TOPOLOGICAL QUANTUM FIELD THEORY AND PURE YANG-MILLS DYNAMICS

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By considering specific limits in the gauge coupling constant of pure Yang–Mills dynamics, it is shown how there exist topological quantum field theory sectors in such systems defining nonperturbative topological configurations of the gauge fields which could well play a vital role in the confinement and chiral symmetry breaking phenomena of phenomenologically realistic theories such as quantum chromodynamics, the theory for the strong interactions of quarks and gluons. A general research programme along such lines is outlined. A series of other topics of possible relevance, ranging from particle phenomenology to the quest for the ultimate unification of all interactions and matter including quantum gravity, are raised in passing. The general discussion is illustrated through some simple examples in 0+1 and 1+1 dimensions, clearly showing the importance of properly accounting for quantum topological properties of the physical and configuration spaces in gauge theories based on compact gauge groups.

1. Introduction

Topological field theories[1,2,3,4] are dynamical systems possessing local or gauge invariances so large and powerful that their physical or gauge invariant sector solely depends on the topology — more precisely, the differentiable topology class — of the space on which these theories are defined. In other words, gauge invariant states and observables in such systems determine topological invariants of the underlying manifold. Once quantised, these features survive, possibly modulo some global aspects related to quantum anomalies.[3] Such topological quantum field theories (TQFT's) have become of great interest in the past fifteen years,[4] and are called to play a vital role in a variety of fields in pure mathematics, and in mathematical and theoretical physics, including the quest for a fundamental quantum unification of all particles and interactions. Even in the case of quantum mechanical systems — of which the degrees of freedom are function of time only —, there exist examples of such topological systems of interest. However, it is in the context of gauge field theories, whether of the Yang–Mills or gravity type, that TQFT's have proved to be of great value.

Among TQFT's, one distinguishes[4] the so-called theories of Schwarz type and of Witten type. Theories of Schwarz type may be defined through a local action principle independent of a metric structure on the underlying manifold. Theories of Witten type require such a metric structure to be specified in order to define their dynamics through an action principle, but their gauge symmetries are such that the physical or gauge invariant observables are independent of the metric structure nonetheless. A favourite example of a TQFT of Schwarz type is 2+1-dimensional pure Chern-Simons theory, whether for an abelian or nonabelian gauge group.[3,5] A favourite example of a TQFT of Witten type is Witten's original construction related to Donaldson's topological classification of 4-dimensional differential structures based on nonabelian (anti)self-dual instanton solutions to the Yang–Mills equations.[1] Another reason for the fascination with TQFT is that, for instance, pure quantum gravity in 2+1 dimensions is, as a matter fact, a Chern-Simons theory based on a noncompact gauge group,[6] possibly hinting at a deeper connection with a fundamental unification of all interactions including quantum gravity.[1,2]

Another source of challenging open problems is that of confinement and dynamical chiral symmetry breaking in quantum nonabelian Yang–Mills theories. Even though all the available evidence[7] concurs with the expectation that in such theories, indeed quarks and gluons condense and remain confined into massive colourless bound states, whatever the excitation energies applied onto these systems, there is no clear and definite understanding of the actual dynamics responsible for this feature quite unique to this large class of gauge theories, of direct relevance to the world of elementary particles and their strong interactions. Many a suggestion has been made,[8] involving either monopole- or instanton-like configurations,[9,10,11] but the verdict is still to be reached as to the actual culprits responsible for the confinement and chiral symmetry breaking phenomena. However, monopoles and instantons are specific gauge field configurations possessing nontrivial topological properties in configuration space made manifest through a nontrivial topology in space(time).

Consequently, with the discovery of TQFT's, the suggestion arises that the dynamics responsible for confinement and chiral symmetry breaking in Yang–Mills theories could possibly be dominated by a purely topological sector of Yang–Mills configuration space, with further corrections induced by topologically trivial quantum fluctuations around the topological non-perturbative sectors to be accounted for in a complete dynamical quantum treatment. The purpose of the present contribution is to try define a programme into that direction, by pointing out a series of puzzling properties of pure Yang–Mills theories such that in specific limits of the gauge coupling constant, one ends up with purely TQFT descriptions. It appears that indeed, there are TQFT's at work behind the scenes of the complete Yang–Mills dynamics. What could the physical consequences of these TQFT sectors be? Do they have any bearing on the confinement and chiral symmetry breaking issues?

The purpose of this contribution is foremost to outline a possible research programme along such lines, raising as we proceed some other questions reaching beyond that specific goal, hoping to sufficiently entice some of the younger participants to the Workshop to launch their own research work into any such direction. Only a few simple illustrative examples are also presented, even though much more could be said already at this stage. In Sec. 2, the general ideas of the programme are described. Section 3 briefly touches onto the possibility of defining generalised Yang–Mills dynamics and their eventual physics interest. Then Sec. 4 presents the simplest illustration of our discussion in 0+1 dimensions, while in Sec. 5 pure abelian U(1) gauge theory in 1+1 dimensions is explicitly solved in the spirit of this contribution, to conclude with some comments in Sec. 6.

A word of apology may be in order. Given the purpose with which this contribution is written, no serious attempt has been made to provide a comprehensive list of references to the literature in which topics are studied that are also related to the present discussion. Only a few, and hopefully useful "entry points" are provided. We apologize to any author who should feel that his/her contribution has been unduly left unmentioned.

