them and is indeed a morphism of $\mathbb{D}$-**LP**. This completes the definition of

$$\varepsilon : \mathbb{D}\text{-}\mathbf{Th}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{LP}$$

on objects and morphisms. It is defined in the obvious (covariant) way on 2-cells, namely for any natural transformation $\alpha : F \longrightarrow G : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathbb{D}$-**Th**, $\varepsilon(\alpha)$ is defined to be the morphism

$$M.\alpha : M.F \longrightarrow M.G$$

of $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, for any $M \in \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{set}]$. It is an easy proof to show that ε is a 2-functor.

The objects of $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$

We present some useful results concerning the objects of the category $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, which we state here in their most general setting.

Proposition 37 *Let $\mathcal{A}$ be a locally small and $\mathbb{D}$-complete category. Any representable functor is a $\mathbb{D}$-presentable object of $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$.*

Proof. We have already shown in Lemma 31 that, for any $A \in \mathcal{A}$, the corresponding representable functor $\mathcal{A}(A, -)$ is $\mathbb{D}$-continuous and proved in Lemma 29 that the category $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$ has $\mathbb{D}$-filtered colimits. We wish now to show that the representable functor $\mathbf{Nat}(M, -) : \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}] \longrightarrow \mathbf{set}$ preserves $\mathbb{D}$-filtered colimits, where M is precisely $\mathcal{A}(A, -)$. For it, take $\mathcal{J}$ a $\mathbb{D}$-filtered category and a functor $H : \mathcal{J} \longrightarrow \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$. Then, we get:

$$\begin{array}{rcll}
\mathbf{Nat}(M, \operatorname*{colim}_{J \in \mathcal{J}} H(J)) & = & \mathbf{Nat}(\mathcal{A}(A, -), \operatorname*{colim}_{J \in \mathcal{J}} H(J)) & \\
& \simeq & (\operatorname*{colim}_{J \in \mathcal{J}} H(J))(A) & (1) \\
& \simeq & \operatorname*{colim}_{J \in \mathcal{J}}(H(J)(A)) & (2) \\
& \simeq & \operatorname*{colim}_{J \in \mathcal{J}} \mathbf{Nat}(\mathcal{A}(A, -), H(J)) & (3)
\end{array}$$

(where (1) and (3) are just instances of the Yoneda Lemma and (2) follows from the fact that $\mathbb{D}$-filtered colimits are computed pointwise in $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, by Corollary 30). That is, $\mathbf{Nat}(M, -)$ is a $\mathbb{D}$-accessible functor, as desired. ∎

Notice that not only any representable object in $\mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$ is $\mathbb{D}$-presentable, but also every $\mathbb{D}$-presentable object in $\mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$ is a retract of a representable one, as we show in the following:

Proposition 38 *Let $\mathcal{A}$ be a locally small and $\mathbb{D}$-complete category. The $\mathbb{D}$-presentable objects of $\mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$ are precisely the retracts of the representable functors.*

Proof. One implication follows directly from Proposition 37 and Corollary 21. Conversely, let $N \in \mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$ be a $\mathbb{D}$-presentable object. By Theorem 26, the category $\mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$ is a free completion of $\mathcal{A}^{op}$ under $\mathbb{D}$-filtered colimits, so, as a $\mathbb{D}$-continuous functor, N can be expressed as the colimit of a $\mathbb{D}$-filtered diagram $Y_{\mathcal{A}^{op}}.H : \mathcal{J} \longrightarrow [\mathcal{A},\mathbf{set}]$, where $H : \mathcal{J} \longrightarrow \mathcal{A}^{op}$ and $\mathcal{J}$ is a $\mathbb{D}$-filtered category; namely $N \simeq \mathrm{colim}_{J\in\mathcal{J}}\, Y_{\mathcal{A}^{op}}.H(J) = \mathrm{colim}_{J\in\mathcal{J}}\, \mathcal{A}(H(J),-)$. Since, by hypothesis the functor $\mathbf{Nat}(N,-)$ preserves $\mathbb{D}$-filtered colimits, we have $\mathbf{Nat}(N,N) \simeq \mathbf{Nat}(N, \mathrm{colim}_{J\in\mathcal{J}}\, \mathcal{A}(H(J),-)) \simeq \mathrm{colim}_{J\in\mathcal{J}}\, \mathbf{Nat}(N, \mathcal{A}(H(J),-))$. Then, the identity natural transformation 1_N on N has to factor through one of the colimit injection σ_J, for some $J \in \mathcal{J}$, as follows:

$$\begin{tikzcd}
N \arrow[r, "1_N"] \arrow[dr, dotted, "\gamma_J"'] & N \simeq \underset{J\in\mathcal{J}}{\mathrm{colim}}\, \mathcal{A}(H(J),-) \\
& \mathcal{A}(H(J),-) \arrow[u, "\sigma_J"']
\end{tikzcd}$$

Namely, we get $\sigma_J.\gamma_J = 1_N$, for some morphism $\gamma_J : N \longrightarrow \mathcal{A}(H(J),-)$ in $\mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$. Therefore, N is a retract of the representable functor $\mathcal{A}(H(J),-) \in \mathbb{D}$-$\mathbf{cont}[\mathcal{A},\mathbf{set}]$. ■

We also point out here two important properties of these special arrows between categories of $\mathbb{D}$-continuous set-valued functors that are morphisms of $\mathbb{D}$-$\mathbf{LP}$.

Corollary 39 *Let $\mathcal{K}$ be a locally small category with $\mathbb{D}$-filtered colimits. Let $F, G : \mathcal{K} \longrightarrow$ **set** be two functors. If F is $\mathbb{D}$-accessible and G is a retract of F, then G is itself $\mathbb{D}$-accessible.*

Proof. It is a simple consequence of Proposition 20, while choosing as diagrams S functors defined on $\mathbb{D}$-filtered categories. ■

Proposition 40 *Let $\mathcal{K}$ and $\mathcal{K}'$ be locally small categories with $\mathbb{D}$-filtered colimits, and let $R : \mathcal{K} \longrightarrow \mathcal{K}'$, and $L : \mathcal{K}' \longrightarrow \mathcal{K}$ be two functors such that $L \dashv R$. If R is $\mathbb{D}$-accessible, then L preserves $\mathbb{D}$-presentable objects.*

Proof. Let $M \in \mathcal{K}'$ be a $\mathbb{D}$-presentable object. To prove that $\mathcal{K}(L(M), -) : \mathcal{K} \longrightarrow$ **set** preserves $\mathbb{D}$-filtered colimits, consider a $\mathbb{D}$-filtered category $\mathcal{J}$ and a functor $H : \mathcal{J} \longrightarrow \mathcal{K}$. We get:

$$\begin{array}{rcll} \mathcal{K}(L(M), \operatorname*{colim}_{J\in\mathcal{J}} H(J)) & \simeq & \mathcal{K}'(M, R(\operatorname*{colim}_{J\in\mathcal{J}} H(J))) & (1) \\ & \simeq & \mathcal{K}'(M, (\operatorname*{colim}_{J\in\mathcal{J}} R.H(J))) & (2) \\ & \simeq & \operatorname*{colim}_{J\in\mathcal{J}} \mathcal{K}'(M, R.H(J)) & (3) \\ & \simeq & \operatorname*{colim}_{J\in\mathcal{J}} \mathcal{K}(L(M), H(J)) & (4) \end{array}$$

(where (1) and (4) hold by $L \dashv R$, (2) follows from R being $\mathbb{D}$-accessible and (3) comes from M being $\mathbb{D}$-presentable). That is, $L(M)$ is a $\mathbb{D}$-presentable object of $\mathcal{K}$, as desired. ∎

Corollary 41 *Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category and let $i : \mathcal{K}' \longrightarrow \mathcal{K}$ be a full reflective subcategory of $\mathcal{K}$. If i is $\mathbb{D}$-accessible, then $\mathcal{K}'$ is locally $\mathbb{D}$-presentable.*

Proof. To prove that $\mathcal{K}'$ is a locally $\mathbb{D}$-presentable category we need to show that it is cocomplete and that there exists a small set $\mathcal{A}'$ of $\mathbb{D}$-presentable objects of $\mathcal{K}'$ such that any object of $\mathcal{K}'$ is a $\mathbb{D}$-filtered colimit of objects of $\mathcal{A}'$. Obviously, $\mathcal{K}'$ is cocomplete (and complete) being a full reflective subcategory of $\mathcal{K}$ cocomplete (and complete). Since i is $\mathbb{D}$-accessible, then the reflection $r : \mathcal{K} \longrightarrow \mathcal{K}'$, that is, its left adjoint functor, preserves $\mathbb{D}$-presentable objects, by Proposition 40. So, denoted by $\mathcal{A}$ the small set of $\mathbb{D}$-presentable objects of $\mathcal{K}$ such that any object of $\mathcal{K}$ can be regarded as a $\mathbb{D}$-filtered colimit of those of $\mathcal{A}$, we can define $\mathcal{A}'$ as the set of all $\{rA | A \in \mathcal{A}\}$. It only remains to prove that any object $K' \in \mathcal{K}'$ is a $\mathbb{D}$-filtered colimit of objects of $\mathcal{A}'$. For it, consider its image $iK' \in \mathcal{K}$ through i. As an object of $\mathcal{K}$, it can be expressed as $\mathbb{D}$-filtered colimit of objects of $\mathcal{A}$, namely $iK' = \operatorname{colim}_{J\in\mathcal{J}} A_J$, where $\mathcal{J}$ is a $\mathbb{D}$-filtered category and the diagram, which we take the colimit of, associates with any $J \in \mathcal{J}$ an object $A_J \in \mathcal{A}$. Since the counit for the adjunction $r \dashv i$ is an isomorphism at any component, i being fully faithful, we can establish a chain of isomorphism as follows:

$$K' \simeq riK' = r(\operatorname*{colim}_{J\in\mathcal{J}} A_J) \simeq \operatorname*{colim}_{J\in\mathcal{J}} rA_J$$

where the last one holds because r, being a left adjoint functor, preserves all existing colimits. That is, K' can be written as a $\mathbb{D}$-filtered colimit of objects of $\mathcal{A}'$. This ends the proof of $\mathcal{K}' \in \mathbb{D}$-**LP**. ∎

The 2-functor ε is a biequivalence

So, we are finally able to state one of the two main theorems of this chapter. This result can be found in [18]. We follow the lines of the remarkable exposition of Gabriel-Ulmer duality given in [3].

Theorem 42 *The 2-functor*

$$\varepsilon : \mathbb{D}\text{-}\mathbf{Th}^{op} \longrightarrow \mathbb{D}\text{-}\mathbf{LP}$$

is a biequivalence of 2-categories.

Proof. We need to prove that the 2-functor ε, defined at Section 2.1, is (a) surjective on objects up to equivalence and (b) locally an equivalence of categories. We divide then the proof into two steps.
(a) Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category and consider its full subcategory $\mathcal{K}_{\mathbb{D}}$ of $\mathbb{D}$-presentable objects. By Proposition 18 and Remark 19, the category ${\mathcal{K}_{\mathbb{D}}}^{op}$ is essentially small and $\mathbb{D}$-complete, so we can apply Remark 27 (Condition 2) and get:

$$\mathcal{K} \simeq \mathbb{D}\text{-}\mathbf{cont}[{\mathcal{K}_{\mathbb{D}}}^{op}, \mathbf{set}]$$

It only remains to check, then, that ${\mathcal{K}_{\mathbb{D}}}^{op}$ is Cauchy complete: but this is the dual of Corollary 22. Therefore, ${\mathcal{K}_{\mathbb{D}}}^{op}$ is an object of $\mathbb{D}$-**Th**. Therefore, we have shown that such a $\mathcal{K}$ is equivalent to the category $\varepsilon({\mathcal{K}_{\mathbb{D}}}^{op})$, as desired.

(b) For each pair of objects $\mathcal{A}$, $\mathcal{B}$ in $\mathbb{D}$-**Th** the functor

$$\varepsilon_{\mathcal{A},\mathcal{B}} : \mathbb{D}\text{-}\mathbf{Th}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathbb{D}\text{-}\mathbf{LP}(\varepsilon(\mathcal{B}), \varepsilon(\mathcal{A}))$$

is an equivalence of categories.

(b1) $\varepsilon_{\mathcal{A},\mathcal{B}}$ is faithful. For it, let $\alpha, \beta : F \longrightarrow G : \mathcal{A} \longrightarrow \mathcal{B}$ be natural transformations such that $\varepsilon_{\mathcal{A},\mathcal{B}}(\alpha) = \varepsilon_{\mathcal{A},\mathcal{B}}(\beta)$ in $\mathbb{D}$-**LP**, that is, $- \cdot \alpha = - \cdot \beta : - \cdot F \longrightarrow - \cdot G$. For each object $A \in \mathcal{A}$, we have to show $\alpha_A = \beta_A$. This follows from the fact that the component of $- \cdot \alpha$ at $M = \mathcal{B}(FA, -)$ is $M.\alpha : \mathcal{B}(FA, F-) \longrightarrow \mathcal{B}(FA, G-)$ and $\alpha_A = (M.\alpha)_A(1_{FA})$; analogously for β. Thus, the equality $M.\alpha = M.\beta$, for every $M \in \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{set}]$, implies in particular $\alpha_A = \beta_A$, as desired.

(b2) $\varepsilon_{\mathcal{A},\mathcal{B}}$ is full. That is, provided $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathbb{D}$-**Th**, then, every natural transformation $\tau : \varepsilon(F) \longrightarrow \varepsilon(G)$, namely every collection of natural transformations $\tau_M : M.F \longrightarrow M.G$, for $M \in \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{set}]$, which is also natural in the variable M, is of the form $\tau = - \cdot \alpha$, for some 2-cell $\alpha : F \longrightarrow G$ in $\mathbb{D}$-**Th**. To prove this, we already know that, for any object $A \in \mathcal{A}$, the functor $M = \mathcal{B}(FA, -)$ is $\mathbb{D}$-continuous, and hence we can consider $\tau_{\mathcal{B}(FA,-)} : \mathcal{B}(FA, F-) \longrightarrow \mathcal{B}(FA, G-)$ and define $\alpha_A = (\tau_{\mathcal{B}(FA,-)})_A(1_{FA}) : FA \longrightarrow GA$. We have to show that α is natural in A. For it, remark first that, for any $f : A \longrightarrow B$ in $\mathcal{A}$, by naturality of $\tau_{\mathcal{B}(FA,-)}$ we obtain:

$$(\tau_{\mathcal{B}(FA,-)})_B(Ff) = Gf.\alpha_A \tag{2.1}$$

Now, given such a $f : A \longrightarrow B$ in $\mathcal{A}$, the corresponding Yoneda transformation $- \cdot Ff = \mathcal{B}(Ff, -) : \mathcal{B}(FB, -) \longrightarrow \mathcal{B}(FA, -)$ is a map of $\mathbb{D}$-$\mathbf{cont}[\mathcal{B}, \mathbf{set}]$ and, since τ_M is natural in $M \in \mathbb{D}$-$\mathbf{cont}[\mathcal{B}, \mathbf{set}]$, we get:

$$\mathcal{B}(Ff, G-).\tau_{\mathcal{B}(FB,-)} = \tau_{\mathcal{B}(FA,-)}.\mathcal{B}(Ff, F-) \tag{2.2}$$

Evaluating equation 2.2 at B and applying it to 1_{FB} gives:

$$\alpha_B.Ff = (\tau_{\mathcal{B}(FA,-)})_B(Ff) = Gf.\alpha_A$$

(where the last equality holds by using equation 2.1), that is, the transformation $\alpha : F \longrightarrow G$ is natural. It only remains to verify that the equality $\tau = - \cdot \alpha$ holds in $\mathbb{D}$-$\mathbf{Th}$, that is, $\tau_M = M.\alpha$ for every $M \in \mathbb{D}$-$\mathbf{cont}[\mathcal{B}, \mathbf{set}]$; even more explicitly, that is, for any object $A \in \mathcal{A}$, $(\tau_M)_A = M.\alpha_A$ holds in $\mathbf{set}$. For it, any element x of MFA corresponds via the Yoneda Lemma to a unique map $\tilde{x} : \mathcal{B}(FA, -) \longrightarrow M$ in $\mathbb{D}$-$\mathbf{cont}[\mathcal{B}, \mathbf{set}]$ satisfying $\tilde{x}_{FA}(1_{FA}) = x$. From the naturality of τ_M in M, we derive $\tau_M.\tilde{x}.F = \tilde{x}.G.\tau_{\mathcal{B}(FA,-)}$, which we can evaluate at A and apply to 1_{FA} to obtain $(\tau_M)_A(x) = \tilde{x}_{GA}(\alpha_A)$. Now, using further the naturality of $\tilde{x}$ (relative to the arrow $\alpha_A : FA \longrightarrow GA$ of $\mathcal{B}$) we derive $\tilde{x}_{GA}(\alpha_A) = M.\alpha_A(x)$, and therefore, by gluing together the last two equalities, we get $(\tau_M)_A(x) = M.\alpha_A(x)$ for all $x \in MFA$, as desired.

(b3) $\varepsilon_{\mathcal{A},\mathcal{B}}$ is essentially surjective on objects, that is, any morphism $R : \varepsilon(\mathcal{B}) \longrightarrow \varepsilon(\mathcal{A})$ in $\mathbb{D}$-$\mathbf{LP}$ is naturally isomorphic to $- \cdot F$, for some morphism $F : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathbb{D}$-$\mathbf{Th}$. For it, consider such a morphism R in $\mathbb{D}$-$\mathbf{LP}$, together with its left adjoint L and the restricted Yoneda embeddings, as in the following diagram:

$$\begin{tikzcd}
\mathcal{B}^{op} \arrow[r, tail, "Y_{\mathcal{B}^{op}}"] & \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B}, \mathbf{set}] \arrow[d, shift left, "R"] \\
\mathcal{A}^{op} \arrow[u, dotted, "L|"] \arrow[r, tail, "Y_{\mathcal{A}^{op}}"'] & \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}] \arrow[u, shift left, "L"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

We claim that L restricts to a functor $\mathcal{A}^{op} \longrightarrow \mathcal{B}^{op}$, which means that, for any $\mathbb{D}$-continuous representable functor $M = \mathcal{A}(A, -)$, its image

$L(M)$ through L is a representable functor. This restriction will provide the expected functor F, which we prove to be indeed a morphism of $\mathbb{D}$-**Th**.

Consider an object A in $\mathcal{A}$. By Proposition 37 $\mathcal{A}(A,-)$ is a $\mathbb{D}$-presentable object, then by Proposition 40 $L(\mathcal{A}(A,-))$ is $\mathbb{D}$-presentable. Hence, it follows from Proposition 38 that $L(\mathcal{A}(A,-))$ is a retract of a representable functor. Since $\mathcal{B}$ is a Cauchy complete category, $L(\mathcal{A}(A,-))$ is itself representable. Therefore, for any $A \in \mathcal{A}$, we can choose an object $B := FA$ in $\mathcal{B}$ such that $\mathcal{B}(B,-) \simeq L(\mathcal{A}(A,-))$. Let us fix a natural isomorphism $l_A : L(\mathcal{A}(A,-)) \longrightarrow \mathcal{B}(FA,-)$, for any $A \in \mathcal{A}$. For a morphism $f : A \longrightarrow A'$ in $\mathcal{A}$, the Yoneda Lemma guarantees that there exists a unique map $Ff : FA \longrightarrow FA'$, for which the natural transformation

$$l_A.L(\mathcal{A}(f,-)).l_{A'}^{-1} : \mathcal{B}(FA',-) \longrightarrow \mathcal{B}(FA,-)$$

is given by composite with Ff. It is easy to verify that this assignment defines a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$, which we will show to be $\mathbb{D}$-continuous.

Since L is a left adjoint functor and, by Proposition 32, $Y_{\mathcal{A}^{op}}$ preserves $\mathbb{D}^{op}$-colimits, the composite $L.Y_{\mathcal{A}^{op}}$ preserves $\mathbb{D}^{op}$-colimits, but this composite is isomorphic (via l_A) to $Y_{\mathcal{B}^{op}}.F^{op}$. Thus, $Y_{\mathcal{B}^{op}}.F^{op}$ preserves $\mathbb{D}^{op}$-colimits and, by Proposition 32 again, $Y_{\mathcal{B}^{op}}$ reflects them. So, F^{op} also preserves them: that is, F preserves $\mathbb{D}$-limits.

Finally, it only remains to prove that the functor $\varepsilon(F) : \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B},\mathbf{set}] \longrightarrow \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A},\mathbf{set}]$ is isomorphic to R. Since $L \dashv R$, it is sufficient to prove that $\varepsilon(F)$ is right adjoint to L, namely that there is an isomorphism $\mathbf{Nat}(L(M),G) \simeq \mathbf{Nat}(M,G\cdot F)$ natural in both the variables $M \in \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A},\mathbf{set}]$ and $G \in \mathbb{D}\text{-}\mathbf{cont}[\mathcal{B},\mathbf{set}]$. First, remark that for a representable functor $M = \mathcal{A}(A,-)$, where $A \in \mathcal{A}$, this is just the Yoneda Lemma:

$$\begin{array}{rcll}\mathbf{Nat}(\mathcal{A}(A,-),G.F) & \simeq & G.F(A) & (1)\\ & \simeq & \mathbf{Nat}(\mathcal{B}(FA,-),G) & (2)\\ & \simeq & \mathbf{Nat}(L(\mathcal{A}(A,-)),G) & (3)\end{array}$$

(where (1) and (2) are just instances of the Yoneda Lemma and (3) is the isomorphism l_A). Then, for an arbitrary $\mathbb{D}$-continuous functor, we just use this fact together with Lemma 5 (or, equivalently, that $\mathbb{D}\text{-}\mathbf{cont}[\mathcal{A},\mathbf{set}]$ is a free completion of $\mathcal{A}^{op}$ under $\mathbb{D}$-filtered colimits). Explicitly, if we write $M \in \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A},\mathbf{set}]$ as a $\mathbb{D}$-filtered colimit of representable functors, namely $M = \mathrm{colim}_{J\in\mathcal{J}}\,\mathcal{A}(H(J),-)$, where $\mathcal{J}$ is a $\mathbb{D}$-filtered category and H is the appropriate diagram $Y_{\mathcal{A}^{op}}.H : \mathcal{J} \longrightarrow$

$[\mathcal{A}, \mathbf{set}]$, we get the following chain of natural isomorphisms:

$$\begin{aligned}
\mathbf{Nat}(\operatorname*{colim}_{J\in\mathcal{J}} \mathcal{A}(H(J),-), G.F) &\simeq \lim_{J\in\mathcal{J}} \mathbf{Nat}(\mathcal{A}(H(J),-), G.F) \\
&\simeq \lim_{J\in\mathcal{J}} \mathbf{Nat}(L(\mathcal{A}(H(J),-)), G) \quad (1) \\
&\simeq \mathbf{Nat}(\operatorname*{colim}_{J\in\mathcal{J}} L(\mathcal{A}(H(J),-)), G) \\
&\simeq \mathbf{Nat}(L(\operatorname*{colim}_{J\in\mathcal{J}} \mathcal{A}(H(J),-)), G) \quad (2)
\end{aligned}$$

(where (1) is just the previous case and (2) follows from L being left adjoint). The proof of the Theorem is now complete. ■

Example 43

1. If we take as $\mathbb{D}$ the doctrine of finite limits, then Theorem 42 is exactly Gabriel-Ulmer duality. This celebrated duality asserts that there is a contravariant biequivalence $\mathbf{Lex}^{op} \longrightarrow \mathbf{LFP}$, where **Lex** is the 2-category of small finitely complete categories, finite limit preserving functors and natural transformations, and **LFP** is the 2-categories of locally finitely presentable categories, finitary right adjoint functors and natural transformations (see [23]).

2. If we take as $\mathbb{D}$ the doctrine of finite products, then Theorem 42 is exactly Adámek-Lawvere-Rosický duality between algebraic theories and multisorted finitary varieties. The established contravariant biequivalence is $\mathbf{Th}^{op} \longrightarrow \mathbf{VAR}$, where **Th** is the 2-category of small Cauchy complete categories with finite products, finite product preserving functors and natural transformations, and **VAR** is the 2-category of multisorted finitary varieties, algebraically exact functors and natural transformations (see [2]).

3. If we take as $\mathbb{D}$ the empty doctrine, we have a duality between the 2-category of small Cauchy complete categories, functors and natural transformations, and the 2-category of categories of the form $[\mathcal{A}, \mathbf{set}]$, for $\mathcal{A}$ small, cocontinuous right adjoint functors and natural transformations. This duality refines the well-known fact that two small categories $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, that is, the corresponding categories of set-valued functors are equivalent, namely $[\mathcal{A}, \mathbf{set}] \simeq [\mathcal{B}, \mathbf{set}]$, if and only if they are Cauchy equivalent, that is, they have equivalent Cauchy completions. We recall that a Cauchy completion of a small category $\mathcal{A}$ is given by the full subcategory of the presheaf category $[\mathcal{A}, \mathbf{set}]$ of retracts of representable functors, see [10].

2.2 Some applications

Comparing sound doctrines

As an application of the duality Theorem 42, we can think of comparing different sound doctrines. More precisely, the idea is to answer to such a question as "under which conditions is it possible that a locally finitely presentable category is a (multisorted finitary) variety?" or "when can a variety be thought of as a presheaf category?". Some work has already been done toward this direction. In [37, 38], Pedicchio and Wood study the case of when locally finitely presentable categories are varieties. In fact, the converse statement is always true: as we know, a (multisorted finitary) variety $\mathcal{K}$ is always of the form $\mathbf{Finprod}[\mathcal{A}^{op}, \mathbf{set}]$, for a small Cauchy complete category $\mathcal{A}$ with finite coproducts, just by taking as $\mathcal{A}$ the full subcategory determined by the strongly finitely presentable objects of $\mathcal{K}$ (that is, those objects $K \in \mathcal{K}$ such that the corresponding representable functor $\mathcal{K}(K, -) : \mathcal{K} \longrightarrow \mathbf{set}$ preserves sifted colimits). Moreover, we can regard it as the category $\mathbf{Lex}[\mathcal{B}^{op}, \mathbf{set}]$ of set-valued functors which preserve all finite limits, where $\mathcal{B}$ is the full subcategory given by all finitely presentable objects of $\mathcal{K}$ and $\mathcal{B}$ is closed in $\mathcal{K}$ under finite colimits. Therefore, $\mathcal{K}$ is a locally finitely presentable category. Pedicchio and Wood establish the following precise comparison between the 2-category **VAR** of varieties and the 2-category **LFP** of locally finitely presentable categories:

Proposition 44

1. *The (not full) inclusion of* **VAR** *into* **LFP** *has a left biadjoint;*

2. *A small category with finite limits $\mathcal{C}$ is such that $\mathbf{Lex}[\mathcal{C}, \mathbf{set}]$ is equivalent to a variety if and only if every object of $\mathcal{C}$ is a regular subobject of an effective injective.*

We recall that an object $C \in \mathcal{C}$ is *effective injective* if the corresponding representable functor $\mathcal{C}(-, C) : \mathcal{C}^{op} \longrightarrow \mathbf{set}$ preserves coequalizers of reflexive graphs. Explicitly, if

$$E \rightarrowtail^{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

is an equalizer in $\mathcal{C}$ of a *reflexive graph* (f, g), that is, there exists an arrow $r : Y \longrightarrow X$ in $\mathcal{C}$ satisfying $r.f = 1_X = r.g$, then,

$$\mathcal{C}(Y, -) \overset{\mathcal{C}(f,-)}{\underset{\mathcal{C}(g,-)}{\rightrightarrows}} \mathcal{C}(X, -) \xrightarrow{\mathcal{C}(e,-)} \mathcal{C}(E, -)$$

is a coequalizer in **set**.

We wish to present here the general setting needed to compare sound doctrines. Consider two sound doctrines $\mathbb{D}_1$ and $\mathbb{D}_2$ such that $\mathbb{D}_1 \hookrightarrow \mathbb{D}_2$. Clearly, there is a (not full) inclusion $i : \mathbb{D}_2\textbf{-Th} \longrightarrow \mathbb{D}_1\textbf{-Th}$, which is just the way of looking at small Cauchy complete categories with $\mathbb{D}_2$-limits as categories which, in particular, have $\mathbb{D}_1$-limits. It is natural, then, to ask if there is a way to associate with any $\mathbb{D}_1$-theory a $\mathbb{D}_2$-theory and by looking for an answer to this question we are led to the study of $\mathbf{F}$-conservative $\mathbf{C}$-completions, for any $\mathbf{F}$ and $\mathbf{C}$ sets of small categories (for this matter see [40], where they work in the dual situation of a cocompletion). In details, a free $\mathbf{F}$-conservative $\mathbf{C}$-completion of a category $\mathcal{A}$ is a full embedding $\xi_{\mathcal{A}} : \mathcal{A} \longrightarrow \overline{\mathcal{A}}$ for which:

- $\overline{\mathcal{A}}$ is a $\mathbf{C}$-complete category;
- $\xi_{\mathcal{A}}$ preserves existing $\mathbf{F}$-limits;
- for any $\mathbf{F}$-continuous functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ into a $\mathbf{C}$-complete category $\mathcal{B}$, there exists a unique (up to isomorphism) functor $\overline{F} : \overline{\mathcal{A}} \longrightarrow \mathcal{B}$ satisfying $F \simeq \overline{F}.\xi_{\mathcal{A}}$ and being $\mathbf{C}$-continuous.

As an example we can mention, once again, locally finitely presentable categories: they are precisely the (finite-colimits)-conservative (filtered-colimit)-cocompletion of small finitely cocomplete categories.
In [40] they investigate the size of these kinds of completions and they can prove the following result:

Proposition 45 *Every category $\mathcal{A}$ has a free $\mathbf{F}$-conservative $\mathbf{C}$-completion. It is a locally small category if $\mathcal{A}$ is $\mathbf{F}$-locally small.*

We recall that a category $\mathcal{A}$ is $\mathbf{F}$-*locally small* provided it is locally small and, for any object $A \in \mathcal{A}$, there is only a small set of morphisms in $\mathcal{A}$ from a nonabsolute $\mathbf{F}$-limit to A. (Explicitly: an absolute $\mathbf{F}$-limit is a limit of a diagram $H : \mathcal{F} \longrightarrow \mathcal{A}$, where $\mathcal{F} \in \mathbf{F}$, which is preserved by any functor; all other limit cones are said to be nonabsolute $\mathbf{F}$-limit.) Let us dwell up on these notion for a moment, although they are a bit marginal to our context. We underline that not only local smallness of the category, in general, does not suffice to imply local smallness of its $\mathbf{F}$-conservative $\mathbf{C}$-completion, as expressed in the previous Proposition, but not even smallness is enough. Namely, starting with a small category $\mathcal{A}$ its $\mathbf{F}$-conservative $\mathbf{C}$-completion $\overline{\mathcal{A}}$ need not to be small, in general. It is interesting to point out, though, that under the further assumption on $\mathbf{C}$ being a small set, then, this is the case, that is:

Remark 46 Let $\mathcal{A}$ be a small category and $\mathbf{C}$ be a small set (of small categories), then the $\mathbf{F}$-conservative $\mathbf{C}$-completion $\overline{\mathcal{A}}$ of $\mathcal{A}$ is a small category.

Therefore, concerning our setting, where we deal with objects of $\mathbb{D}$-$\mathbf{Th}$, that is, (essentially) small categories, we can claim that there always exists a way of associating with a $\mathbb{D}_1$-theory $\mathcal{A}$ a small $\mathbb{D}_2$-complete category $\overline{\mathcal{A}}$, just by taking a $\mathbb{D}_1$-conservative $\mathbb{D}_2$-completion of $\mathcal{A}$. Moreover, one could prove that the Cauchy completion of a $\mathbb{D}_2$-complete category is itself $\mathbb{D}_2$-complete. This means, in particular, that by taking the Cauchy completion of the category $\overline{\mathcal{A}}$, we get a category belonging to $\mathbb{D}_2$-$\mathbf{Th}$. We need, in fact, a little bit more.

Moving back to the case of sound doctrines, we consider the doctrines $\mathbb{D}_1^+$ and $\mathbb{D}_2^+$ obtained from $\mathbb{D}_1$ and $\mathbb{D}_2$ by joining the category $\mathcal{E}$, consisting of a single object $*$ and two morphisms 1_* and e such that $e.e = e$, as in Section 1.4. In light of Lemma 34, a $\mathbb{D}_1$-theory is a $\mathbb{D}_1$-complete and Cauchy complete small category, namely it is a $\mathbb{D}_1^+$-complete small category. Therefore, Proposition 45 applied to the doctrines $\mathbb{D}_1^+$ and $\mathbb{D}_2^+$ guarantees us that every $\mathcal{A} \in \mathbb{D}_1$-$\mathbf{Th}$ has a $\mathbb{D}_1$-conservative $\mathbb{D}_2^+$-completion, that is, there exist a category $\overline{\mathcal{A}} \in \mathbb{D}_2$-$\mathbf{Th}$ and a $\mathbb{D}_1$-$\mathbf{Th}$ morphism $\xi_{\mathcal{A}} : \mathcal{A} \longrightarrow \overline{\mathcal{A}}$, such that $\overline{\mathcal{A}}$ is universal with respect to $\mathbb{D}_2^+$-complete categories and $\mathbb{D}_1$-continuous functors. More explicitly, for any $\mathbb{D}_1$-continuous functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ into a $\mathbb{D}_2$-complete and Cauchy complete category $\mathcal{B}$, there exists a unique (up to isomorphism) $\mathbb{D}_2$-continuous functor $\overline{F}$ making the triangle:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\xi_{\mathcal{A}}"] \arrow[dr, "F"'] & & \overline{\mathcal{A}} \arrow[dl, dotted, "\overline{F}"] \\
& \mathcal{B} &
\end{tikzcd}$$

commute. Thanks to this completion, we get:

Proposition 47 *The inclusion $i : \mathbb{D}_2$-$\mathbf{Th} \longrightarrow \mathbb{D}_1$-$\mathbf{Th}$ has a left biadjoint $\overline{(-)} : \mathbb{D}_1$-$\mathbf{Th} \longrightarrow \mathbb{D}_2$-$\mathbf{Th}$, which associates with any $\mathbb{D}_1$-theory $\mathcal{A}$ its $\mathbb{D}_1$-conservative $\mathbb{D}_2^+$-completion $\overline{\mathcal{A}}$.*

Proof. The left biadjoint $\overline{(-)}$ is defined on morphisms by the universal property of the completion, as in the diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\xi_{\mathcal{A}}"] \arrow[d, "F"'] & \overline{\mathcal{A}} \arrow[d, dotted, "\overline{F}"] \\
\mathcal{B} \arrow[r, "\xi_{\mathcal{B}}"'] & \overline{\mathcal{B}}
\end{tikzcd}$$

where the composite $\xi_{\mathcal{B}}.F$ is $\mathbb{D}_1$-continuous, by hypothesis. As unit η for the adjunction $\overline{(-)} \dashv i$, we can take at each component $\xi_{\mathcal{A}} : \mathcal{A} \longrightarrow \overline{\mathcal{A}}$, which is $\mathbb{D}_1$-continuous, by definition. In this way, it follows directly that η is natural, by how the functor $\overline{(-)}$ has been defined, and η is universal, by the universal property of the completion. Therefore, we have establish the desired adjunction $\overline{(-)} \dashv i$. ∎

Remark 48 For any $\mathcal{B} \in \mathbb{D}_2$-**Th**, note that, in general, $\mathcal{B} \neq \overline{\mathcal{B}}$, simply because the counit for $\overline{(-)} \dashv i$ is not, in general, an isomorphism (i not being a full inclusion).

Thanks to the duality Theorem 42, which states precisely that there is a biequivalence $\mathbb{D}$-**LP** $\simeq (\mathbb{D}\text{-}\mathbf{Th})^{op}$, we can define the corresponding 2-functor $j : \mathbb{D}_2\text{-}\mathbf{LP} \longrightarrow \mathbb{D}_1\text{-}\mathbf{LP}$ on objects by $j(\mathbb{D}_2\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]) = \mathbb{D}_1\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$: but this is obviously not an inclusion. Moreover, $\overline{(-)}$ gives rise to a right biadjoint to j, as displayed in the following diagram:

$$\begin{tikzcd}
(\mathbb{D}_2\text{-}\mathbf{Th})^{op} \arrow[d, "\simeq"'] \arrow[r, phantom, "\top"] & (\mathbb{D}_1\text{-}\mathbf{Th})^{op} \arrow[l, bend right, "\overline{(-)}"'] \arrow[l, leftarrow, bend left, "i"'] \arrow[d, "\simeq"] \\
\mathbb{D}_2\text{-}\mathbf{LP} \arrow[r, bend right, "j"'] \arrow[r, phantom, "\top"] & \mathbb{D}_1\text{-}\mathbf{LP} \arrow[l, dotted, bend right]
\end{tikzcd}$$

Call it $j^* : \mathbb{D}_1\text{-}\mathbf{LP} \longrightarrow \mathbb{D}_2\text{-}\mathbf{LP}$; it is defined on objects $\mathbb{D}_1\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}] \in \mathbb{D}_1\text{-}\mathbf{LP}$ as $j^*(\mathbb{D}_1\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]) = \mathbb{D}_2\text{-}\mathbf{cont}[\overline{\mathcal{A}}, \mathbf{set}]$; it is such that $j \dashv j^*$.

Proposition 49 *The 2-functor $j^* : \mathbb{D}_1\text{-}\mathbf{LP} \longrightarrow \mathbb{D}_2\text{-}\mathbf{LP}$ is an "inclusion", in the sense that $j^*(\mathcal{K}) \simeq \mathcal{K}$, for any $\mathcal{K} \in \mathbb{D}_1\text{-}\mathbf{LP}$.*

Proof. As we know, any locally $\mathbb{D}_1$-presentable category $\mathcal{K}$ is of the form $\mathbb{D}_1\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, for some $\mathcal{A} \in \mathbb{D}_1\text{-}\mathbf{Th}$; consider the $\mathbb{D}_1$-conservative $\mathbb{D}_2$-completion $\overline{\mathcal{A}}$ of $\mathcal{A}$, which is nothing but the image by $\overline{(-)}$ of $\mathcal{A}$.

Then, by definition, $j^*(\mathbb{D}_1\textbf{-cont}[\mathcal{A},\textbf{set}]) = \mathbb{D}_2\textbf{-cont}[\overline{\mathcal{A}},\textbf{set}]$. We wish to establish that there is, in fact, an equivalence of categories between $\mathbb{D}_2\textbf{-cont}[\overline{\mathcal{A}},\textbf{set}]$ and $\mathbb{D}_1\textbf{-cont}[\mathcal{A},\textbf{set}]$. For it, we can define a map $\phi : \mathbb{D}_2\textbf{-cont}[\overline{\mathcal{A}},\textbf{set}] \longrightarrow \mathbb{D}_1\textbf{-cont}[\mathcal{A},\textbf{set}]$ on objects $F \in \mathbb{D}_2\textbf{-cont}[\overline{\mathcal{A}},\textbf{set}]$ by composing with the embedding $\xi_{\mathcal{A}}$, so that F is sent to $F.\xi_{\mathcal{A}}$. Note that the composite $F.\xi_{\mathcal{A}}$ is, in fact, a $\mathbb{D}_1$-continuous functor, since $\xi_{\mathcal{A}}$ is so and the doctrine $\mathbb{D}_1$ embeds into $\mathbb{D}_2$. On the other hand, the morphism $\psi : \mathbb{D}_1\textbf{-cont}[\mathcal{A},\textbf{set}] \longrightarrow \mathbb{D}_2\textbf{-cont}[\overline{\mathcal{A}},\textbf{set}]$ can be defined on objects $G \in \mathbb{D}_1\textbf{-cont}[\mathcal{A},\textbf{set}]$ by taking its unique (up to isomorphism) extension into a $\mathbb{D}_2$-continuous functor $\overline{G}$. It is an easy exercise to prove that $\phi.\psi(G) \simeq G$, by definition of the $\mathbb{D}_2$-completion, and $\psi.\phi(F) \simeq F$, by the uniqueness (up to isomorphism) of this completion. This ends the proof of $\phi : \mathbb{D}_2\textbf{-cont}[\overline{\mathcal{A}},\textbf{set}] \longrightarrow \mathbb{D}_1\textbf{-cont}[\mathcal{A},\textbf{set}]$ being an equivalence of categories. ∎

In other words, what we have just shown can be stated as follows:

Proposition 50 *A category $\mathcal{B} \in \mathbb{D}_2\textbf{-Th}$ is such that $\mathbb{D}_2\textbf{-cont}[\mathcal{B},\textbf{set}] \in \mathbb{D}_1\textbf{-LP}$ if and only if $\mathcal{B}$ is of the form $\overline{\mathcal{A}}$, for some category $\mathcal{A} \in \mathbb{D}_1\textbf{-Th}$.*

Proof. Consider a category $\mathcal{B} \in \mathbb{D}_2\textbf{-Th}$ of the form $\overline{\mathcal{A}}$, for some $\mathcal{A} \in \mathbb{D}_1\textbf{-Th}$. Then, by Proposition 49 we clearly have:

$$\mathbb{D}_2\textbf{-cont}[\mathcal{B},\textbf{set}] = \mathbb{D}_2\textbf{-cont}[\overline{\mathcal{A}},\textbf{set}] \simeq \mathbb{D}_1\textbf{-cont}[\mathcal{A},\textbf{set}] \in \mathbb{D}_1\textbf{-LP}$$

Conversely, if $\mathcal{B} \in \mathbb{D}_2\textbf{-Th}$ is such that $\mathbb{D}_2\textbf{-cont}[\mathcal{B},\textbf{set}] \simeq \mathbb{D}_1\textbf{-cont}[\mathcal{C},\textbf{set}]$, for some category $\mathcal{C} \in \mathbb{D}_1\textbf{-Th}$, then again by Proposition 49 we get:

$$\mathbb{D}_2\textbf{-cont}[\mathcal{B},\textbf{set}] \simeq \mathbb{D}_1\textbf{-cont}[\mathcal{C},\textbf{set}] \simeq \mathbb{D}_2\textbf{-cont}[\overline{\mathcal{C}},\textbf{set}]$$

From the duality Theorem 42, it follows that $\mathcal{B} \simeq \overline{\mathcal{C}}$. ∎

The case studied in [37, 38] corresponds to choosing as $\mathbb{D}_1$ the doctrine of finite products and as $\mathbb{D}_2$ the doctrine of finite limits. The completion is, then, the completion under equalizers of coreflexive graphs and the interesting result of [37] is the characterization of those finitely complete categories which are completions (under equalizers of coreflexive graphs) of categories with finite products, as presented in Proposition 44.

The other relevant example, then, will be when $\mathbb{D}_1$ is the empty doctrine and $\mathbb{D}_2$ is the doctrine of finite products, so that $\mathbb{D}_1\textbf{-LP}$ is the 2-category **PS** of presheaf categories and $\mathbb{D}_2\textbf{-LP}$ is the 2-category **VAR** of varieties.

In order to approach this study we need to provide some context. Of

course, limit completions are usually studied via the corresponding colimit completions. So, rather than study a free finite product completion $\prod_f \mathcal{A}$ of a small category $\mathcal{A}$, we will study its free finite coproduct completion, usually denoted by $\mathbf{Fam}_f\mathcal{A}$. They are clearly linked, then, by $\prod_f = (\mathbf{Fam}_f(-)^{op})^{op}$. We recall $\mathbf{Fam}_f\mathcal{A}$, the category of finite families of objects of $\mathcal{A}$, has for objects the pairs (I, f), where I a finite set and $f : I \longrightarrow \mathcal{A}$ is a functor, and for morphisms the pairs $(a, \alpha) : (I, f) \longrightarrow (J, g)$, where $a : I \longrightarrow J$ is a functor and $\alpha : f \longrightarrow g.a$ is a natural transformation. It is well-know that $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbf{Fam}_f\mathcal{A}$ is a free completion of $\mathcal{A}$ under finite coproducts.
In order to simplify the reading of the following proofs, we also recall explicitly how finite coproducts are computed in $\mathbf{Fam}_f\mathcal{A}$. Given a finite family $((I_j, f_j))_{j\in J}$ of objects of $\mathbf{Fam}_f\mathcal{A}$, its coproduct in $\mathbf{Fam}_f\mathcal{A}$ is given by the pair $(\coprod_{j\in J} I_j, (f_j)_{j\in J})$, where $\coprod_{j\in J} I_j$ is nothing but the disjoint union of the finite sets I_j and $(f_j)_{j\in J}$ is the functor $\coprod_{j\in J} I_j \longrightarrow \mathcal{A}$ defined by $(f_j)_{j\in J}(x) := f_j(x)$, for the unique $j \in J$ such that $x \in I_j$. Moreover, the coproduct injections $(e_j, \mathbf{1}) : (I_j, f_j) \longrightarrow (\coprod_{j\in J} I_j, (f_j)_{j\in J})$ are clearly given by the embedding $e_j : I_j \longrightarrow \coprod_{j\in J} I_j$ and the identity natural transformation $\mathbf{1} : f_j \longrightarrow (f_j)_{j\in J}.e_j$, which at each component $x \in \coprod_{j\in J} I_j$ is the identity $1_{f_j(x)} : f_j(x) \longrightarrow f_j(x)$ in $\mathcal{A}$.

Proposition 51 *Let $\mathcal{A}$ be an (essentially) small and Cauchy complete category, then $\mathbf{Fam}_f\mathcal{A}$ is small and Cauchy complete.*

Proof. The category $\mathbf{Fam}_f\mathcal{A}$ is small. For it, for a given finite set I, there are $(Ob\mathcal{A})^I$ of such functors $f : I \longrightarrow \mathcal{A}$ and, of course, $(Ob\mathcal{A})^I$ is small, $\mathcal{A}$ being small. So, the objects of $\mathbf{Fam}_f\mathcal{A}$ are in bijection with $\coprod_I (Ob\mathcal{A})^I$, for I varying in the category of finite sets, which is (essentially) small.
Consider now an idempotent $(a, \alpha) : (I, f) \longrightarrow (I, f)$ in $\mathbf{Fam}_f\mathcal{A}$. The condition $(a, \alpha).(a, \alpha) = (a, \alpha)$ is equivalent to:

$$\begin{cases} a.a = a & \text{,that is, } a \text{ is an idempotent in } \mathbf{set} \\ \alpha_{a(i)}.\alpha_i = \alpha_i & \forall i \in I \end{cases} \tag{2.3}$$

The idempotent a splits in **set** and its splitting is given by the equalizer of the pair of arrows a and 1_I in **set**, as follows:

$$\begin{tikzcd} J \arrow[r, tail, "c"] & I \arrow[r, shift left, "a"] \arrow[r, shift right, "1_I"'] & I \\ & I \arrow[ul, dotted, "b"] \arrow[u, "a"'] & \end{tikzcd}$$

Therefore, we get $J = \{j \in I | a(j) = j\}$ and, in fact, the two equalities $c.b = a$ and $b.c = 1_J$ (c being a monomorphism) hold. Now, if we consider just the elements $j \in J$, the second equation in 2.3 precisely says that $\alpha_j : f(j) \longrightarrow f(j)$ is an idempotent in $\mathcal{A}$, and, then, it splits, since $\mathcal{A}$ is Cauchy complete. Let us consider its splitting:

$$\begin{tikzcd} X_j \arrow[rr, tail, "s_j"] & & f(j) \arrow[r, shift left, "\alpha_j"] \arrow[r, shift right, "1_{f(j)}"'] & f(j) \\ & f(j) \arrow[ul, dotted, "t_j"] \arrow[ur, "\alpha_j"'] & & \end{tikzcd}$$

where $s_j.t_j = \alpha_j$ and $t_j.s_j = 1_{X_j}$. Hence, we can define a functor $g : J \longrightarrow \mathcal{A}$ by the assignment $j \mapsto X_j$ and in this way we get an object (J, g) of $\mathbf{Fam}_f\mathcal{A}$ and, moreover, two arrows $(b, \beta) : (I, f) \longrightarrow (J, g)$ and $(c, \gamma) : (J, g) \longrightarrow (I, f)$ in $\mathbf{Fam}_f\mathcal{A}$. The first one is given by the functor $b : I \longrightarrow J$ defined, for any $i \in I$, as $b(i) = a(i)$ and the natural transformation $\beta : f \longrightarrow g.b$, whose component at $i \in I$ is given by the composite $\beta_i = t_{a(i)}.\alpha_i : f(i) \longrightarrow X_{a(i)}$. Whereas, the second is determined by the functor $c : J \longrightarrow I$ defined, for any $j \in J$, as $c(j) = j$ and the natural transformation $\gamma : g \longrightarrow f.c$ whose component at $j \in J$ is simply $\gamma_j = s_j$. It only remains to check that $(c, \gamma).(b, \beta) = (a, \alpha)$ and $(b, \beta).(c, \gamma) = 1_{(J,g)}$; namely, that:

- $c.b = a$ and, for any $i \in I$, $\gamma_{b(i)}.\beta_i = \alpha_i$ (using $s_{a(i)}.t_{a(i)} = \alpha_{a(i)}$ and the second equation in 2.3);
- $b.c = 1_{X_j}$ and, for any $j \in J$, $\beta_{c(j)}.\gamma_j = 1_{X_j}$ (which follows by the fact that $\alpha_j.s_j = s_j$, t_j being an epimorphism).

Therefore, we have just shown that (a, α) splits in $\mathbf{Fam}_f\mathcal{A}$ through the object (J, g), that is, $\mathbf{Fam}_f\mathcal{A}$ is Cauchy complete. ∎

Definition 52 *Let $\mathcal{B}$ be a category with finite coproducts. An object $B \in \mathcal{B}$ is* finitely connected *if the corresponding representable functor $\mathcal{B}(B, -) : \mathcal{B} \longrightarrow \mathbf{set}$ preserves finite coproducts.*

For the notion of connectedness, we refer to [13].
Given a small category $\mathcal{B}$ with finite coproducts, we can always consider its full subcategory $\mathcal{A}$ given by finitely connected objects. By studying the Cauchy completeness of $\mathcal{A}$, we are led to analyse retracts of finite colimit-preserving functors and state the following Proposition, which is a slightly more general version of Corollary 39:

Corollary 53 *Consider two functors $F, G : \mathcal{B} \longrightarrow \mathbf{set}$, with $\mathcal{B}$ a small category with finite colimits. If F preserves finite colimits and G is a retract of F, then G preserves finite colimits.*

Proof. It follows as a consequence of Proposition 20, while choosing as diagrams S functors defined on finite categories. ■

As a consequence of Corollary 53, we can establish the following fact:

Proposition 54 *Let $\mathcal{B}$ be a (essentially) small category with finite coproducts and Cauchy complete, then the full subcategory $\mathcal{A}$ of finitely connected objects of $\mathcal{B}$ is Cauchy complete.*

Proof. Let $e : A \longrightarrow A$ be an idempotent on an object A of $\mathcal{A}$, that is, A is finitely connected and $e.e = e$. Since $\mathcal{B}$ is Cauchy complete, e regarded, in particular, as an idempotent on an object of $\mathcal{B}$, splits through an object $X \in \mathcal{B}$ and two arrows $s : X \longrightarrow A$ and $t : A \longrightarrow X$, satisfying $s.t = e$ and $t.s = 1_X$. In particular, X can be thought of as a retract of A, which is a finitely connected object of $\mathcal{B}$, namely the corresponding representable functor $\mathcal{B}(A, -)$ preserves finite coproducts. Therefore, it follows from Corollary 53 that $\mathcal{B}(X, -)$ does so, that is, X is a finitely connected object of $\mathcal{B}$. This proves that the idempotent e factors in fact through some finitely connected object X; stated otherwise, idempotents in $\mathcal{A}$ split and, therefore, $\mathcal{A}$ is Cauchy complete. ■

The reason why we have treated the notion of finitely connected object in this work is made clear at this point: we can, in fact, describe internally categories of the form $\mathbf{Fam}_f$-completion of some other category in terms of their finitely connected objects. We acknowledge that the following result is simply the finitary version of Proposition 6.1.5 in [13]:

Proposition 55 *A small category $\mathcal{B}$ with finite coproducts is of the form $\mathbf{Fam}_f\mathcal{A}$ if and only if any object is a finite coproduct of finitely connected objects. When this is the case, one can choose as $\mathcal{A}$ the full subcategory of finitely connected objects of $\mathcal{B}$.*

Proof. Suppose that $\mathcal{B} \simeq \mathbf{Fam}_f\mathcal{A}$, for a small category $\mathcal{A}$. An object of $\mathcal{B}$ is, then, a pair (I, f), where $f : I \longrightarrow \mathcal{A}$ is the functor which associates with any i the object X_i of $\mathcal{A}$. Recall that $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbf{Fam}_f\mathcal{A}$ is the discrete functor assigning to any X_i the pair $(\{*\}, X_i)$: here we identify the object X_i with the constant functor $X_i : \{*\} \longrightarrow \mathcal{A}$, from the singleton object to $\mathcal{A}$, defined by $\{*\} \mapsto X_i$. By how finite coproducts are computed in $\mathbf{Fam}_f\mathcal{A}$, one can easily check that $(I, f) \simeq \coprod_{i\in I} \eta_{\mathcal{A}} X_i$. It remains to prove that, for any $X \in \mathcal{A}$, $\eta_{\mathcal{A}} X$ is a finitely connected object of $\mathbf{Fam}_f\mathcal{A}$, that is, the corresponding representable functor $\mathbf{Hom}(\eta_{\mathcal{A}} X, -) : \mathbf{Fam}_f\mathcal{A} \longrightarrow \mathbf{set}$ preserves finite coproducts. For it, consider a finite set J and, for any $j \in J$, an object (I_j, f_j) of

$\mathbf{Fam}_f\mathcal{A}$; take then the finite coproduct $\coprod_{j\in J}(I_j, f_j)$ of all these. As recalled above, this is precisely given by the pair $(\coprod_{j\in J} I_j, (f_j)_{j\in J})$, where $\coprod_{j\in J} I_j$ is the disjoint union in **set** of all I_j and $(f_j)_{j\in J}$ is the functor defined on $x \in \coprod_{j\in J} I_j$ as $f_{j'}(x)$, for the unique $j' \in J$ such that $x \in I_{j'}$. Giving an arrow $(a, \alpha) : \eta_{\mathcal{A}}X \longrightarrow \coprod_{j\in J}(I_j, f_j)$ in $\mathbf{Fam}_f\mathcal{A}$ is the same as giving a functor $a : \{*\} \longrightarrow \coprod_{j\in J} I_j$, that is, picking out some $x \in I_{j'}$, and a natural transformation $\alpha : X \longrightarrow (f_j)_{j\in J}.a$, that is, just an arrow $X \longrightarrow f_{j'}(x)$ of $\mathcal{A}$, call it still α. Therefore, (a, α) is, in particular, a morphism $(\{*\}, X) = \eta_{\mathcal{A}}X \longrightarrow (I_{j'}, f_{j'})$ of $\mathbf{Fam}_f\mathcal{A}$, that is, an element of the disjoint union $\coprod_{j\in J} \mathbf{Hom}(\eta_{\mathcal{A}}X, (I_j, f_j))$, as desired.
Conversely, consider a small category $\mathcal{B}$ with finite coproducts and such that any object $B \in \mathcal{B}$ is a finite coproduct $\coprod_{i\in I} X_i$ of finitely connected objects $X_i \in \mathcal{B}$. Let $\mathcal{A}$ be the full subcategory given by the connected objects of $\mathcal{B}$. By the universal property of the $\mathbf{Fam}_f$-completion applied to the inclusion $\iota : \mathcal{A} \longrightarrow \mathcal{B}$, there exists a unique (up to isomorphism) functor $\bar{\iota} : \mathbf{Fam}_f\mathcal{A} \longrightarrow \mathcal{B}$, which preserves finite coproducts and satisfies $\bar{\iota}.\eta_{\mathcal{A}} = \iota$. It is defined on objects (I, f) of $\mathbf{Fam}_f\mathcal{A}$, where $f : I \longrightarrow \mathcal{A}$ is a functor and any $f(i)$ is a finitely connected object of $\mathcal{B}$, as $\bar{\iota}(I, f) = \coprod_{i\in I} f(i)$ and on morphisms $(a, \alpha) : (I, f) \longrightarrow (J, g)$ of $\mathbf{Fam}_f\mathcal{A}$ as the unique arrow $[\alpha_i]_{i\in I}$ given by the universal property of the coproduct, making the following diagram:

$$\begin{tikzcd}
\coprod_{i\in I} f(i) \arrow[r, "{[\alpha_i]_{i\in I}}"] & \coprod_{j\in J} g(j) \\
f(i) \arrow[u, "\sigma_i"] \arrow[r, "\alpha_i"'] & g(a(i)) \arrow[u, "\delta_{a(i)}"']
\end{tikzcd}$$

commute, where σ_i and $\delta_{a(i)}$ are the corresponding coproduct injections in $\mathcal{A}$. Explicitly, $[\alpha_i]_{i\in I}$ is the unique morphism satisfying $[\alpha_i]_{i\in I}.\sigma_i = \delta_{a(i)}.\alpha_i$, for any $i \in I$. We wish to show that $\bar{\iota} : \mathbf{Fam}_f\mathcal{A} \longrightarrow \mathcal{B}$ is an equivalence of categories, namely, $\bar{\iota}$ is full and faithful and essentially surjective on objects, that is, any object B of $\mathcal{B}$ is isomorphic to $\bar{\iota}(I, f)$, for some $(I, f) \in \mathbf{Fam}_f\mathcal{A}$.
$\bar{\iota}$ is faithful. For it, consider two morphisms $(a, \alpha), (b, \beta) : (I, f) \longrightarrow (J, g)$ in $\mathbf{Fam}_f\mathcal{A}$ such that $\bar{\iota}(a, \alpha) = \bar{\iota}(b, \beta)$. From $[\alpha_i]_{i\in I} = [\beta_i]_{i\in I}$ we obtain, for any $i \in I$, a unique arrow $\delta_{a(i)}.\alpha_i = \delta_{b(i)}.\beta_i : f(i) \longrightarrow \coprod_{j\in J} g(j)$ of $\mathcal{B}$; call it t. Since $f(i)$ is finitely connected, we have $\mathcal{B}(f(i), \coprod_{j\in J} g(j)) \simeq \coprod_{j\in J} \mathcal{B}(f(i), g(j))$ and, therefore, t has to factor through some component of the coproduct $\coprod_{j\in J} g(j)$, which needs actually to be unique, being $\coprod_{j\in J} \mathcal{B}(f(i), g(j))$ a disjoint union in **set**. Since t factors simultaneously through $g(a(i))$, via α_i, and $g(b(i))$, via β_i, we

get, by uniqueness of the factorisation, $a(i) = b(i)$ and $\alpha_i = \beta_i$ for any $i \in I$. This precisely means that $(a, \alpha) = (b, \beta)$.
$\bar{\iota}$ is full. For it, consider two objects (I, f) and (J, g) in $\mathbf{Fam}_f\mathcal{A}$ and a morphism $\varphi : \coprod_{i\in I} f(i) \longrightarrow \coprod_{j\in J} g(j)$ in $\mathcal{B}$. Pre-composing φ with the coproduct injections σ_i gives, for any $i \in I$, an arrow $\varphi.\sigma_i : f(i) \longrightarrow \coprod_{j\in J} g(j)$, with $f(i)$ a finitely connected object of $\mathcal{B}$. Therefore, any $\varphi.\sigma_i$ factors through a unique component $g(j)$ of the coproduct $\coprod_{j\in J} g(j)$ as $\delta_j.\varphi_{ij} = \varphi.\sigma_i$, for a unique $j \in J$ and some arrow $\varphi_{ij} : f(i) \longrightarrow g(j)$. The assignment of such a $j \in J$ to any $i \in I$ induces a functor $I \longrightarrow J$, call it a, such that $a(i) := j$. Moreover, we get a natural transformation $\alpha : f \longrightarrow g.a$ defined at $i \in I$ as $\alpha_i := \varphi_{ia(i)} : f(i) \longrightarrow g(a(i))$. Putting them together, we obtain a morphism $(a, \alpha) : (I, f) \longrightarrow (J, g)$ of $\mathbf{Fam}_f\mathcal{A}$ such that $\bar{\iota}(a, \alpha) = \varphi$, by uniqueness of $\bar{\iota}(a, \alpha)$ as arrow satisfying $\delta_{a(i)}.\varphi_{ia(i)} = \delta_{a(i)}.\alpha_i = \varphi.\sigma_i$, for any $i \in I$.
$\bar{\iota}$ is essentially surjective on objects. For it, consider an object B of $\mathcal{B}$, regarded as a finite coproduct $\coprod_{i\in I} X_i$ of finitely connected objects X_i of $\mathcal{B}$. We can define a functor $f : I \longrightarrow \mathcal{A}$ assigning to any $i \in I$ the corresponding object X_i. Therefore, the pair (I, f) belongs to $\mathbf{Fam}_f\mathcal{A}$ and, moreover, it verifies $\bar{\iota}(I, f) = \coprod_{I\in I} \iota(X_i) = \coprod_{i\in I} X_i \simeq B$, as desired. This ends the proof of $\mathcal{B}$ being equivalent to the category $\mathbf{Fam}_f\mathcal{A}$, where $\mathcal{A}$ is its full subcategory of connected objects of $\mathcal{B}$. ■

