

Exponential moments and piecewise thinning for the Bessel point process

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Abstract

We obtain exponential moment asymptotics for the Bessel point process. As a direct consequence, we improve on the asymptotics for the expectation and variance of the associated counting function, and establish several central limit theorems. We show that exponential moment asymptotics can also be interpreted as large gap asymptotics, in the case where we apply the operation of a piecewise constant thinning on several consecutive intervals. We believe our results also provide important estimates for later studies of the global rigidity of the Bessel point process.

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1 Introduction and statement of results

A point process is a model for a collection of points randomly located on some underlying space Λ . Determinantal point processes [48, 5, 38] are characterized by the fact that their correlation functions have a determinantal structure. More precisely, if ρ_k denotes the k -th correlation function of a given point process X , then X is *determinantal* if there exists a function $K : \Lambda^2 \rightarrow \mathbb{R}$ such that for any $k > 0$, we have

$$\rho_k(x_1, \dots, x_k) = \det \left(K(x_i, x_j) \right)_{i,j=1}^k, \quad \text{for all } x_1, \dots, x_k \in \Lambda.$$

K encodes all the information about the process and is called the *correlation kernel* (or simply the *kernel*). Determinantal point processes appear in various domains of mathematics, such as random matrix theory, orthogonal polynomials and tiling models [22, 6]. These processes exhibit a rich analytic and algebraic structure, and several tools have been developed to study them.

Bessel point process. In this paper, we focus on the Bessel point process. This is a determinantal point process on $\mathbb{R}^+ := [0, +\infty)$ whose kernel is given by

$$K_\alpha^{\text{Be}}(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}, \quad \alpha > -1, \quad (1.1)$$

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where α is a parameter of the process which quantifies the attraction (if $\alpha < 0$) or repulsion (if $\alpha > 0$) between the particles and the origin, and J_α is the Bessel function of the first kind of order α .

The Bessel point process appears typically in repulsive particle systems, when the particles are accumulating along a natural boundary (called “hard edge”). This is one of the canonical point processes from the theory of random matrices. It encodes the behavior of the eigenvalues of certain positive definite matrices near the origin [29, 30]. This point process appears also in, among other applications, non-intersecting squared Bessel paths [40].

Given a Borel set $B \subseteq \mathbb{R}^+$, the *occupancy number* N_B is the random variable defined as the number of points that fall into B . Determinantal point processes are always locally finite, i.e. N_B is finite with probability 1 for B bounded. Moreover, all particles are distinct with probability 1.

Exponential moments and piecewise constant thinning. Let us introduce the parameters

$$m \in \mathbb{N}_{>0}, \quad \vec{s} = (s_1, \dots, s_m) \in \mathbb{C}^m \quad \text{and} \quad \vec{x} = (x_1, \dots, x_m) \in (\mathbb{R}^+)^m, \quad (1.2)$$

where $\vec{x}$ is such that $0 =: x_0 < x_1 < x_2 < \dots < x_m < +\infty$. We are interested in exponential moments of the form

$$F_\alpha(\vec{x}, \vec{s}) := \mathbb{E}_\alpha \left[\prod_{j=1}^m s_j^{N_{(x_{j-1}, x_j]}} \right] = \sum_{k_1, \dots, k_m \geq 0} \mathbb{P}_\alpha \left(\bigcap_{j=1}^m N_{(x_{j-1}, x_j]} = k_j \right) \prod_{j=1}^m s_j^{k_j}. \quad (1.3)$$

If $s_j = 0$ and $k_j = 0$ in (1.3) for a certain $j \in \{1, \dots, m\}$, then $s_j^{k_j}$ should be interpreted as being equal to 1. The function $F_\alpha(\vec{x}, \vec{s})$ is an entire function in $s_1, \dots, s_m$, and can be written as a Fredholm determinant with m discontinuities (by [48, Theorem 2]). This function is also known as the joint probability generating function of occupancy numbers on consecutive intervals, and contains a lot of information about the Bessel process: several probabilistic quantities can be deduced from it, such as the expectation and variance of $N_{(x_{j-1}, x_j]}$ (see also [17] for further applications). Here, we show that if $s_j \in [0, 1]$ for all $j = 1, \dots, m$, $F_\alpha(\vec{x}, \vec{s})$ admits another application related to thinning.