2. Pure Yang–Mills Dynamics

2.1. *The Action Principle*

For the sake of the argument, let us consider flat 4-dimensional Minkowski spacetime (and more generally a D-dimensional spacetime) with a metric of signature $\eta_{\mu\nu} = \mathrm{diag}\,(-+++)$ ($\mu = 0, 1, \cdots, D-1$ and $x^\mu = (ct, \vec{x}\,)$), even though most of the discussion hereafter readily extends to a space of arbitrary topology and geometry. Furthermore, in order to bring to the fore important topological properties, let us also compactify the space coordinates into a d-torus T_d, with $d = D-1$, giving spacetime the topology $\mathbb{R}\times T_d$, keeping in mind that at the very end a decompactification limit must be applied to final results. Even though such a compactification breaks manifest full Poincaré covariance at intermediate stages, the purpose of such a procedure is twofold. On the one hand, since quantisation will be considered following the Hamiltonian operator quantisation path,[12,13] full Poincaré covariance is not manifest anyway. And on the other hand, such a compactification renders the normalisation issue of quantum states and operators better behaved, momentum eigenstates being then labelled by a discrete, rather than a continuous set of indices, while infra-red divergences associated to massless excitations of fields are then avoided altogether. The choice of a torus topology is made for ease of calculation, since space(time) then remains a flat manifold invariant under translations, even though the full Lorentz group is then broken down entirely or, at best, to a discrete finite rotation subgroup depending on the shape of the spatial torus T_d.

Consider now an arbitrary simple compact Lie group G with hermitian generators T^a and Lie algebra $[T^a, T^b] = if^{abc}T^c$, f^{abc} being the corresponding structure constants. As is well-known,[13] associated to this algebraic structure, one may introduce a Yang–Mills (YM) field A^a_μ $(\mu = 0, 1, \cdots, d)$ and its field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc}A^b_\mu A^c_\nu$, both varying under spacetime dependent local gauge transformations such that

$$\begin{aligned} A^{a\prime}_\mu(x)T^a &= U(x)A^a_\mu(x)T^aU^{-1}(x) \, + \, i\partial_\mu U(x)U^{-1}(x) \ , \\ F^{a\prime}_{\mu\nu}(x)T^a &= U(x)F^a_{\mu\nu}(x)T^aU^{-1}(x) \ , \end{aligned} \tag{1}$$

where $U(x) = e^{i\theta^a(x)T^a}$ are arbitrary G-valued continuous functions of the spacetime coordinate $x^\mu = (ct, \vec{x}\,)$.

A gauge invariant dynamics of this pure YM theory is defined by the Lagrangian density,[a]

$$\mathcal{L}^{D=4}_{\rm YM} = -\frac{1}{4e^2}F^a_{\mu\nu}F^{a\mu\nu} \, + \, \frac{1}{4}\theta\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \ . \tag{2}$$

Note well that in the r.h.s. of this expression, the first contribution requires a spacetime metric structure to raise and lower the spacetime indices μ and ν, whereas the second contribution is specific to $D = 4$ dimensions, is independent of any metric structure over spacetime, and is topological in character, corresponding to a Pontryagin/Chern class. As a matter of fact, this last term does not contribute at all to the classical equations of motion,

[a]Note that in this context, it is only the parameter e^2 which is relevant, so that the sign of the gauge coupling constant e is irrelevant for a pure Yang–Mills theory.

being a local surface term, namely the local divergence of the Chern-Simons 3-form,

$$\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}=\partial_\mu\left\{\epsilon^{\mu\nu\rho\sigma}\left[A^a_\nu F^a_{\rho\sigma}+\frac{1}{3}f^{abc}A^a_\nu A^b_\rho A^c_\sigma\right]\right\}\ . \tag{3}$$

Nevertheless, in the presence of a nontrivial spacetime topology, and especially at the quantum level, such a topological contribution to the action has important physical consequences for gauge field configurations of nontrivial topology. For instance, it suffices to mention in this context the role of instanton configurations in an euclidean spacetime formulation of the theory, directly connected to the possibility of this topological term which, as a matter of fact, also measures the instanton winding number, indeed a topological invariant.

In other spacetime dimensions, besides the $F^a_{\mu\nu}F^{a\mu\nu}$ contribution to the above Lagrangian density common to all cases, analogous θ-topological terms are also possible. In 1+1 dimensions, one may consider

$$\frac{1}{2}\theta\,\mathrm{Tr}\,\epsilon^{\mu\nu}F^a_{\mu\nu}T^a\ , \tag{4}$$

leading to a nonvanishing contribution for gauge groups G including U(1) factors. In 2+1 dimensions, one may add the Chern-Simons density itself,

$$\frac{1}{2}\theta\epsilon^{\mu\nu\rho}\left[A^a_\mu F^a_{\nu\rho}+\frac{1}{3}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right]\ . \tag{5}$$

And similarly in still higher dimensions of spacetime.

Note that in the above definition of an action principle, and in comparison with the usual expression for the field strength and its squared-contribution $F^a_{\mu\nu}F^{a\mu\nu}$ in the action, one has performed a rescaling of the gauge field A^a_μ by the gauge coupling constant e, namely $A^a_\mu=e\tilde{A}^a_\mu$ with $\tilde{A}^a_\mu$ denoting the usual field parametrisation used in ordinary perturbation theory. Such a rescaling is quite natural in the context of a pure YM theory in the absence of any coupling to some matter fields. In terms of the specific limits in the gauge coupling constant to be discussed presently, such a rescaling is a nontrivial feature, which enables one to explore nonperturbative aspects of this dynamics from a potentially novel point of view.

For instance, considering the 4-dimensional action (2), it appears obvious that the limit $e\to 0$ requires one to restrict to the sector of field configurations of vanishing field strength,

$$e\to 0\ :\quad F^a_{\mu\nu}=0\ . \tag{6}$$

The classification of solutions to these equations is topological in character, with a set of configurations directly dependent on the choice of space(time) topology. In the case of an abelian gauge group for instance, such configurations may be thought of as corresponding to Aharonov-Bohm magnetic flux lines threading the different 1-cycles associated to the d-dimensional spatial torus. In other words, the limit of a vanishing gauge coupling constant should lead to a topological field theory, as will indeed be made more explicit hereafter, showing that the ensuing theory is a so-called TQFT of BF type.[4] However, note that in terms of the perturbative field $\tilde{A}^A_\mu$, given configurations of finite A^a_μ values the limit $e \to 0$ corresponds to a limit of arbitrarily large values for $\tilde{A}^a_\mu$, namely a nonperturbative regime in terms of the usual parametrisation of field configurations at the basis of ordinary perturbation theory.