Let $\mathcal{K}$ be a presheaf category. Recall that an object $K \in \mathcal{K}$ is called *strongly finitely presentable* if the corresponding representable functor $\mathcal{K}(K, -) : \mathcal{K} \longrightarrow \mathbf{set}$ preserves sifted colimits. Taking the collection of strongly finitely presentable objects in $\mathcal{K}$ determines a (small) full subcategory $\mathcal{A}$ of $\mathcal{K}$ with finite coproducts, such that the equivalence of categories $\mathcal{K} \simeq \mathbf{Finprod}[\mathcal{A}^{op}, \mathbf{set}]$ holds. Therefore, any presheaf category is, in particular, a variety. To investigate when the converse is also true, it suffices to put together the previous Propositions 51, 54, 55. In fact, we have just shown:

Corollary 56 *A small Cauchy complete category $\mathcal{B}$ with finite coproducts is of the form $\mathbf{Fam}_f\mathcal{A}$ if and only if any object of $\mathcal{B}$ is a finite coproduct of finitely connected objects. When this is the case, we can choose as $\mathcal{A}$ the full subcategory of $\mathcal{B}$ determined by the finitely connected objects; moreover, $\mathcal{A}$ is Cauchy complete.*

Definition 57 *Let $\mathcal{B}$ be a category with finite products. An object $B \in \mathcal{B}$ is* finitely coconnected *if the corresponding representable functor $\mathcal{B}(-, B) : \mathcal{B}^{op} \longrightarrow \mathbf{set}$ preserves finite coproducts.*

Therefore, accordingly with the general setting, as presented in Proposition 50, and considering the dual completion of $\mathbf{Fam}_f$, we can state:

Proposition 58

1. *the (not full) inclusion* $j^* : \mathbf{PS} \longrightarrow \mathbf{VAR}$ *is right biadjoint to* $j : \mathbf{VAR} \longrightarrow \mathbf{PS}$*;*

2. *An algebraic theory* $\mathcal{B}$*, which, in particular, is a Cauchy complete category, is such that the corresponding variety is equivalent to a presheaf category if and only if each object of* $\mathcal{B}$ *is a finite product of finitely coconnected objects.*

Finally there is one more case still to analyse, which corresponds to choosing as $\mathbb{D}_1$ the empty doctrine and as $\mathbb{D}_2$ the doctrine of finite limits. Having treated separately the previous two cases, we can finally put together the **PS - VAR** comparison and the **VAR - LFP** comparison and establish under which hypothesis a locally finitely presentable category can be thought of as a category of presheaves. Let us mention, first, a preliminary result (see [33], Exercise V.2.1):

Lemma 59 *Let* $\mathcal{A}$ *be a finitely cocomplete category and* $F : \mathcal{A} \longrightarrow \mathcal{B}$ *a functor. If* F *preserves finite coproducts and coequalizers of reflexive graphs, then* F *preserves finite colimits.*

Proof. Since F does preserve finite coproducts, by assumption, we only need to prove that it preserves coequalizers. For it, consider a coequalizer $q : Y \longrightarrow Z$ of the pair of arrows $f, g : X \longrightarrow Y$. By the universal property of the coproduct $X+Y$, we can define two morphisms $< f, 1_Y >$ and $< g, 1_Y >: X + Y \longrightarrow Y$ by:

$$\begin{cases} < f, 1_Y > . \, i_X = f \\ < f, 1_Y > . \, i_Y = 1_Y \end{cases} \tag{2.4}$$

where $i_X : X \longrightarrow X + Y$ and $i_Y : Y \longrightarrow X + Y$ are the canonical coproduct injections (and $1_Y : Y \longrightarrow Y$ is obviously the identity arrow on Y). Moreover, by taking under consideration the arrow i_Y, we get a reflexive graph, as follows:

$$\begin{tikzcd} X + Y \arrow[r, shift left=2, "<f,1_Y>"] \arrow[r, shift right=2, "<g,1_Y>"'] & Y \arrow[l, "i_Y" description] \end{tikzcd}$$

It is easy to verify that q is the coequalizer of f and g if and only if it is the coequalizer of the reflexive graph $< f, 1_Y >$ and $< g, 1_Y >$. Therefore, since F preserves finite coproducts and coequaliers of reflexive graphs, we obtain that the arrow $F(q)$, given by:

$$\begin{tikzcd} F(X) + F(Y) \arrow[r, shift left=2, "<F(f),1_{F(Y)}>"] \arrow[r, shift right=2, "<F(g),1_{F(Y)}>"'] & F(Y) \arrow[l, "i_{F(Y)}" description] \arrow[r, two heads, "F(q)"] & F(Z) \end{tikzcd}$$

is the coequalizer of the reflexive graph $< F(f), 1_{F(Y)} >$ and $< F(g), 1_{F(Y)} >$, which is precisely equivalent to say that $F(q)$ is the coequalizer of the pair $F(f)$ and $F(g)$, by the same property just mentioned. This ends the proof of the functor F preserving coequalizers, and, therefore, all finite colimits. ∎

We recall that an object $C \in \mathcal{C}$ is said to be an *effective projective* if the corresponding representable functor $\mathcal{C}(C, -) : \mathcal{C} \longrightarrow \mathbf{set}$ preserves coequalizers of reflexive graphs. When the category $\mathcal{C}$ happens to have finite colimits, effective projective objects verify an extra property, as stated in the following:

Lemma 60 *Let $\mathcal{C}$ be a finitely cocomplete category. Then, effective projective objects are regular projectives.*

Proof. Consider an effective projective object C in $\mathcal{C}$. By definition, the corresponding representable functor $\mathcal{C}(C, -) : \mathcal{C} \longrightarrow \mathbf{set}$ preserves coequalizers of reflexive graphs. In order to prove that C is moreover a regular projective object, we have to show that the functor $\mathcal{C}(C, -)$ preserves regular epimorphisms. For it, let $r : Y \longrightarrow Z$ be a regular epimorphism in $\mathcal{C}$, that is, r is a coequalizer of a pair of morphisms $f, g : X \longrightarrow Y$ in $\mathcal{C}$. As mentioned in the previous Proposition, r is a coequalizer of f and g if and only if r is a coequalizer of the reflexive graph given by:

$$\begin{tikzcd} X + Y \arrow[r, shift left=2, "<f,1_Y>"] \arrow[r, shift right=2, "<g,1_Y>"'] & Y \arrow[l, "i_Y"' description] \arrow[r, two heads, "r"] & Z \end{tikzcd}$$

where $i_Y : Y \longrightarrow X+Y$ is the canonical coproduct injection. Therefore, $\mathcal{C}(C, r)$ is a coequalizer of the corresponding reflexive graph:

$$\begin{tikzcd} \mathcal{C}(C, X+Y) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathcal{C}(C, Y) \arrow[l] \arrow[r, two heads, "\mathcal{C}(C,r)"] & \mathcal{C}(C, Z) \end{tikzcd}$$

so, in particular, $\mathcal{C}(C, r)$ is a coequalizer, that is, a regular epimorphism. Therefore, we have just shown that $\mathcal{C}(C, -)$ preserves regular epimorphisms. ∎

We can finally state the following self-contained result giving necessary and sufficient conditions on a locally finitely presentable category to be a category of set-valued functors:

Theorem 61 *Let $\mathcal{C}$ be a small and finitely complete category. The following conditions are equivalent:*

1. *$\mathbf{Lex}[\mathcal{C}, \mathbf{set}]$ is equivalent to a presheaf category;*
2. *$\mathcal{C}$ is equivalent to a free completion under finite limits of a small category;*
3. *Any object of $\mathcal{C}$ is a regular subobject of a finite product $\prod_{i \in I} X_i$ such that, for any $i \in I$, the representable functor $\mathcal{C}(-, X_i) : \mathcal{C}^{op} \longrightarrow \mathbf{set}$ preserves finite colimits.*

Proof. Once again we work with the dual of the theory. We write $\mathcal{B} \longrightarrow \mathcal{B}_{rc}$ for the completion under coequalizers of reflexive graphs of a category $\mathcal{B}$ with finite coproducts. One usually refers to it as to Pitts' construction. Whereas, $\mathcal{A} \longrightarrow \mathbf{Fam}_f\mathcal{A}$ still denotes the completion under finite coproducts of a category $\mathcal{A}$. We want to underline explicitly that when a small category $\mathcal{C}$ is of the form $\mathcal{B}_{rc}$ we can always choose as $\mathcal{B}$ its full subcategory given by effective projective objects; analogously, when a small category $\mathcal{B}$ is of the form $\mathbf{Fam}_f\mathcal{A}$ we can always choose as $\mathcal{A}$ its full subcategory given by finitely connected objects. Therefore, starting with a small finitely complete category $\mathcal{C}$, we know by Proposition 44 that $\mathbf{Lex}[\mathcal{C}, \mathbf{set}]$ is equivalent to a variety (namely, $\mathbf{Finprod}[\mathcal{B}^{op}, \mathbf{set}]$) if and only if $\mathcal{C}^{op}$ is of the form $\mathcal{B}_{rc}$, for some small category $\mathcal{B}$ with finite coproducts. Moreover, we know by Proposition 58 that $\mathbf{Finprod}[\mathcal{B}^{op}, \mathbf{set}]$ is equivalent to a presheaf category (namely, $[\mathcal{A}, \mathbf{set}]$) if and only if $\mathcal{B}$ is of the form $\mathbf{Fam}_f\mathcal{A}$, for some small Cauchy complete category $\mathcal{A}$. Thanks to the universal properties of these completions, we can finally put the two characterisations together, as follows:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\eta_{\mathcal{A}}"] \arrow[dr, "F"'] & \mathbf{Fam}_f\mathcal{A} \arrow[r] \arrow[d, "F'"] & (\mathbf{Fam}_f\mathcal{A})_{rc} \arrow[dl, "F''"] \\
& \mathbf{set} &
\end{tikzcd}$$

and get that $\mathbf{Lex}[\mathcal{C}, \mathbf{set}] \simeq [\mathcal{A}, \mathbf{set}]$ if and only if $\mathcal{C}^{op} \simeq (\mathbf{Fam}_f\mathcal{A})_{rc}$, for a small finitely complete category $\mathcal{C}$ and a small Cauchy complete category $\mathcal{A}$. (Note that we make use here of Lemma 59: the functor F' preserving finite coproducts implies that F'' preserves finite coproducts, moreover, it preserves coequalizers of reflexive graphs, and, therefore, finite colimits.) Provided, as we said, that we are working with the dual of the theory, what we have to prove then is the following fact:

> $\mathcal{C} \simeq (\mathbf{Fam}_f\mathcal{A})_{rc}$ for some small Cauchy complete category $\mathcal{A}$ (call it Condition 1′) if and only if (the dual of) Condition 3 holds.

Indeed, the implication $2 \Rightarrow 1$ is obvious, by definition of the completion under finite limits, and the implication $1' \Rightarrow 2$ is proved precisely in [37].

$1' \Rightarrow 3$. Suppose the category $\mathcal{C}$ is equivalent to $(\mathbf{Fam}_f\mathcal{A})_{rc}$, for some small Cauchy complete category $\mathcal{A}$. Then, any object C of $\mathcal{C}$ is a regular quotient of an object B of $\mathcal{B} = \mathbf{Fam}_f\mathcal{A}$, which is an effective projective in $\mathcal{C}$. Moreover, B itself is a finite coproduct $\coprod_{I\in I} X_i$ of objects X_i of $\mathcal{A}$. Any X_i, regarded as object of $\mathcal{B}$, is effective projective in $\mathcal{C}$ (that is, the corresponding representable functor $\mathcal{C}(X_i, -) : \mathcal{C} \longrightarrow \mathbf{set}$ preserves coequalizers of reflexive graphs). On the other hand, any X_i is finitely connected in $\mathcal{B}$ (that is, the representable $\mathcal{B}(X_i, -) : \mathcal{B} \longrightarrow \mathbf{set}$ preserves finite coproducts), so that, we have only to prove that the embedding $\mathcal{B} \longrightarrow \mathcal{B}_{rc}$ preserves finitely connected objects. When this is the case, any X_i is in fact finitely connected in $\mathcal{C}$ and, hence, by Lemma 59, the associated representable functor $\mathcal{C}(X_i, -) : \mathcal{C} \longrightarrow \mathbf{set}$ preserves finite colimits. For it, consider an object $X \in \mathcal{B}$ which is finitely connected and regard it as an object of $\mathcal{C} = \mathcal{B}_{rc}$; we wish to show that $\mathcal{C}(X, -) : \mathcal{C} \longrightarrow \mathbf{set}$ preserves finite coproducts. It suffices, then, to consider the case of a binary coproduct $C_1 + C_2$ of objects C_1, C_2 of $\mathcal{C}$; namely, we have to show that $\mathcal{C}(X, C_1 + C_2) \simeq \mathcal{C}(X, C_1) + \mathcal{C}(X, C_2)$. C_1 and C_2, being objects of $\mathcal{C}$, are precisely coequalizers of reflexive graphs in $\mathcal{B}$ and one can easily check that finite coproducts commute in $\mathcal{C}$ with coequalizers of reflexive graphs, as displayed in the following diagram:

$$\begin{tikzcd}
B'_1 \arrow[r, "j'_1"] \arrow[d, shift right=2, "g_1"'] \arrow[d, shift left=2, "f_1"] & B'_1 + B'_2 \arrow[d, shift right=2] \arrow[d, shift left=2] & B'_2 \arrow[l, "j'_2"'] \arrow[d, shift right=2, "g_2"'] \arrow[d, shift left=2, "f_2"] \\
B_1 \arrow[r, "j_1"'] \arrow[u, "r_1"] \arrow[d, "q_1"'] & B_1 + B_2 \arrow[u] \arrow[d, "q_1+q_2"] & B_2 \arrow[l, "j_2"] \arrow[u, "r_2"] \arrow[d, "q_2"] \\
C_1 \arrow[r, "i_1"'] & C_1 + C_2 & C_2 \arrow[l, "i_2"]
\end{tikzcd}$$

where the upper-middle arrows are, from left to right, precisely $f_1 + f_2$, $r_1 + r_2$ and $g_1 + g_2$. The coproduct $C_1 + C_2$ is, then, the coequalizer of the pair $(f_1 + f_2, g_1 + g_2)$ in $\mathcal{C}$, which is, in fact, a reflexive graph.

Therefore, we can derive:

$$
\begin{aligned}
&\mathcal{C}(X, C_1 + C_2) \simeq \mathcal{C}\left(X, \operatorname{coeq}\left(B_1' + B_2' \underset{g_1+g_2}{\overset{f_1+f_2}{\rightrightarrows}} B_1 + B_2\right)\right)\\
&\simeq \operatorname{coeq}\left(\mathcal{C}(X, B_1' + B_2') \underset{\mathcal{C}(X,g_1+g_2)}{\overset{\mathcal{C}(X,f_1+f_2)}{\rightrightarrows}} \mathcal{C}(X, B_1 + B_2)\right)\\
&\simeq \operatorname{coeq}\left(\mathcal{C}(X, B_1') + \mathcal{C}(X, B_2') \underset{\mathcal{C}(X,g_1)+\mathcal{C}(X,g_2)}{\overset{\mathcal{C}(X,f_1)+\mathcal{C}(X,f_2)}{\rightrightarrows}} \mathcal{C}(X, B_1) + \mathcal{C}(X, B_2)\right)\\
&\simeq \operatorname{coeq}\left(\mathcal{C}(X, B_1') \underset{\mathcal{C}(X,g_1)}{\overset{\mathcal{C}(X,f_1)}{\rightrightarrows}} \mathcal{C}(X, B_1)\right) +\\
&\qquad\qquad + \operatorname{coeq}\left(\mathcal{C}(X, B_2') \underset{\mathcal{C}(X,g_2)}{\overset{\mathcal{C}(X,f_2)}{\rightrightarrows}} \mathcal{C}(X, B_2)\right)\\
&\simeq \mathcal{C}\left(X, \operatorname{coeq}\left(B_1' \underset{g_1}{\overset{f_1}{\rightrightarrows}} B_1\right)\right) + \mathcal{C}\left(X, \operatorname{coeq}\left(B_2' \underset{g_2}{\overset{f_2}{\rightrightarrows}} B_2\right)\right)\\
&\simeq \mathcal{C}(X, C_1) + \mathcal{C}(X, C_2)
\end{aligned}
$$

(where the second and fifth equalities follow from X being an effective projective, the third one comes from X being finitely connected in $\mathcal{B}$ and, finally, the forth equality holds via the commutativity in **set** of finite coproducts with coequalizers of reflexive graphs). Therefore, we have just shown that any object C of $\mathcal{C}$ is a regular quotient of a finite coproduct of objects X_i such that $\mathcal{C}(X_i, -) : \mathcal{C} \longrightarrow \mathbf{set}$ preserves finite colimits. This is precisely the dual of Condition 3.

$3 \Rightarrow 1'$. Suppose that for any object $C \in \mathcal{C}$ there exist a finite family $(X_i)_{i\in I}$ of objects such that $\mathcal{C}(X_i, -) : \mathcal{C} \longrightarrow \mathbf{set}$ preserves finite colimits, and a regular epimorphism r such that $r : \coprod_{i\in I} X_i \longrightarrow C$. We write $\mathcal{A}$ for the full subcategory of $\mathcal{C}$ given by those objects A such that the representable functor $\mathcal{C}(A, -) : \mathcal{C} \longrightarrow \mathbf{set}$ preserves finite colimits, $\mathcal{B}$ for the full subcategory of $\mathcal{C}$ spanned by finite coproducts of objects of $\mathcal{A}$, and $\mathcal{E}$ for the full subcategory of $\mathcal{C}$ given by effective projectives. First note that $\mathcal{A}$ is small and Cauchy complete, since it is closed in $\mathcal{C}$ under retracts by Corollary 53. Moreover, the category $\mathcal{C}$ has enough projectives (that is, for any $C \in \mathcal{C}$ there exist an effective projective E and a regular epimorphism $r : E \longrightarrow C$). For it, any X_i is, in particular, effective projective and in [37] they prove that finite coproducts of effective projectives are effective projective objects. Therefore, we have $\mathcal{C} \simeq \mathcal{E}_{rc}$, as a consequence of Proposition 44. It suffices to show, then, that $\mathcal{E} \simeq \mathbf{Fam}_f\mathcal{A}$. Once again, since any $A \in \mathcal{A}$ is effective projective and effective projectives are closed under finite coproducts, we have that $\mathcal{B}$ is contained in $\mathcal{E}$. In fact, $\mathcal{B}$ is equivalent to $\mathbf{Fam}_f\mathcal{A}$, by Proposition 55. It only remains to prove the inclusion $\mathcal{E} \rightarrowtail \mathcal{B}$. For it, consider an object $E \in \mathcal{E}$; regarded as object of $\mathcal{C}$, we know, by assumption, there exist an object $B \in \mathcal{B}$ and a regular epimorphism $r : B \longrightarrow E$. Since the category $\mathcal{C}$ has finite colimits, it follows from Lemma 60 that E is not only an effective projective but also a regular projective object of $\mathcal{C}$, from which we deduce the existence of an arrow $s : E \longrightarrow B$ such that the identity on E factors through s, as shown in the following diagram:

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[d, "1_E"] \\ B \arrow[r, two heads, "r"'] & E \end{tikzcd}$$

Clearly, E turns up to be a retract of B, and, therefore, $E \in \mathcal{B}$, since $\mathcal{B}$ (which is precisely $\mathbf{Fam}_f\mathcal{A}$) is Cauchy complete by Proposition 51, $\mathcal{A}$ being Cauchy complete. Then, we have proved that $\mathcal{E} \simeq \mathcal{B}$, which implies that $\mathcal{E} \simeq \mathbf{Fam}_f\mathcal{A}$ and, therefore, that $\mathcal{C} \simeq (\mathbf{Fam}_f\mathcal{A})_{rc}$, as desired. This ends the proof of the equivalence of all conditions listed in the statement of the Theorem. ■

2.3 A characterisation in terms of strong generators

Locally finitely presentable categories can be defined as cocomplete categories with a small (up to isomorphism) set of finitely presentable objects such that any object is a filtered colimit of finitely presentable ones. We have already explained how by replacing the doctrine of finite limits by an arbitrary sound doctrine $\mathbb{D}$ we are led to the definition of locally $\mathbb{D}$-presentable categories, namely those cocomplete categories with a small set of $\mathbb{D}$-presentable objects such that any object is a $\mathbb{D}$-filtered colimit of $\mathbb{D}$-presentable ones. A basic theorem, due to Gabriel and Ulmer, asserts that a cocomplete category is locally finitely presentable if and only if it has a strong generator consisting of finitely presentable objects (see [23, 4, 11]). Not surprisingly, a similar result holds in the framework of (finitary, multisorted) varieties: a category is equivalent to a variety if and only if it is cocomplete and has a strong generator consisting of strongly finitely presentable objects. It is clear that locally finitely presentable categories and varieties are special instances of locally $\mathbb{D}$-presentable categories, so it seems natural to look for a generalisation of the previous result to the case of locally $\mathbb{D}$-presentable categories. This is the aim of the section.

The $\mathbb{D}^{op}$-colimit completion

For what it concerns $\mathbb{D}^{op}$-colimit completions, the further assumption of soundness on the doctrine $\mathbb{D}$ is not needed. The definition can be stated in the general setting of any limit doctrine. Nevertheless, the assumption will be require again as soon as we will be talking about $\mathbb{D}$-presentable objects in the next subsection.

Definition 62 *Let $\mathbb{D}$ be a doctrine and let $\mathcal{K}$ be a $\mathbb{D}^{op}$-cocomplete category. For any subcategory $\mathcal{A}$ of $\mathcal{K}$, we define the $\mathbb{D}^{op}$-*colimit completion* of $\mathcal{A}$ as the smallest full subcategory $\tilde{\mathcal{A}}$ of $\mathcal{K}$ which contains $\mathcal{A}$ and is closed in $\mathcal{K}$ under $\mathbb{D}^{op}$-colimits. (Therefore, $\tilde{\mathcal{A}}$ is $\mathbb{D}^{op}$-cocomplete.)*

In other words, the $\mathbb{D}^{op}$-colimit completion of $\mathcal{A}$ is given by the intersection $\tilde{\mathcal{A}} = \bigcap \mathcal{B}$, where the categories $\mathcal{B}$ are closed in $\mathcal{K}$ under $\mathbb{D}^{op}$-colimits and such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{K}$.
Explicitly, $\tilde{\mathcal{A}}$ is the iterative closure of $\mathcal{A}$ in $\mathcal{K}$ under $\mathbb{D}^{op}$-colimits. The idea is the following. We denote by $\mathcal{A}_0$ the full subcategory of $\mathcal{K}$ generated by $\mathcal{A}$. Since it is not necessarily closed under $\mathbb{D}^{op}$-colimits, for any category $\mathcal{D} \in \mathbb{D}$ and any functor $F : \mathcal{D}^{op} \longrightarrow \mathcal{A}_0 \longrightarrow \mathcal{K}$, we add to $\mathcal{A}_0$ a colimit of F and we define $\mathcal{A}_1$ to be the full subcategory of $\mathcal{K}$ generated by the union of $\mathcal{A}_0$ and all colimits arising in this way. There is no reason a priori why $\mathcal{A}_1$ should be $\mathbb{D}^{op}$-cocomplete, therefore, we add to $\mathcal{A}_1$ a colimit of any diagram of the form $F : \mathcal{D}^{op} \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{K}$, for any $\mathcal{D} \in \mathbb{D}$, to get $\mathcal{A}_2$ as the full subcategory of $\mathcal{K}$ generated by it. So, it is clear that $\tilde{\mathcal{A}}$ can be inductively constructed in the following way:

- $\mathcal{A}_0$ is $\mathcal{A}$, regarded as full subcategory of $\mathcal{K}$;
- $\mathcal{A}_{\lambda+1}$ is obtained by adding to $\mathcal{A}_\lambda$ a colimit of any functor $F : \mathcal{D}^{op} \longrightarrow \mathcal{K}$ which factors through $\mathcal{A}_\lambda$, for any $\mathcal{D} \in \mathbb{D}$;
- if λ is a limit ordinal, then $\mathcal{A}_\lambda = \bigcup_{\xi<\lambda} \mathcal{A}_\xi$. (Note that $\mathcal{A}_\lambda$ is a full subcategory of $\mathcal{K}$ and if $\mathcal{A}_\lambda$ is $\mathbb{D}^{op}$-cocomplete, for some ordinal λ, then $\mathcal{A}_\lambda = \mathcal{A}_\mu$ for any other ordinal μ such that $\lambda \leq \mu$.)

It can be proved that there exists an initial ordinal $\hat{\lambda}$ such that $\tilde{\mathcal{A}} = \mathcal{A}_{\hat{\lambda}}$. We refer to Theorem 5 and Lemma 6 in [9] for more details. We just recall that an ordinal number is called an *initial ordinal* if every smaller ordinal has a smaller cardinality. For instance, the ω_α ordinal numbers are just the transfinite initial ordinals.

The following result is due to Ehresmann (see [21]):

Proposition 63 *For any small subcategory $\mathcal{A}$ of $\mathcal{K}$, the $\mathbb{D}^{op}$-colimit completion $\tilde{\mathcal{A}}$ of $\mathcal{A}$ is essentially small.*

Proof. The proof is done by transfinite induction. $\mathcal{A}_0$ is a small category. If $\mathcal{A}_\lambda$ is small, then $\mathcal{A}_\lambda \cup \{\operatorname{colim} F | F : \mathcal{D}^{op} \longrightarrow \mathcal{A}_\lambda \longrightarrow \mathcal{K}, \mathcal{D} \in \mathbb{D}\}$ has to be so, since the category of functors between $\mathcal{D}^{op}$ and $\mathcal{A}_\lambda$ is small, as long as $\mathcal{A}_\lambda$ is small; therefore, $\mathcal{A}_{\lambda+1}$ is a small category. If λ is a limit ordinal and, for any $\xi \leq \lambda$, $\mathcal{A}_\xi$ is small, then $\mathcal{A}_\lambda$, which is nothing but $\bigcup_{\xi<\lambda} \mathcal{A}_\xi$, is so. By transfinite induction carried to $\hat{\lambda}$, we can establish that $\tilde{\mathcal{A}} = \mathcal{A}_{\hat{\lambda}}$ is a small category. ■

Strong generators play an important role

For an arbitrary category, we recall from [10] the notions of generator, strong generator and dense generator. It is well-known that in general these are related in the following way:

$$\text{dense generator} \Rightarrow \text{strong generator} \tag{2.5}$$

$$\text{strong generator} \Rightarrow \text{generator} \tag{2.6}$$

We will discover that in our setting one can construct a dense generator, starting from a strong one.
Let us make clear that, if S is a set and G an object of a category, then $S \circ G$ is the S-indexed *copower* of G, namely the coproduct $\coprod_S G$ with all factors equal to G, for all $s \in S$.

Definition 64 *Let $\mathcal{A}$ be a category. A small set $\mathcal{G}$ of objects of $\mathcal{A}$ is* small generator *if, for any object $A \in \mathcal{A}$, the two following conditions hold:*

1. *the coproduct $\coprod_{G\in\mathcal{G}} \mathcal{A}(G, A) \circ G$ exists;*

2. *the canonical morphism $\gamma_A : \coprod_{G\in\mathcal{G}} \mathcal{A}(G, A) \circ G \longrightarrow A$ is an epimorphism. Explicitly, the morphism γ_A is defined by the universal property of the coproduct, as in the following diagram in $\mathcal{A}$:*

$$\begin{tikzcd}
\coprod_{G\in\mathcal{G}} \coprod_{g:G\to A} G \arrow[rr, "\gamma_A"] & & A \\
\coprod_{g:G\to A} G \arrow[u, "\rho_G"] & & \\
G \arrow[u, "\delta_g"] \arrow[uurr, "g"'] & &
\end{tikzcd}$$

that is, γ_A is the unique arrow satisfying $\gamma_A.\rho_G.\delta_g = g$, for any object $G \in \mathcal{G}$ and any morphism $g : G \longrightarrow A$ in $\mathcal{A}$.

Note that Condition 2 in the previous definition can be equivalently stated by saying that two parallel arrows $x, y : A \longrightarrow B$ in $\mathcal{A}$ are equal if and only if they are equal when pre-composed with any map $g : G \longrightarrow A$, that is, $x.g = y.g$ for all $g \in \mathcal{A}(G, A)$ and for all $G \in \mathcal{G}$. This formulation has the peculiar advantage of not requiring the existence in $\mathcal{A}$ of those coproducts mentioned in Condition 1.

Definition 65 *Let $\mathcal{A}$ be a category with coproducts and $\mathcal{G}$ a small set of objects of $\mathcal{A}$. The set $\mathcal{G}$ is a* strong generator *if, for any object A of $\mathcal{A}$, the canonical morphism*

$$\gamma_A : \coprod_{G \in \mathcal{G}} \mathcal{A}(G, A) \circ G \longrightarrow A$$

is a strong epimorphism of $\mathcal{A}$.

The definition of strong generator can be equivalently formulated in absence of coproducts, as follows:

Definition 66 *Let $\mathcal{A}$ be a category. A small set $\mathcal{G}$ of objects of $\mathcal{A}$ is a* strong generator *if the corresponding family of representable functors $(\mathcal{A}(G, -) : \mathcal{A} \longrightarrow \mathbf{set})_{G \in \mathcal{G}}$ collectively reflects isomorphisms; namely, given an arrow $f : A \longrightarrow A'$ of $\mathcal{A}$, from the fact that, for any $G \in \mathcal{G}$, $\mathcal{A}(G, f)$ is an isomorphism follows that f is an isomorphism.*

Definition 67 *Let $\mathcal{A}$ be a category and $\mathcal{G}$ a small set of objects of $\mathcal{A}$. We write $\mathcal{G}\downarrow A$ for the full subcategory of $\mathcal{A}\downarrow A$ generated by the pairs (G, g) of the form $g : G \longrightarrow A$, where $G \in \mathcal{G}$. The set $\mathcal{G}$ is a* dense generator *if, for any $A \in \mathcal{A}$, the colimit of the canonical diagram:*

$$\Gamma_A : \mathcal{G}\downarrow A \longrightarrow \mathcal{A}\ ,\ (G, g : G \longrightarrow A) \mapsto G$$

is precisely $(A, (g)_{(G,g) \in \mathcal{G}\downarrow A})$.

It is important to remark that in the case of a full reflective subcategory, (strong) generators are carried on by the reflection to (strong) generators. Unfortunately, this does not work for dense generators.

Lemma 68 *Let $\mathcal{A}$ be a locally small category with coproducts and $i : \mathcal{B} \longrightarrow \mathcal{A}$ a full reflective subcategory of $\mathcal{A}$, whose reflection is denoted by r (namely, $r \dashv i$). If $\mathcal{G}$ is a strong generator for $\mathcal{A}$, then $r\mathcal{G} = \{rG | G \in \mathcal{G}\}$ is a strong generator for $\mathcal{B}$.*

Proof. For any $B \in \mathcal{B}$, we need to show that the canonical morphism

$$\gamma_B : \coprod_{G \in \mathcal{G}} \mathcal{B}(rG, B) \circ rG \longrightarrow B$$

is a strong epimorphism in $\mathcal{B}$. The adjunction $r \dashv i$ already gives $\mathcal{B}(rG, B) \simeq \mathcal{A}(G, iB)$ and, r, being a left adjoint functor, preserves coproducts, which means that $\coprod_{G \in \mathcal{G}} \mathcal{B}(rG, B) \circ rG \simeq \coprod_{G \in \mathcal{G}} \mathcal{A}(G, iB) \circ rG \simeq r(\coprod_{G \in \mathcal{G}} \mathcal{A}(G, iB) \circ G)$. Since the counit for the adjunction $r \dashv i$ is at each component an isomorphism, we have $B \simeq riB$, for any $B \in \mathcal{B}$, and γ_B is nothing but the image through r of the canonical morphism

$$\gamma_{iB} : \coprod_{G \in \mathcal{G}} \mathcal{A}(G, iB) \circ G \longrightarrow iB$$

which is a strong epimorphism in $\mathcal{A}$ by hypothesis. Therefore, $\gamma_B \simeq r\gamma_{iB}$ and again r, being a left adjoint functor, preserves strong epimorphisms. We derive, then, that γ_B is a strong epimorphism in $\mathcal{B}$, as desired. ■

Example 69

1. Let $\mathcal{A}$ be a small category. The set $(\mathcal{A}(A, -))_{A \in \mathcal{A}}$ of representable functors is a small generator for $[\mathcal{A}, \mathbf{set}]$. More precisely, it is a dense generator for $[\mathcal{A}, \mathbf{set}]$.

Having recalled these preliminary facts about generators, we can go back to our treatment of $\mathbb{D}^{op}$-colimit completions and, for a locally small category $\mathcal{K}$ with a strong generator, consider precisely the $\mathbb{D}^{op}$-colimit completion of its full subcategory determined by the generator.

Proposition 70 *Let $\mathbb{D}$ be a doctrine. Let $\mathcal{K}$ be a locally small and cocomplete category, with a strong generator $\mathcal{G}$ consisting of $\mathbb{D}$-presentable objects of $\mathcal{K}$. Write $\mathcal{A}$ for the full subcategory of $\mathcal{K}$ generated by $\mathcal{G}$. Then, the $\mathbb{D}^{op}$-colimit completion $\tilde{\mathcal{A}}$ of $\mathcal{A}$ exists and satisfies the following properties:*

1. *$\tilde{\mathcal{A}}$ is essentially small;*
2. *$\tilde{\mathcal{A}}$ is closed under $\mathbb{D}^{op}$-colimits;*
3. *every object of $\tilde{\mathcal{A}}$ is $\mathbb{D}$-presentable in $\mathcal{K}$.*

Proof. Conditions 1 and Condition 2 follow immediately from the definition of $\tilde{\mathcal{A}}$ as the $\mathbb{D}^{op}$-colimit completion of $\mathcal{A}$, as proved in Proposition 63.

Condition 3. We have seen that a $\mathbb{D}^{op}$-colimit completion $\tilde{\mathcal{A}}$ of $\mathcal{A}$ is of the form $\mathcal{A}_{\hat{\lambda}}$ for a certain $\hat{\lambda}$, namely $\tilde{\mathcal{A}} = \mathcal{A}_{\hat{\lambda}} = \bigcup_{\xi<\hat{\lambda}} \mathcal{A}_\xi$. Moreover, we know that $\mathcal{A}_{\lambda+1}$ is the full subcategory of $\mathcal{K}$ generated by $\mathcal{A}_\lambda \cup \{\operatorname{colim} F | F : \mathcal{D}^{op} \longrightarrow \mathcal{A}_\lambda \longrightarrow \mathcal{K}, \mathcal{D} \in \mathbb{D}\}$: so, any object of $\mathcal{A}$ being $\mathbb{D}$-presentable by hypothesis and using Lemma 5 (which precisely states that a $\mathbb{D}^{op}$-colimit of $\mathbb{D}$-presentable objects is again $\mathbb{D}$-presentable), we deduce that any object of $\mathcal{A}_{\lambda+1}$ is $\mathbb{D}$-presentable in $\mathcal{K}$. If λ is a limit ordinal, the objects of $\mathcal{A}_\xi$ are $\mathbb{D}$-presentable in $\mathcal{K}$, for any $\xi < \lambda$, then $\mathcal{A}_\lambda = \bigcup_{\xi<\lambda} \mathcal{A}_\xi$ (which is nothing but the union of the small sets of objects of all $\mathcal{A}_\xi$) has the property that any of its objects is $\mathbb{D}$-presentable. Therefore, by transfinite induction carried to $\hat{\lambda}$, any object of $\tilde{\mathcal{A}}$ is $\mathbb{D}$-presentable in $\mathcal{K}$. ∎

We underline that until this point while dealing with $\mathbb{D}^{op}$-colimit completions the hypothesis on the doctrine $\mathbb{D}$ to be sound was not necessary. In the following Proposition we finally need it:

Proposition 71 *Under the conditions of Proposition 70 and the further assumption on the doctrine $\mathbb{D}$ to be sound, the following properties hold:*

1. *for any object $K \in \mathcal{K}$, the category $\tilde{\mathcal{A}}\downarrow K$ is $\mathbb{D}$-filtered;*
2. *the set $\tilde{\mathcal{A}}$ is a dense generator in $\mathcal{K}$.*

Proof. Condition 1. By Proposition 11, proving that the category $\tilde{\mathcal{A}} \downarrow K$ is $\mathbb{D}$-filtered reduces to show that it is $\mathbb{D}^{op}$-cocomplete, since the doctrine $\mathbb{D}$ is sound. Therefore, we wish to show that $\tilde{\mathcal{A}} \downarrow K$ is $\mathbb{D}^{op}$-cocomplete, $\tilde{\mathcal{A}}$ being so. For it, consider a category $\mathcal{D} \in \mathbb{D}$ and a functor $F : \mathcal{D}^{op} \longrightarrow \tilde{\mathcal{A}} \downarrow K$, which associates with any $D \in \mathcal{D}^{op}$ a pair $(A_D, a_D : A_D \longrightarrow K)$, where $A_D \in \tilde{\mathcal{A}}$. By taking the $\mathbb{D}^{op}$-colimit of the composite $\Gamma_K.F : \mathcal{D}^{op} \longrightarrow \tilde{\mathcal{A}} \downarrow K \longrightarrow \tilde{\mathcal{A}}$, where $\Gamma_K : \tilde{\mathcal{A}} \downarrow K \longrightarrow \tilde{\mathcal{A}}$ is the canonical forgetful functor, we get an object $\operatorname{colim}_{D\in\mathcal{D}^{op}} A_D \in \tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}$ being $\mathbb{D}^{op}$-cocomplete, and, therefore, a unique arrow $a : \operatorname{colim}_{D\in\mathcal{D}^{op}} A_D \longrightarrow K$ satisfying $a.\sigma_D = a_D$, for any $D \in \mathcal{D}^{op}$. Here the morphisms $\sigma_D : A_D \longrightarrow \operatorname{colim}_{D\in\mathcal{D}^{op}} A_D$ indicate the canonical colimit injections. It is easy to prove that the $\mathbb{D}^{op}$-colimit of F in $\tilde{\mathcal{A}} \downarrow K$ is given by $((\operatorname{colim}_{D\in\mathcal{D}^{op}} A_D, a), (\sigma_D)_{D\in\mathcal{D}^{op}})$, just by using that $\operatorname{colim}_{D\in\mathcal{D}^{op}} A_D$ is a colimit in $\tilde{\mathcal{A}}$ and the definition of the canonical comparison morphism a.

Condition 2. We want to show that for any object $K \in \mathcal{K}$ the colimit of the forgetful functor $\tilde{\mathcal{A}} \downarrow K \longrightarrow \tilde{\mathcal{A}} \longrightarrow \mathcal{K}$ (which will still call Γ_K), which sends the pair $(A, a : A \longrightarrow K)$, where $A \in \tilde{\mathcal{A}}$, to A, is precisely $(K, (a : A \longrightarrow K)_{(A,a)\in\tilde{\mathcal{A}}\downarrow K})$. Consider, then, the colimit

$(L, (s_{(A,a)})_{(A,a)\in\tilde{\mathcal{A}}\downarrow K})$ of such a diagram Γ_K in $\mathcal{K}$. Since the morphisms $a : \Gamma_K(A,a) \longrightarrow K$, for (A,a) varying in $\tilde{\mathcal{A}} \downarrow K$, constitute a cocone on Γ_K, we get an essentially unique factorization $\lambda : L \longrightarrow K$, such that $\lambda.s_{(A,a)} = a$, for any $(A,a) \in \tilde{\mathcal{A}} \downarrow K$. We will prove that λ is an isomorphism. Consider first the diagram:

$$\begin{tikzcd} \coprod_{(G,g)} G \arrow[r, "v"] \arrow[dr, "u"'] & L \arrow[d, "\lambda"] \\ & K \end{tikzcd}$$

where the coproduct is indexed by all pairs (G,g) with $G \in \mathcal{G}$ and $g : G \longrightarrow K$ in $\mathcal{K}$. Denote by $\rho_{(G,g)} : G \longrightarrow \coprod G$ the coproduct injections in $\mathcal{K}$; u is the unique map such that $u.\rho_{(G,g)} = g$ and v is the unique map such that $v.\rho_{(G,g)} = s_{(G,g)}$, both of which we get by the universal property of the coproduct. We remark that u is precisely the canonical morphism $\gamma_K : \coprod_{G\in\mathcal{G}} \mathcal{K}(G,K) \circ G \longrightarrow K$. Since, for any (G,g), it holds $\lambda.v.\rho_{(G,g)} = \lambda.s_{(G,g)} = g = u.\rho_{(G,g)}$, by uniqueness of u, we obtain $\lambda.v = u$. From this equality we derive that λ is a strong epimorphism, u being a strong epimorphism by definition of $\mathcal{G}$ strong generator. It only remains, then, to prove that λ is a monomorphism. Consider two arrows $x, y : X \longrightarrow L$ in $\mathcal{K}$ such that $\lambda.x = \lambda.y$. To prove that $x = y$ is to prove that $x.z = y.z$, for any object $G \in \mathcal{G}$ and any arrow $z : G \longrightarrow X$ in $\mathcal{K}$, $\mathcal{G}$ being a generator for $\mathcal{K}$. By Condition 1, the category $\tilde{\mathcal{A}}\downarrow K$ is $\mathbb{D}$-filtered, which implies that $(L, s_{(A,a)})_{(A,a)\in\tilde{\mathcal{A}}\downarrow K}$ is a $\mathbb{D}$-filtered colimit. Since any $G \in \mathcal{G}$ is a $\mathbb{D}$-presentable object of $\mathcal{K}$, we have:

$$\mathcal{K}(G, \operatorname*{colim}_{(A,a)\in\tilde{\mathcal{A}}\downarrow K} \Gamma_K(A,a)) \simeq \operatorname*{colim}_{(A,a)\in\tilde{\mathcal{A}}\downarrow K} \mathcal{K}(G,A)$$

Therefore, both of the arrows $x.z$ and $y.z$ factor through some term of the colimit, that is, there exist $(A,a) \in \tilde{\mathcal{A}} \downarrow K$ and $x' : G \longrightarrow A$ such that $s_{(A,a)}.x' = x.z$ and analogously, there exist $(B,b) \in \tilde{\mathcal{A}} \downarrow K$ and $y' : G \longrightarrow B$ such that $s_{(B,b)}.y' = y.z$. This gives rise to an object $(G, \lambda.x.z) = (G, \lambda.y.z)$ in $\tilde{\mathcal{A}} \downarrow K$, since $\lambda.x = \lambda.y$, and two arrows $x' : (G, \lambda.x.z) \longrightarrow (A,a)$ and $y' : (G, \lambda.y.z) \longrightarrow (B,b)$ in $\tilde{\mathcal{A}}\downarrow K$, having $a.x' = \lambda.s_{(A,a)}.x' = \lambda.x.z$ and $b.y' = \lambda.s_{(B,b)}.y' = \lambda.y.z$ respectively. Hence, we finally can establish $x.z = s_{(A,a)}.x' = s_{(G,\lambda.x.z)} = s_{(G,\lambda.y.z)} = s_{(B,b)}.y' = y.z$, as desired. ∎

We underline the fact that the objects of $\tilde{\mathcal{A}}$ not only are $\mathbb{D}$-presentable in $\mathcal{K}$, but also that we can derive from them all the $\mathbb{D}$-presentable objects of $\mathcal{K}$, as prescribed in the following:

Corollary 72 *Under the conditions of Proposition 71, any $\mathbb{D}$-presentable object of $\mathcal{K}$ is a retract of an object of $\tilde{\mathcal{A}}$.*

Proof. Let $K \in \mathcal{K}$ be a $\mathbb{D}$-presentable object. By Proposition 71, we can express K as a $\mathbb{D}$-filtered colimit of the corresponding canonical diagram $K \simeq \mathrm{colim}_{(A,a)\in\tilde{\mathcal{A}}\downarrow K} A$. We immediately get that $\mathcal{K}(K,K) \simeq \mathcal{K}(K, \mathrm{colim}_{(A,a)\in\tilde{\mathcal{A}}\downarrow K} A) \simeq \mathrm{colim}_{(A,a)\in\tilde{\mathcal{A}}\downarrow K} \mathcal{K}(K,A)$ and, therefore, the identity $1_K : K \longrightarrow K$ has to factor through some term of the colimit $\mathrm{colim}_{(A,a)\in\tilde{\mathcal{A}}\downarrow K} \mathcal{K}(K,A)$, yielding $(B,b) \in \tilde{\mathcal{A}}\downarrow K$ and $m : K \longrightarrow B$ in $\mathcal{K}$ such that $b.m = 1_K$. Thus, K is a retract of B. ∎

We can finally state our characterisation Theorem for locally $\mathbb{D}$-presentable categories in terms of strong generators. This result can be found in [17].

Theorem 73 *Let $\mathbb{D}$ be a sound doctrine and $\mathcal{K}$ a locally small category. The following conditions are equivalent:*

1. *$\mathcal{K}$ is locally $\mathbb{D}$-presentable;*
2. *$\mathcal{K}$ is cocomplete and has a strong generator $\mathcal{G}$ consisting of $\mathbb{D}$-presentable objects of $\mathcal{K}$.*

Proof. $1 \Rightarrow 2$. Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category. By Theorem 26, $\mathcal{K}$ is of the form $\mathbb{D}$-**cont**$[\mathcal{A}, \mathbf{set}]$, for some small and $\mathbb{D}$-complete category $\mathcal{A}$, that is $\mathcal{K}$ is equivalent to a full reflective subcategory of the functor category $[\mathcal{A}, \mathbf{set}]$. Since the set of representable functors is a strong generator in $[\mathcal{A}, \mathbf{set}]$, the set of their reflections is a strong generator in $\mathcal{K}$, by Lemma 68. Moreover, a representable functor is an absolutely presentable object of $[\mathcal{A}, \mathbf{set}]$ (that is, the corresponding representable functor preserves all colimits, in particular, then, the $\mathbb{D}$-filtered ones) and the inclusion of $\mathcal{K} \simeq \mathbb{D}$-**cont**$[\mathcal{A}, \mathbf{set}]$ in $[\mathcal{A}, \mathbf{set}]$ is a $\mathbb{D}$-accessible functor, by Lemma 29. Therefore, by Lemma 40, we can conclude that the reflection of a representable functor is a $\mathbb{D}$-presentable object in $\mathcal{K}$. Namely, the cocomplete category $\mathcal{K}$ has a strong generator, obtained by taking the reflection of all representable functors of $[\mathcal{A}, \mathbf{set}]$. Moreover, they are $\mathbb{D}$-presentable objects of $\mathcal{K}$.

$2 \Rightarrow 1$. Let $\mathcal{K}$ be a cocomplete category with a strong generator $\mathcal{G}$ consisting of $\mathbb{D}$-presentable objects of $\mathcal{K}$. We denote by $\mathcal{A}$ the full subcategory of $\mathcal{K}$ generated by $\mathcal{G}$ and we consider its $\mathbb{D}^{op}$-colimit completion in $\mathcal{K}$. We get in this way an (essentially) small category $\tilde{\mathcal{A}}$ consisting of $\mathbb{D}$-presentable objects, by Proposition 70, such that any object of $\mathcal{K}$ is a $\mathbb{D}$-filtered colimit of objects of $\tilde{\mathcal{A}}$, by Proposition 71. Thus, $\mathcal{K}$ is locally $\mathbb{D}$-presentable. ∎

Example 74

1. Let $\mathbb{D}$ be the doctrine of finite categories. Then, locally $\mathbb{D}$-presentable precisely means locally finitely presentable, and Theorem 73 is the classical result on locally finitely presentable categories, due to Gabriel and Ulmer and quoted at the beginning of Section 2.3, that we wanted to generalise.

2. Let $\mathbb{D}$ be the doctrine of finite discrete categories. Then, locally $\mathbb{D}$-presentable categories are multisorted finitary varieties (see [5]). Theorem 73 asserts that a category is equivalent to a variety if and only if it is cocomplete and has a strong generator consisting of strongly finitely presentable objects, that is, objects G such that the corresponding representable functor $\mathbf{Hom}(G, -)$ preserves sifted colimits (see [2, 5]).

3. Let $\mathbb{D}$ be the empty doctrine. Then, locally $\mathbb{D}$-presentable categories are precisely presheaf categories. Theorem 73 characterises them as those cocomplete categories having a strong generator consisting of *absolutely presentable* objects, that is, objects G such that the corresponding representable functor $\mathbf{Hom}(G, -)$ preserves all small colimits.

Remark 75 Up to minor modification, the characterisation of presheaf categories presented in Example 74 is due to Bunge [14]. The proof is based on the special adjoint functor Theorem and on Beck monadicity Theorem. Whereas the characterisation of varieties given in Example 74 is due to Diers [20]. This proof is based on general proprieties of dense functors and ours is quite similar to this one. A different proof for the characterisation of varieties has also been proposed by Pedicchio and Wood [38]. This consists in proving that equivalence relations are effective and makes use of the original characterisation of varieties due to Lawvere [30].

Consequences

As simple consequences of the new characterisation of locally $\mathbb{D}$-presentable categories in terms of strong generators given in Theorem 73, we can mention some nice results.

Proposition 76 *Any presheaf category is locally $\mathbb{D}$-presentable.*

Proof. Obviously, any functor category $[\mathcal{A}, \mathbf{set}]$, for a small category $\mathcal{A}$, is cocomplete and by taking the small set of representable functors we provide the strong generator for $[\mathcal{A}, \mathbf{set}]$ consisting of absolutely presentable objects, which are, in particular, $\mathbb{D}$-presentable ones. It follows from Theorem 73, then, that the category $[\mathcal{A}, \mathbf{set}]$ is locally $\mathbb{D}$-presentable. ■

Remark 77 Of course, we could have proved directly by the definition of a locally $\mathbb{D}$-presentable category that any presheaf category is of that kind. We only want here to underline that the proof follows immediately, once we have achieved the characterisation Theorem 73.

The previous result leads naturally to a description of locally $\mathbb{D}$-presentable categories as full reflective subcategories of presheaf ones. More precisely, we can realise any locally $\mathbb{D}$-presentable category as follows:

Proposition 78 *Locally $\mathbb{D}$-presentable categories are precisely the $\mathbb{D}$-accessible reflections of presheaf categories.*

Proof. Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category, then $\mathcal{K}$ is of the form $\mathcal{K} \simeq \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}]$, for a small and $\mathbb{D}$-complete category $\mathcal{A}$, by Theorem 26. We know that $i_{\mathcal{A}} : \mathbb{D}\text{-}\mathbf{cont}[\mathcal{A}, \mathbf{set}] \longrightarrow [\mathcal{A}, \mathbf{set}]$ is a full reflective subcategory, and, moreover, by Lemma 29, this inclusion $i_{\mathcal{A}}$ is $\mathbb{D}$-accessible. Therefore, $\mathcal{K}$ is a $\mathbb{D}$-accessible reflection of $[\mathcal{A}, \mathbf{set}]$.
Conversely, by Proposition 76, the category $[\mathcal{A}, \mathbf{set}]$ is locally $\mathbb{D}$-presentable, for any small category $\mathcal{A}$. Then, any full reflective subcategory of $[\mathcal{A}, \mathbf{set}]$, whose inclusion is a $\mathbb{D}$-accessible functor, is a locally $\mathbb{D}$-presentable category, by Corollary 41. ■

Given a locally $\mathbb{D}$-presentable category $\mathcal{K}$, there are many ways of producing other locally $\mathbb{D}$-presentables categories starting from $\mathcal{K}$, by taking, for instance, slice categories of $\mathcal{K}$ or categories of functors with codomain $\mathcal{K}$. A preliminary result is recalled first:

Lemma 79 *Let $\mathcal{K}$ be a finitely complete category and A an object of $\mathcal{K}$. An arrow $f : (X, x) \longrightarrow (Y, y)$ is a strong epimorphism of $\mathcal{K}\downarrow A$ if and only if $f : X \longrightarrow Y$ is a strong epimorphism of $\mathcal{K}$.*

Proof. For the necessary condition, suppose that the arrow $f : (X, x) \longrightarrow (Y, y)$ is a strong epimorphism of $\mathcal{K}\downarrow A$. Since $\mathcal{K}$ is finitely complete, the category $\mathcal{K}\downarrow A$ is finitely complete and the canonical forgetful functor $\Gamma_A : \mathcal{K}\downarrow A \longrightarrow \mathcal{K}$, assigning to any pair $(X, x) \in \mathcal{K}\downarrow A$ the object $X \in \mathcal{K}$, admits a right adjoint $\Lambda_A : \mathcal{K} \longrightarrow \mathcal{K}\downarrow A$, associating with any object $X \in \mathcal{K}$ the pair $(X \times A, \pi_A)$, given by the product $X \times A$ in $\mathcal{K}$ and the second product projection $\pi_A : X \times A \longrightarrow A$. Clearly, the functor Λ_A does preserve monomorphisms. Then, consider the following commutative square in $\mathcal{K}$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ Z \arrow[r, tail, "m"'] & W \end{tikzcd}$$

Diagram 2.3

where m is a monomorphism of $\mathcal{K}$. Thanks to the functor Λ_A, we are able to lift the previous square in $\mathcal{K}\downarrow A$; more explicitly, we can define two arrows $< u, x >$: $(X, x) \longrightarrow (Z \times A, \pi_A)$ and $< v, y >$: $(Y, y) \longrightarrow (W \times A, \pi_A)$ in $\mathcal{K} \downarrow A$, by the universal property of the product in $\mathcal{K}$, and obtain a commutative diagram, as follows:

$$\begin{tikzcd} (X, x) \arrow[r, "f"] \arrow[d, "<u,x>"'] & (Y, y) \arrow[d, "<v,y>"] \\ (Z \times A, \pi_A) \arrow[r, tail, "m\times 1_A"'] & (W \times A, \pi_A) \end{tikzcd}$$

where the map $\Lambda_A(m) = m \times 1_A : (Z \times A, \pi_A) \longrightarrow (W \times A, \pi_A)$ is, moreover, a monomorphism of $\mathcal{K} \downarrow A$. Therefore, from f being a strong epimorphism of $\mathcal{K} \downarrow A$, it follows the existence of an arrow $h : (Y, y) \longrightarrow (Z \times A, \pi_A)$ in $\mathcal{K} \downarrow A$ making the two corresponding triangles in previous diagram commute. In this way, we get a fill-in for Diagram 2.3, namely the composite $\pi_Z.h : Y \longrightarrow Z$, since the morphism $\pi_Z.h$ satisfies both conditions $\pi_Z.h.f = u$ and $m.\pi_Z.h = v$. In order to verify these properties, it suffices to compose with the appropriate product projections. This ends the proof of the arrow $f : X \longrightarrow Y$ being a strong epimorphism of $\mathcal{K}$.

Conversely, suppose that the map $f : X \longrightarrow Y$ is a strong epimorphism of $\mathcal{K}$ and consider the following commutative square in $\mathcal{K}\downarrow A$:

$$\begin{tikzcd} (X, x) \arrow[r, "f"] \arrow[d, "u"'] & (Y, y) \arrow[d, "v"] \\ (Z, z) \arrow[r, tail, "m"'] & (W, w) \end{tikzcd}$$

where m is a monomorphism of $\mathcal{K}\downarrow\mathcal{A}$. Having, in particular, $v.f = m.u$, there exists a unique arrow $\varphi : Y \longrightarrow Z$ such that $\varphi.f = u$ and

$m.\varphi = v$. The only thing remaining to check is whether the composites $z.\varphi$ and y coincide, that is, whether φ actually provides a morphism $(Y, y) \longrightarrow (Z, z)$ in $\mathcal{K}\downarrow A$. For it, we have $z.\varphi = w.m.\varphi = w.v = y$, by definition of m and v arrows of $\mathcal{K}\downarrow A$. Therefore, we have a morphism $\varphi : (Y, y) \longrightarrow (Z, z)$ of $\mathcal{K}\downarrow A$ making the two corresponding triangles in the previous diagram commute, that is, we have shown that the arrow f is a strong epimorphism in $\mathcal{K}\downarrow A$. ∎

So, we are able to state the result concerning slice categories:

Proposition 80 *Let $\mathcal{K}$ be a locally $\mathbb{D}$-presentable category and A be an object of $\mathcal{K}$. Then, the slice category $\mathcal{K}\downarrow A$ is locally $\mathbb{D}$-presentable.*

Proof. Following the lines of the proof contained in Proposition 71, we could show that, more generally, the category $\mathcal{K}$ being cocomplete implies that $\mathcal{K}\downarrow A$ is cocomplete; explicitly, for any diagram $F : \mathcal{M} \longrightarrow \mathcal{K}\downarrow A$, which associates with an object M the pair $(K_M, k_M : K_M \longrightarrow A)$, its colimit in $\mathcal{K}\downarrow A$ is given by $((\operatorname{colim}_{M\in\mathcal{M}} K_M, a), (\sigma_M)_{M\in\mathcal{M}})$, where the object $\operatorname{colim}_{M\in\mathcal{M}} K_M$ is a colimit in $\mathcal{K}$ of the diagram $\Gamma_A.F$ (we denote by $\Gamma_A : \mathcal{K}\downarrow A \longrightarrow \mathcal{K}$ the canonical forgetful functor), the arrows $\sigma_M : K_M \longrightarrow \operatorname{colim}_{M\in\mathcal{M}} K_M$ are the colimit injections, for any $M \in \mathcal{M}$, and $a : \operatorname{colim}_{M\in\mathcal{M}} K_M \longrightarrow A$ is the canonical morphism induced by the maps k_M (that is, for any $M \in \mathcal{M}$, it holds $a.\sigma_M = k_M$). We wish to show that, given the strong generator $\mathcal{G}$ for $\mathcal{K}$, the set of all morphisms $\{g : G \longrightarrow A | G \in \mathcal{G}\}$ in $\mathcal{K}$ is a strong generator for $\mathcal{K}\downarrow A$. For it, consider any object (H, h) of $\mathcal{K}\downarrow A$, that is, an arrow $h : H \longrightarrow A$ of $\mathcal{K}$. We need to show that:

$$\gamma_{(H,h)} : \coprod_{G\in\mathcal{G}} \mathbf{Hom}((G, g), (H, h)) \circ (G, g) \longrightarrow (H, h)$$

is a strong epimorphism of $\mathcal{K}\downarrow A$. By how colimits are computed in $\mathcal{K}\downarrow A$, $\coprod_{G\in\mathcal{G}} \mathbf{Hom}((G, g), (H, h)) \circ (G, g) \simeq (\coprod_{G\in\mathcal{G}} \coprod_{f\in\mathbf{Hom}((G,g),(H,h))} G, \mu)$, where μ can be defined as the unique arrow making the following diagram commute:

$$\begin{tikzcd}
\coprod\limits_{G\in\mathcal{G}}\coprod\limits_{f:G\to H,h.f=g} G \arrow[rr, "\mu"] & & A \\
\coprod\limits_{f:G\to H,h.f=g} G \arrow[u, "\sigma_G"] & & \\
G \arrow[u, "s_f"] \arrow[uurr, "g"'] & &
\end{tikzcd}$$

that is, $\mu.\sigma_G.s_f = g$, for any $G \in \mathcal{G}$ and any $f \in \mathbf{Hom}((G,g),(H,h))$, namely an arrow $f : G \longrightarrow H$ such that $h.f = g$. By definition of the arrows in $\mathcal{K}\!\downarrow\! A$,

$$\gamma_{(H,h)} : \coprod_{G\in\mathcal{G}} \coprod_{f\in\mathbf{Hom}((G,g),(H,h))} (G,\mu) \longrightarrow (H,h)$$

is a morphism

$$\theta : \coprod_{f:G\to H, h.f=g} G \longrightarrow H$$

such that $h.\theta = \mu$. Since $H \in \mathcal{K}$ and $\mathcal{G}$ is a strong generator for $\mathcal{K}$, we know that the canonical morphism:

$$\begin{tikzcd}
\coprod_{G\in\mathcal{G}} \coprod_{f:G\to H} G \arrow[rr, two heads, "\gamma_H"] & & H \\
\coprod_{f:G\to H} G \arrow[u, "\rho_G"] & & \\
G \arrow[u, "\delta_f"] \arrow[uurr, "f"'] & &
\end{tikzcd}$$

such that $\gamma_H.\rho_G.\delta_f = f$, for any $G \in \mathcal{G}$ and any $f : G \longrightarrow H$ in $\mathcal{K}$, is a strong epimorphism. It can be shown that the two arrows γ_H and θ coincide. For it, it suffices to pre-compose them with the corresponding coproduct injections getting f, as result, in both cases. Therefore, θ is a strong epimorphism of $\mathcal{K}$ and, applying Lemma 79, we get that $\gamma_{(H,h)}$ is a strong epimorphism of $\mathcal{K}\!\downarrow\! A$.