The operation of thinning is well-known in the theory of point processes, see e.g. [34]. It has been first studied in the context of random matrices by Bohigas and Pato [3, 4], and first set out rigorously in [7] (see also [8, 14, 10, 12, 9, 18] for further results). A constant thinning consists of removing each particle independently with the same probability $s \in [0, 1]$. As s increases, the level of correlation decreases, so the thinned point process interpolates between the original point process (when $s = 0$) and an uncorrelated process (when $s \rightarrow 1$ at a certain speed) [39]. Here, we consider a more general situation and apply a piecewise constant thinning as follows: for each $j \in \{1, 2, \dots, m\}$, each particle on the interval $[x_{j-1}, x_j]$ is removed with probability s_j . The probability to observe a gap on $(0, x_m)$ in this thinned point process is given by

$$\begin{aligned} \sum_{k_1, \dots, k_m \geq 0} \mathbb{P}_\alpha \left(\bigcap_{j=1}^m N_{[x_{j-1}, x_j]} = k_j \right) \prod_{j=1}^m \mathbb{P} \left(\begin{array}{l} \text{the } k_j \text{ points on } [x_{j-1}, x_j] \text{ have} \\ \text{been removed by the thinning} \end{array} \right) \\ = \sum_{k_1, \dots, k_m \geq 0} \mathbb{P}_\alpha \left(\bigcap_{j=1}^m N_{(x_{j-1}, x_j]} = k_j \right) \prod_{j=1}^m s_j^{k_j}, \end{aligned} \quad (1.4)$$

which is precisely $F_\alpha(\vec{x}, \vec{s})$.¹

¹To be precise, we have used the fact that the boundary of the successive intervals (x_{j-1}, x_j) carry particles with probability 0.

In [49], Tracy and Widom have studied $F_\alpha(x_1, s_1)$, i.e. the case $m = 1$. If $s_1 \in [0, 1]$, $F_\alpha(x_1, s_1)$ is the gap probability on $(0, x_1)$ in the constant thinned Bessel point process, with the thinning parameter given by s_1 . They expressed $F_\alpha(x_1, s_1)$ in terms of the solution of a Painlevé V equation. The generalization of this result for an arbitrary $m \geq 1$ was recently obtained in [17], where it is shown that $F_\alpha(\vec{x}, \vec{s})$ can be expressed identically in terms of a solution to a system of m coupled Painlevé V equations.

Exponential moment asymptotics. Let us now scale the size of the intervals with a new parameter $r > 0$, that is, we consider $F_\alpha(r\vec{x}, \vec{s})$. As r decreases and tends to 0, the intervals (rx_{j-1}, rx_j) shrink and asymptotics for $F_\alpha(r\vec{x}, \vec{s})$ can be obtained rather straightforwardly from its representation as a Fredholm series. In this paper, we address the harder problem of finding the so-called *exponential moment asymptotics*, which are asymptotics for $F_\alpha(r\vec{x}, \vec{s})$ as $r \rightarrow +\infty$. If $s_j \in [0, 1]$ for all $j = 1, \dots, m$, it follows from (1.4) that exponential moment asymptotics can be interpreted as *large gap asymptotics* in the piecewise thinned point process.

We also elaborate here on another main motivation to the study of exponential moment asymptotics. For a large class of point processes taken from the theory of random matrices, the fluctuations of an *individual* point around its typical position are well-understood [37, 32]. An important problem in recent years has been to estimate the *maximal* fluctuation of the points. A bound for the maximal fluctuation that holds true with high probability is called a *global rigidity bound*. The first results of this nature can be found in [28] (for an upper bound) and in [32] (for a lower bound). Sharp bounds have now been obtained for several point processes [1, 19, 33, 46, 42, 43, 20]. As it turns out, some of the techniques that have been used successfully to derive *optimal* rigidity bounds rely heavily on exponential moment asymptotics.