Likewise, the other extreme limit of an infinite gauge coupling constant, $e \to \infty$, leads to a purely topological field theory of which the action principle is metric independent and purely of topological character, reducing to the θ-topological contribution. In the case of the 4-dimensional theory (2), one is simply left with the instanton density, precisely the system used by Witten as the TQFT related to Donaldon's invariants for 4-dimensional differentiable manifolds, leading to (anti)self-dual instanton configurations of the associated YM theory.[1,14] Note again that the limit $e \to \infty$ requires vanishing $\tilde{A}^a_\mu$ configurations in order to maintain finite the A^a_μ ones. For spacetimes of other dimensions, in a similar manner, the limit $e \to \infty$ leads to the associated θ-topological density, multiplied by a free parameter θ. However, in the topological limit, often this free parameter requires to take on a quantised value in order to maintain a nonempty set of gauge invariant physical states when proper account is given of the large components of the group of local gauge transformations, the so-called modular group.

Hence, the two extreme limits in the gauge coupling constant lead indeed to TQFT's of interest and relevance. In the physical sectors of gauge invariant quantum states of definite mass and energy, one should also expect that with either limit, some of these states become infinitely massive and thus decouple, whereas the remainder become massless. Indeed, in either limit leading to a TQFT, the associated Hamiltonian must vanish identically for gauge invariant states, since TQFT's are also reparametrisation invariant theories in the spacetime coordinates — being purely topological in the gauge invariant sector — so that states should also be independent of the time coordinate along which dynamical evolution is generated by the Hamiltonian. As is well known, typically, monopole masses vary according to the inverse of the gauge coupling constant squared, so that in either limit indeed one should expect that some states decouple by acquiring an infinite mass, while others survive by becoming perfectly massless, both types of behaviours being dual to one another through each of the two possible limits. Such a dual situation is encountered with YM theories possessing further supersymmetries, in which the roles of monopoles and instantons in the confinement and chiral symmetry breaking phenomena have been unravelled in an analytic way.[15,16]

This expectation thus also suggests an alternative to ordinary perturbation theory based on the trivial field configuration $\tilde{A}^a_\mu = 0$. Indeed, starting from the sector of gauge invariant states in either of the two topological limits, which are thus nonperturbative in character, one may turn on either the coupling constant e or its inverse $1/e$ and start constructing a perturbation theory based on nonperturbative background configurations which reduce to those of a TQFT in the associated limit. Hopefully, the two limits could then possibly be merged at intermediate coupling values, shedding new light, in a nonperturbative manner, on the dynamical issues raised by the nonperturbative phenomena at the origin of confinement and chiral symmetry breaking in strongly interaction theories such as quantum chromodynamics (QCD). Note well than such a procedure relies from the outset on a sector of quantum states which is gauge invariant and physical, in contradistinction to ordinary perturbation theory based on the non gauge invariant states of the gluon and quark field degrees of freedom.

2.2. *The Hamiltonian First-Order Formulation*

As made explicit above, the limit $e \to \infty$ of pure YM theory leads to a TQFT of a type which is dependent on the spacetime dimension. In 1+1 dimensions, the Lagrangian density reduces purely to the abelian field strength $\theta\epsilon^{\mu\nu}F_{\mu\nu}$. In 2+1 dimensions, it reduces to a pure Chern-Simons dynamics of which the physics resides purely in the gauge field zero-modes.[3,5] In 3+1 dimensions, it should reduce to fields related to some (anti)self-dual instanton configurations.[1] Hence, the limit $e \to \infty$ must be explored for each of these dimensions separately.

On the other hand, in order to confirm that the limit $e \to 0$ leads to a TQFT of BF type, it is preferable to turn to the Hamiltonian formulation, ignoring for the time being the possible θ-topological contribution proportional to the parameter θ. This is a standard exercise in constrained dynamics,[12,13] the outcome of which is as follows, once the physical time coordinate $x^0 = ct$ is taken to be the evolution variable for the dynamics.

The time component of the gauge field, $A_0^a(t,\vec{x}\,)$, is the Lagrange multiplier for the generators of small gauge transformations, themselves represented through first-class constraints expressing Gauss' law,

$$\phi^a = D_i\pi^{ai} = \partial_i\pi^{ai} - f^{abc}A_i^b\pi^{ci} = 0\ , \tag{7}$$

where $\pi^{ai}(ct,\vec{x}\,)$ are the momenta conjugate to the space components $A_i^a(ct,\vec{x}\,)$ of the gauge field, obeying the canonical Poisson brackets,

$$\left\{A_i^a(ct,\vec{x}\,)\,,\,\pi^{bj}(ct,\vec{y}\,)\right\} = \delta^{ab}\delta_i^j\,\delta^{(d)}(\vec{x}-\vec{y}\,)\ , \tag{8}$$

and D_i denoting the usual gauge covariant derivative, defined here to act in the adjoint representation of the gauge group G. Up to some factor, π^{ai} defines the chromoelectric field. The first-order (gauge invariant) Hamiltonian density then writes as

$$\mathcal{H} = \frac{1}{2}e^2\pi^{ai}\,\pi^{ai}\ +\ \frac{1}{4e^2}F_{ij}^aF^{aij}\ , \tag{9}$$

so that finally the first-order Lagrangian density of the Hamiltonian formulation of the system is given as

$$\mathcal{L}_1 = \partial_0A_i^a\,\pi^{ai}\ -\ \frac{1}{2}e^2\pi^{ai}\pi^{ai}\ -\ \frac{1}{4e^2}F_{ij}^aF^{aij}\ +\ A_0^a\left(D_i\pi^{ai}\right)\ . \tag{10}$$