It only remains, then, to prove that the objects $(G,g) \in \mathcal{K}\!\downarrow\! A$ are $\mathbb{D}$-presentable, that is, the corresponding representable functor $\mathbf{Hom}((G,g),-) : \mathcal{K}\downarrow A \longrightarrow \mathbf{set}$ is $\mathbb{D}$-accessible. For it, consider a $\mathbb{D}$-filtered category $\mathcal{J}$ and a diagram $H : \mathcal{J} \longrightarrow \mathcal{K}\!\downarrow\! A$, which associates with any $J \in \mathcal{J}$ a pair (H_J, h_J), and compose it with the canonical forgetful functor $\Gamma_A : \mathcal{K}\!\downarrow\! A \longrightarrow \mathcal{K}$. As we know, a colimit of H in $\mathcal{K}\!\downarrow\! A$ is given by $((\mathrm{colim}_{J\in\mathcal{J}} H_J, h), (\sigma_J)_{J\in\mathcal{J}})$, where $(\mathrm{colim}_{J\in\mathcal{J}} H_J, (\sigma_J)_{J\in\mathcal{J}})$ is a colimit in $\mathcal{K}$ of the composite $\Gamma_A.H$ and h is the unique arrow such that $h.\sigma_J = h_J$, for any $J \in \mathcal{J}$. So, giving a morphism $f : (G,g) \longrightarrow (\mathrm{colim}_{J\in\mathcal{J}} H_J, h)$ in $\mathcal{K}\downarrow A$ is the same as giving an arrow $f : G \longrightarrow \mathrm{colim}_{J\in\mathcal{J}} H_J$ in $\mathcal{K}$ satisfying $h.f = g$. Clearly, f has to factor through some component of the colimit, say it $f_J : G \longrightarrow H_J$, G being $\mathbb{D}$-presentable in $\mathcal{K}$; namely, $\sigma_J.f_J = f$. This precisely means that f, as arrow of $\mathcal{K}\!\downarrow\! A$, factors through $f_J : (G,g) \longrightarrow (H_J, h_J)$, provided that

$h_J.f_J = g$. But, indeed, we have $g = h.f = h.\sigma_J.f_J = h_J.f_J$, by definition of the arrows f_J and h. Stated otherwise, we have just shown that $\mathbf{Hom}((G,g), \mathrm{colim}_{J\in\mathcal{J}}(H_J,h_J)) \simeq \mathrm{colim}_{J\in\mathcal{J}}\,\mathbf{Hom}((G,g),(H_J,h_J))$, as desired. Hence, by Theorem 73, we can affirm that the category $\mathcal{K}\downarrow A$ is locally $\mathbb{D}$-presentable. ∎

Moving to the case of functor categories now, to prove their locally $\mathbb{D}$-presentability, we make use of the characterisation of locally $\mathbb{D}$-presentable categories as $\mathbb{D}$-accessible reflections of presheaf categories, as we discussed in Corollary 78, and of the following:

Lemma 81 *Let $\mathcal{K}$ and $\mathcal{K}'$ be two locally small categories with $\mathbb{D}$-filtered colimits. If $i : \mathcal{K}' \longrightarrow \mathcal{K}$ is a $\mathbb{D}$-accessible reflection of $\mathcal{K}$, then, for any locally small category $\mathcal{A}$, $\hat{i} : [\mathcal{A},\mathcal{K}'] \longrightarrow [\mathcal{A},\mathcal{K}]$ is a $\mathbb{D}$-accessible reflection of the corresponding presheaf category $[\mathcal{A},\mathcal{K}]$.*

Proof. We denote by $L : \mathcal{K} \longrightarrow \mathcal{K}'$ the left adjoint functor to the fully faithful $\mathbb{D}$-accessible inclusion $i : \mathcal{K}' \longrightarrow \mathcal{K}$. We define two new functors $\hat{i} : [\mathcal{A},\mathcal{K}'] \longrightarrow [\mathcal{A},\mathcal{K}]$ and $\hat{L} : [\mathcal{A},\mathcal{K}] \longrightarrow [\mathcal{A},\mathcal{K}']$, which are composing with i and composing with L, respectively. We can establish the adjunction $\hat{L} \dashv \hat{i}$ simply using the adjunction $L \dashv i$. The fact that $\hat{i}$ is fully faithful follows from the same property of the inclusion i. Finally, in order to prove that the functor $\hat{i}$ is $\mathbb{D}$-accessible, consider a $\mathbb{D}$-filtered category $\mathcal{J}$ and a diagram $H : \mathcal{J} \longrightarrow [\mathcal{A},\mathcal{K}']$, which associates with any $J \in \mathcal{J}$ a functor $H_J : \mathcal{A} \longrightarrow \mathcal{K}'$, and take its colimit $(\mathrm{colim}_{J\in\mathcal{J}}\,H_J, (\sigma_J)_{J\in\mathcal{J}})$ in $[\mathcal{A},\mathcal{K}']$. Since colimits are computed pointwise in any presheaf category and, moreover, the functor i is $\mathbb{D}$-accessible, we can deduce that $(i.(\mathrm{colim}_{J\in\mathcal{J}}\,H_J))(A) \simeq (\mathrm{colim}_{J\in\mathcal{J}}\,i.H_J)(A)$, for any object $A \in \mathcal{A}$. That is, $\hat{i}(\mathrm{colim}_{J\in\mathcal{J}}\,H_J) \simeq \mathrm{colim}_{J\in\mathcal{J}}\,\hat{i}(H_J)$: this is precisely the condition on $\hat{i}$ of preserving $\mathbb{D}$-filtered colimits, as desired. ∎

Finally, we can state the result concerning the functor category $[\mathcal{A},\mathcal{K}]$:

Proposition 82 *Let $\mathcal{A}$ be a small category and $\mathcal{K}$ a locally $\mathbb{D}$-presentable category. Then, the functor category $[\mathcal{A},\mathcal{K}]$ is locally $\mathbb{D}$-presentable.*

Proof. We know, by Corollary 78, that any locally $\mathbb{D}$-presentable category $\mathcal{K}$ is a $\mathbb{D}$-accessible reflection of a presheaf category, as in the following diagram:

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift right, tail, "i"'] \arrow[r, phantom, "\perp"] & {[\mathcal{C},\mathbf{set}]} \arrow[l, shift right, "L"'] \end{tikzcd}$$

for some small category $\mathcal{C}$. Therefore, by Lemma 81, we get that:

$$\begin{tikzcd} {[\mathcal{A},\mathcal{K}]} \arrow[r, tail, shift right, "\hat{\imath}"'] \arrow[r, phantom, "\perp"] & {[\mathcal{A},[\mathcal{C},\mathbf{set}]]} \arrow[l, shift right, "\hat{L}"'] \end{tikzcd}$$

is a $\mathbb{D}$-accessible reflection of the category $[\mathcal{A},[\mathcal{C},\mathbf{set}]]$. Clearly, there exists a bijection $[\mathcal{A},[\mathcal{C},\mathbf{set}]] \simeq [\mathcal{A}\times\mathcal{C},\mathbf{set}]$. Hence, the category $[\mathcal{A},\mathcal{K}]$ is, in fact, a $\mathbb{D}$-accessible reflection of a presheaf category, and, being such, it is a locally $\mathbb{D}$-presentable category, again by Corollary 78. ■

In analogy to the case of accessible categories, one could be tempt to believe that coslice categories $A \downarrow \mathcal{K}$ of a locally $\mathbb{D}$-presentable category $\mathcal{K}$, for some object $A \in \mathcal{K}$, are locally $\mathbb{D}$-presentable. More precisely, slice and coslice categories of an accessible category are accessible categories (see, for instance, [5]). This is no longer the case for locally $\mathbb{D}$-presentable categories, as shown in the following counterexample provided by Francis Borceux:

Remark 83 Choose as $\mathbb{D}$ the empty doctrine; take the category $\mathcal{K}$ to be the category **set** of sets and the object A to be the singleton $\{*\} = 1$. Then, the coslice category $1 \downarrow \mathbf{set}$ is precisely the category of pointed sets, which has a zero object, and therefore it is not a presheaf category. By Example 74.3, one concludes that the category $1 \downarrow \mathbf{set}$ is not locally $\mathbb{D}$-presentable.

3

On geometric morphisms and localizations of presheaf categories

The definitions of geometric morphism and localization are recalled. We analyse geometric morphisms, and, in particular, localizations, from the point of view of the domain category by giving explicit conditions on it which fully characterise the localization. Some general conditions on functors are introduced to enlighten our characterisation of geometric morphisms. The theorem concerning localizations of presheaf categories is achieved. Two proofs of it are offered, the second of which requires some sheaf theoretical techniques.

3.1 Rambling about the concept of localization

Definition 84 *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with finite limits.*

- *A* geometric morphism *$F : \mathcal{B} \longrightarrow \mathcal{A}$ is a functor F together with a left adjoint $L : \mathcal{A} \longrightarrow \mathcal{B}$ which preserves finite limits.*
- *Let $F_1, F_2 : \mathcal{B} \longrightarrow \mathcal{A}$ be geometric morphisms. A* geometric transformation *$\alpha : F_1 \longrightarrow F_2$ is defined to be a natural transformation $\alpha : L_1 \longrightarrow L_2$, where the functors L_1 and L_2 denote the left adjoints to F_1 and to F_2, respectively.*

For a complete treatment of geometric morphisms and transformations, we refer to [25, 26]. Amongst geometric morphisms we distinguish those special ones which are full embeddings, as in the following:

Definition 85 *A* localization *of a finitely complete category $\mathcal{A}$ is a full and faithful geometric morphism $i : \mathcal{B} \rightarrowtail \mathcal{A}$.*

We recall, as well, that we say $\mathcal{B}$ is a *full reflective subcategory* of $\mathcal{A}$ if the inclusion $i : \mathcal{B} \rightarrowtail \mathcal{A}$ is a full and faithful right adjoint functor.
We discuss geometric morphisms by analysing the domain category $\mathcal{B}$. More precisely, we follow the suggestion contained in [32], where they study Grothendieck categories (namely, cocomplete abelian categories with a generator and filtered colimits commuting with finite limits) as localizations of the category of additive presheaves defined on a (small) pre-additive category. In this chapter, we characterise those functors $k : \mathcal{C} \longrightarrow \mathcal{E}$, from a small category $\mathcal{C}$ into a cocomplete, exact and extensive category $\mathcal{E}$, which give rise to a localization. In particular, we want to distinguish hypotheses concerning the fully faithful character of the embedding $\mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$ into the category of set-valued functors defined on $\mathcal{C}^{op}$, and hypotheses concerning the left exactness of its left adjoint. These properties are stated in terms of the functor $k : \mathcal{C} \longrightarrow \mathcal{E}$. They are independent from each other and self-contained. In this way, we achieve distinct characterisations of geometric morphisms and localizations. We study localizations $\mathcal{E}$ of presheaf categories $[\mathcal{C}^{op}, \mathbf{set}]$ of set-valued functors and give separately conditions on the functor $k : \mathcal{C} \longrightarrow \mathcal{E}$ for the functor $i : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$ to be just a geometric morphism or, pursuing the analysis further, for the inclusion i to be indeed a fully faithful functor, that is $i : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$ is a localization. This is a major difference with the work contained in [32]. One last remark: we make no distinction between a finite limit-preserving functor and a left exact functor.

Exact and/or extensive categories

We need to provide some context, before presenting the main theorems of this section.

Definition 86 *We say that a category $\mathcal{A}$ is* regular *if it satisfies the following conditions:*

1. *$\mathcal{A}$ has finite limits;*
2. *$\mathcal{A}$ has coequalizers of kernel pairs;*
3. *the pullback of a regular epimorphism along any morphism is a regular epimorphism.*

Or, equivalently:

Definition 87 *We say that a category $\mathcal{A}$ is* regular *if it satisfies the following conditions:*

1. *$\mathcal{A}$ has finite limits;*

2. *any arrow $f : X \longrightarrow Y$ in $\mathcal{A}$ can be factored as a regular epimorphism q followed by a monomorphism m, as in the diagram:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & I \arrow[ur, tail, "m"'] & \end{tikzcd}$$

3. *the pullback of a regular epimorphism along any morphism is a regular epimorphism.*

Usually one refers to the factorisation of morphisms in a regular category as the *regular epi-mono factorisation*. Moreover, one can prove that such a factorisation is essentially unique. Amongst regular categories, the appropriate definition of functor is given by the following:

Definition 88 *Let $\mathcal{A}$ and $\mathcal{B}$ be regular categories. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be* exact *if it preserves:*

1. *finite limits;*

2. *regular epimorphisms.*

Definition 89 *A category $\mathcal{A}$ is said to be* exact *if it is regular and any equivalence relation in $\mathcal{A}$ is effective.*

We recall that an equivalence relation $< r_0, r_1 >: R \rightarrowtail X \times X$, on an object X of $\mathcal{A}$, is *effective* if the coequalizer q of $< r_0, r_1 >$ exists and $< r_0, r_1 >$ is the kernel pair of q. Of course, any kernel pair is an equivalence relation. As examples of exact categories we can briefly mention: **set**, any presheaf category $[\mathcal{A}, \mathbf{set}]$, any category of $\mathbb{T}$-algebras, for a monad $\mathbb{T}$ over **set**. For a complete treatment of regular and exact categories, see, for instance, [11].

The following is an elegant formulation of disjointness and universality of coproducts:

Definition 90 *We say that a category $\mathcal{A}$ with coproducts is* extensive *if, for any small family of objects $(X_i)_{i \in I}$ of $\mathcal{A}$, the canonical functor*

$$\coprod : \prod_{i \in I} \mathcal{A} \downarrow X_i \longrightarrow \mathcal{A} \downarrow \coprod_{i \in I} X_i$$

is an equivalence. In details, the functor $\coprod$ is defined, by the universal property of the coproduct, at any family $(f_i : A_i \longrightarrow X_i)_{i\in I}$, that is, an object of $\prod_{i\in I} \mathcal{A}\downarrow X_i$, as the essentially unique object φ in $\mathcal{A}\downarrow \coprod_{i\in I} X_i$ making the following square:

$$\begin{tikzcd} \coprod_{i\in I} A_i \arrow[r, "\varphi"] & \coprod_{i\in I} X_i \\ A_i \arrow[u, "\tau_i"] \arrow[r, "f_i"'] & X_i \arrow[u, "\sigma_i"'] \end{tikzcd}$$

commute, for any $i \in I$. Here, τ_i and σ_i stand for the corresponding coproduct injections in $\mathcal{A}$.

Definition 91 *A category $\mathcal{A}$ is said to be* lextensive *if it is extensive and has finite limits.*

As examples of lextensive categories, we mention **set**, any presheaf category $[\mathcal{A}^{op}, \mathbf{set}]$ for a small category $\mathcal{A}$. On the other hand, for any category $\mathcal{A}$, $\mathbf{Fam}\mathcal{A}$ is an example of a category which is extensive, but, in general, not lextensive.
We recall the following equivalent, probably more familiar, definition of extensivity:

Definition 92 *A category $\mathcal{A}$ with coproducts is* extensive *if it has pullbacks along the canonical coproduct injections and the following two conditions hold:*

1. *(coproducts are* universal*) given an arrow $f : Y \longrightarrow \coprod_{i\in I} X_i$ and, for any $i \in I$, the pullback:*

$$\begin{tikzcd} Y_i \arrow[r, "\theta_i"] \arrow[d] & Y \arrow[d, "f"] \\ X_i \arrow[r, "\sigma_i"'] & \coprod_{i\in I} X_i \end{tikzcd}$$

   then, the comparison morphism $\coprod_{i\in I} Y_i \longrightarrow Y$, induced by the arrows θ_i, is an isomorphism;

2. *(coproducts are* disjoint*) given a family of objects* $(X_i)_{i\in I}$ *of* $\mathcal{A}$*, for any* $i, j \in I$ *such that* $i \neq j$*, the following square:*

$$\begin{tikzcd} 0 \arrow[r, "!"] \arrow[d, "!"'] & X_j \arrow[d, "\sigma_j"] \\ X_i \arrow[r, "\sigma_i"'] & \coprod_{i\in I} X_i \end{tikzcd}$$

is a pullback (here, we denote by 0 *the initial object of* $\mathcal{A}$*).*

In an extensive category, the coproduct injections have an extra feature, as described in the following:

Proposition 93 *Let* $\mathcal{A}$ *be an extensive category. The canonical coproduct injection* $\sigma_i : X_i \rightarrowtail \coprod_{i\in I} X_i$ *is a monomorphism, for any* $i \in I$*.*

Proposition 94 *Let* $\mathcal{A}$ *be an extensive category. The initial object* 0 *of* $\mathcal{A}$ *is strict, namely every arrow* $X \longrightarrow 0$*, for any* $X \in \mathcal{A}$*, is an isomorphism.*

For more details concerning extensive categories we refer to [15, 36].

A quite surprising result concerning categories that are simultaneously exact and extensive is the following Proposition, which appears, for instance, in [36]:

Proposition 95 *In an exact and extensive category* $\mathcal{A}$*, all epimorphisms are regular epimorphisms.*

Proof. Using the calculus of relations, one can prove that in $\mathcal{A}$ cokernel pairs of monomorphisms always exist and they are pullbacks (see [22]). Therefore, any monomorphism in $\mathcal{A}$ is the equalizer of its cokernel pair: this means that every monomorphism is, in fact, regular in $\mathcal{A}$. Finally, we just need to remark that in any category $\mathcal{A}$ with regular epi-mono factorisations, the fact that any monomorphism is regular implies that any epimorphism is regular too, since when an epimorphism f in $\mathcal{A}$ factors as a regular epimorphism followed by a monomorphism, as $f = m.q$, it follows that the arrow m is an epimorphism, but m is, moreover, a regular monomorphism, that is, an isomorphism in $\mathcal{A}$. ∎

By *epimorphic family* in an arbitrary category $\mathcal{A}$ we mean a collection $(f_i : X_i \longrightarrow X)_{i\in I}$ of arrows of $\mathcal{A}$ such that, for any given pair of arrows $u, v : X \longrightarrow U$, the condition $u.f_i = v.f_i$ for every $i \in I$ implies the equality $u = v$.

Definition 96 *Let $\mathcal{A}$ be a small category and $\mathcal{B}$ a finitely complete category. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be* filtering *if the following conditions hold:*

(F1) *the family $\{F(X) \longrightarrow 1 | X \in \mathcal{A}\}$ of all arrows in $\mathcal{B}$, where 1 is the terminal object of $\mathcal{B}$, is epimorphic;*

(F2) *for any pair of objects $A, A' \in \mathcal{A}$, the family*

$$\{< F(u), F(v) >: F(X) \longrightarrow F(A) \times F(A') | A \xleftarrow{u} X \xrightarrow{v} A' \text{ in } \mathcal{A}\}$$

of all arrows in $\mathcal{B}$, while X varies in $\mathcal{A}$, is epimorphic;

(F3) *for any pair of arrows $A \underset{v}{\overset{u}{\rightrightarrows}} A'$ in $\mathcal{A}$, the family*

$$\{F(X) \longrightarrow E_{u,v} | X \xrightarrow{w} A \underset{v}{\overset{u}{\rightrightarrows}} A' \text{ in } \mathcal{A} \text{ such that } u.w = v.w\}$$

of all arrows in $\mathcal{B}$, while X varies in $\mathcal{A}$, is epimorphic (where $E_{u,v}$ is the equalizer $E_{u,v} \rightarrowtail F(A) \underset{F(v)}{\overset{F(u)}{\rightrightarrows}} F(A')$ of the pair $(F(u), F(v))$ in $\mathcal{B}$).

Remark 97 When we take as $\mathcal{B}$ the category **set**, then, the definition of a filtering functor F is equivalent to state that the category $\mathcal{A} \downarrow F$ is filtering, which means that the functor F is a filtered colimit of representable functors.

We refer to [34, 16] for what concerns filtering functors.

We recall that a subcategory $\mathcal{A}$ of $\mathcal{B}$ is said to be *dense* in $\mathcal{B}$ if any object of $\mathcal{B}$ is a colimit of objects of $\mathcal{A}$; more exactly, a colimit in a canonical way, for which the colimit cone consists of $(B, (a : A \longrightarrow B)_{A \in \mathcal{A}})$. More generally, density can be defined not only for an inclusion $\mathcal{A} \subseteq \mathcal{B}$, but for any functor $F : \mathcal{A} \longrightarrow \mathcal{B}$. The arrows $a : A \longrightarrow B$ are, then, replaced by the objects of the comma category $F \downarrow B$.

Definition 98 *A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is* dense *if, for any $B \in \mathcal{B}$, the colimit of the composite diagram:*

$$F \downarrow B \xrightarrow{\Gamma_B} \mathcal{A} \xrightarrow{F} \mathcal{B} \ , \ (A, a : F(A) \longrightarrow B) \mapsto F(A)$$

is precisely $(B, (a)_{(A,a) \in F \downarrow B})$, where $\Gamma_B : F \downarrow B \longrightarrow \mathcal{A}$, defined by the assignment $(A, a) \mapsto A$, denotes the canonical forgetful functor. In particular, a subcategory $\mathcal{A}$ of $\mathcal{B}$ is dense in $\mathcal{B}$ when the inclusion functor $\mathcal{A} \rightarrowtail \mathcal{B}$ is dense in the sense just defined.

For a complete treatment of dense functors, see [33].

The exact completion of Fam

We devote this section to a brief overview of the exact completion of a category with *weak limits*, in order to present any presheaf category as a free exact completion of a *projective cover* of itself. We recall the coproduct completion $\mathbf{Fam}\mathcal{A}$ of a category $\mathcal{A}$, which is nothing but the infinitary version of the $\mathbf{Fam}_f$-completion, presented in Chapter 2 (see Section 2.2). We refer to [16] for all notions and results contained in this section.

Definition 99 *Let $\mathcal{M}$ be a small category and $T : \mathcal{M} \longrightarrow \mathcal{A}$ a functor. A* weak limit *of T is a cone $(L, (\lambda_M : L \longrightarrow T(M))_{M\in\mathcal{M}})$ in $\mathcal{A}$ such that, for any other cone $(X, (\xi_M : X \longrightarrow T(M))_{M\in\mathcal{M}})$ in $\mathcal{A}$, there exists a morphism $\varphi : X \longrightarrow L$ satisfying $\lambda_M.\varphi = \xi_M$, for any $M \in \mathcal{M}$.*

Basically, the only difference with the usual definition of a limit for a diagram is that, in the definition of a weak limit, one requires only the existence of the canonical factorisation and not its uniqueness. As an immediate consequence, we have that a functor can admit several not isomorphic weak limits. Clearly, we have:

Definition 100 *A category $\mathcal{A}$ is said to be* weakly lex *(where, lex stands for left exact, meaning: finitely complete) if, for any diagram $T : \mathcal{M} \longrightarrow \mathcal{A}$, defined on a finite category $\mathcal{M}$, there exists a weak limit of T in $\mathcal{A}$.*

We are able to present, then, the key notion of functor, for what concerns exact completions:

Definition 101 *Let $\mathcal{A}$ be a weakly lex category, $\mathcal{B}$ a finitely complete category and $F : \mathcal{A} \longrightarrow \mathcal{B}$ a functor. We say that F is* left covering *if, for any functor $T : \mathcal{M} \longrightarrow \mathcal{A}$ defined on a finite category $\mathcal{M}$, and for any weak limit $(L, (\lambda_M : L \longrightarrow T(M))_{M\in\mathcal{M}})$ in $\mathcal{A}$, the canonical factorisation $p : F(L) \longrightarrow \tilde{L}$ is a strong epimorphism of $\mathcal{B}$, where $(\tilde{L}, (\tilde{\lambda}_M : L \longrightarrow F(T(M)))_{M\in\mathcal{M}})$ is a (honest) limit in $\mathcal{B}$ of the composite diagram $F.T$ and p is the unique arrow with the property $\tilde{\lambda}_M.p = F(\lambda_M)$, for any $M \in \mathcal{M}$, as in the following diagram:*

$$\begin{tikzcd}
F(L) \arrow[rr, two heads, "p"] \arrow[dr, "F(\lambda_M)"'] & & \tilde{L} \arrow[dl, "\tilde{\lambda}_M"] \\
& F(T(M)) &
\end{tikzcd}$$

The relation between left exact (that is, finite limit-preserving) functors and left covering functors is explained in the following Proposition. We recall that a functor, defined on a weakly lex category, is said to be *weakly lex* if it preserves weak limits.

Proposition 102 *Let $\mathcal{A}$ be a weakly lex category, $\mathcal{B}$ a finitely complete category and $F : \mathcal{A} \longrightarrow \mathcal{B}$ a functor. Consider the following conditions:*

1. *F is left covering;*

2. *F is weakly lex;*

3. *F is left exact.*

Then, Condition 2 implies Condition 1. If, moreover, the category $\mathcal{A}$ is finitely complete, then, the three conditions are equivalent.

We mention, as well, the following peculiar result concerning composition with a left covering functor:

Proposition 103 *Let $\mathcal{A}$ be a weakly lex category and $\mathcal{B}$ and $\mathcal{C}$ two finitely complete categories. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ be two functors. If F is left covering, G is left exact and, moreover, G sends strong epimorphisms into strong epimorphisms, then, the composite $G.F : \mathcal{A} \longrightarrow \mathcal{C}$ is left covering.*

We recall that an object P in a category $\mathcal{A}$ is said to be *regular projective* (often just *projective*) if, for any arrow $f : P \longrightarrow X$ of $\mathcal{A}$ and for any regular epimorphism $q : Y \longrightarrow X$ in $\mathcal{A}$, there exists an arrow $f' : P \longrightarrow Y$ such that $q.f' = f$.

Definition 104 *Let $\mathcal{P}$ be a full subcategory of a category $\mathcal{A}$. We say that $\mathcal{P}$ is a* projective cover *of $\mathcal{A}$ if*

- *any object of $\mathcal{P}$ is projective in $\mathcal{A}$;*

- *for any object $A \in \mathcal{A}$, there exist an object $P \in \mathcal{P}$ and a regular epimorphism $p : P \twoheadrightarrow A$.*

Clearly, a category $\mathcal{A}$ admits a projective cover if and only if it has *enough projectives* (that is, any object of $\mathcal{A}$ is the codomain of a regular epimorphism whose domain is projective).

Firstly, we discuss the *exact completion* of a category with weak limits. Explicitly, for any weakly lex category $\mathcal{A}$, there exist an exact category $\mathcal{A}_{ex}$ and a morphism $\xi_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}_{ex}$, which is a fully faithful left covering functor, such that $\mathcal{A}_{ex}$ is universal with respect to exact categories and left covering functors. Namely, for any left covering functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ into an exact category $\mathcal{B}$, there exists a unique (up to isomorphism) exact functor $\hat{F}$ making the triangle:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\xi_{\mathcal{A}}"] \arrow[dr, "F"'] & & \mathcal{A}_{ex} \arrow[dl, "\hat{F}"] \\
& \mathcal{B} &
\end{tikzcd}$$

commute. Equivalently, the universal property of the exact completion can be stated in terms of an equivalence between the category of exact functors, from $\mathcal{A}_{ex}$ to $\mathcal{B}$, and the category of left covering functors, from $\mathcal{A}$ to $\mathcal{B}$, which is induced by composing with $\xi_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}_{ex}$:

$$\mathbf{Ex}[\mathcal{A}_{ex}, \mathcal{B}] \simeq \mathbf{Lco}[\mathcal{A}, \mathcal{B}]$$

The functor $\xi_{\mathcal{A}}$ verifies the extra property of preserving all finite limits, which turn out to exist in $\mathcal{A}$; moreover:

Proposition 105 *The image $\xi_{\mathcal{A}}(\mathcal{A})$ of the functor $\xi_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}_{ex}$ is a projective cover of $\mathcal{A}_{ex}$, so that $\mathcal{A}_{ex}$ has enough projectives.*

In fact, one can prove that the exact extension of a left covering functor F is a left Kan extension, as stated in the following:

Proposition 106 *In the above-mentioned setting, the (essentially unique) exact extension $\hat{F} : \mathcal{A}_{ex} \longrightarrow \mathcal{B}$ of the left covering functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is the left Kan extension of F along $\xi_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}_{ex}$.*

We can finally state the main Theorem, which characterises free exact categories over weakly left exact ones:

Theorem 107 *Exact categories with enough projectives are precisely the exact completions of the weakly lex categories of their projective covers.*

Since a left covering functor defined on a finitely complete category is precisely a left exact functor, by Proposition 102, we have, as a particular instance of the universal property of the exact completion, that for any finitely complete category $\mathcal{A}$, there exist an exact category $\mathcal{A}_{ex}$ and a left exact functor $\xi_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}_{ex}$ such that, for any exact category $\mathcal{B}$, composing with $\xi_{\mathcal{A}}$ induces an equivalence of categories:

$$\mathbf{Ex}[\mathcal{A}_{ex}, \mathcal{B}] \simeq \mathbf{Lex}[\mathcal{A}, \mathcal{B}]$$

where $\mathbf{Ex}[\mathcal{A}_{ex}, \mathcal{B}]$ denotes the category of exact functors between $\mathcal{A}_{ex}$ and $\mathcal{B}$, whereas $\mathbf{Lex}[\mathcal{A}, \mathcal{B}]$ denotes the category of finite limit-preserving functors between $\mathcal{A}$ and $\mathcal{B}$.

Secondly, we exhibit the *coproduct completion*. The free coproduct completion of a small category $\mathcal{A}$, which we denote simply by $\mathbf{Fam}\mathcal{A}$, is nothing but the infinitary version of the $\mathbf{Fam}_f$-completion of $\mathcal{A}$ treated in Chapter 2. The category $\mathbf{Fam}\mathcal{A}$ has for objects the pairs (I, f), where I is a discrete category (a set) and $f : I \longrightarrow \mathcal{A}$ is a functor, and for morphisms the pairs $(a, \alpha) : (I, f) \longrightarrow (J, g)$, where $a : I \longrightarrow J$ is a functor and $\alpha : f \longrightarrow g.a$ is a natural transformation. One can easily check that $\mathbf{Fam}\mathcal{A}$ has coproducts and define a functor $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbf{Fam}\mathcal{A}$ via the assignment $X \mapsto (\{*\}, X : \{*\} \longrightarrow \mathcal{A})$, where $\{*\}$ is the singleton set. We talk about $\mathbf{Fam}\mathcal{A}$ in terms of a free coproduct completion of $\mathcal{A}$ because $\mathbf{Fam}\mathcal{A}$ is universal with respect to categories with coproducts. Explicitly, for any functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ into a category $\mathcal{B}$ with coproducts, there exists a unique (up to isomorphism) coproduct-preserving functor F' making the triangle:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\eta_{\mathcal{A}}"] \arrow[dr, "F"'] & & \mathbf{Fam}\mathcal{A} \arrow[dl, "F'"] \\
& \mathcal{B} &
\end{tikzcd}$$

commute. As for the exact completion, we know that, in fact, this (essentially unique) coproduct-preserving extension $F' : \mathbf{Fam}\mathcal{A} \longrightarrow \mathcal{B}$ of the functor F coincides with the left Kan extension of the functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ along $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbf{Fam}\mathcal{A}$. Moreover, an alternative description of $\mathbf{Fam}\mathcal{A}$ is given by the following:

Proposition 108 *Let $\mathcal{A}$ be a small category. The category* $\mathbf{Fam}\mathcal{A}$ *is equivalent to the full subcategory of* $[\mathcal{A}^{op}, \mathbf{set}]$ *spanned by coproducts of representable functors.*

This, in particular, means that $\mathbf{Fam}\mathcal{A}$ is equivalent to a projective cover of $[\mathcal{A}^{op}, \mathbf{set}]$. Therefore, by Theorem 107, we can derive the following fundamental (at least, for our purposes!) result:

Theorem 109 *Let $\mathcal{A}$ be a small category and $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow$* $\mathbf{Fam}\mathcal{A}$ *its coproduct completion. Then, by taking the exact completion of* $\mathbf{Fam}\mathcal{A}$ *we get an equivalence of categories:*

$$(\mathbf{Fam}\mathcal{A})_{ex} \simeq [\mathcal{A}^{op}, \mathbf{set}]$$

The result which explains, in some sense, the nature of filtering functors is stated in the following Proposition: filtering functors "induce" left exact functors.

Proposition 110 *Let $\mathcal{A}$ be a small category and $Y_{\mathcal{A}} : \mathcal{A} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$ its Yoneda embedding. Let $\mathcal{B}$ be a cocomplete, exact and extensive category and $F : \mathcal{A} \longrightarrow \mathcal{B}$ a functor. Consider the following diagram:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\eta_{\mathcal{A}}"] \arrow[dr, "F"'] & \mathbf{Fam}\mathcal{A} \arrow[r, "\xi_{\mathbf{Fam}\mathcal{A}}"] \arrow[d, "F'"] & (\mathbf{Fam}\mathcal{A})_{ex} = [\mathcal{A}^{op}, \mathbf{set}] \arrow[dl, "\mathbf{Lan}_{Y_{\mathcal{A}}}(F)"] \\ & \mathcal{B} & \end{tikzcd}$$

The functor F is filtering if and only if its left Kan extension along Yoneda $L := \mathbf{Lan}_{Y_{\mathcal{A}}}(F)$ is (left) exact.

Proof. The proof proceeds in two steps.

Step 1. We prove that $F : \mathcal{A} \longrightarrow \mathcal{B}$ is such that $\mathbf{Lan}_{Y_{\mathcal{A}}}(F)$ is exact if and only if its coproduct-preserving extension $F' : \mathbf{Fam}\mathcal{A} \longrightarrow \mathcal{B}$ is left covering. (Remark that here it suffices that the category $\mathcal{B}$ is cocomplete and exact.)
The requirement of L being left exact coincides, in fact, with L being an exact functor, since $\mathbf{Lan}_{Y_{\mathcal{A}}}(F)$ has always a right adjoint, given by $\mathcal{B}(F(-), -) : \mathcal{B} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$. Suppose, then, L is exact. It follows, by Proposition 103, that the composite $L.\xi_{\mathbf{Fam}\mathcal{A}}$ is a left covering functor. Moreover, one can easily verify that computing $\eta_{\mathcal{A}}$ followed by $\xi_{\mathbf{Fam}\mathcal{A}}$ does give the Yoneda embedding $Y_{\mathcal{A}}$. In light of this equality, the composite $L.\xi_{\mathbf{Fam}\mathcal{A}}$ preserves coproducts (since both functors do so) and satisfies $L.\xi_{\mathbf{Fam}\mathcal{A}}.\eta_{\mathcal{A}} \simeq L.Y_{\mathcal{A}} \simeq F$, since, $Y_{\mathcal{A}}$ being fully faithful, the universal arrow for the left Kan extension is an isomorphism. Therefore, we can derive that $F' \simeq L.\xi_{\mathbf{Fam}\mathcal{A}}$, by the universal property of the coproduct completion, which, we have just shown, is left covering. Conversely, suppose that F' is left covering. By the universal property of the exact completion, $\mathbf{Lan}_{\xi_{\mathbf{Fam}\mathcal{A}}}(F')$ is an exact functor. So, using the definition of the coproduct completion, we get the following chain of isomorphisms: $\mathbf{Lan}_{\xi_{\mathbf{Fam}\mathcal{A}}}(F') \simeq \mathbf{Lan}_{\xi_{\mathbf{Fam}\mathcal{A}}}(\mathbf{Lan}_{\eta_{\mathcal{A}}}(F)) \simeq \mathbf{Lan}_{\xi_{\mathbf{Fam}\mathcal{A}}.\eta_{\mathcal{A}}}(F) \simeq \mathbf{Lan}_{Y_{\mathcal{A}}}(F)$, which is L and has to be exact.

Step 2. We prove that $F : \mathcal{A} \longrightarrow \mathcal{B}$ is filtering if and only if its coproduct-preserving extension $F' : \mathbf{Fam}\mathcal{A} \longrightarrow \mathcal{B}$ is left covering. (Here, it suffices that the category $\mathcal{B}$ is exact and extensive.)
Without going into details, knowing, from Proposition 95, that in such a category $\mathcal{B}$ epimorphisms are regular, the three Conditions (F1), (F2), (F3) of a filtering functor F (as in Definition 96) are equivalent to require the functor F' being left covering with respect to the terminal object, binary products of objects coming from $\mathcal{A}$ and equalizers of arrows coming from $\mathcal{A}$, respectively. This fact precisely coincides with the functor

F' being left covering, by exactness and extensivity of the category $\mathcal{B}$. (For a detailed proof of this result, see [16].) ■

Let $\mathcal{A}$ be a small category and $\mathcal{B}$ a cocomplete, exact and extensive category. We denote by $\mathbf{GeoMor}[\mathcal{B}, [\mathcal{A}^{op}, \mathbf{set}]]$ the category of geometric morphisms between $\mathcal{B}$ and $[\mathcal{A}^{op}, \mathbf{set}]$ and geometric transformations between those (recall Definition 84). Whereas, we denote by $\mathbf{Filt}[\mathcal{A}, \mathcal{B}]$ the category of filtering functors from $\mathcal{A}$ into $\mathcal{B}$ and all natural transformations between them. The following result enlightens the relation between these two notions, namely geometric morphisms can be classified by filtering functors:

Corollary 111 *Let $\mathcal{A}$ be a small category and $\mathcal{B}$ a cocomplete, exact and extensive category. The functor*

$$\varepsilon : \mathbf{GeoMor}[\mathcal{B}, [\mathcal{A}^{op}, \mathbf{set}]] \longrightarrow \mathbf{Filt}[\mathcal{A}, \mathcal{B}]$$

defined by pre-composing with the Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$, gives an equivalence of categories.

Proof. Let

$$\begin{tikzcd} \mathcal{B} \arrow[r, shift right, "R"'] \arrow[r, phantom, "\perp"] & {[\mathcal{A}^{op}, \mathbf{set}]} \arrow[l, shift right, "L"'] \end{tikzcd}$$

be a geometric morphism and consider the pre-composition $L.Y_{\mathcal{A}}$ of its left adjoint L with Yoneda $Y_{\mathcal{A}} : \mathcal{A} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$. First of all, we need to show that the functor $L.Y_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{B}$ is filtering. By Proposition 110, this is equivalent to prove that the left Kan extension $\mathbf{Lan}_{Y_{\mathcal{A}}}(L.Y_{\mathcal{A}})$ of the composite $L.Y_{\mathcal{A}}$ along Yoneda is left exact, as in the following diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "Y_{\mathcal{A}}"] \arrow[dr, "L.Y_{\mathcal{A}}"'] & & {[\mathcal{A}^{op}, \mathbf{set}]} \arrow[dl, "\mathbf{Lan}_{Y_{\mathcal{A}}}(L.Y_{\mathcal{A}})"] \\ & \mathcal{B} & \end{tikzcd}$$

Since $Y_{\mathcal{A}}$ is fully faithful, the universal arrow for such a left Kan extension is an isomorphism, namely $L.Y_{\mathcal{A}} \simeq \mathbf{Lan}_{Y_{\mathcal{A}}}(L.Y_{\mathcal{A}}).Y_{\mathcal{A}}$. Moreover, $Y_{\mathcal{A}}$ being a dense functor (see Definition 98) and L and $\mathbf{Lan}_{Y_{\mathcal{A}}}(L.Y_{\mathcal{A}})$ preserving all existing colimits, we can derive that $L \simeq \mathbf{Lan}_{Y_{\mathcal{A}}}(L.Y_{\mathcal{A}})$ and we know, by hypothesis, that L preserves finite limits. This ends the proof of $L.Y_{\mathcal{A}}$ being an object of $\mathbf{Filt}[\mathcal{A}, \mathcal{B}]$. In order to prove that ε is an equivalence of categories, we need to verify it is (a) fully faithful and (b) essentially surjective on objects.

(a) ε is full and faithful. Let $R_1, R_2 : \mathcal{B} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$ be two geometric morphisms and take their images through the functor ε. Consider a natural transformation $\alpha : L_1.Y_{\mathcal{A}} \longrightarrow L_2.Y_{\mathcal{A}}$ between these images, where L_1 and L_2 denote the left adjoint functors to R_1 and to R_2, respectively. Namely, for any object $A \in \mathcal{A}$, we have a morphism $\alpha_A : L_1\mathcal{A}(-, A) \longrightarrow L_2\mathcal{A}(-, A)$ in $\mathcal{B}$. We wish to show there exists a unique natural transformation $\beta : L_1 \longrightarrow L_2$ such that $\varepsilon(\beta) = \alpha$, that is, $\beta.Y_{\mathcal{A}} = \alpha$. Consider an arbitrary functor $F \in [\mathcal{A}^{op}, \mathbf{set}]$, which can always be thought of as a (small) colimit of representable functors, namely $F = \mathrm{colim}_{i\in I}\, \mathcal{A}(-, A_i)$. Since the functors L_1 and L_2 preserve all existing colimits, in order to define the natural transformation β, we take the canonical morphism in $\mathcal{B}$ induced by all components α_{A_i} of α, as follows:

$$\begin{tikzcd}
L_1(F) \simeq \underset{i\in I}{\mathrm{colim}}\, L_1\mathcal{A}(-, A_i) \arrow[r, dotted, "\theta"] & \underset{i\in I}{\mathrm{colim}}\, L_2\mathcal{A}(-, A_i) \simeq L_2(F) \\
L_1\mathcal{A}(-, A_i) \arrow[u, "L_1 s_i"] \arrow[r, "\alpha_{A_i}"'] & L_2\mathcal{A}(-, A_i) \arrow[u, "L_2 s_i"']
\end{tikzcd}$$

namely, θ is the unique arrow of $\mathcal{B}$ satisfying $\theta.L_1 s_i = L_2 s_i.\alpha_{A_i}$, for any $i \in I$. (Here we denote by $s_i : \mathcal{A}(-, A_i) \longrightarrow \mathrm{colim}_{i\in I}\, \mathcal{A}(-, A_i)$ the canonical colimit injection in $[\mathcal{A}^{op}, \mathbf{set}]$.) We define the component of β at F to be $\beta_F := \theta$. First of all, the transformation β is natural. For it, let $\gamma : F \longrightarrow G$ be an arrow in $[\mathcal{A}^{op}, \mathbf{set}]$ and, for any $i \in I$, compute the composite $\gamma.s_i : \mathcal{A}(-, A_i) \longrightarrow G = \mathrm{colim}_{j\in J}\, \mathcal{A}(-, B_j)$, provided G, as an object of $[\mathcal{A}^{op}, \mathbf{set}]$, is given by the colimit $(\mathrm{colim}_{j\in J}\, \mathcal{A}(-, B_j), t_j)$ in $[\mathcal{A}^{op}, \mathbf{set}]$. We have:

$$\begin{aligned}
\mathbf{Nat}(\mathcal{A}(-, A_i), \underset{j\in J}{\mathrm{colim}}\, \mathcal{A}(-, B_j)) &\simeq \underset{j\in J}{\mathrm{colim}}\, \mathcal{A}(-, B_j)(A_i) && (1)\\
&\simeq \underset{j\in J}{\mathrm{colim}}\, \mathcal{A}(A_i, B_j) && (2)\\
&\simeq \underset{j\in J}{\mathrm{colim}}\, \mathbf{Nat}(\mathcal{A}(-, A_i), \mathcal{A}(-, B_j)) && (3)
\end{aligned}$$

(where (1) and (3) are just instances of the Yoneda Lemma and (2) follows from the fact that colimits are computed pointwise in $[\mathcal{A}^{op}, \mathbf{set}]$). Therefore, the morphism $\gamma.s_i$ has to factor in $[\mathcal{A}^{op}, \mathbf{set}]$ through a colimit component, as in the following diagram:

$$\begin{tikzcd}
\mathcal{A}(-, A_i) \arrow[r, "\gamma.s_i"] \arrow[dr, dotted, "\gamma'_i"'] & \underset{j\in J}{\mathrm{colim}}\, \mathcal{A}(-, B_j) \\
& \mathcal{A}(-, B_j) \arrow[u, "t_j"']
\end{tikzcd}$$

for some $j \in J$ and some arrow $\gamma_i' : \mathcal{A}(-, A_i) \longrightarrow \mathcal{A}(-, B_j)$ in $[\mathcal{A}^{op}, \mathbf{set}]$. To prove that the transformation β is natural is to prove the identity $L_2\gamma.\beta_F.L_1 s_i = \beta_G.L_1\gamma.L_1 s_i$, for any $i \in I$: this is given by diagram-chasing on the following cube:

$$\begin{tikzcd}
 & L_1\mathcal{A}(-,A_i) \arrow[rr, "\alpha_{A_i}"] \arrow[dl, "L_1 s_i"] \arrow[dd, "L_1\gamma_i'"] & & L_2\mathcal{A}(-,A_i) \arrow[dl, "L_2 s_i"] \arrow[dd, "L_2\gamma_i'"] \\
\operatorname*{colim}_{i\in I} L_1\mathcal{A}(-,A_i) \arrow[rr, "\beta_F"] \arrow[dd, "L_1\gamma"] & & \operatorname*{colim}_{i\in I} L_2\mathcal{A}(-,A_i) \arrow[dd, "L_2\gamma"] & \\
 & L_1\mathcal{A}(-,B_j) \arrow[rr, "\alpha_{B_j}"'] \arrow[dl, "L_1 t_j"] & & L_2\mathcal{A}(-,B_j) \arrow[dl, "L_2 t_j"] \\
\operatorname*{colim}_{j\in J} L_1\mathcal{A}(-,B_j) \arrow[rr, "\beta_G"'] & & \operatorname*{colim}_{j\in J} L_2\mathcal{A}(-,B_j) &
\end{tikzcd}$$

Clearly, we have $\beta.Y_\mathcal{A} = \alpha$, by definition of the natural transformation β. Moreover, β is the unique natural transformation with this property. For it, suppose there exists another natural transformation $\delta : F \longrightarrow G$ such that $\delta.Y_\mathcal{A} = \alpha$. By the fact that β and δ agree when pre-composed with $Y_\mathcal{A}$, it follows that they coincide, since β is entirely determined by its naturality, once we know its component at A, for any $A \in \mathcal{A}$.
(b) ε is essentially surjective on objects. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a filtering functor. Since the category $\mathcal{B}$ is cocomplete, we can always define a pair of adjoint functors $\mathcal{B}(F(-), -) : \mathcal{B} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$, which assigns to any object $B \in \mathcal{B}$ the set-valued functor $\mathcal{B}(F(-), B)$, and $\mathbf{Lan}_{Y_\mathcal{A}}(F) : [\mathcal{A}^{op}, \mathbf{set}] \longrightarrow \mathcal{B}$, which is given by the left Kan extension of F along the Yoneda embedding $Y_\mathcal{A}$. By Proposition 110, the fact that the functor F is filtering is equivalent to the left Kan extension $\mathbf{Lan}_{Y_\mathcal{A}}(F)$ being left exact. Therefore, we get a geometric morphism

$$\begin{tikzcd}
\mathcal{B} \arrow[r, shift right=1.5ex, "\mathcal{B}(F(-),-)"'] \arrow[r, phantom, "\perp"] & {[\mathcal{A}^{op}, \mathbf{set}]} \arrow[l, shift right=1.5ex, "\mathbf{Lan}_{Y_\mathcal{A}}(F)"']
\end{tikzcd}$$

which, in particular, satisfies the condition $F \simeq \mathbf{Lan}_{Y_\mathcal{A}}(F).Y_\mathcal{A}$, since $Y_\mathcal{A}$ is fully faithful. Namely, $\varepsilon(\mathbf{Lan}_{Y_\mathcal{A}}(F)) \simeq F$ and this ends the proof of the functor ε being essentially surjective on objects. ■

Some conditions our functors might satisfy

Let $\mathcal{C}$ be a small category and $k : \mathcal{C} \longrightarrow \mathcal{E}$ a functor into an arbitrary category $\mathcal{E}$.

Definition 112

- *A collection of maps* $(x_i : X_i \longrightarrow X)_{i \in I}$ *in* $\mathcal{E}$ *is said to be* epimorphic *if, given a pair of arrows* $u, v : X \longrightarrow U$*, the condition* $u.x_i = v.x_i$ *for every* $i \in I$ *implies the equality* $u = v$*.*

- *Moreover, a collection of maps* $(f_i : C_i \longrightarrow C)_{i \in I}$ *in* $\mathcal{C}$ *is called* k-epimorphic *if the collection* $(k(f_i))_{i \in I}$ *is epimorphic in* $\mathcal{E}$*.*

Stated otherwise, $(x_i : X_i \longrightarrow X)_{i \in I}$ is an epimorphic family in $\mathcal{E}$ if and only if the canonical morphism x, defined by the universal property of the coproduct, as the unique arrow with the property $x.\sigma_i = x_i$, for any $i \in I$, as in the following diagram:

$$
\begin{tikzcd}
\coprod_{I \in I} X_i \arrow[r, dotted, "x"] & X \\
X_i \arrow[u, "\sigma_i"] \arrow[ur, "x_i"'] &
\end{tikzcd}
$$

is an epimorphism in $\mathcal{E}$. (Here we denote by σ_i the canonical coproduct injection.) This second formulation allows us to introduce more easily the notion of a *regular epimorphic* family, which is nothing but a collection $(x_i : X_i \longrightarrow X)_{i \in I}$ of maps in $\mathcal{E}$ such that the above-mentioned canonical morphism x is, in fact, a regular epimorphism in $\mathcal{E}$. Note that, in general, there is no reason why I should be finite and it is implicit that the category $\mathcal{E}$ has (at least) coproducts in order for the above definitions to make sense.

Lemma 113 *Let* $(x_i : X_i \longrightarrow X)_{i \in I}$ *be a family of maps in* $\mathcal{E}$ *such that there exists a subset* $J \subseteq I$ *for which* $(x_j : X_j \longrightarrow X)_{j \in J}$ *is an epimorphic family. Then, the entire family* $(x_i : X_i \longrightarrow X)_{i \in I}$ *is epimorphic.*

Finally, we still need to introduce the notion of a k-epimorphic subfunctor. We should say explicitly that, for any subfunctor $r : R \rightarrowtail \mathcal{C}(-, C)$, we identify R with the set-theoretic union of all subsets $R(D) \subseteq \mathcal{C}(D, C)$, for D varying in $\mathcal{C}$; as well as, in the proofs contained in the next sections, we allow ourselves to make no notational distinction between a functor $F \in [\mathcal{C}^{op}, \mathbf{set}]$ and the set-theoretic union of all sets $F(C)$, when C varies in $\mathcal{C}$.

Definition 114

- *We call the subfunctor* $r : R \rightarrowtail \mathcal{E}(k(-), X)$ epimorphic *if the corresponding collection of maps* $\{c : k(C) \longrightarrow X | c \in R(C), C \in \mathcal{C}\}$ *in* $\mathcal{E}$ *is epimorphic.*
- *We call* $r : R \rightarrowtail \mathcal{C}(-, C)$ *a* k-epimorphic *subfunctor if the corresponding collection of maps* $\{d : D \longrightarrow C | d \in R(D), D \in \mathcal{C}\}$ *in* $\mathcal{C}$ *is* k*-epimorphic.*

Following the suggestion of [32], we reformulate the main result (for our purposes) about k-epimorphic subfunctors in terms of any cocomplete, exact and extensive category $\mathcal{E}$:

Lemma 115 *Let* $\mathcal{C}$ *be a small category,* $\mathcal{E}$ *a cocomplete, exact and extensive category and* $k : \mathcal{C} \longrightarrow \mathcal{E}$ *a functor. Suppose the functor* $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$*, defined on an object* $X \in \mathcal{E}$ *as* $\mathcal{E}(k(-), X)$*, is a geometric morphism, whose left exact left adjoint functor we denote by* L*. Let* $r : R \rightarrowtail \mathcal{C}(-, C)$ *be a subfunctor in* $[\mathcal{C}^{op}, \mathbf{set}]$*. The following conditions are equivalent:*

1. $L(r)$ *is an isomorphism of* $\mathcal{E}$*;*
2. $L(r)$ *is an epimorphism of* $\mathcal{E}$*;*
3. $r : R \rightarrowtail \mathcal{C}(-, C)$ *is a* k*-epimorphic subfunctor.*

Proof. The functor L being left exact, the arrow $L(r)$ is always a monomorphism; therefore, since by Proposition 95 in any exact and extensive category epimorphisms coincide with regular epimorphisms, Conditions 1 and 2 are equivalent. To prove the equivalence between Conditions 2 and 3, consider the canonical morphism r' induced in $[\mathcal{C}^{op}, \mathbf{set}]$, via the universal property of the coproduct, by all arrows $f \in R$, as follows:

$$\begin{tikzcd} \coprod_{f \in R} \mathcal{C}(-, C_f) \arrow[r, dotted, "r'"] & \mathcal{C}(-, C) \\ \mathcal{C}(-, C_f) \arrow[u, "\rho_f"] \arrow[ur, "\mathcal{C}(-,f)"'] & \end{tikzcd}$$

where the ρ_f denote the canonical coproduct injections. By taking in $[\mathcal{C}^{op}, \mathbf{set}]$ the regular epi-mono factorisation of r', we can prove that its image is given precisely by R, as in the following diagram:

$$\begin{tikzcd} \coprod_{f \in R} \mathcal{C}(-, C_f) \arrow[rr, "r'"] \arrow[dr, two heads] & & \mathcal{C}(-, C) \\ & R \arrow[ur, tail, "r"'] & \end{tikzcd}$$

For it, it suffices to verify the statement pointwise in **set**. For any $A \in \mathcal{C}$, the image of r' is defined as the set $I(A) = \{h : A \longrightarrow C | \exists f \in R(C_f)$ and $h_A : A \longrightarrow C_f : h = f.h_A\}$. On the other hand, R being a subfunctor satisfies the property that, for any $a : A \longrightarrow A'$ in $\mathcal{C}$, provided g is an arrow in $R(A')$, the composite $g.a$ belongs to $R(A)$. It is easy to verify, then, that $I(A)$ and $R(A)$ coincide at any object $A \in \mathcal{C}$. As we will make explicit in the next section, L is nothing but the left Kan extension of the functor k along the Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$. This is the reason why, $Y_{\mathcal{C}}$ being fully faithful, we have the isomorphism $k \simeq L.Y_{\mathcal{C}}$, that is, for any $C \in \mathcal{C}$, we get $k(C) \simeq L.\mathcal{C}(-, C)$. Since L is an exact functor, considering the image through L of the previous factorisation gives the following triangle in $\mathcal{E}$:

$$\begin{tikzcd} \coprod_{f \in R} k(C_f) \arrow[rr, "L(r')"] \arrow[dr] & & k(C) \\ & L(R) \arrow[ur, tail, "L(r)"'] & \end{tikzcd}$$

Moreover, the morphism $L(r')$ is precisely the one induced by the maps $k(f)$, for $f \in R$, that is:

$$\begin{tikzcd} \coprod_{f \in R} k(U_f) \arrow[rr, dotted, "\theta"] & & k(C) \\ k(C_f) \arrow[u, "\sigma_f"] \arrow[urr, "k(f)"'] & & \end{tikzcd}$$

(where the coproduct injections σ_f are nothing but the images of the ρ_f by L, since L does preserve coproducts). For it, from the fact that $r'.\rho_f = \mathcal{C}(-, f)$, for any $f \in R$, it follows that $L(r').\sigma_f \simeq L(r').L(\rho_f) = L(\mathcal{C}(-, f)) \simeq k(f)$. Therefore, $L(r')$ has to coincide with the unique arrow θ, as in the previous diagram, satisfying this property. Then, assuming $L(r)$ is an epimorphism is equivalent to say that $L(r')$ is an epimorphism (even, regular epimorphism). So, for any pair of morphisms $u, v : k(C) \longrightarrow U$ in $\mathcal{E}$ verifying $u.k(f) = v.k(f)$, for any $f \in R$, we have $u.L(r').\sigma_f = v.L(r').\sigma_f$, for any $f \in R$, which means that $u.L(r') = v.L(r')$, by definition of the coproduct $\coprod_{f \in R} k(U_f)$. It follows, from $L(r')$ being an epimorphism, that $u = v$. This is precisely the condition on $R \rightarrowtail \mathcal{C}(-, C)$ of being a k-epimorphic subfunctor. Clearly, the converse holds too: $r : R \rightarrowtail \mathcal{C}(-, C)$ being a k-epimorphic subfunctor is equivalent to the fact that the condition $u.L(r') = v.L(r')$ implies $u = v$, since the morphism $L(r')$ is precisely the one induced by the arrows $k(f)$, for f varying in R; namely, $L(r')$ is an epimorphism.■

It is convenient to list a priori some properties about the functors we are going to deal with in the following sections. The purpose is to present here the conditions which allow us to characterise those functors that induce localizations.

(A) For an object $X \in \mathcal{E}$, consider the collection R_X of all maps

$$\{c : k(C) \longrightarrow X | C \in \mathcal{C}\}$$

in $\mathcal{E}$. The functor k satisfies (A) if and only if, for any object $X \in \mathcal{E}$, the family R_X is epimorphic.

(A_{reg}) – A slightly different version of (A) – For an object $X \in \mathcal{E}$, consider the collection R_X of all maps $\{c : k(C) \longrightarrow X | C \in \mathcal{C}\}$ in $\mathcal{E}$. The functor k satisfies (A_{reg}) if and only if, for any object $X \in \mathcal{E}$, the family R_X is regular epimorphic.

(B) For an object $X \in \mathcal{E}$ and a pair of arrows $k(A) \xleftarrow{a} X \xrightarrow{b} k(B)$ in $\mathcal{E}$, consider the collection $R_{a,b}$ of all maps

$$\left\{ c_i : k(C_i) \longrightarrow X \,\middle|\, \left\{ \begin{array}{l} a.c_i = k(f_A) \\ b.c_i = k(f_B) \end{array} \right. \text{for some } A \xleftarrow{f_A} C_i \xrightarrow{f_B} B \right\}$$

in $\mathcal{E}$. The functor k satisfies (B) if and only if, for any such a pair (a, b), the family $R_{a,b}$ is epimorphic.

(B') – A particular instance of (B) – For any arrow $f : k(A) \longrightarrow k(B)$ in $\mathcal{E}$, consider the collection R_f of all maps

$$\{c_i : C_i \longrightarrow A | f.k(c_i) = k(g_i) \text{ for some } g_i : C_i \longrightarrow B\}$$

in $\mathcal{C}$. The functor k satisfies (B') if and only if, for any such an arrow f, the family R_f is k-epimorphic.

(C) For a pair of arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{C}$, consider the collection $R_{f\&g}$ of all maps

$$\{c_i : C_i \longrightarrow A | f.c_i = g.c_i\}$$

in $\mathcal{C}$. The functor k satisfies (C) if and only if, for any such a pair (f, g), if $k(f) = k(g)$, then, the family $R_{f\&g}$ is k-epimorphic.

3.2 The localizations of presheaf categories

Let $k : \mathcal{C} \longrightarrow \mathcal{E}$ be a functor from a small category $\mathcal{C}$ into a cocomplete category $\mathcal{E}$. We can always define a new functor $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$ from k, by assigning to any object $X \in \mathcal{E}$ the set-valued functor $\mathcal{E}(k(-), X) : \mathcal{C}^{op} \longrightarrow \mathbf{set}$ and, analogously, to any morphism $f : X \longrightarrow Y$ in $\mathcal{E}$, the corresponding natural transformation $\mathcal{E}(k(-), f) : \mathcal{E}(k(-), X) \longrightarrow \mathcal{E}(k(-), Y)$ given by post-composing with f. Moreover, $\mathcal{E}$ being cocomplete, the functor k' has always a left adjoint, defined by taking the left Kan extension of k along the Yoneda embedding $Y_{\mathcal{C}}$, as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Y_{\mathcal{C}}"] \arrow[d, "k"'] & {[\mathcal{C}^{op}, \mathbf{set}]} \arrow[dl, dotted, "\mathbf{Lan}_{Y_{\mathcal{C}}}(k)"] \\ \mathcal{E} & \end{tikzcd}$$

Therefore, provided $L := \mathbf{Lan}_{Y_{\mathcal{C}}}(k)$, we have $L \dashv k'$.
We want to give necessary and sufficient conditions on k to realise $\mathcal{E}$ as a localization of the category of presheaves on $\mathcal{C}$. To simplify our presentation, we split the result in two independent parts and start by introducing the following preliminary:

Lemma 116 *Let $\mathcal{C}$ be a small category, $\mathcal{E}$ a cocomplete, exact and extensive category and $k : \mathcal{C} \longrightarrow \mathcal{E}$ a functor. Suppose k satisfies Conditions* (A) *and* (B)*, then the functor $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$, defined by the assignment $X \mapsto \mathcal{E}(k(-), X)$, is full and faithful.*

Proof. Let X, Y be two objects of $\mathcal{E}$ and $\alpha : \mathcal{E}(k(-), X) \longrightarrow \mathcal{E}(k(-), Y)$ an arrow in $[\mathcal{C}^{op}, \mathbf{set}]$. We wish to show there exists a unique morphism $b : X \longrightarrow Y$ of $\mathcal{E}$ such that $\mathcal{E}(k(-), b) = \alpha$. Consider, first, the collection of maps $R_X = \{h : k(C_h) \longrightarrow X | C_h \in \mathcal{C}\}$ in $\mathcal{E}$ and the canonical morphism λ_X induced by these arrows $h \in R_X$, because of the universal property of the coproduct, as follows:

$$\begin{tikzcd} \coprod_{h \in R_X} k(C_h) \arrow[r, "\lambda_X"] & X \\ k(C_h) \arrow[u, "\sigma_h"] \arrow[ur, "h"'] & \end{tikzcd}$$

Namely, λ_X is the unique arrow in $\mathcal{E}$ satisfying $\lambda_X.\sigma_h = h$, for any $h \in R_X$. (Here we denote by σ_h the canonical coproduct injection in $\mathcal{E}$.)