By studying asymptotics for $F_\alpha(r\vec{x}, \vec{s})$ as $r \rightarrow +\infty$, another main objective of this paper is precisely to provide important estimates for later studies of the global rigidity of the Bessel point process. We comment more on that in Remark 3 below. We should mention that, while the interpretation of $F_\alpha(r\vec{x}, \vec{s})$ as a gap probability only makes sense for $s_1, \dots, s_m \in [0, 1]$, the estimates needed to study the global rigidity require to consider the more general situation $s_1, \dots, s_m \in [0, +\infty)$.

Known results. For $m = 1$, the large r asymptotics of $F_\alpha(r\vec{x}, \vec{s}) = F_\alpha(rx_1, s_1)$ are already known. There are two distinct regimes: 1) $s_1 = 0$ and 2) $s_1 > 0$.

1) *The case $s_1 = 0$.* Using a connection between $F_\alpha(rx_1, 0)$ and a solution to the Painlevé V equation, Tracy and Widom [49] gave an heuristic derivation of the following

$$F_\alpha(rx_1, 0) = \tau_\alpha (rx_1)^{-\frac{\alpha^2}{4}} e^{-\frac{rx_1}{4} + \alpha\sqrt{rx_1}} \left(1 + \mathcal{O}(r^{-1/2})\right), \quad r \rightarrow +\infty, \quad (1.5)$$

for some constant τ_α . They also noted that for $\alpha = \mp \frac{1}{2}$, K_α^{Be} reduces to sine-kernels appearing in orthogonal and symplectic ensembles for which large gap asymptotics are known from the work of Dyson [26]. Using this observation and supported with numerical calculations, they conjectured that $\tau_\alpha = G(1 + \alpha)/(2\pi)^{\frac{\alpha}{2}}$, where G is Barnes' G -function. A proof of the asymptotics (1.5) (including the constant) was first given by Ehrhardt in [27] for $\alpha \in (-1, 1)$ using operator theory methods and finally for all values of $\alpha > -1$ by Deift, Krasovsky and Vasilevska in [21] by performing a Deift/Zhou [25] steepest descent on a Riemann-Hilbert (RH) problem.

2) *The case $s_1 = e^{u_1} > 0$.* Bothner, Its and Prokhorov in [11, Eq (1.35)] recently proved that

$$F_\alpha(rx_1, e^{u_1}) = G\left(1 + \frac{u_1}{2\pi i}\right) G\left(1 - \frac{u_1}{2\pi i}\right) e^{-\frac{\alpha}{2}u_1} (4\sqrt{rx_1})^{\frac{u_1^2}{4\pi^2}} e^{\frac{u_1}{\pi}\sqrt{rx_1}} \left(1 + \mathcal{O}(r^{-1/2})\right), \quad r \rightarrow +\infty, \quad (1.6)$$

uniformly for $s_1 = e^{u_1}$ in compact subsets of $(0, +\infty)$.

The main contribution of this paper is to obtain large r asymptotics for $F_\alpha(r\vec{x}, \vec{s})$ up to and including the constant term in two different situations: in Theorem 1.1 below, we assume $s_1, \dots, s_m \in (0, +\infty)$ and in Theorem 1.4, we assume $s_1 = 0$ and $s_2, \dots, s_m \in (0, +\infty)$. The cases $m = 2$ of Theorem 1.1 and $m = 2$ of Theorem 1.4 are already new results. These theorems contain a lot of information about the Bessel process, and we illustrate this by providing some direct consequences: in Corollaries 1.2 and 1.5, we obtain new asymptotics for the expectation, variance and covariance of the counting function, and in Corollaries 1.3 and 1.6, we establish several central limit theorems.

We mention that large gap asymptotics in the two cases of Theorems 1.1 and 1.4 cannot be treated both at once. In fact, a critical transition occurs as $s_1 \rightarrow 0$ and simultaneously $r \rightarrow +\infty$. This transition is expected to be described in terms of elliptic θ -functions and is not addressed in this paper. For a heuristic discussion, see Section 6.3.