From this point of view, the conjugate momenta fields π^{ai} appear simply as auxiliary gaussian fields for which the equations of motion read

$$\pi^{ai} = \frac{1}{e^2}F_{0i}^a\ . \tag{11}$$

By direct substitution back in (10), one recovers the original Lagrangian action of the system. However, this remark may also suggest to introduce further auxiliary gaussian fields this time associated to the chromomagnetic sector $F_{ij}^a = \partial_iA_j^a - \partial_jA_i^a - f^{abc}A_i^bA_j^c$, in order to linearise the chromomagnetic energy contribution to the Hamiltonian density, namely

$$\frac{1}{4}e^2\phi_{ij}^a\phi^{aij} + \frac{1}{2}\phi_{ij}^aF_{ij}^a = \frac{1}{4}e^2\left(\phi_{ij}^a + \frac{1}{e^2}F_{ij}^a\right)^2 - \frac{1}{4e^2}F_{ij}^aF^{aij}\ , \tag{12}$$

thus with the equation of motion

$$\phi_{ij}^a = -\frac{1}{e^2}F_{ij}^a\ . \tag{13}$$

Consequently, combining the fields π^{ai} and ϕ_{ij}^a into a covariant tensor $\phi^{a\mu\nu}$ transforming in the adjoint representation of the gauge group, with the mixed time-space components $\phi^{a0i} = \pi^{ai}$, the complete Hamiltonian first-order action acquires the manifestly covariant form

$$\mathcal{L}_2 = \frac{1}{2}\phi^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{4}e^2\phi^a_{\mu\nu}\phi^{a\mu\nu} \ , \tag{14}$$

where in the second contribution on the r.h.s. the spacetime metric is to be used to lower the spacetime indices of the antisymmetric field $\phi^{a\mu\nu}$.

Finally, performing a spacetime duality transformation exchanging the chromo-electric and -magnetic sectors of the dynamics,

$$\phi^{a\mu\nu} = \frac{1}{(d-1)!}\epsilon^{\mu\nu\mu_1\mu_2\cdots\mu_{d-1}}\,\phi^a_{\mu_1\mu_2\cdots\mu_{d-1}} \ , \tag{15}$$

in terms of a totally antisymmetric tensor field $\phi^a_{\mu_1\mu_2\cdots\mu_{d-1}}$ transforming in the adjoint representation of the gauge group, the original YM action is equivalent to the following first-order action density in covariant Hamiltonian form,

$$\begin{aligned}\mathcal{L}_3 = &\tfrac{1}{2!(d-1)!}\epsilon^{\mu_1\mu_2\cdots\mu_{d-1}\mu\nu}\phi^a_{\mu_1\mu_2\cdots\mu_{d-1}}\,F^a_{\mu\nu} + \\ &+\tfrac{(-1)^d}{2!(d-1)!}e^2\phi^a_{\mu_1\mu_2\cdots\mu_{d-1}}\phi^{a\mu_1\mu_2\cdots\mu_{d-1}} \ .\end{aligned} \tag{16}$$

It should be emphasized that in this expression, the fields A^a_μ, in terms of which $F^a_{\mu\nu}$ is defined, and $\phi^a_{\mu_1\cdots\mu_{d-1}}$ are independent. However, it is clear that the latter field is an auxiliary gaussian one. The variational principle sets this field proportional to the spacetime dual of the field strength $F^a_{\mu\nu}$, whereupon substitution into the Lagrangian density $\mathcal{L}_3$ one immediately recovers the original YM Lagrangian density.

However, the form (16) of the dynamics makes a few interesting facts explicit. Note that the first contribution in the r.h.s. of (16) does not involve the spacetime metric structure. This term is purely topological in character, and indeed defines the TQFT of so-called BF type (the B field being in this instance the $\phi^a_{\mu_1\cdots\mu_{d-1}}$ one).[17] The second contribution on the r.h.s. of (16) combines both the gauge coupling constant e with the spacetime metric required to raise the spacetime indices of the antisymmetric field $\phi^a_{\mu_1\cdots\mu_{d-1}}$.[b] Again, this very fact hints at the possibility that geometry — namely, the essence of the gravitational interaction — acquires its full physical meaning only in the presence of the other (gauge) interactions, whereas a theory of pure quantum gravity need not necessarily be

[b]Had one from the outset coupled the YM action to a curved spacetime metric, the same remark would have been of application.

a quantum dynamics of geometry but possibly only a theory of quantum topology, namely a TQFT.[1,2,6] Finally, note that by having introduced the field $\phi^a_{\mu_1\cdots\mu_{d-1}}$ dual to the field strength $F^a_{\mu\nu}$, the limit $e \to 0$ is now well defined, and indeed reduces solely to the BF TQFT term,[17] confirming the previous expectation based on a formal argument. In particular, in that limit, the equations of motion of the BF theory are such that

$$F^a_{\mu\nu} = 0 \quad , \quad \epsilon^{\mu\nu\mu_1\mu_2\cdots\mu_{d-1}}\, D_\nu \phi^a_{\mu_1\mu_2\cdots\mu_{d-1}} = 0 \ , \tag{17}$$

thus including the condition of vanishing field strengths in the gauge sector.

2.3. *The Topological Quantum Field Theory Limits*

Given the above considerations, the two topological limits of YM dynamics identifying TQFT sectors are best characterised as follows. For what concerns the $e \to \infty$ limit, the original Lagrangian density formulation is best suited, showing that only the θ-topological contribution related to Pontryagin/Chern-Simons densities of given order, depending on the spacetime dimension, survives in this limit, generally also with a further quantisation restriction on the normalisation factor of such topological actions, dependent on the spacetime topology itself.

For what concerns the $e \to 0$ limit, the first-order Hamiltonian formulation is better suited, in terms of the original gauge field A^a_μ and its field strength $F^a_{\mu\nu}$ together with the $(D-2)$-form $\phi^a_{\mu_1\cdots\mu_{D-2}}$ transforming linearly in the adjoint representation of the gauge group. The topological limit then leads to a TQFT of BF type.

Each of these two possible limits thus identifies a topological sector of gauge invariant physical states of YM dynamics. Possibly, these states could serve as a starting point for a manifestly gauge invariant perturbation expansion of the full YM Hamiltonian above a nonperturbative set of physical states. At this stage, we leave this question open, as a possible avenue for further investigation of such an approach to the nonperturbative dynamics even of pure YM theories.