Condition (A) implies that the family R_X is epimorphic; in other words, λ_X is an epimorphism in $\mathcal{E}$. Therefore, since the category $\mathcal{E}$ is exact and extensive, by Proposition 95, λ_X is, actually, a regular epimorphism. So, by taking the kernel pair of λ_X in $\mathcal{E}$:

$$\begin{tikzcd} N(\lambda_X) \arrow[r, shift left, "\lambda_0"] \arrow[r, shift right, "\lambda_1"'] & \coprod\limits_{h\in R_X} k(C_h) \arrow[r, two heads, "\lambda_X"] & X \end{tikzcd}$$

we already know that λ_X is the coequalizer of its kernel pair (λ_0, λ_1). When considering the pullback $P(h,h')$ of h along h' in $\mathcal{E}$, for any pair of arrows $(h,h') \in R_X \times R_X$, we get a canonical comparison morphism $s_{(h,h')}$, as in the following diagram (provided, as we know, that $\lambda_X.\sigma_h = h$, for any $h \in R_X$):

$$\begin{tikzcd} P(h,h') \arrow[rr, "p_h"] \arrow[dd, "p_{h'}"'] \arrow[dr, dotted, "s_{(h,h')}"] & & k(C_h) \arrow[d, "\sigma_h"] \\ & N(\lambda_X) \arrow[r, "\lambda_0"] \arrow[d, "\lambda_1"'] \arrow[dr, phantom, "(1)"] & \coprod\limits_{h\in R_X} k(C_h) \arrow[d, "\lambda_X"] \\ k(C_{h'}) \arrow[r, "\sigma_{h'}"'] & \coprod\limits_{h\in R_X} k(C_h) \arrow[r, "\lambda_X"'] & X \end{tikzcd}$$

Diagram 3.2

where the outer square and the inner right-hand square (1) are pullbacks and the morphism $s_{(h,h')} : P(h,h') \longrightarrow N(\lambda_X)$ is the unique one with the property $\lambda_0.s_{(h,h')} = \sigma_h.p_h$ and $\lambda_1.s_{(h,h')} = \sigma_{h'}.p_{h'}$, given by the universal property of the pullback (1). First, we want to show that $N(\lambda_X)$, in fact, coincides with the coproduct $\coprod_{(h,h')\in R_X\times R_X} P(h,h')$, with the morphisms $s_{(h,h')} : P(h,h') \longrightarrow N(\lambda_X)$ as coproduct injections, for any pair $(h,h') \in R_X \times R_X$. For it, we need to analyse in more details the pullback in Diagram 3.2. This is obtained by the gluing of four pullbacks in $\mathcal{E}$, as follows:

$$\begin{tikzcd}
P(h,h') \arrow[r] \arrow[d] \arrow[dr, phantom, "(3)"] & N^0_h \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & k(C_h) \arrow[d, "\sigma_h"] \\
N^1_{h'} \arrow[r, "\tau_{h'}"] \arrow[d] \arrow[dr, phantom, "(4)"] & N(\lambda_X) \arrow[r, "\lambda_0"] \arrow[d, "\lambda_1"'] \arrow[dr, phantom, "(1)"] & \coprod_{h\in R_X} k(C_h) \arrow[d, "\lambda_X"] \\
k(C_{h'}) \arrow[r, "\sigma_{h'}"'] & \coprod_{h\in R_X} k(C_h) \arrow[r, "\lambda_X"'] & X
\end{tikzcd}$$

Using the extensivity of the category $\mathcal{E}$, we derive, from pullback (2), that $N(\lambda_X) \simeq \coprod_{h\in R_X} N^0_h$ and, from pullback (4), that $N(\lambda_X) \simeq \coprod_{h\in R_X} N^1_h$. In light of this second isomorphism and considering the index $h \in R_X$ as fixed, we can regard pullback (3) as:

$$\begin{tikzcd}
P(h,h') \arrow[r] \arrow[d] & N^0_h \arrow[d] \\
N^1_{h'} \arrow[r, "\tau_{h'}"'] & N(\lambda_X) \simeq \coprod_{h\in R_X} N^1_h
\end{tikzcd}$$

and, then, from the extensivity of $\mathcal{E}$ again, it follows that $N^0_h \simeq \coprod_{h'\in R_X} P(h,h')$. Therefore, by putting all these isomorphisms together, we get:

$$N(\lambda_X) \simeq \coprod_{h\in R_X} N^0_h \simeq \coprod_{h\in R_X} \coprod_{h'\in R_X} P(h,h') \simeq \coprod_{(h,h')\in R_X\times R_X} P(h,h')$$

That is, the kernel pair of λ_X is given by the coproduct of all pullbacks $P(h,h')$ over the pairs of arrows (h,h') varying in $R_X \times R_X$. Concerning the coproduct injections, it is clear from the context that they have to be, at each component (h,h'), precisely the diagonal morphisms in the pullbacks of the kind (3), namely $s_{(h,h')}$. Note that for any $h \in R_X$, which is nothing but an arrow $h : k(C_h) \longrightarrow X$, the natural transformation $\alpha : \mathcal{E}(k(-),X) \longrightarrow \mathcal{E}(k(-),Y)$ produces a morphism $\alpha_{C_h}(h) : k(C_h) \longrightarrow Y$ in $\mathcal{E}$. All these maps $\alpha_{C_h}(h)$ induce, in fact, a canonical morphism, by the universal property of the coproduct, as follows:

$$
\begin{tikzcd}
\coprod_{h\in R_X} k(C_h) \arrow[r, "\mu_X"] & Y \\
k(C_h) \arrow[u, "\sigma_h"] \arrow[ur, "\alpha_{C_h}(h)"'] &
\end{tikzcd}
$$

that is, μ_X is the unique map in $\mathcal{E}$ satisfying $\mu_X.\sigma_h = \alpha_{C_h}(h)$, for any $h \in R_X$. Suppose that $\mu_X.\lambda_0 = \mu_X.\lambda_1$.

$$
\begin{tikzcd}
N(\lambda_X) \arrow[r, shift left, "\lambda_0"] \arrow[r, shift right, "\lambda_1"'] & \coprod_{h\in R_X} k(C_h) \arrow[r, two heads, "\lambda_X"] \arrow[dr, "\mu_X"'] & X \arrow[d, dotted, "b"] \\
& & Y
\end{tikzcd}
$$

Since λ_X is the coequalizer of the pair (λ_0, λ_1), we obtain a unique morphism $b : X \longrightarrow Y$ such that $b.\lambda_X = \mu_X$, namely $b.\lambda_X.\sigma_h = \mu_X.\sigma_h$, for any $h \in R_X$. This condition is equivalent to $\mathcal{E}(k(C_h), b)(h) = b.h = b.\lambda_X.\sigma_h = \mu_X.\sigma_h = \alpha_{C_h}(h)$, for any morphism $h : k(C_h) \longrightarrow X$ in $\mathcal{E}$, by definition, respectively, of the arrow λ_X and μ_X. This is precisely fully faithfulness of the functor k'. So, it only remains to show that $\mu_X.\lambda_0 = \mu_X.\lambda_1$, or equivalently, that $\mu_X.\lambda_0.s_{(h,h')} = \mu_X.\lambda_1.s_{(h,h')}$, for any pair $(h, h') \in R_X \times R_X$, since we have just proved that $N(\lambda_X)$ is the coproduct of all pullbacks of the kind $P(h, h')$. In other words, we need to prove that $\alpha_{C_h}(h).p_h = \alpha_{C_{h'}}(h').p_{h'}$, for any $(h, h') \in R_X \times R_X$, by definition of the morphisms $s_{(h,h')}$ and μ_X. Consider, then, the span in $\mathcal{E}$:

$$k(C_h) \xleftarrow{p_h} P(h, h') \xrightarrow{p_{h'}} k(C_{h'})$$

By Condition (B), the family $R_{p_h,p_{h'}}$ of all arrows in $\mathcal{E}$

$$\left\{ c_i : k(C_i) \longrightarrow P(h, h') \,\middle|\, \left\{ \begin{array}{l} p_h.c_i = k(f) \\ p_{h'}.c_i = k(g) \end{array} \right. \text{for some}\, C_h \xleftarrow{f} C_i \xrightarrow{g} C_{h'} \right\}$$

is epimorphic. This precisely means that the arrows $\alpha_{C_h}(h).p_h$ and $\alpha_{C_{h'}}(h').p_{h'}$ coincide if and only if the identity $\alpha_{C_h}(h).p_h.c_i = \alpha_{C_{h'}}(h').p_{h'}.c_i$ holds for any $c_i \in R_{p_h,p_{h'}}$. Using the naturality of the transformation α, relative to the maps $f : C_i \longrightarrow C_h$ and $g : C_i \longrightarrow C_{h'}$

in $\mathcal{C}$, we get, for any $c_i \in R_{p_h,p_{h'}}$:

$$\begin{array}{rcll} \alpha_{C_h}(h).p_h.c_i & = & \alpha_{C_h}(h).k(f) & \\ & = & \alpha_{C_i}(h.k(f)) & (1) \\ & = & \alpha_{C_i}(h'.k(g)) & (2) \\ & = & \alpha_{C_{h'}}(h').k(g) & (3) \\ & = & \alpha_{C_{h'}}(h').p_{h'}.c_i & \end{array}$$

(where (1) and (3) are instances of the naturality of α and (2) follows from the definition of $P(h,h')$ as the pullback of h along h' and the fact that $c_i \in R_{p_h,p_{h'}}$). This ends the proof of $\mathcal{E}(k(C_h),b)(h)$ being equal to $\alpha_{C_h}(h)$, for any arrow $h : k(C_h) \longrightarrow X$ in $\mathcal{E}$, for some $C_h \in \mathcal{C}$; in other words, we have shown that $\mathcal{E}(k(-),b) = \alpha$ for a unique morphism $b : X \longrightarrow Y$ of $\mathcal{E}$, as desired. ■

Finally, we state the expected Theorem which classifies the localizations of presheaf categories:

Theorem 117 *Let $\mathcal{C}$ be a small category, $\mathcal{E}$ a cocomplete, exact and extensive category and $k : \mathcal{C} \longrightarrow \mathcal{E}$ a functor. The following conditions are equivalent:*

1. *the functor $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$, defined by the assignment $X \mapsto \mathcal{E}(k(-),X)$, is a localization;*

2. *the functor k satisfies the Conditions* (A), (B) *and it is filtering.*

Proof. $1 \Rightarrow 2$. Suppose the functor $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$, defined by $X \mapsto \mathcal{E}(k(-),X)$, is a localization of the presheaf category $[\mathcal{C}^{op}, \mathbf{set}]$, whose left adjoint we denote by $L : [\mathcal{C}^{op}, \mathbf{set}] \longrightarrow \mathcal{E}$, so we have that k' is fully faithful, L preserves finite limits and $L \dashv k'$. As we know, since $\mathcal{E}$ is cocomplete, there always exists the left Kan extension of k along the Yoneda embedding $Y_{\mathcal{C}}$, which produces a left adjoint functor $\mathbf{Lan}_{Y_{\mathcal{C}}}(k)$ to k'. By uniqueness (up to isomorphism) of the left adjoint functor, we have $L \simeq \mathbf{Lan}_{Y_{\mathcal{C}}}(k)$. Moreover, note that k' being fully faithful is equivalent to the counit for the adjunction $L \dashv k'$ being an isomorphism, namely $L.k' \simeq 1_{\mathcal{E}}$. On the other hand, we denote by $\eta : 1_{[\mathcal{C}^{op},\mathbf{set}]} \longrightarrow k'.L$ the unit for the adjunction. Since the Yoneda embedding $Y_{\mathcal{C}}$ is fully faithful, the universal arrow for the left Kan extension L of k along Yoneda is an isomorphism, namely $k \simeq L.Y_{\mathcal{C}}$, that is $L.\mathcal{C}(-,C) \simeq k(C)$, for any object $C \in \mathcal{C}$; so, when evaluated at representable functors $F = \mathcal{C}(-,C)$, the component of η is the natural transformation $\eta_C : \mathcal{C}(-,C) \longrightarrow \mathcal{E}(k(-),k(C))$, defined on any object $A \in \mathcal{C}$ as the application of functor k, by the assignment

$\mathcal{C}(A,C) \longrightarrow \mathcal{E}(k(A),k(C)) : f \mapsto k(f)$. Therefore, the current situation can be summarised as in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr, "k"'] & \mathbf{Fam}\mathcal{C} \arrow[r] \arrow[d, "\overline{k}"] & (\mathbf{Fam}\mathcal{C})_{ex} \simeq [\mathcal{C}^{op}, \mathbf{set}] \arrow[dl, "L"] \\ & \mathcal{E} & \end{tikzcd}$$

where the relation between these functors is established in Proposition 110, since the category $\mathcal{E}$ is exact and extensive. Explicitly, k is filtering if and only if $\overline{k}$ is left covering if and only if L is left exact. Then, it only remains to prove Conditions (A) and (B) on the functor k.

k satisfies Condition (A). For it, consider an object $X \in \mathcal{E}$ and the collection $R_X = \{c : k(C) \longrightarrow X | C \in \mathcal{C}\}$ of all arrows in $\mathcal{E}$. We want to prove that R_X is an epimorphic family. Let $u, v : X \longrightarrow U$ be a pair of arrows in $\mathcal{E}$ such that $u.c = v.c$, for any $c \in R_X$. This is equivalent to say that the two functions $\mathcal{E}(k(C),u), \mathcal{E}(k(C),v) : \mathcal{E}(k(C),X) \longrightarrow \mathcal{E}(k(C),U)$ coincide at any object $C \in \mathcal{C}$, on any morphism $c : k(C) \longrightarrow X$, that is, $k'(u) = k'(v)$. Since k' is faithful, it follows that $u = v$, as desired.

k satisfies Condition (B). For it, consider an object $X \in \mathcal{E}$, a pair of arrows $k(A) \xleftarrow{a} X \xrightarrow{b} k(B)$ in $\mathcal{E}$ and the corresponding collection $R_{a,b}$ of maps in $\mathcal{E}$

$$\left\{ c_i : k(C_i) \longrightarrow X \,\middle|\, \begin{cases} a.c_i = k(f_A) \\ b.c_i = k(f_B) \end{cases} \text{for some } A \xleftarrow{f_A} C_i \xrightarrow{f_B} B \right\}$$

We want to prove that $R_{a,b}$ is an epimorphic family. Let R_X be the collection of maps $\{c : k(C) \longrightarrow X | C \in \mathcal{C}\}$ and, for any $c \in R_X$, consider, first, the composite $a.c : k(C) \longrightarrow k(A)$ in $\mathcal{E}$. Take the pullback of the unit component η_A along the composite $k'(a.c).\eta_C$ in $[\mathcal{C}^{op}, \mathbf{set}]$, as follows:

$$\begin{tikzcd} P_{a.c} \arrow[rr] \arrow[d, "p"'] & & \mathcal{C}(-,A) \arrow[d, "\eta_A"] \\ \mathcal{C}(-,C) \arrow[r, "\eta_C"'] & \mathcal{E}(k(-),k(C)) \arrow[r, "k'(a.c)"'] & \mathcal{E}(k(-),k(A)) \end{tikzcd}$$

Explicitly, since limits are computed pointwise in $[\mathcal{C}^{op}, \mathbf{set}]$, this pullback is given, at any object $C_x \in \mathcal{C}$, by the collection $P_{a.c}(C_x) = \{(x, f_x) \in \mathcal{C}(C_x,C) \times \mathcal{C}(C_x,A) | a.c.k(x) = k(f_x)\}$. Take, then, the regular epi-mono factorisation of the morphism p in the presheaf category $[\mathcal{C}^{op}, \mathbf{set}]$ (which is regular), as in the diagram:

$$\begin{tikzcd} P_{a.c} \arrow[rr, "p"] \arrow[dr] & & \mathcal{C}(-,C) \\ & R_{a.c} \arrow[ur, tail, "r"'] & \end{tikzcd}$$

Since k' is fully faithful, we have already remarked that the counit for the adjunction $L \dashv k'$ is an isomorphism and this, in particular, implies that $L(\eta_A)$ is an isomorphism. Therefore, since both the pullback and the regular epi-mono factorisation are preserved by L, which is an exact functor, we get that $L(p)$ is an isomorphism, and so is $L(r)$. From Lemma 115 it follows, then, that the subfunctor $r : R_{a.c} \rightarrowtail \mathcal{C}(-, C)$ is k-epimorphic. But regular epi-mono factorisations are computed pointwise in $[\mathcal{C}^{op}, \mathbf{set}]$, that is, the image $R_{a.c}$ of the pullback $P_{a.c}$, at any object $C_x \in \mathcal{C}$, is given by the collection $R_{a.c}(C_x) = \{x : C_x \longrightarrow C | a.c.k(x) = k(f_x) \text{ for some } f_x : C_x \longrightarrow A\}$. For C_x varying in $\mathcal{C}$, this is precisely the family of arrows induced by the k-epimorphic subfunctor $r : R_{a.c} \rightarrowtail \mathcal{C}(-, C)$, and, therefore, $R_{a.c}$ is a k-epimorphic family. Now, for any morphism $x : C_x \longrightarrow C$ in $R_{a.c}$, consider the composite $b.c.k(x) : k(C_x) \longrightarrow k(C) \longrightarrow X \longrightarrow k(B)$. By the same reasoning, one can prove that the subfunctor $R_{b.c.k(x)} \rightarrowtail \mathcal{C}(-, C_x)$ is k-epimorphic, that is, the corresponding collection of morphisms $\{y : C_y \longrightarrow C_x | b.c.k(x).k(y) = k(f_y) \text{ for some } f_y : C_y \longrightarrow B\}$ in $\mathcal{C}$ is a k-epimorphic family. By composing together all arrows of the kind x in $R_{a.c}$ with all arrows of the kind y in $R_{b.c.k(x)}$, we obtain a collection (depending on $c \in R_X$) $S_c = \{k(x).k(y) = k(x.y) : k(C_y) \longrightarrow k(C) | x \in R_{a.c}, y \in R_{b.c.k(x)}\}$, which we wish to prove to be an epimorphic family itself. (Note that here we abuse notation and write $x \in R_{a.c}$ to mean, in fact, $x \in R_{a.c}(C_x)$, for some $C_x \in \mathcal{C}$.) For it, let $u, v : k(C) \longrightarrow U$ be a pair of arrows in $\mathcal{E}$ satisfying $u.k(x).k(y) = v.k(x).k(y)$, for any $x \in R_{a.c}$ and $y \in R_{b.c.k(x)}$. Since the family $R_{b.c.k(x)}$ is k-epimorphic, this amounts to $u.k(x) = v.k(x)$, for any $x \in R_{a.c}$ and using now that $R_{a.c}$ is k-epimorphic, this last statement is further equivalent to $u = v$, as desired. In other words, we have just shown that the family $\{x.y : C_y \longrightarrow C | x \in R_{a.c}, y \in R_{b.c.k(x)}\}$, which we still denote by S_c, is k-epimorphic. Moreover, since $x \in R_{a.c}$, we have $a.c.k(x.y) = a.c.k(x).k(y) = k(f_x).k(y) = k(f_x.y) = k(g_A)$, for some $g_A : C_y \longrightarrow A$ and, since $y \in R_{b.c.k(x)}$, we also have $b.c.k(x.y) = b.c.k(x).k(y) = k(g_B)$, for some $g_B : C_y \longrightarrow B$. So, we have just shown that the collection S_c

is contained in the family of all maps

$$\left\{ z : C_z \longrightarrow C \,\middle|\, \left\{ \begin{array}{l} a.c.k(z) = k(g_A) \\ b.c.k(z) = k(g_B) \end{array} \right. \text{for some } A \xleftarrow{g_A} C_z \xrightarrow{g_B} B \right\}$$

in $\mathcal{C}$. In particular, for any of these arrows $z \in S_c$, the composite $c.k(z) : k(C_z) \longrightarrow X$ does belong to the family $R_{a,b}$, by definition of S_c. So, by considering the collection of all arrows obtained in this way we form a subcollection of $R_{a,b}$; call it $M = \{c.k(z) : k(C_z) \longrightarrow X | c \in R_X, z \in S_c\}$. Finally, it suffices to prove that M is an epimorphic family. For it, let $u, v : X \longrightarrow U$ be a pair of arrows in $\mathcal{E}$ such that $u.c.k(z) = v.c.k(z)$, for any $c \in R_X$ and $z \in S_c$. Since the family S_c is k-epimorphic, this is equivalent to $u.c = v.c$, for any $c \in R_X$, and using now that R_X is epimorphic, by Condition (A), the last statement is further equivalent to $u = v$, as desired. Therefore, M is an epimorphic family which is contained in $R_{a,b}$: it follows from Lemma 113 that $R_{a,b}$ is epimorphic, that is, Condition (B) is verified.

$2 \Rightarrow 1$. Suppose the functor k satisfies the Conditions (A) and (B) and, moreover, k is filtering. As mentioned above, since the category $\mathcal{E}$ is cocomplete, there always exists the left Kan extension of k along the Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$, call it $L := \mathbf{Lan}_{Y_{\mathcal{C}}}(k)$, which is left adjoint to k'. By Proposition 110, we know that k being filtering is equivalent to the functor L being left exact, that is, L preserving finite limits. Concerning the functor k', we have just shown in Lemma 116 that if the functor k satisfies Conditions (A) and (B), then k' is fully faithful. Hence, this ends the proof of $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$ being a localization of the presheaf category $[\mathcal{C}^{op}, \mathbf{set}]$, k' being a fully faithful functor with a left exact left adjoint. ∎

Comparing conditions

Consider a functor $k : \mathcal{C} \longrightarrow \mathcal{E}$ from a small category $\mathcal{C}$ into a cocomplete, exact and extensive category $\mathcal{E}$. We study the relations which occur between Conditions (F1), (F2), (F3) for a filtering functor (see Definition 96) and Conditions (A) and (B) introduced in Section 3.1. There are certain implications that we want to make explicit here, in order to refine the result contained in Theorem 117. The second group of conditions, we have defined, is, indeed, stronger than that giving a filtering functor.

Lemma 118 *Condition* (A) *implies Condition* (F1).

Proof. Since for any object $X \in \mathcal{E}$, the collection of arrows $\{k(C) \longrightarrow X | C \in \mathcal{C}\}$ in $\mathcal{E}$ has to be epimorphic, in particular, it is so when taking

$X = 1$, the terminal object of $\mathcal{E}$ (which exists, $\mathcal{E}$ being, in particular, finitely complete). ■

Note that the converse is not true in general.

Lemma 119 *Condition* (B) *implies Condition* (F2).

Proof. For any pair of objects $A, B \in \mathcal{C}$, consider the product of their images by k, that is, $X := k(A) \times k(B) \in \mathcal{E}$, since $\mathcal{E}$ has binary products. So, we get a span $k(A) \xleftarrow{\pi_A} k(A) \times k(B) \xrightarrow{\pi_B} k(B)$ in $\mathcal{E}$ and, knowing that the functor k satisfies Condition (B), we have a collection R_{π_A,π_B} of maps of $\mathcal{E}$:

$$\left\{ c_i : k(C_i) \longrightarrow X \,\middle|\, \left\{ \begin{array}{l} \pi_A.c_i = k(f_A) \\ \pi_B.c_i = k(f_B) \end{array} \right. \text{for some } A \xleftarrow{f_A} C_i \xrightarrow{f_B} B \right\}$$

which is epimorphic. By definition of X as the binary product of $k(A)$ and $k(B)$, there exists a unique arrow $k(C_i) \longrightarrow k(A) \times k(B)$, whose compositions with the canonical (product) projections give, respectively, $k(f_A)$ and $k(f_B)$; we usually denote it by $< k(f_A), k(f_B) >$. Therefore, for any pair of morphisms $k(f_A)$ and $k(f_B)$, there exists a unique arrow $c_i : k(C_i) \longrightarrow X$ and the collection R_{π_A,π_B} can, in fact, be written as:

$$\{< k(f_A), k(f_B) >: k(C_i) \longrightarrow k(A) \times k(B) | A \xleftarrow{f_A} C_i \xrightarrow{f_B} B \text{ in } \mathcal{C}\}$$

which we have just shown to be epimorphic, by Condition (B). This is precisely Condition (F2). ■

Once again, note that the converse is not always true.

Clearly, Condition (A_{reg}) is stronger than Condition (A): it does not require any proof the fact that Condition (A_{reg}) implies Condition (A). Moreover, to justify why Condition (B') is a special instance of Condition (B), we make the link between the two explicit in the following:

Lemma 120 *Condition* (B) *implies Condition* (B').

Proof. For any arrow $f : k(A) \longrightarrow k(B)$ in $\mathcal{E}$, we wish to show that the collection of all maps $R_f = \{c_i : C_i \longrightarrow A | f.k(c_i) = k(g_i)$ for some $g_i : C_i \longrightarrow B\}$ in $\mathcal{C}$ is k-epimorphic. Consider the span in $\mathcal{E}$ given by the morphism f and the identity on $k(A)$, that is, $k(B) \xleftarrow{f} k(A) \xrightarrow{1_{k(A)}} k(A)$. Since the functor k satisfies Condition (B), we know that the collection $R_{f,1_{k(A)}}$ of morphisms of $\mathcal{E}$

$$\left\{ c_i : k(C_i) \longrightarrow k(A) \,\middle|\, \left\{ \begin{array}{c} f.c_i = k(f_B) \\ 1_{k(A)}.c_i = k(f_A) \end{array} \right. \text{for some } B \xleftarrow{f_B} C_i \xrightarrow{f_A} A \right\}$$

is epimorphic. This family $R_{f,1_{k(A)}}$ is precisely $\{k(f_A) : k(C_i) \longrightarrow k(A) | f.k(f_A) = k(f_B)\,\text{for some}\, B \xleftarrow{f_B} C_i \xrightarrow{f_A} A\}$, which corresponds to the collection $\{c_i : C_i \longrightarrow A | f.k(c_i) = k(g_i)\,\text{for some}\, g_i : C_i \longrightarrow B\}$, which is, therefore, k-epimorphic. But this is just the collection R_f and, then, R_f is k-epimorphic, as desired. ■

In light of the previous Lemmas 118 and 119, we can refine Theorem 117 and state the following:

Corollary 121 *Let $\mathcal{C}$ be a small category, $\mathcal{E}$ a cocomplete, exact and extensive category and $k : \mathcal{C} \longrightarrow \mathcal{E}$ a functor. The following conditions are equivalent:*

1. *the functor $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$, defined by the assignment $X \mapsto \mathcal{E}(k(-), X)$, is a localization;*

2. *the functor k satisfies the Conditions* (A), (B) *and* (F3).

Let $\mathcal{A}$ be a small category and $\mathcal{B}$ a cocomplete, exact and extensive category. We denote by $\mathbf{Loc}[\mathcal{B}, [\mathcal{A}^{op}, \mathbf{set}]]$ the full subcategory of $\mathbf{GeoMor}[\mathcal{B}, [\mathcal{A}^{op}, \mathbf{set}]]$ given by all localizations between $\mathcal{B}$ and the presheaf category $[\mathcal{A}^{op}, \mathbf{set}]$. Whereas, we denote by $\mathbf{Filt}^*[\mathcal{A}, \mathcal{B}]$ the category of functors from $\mathcal{A}$ into $\mathcal{B}$ which are filtering and satisfy Conditions (A) and (B) (or, equivalently, which satisfy Conditions (A), (B) and (F3)) and all natural transformations between them. The following result relates these two notions, namely localizations can be classified by functors belonging to categories of the kind $\mathbf{Filt}^*$:

Corollary 122 *Let $\mathcal{A}$ be a small category and $\mathcal{B}$ a cocomplete, exact and extensive category. The functor*

$$\xi : \mathbf{Loc}[\mathcal{B}, [\mathcal{A}^{op}, \mathbf{set}]] \longrightarrow \mathbf{Filt}^*[\mathcal{A}, \mathcal{B}]$$

defined by pre-composing with the Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$, gives an equivalence of categories.

Proof. Let

$$\begin{tikzcd} \mathcal{B} \arrow[r, shift right, tail, "i"'] \arrow[r, phantom, "\perp"] & {[\mathcal{A}^{op}, \mathbf{set}]} \arrow[l, shift right, "L"'] \end{tikzcd}$$

be a localization and consider the pre-composition $L.Y_{\mathcal{A}}$ of its left adjoint L with Yoneda $Y_{\mathcal{A}} : \mathcal{A} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$. First of all, we need to show that the functor $L.Y_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{B}$ verifies Conditions (A), (B) and (F3). By Proposition 110, to prove $L.Y_{\mathcal{A}}$ being filtering is equivalent to prove that the left Kan extension $\mathbf{Lan}_{Y_\mathcal{A}}(L.Y_\mathcal{A})$ of the composite $L.Y_\mathcal{A}$ along Yoneda is left exact, as in the following diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "Y_\mathcal{A}"] \arrow[dr, "L.Y_\mathcal{A}"'] & & {[\mathcal{A}^{op}, \mathbf{set}]} \arrow[dl, "\mathbf{Lan}_{Y_\mathcal{A}}(L.Y_\mathcal{A})"] \\
& \mathcal{B} &
\end{tikzcd}$$

Since $Y_\mathcal{A}$ is fully faithful, the universal arrow for such a left Kan extension is an isomorphism, namely $L.Y_\mathcal{A} \simeq \mathbf{Lan}_{Y_\mathcal{A}}(L.Y_\mathcal{A}).Y_\mathcal{A}$. Moreover, $Y_\mathcal{A}$ being a dense functor (see Definition 98) and L and $\mathbf{Lan}_{Y_\mathcal{A}}(L.Y_\mathcal{A})$ preserving all existing colimits, we can derive that $L \simeq \mathbf{Lan}_{Y_\mathcal{A}}(L.Y_\mathcal{A})$. Clearly, the functor $\mathbf{Lan}_{Y_\mathcal{A}}(L.Y_\mathcal{A})$ is left adjoint to the functor $\mathcal{B}(L.Y_\mathcal{A}(-), -) : \mathcal{B} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$ and, therefore, by uniqueness (up to isomorphism) of the right adjoint, $\mathcal{B}(L.Y_\mathcal{A}(-), -)$ is isomorphic to i, which is a right adjoint functor to L. In particular, then, $\mathcal{B}(L.Y_\mathcal{A}(-), -)$ $\simeq i$ is full and faithful, which means that

$$\begin{tikzcd}
\mathcal{B} \arrow[r, shift right, "\mathcal{B}(L.Y_\mathcal{A}(-),-)"'] \arrow[r, phantom, "\perp"] & {[\mathcal{A}^{op}, \mathbf{set}]} \arrow[l, shift right, "\mathbf{Lan}_{Y_\mathcal{A}}(L.Y_\mathcal{A})"']
\end{tikzcd}$$

is a localization of the presheaf category $[\mathcal{A}^{op}, \mathbf{set}]$. Then, from Corollary 121, it follows immediately that the functor $L.Y_\mathcal{A}$ does satisfy Conditions (A), (B) and (F3), as desired. This ends the proof of $L.Y_\mathcal{A}$ being an object of $\mathbf{Filt}^*[\mathcal{A}, \mathcal{B}]$. In order to prove that ξ is an equivalence of categories, we need to verify it is (a) fully faithful and (b) essentially surjective on objects.
(a) The proof that ξ is full and faithful follows the same lines of the proof that the functor $\varepsilon : \mathbf{GeoMor}[\mathcal{B}, [\mathcal{A}^{op}, \mathbf{set}]] \longrightarrow \mathbf{Filt}[\mathcal{A}, \mathcal{B}]$ introduced in Corollary 111 is fully faithful, once we have underlined that both the categories $\mathbf{Loc}[\mathcal{B}, [\mathcal{A}^{op}, \mathbf{set}]]$ and $\mathbf{Filt}^*[\mathcal{A}, \mathcal{B}]$ are full subcategories of $\mathbf{GeoMor}[\mathcal{B}, [\mathcal{A}^{op}, \mathbf{set}]]$ and $\mathbf{Filt}[\mathcal{A}, \mathcal{B}]$, respectively.
(b) It only remains to show, then, that ξ is essentially surjective on objects. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor verifying Conditions (A), (B) and (F3). Since the category $\mathcal{B}$ is cocomplete, we can always define a pair of adjoint functors $\mathcal{B}(F(-), -) : \mathcal{B} \longrightarrow [\mathcal{A}^{op}, \mathbf{set}]$, which assigns to any object $B \in \mathcal{B}$ the set-valued functor $\mathcal{B}(F(-), B)$, and $\mathbf{Lan}_{Y_\mathcal{A}}(F) : [\mathcal{A}^{op}, \mathbf{set}] \longrightarrow \mathcal{B}$, which is given by the left Kan extension of F along the Yoneda embedding $Y_\mathcal{A}$. By Proposition 110, the fact that the functor F is filtering is equivalent to the left Kan extension $\mathbf{Lan}_{Y_\mathcal{A}}(F)$ being left exact. Moreover, by Lemma 116, the fact that F satisfies Condition (A) and (B) implies that the functor $\mathcal{B}(F(-), -)$ is

full and faithful. Therefore, we get a localization

$$\begin{tikzcd} \mathcal{B} \arrow[r, shift right, "\mathcal{B}(F(-),-)"'] \arrow[r, phantom, "\perp"] & {[\mathcal{A}^{op}, \mathbf{set}]} \arrow[l, shift right, "\mathbf{Lan}_{Y_\mathcal{A}}(F)"'] \end{tikzcd}$$

which, in particular, satisfies the condition $F \simeq \mathbf{Lan}_{Y_\mathcal{A}}(F).Y_\mathcal{A}$, since $Y_\mathcal{A}$ is fully faithful. Namely, $\xi(\mathbf{Lan}_{Y_\mathcal{A}}(F)) \simeq F$ and this ends the proof of the functor ξ being essentially surjective on objects. ■

A sheaf-theoretical proof

It is a well-known fact that all localizations of the category $[\mathcal{C}^{op}, \mathbf{set}]$ of presheaves are obtained, up to equivalence, as categories of sheaves $\mathbf{Sh}(\mathcal{T})$, for Grothendieck topologies $\mathcal{T}$ on $\mathcal{C}$ (see [25, 34, 36]). A category equivalent to such a category of sheaves is called a Grothendieck topos and the celebrated Giraud Theorem gives an intrinsic characterisation of Grothendieck toposes. We want to present here a new proof of Corollary 121, which makes use of sheaf-theoretic techniques. The main difference with the previous one, then, will be the total abandonment of the notion of filtering functor and exact completion, to dive in a more "topological" context. We recall, first, the axioms for a Grothendieck topology on a small category. Roughly speaking, the notion of Grothendieck topology describes the behaviour of hereditary open covers in a topological space.

Definition 123 *Let $\mathcal{C}$ be a small category. A* Grothendieck topology $\mathcal{T}$ *on $\mathcal{C}$ is given by specifying for any object C in $\mathcal{C}$ a collection $\mathcal{T}(C)$ of subfunctors of $\mathcal{C}(-, C)$ in $[\mathcal{C}^{op}, \mathbf{set}]$, satifying the following axioms:*

(T1) $\mathcal{C}(-, C) \in \mathcal{T}(C)$;

(T2) *for $R \in \mathcal{T}(C)$ and an arrow $f : D \longrightarrow C$ in $\mathcal{C}$, the pullback $f^*(R)$ in $[\mathcal{C}^{op}, \mathbf{set}]$ of R along $\mathcal{C}(-, f) : \mathcal{C}(-, D) \longrightarrow \mathcal{C}(-, C)$ is in $\mathcal{T}(D)$;*

(T3) *consider $S \in \mathcal{T}(C)$ and an arbitrary subfunctor R of $\mathcal{C}(-, C)$, if, for any D in $\mathcal{C}$ and for any $f \in S(D)$, the pullback $f^*(R)$ is in $\mathcal{T}(D)$, it follows that R is in $\mathcal{T}(C)$.*

It can been easily proved that, for any object $A \in \mathcal{C}$, the pullback's component at A is given by $f^*(R)(A) = \{a : A \longrightarrow D | f.a : A \longrightarrow D \longrightarrow C \text{ is in } R(A)\}$.

Definition 124 *Let $\mathcal{T}$ be a Grothendieck topology on $\mathcal{C}$. A presheaf functor $F \in [\mathcal{C}^{op}, \mathbf{set}]$ is said to be a* sheaf (for $\mathcal{T}$) *if, for any object $C \in \mathcal{C}$ and any subfunctor $r : R \rightarrowtail \mathcal{C}(-, C)$ in $\mathcal{T}(C)$, the inclusion r induces a bijection $\mathbf{Nat}(\mathcal{C}(-, C), F) \simeq \mathbf{Nat}(R, F)$.*

Stated otherwise, $F \in [\mathcal{C}^{op}, \mathbf{set}]$ is a sheaf if, for any $C \in \mathcal{C}$, for any $R \in \mathcal{T}(C)$ and for any R-compatible family $(f_h)_{h \in R}$ (that is, a family $\{f_h \in F(C_h) | h : C_h \longrightarrow C \text{ in } R\}$ in **set** such that, for any arrow $c : C'_h \longrightarrow C_h$ of $\mathcal{C}$, $F(c)(f_h) = f_{h.c}$), there exists a unique element $f \in F(C)$ satisfying $F(h)(f) = f_h$, for any $h \in R$.
Given a Grothendieck topology $\mathcal{T}$ on a small category $\mathcal{C}$, the sheaves for $\mathcal{T}$ form a category $\mathbf{Sh}(\mathcal{T})$, where the maps are the natural transformations, namely maps of presheaves. So, $i : \mathbf{Sh}(\mathcal{T}) \rightarrowtail [\mathcal{C}^{op}, \mathbf{set}]$ is a full subcategory. Moreover, the inclusion i has a left adjoint $a : [\mathcal{C}^{op}, \mathbf{set}] \longrightarrow \mathbf{Sh}(\mathcal{T})$, called the *associated sheaf functor*, which preserves finite limits. In other words, $i : \mathbf{Sh}(\mathcal{T}) \rightarrowtail [\mathcal{C}^{op}, \mathbf{set}]$ is a localization of the presheaf category $[\mathcal{C}^{op}, \mathbf{set}]$.

Theorem 125 *Let $\mathcal{C}$ be a small category, $\mathcal{E}$ a cocomplete, exact and extensive category and $k : \mathcal{C} \longrightarrow \mathcal{E}$ a functor. The following conditions are equivalent:*

1. *the functor $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$, defined by the assignment $X \mapsto \mathcal{E}(k(-), X)$, is a localization;*

2. *the functor k satisfies the Conditions* (A), (B) *and* (C).

Proof. $1 \Rightarrow 2$. Suppose the functor $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$ is a localization of the functor category $[\mathcal{C}^{op}, \mathbf{set}]$, whose left exact left adjoint we denote by L. In fact, L is nothing but the left Kan extension of the functor k along the Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$. We have already proved in Theorem 117, that this implies the functor k satisfying Conditions (A) and (B). It only remains to show, then, that the functor k satisfies the additional Condition (C). For it, let $f, g : A \longrightarrow B$ be a pair of arrows in $\mathcal{C}$ and consider the equalizer of the corresponding natural transformation $\mathcal{C}(-, f)$ and $\mathcal{C}(-, g)$ in $[\mathcal{C}^{op}, \mathbf{set}]$:

$$\Omega \overset{\omega}{\rightarrowtail} \mathcal{C}(-, A) \underset{\mathcal{C}(-,g)}{\overset{\mathcal{C}(-,f)}{\rightrightarrows}} \mathcal{C}(-, B)$$

Since limits are computed pointwise in $[\mathcal{C}^{op}, \mathbf{set}]$, for any object $C \in \mathcal{C}$, each component ω_C of ω is the equalizer of the pair $(\mathcal{C}(C, f), \mathcal{C}(C, g))$ in **set**, as follows:

$$\Omega(C) \overset{\omega_C}{\rightarrowtail} \mathcal{C}(C, A) \underset{\mathcal{C}(C,g)}{\overset{\mathcal{C}(C,f)}{\rightrightarrows}} \mathcal{C}(C, B)$$

that is, the set $\Omega(C) = \{h : C \longrightarrow A | f.h = g.h\}$. We point out, explicitly, that we make no notational distinction between the functor Ω

and the set-theoretic union of all $\Omega(C)$, when C varies in $\mathcal{C}$. Moreover, L being left exact, we get that the morphism $L(\omega)$ is the equalizer in $\mathcal{E}$ of the pair $(k(f), k(g))$, as follows:

$$\begin{tikzcd} L(\Omega) \arrow[r, tail, "L(\omega)"] & k(A) \arrow[r, shift left, "k(f)"] \arrow[r, shift right, "k(g)"'] & k(B) \end{tikzcd}$$

(here we use the definition of $L = \mathbf{Lan}_{Y_{\mathcal{C}}}(k)$ as the left Kan extension of k along Yoneda: since $Y_{\mathcal{C}}$ is fully faithful, we derive $k \simeq L.Y_{\mathcal{C}}$). Suppose, then, that $k(f) = k(g)$. We need to prove that the family $R_{f\&g}$ of arrows in $\mathcal{C}$

$$\{c_i : C_i \longrightarrow A | f.c_i = g.c_i\}$$

is k-epimorphic. Since $k(f) = k(g)$, the equalizer $L(\omega)$ is isomorphic to the identity $1_{k(A)}$ on $k(A)$. By considering all morphisms $h \in \Omega$, that is, arrows $h : C_h \longrightarrow A$ in $\mathcal{C}$ such that $f.h = g.h$, we get an induced canonical map r' in $[\mathcal{C}^{op}, \mathbf{set}]$, by the universal property of the coproduct:

$$\begin{tikzcd} \coprod_{h\in\Omega} \mathcal{C}(-, C_h) \arrow[r, "r'"] & \mathcal{C}(-, A) \\ \mathcal{C}(-, C_h) \arrow[u, "\sigma_h"] \arrow[ur, "\mathcal{C}(-,h)"'] & \end{tikzcd}$$

The morphism r' is unique with the property that $r'.\sigma_h = \mathcal{C}(-, h)$, for any $h \in \Omega$, where the morphisms σ_h denote the coproduct injections. We wish to show that the regular epi-mono factorisation in $[\mathcal{C}^{op}, \mathbf{set}]$ of r' is precisely Ω. For it, for any $h \in \Omega$, namely an arrow $h : C_h \longrightarrow A$ in $\mathcal{C}$ such that $f.h = g.h$, we get $\mathcal{C}(-, f).\mathcal{C}(-, h) = \mathcal{C}(-, g).\mathcal{C}(-, h)$. Since $\omega : \Omega \longrightarrow \mathcal{C}(-, A)$ is the equalizer of such a pair of arrows, there exists a unique map $\varphi_h : \mathcal{C}(-, C_h) \longrightarrow \Omega$ such that $\omega.\varphi_h = \mathcal{C}(-, h)$, for any $h \in \Omega$, and therefore, by the universal property of coproduct, there exists a unique morphism $\varphi : \coprod_{h\in\Omega} \mathcal{C}(-, C_h) \longrightarrow \Omega$ satisfying $\varphi.\sigma_h = \varphi_h$, for any $h \in \Omega$. That is, for any arrow $c : C \longrightarrow C_h$, for some $h \in \Omega(C_h)$, $\varphi_C(c)$ is precisely $h.c \in \Omega(C)$. It is easy to verify, by composing with the coproduct injections σ_h, that $\omega.\varphi = r'$. Moreover, the map φ is a regular epimorphism, or, equivalently, for any $C \in \mathcal{C}$, $\varphi_C : \coprod_{h\in\Omega} \mathcal{C}(C, C_h) \longrightarrow \Omega(C)$ is a surjection in **set**. For it, consider an element $c \in \Omega(C)$, then, taking $h = c$ and $C_h = C$ reveals the identity 1_C on C as an element of the coproduct and, clearly, we have $\varphi_C(1_C) = c.1_C = c$, as desired. Therefore, we get a regular epi-mono factorisation in $[\mathcal{C}^{op}, \mathbf{set}]$:

$$\begin{tikzcd} \coprod_{h\in\Omega} \mathcal{C}(-,C_h) \arrow[rr, "r'"] \arrow[dr, two heads, "\varphi"'] & & \mathcal{C}(-,A) \\ & \Omega \arrow[ur, tail, "\omega"'] & \end{tikzcd}$$

and applying the functor L, which is exact, to this diagram gives in $\mathcal{E}$:

$$\begin{tikzcd} \coprod_{h\in\Omega} k(C_h) \arrow[rr, "L(r')"] \arrow[dr, two heads, "L(\varphi)"'] & & k(A) \\ & L(\Omega) \arrow[ur, tail, "L(\omega)"'] & \end{tikzcd}$$

where $L(r')$ is the canonical map induced by the arrows $k(h)$, for $h \in \Omega$, since we do have $L(r').\rho_h = L(r').L(\sigma_h) = L(r'.\sigma_h) = L(\mathcal{C}(-,h)) \simeq k(h)$. (Here the morphisms ρ_h denote the corresponding coproduct injections in $\mathcal{E}$.) From the fact that $L(\omega)$ is an isomorphism, it follows that $L(r')$ is a regular epimorphism. By Lemma 115, we derive this is equivalent to the subfunctor $\omega : \Omega \rightarrowtail \mathcal{C}(-,A)$ being k-epimorphic. But Ω, regarded as the set-theoretic union over $C \in \mathcal{C}$ of all sets $\Omega(C) = \{h : C \longrightarrow A | f.h = g.h\}$, is nothing but the collection $R_{f\&g}$, and the property of Ω being a k-epimorphic subfunctor evolves to $R_{f\&g}$ being a k-epimorphic family. Namely, the functor k satisfies Condition (C).

$2 \Rightarrow 1$. Suppose the functor k satisfies Conditions (A), (B) and (C). We have already proved in Lemma 116 that Conditions (A) and (B) are sufficient for the functor k' to be fully faithful. We should show now that, under the additional Condition (C), the fully faithful inclusion $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$ is, in fact, a localization. This is where sheaf theory will come to hand. Since the length of this proof is quite remarkable, we prefer to clearly divide it into a sequence of steps, each constituting a self-contained result on its own.

Step 1. *The collection $\mathcal{T} = \{r : R \rightarrowtail \mathcal{C}(-,C)$ k-epimorphic $|C \in \mathcal{C}\}$ of all k-epimorphic subfunctors of representable functors gives rise to a Grothendieck topology on $\mathcal{C}$.*

Proof. Axiom (T1) is satisfied. For it, we need to show that, for any $C \in \mathcal{C}$, the representable functor $\mathcal{C}(-,C)$ is k-epimorphic . If we regard the functor $\mathcal{C}(-,C)$ as the set-theoretic union of all sets $\mathcal{C}(D,C)$, for D varying in $\mathcal{C}$, we get the collection of all arrows in $\mathcal{C}$ with codomain C, that is, $\mathcal{S} = \{h : D \longrightarrow C | D \in \mathcal{C}\}$. For any pair of arrows $u, v :$

$k(C) \longrightarrow U$ in $\mathcal{C}$ such that $u.k(h) = v.k(h)$, for any $h \in \mathcal{S}$, we obtain the equality $u = v$ just by taking, in particular, $h = 1_C$. This proves that $\mathcal{S}$ is a k-epimorphic family.

Axiom (T2) is satisfied. For it, let $r : R \rightarrowtail \mathcal{C}(-, C)$ be a k-epimorphic subfunctor and $f : D \longrightarrow C$ a morphism of $\mathcal{C}$ and compute the pullback of r along $\mathcal{C}(-, f)$ in $[\mathcal{C}^{op}, \mathbf{set}]$, as follows:

$$\begin{tikzcd} f^*(R) \arrow[r] \arrow[d, tail, "r^*"'] & R \arrow[d, tail, "r"] \\ \mathcal{C}(-,D) \arrow[r, "\mathcal{C}(-,f)"'] & \mathcal{C}(-,C) \end{tikzcd}$$

We wish to prove that $r^* : f^*(R) \rightarrowtail \mathcal{C}(-, D)$ is a k-epimorphic subfunctor, that is, the set-theoretic union of all collections $f^*(R)(A) = \{a : A \longrightarrow D | f.a \in R(A)\}$ of arrows in $\mathcal{C}$, for A varying in $\mathcal{C}$, constitutes a k-epimorphic family in $\mathcal{C}$. For it, denote by $\mathcal{R}$ the collection $\{h : C_h \longrightarrow C | h \in R(C_h), C_h \in \mathcal{C}\}$ and consider the canonical map $\lambda_{\mathcal{R}}$ induced by all arrows $k(h)$, via the universal property of the coproduct, as follows:

$$\begin{tikzcd} \coprod_{h\in\mathcal{R}} k(C_h) \arrow[r, two heads, "\lambda_{\mathcal{R}}"] & k(C) \\ k(C_h) \arrow[u, "\sigma_h"] \arrow[ur, "k(h)"'] & \end{tikzcd}$$

namely, $\lambda_{\mathcal{R}}$ is the unique map in $\mathcal{E}$ satisfying $\lambda_{\mathcal{R}}.\sigma_h = k(h)$, for any $h \in \mathcal{R}$. The fact that R is a k-epimorphic subfunctor precisely means that the family $\mathcal{R}$ is k-epimorphic, that is, $\lambda_{\mathcal{R}}$ is an epimorphism, in fact, a regular epimorphism by Proposition 95, the category $\mathcal{E}$ being exact and extensive. For any $h \in \mathcal{R}$, the pullback P_h in $\mathcal{E}$ of $k(h)$ (which is equal to $\lambda_{\mathcal{R}}.\sigma_h$) along $k(f)$ factors through the pullback $P_{\mathcal{R}}$ of $\lambda_{\mathcal{R}}$ along $k(f)$, as displayed in the following diagram:

$$\begin{tikzcd} P_h \arrow[rr, "\rho_h"] \arrow[dr, dotted, "\nu_h"] \arrow[dd, "\mu_h"'] & \arrow[d, phantom, "(2)"] & k(C_h) \arrow[d, "\sigma_h"] \\ & P_{\mathcal{R}} \arrow[r, "\rho_{\mathcal{R}}"] \arrow[d, "\mu_{\mathcal{R}}"'] \arrow[dr, phantom, "(1)"] & \coprod_{h\in\mathcal{R}} k(C_h) \arrow[d, two heads, "\lambda_{\mathcal{R}}"] \\ k(D) \arrow[r, "1_{k(D)}"] & k(D) \arrow[r, "k(f)"'] & k(C) \end{tikzcd}$$

where ν_h is the comparison morphism given by the universal property of the pullback (1). Since both the outer square and the inner right-hand square (1) are pullbacks, it follows that (2) is a pullback and, therefore, $\mathcal{E}$ being an extensive category, we can derive $\coprod_{h\in\mathcal{R}} P_h \simeq P_{\mathcal{R}}$. Therefore, for any $h \in \mathcal{R}$, we get a span $k(C_h) \xleftarrow{\rho_h} P_h \xrightarrow{\mu_h} k(D)$ in $\mathcal{E}$, to which we can apply Condition (B) and obtain that the collection R_{ρ_h,μ_h} of all morphisms of $\mathcal{E}$

$$\left\{ c_i : k(C_i) \longrightarrow P_h \,\middle|\, \left\{ \begin{array}{c} \rho_h.c_i = k(l_i) \\ \mu_h.c_i = k(m_i) \end{array} \right. \text{for some}\, C_h \xleftarrow{l_i} C_i \xrightarrow{m_i} D \right\}$$

is epimorphic. Consider, then, the collection $\mathcal{S} = \{g : C_i \longrightarrow D | k(g) = \mu_h.c_i \;,\; h \in \mathcal{R} \;,\; c_i \in R_{\rho_h,\mu_h}\}$. Note that $\lambda_{\mathcal{R}}$ being a regular epimorphism and $\mathcal{E}$ being a regular category imply $\mu_{\mathcal{R}}$ is a regular epimorphism and, obviously, we have $\mu_h = \mu_{\mathcal{R}}.\nu_h$. First, we show the family $\mathcal{S}$ is k-epimorphic. For it, let $u, v : k(D) \longrightarrow U$ be a pair of arrows in $\mathcal{E}$ such that $u.k(g) = v.k(g)$, for any $g \in \mathcal{S}$, that is, $u.\mu_h.c_i = v.\mu_h.c_i$, for any $h \in \mathcal{R}$ and for any $c_i \in R_{\rho_h,\mu_h}$. Equivalently, we have $u.\mu_{\mathcal{R}}.\nu h.c_i = v.\mu_{\mathcal{R}}.\nu_h.c_i$, for any $h \in \mathcal{R}$ and for any $c_i \in R_{\rho_h,\mu_h}$, from which it follows that $u.\mu_{\mathcal{R}}.\nu h = v.\mu_{\mathcal{R}}.\nu_h$, for any $h \in \mathcal{R}$, the family R_{ρ_h,μ_h} being epimorphic. This condition is equivalent to $u.\mu_{\mathcal{R}} = v.\mu_{\mathcal{R}}$, since we have just shown $P_{\mathcal{R}}$ is, indeed, the coproduct $\coprod_{h\in\mathcal{R}} P_h$ (with coproduct injections given precisely by the morphisms ν_h). Finally, we can derive $u = v$, $\mu_{\mathcal{R}}$ being an epimorphism. When taking an arrow $g : C_i \longrightarrow D$ in $\mathcal{S}$, we have $k(g) = \mu_h.c_i$, for some $h : C_h \longrightarrow C$ in $\mathcal{R}$ and some $c_i : k(C_i) \longrightarrow P_h$ in R_{ρ_h,μ_h}, so, in particular, we also have $\rho_h.c_i = k(l_{h,i})$, for some arrow $l_{h,i} : C_i \longrightarrow C_h$. Then, by definition of the pullback P_h of $k(h)$ along $k(f)$, we obtain $k(f.g) = k(f).k(g) = k(f).\mu_h.c_i = k(h).\rho_h.c_i = k(h).k(l_{h,i}) = k(h.l_{h,i})$. From this last equality it follows, by Condition (C), that the collection of arrows $R_{f.g\&h.l_{h,i}} = \{x : C_x \longrightarrow C_i | f.g.x = h.l_{h,i}.x\}$ in $\mathcal{C}$ is k-epimorphic. So, by taking all composites of the kind $g.x$ we get a k-epimorphic family, which is, in fact, contained in $f^*(R)$. For it, let $\mathcal{S}' = \{C_x \xrightarrow{x} C_i \xrightarrow{g} D | g \in \mathcal{S}, x \in R_{f.g\&h.l_{h,i}}\}$ be the family of all these composites in $\mathcal{C}$ and consider a pair of arrows $u, v : k(D) \longrightarrow U$ satisfying $u.k(g).k(x) = v.k(g).k(x)$, for any $g \in \mathcal{S}$ and for any $x \in R_{f.g\&h.l_{h,i}}$. This amounts to saying that $u.k(g) = v.k(g)$, for any $g \in \mathcal{S}$, $R_{f.g\&h.l_{h,i}}$ being k-epimorphic, which is further equivalent to $u = v$, since $\mathcal{S}$ is also k-epimorphic. Then, it only remains to prove that $\mathcal{S}' \subseteq f^*(R)$. Consider an arbitrary $g.x$, where $g \in \mathcal{S}$ and $x \in R_{f.g\&h.l_{h,i}}$; this composite satisfies $f.g.x = h.l_{h,i}.x$, where $h : C_h \longrightarrow C$ is in $R(C_h)$. Therefore, by definition of R as a subfunctor of $\mathcal{C}(-, C)$, we get that the composite

$h.l_{h,i}.x$ is in $R(C_x)$, namely $f.g.x \in R(C_x)$; in other words, $g.x$ belongs to $f^*(R)(C_x)$, as desired. We conclude, by Lemma 113, that the subfunctor $r^* : f^*(R) \rightarrowtail \mathcal{C}(-, D)$, which does contain a k-epimorphic family, namely $\mathcal{S}'$, is itself k-epimorphic.
Axiom (T3) is satisfied. For it, let $S \in \mathcal{T}(C)$, that is, S is a k-epimorphic subfunctor of $\mathcal{C}(-, C)$, $R \rightarrowtail \mathcal{C}(-, C)$ be a subfunctor such that, for any object $D \in \mathcal{C}$ and for any morphism $f \in S(D)$, the pullback $f^*(R)$ is in $\mathcal{T}(D)$. We need to show, then, that $R \in \mathcal{T}(C)$. For any morphism $f : D \longrightarrow C$ in $S(D)$, the pullback $f^*(R)$ is given by the set-theoretic union of all $f^*(R)(A) = \{a : A \longrightarrow D | f.a \in R(A)\}$, for A varying in $\mathcal{C}$. Consider the collection of all such composites $f.a$ in $\mathcal{C}$, which we denote by $\mathcal{S} = \{f.a : A \longrightarrow C | f \in S, a \in f^*(R)\}$: this represents a k-epimorphic family. For it, let $u, v : k(C) \longrightarrow U$ be a pair of arrows in $\mathcal{E}$ such that $u.k(f).k(a) = v.k(f).k(a)$, for any $f \in S$ and for any $a \in f^*(R)$. This amounts to saying that $u.k(f) = v.k(f)$, since $f^*(R) \rightarrowtail \mathcal{C}(-, D)$ is a k-epimorphic subfunctor, which is further equivalent to $u = v$, $S \rightarrowtail \mathcal{C}(-, C)$ being itself a k-epimorphic subfunctor. Therefore, the family $\mathcal{S}$ is k-epimorphic and, moreover, it is contained in R, by definition of the pullback $f^*(R)$. By Lemma 113, we can derive, then, that R is a k-epimorphic subfunctor of $\mathcal{C}(-, C)$, that is, $R \in \mathcal{T}(C)$, as desired.

This ends the proof of

$$\mathcal{T} : C \mapsto \mathcal{T}(C) = \{R \rightarrowtail \mathcal{C}(-, C) | R \text{ } k\text{-epimorphic subfunctor}\}$$

being a Grothendieck topology on $\mathcal{C}$. We consider the full subcategory $\mathbf{Sh}(\mathcal{T})$ of the functor category $[\mathcal{C}^{op}, \mathbf{set}]$ given by the $\mathcal{T}$-sheaves, which, as we have already recalled, gives rise to a localization:

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{T}) \arrow[r, tail, shift right, "i"'] \arrow[r, phantom, "\perp"] & {[\mathcal{C}^{op}, \mathbf{set}]} \arrow[l, shift right, "a"'] \end{tikzcd}$$

We wish to prove that the inclusion $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$ factors through an equivalence of categories $\varsigma : \mathcal{E} \longrightarrow \mathbf{Sh}(\mathcal{T})$, as follows:

$$\begin{tikzcd} \mathcal{E} \arrow[rr, tail, "k'"] \arrow[dr, dotted, "\varsigma"'] & & {[\mathcal{C}^{op}, \mathbf{set}]} \\ & \mathbf{Sh}(\mathcal{T}) \arrow[ur, tail, "i"'] & \end{tikzcd}$$

where the associated sheaf functor $a : [\mathcal{C}^{op}, \mathbf{set}] \longrightarrow \mathbf{Sh}(\mathcal{T})$ is given by $\mathcal{C}(-, C) \mapsto \mathcal{E}(k(-), k(C))$. This will end our proof of $k' : \mathcal{E} \simeq \mathbf{Sh}(\mathcal{T})$

$\rightarrowtail [\mathcal{C}^{op}, \mathbf{set}]$ being a localization.