Main results for $s_1 > 0$. Asymptotics for $F_\alpha(r\vec{x}, \vec{s})$ as $r \rightarrow +\infty$ with $\vec{s} = (s_1, \dots, s_m) \in (0, +\infty)^m$ are more elegantly described after making the following change of variables

$$u_j = \begin{cases} \log \frac{s_j}{s_{j+1}} & \text{for } j = 1, \dots, m-1, \\ \log s_m & \text{for } j = m. \end{cases} \quad (1.7)$$

Note that $u_j \in (-\infty, +\infty)$. Then, with the notation $\vec{u} = (u_1, \dots, u_m)$, we define

$$E_\alpha(\vec{x}, \vec{u}) := \mathbb{E}_\alpha \left[\prod_{j=1}^m e^{u_j N(0, x_j)} \right] = F_\alpha(\vec{x}, \vec{s}). \quad (1.8)$$

Theorem 1.1. *Let $\alpha > -1$, $m \in \mathbb{N}_{>0}$,*

$$\vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m \quad \text{and} \quad \vec{x} = (x_1, \dots, x_m) \in (\mathbb{R}^+)^m$$

be such that $0 < x_1 < x_2 < \dots < x_m < +\infty$. As $r \rightarrow +\infty$, we have

$$E_\alpha(r\vec{x}, \vec{u}) = \exp \left(\sum_{j=1}^m u_j \mu_\alpha(r x_j) + \sum_{j=1}^m \frac{u_j^2}{2} \sigma^2(r x_j) + \sum_{1 \leq j < k \leq m} u_j u_k \Sigma(x_k, x_j) + \sum_{j=1}^m \log G(1 + \frac{u_j}{2\pi i}) G(1 - \frac{u_j}{2\pi i}) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) \right), \quad (1.9)$$

where G is Barnes' G -function, and μ_α , σ^2 and Σ are given by

$$\mu_\alpha(x) = \frac{\sqrt{x}}{\pi} - \frac{\alpha}{2}, \quad \sigma^2(x) = \frac{\log(4\sqrt{x})}{2\pi^2}, \quad \Sigma(x_k, x_j) = \frac{1}{2\pi^2} \log \frac{\sqrt{x_k} + \sqrt{x_j}}{\sqrt{x_k} - \sqrt{x_j}}. \quad (1.10)$$

Alternatively, the asymptotics (1.9) can be rewritten as

$$E_\alpha(r\vec{x}, \vec{u}) = \exp \left(\sum_{1 \leq j < k \leq m} u_j u_k \Sigma(x_k, x_j) \right) \prod_{j=1}^m E_\alpha(r x_j, u_j) \left(1 + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) \right). \quad (1.11)$$

Furthermore, the error terms in (1.9) and (1.11) are uniform in $u_1, \dots, u_m$ in compact subsets of $(-\infty, +\infty)$ and uniform in $x_1, \dots, x_m$ in compact subsets of $(0, +\infty)$, as long as there exists $\delta > 0$ such that

$$\min_{1 \leq j < k \leq m} x_k - x_j \geq \delta. \quad (1.12)$$

Remark 1. The Barnes G -function is a well-known special function defined by

$$G(1+z) = (2\pi)^{z/2} \exp\left(-\frac{z+z^2(1+\gamma_E)}{2}\right) \prod_{k=1}^{\infty} \left[\left(1+\frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right)\right],$$

where $\gamma_E \approx 0.5772$ is the Euler's gamma constant. It is related to the Γ function by the functional relation $G(z+1) = \Gamma(z)G(z)$ (see [45] for further properties).