3. Generalised Pure Yang–Mills Dynamics

Besides the original motivation for the present programme, the above rewriting of pure YM dynamics in first-order form suggests possible generalisations of the usual YM theories, which may well deserve further study, at least in the cases of 2+1 and 3+1 dimensions.

Let us first consider the case of 1+1 dimensions. The antisymmetric field then reduces to a single scalar field in the adjoint representation, so that the first-order Lagrangian density reads,

$$\mathcal{L}^{1+1}_{\rm total}=\frac{1}{2}\epsilon^{\mu\nu}\phi^a F^a_{\mu\nu}-\frac{1}{2}e^2\phi^a\phi^a+\frac{1}{2}N\epsilon^{\mu\nu}{\rm Tr}\,F^a_{\mu\nu}T^a\ . \tag{18}$$

Clearly, the quadratic invariant term $\phi^a\phi^a$ may be extended to an arbitrary G-invariant potential for the field ϕ^a, hence leading to generalised pure YM dynamics in 1+1 dimensions,[18]

$$\mathcal{L}^{1+1}_{\rm total}=\frac{1}{2}\epsilon^{\mu\nu}\phi^a F^a_{\mu\nu}-V(\phi^a)+\frac{1}{2}N\epsilon^{\mu\nu}{\rm Tr}\,F^a_{\mu\nu}T^a\ . \tag{19}$$

It is only in the case of a quadratic potential, associated to the quadratic Casimir invariant of the gauge group G, that the field ϕ^a is gaussian and may easily be integrated out. Clearly, for other choices of the potential contribution, the physics of the system is different, as may be seen already from a different energy eigenspectrum. Note also that by extending these considerations to noncompact gauge groups,[19] one could possibly design theories for pure quantum gravity in 1+1 dimensions possessing a topological sector. Let us also recall that pure YM dynamics in 1+1 dimensions is an example of a totally integrable dynamics,[20] which has not yet yielded all its riches though.[21]

With the situation in 2+1 dimensions, there appear already further possibilities, of which the consequences may well be worth exploring. The antisymmetric field is then a vector field ϕ^a_μ, hence with the YM dynamics,

$$\mathcal{L}^{2+1}_{\rm total}=\frac{1}{2}\epsilon^{\mu\nu\rho}\phi^a_\mu F^a_{\nu\rho}+\frac{1}{2}e^2\phi^a_\mu\phi^{a\mu}+\frac{1}{2}N\epsilon^{\mu\nu\rho}A^a_\mu\left[F^a_{\nu\rho}+\frac{1}{3}f^{abc}A^b_\nu A^c_\rho\right]\ . \tag{20}$$

Further topological terms may then be added to this action, such as

$$\epsilon^{\mu\nu\rho}{\rm Tr}\,\phi_\mu\phi_\nu\phi_\rho\quad,\quad\epsilon^{\mu\nu\rho}{\rm Tr}\,\phi_\mu D_\nu\phi_\rho\ , \tag{21}$$

while the metric-dependent quadratic contribution in the vector field ϕ^a_μ may be generalised to an arbitrary G-invariant potential $V(\phi^a_\mu\phi^{a\mu})$.

As a matter of fact, the case of pure gravity in 2+1 dimensions is given by the sole θ-topological action, namely the Chern-Simons density, for a specific noncompact gauge group depending on the sign of the cosmological constant.[6] The above generalised action could thus provide extensions to the quantum dynamics of pure quantum gravity in 2+1 dimensions, a topic in itself worth exploring including its eventual physical interpretation.

Finally, one has the situation in 3+1 dimensions, with the antisymmetric field $\phi^a_{\mu\nu}$ in the adjoint G-representation. The original action reads, in first-order form,

$$\mathcal{L}^{3+1}_{\rm total}=\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\phi^a_{\mu\nu}F^a_{\rho\sigma}-\frac{1}{4}e^2\phi^a_{\mu\nu}\phi^{a\mu\nu}+\frac{1}{4}N\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\ . \tag{22}$$

Generalisations of this dynamics include the possibility to add further topological terms, such as

$$\epsilon^{\mu\nu\rho\sigma}\phi^a_{\mu\nu}\phi^a_{\rho\sigma}\ , \tag{23}$$

as well as metric-dependent terms extending the quadratic contribution of the antisymmetric field, such as

$$\phi^a_{\mu\nu}F^{a\mu\nu}\quad ,\quad F^a_{\mu\nu}F^{a\mu\nu}\quad ,\quad V(\phi^a_{\mu\nu}\phi^{a\mu\nu})\ . \tag{24}$$

Such possibilities may well be of interest from the point of view of particle phenomenology. Indeed, such generalisations of pure YM dynamics extend the dynamics of the gluons of QCD with further interactions and contributions accompanied by their own coupling parameters. Hence, such generalised YM theories provide generalised QCD theories amenable to experimental tests. In particular, the whole programme of renormalisation theory, including a calculation of the β-functions for the relevant coupling constants and operators including the gauge coupling constant, and how they could possibly affect quark and gluon distribution functions in hadrons and jet production distributions, could then be subjected to experimental tests and constraints, helping in narrowing down the actual relevance of ordinary QCD dynamics to the world of particle physics. One of the uneasy aspects to QCD is that there is no serious contender in competition to it, against which to assess the merits and discriminate the differences and difficulties of each theory, in confrontation with the experimental facts. With the above generalised YM dynamics in 3+1 dimensions may well reside such a possibility, thus worth exploring further.