Step 2. *Let $r : R \rightarrowtail \mathcal{E}(k(-), X)$ be an epimorphic subfunctor and $f' : k(D) \longrightarrow X$ an arrow of $\mathcal{E}$. Then, the pullback $r^* : f'^*(R) \rightarrowtail \mathcal{C}(-, D)$ in $[\mathcal{C}^{op}, \mathbf{set}]$ of r along $\mathcal{E}(k(-), f')$ is a k-epimorphic subfunctor.*

Proof. We recall that $f'^*(R)$ is defined to be the set-theoretic union of all collections $f'^*(R)(C_h) = \{h : C_h \longrightarrow D | f'.k(h) \in R(C_h)\}$ of morphisms in $\mathcal{C}$, for C_h varying in $\mathcal{C}$. This proof is analogous to the one of Axiom (T2) in Step 1. Denote by $\mathcal{R}$ the collection $\{h : k(C_h) \longrightarrow X | h \in R(C_h), C_h \in \mathcal{C}\}$ and consider the canonical map $\lambda_{\mathcal{R}}$ induced by all arrows h, via the universal property of the coproduct, as follows:

$$\begin{tikzcd}
\coprod_{h \in \mathcal{R}} k(C_h) \arrow[r, two heads, "\lambda_{\mathcal{R}}"] & X \\
k(C_h) \arrow[u, "\sigma_h"] \arrow[ur, "h"'] &
\end{tikzcd}$$

namely, $\lambda_{\mathcal{R}}$ is the unique arrow with the property $\lambda_{\mathcal{R}}.\sigma_h = h$, for any $h \in \mathcal{R}$. The fact that R is an epimorphic subfunctor precisely means that the family $\mathcal{R}$ is epimorphic, that is, $\lambda_{\mathcal{R}}$ is an epimorphism, in fact, a regular epimorphism by Proposition 95, the category $\mathcal{E}$ being exact and extensive. For any $h \in \mathcal{R}$, the pullback P_h in $\mathcal{E}$ of h (which is equal to $\lambda_{\mathcal{R}}.\sigma_h$) along f' factors through the pullback $P_{\mathcal{R}}$ of $\lambda_{\mathcal{R}}$ along f', as displayed in the following diagram:

$$\begin{tikzcd}
P_h \arrow[rr, "\rho_h"] \arrow[dr, dotted, "\nu_h"] \arrow[dd, "\mu_h"'] \arrow[drr, phantom, "(2)"] & & k(C_h) \arrow[d, "\sigma_h"] \\
& P_{\mathcal{R}} \arrow[r, "\rho_{\mathcal{R}}"] \arrow[d, two heads, "\mu_{\mathcal{R}}"'] \arrow[dr, phantom, "(1)"] & \coprod_{h \in \mathcal{R}} k(C_h) \arrow[d, two heads, "\lambda_{\mathcal{R}}"] \\
k(D) \arrow[r, "1_{k(D)}"] & k(D) \arrow[r, "f'"'] & X
\end{tikzcd}$$

where ν_h is the comparison morphism given by the universal property of the pullback (1). Since both the outer square and the inner right-hand square (1) are pullbacks, it follows that (2) is a pullback and, therefore, $\mathcal{E}$ being an extensive category, we can derive $\coprod_{h \in \mathcal{R}} P_h \simeq P_{\mathcal{R}}$. Therefore, for any $h \in \mathcal{R}$, we get a span $k(C_h) \xleftarrow{\rho_h} P_h \xrightarrow{\mu_h} k(D)$ in $\mathcal{E}$, to which we can apply Condition (B) and obtain that the collection R_{ρ_h,μ_h} of all morphisms of $\mathcal{E}$

$$\left\{c_i : k(C_i) \longrightarrow P_h \,\middle|\, \left\{\begin{array}{c} \rho_h.c_i = k(l_i) \\ \mu_h.c_i = k(m_i)\end{array}\right. \text{for some}\, C_h \xleftarrow{l_i} C_i \xrightarrow{m_i} D\right\}$$

is epimorphic. Consider, then, the collection $\mathcal{S} = \{g : C_i \longrightarrow D | k(g) = \mu_h.c_i \;,\; h \in \mathcal{R} \;,\; c_i \in R_{\rho_h,\mu_h}\}$. Note that $\lambda_{\mathcal{R}}$ being a regular epimorphism and $\mathcal{E}$ being a regular category imply that $\mu_{\mathcal{R}}$ is a regular epimorphism and, obviously, we have $\mu_h = \mu_{\mathcal{R}}.\nu_h$. First, we show the family $\mathcal{S}$ is k-epimorphic. For it, let $u, v : k(D) \longrightarrow U$ be a pair of arrows in $\mathcal{E}$ such that $u.k(g) = v.k(g)$, for any $g \in \mathcal{S}$, that is, $u.\mu_h.c_i = v.\mu_h.c_i$, for any $h \in \mathcal{R}$ and for any $c_i \in R_{\rho_h,\mu_h}$. Equivalently, we have $u.\mu_{\mathcal{R}}.\nu h.c_i = v.\mu_{\mathcal{R}}.\nu_h.c_i$, for any $h \in \mathcal{R}$ and for any $c_i \in R_{\rho_h,\mu_h}$, from which it follows that $u.\mu_{\mathcal{R}}.\nu h = v.\mu_{\mathcal{R}}.\nu_h$, for any $h \in \mathcal{R}$, the family R_{ρ_h,μ_h} being epimorphic. This condition is equivalent to $u.\mu_{\mathcal{R}} = v.\mu_{\mathcal{R}}$, since we have just shown $P_{\mathcal{R}}$ is the coproduct $\coprod_{h\in\mathcal{R}} P_h$ (with coproduct injections given precisely by the morphisms ν_h). Therefore, we derive $u = v$, $\mu_{\mathcal{R}}$ being an epimorphism. When taking an arrow $g : C_i \longrightarrow D$ in $\mathcal{S}$, we have $k(g) = \mu_h.c_i$, for some $h : k(C_h) \longrightarrow X$ in $R(C_h)$ and some $c_i : k(C_i) \longrightarrow P_h$ in R_{ρ_h,μ_h}, so, in particular, we also have $\rho_h.c_i = k(l_{h,i})$, for some arrow $l_{h,i} : C_i \longrightarrow C_h$. Then, by definition of the pullback P_h of h along f', we obtain $f'.k(g) = f'.\mu_h.c_i = h.\rho_h.c_i = h.k(l_{h,i})$. Since R is a subfunctor of $\mathcal{E}(k(-), X)$, from the fact that $h \in R(C_h)$ it follows that the composite $h.k(l_{h,i}) \in R(C_i)$, namely $f'.k(g) \in R(C_i)$; in other words, g belongs to $f'^*(R)(C_i)$. Therefore, we have just proved that the family $\mathcal{S}$ is contained in $f'^*(R)$. We conclude, by Lemma 113, that the subfunctor $r^* : f'^*(R) \rightarrowtail \mathcal{C}(-, D)$, which does contain the k-epimorphic family $\mathcal{S}$, is itself k-epimorphic .

Step 3. *Let $r : R \rightarrowtail \mathcal{E}(k(-), X)$ be an epimorphic subfunctor. Then, its image $a(r)$ through the associated sheaf functor a is an isomorphism in* $\mathbf{Sh}(\mathcal{T})$.

Proof. First of all, we wish to point out that, for any natural transformation $\omega : \mathcal{C}(-, C) \longrightarrow \mathcal{E}(k(-), X)$, the pullback in $[\mathcal{C}^{op}, \mathbf{set}]$ of r along ω produces an element of the topology $\mathcal{T}(C)$. For it, consider a natural transformation $\omega : \mathcal{C}(-, C) \longrightarrow \mathcal{E}(k(-), X)$ and the pullback in

$[\mathcal{C}^{op}, \mathbf{set}]$ of r along ω, as follows:

$$\begin{tikzcd} P \arrow[r, "\omega'"] \arrow[d, tail, "p"'] & R \arrow[d, tail, "r"] \\ \mathcal{C}(-,C) \arrow[r, "\omega"'] & \mathcal{E}(k(-),X) \end{tikzcd}$$

By the Yoneda Lemma, any natural transformation $\omega : \mathcal{C}(-,C) \longrightarrow \mathcal{E}(k(-),X)$ corresponds to a morphism $f' : k(C) \longrightarrow X$ in $\mathcal{E}$; more precisely, the correspondence is given by $\omega \simeq \mathcal{E}(k(-),f')$. Then, by Step 2, the pullback $p : P \rightarrowtail \mathcal{C}(-,C)$, which is nothing but $f'^{*}(R) = \{h : C_h \longrightarrow C | f'.k(h) \in R(C_h), C_h \in \mathcal{C}\}$, is a k-epimorphic subfunctor. In other words, P belongs to $\mathcal{T}(C)$.
Therefore, for any F sheaf (for the topology $\mathcal{T}$) and for any pair of arrows $\alpha, \beta : \mathcal{E}(k(-),X) \longrightarrow F$ in $[\mathcal{C}^{op}, \mathbf{set}]$ satisfying $\alpha.r = \beta.r$, by pre-composing with ω', we get $\alpha.r.\omega' = \beta.r.\omega'$, that is, $\alpha.\omega.p = \beta.\omega.p$. Since $P \in \mathcal{T}(C)$, from this last equality it follows, by definition of the sheaf F, that $\alpha.\omega = \beta.\omega$. So, the natural transformations α and β coincide when pre-composed with any $\omega : \mathcal{C}(-,C) \longrightarrow \mathcal{E}(k(-),X)$, whenever they coincide when pre-composed with r. In fact, we want to show that if they coincide when pre-composed with r, then they always coincide. For it, in order to prove that $\alpha, \beta : \mathcal{E}(k(-),X) \longrightarrow F$ are the same natural transformation, we need to show that, for any object $A \in \mathcal{C}$ and for any arrow $h : k(A) \longrightarrow X$ in $\mathcal{E}$, $\alpha_A(h) = \beta_A(h)$. But as just remarked, by the Yoneda Lemma, any such an arrow $h : k(A) \longrightarrow X$ corresponds to a natural transformation $\omega : \mathcal{C}(-,A) \longrightarrow \mathcal{E}(k(-),X)$, via $\omega_A(1_A) = h$. Then, for any such a morphism ω, we already know that $\alpha.\omega = \beta.\omega : \mathcal{C}(-,A) \longrightarrow F$. In particular, when evaluated at A, applying this equality to the identity 1_A on A gives the same element $\alpha_A(\omega_A(1_A)) = \beta_A(\omega_A(1_A))$ of $F(A)$, that is, $\alpha_A(h) = \beta_A(h)$, as desired. We can derive that $\alpha = \beta$. Hence, we have just shown that any pair of natural transformations $\alpha, \beta : \mathcal{E}(k(-),X) \longrightarrow F$, from the functor $\mathcal{E}(k(-),X)$ into a sheaf F (for the topology $\mathcal{T}$), coincide whenever $\alpha.r = \beta.r$.
Since the associated sheaf functor a is left exact, the image $a(r)$ of the monomorphism $r : R \rightarrowtail \mathcal{E}(k(-),X)$ is always a monomorphism in $\mathbf{Sh}(\mathcal{T})$; hence, it only remains to prove that $a(r) : a(R) \longrightarrow a(\mathcal{E}(k(-),X))$ is an epimorphism, because in $\mathbf{Sh}(\mathcal{T})$, which is an exact and extensive category, epimorphisms coincide with regular epimorphisms, by Proposition 95. For it, let $\lambda, \mu : a(\mathcal{E}(k(-),X)) \longrightarrow F$ be a pair of morphisms in $\mathbf{Sh}(\mathcal{T})$, namely just two natural transformations,

and suppose that $\lambda.a(r) = \mu.a(r)$. Via the adjunction $a \dashv i$, the map $\lambda.a(r) : a(R) \longrightarrow F$ corresponds to a map $R \longrightarrow i(F)$, which is defined as the composite $i(\lambda.a(r)).\zeta_R = i(\lambda).i(a(r)).\zeta_R$, where $\zeta : 1_{[\mathcal{C}^{op},\mathbf{set}]} \longrightarrow i.a$ is the unit for the adjunction $a \dashv i$. By naturality of the transformation ζ relative to the morphism r, we have $i(a(r)).\zeta_R = \zeta_{\mathcal{E}(k(-),X)}.r$; so that, the identity $\lambda.a(r) = \mu.a(r)$ is equivalent to $i(\lambda).\zeta_{\mathcal{E}(k(-),X)}.r = i(\mu).\zeta_{\mathcal{E}(k(-),X)}.r$, where $i(\lambda).\zeta_{\mathcal{E}(k(-),X)}, i(\mu).\zeta_{\mathcal{E}(k(-),X)} : \mathcal{E}(k(-),X) \longrightarrow i(a(\mathcal{E}(k(-),X))) \longrightarrow i(F)$ are morphisms in $[\mathcal{C}^{op},\mathbf{set}]$. Then, by applying our criteria concerning natural transformations from the functor $\mathcal{E}(k(-),X)$ into a sheaf F (for the topology $\mathcal{T}$), we derive $i(\lambda).\zeta_{\mathcal{E}(k(-),X)} = i(\mu).\zeta_{\mathcal{E}(k(-),X)}$, since they already agree when pre-composed with r. The last equality corresponds to $\epsilon_F.a(i(\lambda)).a(\zeta_{\mathcal{E}(k(-),X)}) = \epsilon_F.a(i(\mu)).a(\zeta_{\mathcal{E}(k(-),X)}) : a(\mathcal{E}(k(-),X)) \longrightarrow F$, via the adjunction $a \dashv i$, where we denote by $\epsilon : a.i \longrightarrow 1_{\mathbf{Sh}(\mathcal{T})}$ the counit for the adjunction, which is actually an isomorphism, since i is fully faithful. This is precisely the equality $\lambda = \mu$, by using the triangular identities for the adjunction. This ends the proof of $a(r)$ being an epimorphism, and, therefore, an isomorphism in $\mathbf{Sh}(\mathcal{T})$.

Step 4. *Let $k' : \mathcal{E} \longrightarrow [\mathcal{C}^{op},\mathbf{set}]$ be the functor defined by $X \mapsto \mathcal{E}(k(-),X)$. Then, for any object $X \in \mathcal{E}$, $k'(X) = \mathcal{E}(k(-),X)$ is a sheaf for the topology $\mathcal{T}$ on $\mathcal{C}$.*

Proof. Let C be an object of $\mathcal{C}$ and consider a k-epimorphic subfunctor $r : R \rightarrowtail \mathcal{C}(-,C)$ and a natural transformation $\alpha : R \longrightarrow \mathcal{E}(k(-),X)$. We wish to show that there exists a unique natural transformation $\beta : \mathcal{C}(-,C) \longrightarrow \mathcal{E}(k(-),X)$, namely an arrow $b \in \mathcal{E}(k(C),X)$ such that $\mathcal{E}(k(-),b) \simeq \beta$, satisfying $\beta.r \simeq \alpha$, that is, $\mathcal{E}(k(-),b) \simeq \alpha$. The collection $\mathcal{R}$ of all morphisms $\{h : C_h \longrightarrow C | h \in R(C_h), C_h \in \mathcal{C}\}$ in $\mathcal{C}$ induces a canonical map $\lambda_{\mathcal{R}}$, by the universal property of the coproduct, as follows:

$$\begin{tikzcd} \coprod_{h\in\mathcal{R}} k(C_h) \arrow[r, "\lambda_{\mathcal{R}}"] & k(C) \\ k(C_h) \arrow[u, "\sigma_h"] \arrow[ur, "k(h)"'] & \end{tikzcd}$$

Namely, $\lambda_{\mathcal{R}}$ is the unique arrow with the property $\lambda_{\mathcal{R}}.\sigma_h = k(h)$, for any $h \in \mathcal{R}$, where we denote by σ_h the canonical coproduct injection in $\mathcal{E}$. The fact that the subfunctor R is k-epimorphic means that the family $\mathcal{R}$ is k-epimorphic, that is, $\lambda_{\mathcal{R}}$ is an epimorphism, in fact, a regular epimorphism by Proposition 95, the category $\mathcal{E}$ being exact and extensive. So, following the same reasoning as in the proof of Lemma 116, we can prove that the kernel pair $N(\lambda_{\mathcal{R}})$ of $\lambda_{\mathcal{R}}$, of which $\lambda_{\mathcal{R}}$ is the coequalizer, is given by the coproduct of all pullbacks $P(h,h')$ of $k(h)$ along $k(h')$ in $\mathcal{E}$, over all pairs of arrows $(h,h') \in \mathcal{R} \times \mathcal{R}$, as in the diagram:

$$\begin{tikzcd} \coprod\limits_{(h,h') \in \mathcal{R}\times\mathcal{R}} P(h,h') \simeq N(\lambda_{\mathcal{R}}) \arrow[r, shift left, "\lambda_0"] \arrow[r, shift right, "\lambda_1"'] & \coprod\limits_{h \in \mathcal{R}} k(C_h) \arrow[r, two heads, "\lambda_{\mathcal{R}}"] & k(C) \end{tikzcd}$$

where

$$\begin{tikzcd} P(h,h') \arrow[r, "p_h"] \arrow[d, "p_{h'}"'] & k(C_h) \arrow[d, "k(h)"] \\ k(C_{h'}) \arrow[r, "k(h')"'] & k(C) \end{tikzcd}$$

is the above-mentioned pullback of $k(h)$ along $k(h')$ in $\mathcal{E}$. For any $h \in \mathcal{R}$, that is, an arrow $h : C_h \longrightarrow C$ in $\mathcal{C}$ belonging to $R(C_h)$, the natural transformation $\alpha : R \longrightarrow \mathcal{E}(k(-), X)$ produces a morphism $\alpha_{C_h}(h) : k(C_h) \longrightarrow X$; moreover, all these maps $\alpha_{C_h}(h)$ induce a canonical morphism $\mu_{\mathcal{R}}$ in $\mathcal{E}$, by the universal property of the coproduct, as follows:

$$\begin{tikzcd} \coprod\limits_{h \in \mathcal{R}} k(C_h) \arrow[r, "\mu_{\mathcal{R}}"] & X \\ k(C_h) \arrow[u, "\sigma_h"] \arrow[ur, "\alpha_{C_h}(h)"'] & \end{tikzcd}$$

Explicitly, $\mu_{\mathcal{R}}$ is the unique map satisfying $\mu_{\mathcal{R}}.\sigma_h = \alpha_{C_h}(h)$, for any $h \in \mathcal{R}$. Suppose that $\mu_{\mathcal{R}}.\lambda_0 = \mu_{\mathcal{R}}.\lambda_1$. Since $\lambda_{\mathcal{R}}$ is the coequalizer of the pair (λ_0, λ_1), we obtain a unique morphism $b : k(C) \longrightarrow X$ such that $b.\lambda_{\mathcal{R}} = \mu_{\mathcal{R}}$, that is, $b.\lambda_{\mathcal{R}}.\sigma_h = \mu_{\mathcal{R}}.\sigma_h$ for any $h \in \mathcal{R}$. This condition is equivalent to $\mathcal{E}(k(C_h), b)(h) = b.k(h) = b.\lambda_{\mathcal{R}}.\sigma_h = \mu_{\mathcal{R}}.\sigma_h = \alpha_{C_h}(h)$, for any morphism $h : C_h \longrightarrow C$ in $\mathcal{R}$, by definition of the arrows $\lambda_{\mathcal{R}}$ and $\mu_{\mathcal{R}}$. This is precisely $\mathcal{E}(k(-), b) \simeq \alpha$, as desired. So, it only remains to show that $\mu_{\mathcal{R}}.\lambda_0 = \mu_{\mathcal{R}}.\lambda_1$, or, equivalently, that $\mu_{\mathcal{R}}.\lambda_0.s_{(h,h')} = \mu_{\mathcal{R}}.\lambda_1.s_{(h,h')}$, for any pair $(h,h') \in \mathcal{R} \times \mathcal{R}$, where the maps $s_{(h,h')} : P(h,h') \longrightarrow N(\lambda_{\mathcal{R}}) \simeq \coprod_{(h,h') \in \mathcal{R}\times\mathcal{R}} P(h,h')$ are the canonical coproduct injections in $\mathcal{E}$. In other words, we need to prove that $\alpha_{C_h}(h).p_h = \alpha_{C_{h'}}(h').p_{h'}$, for any $(h,h') \in \mathcal{R} \times \mathcal{R}$, by definition of the morphisms $s_{(h,h')}$ (they are the unique ones such that $\lambda_0.s_{(h,h')} = \sigma_h.p_h$ and $\lambda_1.s_{(h,h')} = \sigma_{h'}.p_{h'}$) and $\mu_{\mathcal{R}}$. Consider, then, for any $(h,h') \in \mathcal{R} \times \mathcal{R}$, the span in $\mathcal{E}$:

$$\begin{tikzcd} k(C_h) & P(h,h') \arrow[l, "p_h"'] \arrow[r, "p_{h'}"] & k(C_{h'}) \end{tikzcd}$$

By Condition (B), the family $R_{p_h,p_{h'}}$ of all arrows in $\mathcal{E}$:

$$\left\{ c_i : k(C_i) \longrightarrow P(h,h') \,\middle|\, \left\{ \begin{array}{l} p_h.c_i = k(f) \\ p_{h'}.c_i = k(g) \end{array} \right. \text{for some}\, C_h \xleftarrow{\;f\;} C_i \xrightarrow{\;g\;} C_{h'} \right\}$$

is epimorphic. Explicitly, this means that the arrows $\alpha_{C_h}(h).p_h$ and $\alpha_{C_{h'}}(h').p_{h'}$ coincide if and only if the identity $\alpha_{C_h}(h).p_h.c_i = \alpha_{C_{h'}}(h').p_{h'}.c_i$ holds, for any $c_i \in R_{p_h,p_{h'}}$. Using the naturality of the transformation $\alpha : R \longrightarrow \mathcal{E}(k(-), X)$, relative to the map $f : C_i \longrightarrow C_h$ and the map $g : C_i \longrightarrow C_{h'}$ in $\mathcal{C}$, respectively, we get:

$$\begin{array}{lllll} \alpha_{C_h}(h).p_h.c_i & = & \alpha_{C_h}(h).k(f) & = & \alpha_{C_i}(h.f) \\ \alpha_{C_{h'}}(h').p_{h'}.c_i & = & \alpha_{C_{h'}}(h').k(g) & = & \alpha_{C_i}(h'.g) \end{array}$$

for any $c_i \in R_{p_h,p_{h'}}$, since $h \in R(C_h)$ and $h' \in R(C_{h'})$. Then, it only remains to show that $\alpha_{C_i}(h.f)$ and $\alpha_{C_i}(h'.g)$ coincide, for any $c_i \in R_{p_h,p_{h'}}$. Since $k(h.f) = k(h).k(f) = k(h).p_h.c_i = k(h').p_{h'}.c_i = k(h').k(g) = k(h'.g)$, for any $c_i \in R_{p_h,p_{h'}}$, by Condition (C), we know that the family of arrows $R_{h.f\&h'.g} = \{x : C_x \longrightarrow C_i | h.f.x = h'.g.x\}$ in $\mathcal{C}$ is k-epimorphic. The naturality of the transformation α relative to any element $x \in R_{h.f\&h'.g}$, namely a map $x : C_x \longrightarrow C_i$ in $\mathcal{C}$, gives $\alpha_{C_i}(h.f).k(x) = \alpha_{C_x}(h.f.x)$, when applied to the composite $h.f \in R(C_i)$, and $\alpha_{C_i}(h'.g).k(x) = \alpha_{C_x}(h'.g.x)$, when applied to $h'.g \in R(C_i)$. Therefore, the pair of morphisms $\alpha_{C_i}(h.f), \alpha_{C_i}(h'.g) : k(C_i) \longrightarrow X$, for any $c_i \in R_{p_h,p_{h'}}$, coincide if and only if $\alpha_{C_i}(h.f).k(x) = \alpha_{C_i}(h'.g).k(x)$, for any $x \in R_{h.f\&h'.g}$, since the collection $R_{h.f\&h'.g}$ is k-epimorphic. This last equality obviously holds by the naturality of α applied to the maps $x \in R_{h.f\&h'.g}$, which, by definition, satisfy $h.f.x = h'.g.x$. This ends the proof of $\mathcal{E}(k(C_h), b)(h)$ being equal to $\alpha_{C_h}(h)$, for any arrow $h : C_h \longrightarrow C$ in $\mathcal{R}$; in other words, we have $\mathcal{E}(k(-), b) \simeq \alpha$ for a unique arrow $b : k(C) \longrightarrow X$ of $\mathcal{E}$, as desired.

Step 5. *For any object $C \in \mathcal{C}$, $\eta_C : \mathcal{C}(-, C) \longrightarrow \mathcal{E}(k(-), k(C))$ is a map in the presheaf category $[\mathcal{C}^{op}, \mathbf{set}]$ (that is, η_C is a natural transformation) and, moreover, the pair $(\mathcal{E}(k(-), k(C)), \eta_C)$ is a reflection of $\mathcal{C}(-, C)$ along the inclusion $i : \mathbf{Sh}(\mathcal{T}) \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$.*

Proof. For any object $C \in \mathcal{C}$, the map $\eta_C : \mathcal{C}(-, C) \longrightarrow \mathcal{E}(k(-), k(C))$ is defined, at any $A \in \mathcal{C}$, as $(\eta_C)_A : \mathcal{C}(A, C) \longrightarrow \mathcal{E}(k(A), k(C))$, via the assignment $f \mapsto k(f)$. We know, by Step 4, that each functor $\mathcal{E}(k(-), k(C))$ is a sheaf (for the topology $\mathcal{T}$), since $k(C) \in \mathcal{E}$. First, we wish to show that $\eta_C : \mathcal{C}(-, C) \longrightarrow \mathcal{E}(k(-), k(C))$ is a natural transformation, that is, for any morphism $a : A \longrightarrow A'$ in $\mathcal{C}$, the equality $k(f).k(a) = k(f.a)$ holds, for any $f \in \mathcal{C}(A', C)$: but this is just the condition on k of being a functor. Concerning the second part of the statement, consider a sheaf F (for the topology $\mathcal{T}$) and a natural transformation $\alpha : \mathcal{C}(-, C) \longrightarrow F$; we wish to prove that there exists a unique natural transformation $\beta : \mathcal{E}(k(-), k(C)) \longrightarrow F$ making the following triangle:

$$\begin{tikzcd} \mathcal{C}(-,C) \arrow[rr, "\eta_C"] \arrow[dr, "\alpha"'] & & \mathcal{E}(k(-),k(C)) \arrow[dl, dotted, "\beta"] \\ & F & \end{tikzcd}$$

commute. Defining such a β is defining, at any object $A \in \mathcal{C}$, a map $\beta_A : \mathcal{E}(k(A), k(C)) \longrightarrow F(A)$, which associates with any morphism $f : k(A) \longrightarrow k(C)$ in $\mathcal{E}$ an element of $F(A)$. Let $f : k(A) \longrightarrow k(C)$ be such an arrow in $\mathcal{E}$ and R_f be the collection of all maps in $\mathcal{C}$ given by $\{c_i : C_i \longrightarrow A | f.k(c_i) = k(g_i) \text{ for some } g_i : C_i \longrightarrow C\}$. By Lemma 120, since the functor k satisfies Condition (B), it has to satisfy Condition (B'), we get that the family R_f is k-epimorphic. Consider, then, the subfunctor corresponding to this collection, which we still denote by $R_f \rightarrowtail \mathcal{C}(-, A)$ and is k-epimorphic; in other words, R_f belongs to $\mathcal{T}(A)$. Moreover, for any morphism $g_i : C_i \longrightarrow C$ corresponding to an arrow c_i in R_f, the natural transformation $\alpha : \mathcal{C}(-, C) \longrightarrow F$ produces an element $\alpha_{C_i}(g_i) \in F(C_i)$. We claim that the collection $\mathcal{F} := \{\alpha_{C_i}(g_i) \in F(C_i) | c_i : C_i \longrightarrow A \text{ in } R_f\}$ is an R_f-compatible family. For it, let $\theta : C_j \longrightarrow C_i$ be an arrow in $\mathcal{C}$, we wish to show that $F(\theta)(\alpha_{C_i}(g_i)) = \alpha_{C_j}(g_i.\theta)$, where $F(\theta) : F(C_i) \longrightarrow F(C_j)$ is a map in **set**, and that $\alpha_{C_j}(g_i.\theta)$ is an element of the family $\mathcal{F}$. By naturality of the transformation α relative to the morphism $\theta : C_j \longrightarrow C_i$ of $\mathcal{C}$, we obtain the commutativity of the following square:

$$\begin{tikzcd} \mathcal{C}(C_i, C) \arrow[r, "\alpha_{C_i}"] \arrow[d, "\mathcal{C}(\theta, C)"'] & F(C_i) \arrow[d, "F(\theta)"] \\ \mathcal{C}(C_j, C) \arrow[r, "\alpha_{C_j}"'] & F(C_j) \end{tikzcd}$$

Applying this commutativity to the morphism $g_i : C_i \longrightarrow C$ of $\mathcal{C}$ gives, precisely, the desired equality $F(\theta)(\alpha_{C_i}(g_i)) = \alpha_{C_j}(g_i.\theta)$. Moreover, since $R_f \rightarrowtail \mathcal{C}(-, A)$ is a subfunctor, from the fact that $c_i \in R_f(C_i)$, we can derive that $c_i.\theta \in R_f(C_j)$. In fact, the arrow $c_i.\theta : C_j \longrightarrow A$ is such that $f.k(c_i.\theta) = f.k(c_i).k(\theta) = k(g_i).k(\theta) = k(g_i.\theta)$, where the composite $g_i.\theta : C_j \longrightarrow C$ is in $\mathcal{C}$. Namely, $\alpha_{C_j}(g_i.\theta) \in F(C_j)$, for $c_i.\theta$ in R_f, as desired. Therefore, F being a sheaf (for the topology $\mathcal{T}$), there exists a unique element $\varphi \in F(A)$ satisfying $F(c_i)(\varphi) = \alpha_{C_i}(g_i)$, for any c_i in R_f. We can, then, define β, at $A \in \mathcal{C}$, of the morphism $f \in \mathcal{E}(k(A), k(C))$ as $\beta_A(f) := \varphi$. First of all, the transformation $\beta : \mathcal{E}(k(-), k(C)) \longrightarrow F$, defined by this assignment, is natural. For it, let $h : B \longrightarrow A$ be an arrow in $\mathcal{C}$ and consider the following square:

$$\begin{tikzcd}
\mathcal{E}(k(A),k(C)) \arrow[r, "\beta_A"] \arrow[d, "\mathcal{E}(k(h),k(C))"'] & F(A) \arrow[d, "F(h)"] \\
\mathcal{E}(k(B),k(C)) \arrow[r, "\beta_B"'] & F(B)
\end{tikzcd}$$

It commutes if and only if, for any morphism $a : k(A) \longrightarrow k(C)$ in $\mathcal{E}$, we get $F(h)(\beta_A(a)) = \beta_B(a.k(h))$. Since $a.k(h) : k(B) \longrightarrow k(C)$ is an arrow in $\mathcal{E}$, we can consider the family $R_{a.k(h)}$, which is given by $\{c_x : C_x \longrightarrow B | a.k(h).k(c_x) = k(g_x)$ for some $g_x : C_x \longrightarrow C\}$ and we know that $\beta_B(a.k(h)) \in F(B)$ has be defined to be the unique element satisfying $F(c_x)(\beta_B(a.k(h))) = \alpha_{C_x}(g_x)$, for any $c_x \in R_{a.k(h)}$. Then, for any $c_x \in R_{a.k(h)}$, consider $F(c_x)(F(h)(\beta_A(a))) = F(h.c_x)(\beta_A(a))$: from the fact that $c_x \in R_{a.k(h)}$, it follows that $a.k(h.c_x) = a.k(h).k(c_x) = k(g_x)$, where $g_x : C_x \longrightarrow C$ is an arrow in $\mathcal{C}$; therefore, $h.c_x : C_x \longrightarrow A$ is an arrow in $R_a = \{d_y : D_y \longrightarrow A | a.k(d_y) = k(l_y)$ for some $l_y : D_y \longrightarrow C\}$. By definition, the element $\beta_A(a) \in F(A)$ is the unique such that $F(d_y)(\beta_A(a)) = \alpha_{D_y}(l_y)$, for any $d_y \in R_a$. So, we do have $F(h.c_x)(\beta_A(a)) = \alpha_{C_x}(g_x)$, for any $c_x \in R_{a.k(h)}$, and, therefore, $F(h)(\beta_A(a))$ and $\beta_B(a.k(h))$ must coincide, as desired. Secondly, the natural transformation β satisfies the condition $\beta.\eta_C \simeq \alpha$. For it, for any object $A \in \mathcal{C}$, we need to show that $\beta_A.(\eta_C)_A(f)$, which is $\beta_A(k(f))$, coincides with $\alpha_A(f)$, for any morphism $f : A \longrightarrow C$ in $\mathcal{C}$. Consider the corresponding collection of maps $R_{k(f)} = \{c_i : C_i \longrightarrow A | k(f).k(c_i) = k(g_i)$ for some $g_i : C_i \longrightarrow C\}$, which, in this peculiar case of a map of the kind $k(f) : k(A) \longrightarrow k(B)$, coincides with the family of all arrows in $\mathcal{C}$ with codomain A, namely $\{c_i : C_i \longrightarrow A | C_i \in \mathcal{C}\}$. By definition, $\beta_A(k(f)) \in F(A)$ is the unique element such that $F(c_i)(\beta_A(k(f))) = \alpha_{C_i}(f.c_i)$, for any $c_i : C_i \longrightarrow A$. From the naturality of the transformation α relative to such a morphism $c_i : C_i \longrightarrow A$, it follows that $F(c_i)(\alpha_A(f)) = \alpha_{C_i}(f.c_i)$, where $\alpha_A(f) \in F(A)$. By uniqueness of the element $\beta_A(k(f))$ with such a property, we do have $\beta_A(k(f)) = \alpha_A(f)$, as desired. Finally, the natural transformation β is unique with the property $\beta.\eta_C \simeq \alpha$. For it, suppose there exists another natural transformation $\beta' : \mathcal{E}(k(-), k(C)) \longrightarrow F$ verifying $\beta'.\eta_C \simeq \alpha$, that is, for any object $A \in \mathcal{C}$ and for any morphism $f : A \longrightarrow C$ in $\mathcal{C}$, we have $\beta'_A(k(f)) = \alpha_A(f)$. By definition of β, the element $\beta_A(k(f)) \in F(A)$ is the unique one satisfying, for any $c_i \in R_{k(f)}$, the condition $F(c_i)(\beta_A(k(f))) = \alpha_{C_i}(f.c_i)$, and therefore, we have just shown, $\beta_A(k(f))$ coincides with $\alpha_A(f)$, by naturality of α. Therefore, β and β', whose components agree at any morphism $f : A \longrightarrow C$ in $\mathcal{C}$, have to be the same natural transformation, as desired.

So, by Step 4, there is a functor $\varsigma : \mathcal{E} \longrightarrow \mathbf{Sh}(\mathcal{T})$, defined by the assignment $X \mapsto \mathcal{E}(k(-), X)$, which satisfies $i.\varsigma \simeq k'$. Since both functors i and k' are fully faithful, it follows that ς is full and faithful. In order to show that ς is an equivalence of categories, it only remains to prove, then, that ς is essentially surjective. To this aim, we devote the following steps.

Step 6. *The functor $\varsigma : \mathcal{E} \longrightarrow \mathbf{Sh}(\mathcal{T})$ preserves coproducts.*
Proof. Let $(X_i)_{i\in I}$ be a collection of objects of $\mathcal{E}$ and consider their coproduct $\coprod_{i\in I} X_i$ in $\mathcal{E}$, with coproduct injections denoted by $\sigma_i : X_i \longrightarrow \coprod_{i\in I} X_i$. For any of these arrows σ_i, we get a functor $\mathcal{E}(k(-), \sigma_i)$: $\mathcal{E}(k(-), X_i) \longrightarrow \mathcal{E}(k(-), \coprod_{i\in I} X_i)$ in $[\mathcal{C}^{op}, \mathbf{set}]$; otherwise written, $k'(\sigma_i)$: $k'(X_i) \longrightarrow k'(\coprod_{i\in I} X_i)$. These functors induce a canonical morphism $\lambda : \coprod_{i\in I} \mathcal{E}(k(-), X_i) \longrightarrow \mathcal{E}(k(-), \coprod_{i\in I} X_i)$ in $[\mathcal{C}^{op}, \mathbf{set}]$, by the universal property of the coproduct, as follows:

$$\begin{tikzcd}
\coprod\limits_{i\in I} \mathcal{E}(k(-), X_i) \arrow[r, "\lambda"] & \mathcal{E}(k(-), \coprod\limits_{i\in I} X_i) \\
\mathcal{E}(k(-), X_i) \arrow[u, "\rho_i"] \arrow[ur, "\mathcal{E}(k(-),\sigma_i)"'] &
\end{tikzcd}$$

Namely, λ is the unique arrow satisfying $\lambda.\rho_i = \mathcal{E}(k(-), \sigma_i)$, for any $i \in I$, where we denote by ρ_i the canonical coproduct injection in $[\mathcal{C}^{op}, \mathbf{set}]$. Note that, since $i : \mathbf{Sh}(\mathcal{T}) \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$ is fully faithful, the counit for the adjunction $a \dashv i$ is an isomorphism, that is $a.i \simeq 1_{\mathbf{Sh}(\mathcal{T})}$; moreover, by definition of ς, we have $i.\varsigma \simeq k'$. Therefore, we get $a.k' \simeq a.i.\varsigma \simeq \varsigma$. Applying the functor $a : [\mathcal{C}^{op}, \mathbf{set}] \longrightarrow \mathbf{Sh}(\mathcal{T})$, which preserves coproducts, being a left adjoint, to the previous diagram gives in $\mathbf{Sh}(\mathcal{T})$:

$$\begin{tikzcd}
\coprod\limits_{i\in I} \varsigma(X_i) \arrow[r, "a(\lambda)"] & \varsigma(\coprod\limits_{i\in I} X_i) \\
\varsigma(X_i) \arrow[u, "\delta_i \simeq a(\rho_i)"] \arrow[ur, "\varsigma(\sigma_i)"'] &
\end{tikzcd}$$

To prove that ς preserves coproducts is to show, then, that $a(\lambda)$ is an isomorphism in $\mathbf{Sh}(\mathcal{T})$. First, we show that $\lambda : \coprod_{i\in I}\mathcal{E}(k(-),X_i) \longrightarrow \mathcal{E}(k(-),\coprod_{i\in I}X_i)$ is a monomorphism in $[\mathcal{C}^{op},\mathbf{set}]$, that is, for any $A\in\mathcal{C}$, the map $\lambda_A : \coprod_{i\in I}\mathcal{E}(k(A),X_i) \longrightarrow \mathcal{E}(k(A),\coprod_{i\in I}X_i)$, defined on an arrow $f : k(A)\longrightarrow X_i$ in $\mathcal{E}$ as $\lambda_A((\rho_i)_A(f)) = \sigma_i.f$, is an injection in $\mathbf{set}$. For it, take two elements $x,y\in\coprod_{i\in I}\mathcal{E}(k(A),X_i)$, that is, an element $x_i\in\mathcal{E}(k(A),X_i)$, for some $i\in I$, and an element $y_j\in\mathcal{E}(k(A),X_j)$, for some $j\in I$, such that $(\rho_i)_A(x_i)=x$ and $(\rho_j)_A(y_j)=y$. Suppose that $\lambda_A(x)=\lambda_A(y)$, namely $\sigma_i.x_i=\sigma_j.y_j$, by definition of λ_A. The category $\mathcal{E}$ being extensive, two cases are possible: either $i=j$ and we conclude immediately that $x_i=y_i$, since the coproduct injection σ_i is a monomorphism, by Proposition 93; or $i\neq j$ and, since coproducts in $\mathcal{E}$ are disjoint, the pair (x_i,y_j) factors through the pullback of σ_i along σ_j, which is given by the initial object 0 of $\mathcal{E}$. So, there exists an unique arrow $\varphi : k(A)\longrightarrow 0$ which, moreover, is an isomorphism, since initial objects are strict in an extensive category. Applying the functor $\mathcal{E}(k(A),-) : \mathcal{E}\longrightarrow\mathbf{set}$, which preserves finite limits, to the pullback in $\mathcal{E}$ of σ_i along σ_j, gives:

$$\begin{tikzcd}
\mathcal{E}(k(A),0) \arrow[r, tail] \arrow[d, tail] & \mathcal{E}(k(A),X_i) \arrow[d, tail, "\mathcal{E}(k(A),\sigma_i)"] \\
\mathcal{E}(k(A),X_j) \arrow[r, tail, "\mathcal{E}(k(A),\sigma_j)"'] & \mathcal{E}(k(A),\coprod_{i\in I}X_i)
\end{tikzcd}$$

which is a pullback in $\mathbf{set}$. From the fact that, by Proposition 93, σ_i is a monomorphism, for any $i\in I$, follows easily that $\mathcal{E}(k(A),\sigma_i)$ is a monomorphism, for any $i\in I$. But we know that the pullback of two injections in $\mathbf{set}$ is given by their intersection, so, we get $\mathcal{E}(k(A),X_i)\cap\mathcal{E}(k(A),X_j)\simeq\mathcal{E}(k(A),0)\simeq\{*\}$, $k(A)$ being isomorphic to 0, from which we can conclude that x_i and y_j have to coincide, since they do belong to the intersection. This ends the proof of λ being a monomorphism in $[\mathcal{C}^{op},\mathbf{set}]$; in other words, $\lambda : \coprod_{i\in I}\mathcal{E}(k(-),X_i) \rightarrowtail \mathcal{E}(k(-),\coprod_{i\in I}X_i)$ is a subfunctor. It only remains to prove, then, that λ is an epimorphic subfunctor; then, by Step 3, we derive that $a(\lambda)$ is an isomorphism in $\mathbf{Sh}(\mathcal{T})$. Proving that λ is an epimorphic subfunctor is to prove that the set-theoretic union of all collections

$$\mathcal{I}(A)=\left\{x : k(A)\longrightarrow\coprod_{i\in I}X_i \,|\, \exists i\in I \text{ and } x' : k(A)\longrightarrow X_i : x=\sigma_i.x'\right\}$$

of arrows of $\mathcal{E}$, for A varying in $\mathcal{C}$, call it $\mathcal{I}$, is an epimorphic family. For it, consider, for any $i\in I$, all composites of the kind $\sigma_i.h$, of the coproduct injection σ_i with the arrows h, for $h \in R_{X_i} = \{h : k(C_h) \longrightarrow X_i | C_h \in \mathcal{C}\}$. Denote by $\mathcal{S}$ the collection $\{\sigma_i.h : k(C_h) \longrightarrow \coprod_{i\in I} X_i | i \in I, h \in R_{X_i}\}$ of morphisms in $\mathcal{E}$ arising in this way. $\mathcal{S}$ is epimorphic: let $u, v : \coprod_{i\in I} X_i \longrightarrow U$ be a pair of arrows in $\mathcal{E}$ such that $u.\sigma_i.h = v.\sigma_i.h$, for any $i \in I$ and for any $h \in R_{X_i}$. This amounts to saying that $u.\sigma_i = v.\sigma_i$, for any $i \in I$, R_{X_i} being an epimorphic family by Condition (A). By definition of the coproduct, the last identity is equivalent to $u = v$. Moreover, $\mathcal{S}$ is contained in $\mathcal{I}$, by mere definition of any composite $\sigma_i.h : k(C_h) \longrightarrow \coprod_{i\in I} X_i$. We conclude, by Lemma 113, that the subfunctor $\lambda : \coprod_{i\in I} \mathcal{E}(k(-), X_i) \rightarrowtail \mathcal{E}(k(-), \coprod_{i\in I} X_i)$, which does contain an epimorphic family, namely $\mathcal{S}$, is itself epimorphic. Therefore, it follows from Step 3 that $a(\lambda)$ is an isomorphism, that is, the functor ς preserves coproducts.

Step 7. *The functor $\varsigma : \mathcal{E} \longrightarrow \mathbf{Sh}(\mathcal{T})$ is exact.*
Proof. First of all, ς preserves all limits, since we have $i.\varsigma \simeq k'$, where both the functors i and k', being right adjoints, preserve all limits and i, being fully faithful, reflects them too. Then, it only remains to prove that $\varsigma : \mathcal{E} \longrightarrow \mathbf{Sh}(\mathcal{T})$ preserves regular epimorphisms. For it, let $f : X \longrightarrow Y$ be a regular epimorphism in $\mathcal{E}$ and consider the corresponding functor $\mathcal{E}(k(-), f) : \mathcal{E}(k(-), X) \longrightarrow \mathcal{E}(k(-), Y)$, which is defined by composition with f. Explicitly, for any $A \in \mathcal{C}$, it associates with any morphism $a : k(A) \longrightarrow X$ in $\mathcal{E}$ its composite $f.a$ with the arrow f. Consider the regular epi-mono factorisation of $\mathcal{E}(k(-), f)$ in $[\mathcal{C}^{op}, \mathbf{set}]$:

$$\begin{tikzcd} \mathcal{E}(k(-),X) \arrow[rr, "\mathcal{E}(k(-),f)"] \arrow[dr, two heads] & & \mathcal{E}(k(-),Y) \\ & I \arrow[ur, tail, "r"'] & \end{tikzcd}$$

This is computed pointwise in **set**, namely, for any $A \in \mathcal{C}$, the image of the map $\mathcal{E}(k(A), f) : \mathcal{E}(k(A), X) \longrightarrow \mathcal{E}(k(A), Y)$ is defined as the set $I(A) = \{y : k(A) \longrightarrow Y | \exists x : k(A) \longrightarrow X \text{ such that } f.x = y\}$. We denote by $\mathcal{I}$ the set-theoretic union of all families $I(A)$, for A varying in $\mathcal{C}$: the collection $\mathcal{I}$ is an epimorphic family. For it, consider all composites $f.h$, for any morphism h in $R_X = \{h : k(C_h) \longrightarrow X | C_h \in \mathcal{C}\}$: they form a collection $\mathcal{S} = \{f.h : k(C_h) \longrightarrow Y | h \in R_X\}$ of morphisms in $\mathcal{E}$. $\mathcal{S}$ is epimorphic: let $u, v : Y \longrightarrow U$ be a pair of arrows in $\mathcal{E}$ such that $u.f.h = v.f.h$, for any $h \in R_X$, which is an epimorphic family by Condition (A). So, the previous equality is equivalent to $u.f = v.f$, from which it follows that $u = v$, f being an epimorphism. Moreover, $\mathcal{S}$ is contained in $\mathcal{I}$, by mere definition of any composite $f.h : k(C_h) \longrightarrow Y$.

We conclude, by Lemma 113, that the collection $\mathcal{I}$, which does contain an epimorphic family, namely $\mathcal{S}$, is itself epimorphic. This precisely means that the corresponding $r : I \rightarrowtail \mathcal{E}(k(-), Y)$ is an epimorphic subfunctor. Then, by Step 3, we obtain that the image $a(r)$ of r through the associated sheaf functor a is an isomorphism. As shown in Step 6, the equivalence $a.k' \simeq \varsigma$ holds, so, applying the functor a to the regular epi-mono factorisation of the morphism $\mathcal{E}(k(-), f)$, otherwise written, $k'(f) : k'(X) \longrightarrow k'(Y)$, gives in $\mathbf{Sh}(\mathcal{T})$:

$$\begin{tikzcd}
\varsigma(X) \arrow[rr, "\varsigma(f)"] \arrow[dr] & & \varsigma(Y) \\
& a(I) \arrow[ur, tail, "a(r)"'] &
\end{tikzcd}$$

The arrow $a(r)$ being an isomorphism, it implies that $\varsigma(f)$ is a regular epimorphism, as desired.

Step 8. *For any object $C \in \mathcal{C}$, there exists an isomorphism $\delta_C : a(\mathcal{C}(-, C)) \simeq \mathcal{E}(k(-), k(C))$ in $\mathbf{Sh}(\mathcal{T})$.*

Proof. Let $\zeta : 1_{[\mathcal{C}^{op},\mathbf{set}]} \longrightarrow i.a$ be the unit for the adjunction $a \dashv i$ and, for any $C \in \mathcal{C}$, consider the sheaf $a(\mathcal{C}(-, C))$ in $\mathbf{Sh}(\mathcal{T})$. In Step 5, we establish that η_C is a reflection of $\mathcal{C}(-, C)$ along $i : \mathbf{Sh}(\mathcal{T}) \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$; by the universal property of η_C, there exists a unique natural transformation $\gamma_C : \mathcal{E}(k(-), k(C)) \longrightarrow a(\mathcal{C}(-, C))$ making the following diagram in $\mathbf{Sh}(\mathcal{T})$:

$$\begin{tikzcd}
\mathcal{C}(-, C) \arrow[rr, "\eta_C"] \arrow[dr, "\zeta_{\mathcal{C}(-,C)}"'] & & \mathcal{E}(k(-), k(C)) \arrow[dl, dotted, "\gamma_C"] \\
& a(\mathcal{C}(-, C)) &
\end{tikzcd}$$

commute. On the other hand, by the universal property of ζ being the unit for the adjunction $a \dashv i$, there exists a unique natural transformation $\delta_C : i(a(\mathcal{C}(-, C))) \simeq a(\mathcal{C}(-, C)) \longrightarrow i(\mathcal{E}(k(-), k(C))) \simeq \mathcal{E}(k(-), k(C))$ making the following triangle in $[\mathcal{C}^{op}, \mathbf{set}]$:

$$\begin{tikzcd}
\mathcal{C}(-, C) \arrow[rr, "\zeta_{\mathcal{C}(-,C)}"] \arrow[dr, "\eta_C"'] & & a(\mathcal{C}(-, C)) \arrow[dl, dotted, "\delta_C"] \\
& \mathcal{E}(k(-), k(C)) &
\end{tikzcd}$$

commute. It is easy to verify that the two equalities $\gamma_C.\delta_C \simeq 1_{a(\mathcal{C}(-,C))}$ and $\delta_C.\gamma_C \simeq 1_{\mathcal{E}(k(-),k(C))}$ hold, simply by definitions of γ_C, as the unique map satisfying $\gamma_C.\eta_C \simeq \zeta_{\mathcal{C}(-,C)}$, and of δ_C, as the unique one such that $\delta_C.\zeta_{\mathcal{C}(-,C)} \simeq \eta_C$, respectively. Therefore, we get that $\delta_C : a(\mathcal{C}(-,C)) \simeq \mathcal{E}(k(-),k(C))$ is an isomorphism of functors, for any $C \in \mathcal{C}$, as desired.

So, having $i : \mathbf{Sh}(\mathcal{T}) \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$ a localization of the presheaf category $[\mathcal{C}^{op}, \mathbf{set}]$ with associated sheaf functor a, and, knowing that the small set $\mathcal{G} = \{\mathcal{C}(-,C) | C \in \mathcal{C}\}$ is a (strong) generator for $[\mathcal{C}^{op}, \mathbf{set}]$, we derive, from Lemma 68, that $a\mathcal{G} = \{a(\mathcal{C}(-,C)) | C \in \mathcal{C}\}$ is a (strong) generator for $\mathbf{Sh}(\mathcal{T})$; but this small set is precisely $\{\mathcal{E}(k(-),k(C)) | C \in \mathcal{C}\}$, by Step 8. More explicitly, for any sheaf F (for the topology $\mathcal{T}$), which can be regarded, in particular, as a presheaf $i(F) \simeq F : \mathcal{C}^{op} \longrightarrow \mathbf{set}$, it is well-known that we can write $i(F)$ as the coequalizer in $[\mathcal{C}^{op}, \mathbf{set}]$ of the pair arrows of e_0 and e_1, in a canonical way, as follows:

$$\begin{tikzcd}
\coprod\limits_{f \in \mathbf{Elts}(i(F))} \mathcal{C}(-,D) \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & \coprod\limits_{C \in \mathbf{Elts}(i(F))} \mathcal{C}(-,C) \arrow[r, two heads, "q"] & i(F) \\
\mathcal{C}(-,s(f)) \arrow[u, "\rho_f"] \arrow[ur, "\sigma_{s(f)}"] \arrow[r, "\mathcal{C}(-,f)"'] & \mathcal{C}(-,t(f)) \arrow[u, "\sigma_{t(f)}"'] &
\end{tikzcd}$$

where, for any morphism $f : s(f) \longrightarrow t(f)$ in $\mathcal{C}$, e_0 is defined as the unique map satisfying $e_0.\rho_f = \sigma(s(f))$, whereas, e_1 is defined as the unique map such that $e_1.\rho_f = \sigma_{t(f)}.\mathcal{C}(-,f)$. Obviously, the categories $\mathbf{Elts}(i(F))$ and $\mathbf{Elts}(F)$ coincide. So, applying the associated sheaf functor a, which is a left adjoint, and, hence, it preserves colimits, to the previous diagram, gives in $\mathbf{Sh}(\mathcal{T})$:

$$\begin{tikzcd}
\coprod\limits_{f \in \mathbf{Elts}(F)} a(\mathcal{C}(-,D)) \arrow[r, shift left, "a(e_0)"] \arrow[r, shift right, "a(e_1)"'] & \coprod\limits_{C \in \mathbf{Elts}(F)} a(\mathcal{C}(-,C)) \arrow[r, two heads, "a(q)"] & a(i(F)) \simeq F \\
a(\mathcal{C}(-,s(f))) \arrow[u, "a(\rho_f)"] \arrow[ur, "a(\sigma_{s(f)})"] \arrow[r, "a(\mathcal{C}(-,f))"'] & a(\mathcal{C}(-,t(f))) \arrow[u, "a(\sigma_{t(f)})"'] &
\end{tikzcd}$$

We underline the fact that the morphisms $a(\rho_f)$ are precisely the coproduct injections $a(\mathcal{C}(-,s(f))) \longrightarrow \coprod_{f \in \mathbf{Elts}(F)} a(\mathcal{C}(-,D))$, since a preserves coproducts; and analogously, for the maps $a(\sigma_C)$, which are the coproduct injections $a(\mathcal{C}(-,C)) \longrightarrow \coprod_{C \in \mathcal{C}} a(\mathcal{C}(-,C))$. The isomorphisms $\delta_C : a(\mathcal{C}(-,C)) \simeq \mathcal{E}(k(-),k(C))$, given at Step 8, induce two canonical arrows $\coprod_{f \in \mathbf{Elts}(F)} \delta_D$ and $\coprod_{C \in \mathcal{C}} \delta_C$, defined by the commutativity of the following diagram:

$$
\begin{tikzcd}
\coprod\limits_{f\in\mathbf{Elts}(F)} a(\mathcal{C}(-,D)) \arrow[r, "\coprod\delta_D"] & \coprod\limits_{f\in\mathbf{Elts}(F)} \mathcal{E}(k(-),k(D)) \\
a(\mathcal{C}(-,s(f))) \arrow[u, "a(\rho_f)"] \arrow[r, "\delta_{s(f)}"'] & \mathcal{E}(k(-),k(s(f))) \arrow[u, "\varsigma(\rho'_f)"']
\end{tikzcd}
$$

(and analogously for $\coprod_{C\in\mathcal{C}}\delta_C$). Clearly, these two maps $\coprod_{f\in\mathbf{Elts}(F)}\delta_D$ and $\coprod_{C\in\mathcal{C}}\delta_C$ are isomorphisms themselves. On the other hand, we recall that, by definition of the functor ς, $\varsigma(k(C)) = \mathcal{E}(k(-),k(C))$, for any object $C\in\mathcal{C}$; so, we can consider in $\mathbf{Sh}(\mathcal{T})$:

$$
\begin{tikzcd}
\coprod\limits_{f\in\mathbf{Elts}(F)} \varsigma(k(D)) \arrow[r, shift left, "\varsigma(l_0)"] \arrow[r, shift right, "\varsigma(l_1)"'] & \coprod\limits_{C\in\mathbf{Elts}(F)} \varsigma(k(C))
\end{tikzcd}
$$

which are the images through the functor ς of pair of arrows l_0 and l_1 in $\mathcal{E}$. (Note that we have used here that, by Step 6, ς preserves coproducts.) In details, for any morphism $f : s(f) \longrightarrow t(f)$ in $\mathcal{C}$, the morphism l_0 is defined as the unique map such that $l_0.\rho'_f = \sigma'_{s(f)}$, whereas l_1 is defined as the unique arrow satisfying $l_1.\rho'_f = \sigma'_{t(f)}.k(f)$, as displayed in the following diagram in $\mathcal{E}$:

$$
\begin{tikzcd}
\coprod\limits_{f\in\mathbf{Elts}(F)} k(D) \arrow[r, shift left, "l_0"] \arrow[r, shift right, "l_1"'] & \coprod\limits_{C\in\mathbf{Elts}(F)} k(C) \arrow[r, two heads, "p"] & X \\
k(s(f)) \arrow[u, "\rho'_f"] \arrow[ur, "\sigma'_{s(f)}"] \arrow[r, "k(f)"'] & k(t(f)) \arrow[u, "\sigma'_{t(f)}"']
\end{tikzcd}
$$

The same remark concerning coproduct injections holds for the morphisms $\varsigma(\rho'_f)$ and $\varsigma(\sigma'_C)$: ς preserving coproducts, they are, respectively, the coproduct injections $\varsigma(k(s(f))) \longrightarrow \coprod_{f\in\mathbf{Elts}(F)}\varsigma(k(D))$ and $\varsigma(k(C)) \longrightarrow \coprod_{C\in\mathcal{C}}\varsigma(k(C))$. We wish to show that, for any $f : s(f) \longrightarrow t(f)$ in $\mathcal{C}$, we have an isomorphism of functors $\delta_{t(f)}.a(\mathcal{C}(-,f)) \simeq \mathcal{E}(k(-),k(f)).\delta_{s(f)} : a(\mathcal{C}(-,s(f))) \longrightarrow \mathcal{E}(k(-),k(t(f)))$. For it, it suffices to notice, first of all, that the transformation $\eta_C : \mathcal{C}(-,C) \longrightarrow \mathcal{E}(k(-),k(C))$, introduced at Step 5, is, in fact, natural in C, that is, for any morphism $c : C \longrightarrow C'$ in $\mathcal{C}$, we have $\mathcal{E}(k(-),k(c)).\eta_C = \eta_{C'}.\mathcal{C}(-,c)$, which follows directly by the fact that k is a functor. Then, by the universal property of the unit ζ for the adjunction $a \dashv i$, there exists a unique arrow θ making the following triangle commute:

$$\begin{tikzcd} \mathcal{C}(-,s(f)) \arrow[rr, "\zeta_{\mathcal{C}(-,s(f))}"] \arrow[dr, "\eta_{t(f)}.\mathcal{C}(-,f)"'] & & a(\mathcal{C}(-,s(f))) \arrow[dl, dotted, "\theta"] \\ & \mathcal{E}(k(-),k(t(f))) & \end{tikzcd}$$

It is easy to verify that both morphisms $\delta_{t(f)}.a(\mathcal{C}(-,f))$ and $\mathcal{E}(k(-),k(f)).\delta_{s(f)}$ do so, and, therefore, they have to coincide. As a consequence, the isomorphism $\varsigma(l_0).\coprod_{f\in\mathbf{Elts}(F)}\delta_D \simeq \coprod_{C\in\mathcal{C}}\delta_C.a(e_0)$ is established, just by composing with the coproduct injections; and, analogously, $\varsigma(l_1).\coprod_{f\in\mathbf{Elts}(F)}\delta_D \simeq \coprod_{C\in\mathcal{C}}\delta_C.a(e_1)$ holds. Since $\mathcal{E}$ is cocomplete, we can always consider the coequalizer $p:\coprod_{C\in\mathbf{Elts}(F)}k(C)\longrightarrow X$ in $\mathcal{E}$ of the pair (l_0,l_1). By Step 7, ς being exact, we get that $\varsigma(p)$ is the coequalizer of the pair $(\varsigma(l_0),\varsigma(l_1))$, as in the following diagram in $\mathbf{Sh}(\mathcal{T})$:

$$\varsigma\left(\coprod_{f\in\mathbf{Elts}(F)}k(D)\right) \overset{\varsigma(l_0)}{\underset{\varsigma(l_1)}{\rightrightarrows}} \varsigma\left(\coprod_{C\in\mathbf{Elts}(F)}k(C)\right) \xrightarrow{\varsigma(p)} \varsigma(X)$$

Therefore, we obtain that $\varsigma(p)=\mathrm{coeq}(\varsigma(l_0),\varsigma(l_1))\simeq \mathrm{coeq}(a(e_0),a(e_1))=a(q)$; in other words, we have $\varsigma(X)\simeq F$, where X is the coequalizer in $\mathcal{E}$ of the pair (l_0,l_1). This ends the proof of the functor ς being essentially surjective.

Finally, we have established an equivalence of categories ς, making the following triangle:

$$\begin{tikzcd} \mathcal{E} \arrow[rr, shift right, "k'"'] \arrow[dr, "\varsigma"'] & & {[\mathcal{C}^{op},\mathbf{set}]} \arrow[ll, shift right, "L"'] \arrow[dl, shift right, "a"'] \\ & \mathbf{Sh}(\mathcal{T}) \arrow[ur, shift right, "i"'] & \end{tikzcd}$$

commute; namely $i.\varsigma \simeq k'$. Stated otherwise, the category $\mathcal{E}$ is equivalent to the Grothendieck topos $\mathbf{Sh}(\mathcal{T})$, and, therefore, $\mathcal{E}$ is a localization of the presheaf category $[\mathcal{C}^{op},\mathbf{set}]$, as desired.

Moreover, we can prove that the associated sheaf functor a, in fact, is given by the assignment $\mathcal{C}(-,C)\mapsto\mathcal{E}(k(-),k(C))$ For it, we have to establish, for any $C\in\mathcal{C}$, that $a(\mathcal{C}(-,C))\simeq\varsigma.L(\mathcal{C}(-,C))$, since, $Y_{\mathcal{C}}$ being fully faithful, the universal arrow for the left Kan extension L of k along Yoneda is an isomorphism, namely $k\simeq L.Y_{\mathcal{C}}$, and we know that $k'\simeq i.\varsigma$.

That is, we want to show that $a \simeq \varsigma.L$: this amounts to showing that $\varsigma.L$ is left adjoint to $i : \mathbf{Sh}(\mathcal{T}) \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$, by uniqueness (up to isomorphism) of the left adjoint functor. This follows immediately by the universal property of $(\mathcal{E}(k(-), k(C)), \eta_C)$ being a reflection of $\mathcal{C}(-, C)$ along i, as claimed in Step 5. In fact, $\eta_C : \mathcal{C}(-, C) \longrightarrow \mathcal{E}(k(-), k(C))$ turns out to be the component at C of the unit for the adjunction $a \dashv i$. Finally, we can extend the definition of a to an arbitrary $F \in [\mathcal{C}^{op}, \mathbf{set}]$, by its property of preserving existing colimits.