Remark 2. As can be seen by comparing (1.9) with (1.5), there is a notable difference between the cases $s_1 = 0$ and $s_1 > 0$ in the large r asymptotics of $\log F_\alpha(rx_1, s_1)$: the leading term is of order $\mathcal{O}(r)$ if $s_1 = 0$, while it is of order $\mathcal{O}(\sqrt{r})$ if s_1 is bounded away from 0. We also recall that $F_\alpha(rx_1, s_1)$ can be interpreted as a gap probability for $s_1 \in [0, 1]$. In this case, the difference between the leading terms means that it is significantly more likely to observe a large gap in the thinned Bessel point process (roughly $\sim e^{-c\sqrt{r}}$, $c > 0$) rather than in the unthinned Bessel point process ($\sim e^{-cr}$). From (1.3), for general values of $s_1 \in [0, +\infty)$ we have

$$F_\alpha(rx_1, s_1) = \sum_{k \geq 0} \mathbb{P}_\alpha(N_{(0, rx_1)} = k) s_1^k. \quad (1.13)$$

Using Theorem 1.1, it is possible to understand which terms in the sum contribute the most when s_1 is bounded away from 0. The first term is simply $\mathbb{P}_\alpha(N_{(0, rx_1)} = 0)$, which we know by (1.5) to be roughly $\sim e^{-cr}$ for large r . In fact, for typical realizations of the Bessel point process, the number of points lying on $(0, rx_1)$ is of order $\mathcal{O}(\sqrt{r})$ as $r \rightarrow +\infty$, see (1.15). Thus, when k is of order $\mathcal{O}(\sqrt{r})$, $\mathbb{P}_\alpha(N_{(0, rx_1)} = k)$ is of order $\mathcal{O}(1)$ and s_1^k is $\mathcal{O}(e^{-c\sqrt{r}})$. Hence by Theorem 1.1, for large r and for s_1 bounded away from 0, the dominant terms in (1.13) are those for which k is of order $\mathcal{O}(\sqrt{r})$.

Remark 3. By combining Theorem 1.1 with [16, Theorem 1.2], the following global rigidity upper bound for the Bessel point process has been established in [16]: for any $\epsilon > 0$, we have

$$\lim_{k_0 \rightarrow \infty} \mathbb{P} \left(\sup_{k \geq k_0} \frac{|\frac{1}{\pi} b_k^{1/2} - k|}{\log k} \leq \frac{1}{\pi} + \epsilon \right) = 1, \quad (1.14)$$

where b_k is the k -th smallest point in the Bessel point process. In fact, the proof of (1.14) uses only the first exponential moment asymptotics (1.6), or equivalently, (1.9) with $m = 1$. This upper bound is expected to be sharp, although this has not been proved (see also [16, Remark 1.3] for a heuristic discussion, and [16, Figure 2] for numerical simulations). Sharp lower bounds are in general very difficult to obtain, and require, among other things, the asymptotics of the second exponential moment (see e.g. [20]). These asymptotics are also provided by Theorem 1.1 with $m = 2$.

Remark 4. From (1.11) we see that, asymptotically, the m -th exponential moment can be written as the product of two terms: the first term is a constant pre-factor which depends only on the constants $\Sigma(x_k, x_j)$, and the second term is the product of m first exponential moment. Note that this phenomenon holds also for the Airy point process, see [15].

In [47, Theorem 2], Soshnikov obtained the following asymptotics for the expectation and variance of the counting function as $r \rightarrow +\infty$

$$\mathbb{E}_\alpha[N_{(0, rx)}] = \frac{\sqrt{rx}}{\pi} + \mathcal{O}(1), \quad \text{and} \quad \text{Var}_\alpha[N_{(0, rx)}] = \frac{\log rx}{4\pi^2} + \mathcal{O}(1). \quad (1.15)$$

Theorem 1.1 allows to improve on these asymptotics.

Corollary 1.2. *Let $x > 0$ and $x_2 > x_1 > 0$ be fixed. As $r \rightarrow +\infty$, we have*

$$\mathbb{E}_\alpha[N_{(0,rx)}] = \mu_\alpha(rx) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) = \frac{\sqrt{rx}}{\pi} - \frac{\alpha}{2} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \quad (1.16)$$

$$\begin{aligned} \text{Var}_\alpha[N_{(0,rx)}] &= \sigma^2(rx) + \frac{1 + \gamma_E}{2\pi^2} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) \\ &= \frac{\log rx}{4\pi^2} + \frac{1 + \log 4 + \gamma_E}{2\pi^2} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \end{aligned} \quad (1.17)$$

$$\text{Cov}_\alpha[N_{(0,rx_1)}, N_{(0,rx_2)}] = \Sigma(x_2, x_1) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) = \frac{1}{2\pi^2} \log \frac{\sqrt{x_2} + \sqrt{x_1}}{\sqrt{x_2} - \sqrt{x_1}} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \quad (1.18)$$

where $\gamma_E \approx 0.5772$ is Euler's gamma constant.