As a last remark, one may wonder how to define similar considerations for quantum mechanical systems, namely theories defined in 0+1 dimensions. The most efficient procedure is to consider the dimensional reduction from 1+1 dimensions to 0+1 dimensions, namely consider the 1+1 dimensional case with field degrees of freedom of which the space dependence is turned off. Consequently, one has the degrees of freedom $\phi^a(t)$ together with coordinates $q^a(t)$ corresponding to the space component of the original gauge field $A^a_\mu(t,x)$ taken to be x-independent, for which the dynamics is governed by the Lagrange function,

$$L^{0+1}_{\rm total} = \phi^a D_t q^a\ -\ \frac{1}{2}e^2\phi^a\phi^a + N{\rm Tr}\,D_t q^a T^a\ , \tag{25}$$

where D_t denotes the usual gauge covariant derivative in the time direction, $D_t\cdot\ = \partial_t + i[A^a_0(t)T^a,\cdot]$, in the adjoint G-representation. Note that here as well the term quadratic in ϕ^a may be generalised to an arbitrary G-invariant potential $V(\phi^a)$. Furthermore, for reasons detailed hereafter in the abelian case, the coordinates $q^a(t)$ are restricted to a specific torus topology depending on the considered gauge group, as a consequence of large gauge transformations in the original 1+1 dimensional theory. Finally, in 0+1 dimensions, an arbitrary G-invariant potential in the coordinates $q^a(t)$ may also be added to the above action.[24,25]

For the sake of illustration, as simple examples of topological theories we shall consider hereafter systems of the above type, in limits equivalent to the limits discussed above in terms of the gauge coupling constant e. Furthermore, we shall restrict to the abelian U(1) gauge group only with a compact topology.

4. The Abelian Case in 0+1 Dimensions

As the simplest illustration of the general discussion, let us turn to the case of a free particle of mass m moving on a circle of radius R, described by the following Lagrange function,

$$L = \frac{1}{2} m \dot{q}^2 + N \dot{q} \ , \tag{26}$$

where $q(t)$ is the particle coordinate, with values identified modulo integer multiples of $L = 2\pi R$, and N an arbitrary constant. The second term on the r.h.s. of this expression is a total time derivative, *i.e.*, a θ-topological contribution, and thus does not affect the classical equations of motion. However at the quantum level, because of the nontrivial circle topology of the configuration space, the presence of such a term leads to specific physical consequences. Furthermore, in the limit that $m = 0$, even though the Lagrange reduces to a simple total time derivative, hence leading to a trivial equation of motion, $0 = 0$, still at the quantum level one has a nontrivial quantum situation, albeit then a purely topological quantum system. Finally, note that the arbitrary normalisation constant N of the topological term $N\dot{q}$ is physically tantamount to having a constant U(1) vector potential $A(q)$ present on the circle and to which the particle would be magnetically coupled.

In terms of the systems discussed previously, it may readily be seen that (26) is equivalent to the dynamics described by (25) in the case of the abelian U(1) gauge group, and once the auxiliary degree of freedom $\phi(t)$ is integrated out, with the mass parameter m corresponding to the inverse gauge coupling constant squared, $m = 1/e^2$. The restriction to the circle topology for the coordinate $q(t)$ stems from the dimensional reduction of the compact U(1) gauge field A_μ, as will become clear in the next section. Indeed, the modular group or group of large U(1) gauge transformations effectively restricts the configuration space of the constant mode of the space component of the gauge field — indeed the coordinate $q(t)$ — to the circle topology.

The first-order Hamiltonian formulation of the system is straightforward. The associated action principle writes as

$$S[q,p] = \int dt\, \left[\dot{q}p - \frac{1}{2m}\,(p-N)^2\right]\ , \tag{27}$$

which is indeed equivalent in form to (25). Here, $p(t)$ denotes the momentum conjugate to $q(t)$, with the canonical Poisson bracket $\{q(t),p(t)\}=1$.

The canonical quantisation of this system is immediate, provided, however, one takes due account of the circle topology of the configuration space $q(t)$. As discussed in Refs. 22 and 23, unitarily inequivalent representations of the Heisenberg algebra $[\hat{q},\hat{p}]=i\hbar$ on the circle are distinguished by a U(1) holonomy parameter λ defined modulo the integers, $0\le\lambda<1$, contributing to the $\hat{p}$-eigenspectrum which itself is discrete given the compact configuration space in $q(t)$. Hence, a basis of Hilbert space is spanned by all orthonormalised $\hat{p}$-eigenstates $|n\rangle$ ($n\in\mathbb{Z}$),

$$\hat{p}|n\rangle = \frac{\hbar}{R}\,(n+\lambda)\quad,\quad \langle n|m\rangle = \delta_{nm}\ . \tag{28}$$

In the configuration space representation with $0\le q<2\pi R$, one has,

$$\begin{gathered}\hat{q}|q\rangle = q|q\rangle\quad,\quad \langle q|q'\rangle=\delta(q-q')\quad,\quad \langle q|n\rangle = \tfrac{1}{\sqrt{2\pi R}}e^{inq/R}\ ,\\ \langle q|\hat{q}|\psi\rangle = q\langle q|\psi\rangle\quad,\quad \langle q|\hat{p}|\psi\rangle = \left[-i\hbar\tfrac{d}{dq}+\tfrac{\hbar}{R}\lambda\right]\langle q|\psi\rangle\ .\end{gathered} \tag{29}$$

Consequently, the energy spectrum is given by

$$\hat{H}|n\rangle = \frac{1}{2m}\left[\hat{p}-N\right]^2|n\rangle = E_n|n\rangle\quad,\quad E_n=\frac{\hbar^2}{2mR^2}\left[(n+\lambda)-\frac{R}{\hbar}N\right]^2\ . \tag{30}$$

Note the spectral-flow properties of this parabolic spectrum as a function of the holonomy parameter λ defined modulo integer values. Furthermore, this parameter combines with the θ-topological term normalisation constant N into a unique contribution. Both quantities, even though combined in this manner, thus indeed have physical consequences at the quantum level, as they, for instance, affect the physical spectrum of the system. This feature is of purely topological origin, and is made possible by the circle topology of the q-configuration space. All the nontrivial effects of the circle topology of configuration space contribute through both the holonomy parameter λ and the θ-parameter N combined in the above fashion. This is a general feature which survives for systems in higher dimensions. In contradistinction in the decompactification limit $R \to \infty$, the spectrum becomes independent of both λ and N, as indeed it should since the Heisenberg algebra on the real line possesses only a single representation, leading back to the usual system of a free particle on the real line.