This concludes the proof of the Theorem 125 ($2 \Rightarrow 1$). ■

4

On geometric morphisms and localizations of algebraic categories

The project of characterising localizations and classifying geometric morphisms undertaken in the previous chapter finds here its natural continuation: the theorem concerning localizations of (one sorted) algebraic categories is achieved. In order to state it, we develop the theory of "special filtering" functors - as opposed to filtering functors - in the particular case of algebraic theories and relate this notion to the exact completion of the full subcategory of the category of models given by the free ones.

4.1 Algebraic theories and algebraic categories

All definitions and results contained in this section can be found in [11, 30].

Definition 126 *By an* algebraic theory *(also called* Lawvere theory*)* $\mathcal{C}$ *we mean a category* $\mathcal{C}$ *with a denumerable set of distinct objects* $\{0T, 1T, 2T, ..., nT, ...\}$*, where* n *is a natural number and each* nT *is the* n*-th coproduct of the object* $T = 1T$.

In particular, we assume that $0T = 0$ is the initial object of $\mathcal{C}$.

Definition 127 *Let* $\mathcal{C}$ *be an algebraic theory. A* model *of* $\mathcal{C}$ *is a functor* $F : \mathcal{C}^{op} \longrightarrow \mathbf{set}$ *which preserves finite products.*

We write $\mathbf{Mod}(\mathcal{C})$ for the category of $\mathcal{C}$-models and natural transformations between them. Categories of this form are called *algebraic categories* or *finitary varieties*. The reason why we choose the additive notation for an algebraic theory will be enlightened in Section 4.4.

Lemma 128 *Let $\mathcal{C}$ be an algebraic theory and $\alpha : F \longrightarrow G$ a morphism in $\mathbf{Mod}(\mathcal{C})$. Then, the following square commutes:*

$$\begin{tikzcd} F(nT) \arrow[r, "\alpha_{nT}"] \arrow[d, "\simeq"'] & G(nT) \arrow[d, "\simeq"] \\ F(T)^n \arrow[r, "(\alpha_T)^n"'] & G(T)^n \end{tikzcd}$$

where the isomorphisms involved are the canonical ones given by the universal property of finite products in **set**.

Given an algebraic theory $\mathcal{C}$, we can always consider the canonical functor

$$\mathbb{U} : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathbf{set}$$

of *evaluation* at T, that is, $\mathbb{U}$ associates with any model F its value $F(T) \in \mathbf{set}$ at T. In fact, the functor $\mathbb{U}$ has a few interesting features, as stated in the following:

Proposition 129 *The functor $\mathbb{U} : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathcal{C}$ satisfies the following properties:*

1. *$\mathbb{U}$ is a representable functor, represented by $\mathcal{C}(-, T)$;*
2. *$\mathbb{U}$ is faithful;*
3. *$\mathbb{U}$ reflects isomorphisms;*
4. *$\mathcal{C}(-, T)$ is a strong generator for $\mathbf{Mod}(\mathcal{C})$.*

On the other hand, a lot can be said concerning limits and colimits in algebraic categories and we list here their major properties:

Proposition 130 *Let $\mathcal{C}$ be an algebraic theory. The category $\mathbf{Mod}(\mathcal{C})$ is complete and limits are computed pointwise. Moreover, the forgetful functor $\mathbb{U} : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathbf{set}$ preserves and reflects limits.*

Proposition 131 *Let $\mathcal{C}$ be an algebraic theory. The category $\mathbf{Mod}(\mathcal{C})$ has filtered colimits and these are computed pointwise. Moreover, the forgetful functor $\mathbb{U} : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathbf{set}$ preserves and reflects filtered colimits.*

Proposition 132 *Let $\mathcal{C}$ be an algebraic theory. In the category $\mathbf{Mod}(\mathcal{C})$, finite limits commute with filtered colimits.*

Moreover, $\mathbf{Mod}(\mathcal{C}) = \mathbf{Finprod}[\mathcal{C}^{op}, \mathbf{set}]$ can be shown to be a full reflective subcategory of the corresponding category of presheaves, as established in the following:

Proposition 133 *Let $\mathcal{C}$ be an algebraic theory. The category $\mathbf{Mod}(\mathcal{C})$ is reflective in the presheaf category $[\mathcal{C}^{op}, \mathbf{set}]$, and, therefore, it is both complete and cocomplete.*

Together with the forgetful functor $\mathbb{U}$, there is also a "natural way" of associating with any set a $\mathcal{C}$-model, which is described in the following:

Proposition 134 *Let $\mathcal{C}$ be an algebraic theory. The forgetful functor $\mathbb{U} : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathbf{set}$ admits a left adjoint $\mathbb{F} : \mathbf{set} \longrightarrow \mathbf{Mod}(\mathcal{C})$. Explicitly, the functor $\mathbb{F}$ is defined on any set X as*

$$\mathbb{F}(X) = \coprod_{x \in X} \mathcal{C}(-, T)$$

We call $\mathbb{F}$ the free model functor*.*

Amongst the properties the free model functor satisfies, we recall the following:

Proposition 135 *Let $\mathcal{C}$ be an algebraic theory. The free model functor $\mathbb{F} : \mathbf{set} \longrightarrow \mathbf{Mod}(\mathcal{C})$ preserves monomorphisms.*

By a *free $\mathcal{C}$-model* or *free algebra* we mean (up to isomorphism) a model of the form $\mathbb{F}(X)$, for some set X. In particular, when n is a finite set, we refer to $\mathbb{F}(n)$ as a *finitely generated free model*. Whereas, by a *finitely presentable $\mathcal{C}$-model* we mean a model F which can be obtained via a coequalizer diagram in $\mathbf{Mod}(\mathcal{C})$, as follows:

$$\begin{tikzcd} \mathbb{F}(m) \arrow[r, shift left] \arrow[r, shift right] & \mathbb{F}(n) \arrow[r, two heads] & F \end{tikzcd}$$

where m and n are finite sets, that is, a coequalizer of a pair of arrows between finitely generated free models. The relation between free models and finitely generated free models is made clear in the following:

Lemma 136 *Every free model $\mathbb{F}(X)$ can be written as a filtered colimit*

$$\mathbb{F}(X) = \operatorname*{colim}_{n \subseteq X} \mathbb{F}(n)$$

where n runs through the finite subsets of X.

In particular, one can prove that, when n is a finite set, there is an isomorphism $\mathbb{F}(n) \simeq \mathcal{C}(-, nT)$ in $\mathbf{Mod}(\mathcal{C})$, as follows:

Proposition 137 *For any finite set n, the free model functor $\mathbb{F}$ evaluated at n is isomorphic in $\mathbf{Mod}(\mathcal{C})$ to the representable functor $\mathcal{C}(-, nT)$, namely $\coprod_n \mathcal{C}(-,T) \simeq n\mathcal{C}(-,T) \simeq \mathcal{C}(-,nT)$.*

Proof. Clearly, the representable $\mathcal{C}(-,nT)$ is a functor in $\mathbf{Mod}(\mathcal{C})$. It suffices to show, then, that for any finite set n and for any model G of $\mathcal{C}$, there is a bijection $\mathbf{Nat}(\mathcal{C}(-,nT),G) \simeq \mathbf{set}(n,\mathbb{U}(G))$, natural in both the variables n and G, since the adjunction $\mathbb{F} \dashv \mathbb{U}$ is established. For it, we have:

$$
\begin{aligned}
\mathbf{Nat}(\mathcal{C}(-,nT),G) &\simeq G(nT) && (1)\\
&\simeq G(T)^n && (2)\\
&\simeq \mathbf{set}(\{1,2,...,n\},G(T))\\
&\simeq \mathbf{set}(\{1,2,...,n\},\mathbb{U}(G))
\end{aligned}
$$

(where (1) is just an instance of the Yoneda Lemma and (2) follows from the fact that $G \in \mathbf{Mod}(\mathcal{C})$). ■

Therefore:

Proposition 138 *An algebraic theory $\mathcal{C}$ is equivalent to the dual of the full subcategory of $\mathbf{Mod}(\mathcal{C})$ given by finitely generated free models.*

We point out the projective objects in $\mathbf{Mod}(\mathcal{C})$:

Proposition 139 *The free models of an algebraic theory $\mathcal{C}$ are projective in $\mathbf{Mod}(\mathcal{C})$.*

We write $\eta : 1_{\mathbf{set}} \longrightarrow \mathbb{U}.\mathbb{F}$ and $\epsilon : \mathbb{F}.\mathbb{U} \longrightarrow 1_{\mathbf{Mod}(\mathcal{C})}$, respectively, for the unit and the counit for the adjunction $\mathbb{F} \dashv \mathbb{U}$.

Proposition 140 *For any $\mathcal{C}$-model F, the counit component $\epsilon_F : \mathbb{F}.\mathbb{U}(F) \longrightarrow F$ at F is a regular epimorphism in $\mathbf{Mod}(\mathcal{C})$.*

We recall that ϵ_F is defined as the canonical morphism induced in $\mathbf{Mod}(\mathcal{C})$ by all natural transformations $\alpha \in \mathbf{Nat}(\mathcal{C}(-,T),F)$, as in the following diagram:

$$
\begin{tikzcd}
\coprod_{\alpha \in \mathbf{Nat}(\mathcal{C}(-,T),F)} \mathcal{C}(-,T) \arrow[r, two heads, "\epsilon_F"] & F \\
\mathcal{C}(-,T) \arrow[u, "\tau_\alpha"] \arrow[ur, "\alpha"'] &
\end{tikzcd}
$$

where the τ_α denote the canonical coproduct injections in $\mathbf{Mod}(\mathcal{C})$.

Proposition 141 *Let $\mathcal{C}$ be an algebraic theory. The category* $\mathbf{Mod}(\mathcal{C})$ *is regular and exact.*

Proposition 142 *Let $\mathcal{C}$ be an algebraic theory. The forgetful functor* $\mathbb{U} : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathbf{set}$ *preserves and reflects regular epimorphisms.*

4.2 A miscellany of preliminary results

Given an algebraic theory $\mathcal{C}$, we consider the Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow [\mathcal{C}^{op}, \mathbf{set}]$. For any object $C \in \mathcal{C}$, the functor $\mathcal{C}(-, C) : \mathcal{C}^{op} \longrightarrow \mathbf{set}$ preserves finite products, since we have $\mathcal{C}(nT, C) \simeq \mathcal{C}(T, C)^n$. So, the Yoneda embedding, in fact, factors through the full subcategory of the presheaf category $[\mathcal{C}^{op}, \mathbf{set}]$ given by finite product-preserving functors, namely through the algebraic category $\mathbf{Mod}(\mathcal{C}) = \mathbf{Finprod}[\mathcal{C}^{op}, \mathbf{set}]$, as displayed in the following commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "Y_{\mathcal{C}}"] \arrow[dr, dotted, "\tilde{Y}_{\mathcal{C}}"'] & & {[\mathcal{C}^{op}, \mathbf{set}]} \\ & \mathbf{Mod}(\mathcal{C}) \arrow[ur, "i"'] & \end{tikzcd}$$

Diagram 4.2

As a consequence of $i : \mathbf{Mod}(\mathcal{C}) \rightarrowtail [\mathcal{C}^{op}, \mathbf{set}]$ being a full reflective subcategory of $[\mathcal{C}^{op}, \mathbf{set}]$, whose left adjoint we denote by r, we get the following result (we refer to [33], for the notion of dense functor):

Lemma 143 *The restricted Yoneda embedding* $\tilde{Y}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$ *is a dense functor.*

Proof. By definition of dense functor (see Definition 98), for any object $F \in \mathbf{Mod}(\mathcal{C})$, we need to show that the colimit of the composite diagram:

$$\tilde{Y}_{\mathcal{C}} {\downarrow} F \xrightarrow{\Gamma_F} \mathcal{C} \xrightarrow{\tilde{Y}_{\mathcal{C}}} \mathbf{Mod}(\mathcal{C}) \ , \ (C, \gamma : \tilde{Y}_{\mathcal{C}}(C) \longrightarrow F) \mapsto \tilde{Y}_{\mathcal{C}}(C)$$

is precisely $(F, (\gamma)_{(C,\gamma) \in \tilde{Y}_{\mathcal{C}} \downarrow F})$, where Γ_F denotes the canonical forgetful functor. The Yoneda embedding $Y_{\mathcal{C}}$ being dense, we know that the functor $i(F) \in [\mathcal{C}^{op}, \mathbf{set}]$ can be written as a colimit of the canonical diagram:

$$Y_{\mathcal{C}} {\downarrow} i(F) \xrightarrow{\Gamma_{i(F)}} \mathcal{C} \xrightarrow{Y_{\mathcal{C}}} [\mathcal{C}^{op}, \mathbf{set}] \ , \ (A, \alpha : Y_{\mathcal{C}}(A) \longrightarrow i(F)) \mapsto Y_{\mathcal{C}}(A)$$

Since the counit for the adjunction $r \dashv i$ is, at each component, an isomorphism, from the commutativity of Diagram 4.2, we can derive that $\tilde{Y}_{\mathcal{C}} \simeq ri\tilde{Y}_{\mathcal{C}} \simeq rY_{\mathcal{C}}$ and, moreover, r preserving all existing colimits, we get the functor $F \simeq ri(F)$ is isomorphic to:

$$r\left(\operatorname*{colim}_{(A,\alpha)\in Y_{\mathcal{C}}\downarrow i(F)} Y_{\mathcal{C}}(A)\right) \simeq \operatorname*{colim}_{(A,\alpha)\in Y_{\mathcal{C}}\downarrow i(F)} rY_{\mathcal{C}}(A) \simeq \operatorname*{colim}_{(A,\alpha)\in Y_{\mathcal{C}}\downarrow i(F)} \tilde{Y}_{\mathcal{C}}(A)$$

Clearly, the adjunction $r \dashv i$ gives:

$$\mathbf{Nat}(Y_{\mathcal{C}}(A), i(F)) \simeq \mathbf{Nat}(rY_{\mathcal{C}}(A), F) \simeq \mathbf{Nat}(\tilde{Y}_{\mathcal{C}}(A), F)$$

for any object $A \in \mathcal{C}$, which, in particular, implies that the two slice categories $\tilde{Y}_{\mathcal{C}} \downarrow F$ and $Y_{\mathcal{C}} \downarrow i(F)$ coincide. Therefore, we obtain $F \simeq \operatorname{colim}_{(A,\alpha)\in\tilde{Y}_{\mathcal{C}}\downarrow F} \tilde{Y}_{\mathcal{C}}(A)$, as desired. ■

With abuse of notation, we will denote the factorisation $\tilde{Y}_{\mathcal{C}}$ also by

$$Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$$

and call it the (restricted) Yoneda embedding. Moreover:

Lemma 144 *Let $\mathcal{C}$ be an algebraic theory. The restricted Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$ does preserve finite coproducts.*

Proof. This result can be seen as a special instance of Proposition 32 in Chapter 1, when we choose as $\mathbb{D}$ the doctrine of finite products and as $\mathcal{A}$ the dual category $\mathcal{C}^{op}$. ■

For a complete treatment of the restricted Yoneda embedding see [11].

We present some facts concerning copowers (for the definition of a set-indexed copower see Section 2.3):

Lemma 145 *Let $\mathcal{E}$ be a category with coproducts. For any object $X \in \mathcal{E}$, the representable functor $\mathcal{E}(X, -) : \mathcal{E} \longrightarrow \mathbf{set}$ has a left adjoint, which is the* copower functor $- \circ X : \mathbf{set} \longrightarrow \mathcal{E}$.

Proof. It is easy to check there is an isomorphism $\mathcal{E}(A \circ X, Y) \simeq \mathbf{set}(A, \mathcal{E}(X, Y))$, natural in both the variables $A \in \mathbf{set}$ and $Y \in \mathcal{E}$, given by pre-composing with the coproduct injections $i_a : X \longrightarrow \coprod_{a\in A} X = A \circ X$. ■

As a corollary of this result, in any category $\mathcal{E}$ with coproducts, taking an object $X \in \mathcal{E}$, we obtain, for any set A which can be obviously regarded as a (filtered) colimit of its finite subsets $S \subseteq A$, that the copower $A \circ X$ can be thought of as a filtered colimit, as follows:

$$A \circ X = (\operatorname*{colim}_{S\subseteq A} S) \circ X \simeq \operatorname*{colim}_{S\subseteq A}(S \circ X)$$

since, $-\circ X$, being left adjoint, preserves all colimits which happen to exist in **set**.

Remark 146 It is a well-known fact that, for any set A, the category $\mathcal{A} = \{S|S \subseteq A \text{ finite}\}$ is filtered, and, hence, connected. This is the reason why, by taking a constant diagram $H : \mathcal{A} = \{S|S \subseteq A \text{ finite}\} \longrightarrow \mathcal{E}$ into a cocomplete category $\mathcal{E}$, which associates with any $S \in \mathcal{A}$ the object $X \in \mathcal{E}$, and considering the colimit over the category $\mathcal{A}$ of the constant functor H, we obtain:

$$\operatorname*{colim}_{S \subseteq A} H(S) = \operatorname*{colim}_{S \subseteq A} X \simeq X.$$

We refer to [33] for what concerns copowers.
As a direct consequence of Proposition 135, let us mention:

Lemma 147 *Let A be a set and $S \subseteq A$ be a finite subset of A. If S is not the empty set, then, the corresponding colimit injection:*

$$j_S : \coprod_{a \in S \subseteq A} \mathcal{C}(-,T) \rightarrowtail \coprod_{a \in A} \mathcal{C}(-,T)$$

is a monomorphism in $\mathbf{Mod}(\mathcal{C})$.

Proof. For any finite non empty subset $S \subseteq A$, the inclusion $i_S : S \rightarrowtail A$ is a monomorphism in **set**, that is, a split monomorphism. Namely, there exists a retraction $r : A \longrightarrow S$ such that $r.i_S = 1_S$. Therefore, since, by Proposition 135, the free model functor $\mathbb{F} : \mathbf{set} \longrightarrow \mathbf{Mod}(\mathcal{C})$ preserves monomorphisms, the colimit injection

$$j_S \simeq \mathbb{F}(i_S) : \coprod_{a \in S} \mathcal{C}(-,T) = \mathbb{F}(S) \rightarrowtail \coprod_{a \in A} \mathcal{C}(-,T) = \mathbb{F}(A)$$

is a monomorphism in $\mathbf{Mod}(\mathcal{C})$. ∎

4.3 Some more conditions our functors might satisfy

Let $\mathcal{C}$ be an algebraic theory and $k : \mathcal{C} \longrightarrow \mathcal{E}$ a finite coproduct-preserving functor into an arbitrary category $\mathcal{E}$ with coproducts. Following the suggestion of Section 3.1, we reformulate Condition $(A)_{reg}$ in terms of such a functor k defined on an algebraic theory:

(A_{reg}) For an object $X \in \mathcal{E}$, consider the collection R_X of all maps $\{h : k(T) \longrightarrow X\}$ in $\mathcal{E}$. The functor k satisfies (A_{reg}) if and only if, for any object $X \in \mathcal{E}$, the family R_X is regular epimorphic.

Namely, the unique arrow λ in $\mathcal{E}$, induced by the universal property of the coproduct, making the following diagram:

$$\begin{tikzcd} \coprod_{h \in R_X} k(T) \arrow[r, two heads, "\lambda"] & X \\ k(T) \arrow[u, "\sigma_h"] \arrow[ur, "h"'] & \end{tikzcd}$$

commute, for any $h \in R_X$, is a regular epimorphism in $\mathcal{E}$. (Here the σ_h denote the canonical coproduct injections in $\mathcal{E}$.)

We introduce new conditions which will allow us to treat the case of localizations of algebraic categories:

($\tilde{\mathrm{B}}$) For a pair of arrows $X \underset{x_1}{\overset{x_0}{\rightrightarrows}} k(nT)$ in $\mathcal{E}$, consider the collection M_{x_0,x_1} of all maps

$$\left\{ g : k(T) \longrightarrow X \;\middle|\; \left\{ \begin{array}{c} x_0.g = k(f_0) \\ x_1.g = k(f_1) \end{array} \right. \text{for some } T \underset{f_1}{\overset{f_0}{\rightrightarrows}} nT \text{ in } \mathcal{C} \right\}$$

in $\mathcal{E}$. The functor k satisfies ($\tilde{\mathrm{B}}$) if and only if, for any such a pair (x_0, x_1), the family M_{x_0,x_1} is epimorphic.

Remark 148 A useful remark we point out, once for all, is that, when considering arrows of $\mathcal{E}$ into an object $X \in \mathcal{E}$, we just need to take under consideration morphisms defined on $k(T)$, and not on an arbitrary $k(nT)$, where nT is an arbitrary finite coproduct of the object T in $\mathcal{C}$. This follows immediately from the universal property of the coproduct, provided $k : \mathcal{C} \longrightarrow \mathcal{E}$ is a coproduct-preserving functor, as displayed:

$$\begin{tikzcd} k(nT) \simeq nk(T) \arrow[r, dotted] & X \\ k(T) \arrow[u, "\sigma_i"] \arrow[ur] & \end{tikzcd}$$

The notion of filtering functor (recall Definition 96) needs to be refined in terms of the domain category $\mathcal{C}$, which is an algebraic theory in the current situation. That is where the following conditions come from. Note that, here, we work under the further assumption on the category $\mathcal{E}$ of finite completeness.

(SF1) Let 1 be the terminal object of the category $\mathcal{E}$ and consider the unique arrow $! : k(T) \longrightarrow 1$ in $\mathcal{E}$. The functor k satisfies (SF1) if and only if the map ! is a regular epimorphism of $\mathcal{E}$.

(SF2) For a pair of objects $nT, mT \in \mathcal{C}$, consider the collection $S_{n,m}$ of all arrows

$$\{< k(u), k(v) >: k(T) \longrightarrow k(nT) \times k(mT) | \, nT \xleftarrow{u} T \xrightarrow{v} mT\}$$

in $\mathcal{E}$. The functor k satisfies (SF2) if and only if, for any such a pair (nT, mT), the family $S_{n,m}$ is regular epimorphic. (Here $< k(u), k(v) >$ is the unique map in $\mathcal{E}$ satisfying $\pi_1. < k(u), k(v) > = k(u)$ and $\pi_2. < k(u), k(v) >= k(v)$, obtained by the universal property of the product $k(nT) \xleftarrow{\pi_1} k(nT) \times k(mT) \xrightarrow{\pi_2} k(mT)$ in $\mathcal{E}$.) In other words, the canonical map λ induced by all pairs of arrows $nT \xleftarrow{u} T \xrightarrow{v} mT$ in $\mathcal{C}$, as in the following commutative diagram:

$$\begin{tikzcd}
\coprod_{u,v} k(T) \arrow[r, two heads, "\lambda"] & k(nT) \times k(mT) \\
k(T) \arrow[u, "\sigma_{u,v}"] \arrow[ur, "<k(u),k(v)>"'] &
\end{tikzcd}$$

is a regular epimorphism in $\mathcal{E}$.

(SF3) For a pair of arrows $nT \overset{u}{\underset{v}{\rightrightarrows}} mT$ in $\mathcal{C}$, consider the collection $S_{u\&v}$ of all arrows

$$\{e_w : k(T) \longrightarrow E_{u,v} | \, T \xrightarrow{w} nT \overset{u}{\underset{v}{\rightrightarrows}} mT \text{ such that } u.w = v.w\}$$

in $\mathcal{E}$. The functor k satisfies (SF3) if and only if, for any such a pair (u, v), the family $S_{u\&v}$ is regular epimorphic. (Here $(E_{u,v}, l)$ is the equalizer $E_{u,v} \overset{l}{\rightarrowtail} k(nT) \overset{k(u)}{\underset{k(v)}{\rightrightarrows}} k(mT)$ of the pair $(k(u), k(v))$ in $\mathcal{E}$ and e_w is the unique map in $\mathcal{E}$ satisfying $l.e_w = k(w)$, for any such an arrow w, obtained by the universal property of the equalizer.) In other words, the canonical map μ induced by all pairs of arrows $w : T \longrightarrow nT$ in $\mathcal{C}$ satisfying $u.w = v.w$, as in the following commutative diagram:

$$\begin{tikzcd} \coprod_{w|u.w=v.w} k(T) \arrow[r, two heads, "\mu"] & E_{u,v} \\ k(T) \arrow[u, "\sigma_w"] \arrow[ur, "e_w"'] & \end{tikzcd}$$

is a regular epimorphism in $\mathcal{E}$.

Remark 149 Obviously, Condition (SF1) is a special instance of Condition (A_{reg}), when we choose as X the terminal object 1 of $\mathcal{E}$.

We might want to give a name to such a functor k, as in the following:

Definition 150 *Let $\mathcal{C}$ be an algebraic theory and $\mathcal{E}$ a finitely complete category with coproducts. A functor $k : \mathcal{C} \longrightarrow \mathcal{E}$ is said to be* special filtering *if k satisfies the above-mentioned Conditions* (SF1), (SF2) *and* (SF3).

As suggested by Robert Paré, while considering for $\mathcal{E} = \mathbf{set}$, the notion of special filtering functor and that of flat functor coincide, as shown in the following:

Proposition 151 *For any algebraic theory $\mathcal{C}$, a functor $k : \mathcal{C} \longrightarrow \mathbf{set}$ is special filtering if and only if it is flat.*

Proof. Recall from Section 1.1 the definition of flat functor. We wish to show that the dual category $\mathbf{Elts}(k)^{op}$ of the category of elements of k is filtered. For it, Condition (SF1) precisely says that the set $k(T)$ is not empty and, therefore, $\mathbf{Elts}(k)$ is not empty, since there always exists a pair $(T, x) \in \mathbf{Elts}(k)$. Given a pair of objects (nT, x) and (mT, y) of $\mathbf{Elts}(k)$, namely $x \in k(nT)$ and $y \in k(mT)$, Condition (SF2) claims that the canonical arrow $\lambda : \coprod_{u,v} k(T) \twoheadrightarrow k(nT) \times k(mT)$, from the coproduct over all spans $nT \xleftarrow{u} T \xrightarrow{v} mT$ in $\mathcal{C}$ into the binary product of $k(nT)$ and $k(mT)$, is a regular epimorphism in $\mathbf{set}$. Therefore, given $(x, y) \in k(nT) \times k(mT)$, there exist arrows $u : T \longrightarrow nT$ and $v : T \longrightarrow mT$ in $\mathcal{C}$ and an element $z \in k(T)$ such that $x = k(u)(z) \in k(nT)$ and $y = k(v)(z) \in k(mT)$. In other words, we get a cospan

$$(nT, x) \xrightarrow{u} (T, z) \xleftarrow{v} (mT, y)$$

in $\mathbf{Elts}(k)^{op}$, as desired.

Finally, consider a pair of arrows $s, t : (nT, x) \longrightarrow (mT, y)$ in $\mathbf{Elts}(k)^{op}$, namely a pair of arrows $s, t : mT \longrightarrow nT$ in $\mathcal{C}$, such that $k(s)(y) = x = k(t)(y)$. Condition (SF3) states that the canonical morphism $\mu : \coprod_{w|s.w=t.w} k(T) \twoheadrightarrow E_{s,t}$, from the coproduct over all maps $w : T \longrightarrow mT$ in $\mathcal{C}$ satisfying $s.w = t.w$, into the equalizer $E_{s,t} \rightarrowtail k(mT)$ in **set** of the pair (k(s),k(t)), is a regular epimorphism in **set**. Therefore, given $y \in k(mT)$, which, in particular, sits inside $E_{s,t}$, there exist an arrow $w : T \longrightarrow mT$, satisfying $s.w = t.w$, and an element $z \in k(T)$, such that $k(w)(z) = y$. In other words, we get a morphism $w : (mT, y) \longrightarrow (T, z)$ in $\mathbf{Elts}(k)^{op}$ with the property $s.w = t.w$, since it is easy to verify that $k(w.s)(z) = x = k(w.t)(z)$. ■

4.4 The coproduct completion and the exact completion in the case of an algebraic theory

Let $\mathcal{C}$ be an algebraic theory and let $\mathcal{F}(\mathcal{C})$ denote the full subcategory of $\mathbf{Mod}(\mathcal{C})$ consisting of free algebras, that is, functors of the kind $\coprod_{x \in X} \mathcal{C}(-, T)$, for some set X. Consider the embedding $\iota_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{F}(\mathcal{C})$, defined via the assignment $nT \mapsto n\mathcal{C}(-, T)$, for any natural number n. In particular, we have $\iota_{\mathcal{C}}(T) = \mathcal{C}(-, T)$. Note that, by Proposition 137, we clearly have $\iota_{\mathcal{C}}(nT) = n\mathcal{C}(-, T) \simeq \mathcal{C}(-, nT)$.

Proposition 152 *For any algebraic theory $\mathcal{C}$, $\iota_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{F}(\mathcal{C})$ is a free coproduct completion of $\mathcal{C}$, conservative with respect to finite coproducts.*

Proof. First of all, the category $\mathcal{F}(\mathcal{C})$ clearly has coproducts and the functor $\iota_{\mathcal{C}}$ preserves finite coproducts, since one can easily show that $\iota_{\mathcal{C}}(nT+mT) \simeq \iota_{\mathcal{C}}((n+m)T) = (n+m)\mathcal{C}(-, T) \simeq n\mathcal{C}(-, T)+m\mathcal{C}(-, T) = \iota_{\mathcal{C}}(nT) + \iota_{\mathcal{C}}(mT)$. Concerning the universal property of the coproduct completion, let $k : \mathcal{C} \longrightarrow \mathcal{E}$ be a finite coproduct-preserving functor into a category $\mathcal{E}$ with coproducts, and define a functor $k'' : \mathcal{F}(\mathcal{C}) \longrightarrow \mathcal{E}$, at any object $\coprod_{x \in X} \mathcal{C}(-, T)$ of $\mathcal{F}(\mathcal{C})$, as $k''(\coprod_{x \in X} \mathcal{C}(-, T)) = \coprod_{x \in X} k(T)$. In particular, we have $k''(\mathcal{C}(-, T)) = k(T)$. To define the functor k'' at morphisms $f : \coprod_{x \in X} \mathcal{C}(-, T) \longrightarrow \coprod_{y \in Y} \mathcal{C}(-, T)$ of $\mathcal{F}(\mathcal{C})$, consider the composite $f.\tau_x : \mathcal{C}(-, T) \longrightarrow \coprod_{y \in Y} \mathcal{C}(-, T)$, for any $x \in X$, where the morphisms $\tau_x : \mathcal{C}(-, T) \longrightarrow \coprod_{x \in X} \mathcal{C}(-, T)$ denote the canonical coproduct injections in $\mathcal{F}(\mathcal{C})$. For any such a composite $f.\tau_x$, since, by Lemma 136, $\coprod_{y \in Y} \mathcal{C}(-, T)$ is a filtered colimit of all coproducts $\coprod_{y \in S \subseteq Y} \mathcal{C}(-, T)$, over all finite subsets S of Y, and $\mathcal{C}(-, T)$ is finitely presentable in $\mathbf{Mod}(\mathcal{C})$, there exists a morphism $\tilde{f}_x : \mathcal{C}(-, T) \longrightarrow \coprod_{y \in S} \mathcal{C}(-, T)$ in $\mathcal{F}(\mathcal{C})$, for some finite subset S of Y, satisfying $j_S.\tilde{f}_x =$

$f.\tau_x$, as displayed:

$$\begin{tikzcd}
\mathcal{C}(-,T) \arrow[r, "\tau_x"] \arrow[drr, dotted, "\tilde{f}_x"'] & \coprod_{x\in X} \mathcal{C}(-,T) \arrow[r, "f"] & \coprod_{y\in Y} \mathcal{C}(-,T) \simeq \operatorname*{colim}_{S\subseteq Y} \coprod_{y\in S} \mathcal{C}(-,T) \\
& & \coprod_{y\in S} \mathcal{C}(-,T) \simeq \mathcal{C}(-,mT) \arrow[u, "j_S"']
\end{tikzcd}$$

where m denotes the cardinality of the finite subset $S \subseteq Y$. (Note that we made use here of Proposition 137.) By the Yoneda Lemma, such a morphism $\tilde{f}_x$ corresponds to a unique element $f_x \in \mathcal{C}(T, mT)$, such that $\mathcal{C}(-, f_x) = \tilde{f}_x$. Then, taking the unique map α in $\mathcal{E}$ induced by all images $k(f_x)$, for x varying in the set X, as in the diagram:

$$\begin{tikzcd}
\coprod_{x\in X} k(T) = k''(\coprod_{x\in X} \mathcal{C}(-,T)) \arrow[r, dotted, "\alpha"] & \coprod_{y\in Y} k(T) = k''(\coprod_{y\in Y} \mathcal{C}(-,T)) \\
k(T) = k''(\mathcal{C}(-,T)) \arrow[u, "\sigma_x \simeq k''(\tau_x)"] \arrow[r, "k(f_x)"'] & k(mT) = k''(\mathcal{C}(-,mT)) \arrow[u, "k''(j_S)"']
\end{tikzcd}$$

gives the value of the functor k'' at f. It only remains to verify that k'' is well-defined on arrows. For it, note that if there exists another morphism $\tilde{g}_x : \mathcal{C}(-,T) \longrightarrow \coprod_{y\in S'} \mathcal{C}(-,T)$ in $\mathcal{F}(\mathcal{C})$, for another finite subset S' of Y, satisfying $j_{S'}.\tilde{g}_x = f.\tau_x$, then by considering the union of the finite sets S and S', we obtain the following commutative square:

$$\begin{tikzcd}
\mathcal{C}(-,T) \arrow[r, "\tilde{f}_x"] \arrow[d, "\tilde{g}_x"'] & \coprod_{y\in S} \mathcal{C}(-,T) \arrow[d, tail, "\rho_S"] & \\
\coprod_{y\in S'} \mathcal{C}(-,T) \arrow[r, tail, "\rho_{S'}"'] & \coprod_{y\in S\cup S'} \mathcal{C}(-,T) \arrow[r, tail, "j_{S\cup S'}"] & \coprod_{y\in Y} \mathcal{C}(-,T)
\end{tikzcd}$$

since composing with the arrow $j_{S\cup S'}$, which is a monomorphism in $\mathcal{F}(\mathcal{C})$ by Lemma 147, gives $j_{S\cup S'}.\rho_S.\tilde{f}_x = j_S.\tilde{f}_x = f.\tau_x = j_{S'}.\tilde{g}_x = j_{S\cup S'}.\rho_{S'}.\tilde{g}_x$. The commutativity of the above square is precisely the condition on the functor k'' of being well-defined. Finally, to check that the functor k'' does preserve coproducts, we just need to look at how these are constructed in $\mathcal{F}(\mathcal{C})$. For any small discrete category $\mathcal{I}$ and for any functor $H : \mathcal{I} \longrightarrow \mathcal{F}(\mathcal{C})$, which assigns to any object $I \in \mathcal{I}$ the free algebra $\coprod_{x\in X_I}\mathcal{C}(-,T)$, by taking the colimit of the diagram H, we get:

$$\coprod_{I\in\mathcal{I}}\coprod_{x\in X_I}\mathcal{C}(-,T)\simeq\coprod_{x\in\dot\cup X_I}\mathcal{C}(-,T)$$

where $\dot\cup_{I\in\mathcal{I}}X_I$ denotes the disjoint union of all sets X_I in **set**. The preservation of these coproducts by the functor k'' is, then, straightforward. Moreover, the functor k'' makes the triangle:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\iota_{\mathcal{C}}"] \arrow[dr, "k"'] & & \mathcal{F}(\mathcal{C}) \arrow[dl, "k''"] \\
& \mathcal{E} &
\end{tikzcd}$$

commute (up to isomorphism). For it, for any natural number n, we have $k''.\iota_{\mathcal{C}}(nT) = k''(n\mathcal{C}(-,T)) = k''(\coprod_n\mathcal{C}(-,T)) = \coprod_n k(T) = nk(T) = k(nT)$, since the functor k preserves finite coproducts. In fact, the functor k'' is essentially unique with the property $k''.\iota_{\mathcal{C}} = k$, since, by definition, k'' is entirely determined by its value on the representable $\mathcal{C}(-,T)$, via its property of preserving coproducts. ∎

Remark 153 For any algebraic theory $\mathcal{C}$ and $k:\mathcal{C}\longrightarrow\mathcal{E}$ a finite coproduct-preserving functor into a category $\mathcal{E}$ with coproducts, its coproduct extension $k'':\mathcal{F}(\mathcal{C})\longrightarrow\mathcal{E}$ needs to be defined only on the representable $\mathcal{C}(-,T)$. In fact, the definition of k'' on an arbitrary object $\coprod_{x\in X}\mathcal{C}(-,T)$ follows directly from the property of preserving coproducts and its value at $\mathcal{C}(-,T)$.

We underline, first, a useful feature characterising the injections relative to these coproducts in $\mathcal{E}$, via the following:

Lemma 154 *Let $\mathcal{C}$ be an algebraic theory and $k:\mathcal{C}\longrightarrow\mathcal{E}$ a finite coproduct-preserving functor into a category $\mathcal{E}$ with coproducts. Consider a set A and a finite subset of it $S\subseteq A$. If S is not the empty set, then, the corresponding colimit injection:*

$$\tilde{j}_S:\coprod_{a\in S\subseteq A}k(T)\rightarrowtail\coprod_{a\in A}k(T)$$

is a monomorphism in $\mathcal{E}$.

Proof. As pointed out in Lemma 147, for any finite subset $S\subseteq A$, the colimit injections in $\mathcal{F}(\mathcal{C})$ are $j_S\simeq\mathbb{F}(i_S):\coprod_{a\in S}\mathcal{C}(-,T)=\mathbb{F}(S)\longrightarrow$

$\coprod_{a\in A}\mathcal{C}(-,T)=\mathbb{F}(A)$. Applying the functor k'', which preserves coproducts, to these arrows gives in $\mathcal{E}$:

$$k''(\mathbb{F}(i_S))\simeq\tilde{j}_S : k''(\mathbb{F}(S))=\coprod_{a\in S}k(T)\longrightarrow k''(\mathbb{F}(A))=\coprod_{a\in A}k(T)$$

Since we are assuming that the subset S is not-empty, the inclusion $i_S : S\longrightarrow A$ is a split monomorphism in **set**, that is, there exists a retraction $r : A\longrightarrow S$ satisfying $r.i_S=1_S$. Therefore, by applying the composite functor $k''.\mathbb{F}$ to this identity, we can deduce that $k''(\mathbb{F}(r)).k''(\mathbb{F}(i_S))=1_{k''(\mathbb{F}(S))}$ and, further, that $k''(\mathbb{F}(i_S))\simeq\tilde{j}_S$ is a monomorphism in $\mathcal{E}$, as desired. ∎

Lemma 155 *Let $\mathcal{C}$ be an algebraic theory and consider its coproduct completion $\iota_{\mathcal{C}} : \mathcal{C}\longrightarrow\mathcal{F}(\mathcal{C})$ conservative with respect to finite coproducts. For any finite coproduct-preserving functor $k : \mathcal{C}\longrightarrow\mathcal{E}$ into a category $\mathcal{E}$ with coproducts, the coproduct extension $k'' : \mathcal{F}(\mathcal{C})\longrightarrow\mathcal{E}$ of k coincides with the left Kan extension of k along the embedding $\iota_{\mathcal{C}}$, that is, $k''\simeq$* $\mathbf{Lan}_{\iota_{\mathcal{C}}}(k)$.

Proof. By definition of the coproduct completion, $k'' : \mathcal{F}(\mathcal{C})\longrightarrow\mathcal{E}$ is the essentially unique coproduct-preserving functor satisfying $k''.\iota_{\mathcal{C}}\simeq k$, as shown:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\iota_{\mathcal{C}}"] \arrow[dr, "k"'] & & \mathcal{F}(\mathcal{C}) \arrow[dl, "k''"'] \arrow[dl, dotted, shift left=2, "G"] \\
& \mathcal{E} &
\end{tikzcd}$$

Let $G : \mathcal{F}(\mathcal{C})\longrightarrow\mathcal{E}$ be a functor and $\gamma : k\longrightarrow G.\iota_{\mathcal{C}}$ be a natural transformation, whose component at T, is, in particular, a morphism $\gamma_T : k(T)\longrightarrow G(\mathcal{C}(-,T))$ of $\mathcal{E}$. We want to show there exists a unique natural transformation $\alpha : k''\longrightarrow G$, such that $\alpha.\iota_{\mathcal{C}}\simeq\gamma$. Define α, at the arbitrary object $\coprod_{x\in X}\mathcal{C}(-,T)$ of $\mathcal{F}(\mathcal{C})$, as the unique map $\alpha_{\coprod_{x\in X}\mathcal{C}(-,T)}$ in $\mathcal{E}$ induced by all composites $G(\tau_x).\gamma_T$, for x varying in the set X, as in the diagram:

$$\begin{tikzcd}
\coprod_{x\in X}k(T) \arrow[rr, dotted, "\alpha_{\coprod_{x\in X}\mathcal{C}(-,T)}"] & & G(\coprod_{x\in X}\mathcal{C}(-,T)) \\
k(T) \arrow[u, "\sigma_x"] \arrow[urr, "G(\tau_x).\gamma_T"'] & &
\end{tikzcd}$$

where the morphisms $\tau_x : \mathcal{C}(-,T)\longrightarrow\coprod_{x\in X}\mathcal{C}(-,T)$ denote the canonical coproduct injections in $\mathcal{F}(\mathcal{C})$ and the morphisms $\sigma_x\simeq k''(\tau_x)$:

$k(T) \longrightarrow \coprod_{x\in X} k(T)$ those in $\mathcal{E}$. Clearly, by construction, we have that $\alpha.\iota_{\mathcal{C}} \simeq \gamma$, since the coproduct reduces to one single term. So, the non-trivial part of the proof is to show that the transformation α, just defined, is natural. For it, let $f : \coprod_{x\in X} \mathcal{C}(-,T) \longrightarrow \coprod_{y\in Y} \mathcal{C}(-,T)$ be an arrow in $\mathcal{F}(\mathcal{C})$. For any S finite subset of Y, note that, if m denotes its cardinality, then, the isomorphisms $\coprod_{y\in S\subseteq Y} k(T) \simeq k(mT)$ and $\coprod_{y\in S\subseteq Y} \mathcal{C}(-,T) \simeq \mathcal{C}(-,mT)$ are established, the functors k and $\iota_{\mathcal{C}}$ preserving finite coproducts, respectively. Consider the following diagram in $\mathcal{E}$:

$$\begin{tikzcd}
k(T) \arrow[r, "\gamma_T"] \arrow[d, "\sigma_x"'] \arrow[dr, phantom, "(1)"] & G(\mathcal{C}(-,T)) \arrow[d, "G(\tau_x)"] \\
\coprod\limits_{x\in X} k(T) \arrow[r, "\alpha_{\coprod_{x\in X}\mathcal{C}(-,T)}"] \arrow[d, "k''(f)"'] \arrow[dr, phantom, "(2)"] & G(\coprod\limits_{x\in X}\mathcal{C}(-,T)) \arrow[d, "G(f)"] \\
\coprod\limits_{y\in Y} k(T) \arrow[r, "\alpha_{\coprod_{y\in Y}\mathcal{C}(-,T)}"] \arrow[dr, phantom, "(3)"] & G(\coprod\limits_{y\in Y}\mathcal{C}(-,T)) \\
k(mT) \arrow[r, "\alpha_{\coprod_{y\in S}\mathcal{C}(-,T)}"'] \arrow[u, "k''(j_S)\simeq\tilde{j}_S"] & G(\mathcal{C}(-,mT)) \arrow[u, "G(j_S)"']
\end{tikzcd}$$

where, for any finite subset $S \subseteq Y$, $j_S : \coprod_{y\in S} \mathcal{C}(-,T) \longrightarrow \coprod_{y\in Y} \mathcal{C}(-,T)$ in $\mathcal{F}(\mathcal{C})$ and $k''(j_S) \simeq \tilde{j}_S : \coprod_{y\in S} k(T) \longrightarrow \coprod_{y\in Y} k(T)$ in $\mathcal{E}$ still denote the corresponding colimit injections (note that the functor k'' preserves coproducts). Commutativity of square (1) is just the definition of the transformation α. In fact, to prove naturality of α is to prove the commutativity of square (2), that is, by definition of the coproduct, $G(f).\alpha_{\coprod_{x\in X}\mathcal{C}(-,T)}.\sigma_x = \alpha_{\coprod_{y\in Y}\mathcal{C}(-,T)}.k''(f).\sigma_x$, for any element $x \in X$. Consider, then, the composite $f.\tau_x : \mathcal{C}(-,T) \longrightarrow \coprod_{y\in Y} \mathcal{C}(-,T)$ in $\mathcal{F}(\mathcal{C})$, for any $x \in X$. Since, by Lemma 136, $\coprod_{y\in Y} \mathcal{C}(-,T)$ is a filtered colimit of all coproducts $\coprod_{y\in S\subseteq Y} \mathcal{C}(-,T)$ over all finite subsets S of Y, and $\mathcal{C}(-,T)$ is finitely presentable in $\mathbf{Mod}(\mathcal{C})$, there exists a morphism $\tilde{f}_x : \mathcal{C}(-,T) \longrightarrow \coprod_{y\in S} \mathcal{C}(-,T)$ in $\mathcal{F}(\mathcal{C})$, for some finite subset S of Y, satisfying $j_S.\tilde{f}_x = f.\tau_x$, as displayed:

$$
\begin{tikzcd}
\mathcal{C}(-,T) \arrow[r, "\tau_x"] \arrow[drr, dotted, "\tilde{f}_x"'] & \coprod_{x\in X}\mathcal{C}(-,T) \arrow[r, "f"] & \coprod_{y\in Y}\mathcal{C}(-,T) \simeq \operatorname*{colim}_{S\subseteq Y}\coprod_{y\in S}\mathcal{C}(-,T) \\
& & \coprod_{y\in S}\mathcal{C}(-,T) \simeq \mathcal{C}(-,mT) \arrow[u, "j_S"']
\end{tikzcd}
$$

Diagram 4.4

where m denotes the cardinality of the finite subset $S \subseteq Y$. By the Yoneda Lemma, such a morphism $\tilde{f}_x$ corresponds to a unique element $f_x \in \mathcal{C}(T, mT)$, such that $\mathcal{C}(-, f_x) = \tilde{f}_x$. So, using the naturality of the transformation γ relative to the arrow $f_x : T \longrightarrow mT$ in $\mathcal{C}$, we obtain $G(\tilde{f}_x).\gamma_T = G(\mathcal{C}(-, f_x)).\gamma_T = \gamma_{mT}.k(f_x)$ and, therefore, by definition of $\tilde{f}_x$ and commutativity of square (1), we have $G(f).\alpha_{\coprod_{x\in X}\mathcal{C}(-,T)}.\sigma_x = G(f).G(\tau_x).\gamma_T = G(f.\tau_x).\gamma_T = G(j_S.\tilde{f}_x).\gamma_T = G(j_S).G(\tilde{f}_x).\gamma_T = G(j_S).\gamma_{mT}.k(f_x)$. On the other hand, since $\alpha.\iota_{\mathcal{C}} \simeq \gamma$ holds and γ is a natural transformation, we already know that $\alpha_{\coprod_{y\in S}\mathcal{C}(-,T)} \simeq \gamma_{mT}$. Then, applying the coproduct-preserving functor k'' to the Diagram 4.4, gives $k''(f).\sigma_x \simeq k''(f).k''(\tau_x) = k''(j_S).k''(\tilde{f}_x) \simeq \tilde{j}_S.k(f_x)$. Therefore, we have obtained $\alpha_{\coprod_{y\in Y}\mathcal{C}(-,T)}.k''(f).\sigma_x \simeq \alpha_{\coprod_{y\in Y}\mathcal{C}(-,T)}.\tilde{j}_S.k(f_x)$. It only remains to prove, then, that square (3) commutes in order to have the desired inner square (2) being commutative. For it, the composites $\alpha_{\coprod_{y\in Y}\mathcal{C}(-,T)}.k''(j_S)$ and $G(j_S).\gamma_{mT}$ coincide if and only if their pre-compositions with the coproduct injections $k(\delta_y) : k(T) \longrightarrow \coprod_{y\in S\subseteq Y} k(T) \simeq k(mT)$ coincide, where the morphisms $\delta_y : T \longrightarrow mT$ are themselves coproduct injections in $\mathcal{C}$, the functor k being finite coproduct-preserving. So, using that the functors k and k'' preserve finite coproducts and coproducts, respectively, the definition of the transformation α and the naturality of the transformation γ relative to the arrow $\delta_y : T \longrightarrow mT$ in $\mathcal{C}$, we can derive $\alpha_{\coprod_{y\in Y}\mathcal{C}(-,T)}.k''(j_S).k(\delta_y) \simeq \alpha_{\coprod_{y\in Y}\mathcal{C}(-,T)}.\rho_y \simeq G(\nu_y).\gamma_T \simeq G(j_S).G(\iota_{\mathcal{C}}(\delta_y)).\gamma_T \simeq G(j_S).\gamma_{mT}.k(\delta_y)$; here we denote by $\rho_y : k(T) \longrightarrow \coprod_{y\in Y} k(T)$ and $\nu_y : \mathcal{C}(-,T) \longrightarrow \coprod_{y\in Y}\mathcal{C}(-,T)$ the corresponding canonical coproduct injections in $\mathcal{E}$ and in $\mathcal{F}(\mathcal{C})$. This is precisely the commutativity of square (3), from which it follows that of square (2), that is, the naturality of α. Finally, α is the unique natural transformation with the property $\alpha.\iota_{\mathcal{C}} \simeq \gamma$. For it, suppose there exists another one; call it $\beta : k'' \longrightarrow G$. By the simple fact that α and β agree when pre-composed with $\iota_{\mathcal{C}}$, it follows they coincide, since α is entirely determined by its naturality, once we know its component at T (and so is β), as shown:

$$\begin{tikzcd}
\coprod_{x\in X} k(T) = k''(\coprod_{x\in X} \mathcal{C}(-,T)) \arrow[r, "\alpha_{\coprod_{x\in X}\mathcal{C}(-,T)}"] & G(\coprod_{x\in X} \mathcal{C}(-,T)) \\
k(T) = k''(\mathcal{C}(-,T)) \arrow[u, "\sigma_x \simeq k''(\tau_x)"] \arrow[r, "\alpha_{\mathcal{C}(-,T)}"'] & G(\mathcal{C}(-,T)) \arrow[u, "G(\tau_x)"']
\end{tikzcd}$$

This ends the proof of the coproduct extension k'' being the left Kan extension of k along $\iota_{\mathcal{C}}$. ■

On the other hand, we write $\zeta_{\mathcal{F}(\mathcal{C})} : \mathcal{F}(\mathcal{C}) \longrightarrow (\mathcal{F}(\mathcal{C}))_{ex}$ for the exact completion of the full subcategory $\mathcal{F}(\mathcal{C})$ of $\mathbf{Mod}(\mathcal{C})$ given by the free algebras. As recalled in the previous chapter (see Section 3.1), for any left covering functor $k'' : \mathcal{F}(\mathcal{C}) \longrightarrow \mathcal{E}$ into an exact category $\mathcal{E}$, the unique (up to isomorphism) exact extension $\hat{k''} : \mathcal{F}(\mathcal{C})_{ex} \longrightarrow \mathcal{E}$ of k'' is precisely the left Kan extension of k'' along the inclusion $\zeta_{\mathcal{F}(\mathcal{C})}$, namely $\hat{k''} \simeq \mathbf{Lan}_{\zeta_{\mathcal{F}(\mathcal{C})}}(k'')$. Moreover, by putting together Proposition 139 and Proposition 140, we can derive that $\mathcal{F}(\mathcal{C})$ is a projective cover of $\mathbf{Mod}(\mathcal{C})$ and, therefore, we deduce that:

- $\mathcal{F}(\mathcal{C})_{ex} \simeq \mathbf{Mod}(\mathcal{C})$ (see Theorem 107);
- $\mathbf{Ex}[\mathcal{F}(\mathcal{C})_{ex}, \mathcal{E}] \simeq \mathbf{Lco}[\mathcal{F}(\mathcal{C}), \mathcal{E}]$, by the universal property of the exact completion.

Hence, we can summarise the current situation via the following diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\iota_{\mathcal{C}}"] \arrow[dr, "k"'] & \mathcal{F}(\mathcal{C}) \arrow[r, "\zeta_{\mathcal{F}(\mathcal{C})}"] \arrow[d, "k''"] & \mathcal{F}(\mathcal{C})_{ex} \simeq \mathbf{Mod}(\mathcal{C}) \arrow[dl, "L"] \\
& \mathcal{E} &
\end{tikzcd}$$

Note, first of all, that the composite $\zeta_{\mathcal{F}(\mathcal{C})}.\iota_{\mathcal{C}}$ is nothing but the restricted Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$. If we assume, moreover, that the category $\mathcal{E}$ is cocomplete and we denote, as usual, by L the left Kan extension of the functor k along the Yoneda embedding $Y_{\mathcal{C}}$, we have $L = \mathbf{Lan}_{Y_{\mathcal{C}}}(k) \simeq \mathbf{Lan}_{\zeta_{\mathcal{F}(\mathcal{C})}.\iota_{\mathcal{C}}}(k) \simeq \mathbf{Lan}_{\zeta_{\mathcal{F}(\mathcal{C})}}(\mathbf{Lan}_{\iota_{\mathcal{C}}}(k)) \simeq \mathbf{Lan}_{\zeta_{\mathcal{F}(\mathcal{C})}}(k'') \simeq \hat{k''}$, using Lemma 155. Obviously, the functor L is a left adjoint to $k' : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$, defined by the assignment $X \mapsto \mathcal{E}(k(-), X)$. Therefore, since the equivalence of categories $\mathbf{Ex}[\mathcal{F}(\mathcal{C})_{ex}, \mathcal{E}] \simeq \mathbf{Lco}[\mathcal{F}(\mathcal{C}), \mathcal{E}]$ holds, we already know that the functor L, which we wish to prove (for later purposes) to be left exact, is exact if and only if the coproduct extension k'' of k is a left covering functor. So, our aim is now to find out conditions on the functor k which reveal its coproduct extension k'' as a left covering functor. We recall that the category $\mathcal{F}(\mathcal{C})$, being a projective cover of category $\mathbf{Mod}(\mathcal{C})$ of models, has weak limits. We wish to describe explicitly some of them in order to enlighten the notion of a left covering functor defined on $\mathcal{F}(\mathcal{C})$.

Proposition 156 *Let $\mathcal{C}$ be an algebraic theory and consider the full subcategory $\mathcal{F}(\mathcal{C})$ of $\mathbf{Mod}(\mathcal{C})$ consisting of the free $\mathcal{C}$-models. Then:*

1. *$\mathcal{C}(-,T)$ is a weak terminal object of $\mathcal{F}(\mathcal{C})$;*
2. $$\coprod_{nT \xleftarrow{u} T \xrightarrow{v} mT} \mathcal{C}(-,T)$$ *is a weak binary product in $\mathcal{F}(\mathcal{C})$ of the objects $\mathcal{C}(-,nT)$ and $\mathcal{C}(-,mT)$, coming from $\mathcal{C}$;*
3. $$\coprod_{T \xrightarrow{w} nT\,,\, u.w=v.w} \mathcal{C}(-,T)$$ *is a weak equalizer in $\mathcal{F}(\mathcal{C})$ of the pair of arrows $\mathcal{C}(-,nT) \underset{\mathcal{C}(-,v)}{\overset{\mathcal{C}(-,u)}{\rightrightarrows}} \mathcal{C}(-,mT)$, coming from $\mathcal{C}$.*

Proof. Condition 1 follows directly from the universal property of the coproduct. For any object $\coprod_{x\in X}\mathcal{C}(-,T)$ of $\mathcal{F}(\mathcal{C})$, there exists a unique arrow $\varphi : \coprod_{x\in X}\mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,T)$ in $\mathcal{F}(\mathcal{C})$ making the triangle:

$$\begin{tikzcd}
\coprod_{x\in X}\mathcal{C}(-,T) \arrow[r, "\varphi"] & \mathcal{C}(-,T) \\
\mathcal{C}(-,T) \arrow[u, "\tau_x"] \arrow[ur, "1_{\mathcal{C}(-,T)}"'] &
\end{tikzcd}$$

commute, for any $x \in X$. The arrow $\varphi : \coprod_{x\in X}\mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,T)$ suffices to claim that $\mathcal{C}(-,T)$ is a weak terminal object in $\mathcal{F}(\mathcal{C})$.
Concerning Condition 2, let $\coprod_{u,v}\mathcal{C}(-,T)$ be the coproduct in $\mathcal{F}(\mathcal{C})$ over all spans $nT \xleftarrow{u} T \xrightarrow{v} mT$ in $\mathcal{C}$. For any such a span, we

can define two unique arrows $\pi_1 : \coprod_{u,v} \mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,nT)$ and $\pi_2 : \coprod_{u,v} \mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,mT)$ satisfying $\pi_1.\tau_{u,v} = \mathcal{C}(-,u)$ and $\pi_2.\tau_{u,v} = \mathcal{C}(-,v)$, respectively, where the maps $\tau_{u,v} : \mathcal{C}(-,T) \longrightarrow \coprod_{u,v} \mathcal{C}(-,T)$ denote the canonical coproduct injections in $\mathcal{F}(\mathcal{C})$. Moreover, for any other object $\coprod_{x\in X} \mathcal{C}(-,T)$ of $\mathcal{F}(\mathcal{C})$ and for any other pair of arrows $f : \coprod_{x\in X} \mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,nT)$ and $g : \coprod_{x\in X} \mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,mT)$ in $\mathcal{F}(\mathcal{C})$, by pre-composing f and g with the coproduct injection $\tau_x : \mathcal{C}(-,T) \longrightarrow \coprod_{x\in X} \mathcal{C}(-,T)$, for any element $x \in X$, we get two morphisms $f.\tau_x : \mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,nT)$ and $g.\tau_x : \mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,mT)$, which correspond, by the Yoneda Lemma, to two elements $f_x \in \mathcal{C}(T,nT)$ and $g_x \in \mathcal{C}(T,mT)$, via $\mathcal{C}(-,f_x) = f.\tau_x$ and $\mathcal{C}(-,g_x) = g.\tau_x$. Therefore, for any $x \in X$, we obtain a span $nT \xleftarrow{f_x} T \xrightarrow{g_x} mT$ in $\mathcal{C}$, which provides a coproduct injection $\tau_{f_x,g_x} : \mathcal{C}(-,T) \longrightarrow \coprod_{u,v} \mathcal{C}(-,T)$ in $\mathcal{F}(\mathcal{C})$. Then, by the universal property of the coproduct, we get a unique map $\varphi : \coprod_{x\in X} \mathcal{C}(-,T) \longrightarrow \coprod_{u,v} \mathcal{C}(-,T)$ in $\mathcal{F}(\mathcal{C})$, such that $\varphi.\tau_x = \tau_{f_x,g_x}$, for any element $x \in X$. Moreover, the morphism φ satisfies $\pi_1.\varphi = f$, since pre-composing with τ_x, for any $x \in X$, gives $\pi_1.\varphi.\tau_x = \pi_1.\tau_{f_x,g_x} = \mathcal{C}(-,f_x) = f.\tau_x$, by definition of the morphisms φ and π_1; analogously, the equality $\pi_2.\varphi = g$ holds. This ends the proof of $\coprod_{u,v} \mathcal{C}(-,T)$ being a weak binary product of $\mathcal{C}(-,nT)$ and $\mathcal{C}(-,mT)$ in $\mathcal{F}(\mathcal{C})$.