Remark 5. Basic properties of the expectation and the variance imply that

$$\mathbb{E}_\alpha[N_{(rx_1, rx_2)}] = \mu_\alpha(rx_2) - \mu_\alpha(rx_1) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \quad (1.19)$$

$$\begin{aligned} \text{Var}_\alpha[N_{(rx_1, rx_2)}] &= \sigma^2(rx_1) + \sigma^2(rx_2) + \frac{1 + \gamma_E}{\pi^2} - 2\Sigma(x_2, x_1) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) \\ &= \frac{\log r}{2\pi^2} + \frac{\log(16\sqrt{x_1x_2})}{2\pi^2} + \frac{1 + \gamma_E}{\pi^2} - \frac{1}{\pi^2} \log \frac{\sqrt{x_2} + \sqrt{x_1}}{\sqrt{x_2} - \sqrt{x_1}} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (1.20)$$

In [47, Theorem 2], Soshnikov obtained ²

$$\text{Var}_\alpha[N_{(r, 2r)}] = \frac{\log r}{4\pi^2} + \mathcal{O}(1), \quad (1.21)$$

which does not agree with the leading term of (1.20) (a factor 2 is missing). There is a similar error in [47, Theorem 1] for $k \geq 2$, see [15, Remark 1] (and furthermore the constant $\frac{11}{12\pi^2}$ in [47, Theorem 1] should instead be $\frac{3}{4\pi^2}$).

Proof of Corollary 1.2. Expanding (1.8) for $m = 1$ as $u \rightarrow 0$, we obtain

$$E_\alpha(x, u) = 1 + u \mathbb{E}_\alpha[N_{(0,x)}] + \frac{u^2}{2} \mathbb{E}_\alpha[N_{(0,x)}^2] + \mathcal{O}(u^3). \quad (1.22)$$

On the other hand, we can also expand the large r asymptotics of $E_\alpha(rx, u)$ (given by the right-hand side of (1.9) with $m = 1$) as $u \rightarrow 0$, since these asymptotics are valid uniformly for u in compact subsets of $\mathbb{R}$ (in particular in a neighborhood of 0). Comparing this expansion with (1.22), we obtain (1.16) and (1.17), where γ_E comes from the expansion of the Barnes' G -functions (see [45, formula 5.17.3]). The covariance between the two occupancy numbers $N_{(0,x_1)}$ and $N_{(0,x_2)}$ can be obtained from (1.8) with $m = 2$ as follows:

$$\begin{aligned} \frac{E_\alpha((x_1, x_2), (u, u))}{E_\alpha(x_1, u)E_\alpha(x_2, u)} &= \frac{\mathbb{E}_\alpha[e^{uN_{(0,x_1)}}e^{uN_{(0,x_2)}}]}{\mathbb{E}_\alpha[e^{uN_{(0,x_1)}}]\mathbb{E}_\alpha[e^{uN_{(0,x_2)}}]} \\ &= 1 + \text{Cov}_\alpha(N_{(0,x_1)}, N_{(0,x_2)})u^2 + \mathcal{O}(u^3), \quad \text{as } u \rightarrow 0. \end{aligned} \quad (1.23)$$

After the rescaling $(x_1, x_2) \mapsto r(x_1, x_2)$, we can obtain large r asymptotics for the left-hand side of the above expression using Theorem 1.1. By an expansion as $u \rightarrow 0$ of these asymptotics, and a comparison with (1.23), we obtain (1.18). $\square$

²The quantity $\nu_k(r)$ in [47, Theorem 2] corresponds to $N_{(0,kr)} - N_{(0,(k-1)r)} = N_{((k-1)r, kr)}$ in our notation.