The above results may also be used to consider the limits $e \to 0$ and $e \to \infty$ discussed previously, this time in terms of the mass parameter m which determines the parabolic curvature of the parabolic energy spectrum (30). In the limit that $m \to \infty$, or $e \to 0$, the parabola degenerates into a flat spectrum, leaving over all the states in Hilbert space, but with a vanishing energy. This is the static limit of the particle model.

The limit $m \to 0$ or $e \to \infty$ is that of a purely topological particle on the circle. All states acquire then an infinite energy and decouple, except at most for one state $|n_0\rangle$ provide the combination of constants N and λ (which could both depend on m) is chosen such that

$$\lim_{m\to 0} N(m) = N_0 = \frac{\hbar}{R}(n_0 + \lambda_0) \ . \tag{31}$$

In other words, the normalisation of the action for the topological particle needs to be quantised according to this rule, in order to obtain a nonempty spectrum of states. In the present case, this spectrum of states is one dimensional, which is a characteristic feature of TQFT's. Indeed, generally even in quantum mechanics (think of the harmonic oscillator for instance), the space of quantum states is infinite dimensional, albeit sometimes discrete, while for TQFT's it is often finite dimensional, meaning that the physical classical phase space itself is then also compact (which is indeed the case for our example in the limit $m = 0$).

As a matter of fact, the topological particle is a constrained system,[12,13] with the constraint

$$p(t) - N = 0 \ . \tag{32}$$

The gauge invariance associated to this constraint is that of arbitrary "field" redefinitions of the coordinate $q(t)$. Quantum physical states should thus be annihilated by this constraint, hence for the present case indeed leading to a single state in Hilbert space, provided the normalisation constant N obeys the quantisation condition (31). Note however that these features directly related to the nontrivial topology of configuration space, and implying a nontrivial physical content associated to an action which is a pure total derivative, are totally "dissolved" away in the decompactification limit. Indeed, the normalisation factor N then needs to vanish altogether, leaving over a trivially vanishing action for the system. The topology of the real line being trivial, the TQFT of the topological particle does not provide any information. However, for a circle topology, the topological particle leads to a nontrivial quantum system. As a matter of fact, which shall not be detailed further here, given the circle topology, namely precisely the U(1) group manifold, all the possible quantum realisations of the topological particle are in one-to-one correspondence with the irreducible representations of the U(1) group, indeed labelled by the integers n_0 specifying the single physical state surviving in the TQFT limit $m = 0$. The same conclusion applies to the topological particle defined over general compact Lie group manifolds and their compact cosets.[26]

5. The Abelian Case in 1+1 Dimensions

As another simple illustration of our general programme, let us now consider the pure U(1) gauge theory in 1+1 dimensions, defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}N\epsilon^{\mu\nu}F_{\mu\nu}\ , \tag{33}$$

or equivalently

$$\mathcal{L} = \frac{1}{2}\epsilon^{\mu\nu}\phi F_{\mu\nu} - \frac{1}{2}e^2\phi^2 + \frac{1}{2}N\epsilon^{\mu\nu}F_{\mu\nu} = \phi F_{01} - \frac{1}{2}e^2\phi^2 + NF_{01}\ , \tag{34}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ . \tag{35}$$

The space coordinate x is of course assumed to be compactified into a circle S of radius R and circumference $L = 2\pi R$.

By construction, this system is invariant under local compact U(1) gauge transformations, of the form,

$$A'_\mu(t,x) = A_\mu(t,x) + \partial_\mu\theta(t,x) = A_\mu(t,x) + \partial_\mu\theta_0(t,x) + \sum_{k\in\mathbb{Z}}\frac{k}{R}\delta_{\mu,1}\ , \tag{36}$$

where the U(1) phase parameter $\theta(t,x)$ of the transformation includes both a contribution $\theta_0(t,x)$ periodic on the circle in space, associated to small gauge transformations, as well as the large gauge transformation component corresponding to the $\mathbb{Z}$ modular group of topologically nontrivial U(1) transformations on the circle distinguished by the winding number $k \in \mathbb{Z}$,

$$\theta(t,x) = \theta_0(t,x) + \sum_{k\in\mathbb{Z}} k\frac{x}{R} \quad , \quad \theta_0(t,x+2\pi R) = \theta_0(t,x) \ , \tag{37}$$

such that $e^{i\theta(t,x)}$ be single-valued on the considered spacetime topology $\mathbb{R}\times S$.

In explicit form, the first-order action density reads, after integrating by parts in x and assuming the fields $A_\mu(t,x)$ and $\phi(t,x)$ to be $2\pi R$-periodic in x,

$$\mathcal{L} = \partial_t A_1(\phi+N) - \frac{1}{2}e^2(\phi+N-N)^2 + A_0\partial_x(\phi+N) \ . \tag{38}$$

Consequently, it follows that $A_1(t,x)$ and $\phi(t,x)$ are canonically conjugate degrees of freedom, with the Poisson brackets $\{A_1(t,x),\phi(t,y)\} = \delta(x-y)$, while one has the first-class Hamiltonian density $\mathcal{H} = e^2\phi^2/2$ and first-class constraint $\partial_x\phi = 0$, which generates the small U(1) gauge transformations. The time component $A_0(t,x)$ appears to be nothing but the Lagrange multiplier for the first-class constraint.