Finally, about Condition 3, let $u, v : nT \longrightarrow mT$ be a pair of arrow in $\mathcal{C}$ and consider the coproduct $\coprod_{w|u.w=v.w} \mathcal{C}(-,T)$ over all morphisms $w : T \longrightarrow nT$ in $\mathcal{C}$ satisfying the condition $u.w = v.w$. By its universal property, we can define a unique arrow $e : \coprod_{w|u.w=v.w} \mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,nT)$ in $\mathcal{F}(\mathcal{C})$ making the following triangle:

$$\begin{tikzcd}
\coprod\limits_{w|u.w=v.w} \mathcal{C}(-,T) \arrow[rr, "e"] & & \mathcal{C}(-,nT) \arrow[r, shift left, "\mathcal{C}(-,u)"] \arrow[r, shift right, "\mathcal{C}(-,v)"'] & \mathcal{C}(-,mT) \\
\mathcal{C}(-,T) \arrow[u, "\tau_w"] \arrow[urr, "\mathcal{C}(-,w)"'] & & &
\end{tikzcd}$$

commute, namely the map e is such that $\mathcal{C}(-,w) = e.\tau_w$, for any morphism w verifying $u.w = v.w$. In fact, the morphism e satisfies $\mathcal{C}(-,u).e = \mathcal{C}(-,v).e$, since, for any such an arrow w, by pre-composing both of these composites with τ_w, we obtain $\mathcal{C}(-,u).\mathcal{C}(-,w)$ and $\mathcal{C}(-,v).\mathcal{C}(-,w)$, which coincide because of $u.w = v.w$. Moreover, for any other object $\coprod_{x\in X} \mathcal{C}(-,T)$ of $\mathcal{F}(\mathcal{C})$ and for any other arrow $f : \coprod_{x\in X} \mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,nT)$ of $\mathcal{F}(\mathcal{C})$ satisfying $\mathcal{C}(-,u).f = \mathcal{C}(-,v).f$, the composite $f.\tau_x : \mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,nT)$ of f with the coproduct injection $\tau_x : \mathcal{C}(-,T) \longrightarrow \coprod_{x\in X} \mathcal{C}(-,T)$, for any $x \in X$, corresponds, by the Yoneda Lemma, to an element $f_x \in \mathcal{C}(T, nT)$, via $\mathcal{C}(-, f_x) = f.\tau_x$. Therefore, for any $x \in X$, we have $\mathcal{C}(-, u.f_x) = \mathcal{C}(-, u).\mathcal{C}(-, f_x) = \mathcal{C}(-, u).f.\tau_x = \mathcal{C}(-, v).f.\tau_x = \mathcal{C}(-, v).\mathcal{C}(-, f_x) = \mathcal{C}(-, v.f_x)$, from which we can derive $u.f_x = v.f_x$, the embedding $\iota_\mathcal{C}$ being faithful. The morphism $f_x : T \longrightarrow nT$ provides, then, a coproduct injection $\tau_{f_x} : \mathcal{C}(-, T) \longrightarrow \coprod_{w|u.w=v.w} \mathcal{C}(-, T)$ in $\mathcal{F}(\mathcal{C})$, for any $x \in X$. By the universal property of the coproduct, we get a unique map $\varphi : \coprod_{x \in X} \mathcal{C}(-, T) \longrightarrow \coprod_{w|u.w=v.w} \mathcal{C}(-, T)$ in $\mathcal{F}(\mathcal{C})$, such that $\varphi.\tau_x = \tau_{f_x}$, for any element $x \in X$. Clearly, the morphism φ verifies $e.\varphi = f$, since, by pre-composing both of these composites with the coproduct injection τ_x, for any $x \in X$, we get $e.\varphi.\tau_x = e.\tau_{f_x} = \mathcal{C}(-, f_x) = f.\tau_x$, by definition of the morphisms φ and e. This ends the proof of $\coprod_{w|u.w=v.w} \mathcal{C}(-, T)$ being a weak equalizer of the pair $(\mathcal{C}(-, u), \mathcal{C}(-, v))$ in $\mathcal{F}(\mathcal{C})$. ∎

One last preliminary result concerning regular epimorphisms needs to be mentioned:

Lemma 157 *Let $\mathcal{E}$ be a finitely complete and cocomplete category in which filtered colimits commute with finite limits. Consider a collection $\{\lambda_i : X_i \longrightarrow Y_i | i \in I\}$ of regular epimorphisms in $\mathcal{E}$, where the index i varies in a filtered category I. Then, the canonical map λ in $\mathcal{E}$, induced by the universal property of the colimit, as in the following diagram:*

$$\begin{tikzcd}
\operatorname*{colim}_{i \in I} X_i \arrow[r, dotted, "\lambda"] & \operatorname*{colim}_{i \in I} Y_i \\
X_i \arrow[u, "\rho_i"] \arrow[r, two heads, "\lambda_i"'] & Y_i \arrow[u, "\delta_i"']
\end{tikzcd}$$

is a regular epimorphism in $\mathcal{E}$. Namely,λ is the unique arrow in $\mathcal{E}$ satisfying $\lambda.\rho_i = \delta_i.\lambda_i$, for any $i \in I$. Here the maps ρ_i and δ_i stand for the corresponding colimit injections in $\mathcal{E}$. We usually denote this map λ by $\mathrm{colim}_{i \in I} \lambda_i$.

Proof. For any $i \in I$, the arrow $\lambda_i : X_i \longrightarrow Y_i$ is a regular epimorphism in $\mathcal{E}$ and, therefore, the coequalizer of its kernel pair (m_i, n_i) in $\mathcal{E}$. Take a (filtered) colimit of all kernel pairs $N(\lambda_i)$ of all morphisms λ_i, for i varying in I. Since in the category $\mathcal{E}$ filtered colimits commute with finite limits, we obtain the kernel pair of λ (which exists, the category $\mathcal{E}$ being finitely complete) as the filtered colimits of such kernel pairs, that is:

$$N(\lambda) \simeq \operatorname*{colim}_{i \in I} N(\lambda_i)$$

(Here we made use of Remark 146.) By the universal property of the colimit, we get two canonical arrows $m, n : \mathrm{colim}_{i\in I} N(\lambda_i) \longrightarrow \mathrm{colim}_{i\in I} X_i$, unique with the properties $m.\sigma_i = \rho_i.m_i$ and $n.\sigma_i = \rho_i.n_i$, respectively, for any $i \in I$. More explicitly, we get the following commutative diagram in $\mathcal{E}$:

$$\begin{tikzcd}
\underset{i\in I}{\mathrm{colim}}\, N(\lambda_i) \arrow[r, shift left, "m"] \arrow[r, shift right, "n"'] & \underset{i\in I}{\mathrm{colim}}\, X_i \arrow[r, "\lambda"] & \underset{i\in I}{\mathrm{colim}}\, Y_i \\
N(\lambda_i) \arrow[u, "\sigma_i"] \arrow[r, shift left, "m_i"] \arrow[r, shift right, "n_i"'] & X_i \arrow[u, "\rho_i"'] \arrow[r, two heads, "\lambda_i"'] & Y_i \arrow[u, "\delta_i"']
\end{tikzcd}$$

where the morphisms σ_i denote the canonical colimit injections in $\mathcal{E}$. We wish to show that λ is the coequalizer of the pair (m, n) in $\mathcal{E}$. For it, proving $\lambda.m = \lambda.n$ amounts to proving that $\lambda.m.\sigma_i = \lambda.n.\sigma_i$, for any $i \in I$, which is equivalent to $\lambda.\rho_i.m_i = \lambda.\rho_i.n_i$, for any $i \in I$, by definition of the arrows m and n. This equality is further equivalent to $\delta_i.\lambda_i.m_i = \delta_i.\lambda_i.n_i$, for any $i \in I$, by mere definition of the morphism λ. Finally, the last equality follows immediately by the fact that each λ_i is a coequalizer of its kernel pair (m_i, n_i). Concerning the universal property of the coequalizer, consider any other object $Z \in \mathcal{E}$ and any other morphism $z : \mathrm{colim}_{i\in I} X_i \longrightarrow Z$ in $\mathcal{E}$ satisfying $z.m = z.n$, namely $z.m.\sigma_i = z.n.\sigma_i$, for any $i \in I$. This condition is equivalent to $z.\rho_i.m_i = z.\rho_i.n_i$, for any $i \in I$. Therefore, by the universal property of λ_i being the coequalizer of the pair (m_i, n_i) in $\mathcal{E}$, there exists a unique arrow $\varphi_i : Y_i \longrightarrow Z$ in $\mathcal{E}$ such that $\varphi_i.\lambda_i = z.\rho_i$, for any $i \in I$. These morphisms φ_i determine a unique map $\varphi : \mathrm{colim}_{i\in I} Y_i \longrightarrow Z$ satisfying $\varphi.\delta_i = \varphi_i$, for any $i \in I$, by the universal property of the colimit. Clearly, we have $\varphi.\lambda = z$, just by mere definition of the morphisms φ_i, once pre-composed the previous equality with the colimit injections ρ_i. This ends the proof of λ being the coequalizer in $\mathcal{E}$ of the pair (m, n), which is its kernel pair. Hence, λ is a regular epimorphism in $\mathcal{E}$. ∎

The peculiar choice of the name "special filtering functor" in Definition 150 will make sense in light of the following:

Proposition 158 *Let $\mathcal{C}$ be an algebraic theory and consider its coproduct completion $\iota_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{F}(\mathcal{C})$ conservative with respect to finite coproducts. Let $k : \mathcal{C} \longrightarrow \mathcal{E}$ be a finite coproduct-preserving functor into a finitely complete category $\mathcal{E}$ with coproducts and take its coproduct extension $k'' : \mathcal{F}(\mathcal{C}) \longrightarrow \mathcal{E}$, which is the unique (up to isomorphism)* *functor k'' preserving coproducts and making the triangle:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\iota_{\mathcal{C}}"] \arrow[dr, "k"'] & & \mathcal{F}(\mathcal{C}) \arrow[dl, "k''"] \\ & \mathcal{E} & \end{tikzcd}$$

commute. Then, the functor k is special filtering if and only if k'' is left covering with respect to the weak terminal object, weak binary products of objects coming from $\mathcal{C}$ and weak equalizers of parallel arrows coming from $\mathcal{C}$.

Proof. Let 1 be the terminal object of the category $\mathcal{E}$ and consider $\mathcal{C}(-,T)$, which we have shown, in Proposition 156, is a weak terminal object of $\mathcal{F}(\mathcal{C})$. To say that the functor k'' is left covering with respect to the weak terminal object of $\mathcal{F}(\mathcal{C})$ is to say that the canonical arrow $! : k''(\mathcal{C}(-,T)) = k(T) \longrightarrow 1$ is a regular epimorphism in $\mathcal{E}$: this is precisely Condition (SF1), since, by definition of the coproduct completion, $k''.\iota_{\mathcal{C}} = k$ holds.
Let nT, mT be a pair of objects of $\mathcal{C}$ and consider the coproduct $\coprod_{u,v} \mathcal{C}(-,T)$ over all spans $nT \xleftarrow{u} T \xrightarrow{v} mT$ in $\mathcal{C}$: we proved, in Proposition 156, this is a weak binary product of $\mathcal{C}(-,nT)$ and $\mathcal{C}(-,mT)$ in $\mathcal{F}(\mathcal{C})$. To say that k'' is left covering with respect to weak binary products of objects coming from $\mathcal{C}$ is to say that the induced map $\lambda : k''(\coprod_{u,v} \mathcal{C}(-,T)) \longrightarrow k''(\mathcal{C}(-,nT)) \times k''(\mathcal{C}(-,mT))$, into the product of $k''(\mathcal{C}(-,nT))$ and $k''(\mathcal{C}(-,mT))$ in $\mathcal{E}$, is a regular epimorphism of $\mathcal{E}$; namely, the unique morphism λ making the following triangle:

$$\begin{tikzcd} k''\left(\coprod\limits_{u,v} \mathcal{C}(-,T)\right) = \coprod\limits_{u,v} k(T) \arrow[r, "\lambda"] & k(nT) \times k(mT) \\ k(T) \arrow[u, "\sigma_{u,v}"] \arrow[ur, "<k(u),k(v)>"'] & \end{tikzcd}$$

commute, for any span $nT \xleftarrow{u} T \xrightarrow{v} mT$ in $\mathcal{C}$, is a regular epimorphism. (Here we used that the functor k'' preserves coproducts and is defined on representables by $k''.\iota_{\mathcal{C}} \simeq k$.) The map λ being a regular epimorphism is precisely Condition (SF2).
Let $u, v : nT \longrightarrow mT$ be a pair of arrows in $\mathcal{C}$ and consider the coproduct $\coprod_{w|u.w=v.w} \mathcal{C}(-,T)$ over all arrows $w : T \longrightarrow nT$ in $\mathcal{C}$ such that $u.w = v.w$: we have shown, in Proposition 156, that $e : \coprod_{w|u.w=v.w} \mathcal{C}(-,T) \longrightarrow \mathcal{C}(-,nT) \rightrightarrows \mathcal{C}(-,mT)$ is a weak equalizer of $\mathcal{C}(-,u)$ and $\mathcal{C}(-,v)$ in

$\mathcal{F}(\mathcal{C})$. Let $l : E_{u,v} \rightarrowtail k(nT) \rightrightarrows k(mT)$ be the equalizer of the pair $(k''(\mathcal{C}(-,u)), k''(\mathcal{C}(-,v)))$ in $\mathcal{E}$. To say that the functor k'' is left covering with respect to equalizers of parallel arrows coming from $\mathcal{C}$ is to say that the canonical map $p : k''(\coprod_{w|u.w=v.w} \mathcal{C}(-,T)) \longrightarrow E_{u,v}$, into the equalizer in $\mathcal{E}$ of the pair $(k(u), k(v))$, is a regular epimorphism of $\mathcal{E}$; namely, the unique morphism p in $\mathcal{E}$, satisfying $l.p \simeq k''(e)$, as shown:

$$\begin{tikzcd}
 & E_{u,v} \arrow[rr, tail, "l"] & & k(nT) \arrow[r, shift left, "k(u)"] \arrow[r, shift right, "k(v)"'] & k(mT) \\
k''\left(\coprod_{w|u.w=v.w} \mathcal{C}(-,T)\right) = \coprod_{w|u.w=v.w} k(T) \arrow[ur, "p"] \arrow[urrr, "k''(e)"'] & & & &
\end{tikzcd}$$

obtained via the universal property of the equalizer, is a regular epimorphism in $\mathcal{E}$. (Once again, we used that the functor k'' preserves coproducts and is defined on representables by $k''.\iota_{\mathcal{C}} \simeq k$.) On the other hand, Condition (SF3) states that the canonical map $\mu : \coprod_{w|u.w=v.w} k(T) \longrightarrow E_{u,v}$, induced by all arrows $w : T \longrightarrow nT$ in $\mathcal{C}$ such that $u.w = v.w$, is a regular epimorphism in $\mathcal{E}$. Explicitly, μ is the unique morphism in $\mathcal{E}$ making the following diagram:

$$\begin{tikzcd}
\coprod_{w|u.w=v.w} k(T) \arrow[r, "\mu"] & E_{u,v} \\
k(T) \arrow[u, "\sigma_w"] \arrow[ur, "e_w"'] &
\end{tikzcd}$$

commute, where e_w is defined, for any such an arrow w, by the condition $l.e_w \simeq k(w)$, via the universal property of the equalizer, and σ_w denotes the canonical coproduct injection in $\mathcal{E}$. It suffices to prove, then, that the morphisms p and μ coincide; in other words, that they coincide when pre-composed with the coproduct injection σ_w, for any arrow w. For it, the map l being a monomorphism in $\mathcal{E}$, we just need to show that $l.p.\sigma_w \simeq l.\mu.\sigma_w$. We have $l.p.\sigma_w = k''(e).\sigma_w = k''(e).k''(\tau_w) = k''(e.\tau_w) = k''(\mathcal{C}(-,w)) \simeq k(w) \simeq l.e_w \simeq l.\mu.\sigma_w$, by definition of the morphisms p, e, e_w, μ, respectively, and the fact that the functor k'' preserves coproducts. Therefore, Condition (SF3) is equivalent to the functor k'' being left covering with respect to equalizers of parallel arrows coming from $\mathcal{C}$, as desired. ■

Proposition 159 *Under the conditions of Proposition 158 and assuming, moreover, that the category $\mathcal{E}$ is cocomplete and has filtered colimits* *commuting with finite limits, if the coproduct extension k'' is left covering with respect to weak binary products and weak equalizers of objects and arrows coming from $\mathcal{C}$, then, k'' is left covering with respect to weak binary products and weak equalizers.*

Proof. We underline that, in light of Lemma 136, any object $F = \coprod_{x\in X}\mathcal{C}(-,T)$ of $\mathcal{F}(\mathcal{C})$ can be thought of as a filtered colimit of representable functors of the form $\mathcal{C}(-,mT)$, for some natural number m; more precisely, we have:

$$F = \coprod_{x\in X}\mathcal{C}(-,T) = \operatorname*{colim}_{S\subseteq X}\coprod_{x\in S}\mathcal{C}(-,T) \simeq \operatorname*{colim}_{S\subseteq X}\mathcal{C}(-,mT)$$

where the filtered colimit is taken over all finite subsets S of X and m denotes the cardinality of such a S. Therefore, we proceed by induction on the cardinality m of these finite subsets $S \subseteq X$.

Concerning binary products, we suppose, for any free algebra $F = \coprod_{x\in X}\mathcal{C}(-,T)$ and for any natural number m, the canonical map $\lambda_S : k''(F\times\mathcal{C}(-,mT)) \twoheadrightarrow k''(F)\times k''(\mathcal{C}(-,mT)) = k''(F)\times k(mT)$ is a regular epimorphism in $\mathcal{E}$. We wish to show, then, that the canonical map $\nu : k''(F\times\coprod_{y\in Y}\mathcal{C}(-,T)) \longrightarrow k''(F)\times k''(\coprod_{y\in Y}\mathcal{C}(-,T)) = k''(F)\times\coprod_{y\in Y}k(T)$ is a regular epimorphism in $\mathcal{E}$. The idea is to regard $\coprod_{y\in Y}\mathcal{C}(-,T)$ as the filtered colimit over all finite subsets $S\subseteq Y$, namely $\coprod_{y\in Y}\mathcal{C}(-,T) = \operatorname{colim}_{S\subseteq Y}\coprod_{y\in S}\mathcal{C}(-,T) \simeq \operatorname{colim}_{S\subseteq Y}\mathcal{C}(-,mT)$, where m is the cardinality of S. In $\mathcal{F}(\mathcal{C})$, for S varying over the finite subsets of Y, we get an induced arrow ψ, making the following triangle:

$$\begin{tikzcd}
\operatorname*{colim}_{S\subseteq Y}(F\times\mathcal{C}(-,mT)) \arrow[rr, "\psi"] & & F\times(\operatorname*{colim}_{S\subseteq Y}\mathcal{C}(-,mT)) \\
F\times\mathcal{C}(-,mT) \arrow[u, "\rho_S"] \arrow[urr, "1_F\times j_S"'] & &
\end{tikzcd}$$

commute, where $j_S : \mathcal{C}(-,mT) \longrightarrow \operatorname{colim}_{S\subseteq Y}\mathcal{C}(-,mT)$ denotes the canonical colimit injection in $\mathcal{F}(\mathcal{C})$. On the other hand, we can consider the canonical arrow ϕ in $\mathcal{E}$ induced, via the universal property of the colimit, by all morphisms $k''(\rho_S)$, for S varying over the finite subsets of Y, as follows:

$$\begin{tikzcd}
\operatorname*{colim}_{S\subseteq Y}k''(F\times\mathcal{C}(-,mT)) \arrow[rr, "\phi"] & & k''(\operatorname*{colim}_{S\subseteq Y}(F\times\mathcal{C}(-,mT))) \\
k''(F\times\mathcal{C}(-,mT)) \arrow[u, "\gamma_S"] \arrow[urr, "k''(\rho_S)"'] & &
\end{tikzcd}$$

where the maps γ_S denote the corresponding colimit injections in $\mathcal{E}$. Finally, we take the canonical arrow θ in $\mathcal{E}$, obtained by the universal property of the colimit, making the following diagram:

$$\begin{tikzcd}
\operatorname*{colim}_{S\subseteq Y}(k''(F)\times k(mT)) \arrow[r, "\theta"] & k''(F)\times(\operatorname*{colim}_{S\subseteq Y} k(mT)) \\
k''(F)\times k(mT) \arrow[u, "\delta_S"] \arrow[ur, "1_{k''(F)}\times\tilde{j}_S"'] &
\end{tikzcd}$$

commute, for any S. (Once again, we denote by $\tilde{j}_S \simeq k''(j_S) : k(mT) \longrightarrow \operatorname{colim}_{S\subseteq Y} k(mT)$ the corresponding colimit injection in $\mathcal{E}$.) The arrow θ is, moreover, an isomorphism, since filtered colimits commute with finite limits in $\mathcal{E}$. Explicitly, we have:

$$\begin{aligned}
\operatorname*{colim}_{S\subseteq Y}(k''(F)\times k(mT)) &\simeq \operatorname*{colim}_{S\subseteq Y} k''(F)\times \operatorname*{colim}_{S\subseteq Y} k(mT) && (1)\\
&\simeq k''(F)\times \operatorname*{colim}_{S\subseteq Y} k(mT) && (2)
\end{aligned}$$

(where (1) follows from the fact that filtered colimits commute with binary products in $\mathcal{E}$ and (2) is an instance of Remark 146). Therefore, we get a diagram in $\mathcal{E}$, as follows:

$$\begin{tikzcd}
k''(F\times\coprod_{y\in Y}\mathcal{C}(-,T)) = k''(F\times(\operatorname*{colim}_{S\subseteq Y}\mathcal{C}(-,mT))) \arrow[r, "\nu"] & k''(F)\times\operatorname*{colim}_{S\subseteq Y} k(mT) \\
k''(\operatorname*{colim}_{S\subseteq Y} F\times\mathcal{C}(-,mT)) \arrow[u, "k''(\psi)"] & \\
\operatorname*{colim}_{S\subseteq Y} k''(F\times\mathcal{C}(-,mT)) \arrow[u, "\phi"] \arrow[r, "\operatorname{colim}_{S\subseteq Y}\lambda_S"'] & \operatorname*{colim}_{S\subseteq Y}(k''(F)\times k(mT)) \arrow[uu, "\theta"']
\end{tikzcd}$$

where $\operatorname{colim}_{S\subseteq Y}\lambda_S$ is the canonical map in $\mathcal{E}$ induced by all morphisms λ_S, via the universal property of the colimit. This square is commutative since, by pre-composing with the colimit injections γ_S, for any finite subset $S\subseteq Y$, we get:

$$\begin{aligned}
\nu.k''(\psi).\phi.\gamma_S &\simeq \nu.k''(\psi).k''(\rho_S)\\
&= \nu.k''(\psi.\rho_S)\\
&\simeq \nu.k''(1_F\times j_S)\\
&\simeq (1_{k''(F)}\times\tilde{j}_S).\lambda_S && (1)\\
&\simeq \theta.\delta_S.\lambda_S\\
&\simeq \theta.\operatorname{colim}_{S\subseteq Y}\lambda_S.\gamma_S && (2)
\end{aligned}$$

(where (1) follows by definition of the canonical arrows $1_F \times j_S$ and $1_{k''(F)} \times \tilde{j}_S$ and (2) is just the definition of the morphism $\mathrm{colim}_{S\subseteq Y}\, \lambda_S$). From Lemma 157, since for any finite subset $S \subseteq Y$, the corresponding arrow λ_S is a regular epimorphism in $\mathcal{E}$, it follows that the map $\mathrm{colim}_{S\subseteq Y}\, \lambda_S$ is a regular epimorphism in $\mathcal{E}$. Therefore, by the commutativity of the previous diagram, in which the arrow θ is, in particular, an isomorphism, we get $\nu.k''(\psi).\phi \simeq \mathrm{colim}_{S\subseteq Y}\, \lambda_S$, from which it follows immediately that ν is a regular epimorphism in $\mathcal{E}$, the arrow $\mathrm{colim}_{S\subseteq Y}\, \lambda_S$ being so. This ends the proof of k'' being left covering with respect to weak binary products.
Now, concerning equalizers, let $f, g : \coprod_{x\in X} \mathcal{C}(-,T) = \mathrm{colim}_{S\subseteq X} \coprod_{x\in S} \mathcal{C}(-,T) \simeq \mathrm{colim}_{S\subseteq X}\, \mathcal{C}(-,nT) \rightrightarrows \coprod_{y\in Y} \mathcal{C}(-,T)$ be a pair of morphisms in $\mathcal{F}(\mathcal{C})$, where S varies over the finite subsets of X and n denotes its cardinality. Take any such a finite subset $S \subseteq X$ and consider the two composites $f.j_S, g.j_S : \mathcal{C}(-,nT) \longrightarrow \coprod_{y\in Y} \mathcal{C}(-,T)$, with the colimit injection $j_S : \coprod_{x\in S} \mathcal{C}(-,T) \simeq \mathcal{C}(-,nT) \longrightarrow \coprod_{x\in X} \mathcal{C}(-,T)$: they factor in $\mathcal{F}(\mathcal{C})$ through some finite subsets A and B of Y, the object $\mathcal{C}(-,nT)$ being finitely presentable in $\mathcal{F}(\mathcal{C})$ and $\coprod_{y\in Y} \mathcal{C}(-,T)$ being a filtered colimit of all coproducts $\coprod_{y\in S'} \mathcal{C}(-,T)$ taken over the finite subsets S' of Y. In fact, we can always assume $f.j_S$ and $g.j_S$ factor through the same component $S' \subseteq Y$, by taking as S' the set-theoretic union of A and B, if necessary, which is still a finite subset of Y. Moreover, we suppose that the set Y is infinite, and, therefore, there always exists a non-empty subset $S' \subseteq Y$, through which $f.j_S$ and $g.j_S$ factor. In this way, we have two commutative triangles in $\mathcal{F}(\mathcal{C})$:

$$\begin{tikzcd}
\mathcal{C}(-,nT) \arrow[r, shift left, "f.j_S"] \arrow[r, shift right, "g.j_S"'] \arrow[dr, dotted, shift left, "f_S"] \arrow[dr, dotted, shift right, "g_S"'] & \coprod\limits_{y\in Y} \mathcal{C}(-,T) \\
& \coprod\limits_{y\in S'} \mathcal{C}(-,T) \simeq \mathcal{C}(-,mT) \arrow[u, "j'_{S'}"']
\end{tikzcd}$$

where m is the cardinality of the subset $S' \subseteq Y$ and the two conditions $j'_{S'}.f_S = f.j_S$ and $j'_{S'}.g_S = g.j_S$ hold. Let $e_S : E_S \longrightarrow \mathcal{C}(-,nT) \rightrightarrows \mathcal{C}(-,mT)$ be a weak equalizer in $\mathcal{F}(\mathcal{C})$ of the pair (f_S, g_S) in $\mathcal{F}(\mathcal{C})$, which are two parallel morphisms coming from $\mathcal{C}$. By taking the equalizer $d_S : D_S \rightarrowtail k(nT) \rightrightarrows k(mT)$ in $\mathcal{E}$ of the pair $(k''(f_S), k''(g_S))$, we get a comparison morphism p_S, as shown:

$$\begin{tikzcd} D_S \arrow[r, tail, "d_S"] & k(nT) \arrow[r, shift left, "k''(f_S)"] \arrow[r, shift right, "k''(g_S)"'] & k(mT) \\ k''(E_S) \arrow[u, "p_S"] \arrow[ur, "k''(e_S)"'] & & \end{tikzcd}$$

satisfying $d_S.p_S = k''(e_S)$. By hypothesis, the functor k'' is left covering with respect to parallel arrows coming from $\mathcal{C}$; in other words, p_S is a regular epimorphism in $\mathcal{E}$. Finally, let $e : E \longrightarrow \coprod_{x\in X} \mathcal{C}(-,T) \rightrightarrows \coprod_{y\in Y} \mathcal{C}(-,T)$ be a weak equalizer in $\mathcal{F}(\mathcal{C})$ of the pair (f,g). Since the equality $f_S.e_S = g_S.e_S$ holds, we can derive $f.j_S.e_S = j'_{S'}.f_S.e_S = j'_{S'}.g_S.e_S = g.j_S.e_S$. So, by the universal property of the weak equalizer of (f,g), there exists a morphism $\Sigma_S : E_S \longrightarrow E$ in $\mathcal{F}(\mathcal{C})$, satisfying $e.\Sigma_S = j_S.e_S$. Clearly, this construction can be accomplished for any finite subset $S \subseteq X$. Then, by considering the filtered colimit $\mathrm{colim}_{S\subseteq X} E_S$ of all weak equalizers of the pairs (f_S, g_S), over all finite subsets S of X, we obtain a unique comparison map $\Sigma : \mathrm{colim}_{S\subseteq X} E_S \longrightarrow E$ in $\mathbf{Mod}(\mathcal{C})$, such that $\Sigma.\epsilon_S = \Sigma_S$, for any finite subset $S \subseteq X$. Here by $\epsilon_S : E_S \longrightarrow \mathrm{colim}_{S\subseteq X} E_S$ we denote the canonical colimit injection in $\mathbf{Mod}(\mathcal{C})$. Our aim is to prove that the canonical morphism p in $\mathcal{E}$, induced by the universal property of the equalizer $d : D \rightarrowtail \coprod_{x\in X} k(T) \rightrightarrows \coprod_{y\in Y} k(T)$ of the pair of arrows $(k''(f), k''(g))$ in $\mathcal{E}$, as follows:

$$\begin{tikzcd} D \arrow[r, tail, "d"] & \coprod\limits_{x\in X} k(T) \arrow[r, shift left, "k''(f)"] \arrow[r, shift right, "k''(g)"'] & \coprod\limits_{y\in Y} k(T) \\ k''(E) \arrow[u, "p"] \arrow[ur, "k''(e)"'] & & \end{tikzcd}$$

is a regular epimorphism in $\mathcal{E}$. Namely, p is the unique map in $\mathcal{E}$ satisfying $d.p = k''(e)$. First of all, we point out that, since, by Lemma 154, the colimit injection $k''(j'_{S'}) \simeq \tilde{j'}_{S'} : k(mT) \longrightarrow \coprod_{y\in Y} k(T)$ is a monomorphism in $\mathcal{E}$, the equalizer (D_S, d_S) of the pair $(k''(f_S), k''(g_S))$ is, in fact, the equalizer of the pair $(k''(f.j_S), k''(g.j_S))$ as well, since $k''(f.j_S) = k''(j'_{S'}.f_S) = k''(j'_{S'}).k''(f_S) \simeq \tilde{j'}_{S'}.k''(f_S)$ and, analogously, for $k''(g.j_S)$. It suffices to prove, then, the universal property of the equalizer. For it, consider any other object $Z \in \mathcal{E}$ and any other morphism $z : Z \longrightarrow k(nT)$ in $\mathcal{E}$, satisfying $k''(j'_{S'}).k''(f_S).z = k''(j'_{S'}).k''(g_S).z$; from the fact that $k''(j'_{S'})$ is a monomorphism, it follows immediately that $k''(f_S).z =$

$k''(g_S).z$. Therefore, by the universal property of (D_S, d_S) being the equalizer in $\mathcal{E}$ of $(k''(f_S), k''(g_S))$, there exists a unique arrow $\varphi : Z \longrightarrow D_S$ in $\mathcal{E}$, such that $d_S.\varphi = z$, as desired. As a consequence of (D_S, d_S) being the equalizer of the pair $(k''(f.j_S), k''(g.j_S))$, we have $k''(f).k''(j_S).d_S = k''(g).k''(j_S).d_S$, and, by the universal property of the equalizer (D, d) of $(k''(f), k''(g))$, we get a unique arrow $\theta_S : D_S \longrightarrow D$ in $\mathcal{E}$, verifying $d.\theta_S = k''(j_S).d_S$. In fact, since this fact holds for any finite subset $S \subseteq X$, provided $\rho_S : D_S \longrightarrow \mathrm{colim}_{S \subseteq X} D_S$ denotes the colimit injection in $\mathcal{E}$, there exists a unique map $\theta : \mathrm{colim}_{S \subseteq X} D_S \longrightarrow D$ in $\mathcal{E}$, such that $\theta.\rho_S = \theta_S$, for any such a finite subset S of X. We know that the coproduct $\coprod_{x \in X} k(T)$ is the filtered colimit of all coproducts $\coprod_{x \in S} k(T)$, over all finite subsets S of X, and that in $\mathcal{E}$, by hypothesis, filtered colimits commute with finite limits; therefore, the comparison map $\theta : \mathrm{colim}_{S \subseteq X} D_S \longrightarrow D$ is, in fact, an isomorphism of $\mathcal{E}$, (D_S, d_S) and (D, d) being the equalizers in $\mathcal{E}$ of the pairs $(k''(j'_{S'}).k''(f_S), k''(j'_{S'}).k''(g_S))$ and $(k''(f), k''(g))$, respectively. The following diagram summarises the current situation in $\mathcal{E}$:

$$\begin{tikzcd}
L(\underset{S \subseteq X}{\mathrm{colim}}\, E_S) \arrow[r, "L(\Sigma)"] & k''(E) \arrow[d, "p"] \arrow[dr, "k''(e)"] & & \\
 & D \arrow[r, tail, "d"'] & \coprod_{x \in X} k(T) \arrow[r, shift left, "k''(f)"] \arrow[r, shift right, "k''(g)"'] & \coprod_{y \in Y} k(T) \\
\underset{S \subseteq X}{\mathrm{colim}}\, D_S \arrow[ur, "\theta"] & D_S \arrow[l, "\rho_S"] \arrow[u, "\theta_S"] \arrow[r, tail, "d_S"'] & k(nT) \arrow[u, "k''(j_S)"] \arrow[r, shift left, "k''(f_S)"] \arrow[r, shift right, "k''(g_S)"'] & k(mT) \arrow[u, tail, "k''(j'_{S'})"'] \\
\underset{S \subseteq X}{\mathrm{colim}}\, k''(E_S) \arrow[u, two heads, "\mathrm{colim}_{S \subseteq X} p_S"'] \arrow[uuu, dotted, bend left, "\Omega"] & k''(E_S) \arrow[l, "\delta_S"] \arrow[u, two heads, "p_S"'] \arrow[ur, "k''(e_S)"'] \arrow[uuul, dotted, "L(\epsilon_S)"] & &
\end{tikzcd}$$

where the morphisms $\delta_S : k''(E_S) \longrightarrow \mathrm{colim}_{S \subseteq X} k''(E_S)$ denote the canonical colimit injections in $\mathcal{E}$. Note that the filtered colimit $\mathrm{colim}_{S \subseteq X} E_S$ of all equalizers E_S in $\mathcal{F}(\mathcal{C})$ is an object of the category $\mathbf{Mod}(\mathcal{C})$, which is equivalent to $\mathcal{F}(\mathcal{C})_{ex}$. Therefore, we can only take its image through the functor $L = \mathbf{Lan}_{\zeta_{\mathcal{F}(\mathcal{C})}}(k'') : \mathcal{F}(\mathcal{C})_{ex} \longrightarrow \mathcal{E}$, instead of its image through k''. Recall that, since $\zeta_{\mathcal{F}(\mathcal{C})}$ is fully faithful, the universal arrow for the left Kan extension is an isomorphism, namely $k'' \simeq L.\zeta_{\mathcal{F}(\mathcal{C})}$. The same remark holds for the morphisms ϵ_S and Σ of $\mathbf{Mod}(\mathcal{C})$. Moreover, the arrows $L(\epsilon_S) : k''(E_S) \longrightarrow L(\mathrm{colim}_{S \subseteq X} E_S)$, for any finite subset $S \subseteq X$, induce a unique map $\Omega : \mathrm{colim}_{S \subseteq X} k''(E_S) \longrightarrow L(\mathrm{colim}_{S \subseteq X} E_S)$ in $\mathcal{E}$, by the universal property of the colimit, satisfying $\Omega.\delta_S = L(\epsilon_S)$, for any such a S. In order to prove that p is a regular epimorphism in $\mathcal{E}$, it suffices, then, to show that $p.L(\Sigma).\Omega \simeq \theta.\operatorname{colim}_{S\subseteq X} p_S$, since the last composite is, in particular, equivalent to $\operatorname{colim}_{S\subseteq X} p_S$, θ being an isomorphism, and we know, by Lemma 157, that the fact that p_S is a regular epimorphism in $\mathcal{E}$, for any finite subset S of X, implies the arrow $\operatorname{colim}_{S\subseteq X} p_S$ is a regular epimorphism in $\mathcal{E}$; hence, p is so. Furthermore, we need to prove that $d.p.L(\Sigma).\Omega \simeq d.\theta.\operatorname{colim}_{S\subseteq X} p_S$, the arrow d being a monomorphism, that is, $d.p.L(\Sigma).\Omega.\delta_S \simeq d.\theta.\operatorname{colim}_{S\subseteq X} p_S.\delta_S$, by pre-composing with the colimit injection δ_S, for any finite subset $S \subseteq X$. For it, using commutativity of all squares and triangles involved in the previous diagram in $\mathcal{E}$, we get, for any finite subset $S \subseteq X$:

$$\begin{array}{rcll}
d.p.L(\Sigma).\Omega.\delta_S & \simeq & k''(e).L(\Sigma).\Omega.\delta_S & \\
& \simeq & k''(e).L(\Sigma).L(\epsilon_S) & (1)\\
& \simeq & k''(e).L(\Sigma.\epsilon_S) & \\
& \simeq & k''(e).k''(\Sigma_S) & (2)\\
& \simeq & k''(e.\Sigma_S) & \\
& \simeq & k''(j_S.e_S) & (3)\\
& \simeq & k''(j_S).k''(e_S) & \\
& \simeq & k''(j_S).d_S.p_S & \\
& \simeq & d.\theta_S.p_S & \\
& \simeq & d.\theta.\rho_S.p_S & \\
& \simeq & d.\theta.\operatorname{colim}_{S\subseteq X} p_S.\delta_S &
\end{array}$$

(where (1) follows from the definition of the morphism Ω, (2) follows from the definition of the morphism Σ and from the fact $L.\iota_{\mathcal{F}(\mathcal{C})} \simeq k''$ and (3) follows from the definition of the morphism Σ_S). To end this proof, let us just remark that in the case of a set Y which is finite, the reasoning remains the same, by taking as S' the whole set Y and as $j'_{S'}$ the identity 1_Y on Y. Therefore, there is actually no restriction at all in assuming, as we did, that Y is infinite. We have just shown, then, that k'' is left covering with respect to weak equalizers. ■

Corollary 160 *If, moreover, the category $\mathcal{E}$ is regular and k'' is left covering with respect to the weak terminal object, then k'' is left covering.*

Proof. The proof is the one of Proposition 27 contained in [16]. ■

In analogy to the presheaf case, the result which "justifies" the notion of special filtering functor is stated in the following Proposition: special filtering functors "induce" left exact functors.

Proposition 161 *Let $\mathcal{C}$ be an algebraic theory and $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$ its restricted Yoneda embedding. Let $\mathcal{E}$ be a cocomplete and exact category in which filtered colimits commute with finite limits and $k : \mathcal{C} \longrightarrow \mathcal{E}$*

be a finite coproduct-preserving functor. Consider the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\iota_{\mathcal{C}}"] \arrow[dr, "k"'] & \mathcal{F}(\mathcal{C}) \arrow[r, "\zeta_{\mathcal{F}(\mathcal{C})}"] \arrow[d, "k''"] & \mathcal{F}(\mathcal{C})_{ex} = \mathbf{Mod}(\mathcal{C}) \arrow[dl, "\mathbf{Lan}_{Y_{\mathcal{C}}}(k)"] \\ & \mathcal{E} & \end{tikzcd}$$

The functor k is special filtering if and only if its left Kan extension along Yoneda $L := \mathbf{Lan}_{Y_{\mathcal{C}}}(k)$ is (left) exact.

Proof. The proof proceeds in two steps.

Step 1. We prove that $k : \mathcal{C} \longrightarrow \mathcal{E}$ is special filtering if and only if its coproduct extension $k'' : \mathcal{F}(\mathcal{C}) \longrightarrow \mathcal{E}$ is left covering. (Remark that here it suffices the regularity of the category $\mathcal{E}$.)
By putting together the results in Proposition 158, Proposition 159 and Corollary 160, we can claim that the functor k is special filtering if and only if k'' is left covering with respect to the weak terminal object, weak binary products of objects coming from $\mathcal{C}$ and weak equalizers of parallel arrows coming from $\mathcal{C}$ if and only if k'' is left covering.

Step 2. We prove that the coproduct extension k'' of k is left covering if and only if the functor $\mathbf{Lan}_{Y_{\mathcal{C}}}(k)$ is exact.
The requirement of L being left exact coincides, in fact, with L being an exact functor, since $\mathbf{Lan}_{Y_{\mathcal{C}}}(k)$ has always a right adjoint, given by $\mathcal{E}(k(-), -) : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$. Moreover, the composite $\zeta_{\mathcal{F}(\mathcal{C})} . \iota_{\mathcal{C}}$ coincides with the restricted Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$. So, we get $L = \mathbf{Lan}_{Y_{\mathcal{C}}}(k) \simeq \mathbf{Lan}_{\zeta_{\mathcal{F}(\mathcal{C})} . \iota_{\mathcal{C}}}(k) \simeq \mathbf{Lan}_{\zeta_{\mathcal{F}(\mathcal{C})}}(\mathbf{Lan}_{\iota_{\mathcal{C}}}(k)) \simeq \mathbf{Lan}_{\zeta_{\mathcal{F}(\mathcal{C})}}(k'') \simeq \hat{k''}$, by using Lemma 155; namely the functor L coincides with the exact extension $\hat{k''}$ of the coproduct extension k''. Therefore, by the universal property of the exact completion, we get that the functor L is exact if and only if k'' is left covering. ∎

Let $\mathcal{C}$ be an algebraic theory and $\mathcal{E}$ a cocomplete and exact category in which filtered colimits commute with finite limits. We denote by $\mathbf{GeoMor}[\mathcal{E}, \mathbf{Mod}(\mathcal{C})]$ the category of geometric morphisms between $\mathcal{E}$ and $\mathbf{Mod}(\mathcal{C})$ and geometric transformations between those (see Definition 84). Whereas, we denote by $\mathbf{SFilt}[\mathcal{C}, \mathcal{E}]$ the category of special filtering functors which preserve finite coproducts from $\mathcal{C}$ into $\mathcal{E}$ and all natural transformations between them. The following result enlightens the relation between these two notions, namely geometric morphisms can be classified by special filtering functors:

Corollary 162 *Let $\mathcal{C}$ be an algebraic theory and $\mathcal{E}$ a cocomplete and exact category in which filtered colimits commute with finite limits. The functor*

$$\varepsilon : \mathbf{GeoMor}[\mathcal{E}, \mathbf{Mod}(\mathcal{C})] \longrightarrow \mathbf{SFilt}[\mathcal{C}, \mathcal{E}]$$

defined by pre-composing with the restricted Yoneda embedding $Y_{\mathcal{C}}$: $\mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$, gives an equivalence of categories.

Proof. Let

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift right, "R"'] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathcal{C}) \arrow[l, shift right, "L"'] \end{tikzcd}$$

be a geometric morphism and consider the pre-composition $L.Y_{\mathcal{C}}$ of its left adjoint L with Yoneda $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$. First of all, we need to show that the functor $L.Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{E}$ is special filtering. By Proposition 161, this is equivalent to prove that the left Kan extension $\mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}})$ of the composite $L.Y_{\mathcal{C}}$ along Yoneda is left exact, as in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "Y_{\mathcal{C}}"] \arrow[dr, "L.Y_{\mathcal{C}}"'] & & \mathbf{Mod}(\mathcal{C}) \arrow[dl, "\mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}})"] \\ & \mathcal{E} & \end{tikzcd}$$

Since $Y_{\mathcal{C}}$ is fully faithful, the universal arrow for such a left Kan extension is an isomorphism, namely $L.Y_{\mathcal{C}} \simeq \mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}}).Y_{\mathcal{C}}$. Moreover, the restricted Yoneda embedding $Y_{\mathcal{C}}$ being a dense functor, by Lemma 143, and the functors L and $\mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}})$ preserving all existing colimits, we can derive that $L \simeq \mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}})$ and we know, by hypothesis, that L preserves finite limits. It only remains to show, then, that $L.Y_{\mathcal{C}}$ does preserve finite coproducts: this property follows immediately from the fact that, L, being a left adjoint functor, preserves all existing colimits and the restricted Yoneda embedding $Y_{\mathcal{C}}$ is a finite coproduct-preserving functor, as established in Lemma 144. This ends the proof of $L.Y_{\mathcal{C}}$ being an object of $\mathbf{SFilt}[\mathcal{C}, \mathcal{E}]$. In order to prove that ε is an equivalence of categories, we need to verify it is (a) fully faithful and (b) essentially surjective on objects.
(a) ε is full and faithful. Let $R_1, R_2 : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$ be two geometric morphisms and take their images through the functor ε. Consider a natural transformation $\alpha : L_1.Y_{\mathcal{C}} \longrightarrow L_2.Y_{\mathcal{C}}$ between these images, where L_1 and L_2 denote the left adjoint functors to R_1 and to R_2, respectively. We want to show there exists a unique natural transformation $\beta : L_1 \longrightarrow L_2$ such that $\varepsilon(\alpha) = \beta$, namely $\beta.Y_{\mathcal{C}} = \alpha$. We have just proved that, given

a geometric morphism $R : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$, with left adjoint L, we get $L \simeq \mathbf{Lan}_{Y_\mathcal{C}}(L.Y_\mathcal{C})$. Therefore, concerning the two geometric morphisms R_1 and R_2, we have $L_1 \simeq \mathbf{Lan}_{Y_\mathcal{C}}(L_1.Y_\mathcal{C})$ and $L_2 \simeq \mathbf{Lan}_{Y_\mathcal{C}}(L_2.Y_\mathcal{C})$ and we can derive:

$$\begin{array}{rcll} \mathbf{Nat}(L_1, L_2) & \simeq & \mathbf{Nat}(\mathbf{Lan}_{Y_\mathcal{C}}(L_1.Y_\mathcal{C}), \mathbf{Lan}_{Y_\mathcal{C}}(L_2.Y_\mathcal{C})) & (1) \\ & \simeq & \mathbf{Nat}(L_1.Y_\mathcal{C}, \mathbf{Lan}_{Y_\mathcal{C}}(L_2.Y_\mathcal{C}).Y_\mathcal{C}) & (2) \\ & \simeq & \mathbf{Nat}(L_1.Y_\mathcal{C}, L_2.Y_\mathcal{C}) & (3) \end{array}$$

(where (1) is just the above-mentioned property, (2) is the universal property of the left Kan extension of $L_1.Y_\mathcal{C}$ along $Y_\mathcal{C}$ and (3) follows from the fact that, $Y_\mathcal{C}$ being fully faithful, the universal arrow for the left Kan extension $\mathbf{Lan}_{Y_\mathcal{C}}(L_2.Y_\mathcal{C})$ is an isomorphism). In this way, we have achieved a bijection between the sets of natural transformations: for any $\alpha : L_1.Y_\mathcal{C} \longrightarrow L_2.Y_\mathcal{C}$, there exists a unique $\beta : L_1 \longrightarrow L_2$, satisfying $\beta.Y_\mathcal{C} \simeq \alpha$, as desired.
(b) ε is essentially surjective on objects. Let $k : \mathcal{C} \longrightarrow \mathcal{E}$ be a special filtering functor which preserves finite coproducts. Since the category $\mathcal{E}$ is cocomplete, we can always define a pair of adjoint functors $\mathcal{E}(k(-), -) : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$, which assigns to any object $X \in \mathcal{E}$ the set-valued functor $\mathcal{E}(k(-), X)$, and $\mathbf{Lan}_{Y_\mathcal{C}}(k) : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathcal{E}$, which is given by the left Kan extension of k along the restricted Yoneda embedding $Y_\mathcal{C}$. Note that the functor $\mathcal{E}(k(-), X) : \mathcal{C}^{op} \longrightarrow \mathbf{set}$ does preserve finite products, since $\mathcal{E}(k(nT), X) \simeq \mathcal{E}(nk(T), X) \simeq \mathcal{E}(k(T), X)^n$, k being a finite coproduct-preserving functor. Then, by Proposition 161, the fact that the functor k is special filtering is equivalent to the left Kan extension $\mathbf{Lan}_{Y_\mathcal{C}}(k)$ being left exact. Therefore, we get a geometric morphism

$$\begin{tikzcd} \mathcal{E} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathcal{E}(k(-),-)"'] & \mathbf{Mod}(\mathcal{C}) \arrow[l, bend right, "\mathbf{Lan}_{Y_\mathcal{C}}(k)"'] \end{tikzcd}$$

which, in particular, satisfies the condition $k \simeq \mathbf{Lan}_{Y_\mathcal{C}}(k).Y_\mathcal{C}$, since $Y_\mathcal{C}$ is fully faithful. Namely, $\varepsilon(\mathbf{Lan}_{Y_\mathcal{C}}(k)) \simeq k$ and this ends the proof of the functor ε being essentially surjective on objects. ∎

4.5 The localizations of (one sorted) algebraic categories

Let $k : \mathcal{C} \longrightarrow \mathcal{E}$ be a finite coproduct-preserving functor from an algebraic theory $\mathcal{C}$ into a cocomplete category $\mathcal{E}$. We can always define a new functor $k' : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$ from k, by assigning to any object $X \in \mathcal{E}$ the set-valued functor $\mathcal{E}(k(-), X) : \mathcal{C}^{op} \longrightarrow \mathbf{set}$, which, in fact, preserves finite products, since $\mathcal{E}(k(nT), X) \simeq \mathcal{E}(nk(T), X) \simeq \mathcal{E}(k(T), X)^n$. Analogously, the functor k' associates with any morphism $f : X \longrightarrow Y$ in $\mathcal{E}$

the corresponding natural transformation $\mathcal{E}(k(-), f) : \mathcal{E}(k(-), X) \longrightarrow \mathcal{E}(k(-), Y)$ given by post-composing with f. Moreover, $\mathcal{E}$ being cocomplete, the functor k' has always a left adjoint, defined by taking the left Kan extension of k along the restricted Yoneda embedding $Y_{\mathcal{C}}$, as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Y_{\mathcal{C}}"] \arrow[d, "k"'] & \mathbf{Mod}(\mathcal{C}) \arrow[dl, dotted, "\mathbf{Lan}_{Y_{\mathcal{C}}}(k)"] \\ \mathcal{E} & \end{tikzcd}$$

Therefore, provided $L := \mathbf{Lan}_{Y_{\mathcal{C}}}(k)$, we have $L \dashv k'$.

Our purpose is to give necessary and sufficient conditions on the functor k to realise $\mathcal{E}$ as a localization of the algebraic category $\mathbf{Mod}(\mathcal{C})$. We need a preliminary result concerning subfunctors of the representable functor $\mathcal{C}(-, T)$:

Lemma 163 *Let $\mathcal{C}$ be an algebraic theory, $\mathcal{E}$ a cocomplete and exact category and $k : \mathcal{C} \longrightarrow \mathcal{E}$ a finite coproduct-preserving functor. Suppose the functor $k' : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$, defined on an object $X \in \mathcal{E}$ as $\mathcal{E}(k(-), X)$, is a geometric morphism, whose left exact left adjoint we denote by $L : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathcal{E}$. Let $r : R \rightarrowtail \mathcal{C}(-, T)$ be a subfunctor in $\mathbf{Mod}(\mathcal{C})$. If the image $L(r)$ through L of the monomorphism r is an isomorphism in $\mathcal{E}$, then the subfunctor R evaluated at T is k-epimorphic, that is, $R(T) \subseteq \mathcal{C}(T, T)$ is a k-epimorphic family.*

Proof. The collection of morphisms $f \in R(T) \subseteq \mathcal{C}(T, T)$ induces a canonical map $r' : \coprod_{f \in R(T)} \mathcal{C}(-, T) \longrightarrow \mathcal{C}(-, T)$ in $\mathbf{Mod}(\mathcal{C})$, by the universal property of the coproduct, as follows:

$$\begin{tikzcd} \coprod\limits_{f \in R(T)} \mathcal{C}(-, T) \arrow[r, "r'"] & \mathcal{C}(-, T) \\ \mathcal{C}(-, T) \arrow[u, "\tau_f"] \arrow[ur, "\mathcal{C}(-, f)"'] & \end{tikzcd}$$

namely, r' is the unique map satisfying $r'.\tau_f = \mathcal{C}(-, f)$, for any arrow $f : T \longrightarrow T$ in $R(T)$. (Here we denote by τ_f the canonical coproduct injection in $\mathbf{Mod}(\mathcal{C})$.) We wish to show that the regular epi-mono factorisation of the morphism r' in $\mathbf{Mod}(\mathcal{C})$ is given by:

$$\begin{tikzcd} \coprod\limits_{f \in R(T)} \mathcal{C}(-, T) \arrow[rr, "r'"] \arrow[dr, two heads, "\epsilon_R"'] & & \mathcal{C}(-, T) \\ & R \arrow[ur, tail, "r"'] & \end{tikzcd}$$

where ϵ_R is the component at R of the counit $\epsilon : \mathbb{F}.\mathbb{U} \longrightarrow 1_{\mathbf{Mod}(\mathcal{C})}$ for the adjunction $\mathbb{F} \dashv \mathbb{U}$, where $\mathbb{F}$ denotes the free model functor (see Proposition 134). More explicitly, ϵ_R is defined, at any $\mathcal{C}$-model F, as the canonical map in $\mathbf{Mod}(\mathcal{C})$ induced by all natural transformations $\alpha \in \mathbf{Nat}(\mathcal{C}(-,T), F)$, as displayed:

$$
\begin{tikzcd}
\coprod\limits_{\alpha \in \mathbf{Nat}(\mathcal{C}(-,T),F)} \mathcal{C}(-,T) \arrow[r, two heads, "\epsilon_F"] & F \\
\mathcal{C}(-,T) \arrow[u, "\tau_\alpha"] \arrow[ur, "\alpha"'] &
\end{tikzcd}
$$

By Proposition 140, the map ϵ_R is a regular epimorphism in $\mathbf{Mod}(\mathcal{C})$. It only remains to check, then, that the composite $r.\epsilon_R$ is equal to r', r being a monomorphism. In other words, we need to show that $r.\epsilon_R.\tau_f = r'.\tau_f = \mathcal{C}(-,f)$, for any morphism f in $R(T)$. By the Yoneda Lemma, there exists a bijection $\mathbf{Nat}(\mathcal{C}(-,T), R) \simeq R(T)$, which associates with any element $f \in R(T)$ a natural transformation $\gamma(f) : \mathcal{C}(-,T) \longrightarrow R$, defined at any object $nT \in \mathcal{C}$ as $\gamma(f)_{nT} : \mathcal{C}(nT,T) \longrightarrow R(nT)$, via $h \mapsto R(h)(f)$. Moreover, the definition of ϵ_R gives $\epsilon_R.\tau_{\gamma(f)} = \gamma(f)$, for any such an arrow f in $R(T)$. We assume to make no distinction between the coproduct injections $\tau_{\gamma(f)}$ and τ_f. Therefore, to prove that $r.\epsilon_R.\tau_f = \mathcal{C}(-,f)$ is to prove that $r.\gamma(f) = \mathcal{C}(-,f)$, or, equivalently, for any object $nT \in \mathcal{C}$ and for any morphism $h : nT \longrightarrow T$, that $R(h)(f) = f.h$. This last equality follows directly by the naturality of the inclusion $r : R \rightarrowtail \mathcal{C}(-,T)$ relative to the arrow h. hence, the morphism r' admits $r.\epsilon_R$ as regular epi-mono factorisation in $\mathbf{Mod}(\mathcal{C})$. As already remarked, L is nothing but the left Kan extension of the functor k along the restricted Yoneda embedding $Y_\mathcal{C} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$. This is the reason why, $Y_\mathcal{C}$ being fully faithful, we have the isomorphism $k \simeq L.Y_\mathcal{C}$; in particular, for $T \in \mathcal{C}$, we get $k(T) \simeq L.\mathcal{C}(-,T)$. So, applying the exact functor L to the regular epi-mono factorisation of the morphism r' in $\mathbf{Mod}(\mathcal{C})$ gives the following triangle in $\mathcal{E}$:

$$
\begin{tikzcd}
\coprod\limits_{f \in R(T)} k(T) \arrow[rr, "L(r')"] \arrow[dr, two heads, "L(\epsilon_R)"'] & & k(T) \\
& L(R) \arrow[ur, tail, "L(r)"'] &
\end{tikzcd}
$$

where $L(r)$ is a monomorphism and $L(\epsilon_R)$ is a regular epimorphism. Moreover, $L(r')$ is precisely the canonical map induced by all arrows $k(f)$, for f varying in $R(T)$, since r' is the canonical map induced by the morphisms $\mathcal{C}(-, f)$ and L preserves coproducts. By hypothesis, $L(r)$ is an isomorphism in $\mathcal{E}$, which implies that $L(r') \simeq L(\epsilon_R)$, where $L(\epsilon_R)$ is a regular epimorphism, ϵ_R being so. This means that the arrow $L(r')$, as in the following diagram:

$$\begin{tikzcd} \coprod_{f \in R(T)} k(T) \arrow[rr, "L(r') \simeq L(\epsilon_R)"] & & k(T) \simeq L(R) \\ k(T) \arrow[u, "\sigma_f"] \arrow[urr, "k(f)"'] & & \end{tikzcd}$$

is a regular epimorphism in $\mathcal{E}$. In other words, $R(T)$ is a k-epimorphic family. For it, for any pair of morphisms $u, v : k(T) \longrightarrow U$ in $\mathcal{E}$, satisfying $u.k(f) = v.k(f)$, for any $f \in R(T)$, we have $u.L(r').\sigma_f = v.L(r').\sigma_f$, for any $f \in R(T)$, which implies that $u.L(r') = v.L(r')$, by definition of the coproduct $\coprod_{f \in R(T)} k(T)$. It follows from $L(r')$ being a (regular) epimorphism that $u = v$, as desired. ■

In order to make our presentation of the classifying Theorem for localizations of algebraic categories, easier to read, we prove separately the following preliminary fact:

Lemma 164 *Let $\mathcal{C}$ be an algebraic theory, $\mathcal{E}$ a cocomplete and exact category in which filtered colimits commute with finite limits and $k : \mathcal{C} \longrightarrow \mathcal{E}$ a finite coproduct-preserving functor. Suppose k satisfies Conditions* (A_{reg}) *and* ($\tilde{\mathrm{B}}$)*, then the functor $k' : \mathcal{E} \longrightarrow$ **Mod**$(\mathcal{C})$, defined by the assignment $X \mapsto \mathcal{E}(k(-), X)$, is full and faithful.*

Proof. Let X, Y be two objects of $\mathcal{E}$ and $\alpha : \mathcal{E}(k(-), X) \longrightarrow \mathcal{E}(k(-), Y)$ a morphism in **Mod**$(\mathcal{C})$. Namely, for any natural number n, we have a map $\alpha_{nT} : \mathcal{E}(k(nT), X) \longrightarrow \mathcal{E}(k(nT), Y)$, which is just a morphism $<\alpha_T>: \mathcal{E}(k(T), X)^n \longrightarrow \mathcal{E}(k(T), Y)^n$ in **set**, since $\mathcal{E}(k(nT), X) \simeq \mathcal{E}(nk(T), X) \simeq \mathcal{E}(k(T), X)^n$ holds for any n. The map $< \alpha_T >$ is defined, on any arrow $f : k(nT) \simeq nk(T) \longrightarrow X$, via the assignment $(f.\sigma_i)_i \mapsto (\alpha_T(f.\sigma_i))_i$, where the arrows $\sigma_i : k(T) \longrightarrow nk(T)$ denote the canonical coproduct injections in $\mathcal{E}$. It is clear that it suffices to consider $n = 1$ and the corresponding arrow $\alpha_T : \mathcal{E}(k(T), X) \longrightarrow \mathcal{E}(k(T), Y)$, in order to get all arbitrary components of α, via its naturality. We want to show there exists a unique morphism $b : X \longrightarrow Y$ of $\mathcal{E}$ such that $\mathcal{E}(k(-), b) = \alpha$; namely, for any element $f \in \mathcal{E}(k(T), X)$, the equality $b.f = \alpha_T(f)$ holds. Once again, note that it is sufficient to establish this equality at T, then, it will follow at any object nT, by the universal property of the finite coproduct and the naturality of α. Consider, then, the collection $R_X = \{h : k(T) \longrightarrow X\}$ of all maps into X in $\mathcal{E}$ and the canonical morphism λ in $\mathcal{E}$ induced by these arrows $h \in R_X$, because of the universal property of the coproduct, as follows:

$$\begin{tikzcd} \coprod_{h\in R_X} k(T) \arrow[r, "\lambda"] & X \\ k(T) \arrow[u, "\sigma_h"] \arrow[ur, "h"'] & \end{tikzcd}$$

Namely, λ is the unique arrow satisfying $\lambda.\sigma_h = h$, for any $h \in R_X$. (Once again, we denote by σ_h the canonical coproduct injection in $\mathcal{E}$.) Condition (A_{reg}) states that the family R_X is regular epimorphic; in other words, λ is a regular epimorphism in $\mathcal{E}$. By taking the kernel pair (λ_0, λ_1) of λ in $\mathcal{E}$:

$$\begin{tikzcd} N(\lambda) \arrow[r, shift left, "\lambda_0"] \arrow[r, shift right, "\lambda_1"'] & \coprod_{h\in R_X} k(T) \arrow[r, two heads, "\lambda"] & X \end{tikzcd}$$

we already know that, λ, being a regular epimorphism, is the coequalizer of its kernel pair (λ_0, λ_1). When considering any finite subset $S \subseteq R_X$, we can define the canonical map λ_S, induced by the arrows just in S, and its corresponding kernel pair (m_S, n_S) in $\mathcal{E}$, as in the diagram:

$$\begin{tikzcd} N(\lambda_S) \arrow[r, shift left, "m_S"] \arrow[r, shift right, "n_S"'] & \coprod_{h\in S\subseteq R_X} k(T) \arrow[r, "\lambda_S"] & X \\ & k(T) \arrow[u, "\rho_h"] \arrow[ur, "h"'] & \end{tikzcd}$$

that is, $\lambda_S : \coprod_{h\in S} k(T) \longrightarrow X$ is the unique map satisfying $\lambda_S.\rho_h = h$, for any $h \in S$. Obviously, in **set** we have $R_X = \operatorname{colim}_{S\subseteq R_X} S$, for S running through all finite subsets of R_X; moreover, this is a filtered colimit. Therefore, the coproduct $\coprod_{h\in R_X} k(T)$ is the filtered colimit of all coproducts $\coprod_{h\in S} k(T)$, for S varying over all finite subsets of R_X, as established in Lemma 145, and since filtered colimits commute in $\mathcal{E}$ with finite limits, by hypothesis, we obtain the kernel pair of λ as a filtered colimits of the kernel pairs of all morphisms λ_S, that is:

$$N(\lambda) \simeq \operatorname*{colim}_{S\subseteq R_X} N(\lambda_S)$$

(Here we made use of Remark 146.) More explicitly, we get the following commutative diagram:

$$\begin{tikzcd}
N(\lambda) \arrow[r, shift left, "\lambda_0"] \arrow[r, shift right, "\lambda_1"'] & \coprod_{h\in R_X} k(T) \arrow[r, two heads, "\lambda"] & X \\
N(\lambda_S) \arrow[u, "\hat{j}_S"] \arrow[r, shift left, "m_S"] \arrow[r, shift right, "n_S"'] & \coprod_{h\in S\subseteq R_X} k(T) \arrow[u, "\tilde{j}_S"'] \arrow[ur, "\lambda_S"'] &
\end{tikzcd}$$

where we denote by the morphisms $\hat{j}_S$ and $\tilde{j}_S$ the corresponding colimit injections in $\mathcal{E}$. Note that, for any $h \in R_X$, which is nothing but an arrow $h : k(T) \longrightarrow X$ in $\mathcal{E}$, the natural transformation $\alpha : \mathcal{E}(k(-), X) \longrightarrow \mathcal{E}(k(-), Y)$ produces a morphism $\alpha_T(h) : k(T) \longrightarrow Y$. All these maps $\alpha_T(h)$ induce, in fact, a canonical morphism μ in $\mathcal{E}$, by the universal property of the coproduct, as follows:

$$\begin{tikzcd}
\coprod_{h\in R_X} k(T) \arrow[r, "\mu"] & Y \\
k(T) \arrow[u, "\sigma_h"] \arrow[ur, "\alpha_T(h)"'] &
\end{tikzcd}$$

that is, μ is the unique map in $\mathcal{E}$ satisfying $\mu.\sigma_h = \alpha_T(h)$, for any element $h \in R_X$. As before, we can consider the coproducts just over a finite subset $S \subseteq R_X$, for S varying over all such subsets, and define the corresponding induced morphisms μ_S in $\mathcal{E}$, as displayed:

$$\begin{tikzcd}
\coprod_{h\in S\subseteq R_X} k(T) \arrow[r, "\mu_S"] & Y \\
k(T) \arrow[u, "\sigma_h"] \arrow[ur, "\alpha_T(h)"'] &
\end{tikzcd}$$

But S being a finite set, we get $\coprod_{h\in S} k(T) \simeq k(\coprod_{h\in S} T) \simeq k(mT)$, where m denotes the cardinality of S. Moreover, the coproduct injections ρ_h are nothing but the images $k(i_h)$ through the functor k of the canonical coproduct injections $i_h : T \longrightarrow mT = T + T + ... + T$ (m times) in $\mathcal{C}$. Therefore, in the definitions of the morphisms λ_S and μ_S, we can replace the coproduct injections ρ_h and obtain $\lambda_S.k(i_h) = h$ and $\mu_S.k(i_h) = \alpha_T(h)$, for any $h \in S$, respectively. By naturality of the transformation α relative to the morphism $i_h : T \longrightarrow mT$ in $\mathcal{C}$, we can derive that $\alpha_T(h) = \alpha_{mT}(\lambda_S).k(i_h)$: in particular, it follows that $\mu_S = \alpha_{mT}(\lambda_S)$, from the uniqueness of the arrow μ_S satisfying such a property. Suppose that $\mu.\lambda_0 = \mu.\lambda_1$.

$$\begin{tikzcd} N(\lambda) \arrow[r, shift left, "\lambda_0"] \arrow[r, shift right, "\lambda_1"'] & \coprod_{h\in R_X} k(T) \arrow[r, two heads, "\lambda"] \arrow[dr, "\mu"'] & X \arrow[d, dotted, "b"] \\ & & Y \end{tikzcd}$$

Since λ is the coequalizer of the pair (λ_0, λ_1), we obtain a unique map $b : X \longrightarrow Y$, such that $b.\lambda = \mu$, namely $b.\lambda.\sigma_h = \mu.\sigma_h$, for any element $h \in R_X$. This condition is equivalent to $\mathcal{E}(k(T), b)(h) = b.h = b.\lambda.\sigma_h = \mu.\sigma_h = \alpha_T(h)$, for any morphism $h : k(T) \longrightarrow X$ in $\mathcal{E}$, by definition of the arrows λ and μ, respectively. This is precisely fully faithfulness of the functor k'. So, it only remains to show that $\mu.\lambda_0 = \mu.\lambda_1$, or, equivalently, that $\mu.\lambda_0.\hat{j}_S = \mu.\lambda_1.\hat{j}_S$, for any finite subset $S \subseteq R_X$, since we have just proved that $N(\lambda)$ is the filtered colimit of all kernel pairs of the kind $N(\lambda_S)$. In other words, we need to prove that $\mu_S.m_S = \mu_S.n_S$, for any finite subset $S \subseteq R_X$, since we have $\mu.\lambda_0.\hat{j}_S = \mu.\tilde{j}_S.m_S = \mu_S.m_S$ and $\mu.\lambda_1.\hat{j}_S = \mu.\tilde{j}_S.n_S = \mu_S.n_S$, by definition of the morphisms $\hat{j}_S$, $\tilde{j}_S$ and μ_S. So, we get a pair of arrows $m_S, n_S : N(\lambda_S) \longrightarrow k(mT)$ in $\mathcal{E}$, where m is the cardinality of S. By Condition ($\tilde{\mathrm{B}}$), the collection M_{m_S,n_S} of arrows in $\mathcal{E}$

$$\left\{ g : k(T) \longrightarrow N(\lambda_S) \,\middle|\, \left\{ \begin{array}{l} m_S.g = k(f_0) \\ n_S.g = k(f_1) \end{array} \right. \text{for some}\, T \overset{f_0}{\underset{f_1}{\rightrightarrows}} mT \right\}$$

is an epimorphic family. This precisely means that the arrows $\mu_S.m_S$ and $\mu_S.n_S$ coincide if and only if the identity $\mu_S.m_S.g = \mu_S.n_S.g$ holds, for any element $g \in M_{m_S,n_S}$. Using the naturality of the transformation α, relative to the maps $f_0 : T \longrightarrow mT$ and $f_1 : T \longrightarrow mT$ in $\mathcal{C}$, we get, for any $g \in M_{m_S,n_S}$:

$$\begin{array}{rcll} \mu_S.m_S.g & = & \mu_S.k(f_0) & \\ & = & \alpha_T(\lambda_S.k(f_0)) & (1) \\ & = & \alpha_T(\lambda_S.k(f_1)) & (2) \\ & = & \mu_S.k(f_1) & (3) \\ & = & \mu_S.n_S.g & \end{array}$$

(where (1) and (3) are instances of the naturality of α and (2) follows from the definition of (m_S, n_S) as the kernel pair of the morphism λ_S

and the fact that $g \in M_{m_S,n_S}$). This ends the proof of $\mathcal{E}(k(T),b)(h)$ being equal to $\alpha_T(h)$, for any arrow $h : k(T) \longrightarrow X$ in $\mathcal{E}$; in other words, we have shown that $\mathcal{E}(k(-),b) = \alpha$ for a unique morphism $b : X \longrightarrow Y$ of $\mathcal{E}$, as desired. ∎

We can, finally, state our Theorem which classifies the localizations of algebraic categories:

Theorem 165 *Let $\mathcal{C}$ be an algebraic theory, $\mathcal{E}$ a cocomplete and exact category in which filtered colimits commute with finite limits and $k : \mathcal{C} \longrightarrow \mathcal{E}$ a finite coproduct-preserving functor. The following conditions are equivalent:*

1. *the functor $k' : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$, defined by the assignment $X \mapsto \mathcal{E}(k(-),X)$, is a localization;*

2. *the functor k satisfies the Conditions* (A_{reg}), ($\tilde{\mathrm{B}}$), (SF2) *and* (SF3).