We can also deduce from Theorem 1.1 the following central limit theorems. Corollary 1.3 (a) is the most natural central limit theorem that follows from Theorem 1.1, while Corollary 1.3 (b) allows for a comparison with [47, Theorem 2].

Corollary 1.3. (a) Consider the random variables $N_j^{(r)}$ defined by

$$N_j^{(r)} = \frac{N_{(0,rx_j)} - \mu_\alpha(rx_j)}{\sqrt{\sigma^2(rx_j)}}, \quad j = 1, \dots, m.$$

As $r \rightarrow +\infty$, we have

$$(N_1^{(r)}, N_2^{(r)}, \dots, N_m^{(r)}) \xrightarrow{d} \mathcal{N}(\vec{0}, I_m), \quad (1.24)$$

where “ $\xrightarrow{d}$ ” means convergence in distribution, I_m is the $m \times m$ identity matrix, and $\mathcal{N}(\vec{0}, I_m)$ is a multivariate normal random variable of mean $\vec{0} = (0, \dots, 0)$ and covariance matrix I_m .

(b) Consider the random variables $\hat{N}_j^{(r)}$ defined by

$$\hat{N}_1^{(r)} = \frac{N_{(0,rx_1)} - \mu_\alpha(rx_1)}{\sqrt{\sigma^2(rx_1)}}, \quad \hat{N}_j^{(r)} = \frac{N_{(rx_{j-1}, rx_j)} - (\mu_\alpha(rx_j) - \mu_\alpha(rx_{j-1}))}{\sqrt{\sigma^2(rx_j) + \sigma^2(rx_{j-1})}}, \quad j = 2, \dots, m.$$

As $r \rightarrow +\infty$, we have

$$(\hat{N}_1^{(r)}, \hat{N}_2^{(r)}, \dots, \hat{N}_m^{(r)}) \xrightarrow{d} \mathcal{N}(\vec{0}, \Sigma_m), \quad (1.25)$$

where Σ_m is given by

$$\begin{aligned} (\Sigma_m)_{1,j} &= (\Sigma_m)_{j,1} = \delta_{1,j} - \frac{1}{\sqrt{2}}\delta_{2,j}, & 1 \leq j \leq m. \\ (\Sigma_m)_{i,j} &= \delta_{i,j} - \frac{1}{2}\delta_{i,j+1} - \frac{1}{2}\delta_{i,j-1}, & 2 \leq i, j \leq m. \end{aligned}$$

Remark 6. In Corollary 1.3 (b), we have $(\Sigma_m)_{1,2} = (\Sigma_m)_{2,1} = -1/\sqrt{2}$, and this does not agree with [47, Theorem 2] (where it was obtained that $(\Sigma_m)_{1,2} = (\Sigma_m)_{2,1} = -1/2$). This is related to the factor 2 missing in (1.21).

Proof of Corollary 1.3. (a) After changing variables $u_j = t_j/\sqrt{\sigma^2(rx_j)}$ in (1.9) (recall that E_α is given by (1.8)), we obtain the following asymptotics

$$\mathbb{E}_\alpha \left[\exp \left(\sum_{j=1}^m t_j N_j^{(r)} \right) \right] = \exp \left(\sum_{j=1}^m \frac{t_j^2}{2} + \mathcal{O} \left(\frac{1}{\log r} \right) \right), \quad \text{as } r \rightarrow +\infty.$$

Thus, for each $t_1, \dots, t_m \in \mathbb{R}^m$, $(N_1^{(r)}, \dots, N_m^{(r)})_{r \geq 1}$ is a sequence of random variables whose moment generating functions converge to $\exp(\sum_{j=1}^m t_j^2/2)$ as $r \rightarrow +\infty$. The convergence is pointwise in $t_1, \dots, t_m \in \mathbb{R}^m$, but this is sufficient to imply the convergence in distribution (1.24) (see e.g. [2]).