Rather than addressing the full-fledged quantisation of this system, let us consider the short-cut approach by first identifying the gauge invariant degrees of freedom of the physical sector at the classical level. Given the gauge transformations in Hamiltonian form

$$A'_0 = A_0 + \partial_t\theta_0 \quad , \quad A'_1 = A_1 + \partial_x\theta_0 + \frac{2\pi k}{L} \quad , \quad \phi' = \phi \ , \tag{39}$$

it appears that whatever the configurations for $A_0(t,x)$ and $A_1(t,x)$, it is always possible to find a small gauge transformation $\theta_0(t,x)$ such that $A_1(t,x)$ is reduced to its space zero-mode, independent of x, namely $A_1(t,x) = \bar{A}_1(t)$, while the Lagrange multiplier may even be set to vanish, $A_0(t,x) = 0$. This exhausts the entire freedom in performing small gauge transformations parametrised by the x-periodic functions $\theta_0(t,x)$, However, under the large gauge transformation of homotopy class k, the A_1 zero-mode $\bar{A}_1(t)$ is still affected, being shifted by a k-multiple of $2\pi/L = 1/R$. In other words, invariance of the system under the modular group $\mathbb{Z}$ of large gauge transformations implies the circle topology for its physical configuration space $\bar{A}_1(t)$. Finally for what concerns the gauge invariant field $\phi(t,x)$, the first-class constraint of Gauss' law $\partial_x\phi = 0$ is solved by restricting this sector of degrees of freedom also to its space zero-mode $\bar{\phi}(t)$.

Consequently, at the classical level, the gauge invariant physical content of the system reduces to that of the zero-modes $L\bar{A}_1(t)$ and $\bar{\phi}(t)$, which are canonically conjugate to one another. Indeed, upon this reduction to the physical sector, the first-order Hamiltonian action reads,

$$S=\int dt\, L\left[\partial_t\bar{A}_1\bar{\phi}-\frac{1}{2}e^2\bar{\phi}^2+N\partial_t\bar{A}_1\right]\ , \tag{40}$$

in which it must be understood that the coordinate $L\bar{A}_1$ is to take its values in the circle of radius unity. In other words, the physics of pure U(1) abelian gauge theory in 1+1 dimensions, including a θ-topological contribution, is that of a massive particle on a circle including its θ-topological term. As a matter of fact, this conclusion remains valid even for a pure nonabelian YM theory in 1+1 dimensions.[25]

When comparing to the discussion of the previous section, we see that the identification is such that, for instance, $L\bar{A}_1(t)$ corresponds to $q(t)$, $(\bar{\phi}(t)+N)$ corresponds to $p(t)$, while the radius parameter is now given by $R=1$ with e^2L corresponding to $1/m$. Consequently, the spectrum of physical states reduces to the complete set of all states $|n\rangle$ such that $\hat{\phi}|n\rangle=[\hbar(n+\lambda)-N]|n\rangle$, while the physical energy spectrum of the pure U(1) gauge theory is given by

$$E_n=\frac{1}{2}e^2L\hbar^2\left[(n+\lambda)-\frac{N}{\hbar}\right]^2\ . \tag{41}$$

The $e\to 0$ limit is thus tantamount to the totally compactified limit $L\to 0$, leading to states all of vanishing energy, corresponding to a BF TQFT whose Hilbert space is spanned by all momentum eigenstates $|n\rangle$. In the other extreme limit $e\to\infty$, tantamount to the decompactification limit $L\to\infty$, the TQFT with a single quantum physical state $|n_0\rangle$ is recovered, provided however the θ-topological contribution to the action is normalised in such a manner that

$$\lim_{e\to\infty}N(e)=\hbar(n_0+\lambda_0)\ , \tag{42}$$

reproducing again the quantisation condition on this normalisation factor. Note also that the spatial circle in the coordinate x of circumference L implies the circular topology of radius $1/L$ for the modular space of the system defined by the zero-mode $\bar{A}_1(t)$. Thus, these extreme compactification or decompactification limits in physical space correspond to the opposite process in the actual physical configuration space of the system, namely its modular space spanned by the variable $\bar{A}_1(t)$. This feature is reminiscent of T-duality in string theory.[13] Could it be that in fact our physical space is the modular space of some unknown Yang–Mills theory living in some yet undiscovered and very much compactified physical universe?

Note that the above explicit and exact results illustrate a property expected on general grounds, and mentioned previously. Namely the fact that when the gauge coupling constant e varies between 0 and ∞, the spectrum of physical states includes states that either decouple by acquiring an infinite energy, or else degenerate to a vanishing energy. For the simplest systems discussed here, only one of these classes of states exists; there are no quantum states acquiring vanishing or infinite energy while some other subset behaves in the opposite fashion as the gauge coupling constant is varied. However, starting with systems in 2+1 dimensions, the latter behaviour is also observed.[15,16,27]

6. Conclusions

The main purpose of the present communication is to outline a general programme towards a study of the nonperturbative dynamics of Yang–Mills theories which could possibly be at the origin of the challenging problems of confinement and chiral symmetry breaking in theories such as quantum chromodynamics. The approach is motivated by the suggestion that this dynamics could be dominated by sectors of Yang–Mills dynamics which to first approximation are those of Topological Quantum Field Theories. Indeed, topological quantum effects in the configuration space of these systems have for long been suspected to play an essential role in these nonperturbative issues, through monopole and instanton solutions to the Yang–Mills equations. The present approach also presents the advantage of considering from the outset gauge invariant physical quantum states only, defining such nonperturbative topological configurations, in terms of which to set up a novel type of perturbation theory. The TQFT sectors that could be identified from the YM dynamics are reached in the two extreme limits of either a vanishing or an infinite gauge coupling constant.

Some of the features expected on general grounds could also be illustrated in the case of two simple examples of such systems, explicitly showing the importance of properly accounting for the topology of physical and configuration spaces in Yang–Mills dynamics.

In the course of the discussion a series of open and fascinating problems were also suggested, which could well prove to be worth exploring further. These problems span from the phenomenology of the strong interactions and their experimental study to even the quest for a final unification of all particles and interactions including a theory for quantum gravity. It is hoped that some participants to this Workshop will be sufficiently inspired by one or another of these open problems, to launch their own efforts with a curiosity-driven enthusiasm, and contribute to their resolution and to progress in our understanding of the fascinating and beautiful riches of our physical universe.

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