Proof. Note, first of all, that we can omit Condition (SF1) in the statement of the Theorem in light of Remark 149.
$1 \Rightarrow 2$. Suppose the functor $k' : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$, defined by $X \mapsto \mathcal{E}(k(-),X)$, is a localization of the algebraic category $\mathbf{Mod}(\mathcal{C})$, whose left adjoint we denote by $L : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathcal{E}$; so we have that k' is fully faithful, L preserves finite limits and $L \dashv k'$. As we know, since $\mathcal{E}$ is cocomplete, there always exists the left Kan extension of k along the restricted Yoneda embedding $Y_{\mathcal{C}}$, which produces a left adjoint functor $\mathbf{Lan}_{Y_{\mathcal{C}}}(k)$ to k'. By uniqueness (up to isomorphism) of the left adjoint functor, we have $L \simeq \mathbf{Lan}_{Y_{\mathcal{C}}}(k)$. Moreover, note that k' being fully faithful is equivalent to the counit ξ for the adjunction $L \dashv k'$ being an isomorphism, that is $L.k' \simeq 1_{\mathcal{E}}$. On the other hand, we denote by $\eta : 1_{\mathbf{Mod}(\mathcal{C})} \longrightarrow k'.L$ the unit for the adjunction. Since the restricted Yoneda embedding $Y_{\mathcal{C}}$ is fully faithful, the universal arrow for the left Kan extension L of k along Yoneda is an isomorphism, namely $k \simeq L.Y_{\mathcal{C}}$, that is $L.\mathcal{C}(-,mT) \simeq k(mT)$, for any object $C = mT \in \mathcal{C}$; in particular, when evaluated at some representable functor $F = \mathcal{C}(-,mT)$, the component of η is the natural transformation $\eta_{mT} : \mathcal{C}(-,mT) \longrightarrow \mathcal{E}(k(-),k(mT))$, defined on any object $nT \in \mathcal{C}$ as the application of functor k, by the assignment $\mathcal{C}(nT,mT) \longrightarrow \mathcal{E}(k(nT),k(mT)) : f \mapsto k(f)$. Clearly, it suffices to consider $n = 1$ and the corresponding component $\eta_T : \mathcal{C}(-,T) \longrightarrow \mathcal{E}(k(-),k(T))$ of the natural transformation η, since all other components can be derived by its naturality. Therefore, the

current situation can be summarised as in the following diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\iota_{\mathcal{C}}"] \arrow[dr, "k"'] & \mathcal{F}(\mathcal{C}) \arrow[r, "\zeta_{\mathcal{F}(\mathcal{C})}"] \arrow[d, "k''"] & \mathcal{F}(\mathcal{C})_{ex} = \mathbf{Mod}(\mathcal{C}) \arrow[dl, "L"] \\
& \mathcal{E} &
\end{tikzcd}$$

where the relation between these functors is established in Proposition 161, since the category $\mathcal{E}$ is exact and has filtered colimits commuting with finite limits. Explicitly, k is special filtering if and only if k'' is left covering if and only if L is left exact. Then, it only remains to prove Conditions (A$_{reg}$) and ($\tilde{\mathrm{B}}$) on the functor k.

k satisfies Condition (A$_{reg}$). For it, consider an object $X \in \mathcal{E}$ and the collection $R_X = \{h : k(T) \longrightarrow X\}$ of all arrows of $\mathcal{E}$ into X. We want to prove that R_X is a regular epimorphic family, that is, the canonical morphism λ, induced by the universal property of the coproduct, as follows:

$$\begin{tikzcd}
\coprod_{h \in R_X} k(T) \arrow[r, "\lambda"] & X \\
k(T) \arrow[u, "\sigma_h"] \arrow[ur, "h"'] &
\end{tikzcd}$$

is a regular epimorphism in $\mathcal{E}$. Explicitly, λ is the unique arrow satisfying $\lambda.\sigma_h = h$, for any $h \in R_X$ (here we denote by σ_h the canonical coproduct injection in $\mathcal{E}$). As recalled in Proposition 140, when considering the free model functor $\mathbb{F} : \mathbf{set} \longrightarrow \mathbf{Mod}(\mathcal{C})$, which is the left adjoint to the forgetful functor $\mathbb{U} : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathbf{set}$, the counit ϵ for the adjunction $\mathbb{F} \dashv \mathbb{U}$ has the extra feature that each of its component is a regular epimorphism in $\mathbf{Mod}(\mathcal{C})$. Namely, for any $\mathcal{C}$-model F, $\epsilon_F : \mathbb{F}.\mathbb{U}(F) \longrightarrow F$, defined as the canonical morphism induced by all natural transformations $\alpha \in \mathbf{Nat}(\mathcal{C}(-,T), F)$, as in the following diagram:

$$\begin{tikzcd}
\coprod_{\alpha \in \mathbf{Nat}(\mathcal{C}(-,T),F)} \mathcal{C}(-,T) \arrow[r, two heads, "\epsilon_F"] & F \\
\mathcal{C}(-,T) \arrow[u, "\tau_\alpha"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

(here the morphisms τ_α denote the canonical coproduct injections in $\mathbf{Mod}(\mathcal{C})$) is a regular epimorphism in $\mathbf{Mod}(\mathcal{C})$. Then, applying the left exact left adjoint functor L to the previous diagram gives:

$$\begin{tikzcd} \coprod\limits_{\alpha\in\mathbf{Nat}(\mathcal{C}(-,T),F)} k(T) \arrow[r, two heads, "L(\epsilon_F)"] & L(F) \\ k(T) \arrow[u, "L(\tau_\alpha)\simeq\sigma_\alpha"] \arrow[ur, "L(\alpha)"'] & \end{tikzcd}$$

where $L(\epsilon_F)$ is a regular epimorphism in $\mathcal{E}$, ϵ_F being such in $\mathbf{Mod}(\mathcal{C})$ and L being exact. So, when taking as F the representable functor $\mathcal{E}(k(-),X)$, the last diagram becomes:

$$\begin{tikzcd} \coprod\limits_{\alpha\in\mathbf{Nat}(\mathcal{C}(-,T),\mathcal{E}(k(-),X))} k(T) \arrow[r, two heads, "L(\epsilon_{\mathcal{E}(k(-),X)})"] & L(\mathcal{E}(k(-),X))\simeq X \\ k(T) \arrow[u, "L(\tau_\alpha)\simeq\sigma_\alpha"] \arrow[ur, "L(\alpha)"'] & \end{tikzcd}$$

where $L(\epsilon_{\mathcal{E}(k(-),X)})$ is a regular epimorphism in $\mathcal{E}$. (Here we made use of the isomorphism $L.k' \simeq 1_{\mathcal{E}}$.) The coproduct in the previous diagram is taken over all elements $\alpha \in \mathbf{Nat}(\mathcal{C}(-,T),\mathcal{E}(k(-),X)) = \mathbf{Nat}(\mathcal{C}(-,T),k'(X)) \simeq \mathbf{Nat}(L(\mathcal{C}(-,T)),X) \simeq \mathbf{Nat}(k(T),X) = R_X$, using the Yoneda Lemma, the adjunction $L \dashv k'$ and the fact that, $Y_{\mathcal{C}}$ being fully faithful, the universal arrow for the left Kan extension is an isomorphism (that is, $k \simeq L.Y_{\mathcal{C}}$) respectively. More precisely, any morphism $\alpha : \mathcal{C}(-,T) \longrightarrow k'(X)$ corresponds to a unique arrow $h : L(\mathcal{C}(-,T)) \simeq k(T) \longrightarrow X$, via $\xi_X.L(\alpha) = h$, where ξ, which still denotes the counit for the adjunction $L \dashv k'$, is an isomorphism, so that we have $L(\alpha) = h$. Therefore, $L(\epsilon_{\mathcal{E}(k(-),X)})$ is precisely the canonical map induced by all arrows h, namely, $L(\epsilon_{\mathcal{E}(k(-),X)})$ has to be equal to λ, and this ends the proof of λ being a regular epimorphism in $\mathcal{E}$.
k satisfied Condition ($\tilde{\mathrm{B}}$). For it, consider an object $X \in \mathcal{E}$, a pair of arrows $x_0, x_1 : X \longrightarrow k(nT)$ in $\mathcal{E}$ and the corresponding collection M_{x_0,x_1} of maps in $\mathcal{E}$

$$\left\{ g : k(T) \longrightarrow X \;\middle|\; \begin{cases} x_0.g = k(f_0) \\ x_1.g = k(f_1) \end{cases} \text{for some}\, T \overset{f_0}{\underset{f_1}{\rightrightarrows}} nT \text{ in } \mathcal{C} \right\}$$

We want to show that M_{x_0,x_1} is an epimorphic family. Let R_X be the collection $\{h : k(T) \longrightarrow X\}$ of all maps of $\mathcal{E}$ into X, which is regular epimorphic by Condition (A_{reg}), and, for any $h \in R_X$, consider, first, the composite $x_0.h : k(T) \longrightarrow k(nT)$ in $\mathcal{E}$. Take the pullback of the unit component η_{nT} along the composite $k'(x_0.h).\eta_T$ in $\mathbf{Mod}(\mathcal{C})$, as follows:

$$\begin{tikzcd}
P_{x_0.h} \arrow[rr] \arrow[d, "p"'] & & \mathcal{C}(-,nT) \arrow[d, "\eta_{nT}"] \\
\mathcal{C}(-,T) \arrow[r, "\eta_T"'] & \mathcal{E}(k(-),k(T)) \arrow[r, "k'(x_0.h)"'] & \mathcal{E}(k(-),k(nT))
\end{tikzcd}$$

Recall, from Proposition 130, that in $\mathbf{Mod}(\mathcal{C})$ limits are computed pointwise; therefore, the previous pullback is given at T by the collection $P_{x_0.h}(T) = \{(g,f) \in \mathcal{C}(T,T) \times \mathcal{C}(T,nT) | x_0.h.k(g) = k(f)\}$. Take, then, the regular epi-mono factorisation of the morphism p in $\mathbf{Mod}(\mathcal{C})$, which is regular, as in the diagram:

$$\begin{tikzcd}
P_{x_0.h} \arrow[rr, "p"] \arrow[dr, two heads, "e"'] & & \mathcal{C}(-,T) \\
& R_{x_0.h} \arrow[ur, tail, "r"'] &
\end{tikzcd}$$

We wish to prove that the image $R_{x_0.h}$ evaluated at T is precisely the image of the morphism $p(T)$ in **set**. For it, to evaluate the previous diagram at T is to apply the forgetful functor $\mathbb{U} : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathbf{set}$ to it, and we know that $\mathbb{U}$ preserves both monomorphisms, by Proposition 130, and regular epimorphisms, by Proposition 142. Therefore, $r(T)$ being a monomorphism and $e(T)$ being a regular epimorphism, we get the following commutative triangle in **set**:

$$\begin{tikzcd}
P_{x_0.h}(T) \arrow[rr, "p(T)"] \arrow[dr, two heads, "e(T)"'] & & \mathcal{C}(T,T) \\
& R_{x_0.h}(T) \arrow[ur, tail, "r(T)"'] &
\end{tikzcd}$$

which gives the unique (up to isomorphism) regular epi-mono factorisation of the arrow $p(T)$ in **set**; in other words, $R_{x_0.h}(T) = \{g : T \longrightarrow T | x_0.h.k(g) = k(f) \text{ for some } f : T \longrightarrow nT\}$, as desired. Now, since k' is fully faithful, we have already remarked that the counit ξ for the adjunction $L \dashv k'$ is an isomorphism and this, in particular, implies that $L(\eta_{nT})$ is an isomorphism. Therefore, since both the pullback and the regular epi-mono factorisation are preserved by the exact functor L, we get that $L(p)$ is an isomorphism, and so is $L(r)$. Therefore, the subfunctor $r : R_{x_0.h} \rightarrowtail \mathcal{C}(-,T)$ is such that its image $L(r)$ through the functor

L is an isomorphism in $\mathcal{E}$: from Lemma 163, it follows that the family $R_{x_0.h}(T) = \{g : T \longrightarrow T | x_0.h.k(g) = k(f) \text{ for some } f : T \longrightarrow nT\} \subseteq \mathcal{C}(T,T)$ is k-epimorphic. So, for any element $g \in R_{x_0.h}(T)$, consider now the composite $x_1.h.k(g) : k(T) \longrightarrow k(T) \longrightarrow X \longrightarrow k(nT)$ in $\mathcal{E}$. By the same reasoning, one can prove that the family $R_{x_1.h.k(g)}(T) = \{t : T \longrightarrow T | x_1.h.k(g).k(t) = k(s) \text{ for some } s : T \longrightarrow nT\}$ is k-epimorphic. By composing together all arrows of the kind g in $R_{x_0.h}(T)$ with all arrows of the kind t in $R_{x_1.h.k(g)}(T)$, we obtain a collection (depending on $h \in R_X$) $S_h = \{k(g).k(t) = k(g.t) : k(T) \longrightarrow k(T) | g \in R_{x_0.h}(T), t \in R_{x_1.h.k(g)}(T)\}$ of morphisms of $\mathcal{E}$, which we wish to prove to be an epimorphic family itself. For it, let $u, v : k(T) \longrightarrow U$ be a pair of arrows in $\mathcal{E}$ satisfying $u.k(g).k(t) = v.k(g).k(t)$, for any $g \in R_{x_0.h}(T)$ and $t \in R_{x_1.h.k(g)}(T)$. Since the family $R_{x_1.h.k(g)}(T)$ is k-epimorphic, this amounts to $u.k(g) = v.k(g)$, for any $g \in R_{x_0.h}(T)$ and using now that $R_{x_0.h}(T)$ is k-epimorphic, this last statement is further equivalent to $u = v$, as desired. In other words, we have just shown that the family of morphisms $\{g.t : T \longrightarrow T | g \in R_{x_0.h}(T), t \in R_{x_1.h.k(g)}(T)\}$ in $\mathcal{C}$, which we still denote by S_h, is k-epimorphic. Moreover, using that $g \in R_{x_0.h}(T)$, we have $x_0.h.k(g.t) = x_0.h.k(g).k(t) = k(f).k(t) = k(f.t) = k(f')$, for some $f' : T \longrightarrow nT$ and, using that $t \in R_{x_1.h.k(g)}(T)$, we also have $x_1.h.k(g.t) = x_1.h.k(g).k(t) = k(s)$, for some $s : T \longrightarrow nT$. So, we have just shown that the collection S_h is contained in the family of maps

$$\left\{ w : T \longrightarrow T \,\middle|\, \begin{cases} x_0.h.k(w) = k(f_0) \\ x_1.h.k(w) = k(f_1) \end{cases} \text{for some}\, T \underset{f_1}{\overset{f_0}{\rightrightarrows}} nT \right\}$$

in $\mathcal{C}$. In particular, for any of these arrows $w \in S_h$, the composite $h.k(w) : k(T) \longrightarrow X$ does belong to the family M_{x_0,x_1}, by definition of S_h. So, by considering the collection of all arrows obtained in this way, we form a subcollection of M_{x_0,x_1}; call it $N = \{h.k(w) : k(T) \longrightarrow X | h \in R_X, w \in S_h\}$. Finally, it suffices to prove that N is an epimorphic family. For it, let $u, v : X \longrightarrow U$ be a pair of arrows in $\mathcal{E}$ such that $u.h.k(w) = v.h.k(w)$, for any $h \in R_X$ and $w \in S_h$. Since the family S_h is k-epimorphic, this is equivalent to $u.h = v.h$, for any $h \in R_X$, and using now that R_X is (regular) epimorphic, by Condition (A_{reg}), the last statement is further equivalent to $u = v$, as desired. Therefore, N is an epimorphic family which is contained in M_{x_0,x_1}: it follows, then, from Lemma 113 that M_{x_0,x_1} is epimorphic, that is, Condition ($\tilde{\mathrm{B}}$) is verified.

$2 \Rightarrow 1$. Suppose the functor k satisfies the Conditions (A_{reg}), ($\tilde{\mathrm{B}}$), (SF2) and (SF3). As mentioned above, the category $\mathcal{E}$ being cocomplete, there always exists the left Kan extension of the functor k along the restricted Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$, call it $L := \mathbf{Lan}_{Y_{\mathcal{C}}}(k)$, which is left adjoint to k'. By Proposition 161, we know that k being special filtering is equivalent to the functor L being left exact, that is, L preserves finite limits. Concerning the functor k', we have just shown in Lemma 164 that if the functor k satisfies Conditions (A_{reg}) and ($\tilde{\mathrm{B}}$), then k' is fully faithful. Hence, we have just shown that k' is a fully faithful functor with a left exact left adjoint. This ends the proof of $k' : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$ being a localization of the algebraic category $\mathbf{Mod}(\mathcal{C})$. ■

Remark 166 When assuming $k' : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$ is a localization of the algebraic category $\mathbf{Mod}(\mathcal{C})$, an alternative proof of the fact that the functor k satisfies Condition (A_{reg}) can be given as follows. Clearly, $k' : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$ is a full reflective subcategory of $\mathbf{Mod}(\mathcal{C})$, whose left adjoint functor we denote by L. Knowing that the small set $\{\mathcal{C}(-,T)\}$ is a regular generator for $\mathbf{Mod}(\mathcal{C})$, we derive, from Lemma 68, that $\{L(\mathcal{C}(-,T)) \simeq k(T)\}$ is a regular generator for $\mathcal{E}$. This is exactly the condition on $\lambda : \coprod_{h \in \mathcal{E}(k(T),X)=R_X} k(T) \longrightarrow X$ of being a regular epimorphism in $\mathcal{E}$. Note that we can use Lemma 68, which is originally formulated in terms of a strong generator, precisely because in a regular category strong and regular epimorphisms coincide.

Let $\mathcal{C}$ be an algebraic theory and $\mathcal{E}$ a cocomplete and exact category in which filtered colimits commute with finite limits. We denote by $\mathbf{Loc}[\mathcal{E}, \mathbf{Mod}(\mathcal{C})]$ the full subcategory of $\mathbf{GeoMor}[\mathcal{E}, \mathbf{Mod}(\mathcal{C})]$ given by all localizations between $\mathcal{E}$ and the algebraic category $\mathbf{Mod}(\mathcal{C})$. Whereas, we denote by $\mathbf{SFilt}^*[\mathcal{C}, \mathcal{E}]$ the category of functors from $\mathcal{C}$ into $\mathcal{E}$ which are finite coproduct-preserving, special filtering and satisfy Conditions (A_{reg}) and ($\tilde{\mathrm{B}}$) (or, equivalently, which are finite coproduct-preserving and satisfy Conditions (A_{reg}), ($\tilde{\mathrm{B}}$), (SF2) and (SF3)) and all natural transformations between them. The following result relates these two notions, namely localizations can be classified by functors belonging to categories of the kind $\mathbf{SFilt}^*$:

Corollary 167 *Let $\mathcal{C}$ be an algebraic theory and $\mathcal{E}$ a cocomplete and exact category in which filtered colimits commute with finite limits. The functor*

$$\xi : \mathbf{Loc}[\mathcal{E}, \mathbf{Mod}(\mathcal{C})] \longrightarrow \mathbf{SFilt}^*[\mathcal{C}, \mathcal{E}]$$

defined by pre-composing with the restricted Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$, gives an equivalence of categories.

Proof. Let

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift right, tail, "i"'] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathcal{C}) \arrow[l, shift right, "L"'] \end{tikzcd}$$

be a localization and consider the pre-composition $L.Y_{\mathcal{C}}$ of its left adjoint L with Yoneda $Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Mod}(\mathcal{C})$. First of all, we need to show that the functor $L.Y_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{E}$ verifies Conditions (A_{reg}), ($\tilde{\mathrm{B}}$), (SF2) and (SF3). By Proposition 161, to prove $L.Y_{\mathcal{C}}$ being special filtering is equivalent to prove that the left Kan extension $\mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}})$ of the composite $L.Y_{\mathcal{C}}$ along the restricted Yoneda is left exact, as in the following diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "Y_{\mathcal{C}}"] \arrow[dr, "L.Y_{\mathcal{C}}"'] & & \mathbf{Mod}(\mathcal{C}) \arrow[dl, "\mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}})"] \\
& \mathcal{E} &
\end{tikzcd}$$

Since $Y_{\mathcal{C}}$ is fully faithful, the universal arrow for such a left Kan extension is an isomorphism, namely $L.Y_{\mathcal{C}} \simeq \mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}}).Y_{\mathcal{C}}$. Moreover, $Y_{\mathcal{C}}$ being a dense functor, by Lemma 143, and L and $\mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}})$ preserving all existing colimits, we can derive that $L \simeq \mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}})$. Clearly, the functor $\mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}})$ is left adjoint to the functor $\mathcal{E}(L.Y_{\mathcal{C}}(-), -) : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$ and, therefore, by uniqueness (up to isomorphism) of the right adjoint, $\mathcal{E}(L.Y_{\mathcal{C}}(-), -)$ is isomorphic to i, which is a right adjoint functor to L. In particular, then, $\mathcal{E}(L.Y_{\mathcal{C}}(-), -) \simeq i$ is full and faithful, which means that

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift right=1.5ex, "\mathcal{E}(L.Y_{\mathcal{C}}(-),-)"'] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathcal{C}) \arrow[l, shift right=1.5ex, "\mathbf{Lan}_{Y_{\mathcal{C}}}(L.Y_{\mathcal{C}})"']
\end{tikzcd}$$

is a localization of the algebraic category $\mathbf{Mod}(\mathcal{C})$. Then, from Theorem 165, it follows immediately that the functor $L.Y_{\mathcal{C}}$ does satisfy Conditions (A_{reg}), ($\tilde{\mathrm{B}}$), (SF2) and (SF3), as desired. It only remains to show, then, that $L.Y_{\mathcal{C}}$ does preserve finite coproducts: this property follows immediately from the fact that, L, being a left adjoint functor, preserves all existing colimits and the restricted Yoneda embedding $Y_{\mathcal{C}}$ is a finite coproduct-preserving functor, as remarked in Lemma 144. This ends the proof of $L.Y_{\mathcal{C}}$ being an object of $\mathbf{SFilt}^*[\mathcal{C}, \mathcal{E}]$. In order to prove that ξ is an equivalence of categories, we need to verify it is (a) fully faithful and (b) essentially surjective on objects.

(a) The proof that ξ is full and faithful follows the same lines of the proof that the functor $\varepsilon : \mathbf{GeoMor}[\mathcal{E}, \mathbf{Mod}(\mathcal{C})] \longrightarrow \mathbf{SFilt}[\mathcal{C}, \mathcal{E}]$ introduced in Corollary 162 is fully faithful, once we have underlined that both the categories $\mathbf{Loc}[\mathcal{E}, \mathbf{Mod}(\mathcal{C})]$ and $\mathbf{SFilt}^*[\mathcal{C}, \mathcal{E}]$ are full subcategories of $\mathbf{GeoMor}[\mathcal{E}, \mathbf{Mod}(\mathcal{C})]$ and $\mathbf{SFilt}[\mathcal{C}, \mathcal{E}]$, respectively.

(b) It only remains to show, then, that ξ is essentially surjective on objects. Let $k : \mathcal{C} \longrightarrow \mathcal{E}$ be a finite coproduct-preserving functor verifying Conditions (A_{reg}), ($\tilde{\mathrm{B}}$), (SF2) and (SF3). Since the category $\mathcal{E}$ is cocomplete, we can always define a pair of adjoint functors

$\mathcal{E}(k(-),-) : \mathcal{E} \longrightarrow \mathbf{Mod}(\mathcal{C})$, which assigns to any object $X \in \mathcal{E}$ the set-valued functor $\mathcal{E}(k(-),X)$, and $\mathbf{Lan}_{Y_\mathcal{C}}(k) : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathcal{E}$, which is given by the left Kan extension of k along the restricted Yoneda embedding $Y_\mathcal{C}$. Note that the functor $\mathcal{E}(k(-),X) : \mathcal{C}^{op} \longrightarrow \mathbf{set}$ does preserve finite products, since $\mathcal{E}(k(nT),X) \simeq \mathcal{E}(nk(T),X) \simeq \mathcal{E}(k(T),X)^n$, k being a finite coproduct-preserving functor. Then, by Proposition 161, the fact that the functor k is special filtering is equivalent to the left Kan extension $\mathbf{Lan}_{Y_\mathcal{C}}(k)$ being left exact. Moreover, by Lemma 164, the fact that k satisfies Condition (A_{reg}) and ($\tilde{\mathrm{B}}$) implies that the functor $\mathcal{E}(k(-),-)$ is full and faithful. Therefore, we get a localization

$$\begin{tikzcd} \mathcal{E} \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\mathcal{E}(k(-),-)"'] & \mathbf{Mod}(\mathcal{C}) \arrow[l, shift right=1.5ex, "\mathbf{Lan}_{Y_\mathcal{C}}(k)"'] \end{tikzcd}$$

which, in particular, satisfies the condition $k \simeq \mathbf{Lan}_{Y_\mathcal{C}}(k).Y_\mathcal{C}$, since $Y_\mathcal{C}$ is fully faithful. Namely, $\xi(\mathbf{Lan}_{Y_\mathcal{C}}(k)) \simeq k$ and this ends the proof of the functor ξ being essentially surjective on objects. ■

4.6 The link with other characterisation theorems for localizations of algebraic categories

It is widely known that one can compare categories of $\mathcal{C}$-models, for an algebraic theory $\mathcal{C}$, and classical categories of $\mathbb{T}$-algebras for a monad $\mathbb{T}$ (see [31]). We review quickly some basic notions in order to introduce the setting.

Definition 168 *A* monad $\mathbb{T} = (T, \upsilon, \mu)$ *on a category $\mathcal{A}$ consists of a functor $T : \mathcal{A} \longrightarrow \mathcal{A}$ and two natural transformations $\upsilon : 1_\mathcal{A} \longrightarrow T$ and $\mu : TT \longrightarrow T$, which make the following diagrams commute:*

$$\begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\upsilon T"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] & T \arrow[l, "T\upsilon"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd}$$

Proposition 169 *Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. Every adjunction $F \dashv G$, where the functors are $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$, gives rise to a monad on the category $\mathcal{A}$.*

Proof. Just take the composite $T = G.F : \mathcal{A} \longrightarrow \mathcal{A}$ as endofunctor, the unit for the adjunction as υ and the natural transformation given by the composite $G.\varepsilon.F$ as μ, provided $\varepsilon : F.G \longrightarrow 1_{\mathcal{B}}$ is the counit for the adjunction $F \dashv G$. ■

Definition 170 *Let* $\mathbb{T} = (T, \upsilon, \mu)$ *be a monad on a category* $\mathcal{A}$. *A* $\mathbb{T}$-algebra (A, a) *is a pair consisting of an object* $A \in \mathcal{A}$ *and an arrow* $a : TA \longrightarrow A$ *of* $\mathcal{A}$*, which makes both the diagrams:*

$$\begin{tikzcd} TTA \arrow[r, "Ta"] \arrow[d, "\mu A"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\upsilon A"] \arrow[dr, "1_A"'] & TA \arrow[d, "a"] \\ & A \end{tikzcd}$$

commute. A morphism of $\mathbb{T}$-algebras $f : (A, a) \longrightarrow (B, b)$ *is an arrow* $f : A \longrightarrow B$ *of* $\mathcal{A}$*, which renders commutative the diagram:*

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\ TB \arrow[r, "b"'] & B \end{tikzcd}$$

In fact, the collection of $\mathbb{T}$-algebras (and morphisms of $\mathbb{T}$-algebras) form a category, as established in the following:

Proposition 171 *Let* $\mathbb{T} = (T, \upsilon, \mu)$ *be a monad on a category* $\mathcal{A}$. *Then, the collection of all* $\mathbb{T}$*-algebras and their morphisms forms a category, which is usually denoted by* $\mathcal{A}^{\mathbb{T}}$. *Moreover, there is an adjunction* $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$*, where the functors* $F^{\mathbb{T}} : \mathcal{A} \longrightarrow \mathcal{A}^{\mathbb{T}}$ *and* $G^{\mathbb{T}} : \mathcal{A}^{\mathbb{T}} \longrightarrow \mathcal{A}$ *are defined by the following assignments:*

$$\begin{array}{lll} F^{\mathbb{T}} : & A & \mapsto & (TA, \mu_A) \\ G^{\mathbb{T}} : & (A, a) & \mapsto & A \end{array}$$

where $\mu_A : TTA \longrightarrow TA$ *is the morphism in* $\mathcal{A}$ *satisfying the* $\mathbb{T}$*-algebra axioms. The monad arising on* $\mathcal{A}$ *from this adjunction is the given monad* $\mathbb{T} = (T, \upsilon, \mu)$.

For a complete treatment of monads and their algebras, we refer to [33, 36].

Definition 172 *A monad* $\mathbb{T} = (T, \upsilon, \mu)$ *on* **set** *has* finite rank *if the functor* $T : \mathbf{set} \longrightarrow \mathbf{set}$ *preserves filtered colimits.*

We recall the last preliminary result:

Theorem 173 *Let $\mathcal{A}$ be an algebraic category, with associated forgetful functor $\mathbb{U} : \mathcal{A} \longrightarrow \mathbf{set}$ and free model functor $\mathbb{F} : \mathbf{set} \longrightarrow \mathcal{A}$. Consider the following diagram:*

$$\begin{tikzcd}
\mathbf{set} \arrow[r, shift left, "\mathbb{F}"] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "F^{\mathbb{T}}"'] \arrow[d, phantom, "\dashv"] & \mathcal{A} \arrow[l, shift left, "\mathbb{U}"] \arrow[dl, "K"] \\
\mathbf{set}^{\mathbb{T}} \arrow[u, shift right, "G^{\mathbb{T}}"'] &
\end{tikzcd}$$

where $\mathbb{T}$ is the monad on $\mathbf{set}$ *defined by the adjunction $\mathbb{F} \dashv \mathbb{U}$. Then, there is a unique (up to isomorphism) comparison functor $K : \mathcal{A} \longrightarrow \mathbf{set}^{\mathbb{T}}$ satisfying $G^{\mathbb{T}}.K \simeq \mathbb{U}$ and $K.\mathbb{F} \simeq F^{\mathbb{T}}$.*

Therefore, the theorem relating monadic categories over **set** with algebraic categories can be stated in the following terms (see [7, 11]):

Theorem 174 *Let $\mathcal{C}$ be an algebraic theory, then there exists a monad $\mathbb{T}$ on* **set** *with finite rank such that the equivalence of categories $\mathbf{set}^{\mathbb{T}} \simeq \mathbf{Mod}(\mathcal{C})$ is established. Conversely, let $\mathbb{T}$ be a monad on* **set** *with finite rank, then there exists an algebraic theory $\mathcal{C}$ satisfying the equivalence $\mathbf{Mod}(\mathcal{C}) \simeq \mathbf{set}^{\mathbb{T}}$.*

Proof. We just sketch the two constructions. In one direction, given an algebraic theory $\mathcal{C}$, we can always consider the free model functor $\mathbb{F} : \mathbf{set} \longrightarrow \mathbf{Mod}(\mathcal{C})$, which is the left adjoint to the forgetful $\mathbb{U} : \mathbf{Mod}(\mathcal{C}) \longrightarrow \mathbf{set}$. The adjunction $\mathbb{F} \dashv \mathbb{U}$, as any other adjunction, gives rise to a monad $\mathbb{T}$ on **set**, by taking as T the composite $\mathbb{U}.\mathbb{F}$. Moreover, one can prove that the monad $\mathbb{T}$ has finite rank, using that, by Proposition 131, the functor $\mathbb{U}$ reflects filtered colimits. In the other direction, given a monad $\mathbb{T}$ on **set**, we can always construct the canonical adjunction:

$$\begin{tikzcd}
\mathbf{set} \arrow[r, shift left, "F^{\mathbb{T}}"] \arrow[r, phantom, "\perp"] & \mathbf{set}^{\mathbb{T}} \arrow[l, shift left, "G^{\mathbb{T}}"]
\end{tikzcd}$$

and consider the full subcategory $\mathcal{F}$ of $\mathbf{set}^{\mathbb{T}}$ spanned by the finitely generated free algebras $F^{\mathbb{T}}(n)$, where n is a finite set. Then, by taking its dual category $\mathcal{F}^{op}$, we define an algebraic theory $\mathcal{C}$, with $T = F^{\mathbb{T}}(1)$, since we have $F^{\mathbb{T}}(n) = \coprod_n F^{\mathbb{T}}(1)$. (Here 1 denotes the one element set.) By considering the adjunction:

$$\begin{tikzcd}
\mathbf{set} \arrow[r, shift left, "\mathbb{F}"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathcal{C}) \arrow[l, shift left, "\mathbb{U}"]
\end{tikzcd}$$

provided by the free model functor $\mathbb{F}$, and computing the induced monad $\mathbb{T}'$, defined via the composite $\mathbb{U}.\mathbb{F}$, we obtain an equivalence of categories $\mathbf{set}^{\mathbb{T}'} \simeq \mathbf{Mod}(\mathcal{C})$. Note that $\mathbb{T}'$ has, in fact, finite rank. Then, one can prove that the two monads $\mathbb{T}$ and $\mathbb{T}'$ are equivalent, precisely by using that both of them have finite rank. ■

Finally, we can present the celebrated Lawvere characterisation Theorem for algebraic categories (see [30]):

Theorem 175 *A locally small category $\mathcal{A}$ is equivalent to a (one sorted) algebraic category if and only if:*

1. *$\mathcal{A}$ is exact;*
2. *$\mathcal{A}$ has a regular projective regular generator G;*
3. *G is finitely presentable in $\mathcal{A}$.*

We recall that an object $G \in \mathcal{A}$ is called a *regular generator* if G admits all copowers and, for any object $A \in \mathcal{A}$, the canonical morphism $\gamma_A : \coprod_{f \in \mathcal{A}(G,A)} G \longrightarrow A$, induced by the universal property of the coproduct, is a regular epimorphism in $\mathcal{A}$. Whereas, $G \in \mathcal{A}$ is said to be *regular projective* if the corresponding representable functor $\mathcal{A}(G,-) : \mathcal{A} \longrightarrow \mathbf{set}$ preserves regular epimorphisms. We point out that this definition is nothing but the 1-element version of the definition of (strong) generator $\mathcal{G}$, analysed in Section 2.3.

Now, moving back to our topic for this chapter, some work has already been done concerning localizations of algebraic categories by Vitale in [41, 42]. The starting point is the following characterisation of an algebraic category, which is equivalent to the one contained in Theorem 175 and for which we still refer to [30]. We should mention that, in fact, the following Theorem was the original one introduced by Lawvere in his Ph.D. thesis. Whereas, Theorem 175 appeared only afterwards and became the reference for Lawvere's characterisation of algebraic categories, because of the more popular definition of finitely presentable objects. Here is the result:

Theorem 176 *A locally small category $\mathcal{A}$ is equivalent to a (one sorted) algebraic category if and only if:*

1. *$\mathcal{A}$ is exact;*
2. *$\mathcal{A}$ has finite colimits;*
3. *$\mathcal{A}$ has a regular projective regular generator G;*

4. G is abstractly finite.

We recall that an object $A \in \mathcal{A}$ is said to be *abstractly finite* if any arrow $A \longrightarrow S \circ A$ factors through some arrow $S' \circ A \longrightarrow S \circ A$, induced by the inclusion $S' \hookrightarrow S$, for some finite subset $S' \subseteq S$ (provided the object A admits all copowers and $S \circ A$ denotes the S-indexed copower of A). Clearly, any finitely presentable object is abstractly finite. The converse is only true in presence of the other axioms for an algebraic category. The idea of the previous Theorem can be carried out to investigate localizations of algebraic categories and leads to the following characterisation Theorem (see [42]):

Theorem 177 *A locally small category $\mathcal{B}$ is equivalent to a localization of a (one sorted) algebraic category if and only if:*

1. *$\mathcal{B}$ is exact;*

2. *$\mathcal{B}$ has a regular generator G;*

3. *filtered colimits exist and commute with finite limits in $\mathcal{B}$.*

Proof. We only sketch the construction in one direction, since the other one is trivial: just remark that any algebraic category satisfies Condition 1, 2 and 3 and these are stable under localizations.
Therefore, suppose $\mathcal{B}$ is an exact category with a regular generator G and with filtered colimits commuting with finite limits. It suffices that $\mathcal{B}$ satisfies Conditions 1 and 2 to deduce, from Proposition 2.1 in [41], $\mathcal{B}$ is equivalent to a localization of a monadic category over **set**; more explicitly, $\mathcal{B}$ is equivalent to a localization of $\mathbf{set}^{\mathbb{T}}$, where the monad $\mathbb{T}$ is determined by the adjunction $- \circ G \dashv \mathcal{B}(G, -)$, namely the endofunctor T is the composite $\mathcal{B}(G, - \circ G)$. In particular, the proof of Proposition 2.1 establishes that, when taking as $\mathcal{C}$ the full subcategory of $\mathcal{B}$ given by all copowers $S \circ G$ of G, for S varying in **set**, we get $\mathcal{B} \simeq \mathcal{C}_{ex}$. In order to prove that $\mathcal{B}$ is, in fact, a localization of an algebraic category, we need to construct a new monad $\mathbb{T}'$, the finitary part of $\mathbb{T}$, show that $\mathbb{T}'$ has finite rank and $\mathcal{B}$ is still a localization of $\mathbf{set}^{\mathbb{T}'}$. For it, consider the subcategory $\mathcal{D}$ of $\mathcal{C}$ having the same objects of $\mathcal{C}$ and for morphisms $f : R \circ G \longrightarrow S \circ G$ those arrows of $\mathcal{C}$ (that is, of $\mathcal{B}$) such that, for each element $r \in R$, the composite $f.\sigma_r : G \longrightarrow S \circ G$ factors through the arrow $S' \circ G \longrightarrow S \circ G$, induced by the inclusion $S' \hookrightarrow S$, for some finite

subset $S' \subseteq S$, as follows:

$$\begin{tikzcd} G \arrow[r, "\sigma_r"] \arrow[drr, dotted, "f'_r"'] & R \circ G \arrow[r, "f"] & S \circ G \\ & & S' \circ G \arrow[u] \end{tikzcd}$$

where $\sigma_r : G \longrightarrow R \circ G$ denotes the rth coproduct injection in $\mathcal{B}$. We underline that $\mathcal{D}$ is obviously not a full subcategory of $\mathcal{C}$. We obtain, then, a new adjunction $- \circ G \dashv \mathcal{D}(G, -)$ and a corresponding new monad $\mathbb{T}'$ on **set**, by computing the composite $T' = \mathcal{D}(G, - \circ G)$. The monad $\mathbb{T}'$ has finite rank: this property follows from the simple fact that the object G is abstractly finite in $\mathcal{D}$, although the proof is far from being evident. Therefore, we have:

$$\begin{tikzcd} \mathcal{B} \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathbf{set}^{\mathbb{T}} \arrow[l, shift right] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathbf{set}^{\mathbb{T}'} \simeq \mathcal{D}_{ex} \arrow[l, shift right] \end{tikzcd}$$

where the first pair is a localization and the second one an adjunction. Under the further assumption that the category $\mathcal{B}$ verifies Condition 3, one can prove that the composite $\mathcal{B} \longrightarrow \mathbf{set}^{\mathbb{T}'}$ is a fully faithful functor and its left adjoint $\mathbf{set}^{\mathbb{T}'} \longrightarrow \mathcal{B}$ is nothing but the exact extension of the (not full) inclusion $\mathcal{D} \longrightarrow \mathcal{B}$, which still is a left covering functor, and, therefore, its exact extension is left exact. In other words, we have just proved that the category $\mathcal{B}$ is a localization of the monadic category $\mathbf{set}^{\mathbb{T}'}$, where the monad $\mathbb{T}'$ has finite rank, that is, $\mathbf{set}^{\mathbb{T}'}$ is an algebraic category, by Proposition 174 and $\mathcal{B}$ is a localization of it. ∎

Our present aim is to obtain Theorem 177 as a corollary of our characterisation Theorem 165. We need to explicitly describe, then, the algebraic theory $\mathcal{C}$ and the finite coproduct-preserving functor $k : \mathcal{C} \longrightarrow \mathcal{B}$ which will realise the category $\mathcal{B}$ as a localization of the category $\mathbf{Mod}(\mathcal{C})$ of $\mathcal{C}$-models. Given an exact category $\mathcal{B}$ with a regular generator G and with filtered colimits commuting with finite limits, we can always define the subcategory $\mathcal{D}$ of $\mathcal{B}$ having for objects all copowers $S \circ G$ of G, for S varying in **set**, and for morphisms $f : R \circ G \longrightarrow S \circ G$ those arrows f in $\mathcal{B}$ such that, for any element $r \in R$, the composite $f.\sigma_r : G \longrightarrow S \circ G$ factors in $\mathcal{B}$ through the morphism $S' \circ G \longrightarrow S \circ G$, induced by the inclusion $S' \rightarrowtail S$, for some finite subset S' of S, as in the following diagram:

$$\begin{tikzcd} G \arrow[r, "\sigma_r"] \arrow[drr, dotted, "f'_r"'] & R \circ G \arrow[r, "f"] & S \circ G \\ & & S' \circ G \arrow[u] \end{tikzcd}$$

where $\sigma_r : G \longrightarrow R \circ G$ denotes the rth coproduct injection in $\mathcal{B}$. Obviously, $\mathcal{D}$ is not a full subcategory of $\mathcal{B}$. Then, we can take the full subcategory $\mathcal{C}$ of $\mathcal{D}$ given by only those copowers $n \circ G$, for n some finite set. We make no notational distinction between $n \circ G$ and nG, which means that we denote, at the same time, by n the finite set and the natural number corresponding to its cardinality (where this doesn't lead to any confusion). In fact, $\mathcal{C}$ is an algebraic theory with $T = G$. Call $k : \mathcal{C} \longrightarrow \mathcal{B}$ the (not full) inclusion. This is the setting which enables us to state the desired result:

Proposition 178 *Let $\mathcal{B}$ be an exact category with a regular generator G and with filtered colimits commuting with finite limits. Let $k : \mathcal{C} \longrightarrow \mathcal{B}$ be the (not full) inclusion of the subcategory $\mathcal{C}$ determined by the finite copowers nG of G, for any finite set n. Then, the functor $k : \mathcal{C} \longrightarrow \mathcal{B}$ preserves finite coproducts and satisfies the Conditions* (A_{reg}), $(\tilde{\mathrm{B}})$, (SF2) *and* (SF3).

Proof. Clearly, the inclusion k preserves finite coproducts. We simply treat the case of binary ones. For it, let nG and mG be two objects of $\mathcal{C}$, namely n and m are finite sets, and consider their coproduct in $\mathcal{C}$, which is isomorphic to $(n+m)G$, provided $n+m$ denotes the disjoint union in **set** of n and m. Then, we have $k(nG+mG) \simeq k((n+m)G) = (n+m)G \simeq nG + mG = k(nG) + k(mG)$.
k satisfies Condition (A_{reg}). For it, consider an object $X \in \mathcal{B}$ and the collection $R_X = \{g : G \longrightarrow X\}$ of all arrows of $\mathcal{B}$ into X, that is, $R_X = \mathcal{B}(G, X)$. We want to prove that R_X is a regular epimorphic family, that is, the canonical morphism λ, induced by the universal property of the coproduct, as follows:

$$\begin{tikzcd} \coprod_{g \in R_X} G \simeq \mathcal{B}(G,X) \circ G \arrow[r, "\lambda"] & X \\ G \arrow[u, "\sigma_g"] \arrow[ur, "g"'] & \end{tikzcd}$$

is a regular epimorphism in $\mathcal{B}$. This is precisely the condition on G of being a regular generator for $\mathcal{B}$. Note that we only need to take under

consideration morphisms into X with domain G, since any arbitrary arrow $f : nG \longrightarrow X$ in $\mathcal{B}$, for an arbitrary object nG of $\mathcal{C}$, can be entirely determined by its pre-composites with the coproduct injections $\sigma_i : G \longrightarrow nG$ in $\mathcal{C}$.
k satisfies Condition ($\tilde{\mathrm{B}}$). For it, consider an object $X \in \mathcal{B}$, a pair of arrows $x_0, x_1 : X \longrightarrow nG$ in $\mathcal{B}$, where n is a finite set, and the corresponding collection of maps in $\mathcal{B}$

$$M_{x_0,x_1} = \left\{ g : G \longrightarrow X \,\middle|\, \begin{cases} x_0.g = k(f_0) \\ x_1.g = k(f_1) \end{cases} \text{for some}\, G \overset{f_0}{\underset{f_1}{\rightrightarrows}} nG \text{ in } \mathcal{C} \right\}$$

We want to show that M_{x_0,x_1} is an epimorphic family. Note that the arrows $f_0, f_1 : G \longrightarrow nG$ not only are in $\mathcal{C}$ but actually in $\mathcal{B}$, since, n being a finite set, they always factor through a finite subset of n, namely n itself. Therefore, their images via the inclusion k coincide with the arrows f_0 and f_1 themselves. Then, for any morphism $g : G \longrightarrow X$ in $\mathcal{B}$, the composites $x_0.g$ and $x_1.g$ satisfy the condition of being arrows in $\mathcal{C}$, since n is a finite set and, then, this condition reduces to just being arrows of $\mathcal{B}$. It follows that $M_{x_0,x_1} = \mathcal{B}(G, X)$. From G being a regular generator in $\mathcal{B}$, we derive that the canonical morphism λ, induced by the universal property of the coproduct, as follows:

$$\begin{tikzcd} \coprod_{g \in M_{x_0,x_1}} G \simeq \mathcal{B}(G,X) \circ G \arrow[r, two heads, "\lambda"] & X \\ G \arrow[u, "\sigma_g"] \arrow[ur, "g"'] & \end{tikzcd}$$

is a regular epimorphism in $\mathcal{B}$. Explicitly, λ is the unique map in $\mathcal{B}$ verifying $\lambda.\sigma_g = g$, for any arrow g in M_{x_0,x_1}. In particular, we get that M_{x_0,x_1} is an epimorphic family. For it, let $u, v : X \longrightarrow U$ be a pair of arrows in $\mathcal{B}$ satisfying $u.g = v.g$, for any $g \in M_{x_0,x_1}$, or, equivalently $u.\lambda.\sigma_g = v.\lambda.\sigma_g$, for any $g \in M_{x_0,x_1}$. The last equality implies that $u.\lambda = v.\lambda$, by definition of the coproduct $\coprod_{g \in M_{x_0,x_1}} G$. It follows from λ being a (regular) epimorphism that $u = v$, as desired.
k satisfies Condition (SF2). For it, let nG and mG be a pair of objects in $\mathcal{C}$ and consider the collection $S_{n,m}$ of all arrows

$$\{< u, v >: G \longrightarrow nG \times mG | nG \xleftarrow{u} G \xrightarrow{v} mG \text{ in } \mathcal{C}\}$$

in $\mathcal{B}$. We want to prove that the family $S_{n,m}$ is regular epimorphic. The map $< u, v >$ is the unique one in $\mathcal{B}$ satisfying $\pi_1. < u, v >= u$ and $\pi_2. < u, v >= v$, obtained by the universal property of the product nG

$\xleftarrow{\pi_1} nG \times mG \xrightarrow{\pi_2} mG$ in $\mathcal{B}$. Note that the morphisms $u : G \longrightarrow nG$ and $v : G \longrightarrow mG$ not only are in $\mathcal{C}$ but actually in $\mathcal{B}$, since, n and m, being finite sets, always factor through a finite subset of n and m, respectively, namely n and m themselves. Therefore, for any morphism $g : G \longrightarrow nG \times mG$ in $\mathcal{B}$, the composites $\pi_1.g$ and $\pi_2.g$ satisfy the condition of being arrows in $\mathcal{C}$, since n and m are finite sets and, then, this condition reduces to just being arrows of $\mathcal{B}$. It follows that the morphism g can be identified with $< \pi_1.g, \pi_2.g >\in S_{n,m}$; in other words, we have just shown that $S_{n,m} = \mathcal{B}(G, nG\times mG)$. From G being a regular generator in $\mathcal{B}$, we derive that the canonical morphism λ, induced by the universal property of the coproduct, as follows:

$$\begin{tikzcd} \coprod\limits_{g\in S_{n,m}} G \simeq \mathcal{B}(G, nG\times mG)\circ G \arrow[r, two heads, "\lambda"] & nG \times mG \\ G \arrow[u, "\sigma_g"] \arrow[ur, "g"'] & \end{tikzcd}$$

is a regular epimorphism in $\mathcal{B}$. This is precisely the condition on the family $S_{n,m}$ of being regular epimorphic, as desired.
k satisfies Condition (SF3). For it, let $u, v : nG \longrightarrow mG$ be a pair of arrows in $\mathcal{C}$ and consider the collection $S_{u\&v}$ of all arrows

$$\{e_w : G \longrightarrow E_{u,v} | G \xrightarrow{w} nG \underset{v}{\overset{u}{\rightrightarrows}} mG \text{ in } \mathcal{C} \text{ such that } u.w = v.w\}$$

in $\mathcal{B}$. We want to prove that the family $S_{u\&v}$ is regular epimorphic. Here $(E_{u,v}, l)$ is the equalizer $E_{u,v} \overset{l}{\rightarrowtail} nG \underset{v}{\overset{u}{\rightrightarrows}} mG$ of the pair (u, v) in $\mathcal{B}$ and e_w is the unique map in $\mathcal{B}$ satisfying $l.e_w = w$, for any such an arrow w, obtained by the universal property of the equalizer. Therefore, $S_{u\&v}$ is nothing but the collection $\{e_w : G \longrightarrow E_{u,v} | l.e_w = w : G \longrightarrow nG \text{ in } \mathcal{C}\}$. Moreover, any such composite $l.e_w = w : G \longrightarrow nG$ is an arrow in $\mathcal{B}$, since n is a finite set. Therefore, for any morphism $g : G \longrightarrow E_{u,v}$ in $\mathcal{B}$, the composite $l.g : G \longrightarrow nG$ is an arrow in $\mathcal{C}$ such that $u.l.g = v.l.g$, since $(E_{u,v}, l)$ is the equalizer of the pair (u, v) in $\mathcal{B}$. It follows, hence, that $S_{u\&v} = \mathcal{B}(G, E_{u,v})$. From G being a regular generator in $\mathcal{B}$, we derive that the canonical morphism μ, induced by the universal property of the coproduct, as follows:

$$\begin{tikzcd} \coprod\limits_{e_w\in S_{u\&v}} G \simeq \mathcal{B}(G, E_{u,v})\circ G \arrow[r, two heads, "\mu"] & E_{u,v} \\ G \arrow[u, "\sigma_w"] \arrow[ur, "e_w"'] & \end{tikzcd}$$

is a regular epimorphism in $\mathcal{B}$. This is precisely the condition on the family $S_{u\&v}$ of being regular epimorphic, as desired. ■

Therefore, as direct consequence of our Theorem 165, we derive:

Corollary 179 *Any cocomplete, exact category, with a regular generator and with filtered colimits commuting with finite limits, is a localization of a (one sorted) algebraic category.*

The converse of this Corollary is always true since any localization of an algebraic category verifies these conditions. In fact, in any algebraic category filtered colimits commute with finite limits and the condition is stable under localizations.

Acknowledgements

So, where to start ... Enrico M. Vitale, the supervisor of this thesis, deserves many many thanks. He has provided me with four years of support, belief, mathematical ideas and inspiration. His office's door was always open for enriching discussions. He believed – hopefully he still believes – in me and the knowledge of it has motivated me to improve.

I wish to thank the whole of the Category Theory group in Louvain-la-Neuve: Francis Borceux, Mathieu Dupont, Jean-Roger Roisin and Isar Stubbe. I express my gratitude to Francis for several stimulating conversations and, in particular, for having helped me in simplifying part of the proofs contained in this manuscript and in rendering it smoother for the reader: Section 1.4 is due to his helpful observations. I thank Isar for having welcomed me in Louvain-la-Neuve since the earliest days and for having offered me great help in dealing with the typing of this thesis: the layout setting of this manuscript was borrowed from Isar's own thesis. I thank Richard Wood for his kindness in carefully reading this thesis and having pointed out to me great many typos and misuses of the English language. This text wouldn't look as it does without his priceless help.

I am thankful to Paul Sintzoff for so many reasons that trying to list them would be hopeless. Paul was always – and I like to think he always will be – there for me. His friendship led me through four years of gay Belgian life.

Leaving the colleagues aside, I say thank you to my brother Massimo. There are these people in this world who believe in me, regardless of time, choices and words. Massimo is my first, yet, strictest admirer. His unconditional love has inspired me to push myself a little bit further every day, to give my best in life. My mother Laura – a real Italian mum – supports and loves me in her unique wonderful way, which makes me feel warm inside and drives me crazy at once. She is always at my side and she deserves all my gratitude.

And, then, we come to friends, the ones cited above and many others. As you can expect, in four years of somebody's life, many events occur, and plenty of characters cross their path. Many people have entwined my life during this time and contributed – consciously or unconsciously – in depth or just by dropping a touch – to the development of my thesis. My friends provide me with good chats, a laughter, a vent to emotions. They enrich my life and, therefore, determine who I am and what I do. Great thanks go to Mario, Valentina, Monica, Jeannette and Rustam-Jan, Gillian Wood, Cristina Pedicchio, Ikram, Griet, Bert, Jaymeen, Kaat and Alessandra Vitale. They have fed me with pure confidence and belief in myself. Although being quite a self-confident person, these feelings are essential and vitalising for me: thank you.

Finally, going back to the Université catholique de Louvain, I wish to thank the Mathematics Department for having provided me with the means to realise this satisfactory and overwhelming project and, moreover, all my students: the time I spent with them was precious, refreshing and ... fun.

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Index