(b) First, we make the change of variables $u_j = v_j - v_{j+1}$, $j = 1, \dots, m$ with $v_{m+1} := 0$ in (1.9). As $r \rightarrow +\infty$, we then have

$$\begin{aligned} \mathbb{E}_\alpha \left[\exp \left(\sum_{j=1}^m v_j N_{(rx_{j-1}, rx_j)} \right) \right] &= \exp \left(v_1 \mu_\alpha(rx_1) + \sum_{j=2}^m v_j (\mu_\alpha(rx_j) - \mu_\alpha(rx_{j-1})) \right) \\ &+ \frac{v_1^2}{2} \sigma^2(rx_1) + \sum_{j=2}^m \frac{v_j^2}{2} (\sigma^2(rx_{j-1}) + \sigma^2(rx_j)) - \sum_{j=2}^m v_{j-1} v_j \sigma^2(rx_{j-1}) \\ &+ \sum_{1 \leq j < k \leq m} (v_j - v_{j+1})(v_k - v_{k+1}) \Sigma(x_k, x_j) + \sum_{j=1}^m \log G(1 + \frac{v_j - v_{j+1}}{2\pi i}) G(1 - \frac{v_j - v_{j+1}}{2\pi i}) + \mathcal{O} \left(\frac{\log r}{\sqrt{r}} \right). \end{aligned}$$

Second, we apply the rescaling $v_1 = t_1/\sqrt{\sigma^2(rx_1)}$ and $v_j = t_j/\sqrt{\sigma^2(rx_j) + \sigma^2(rx_{j-1})}$ for $j \geq 2$. This gives

$$\mathbb{E}_\alpha \left[\exp \left(\sum_{j=1}^m t_j \widehat{N}_j^{(r)} \right) \right] = \exp \left(\frac{1}{2} \sum_{j=1}^m t_j^2 - \frac{1}{\sqrt{2}} t_1 t_2 - \frac{1}{2} \sum_{j=2}^{m-1} t_j t_{j+1} + \mathcal{O} \left(\frac{1}{\log r} \right) \right), \quad \text{as } r \rightarrow +\infty.$$

As in part (a), the pointwise convergence in $t_1, \dots, t_m \in \mathbb{R}^m$ is sufficient to conclude. $\square$

Main results for $s_1 = 0$. This situation gives information about the Bessel point process, when we condition on the event that no particle lies on the left-most interval, i.e. $N_{(0, x_1)} = 0$. Conditional point processes have been studied in other contexts in e.g. [14]. For $s_1 = 0$ and $s_2, \dots, s_m \in (0, +\infty)$, we define

$$u_j = \begin{cases} \log \frac{s_j}{s_{j+1}} & \text{for } j = 2, \dots, m-1, \\ \log s_m & \text{for } j = m, \end{cases} \quad (1.26)$$

and consider the following conditional expectation

$$E_\alpha^c(\vec{x}, \vec{u}) = \mathbb{E}_\alpha^c \left[\prod_{j=2}^m e^{u_j N_{(x_1, x_j)}} \right] := \mathbb{E}_\alpha \left[\prod_{j=2}^m e^{u_j N_{(x_1, x_j)}} \middle| N_{(0, x_1)} = 0 \right] = \frac{F_\alpha(\vec{x}, \vec{s})}{F_\alpha(x_1, 0)}, \quad (1.27)$$

where we recall (see (1.3)) that $F_\alpha(x_1, 0) = \mathbb{P}_\alpha(N_{(0, x_1)} = 0)$.

Theorem 1.4. Let $m \in \mathbb{N}_{>0}$, $\alpha > -1$,

$$\vec{u} = (u_2, \dots, u_m) \in \mathbb{R}^{m-1} \quad \text{and} \quad \vec{x} = (x_1, \dots, x_m) \in (\mathbb{R}^+)^m$$

be such that $0 < x_1 < x_2 < \dots < x_m < +\infty$. As $r \rightarrow +\infty$, we have

$$E_\alpha^c(r\vec{x}, \vec{u}) = \exp \left(\sum_{j=2}^m u_j \tilde{\mu}_\alpha(r, x_j) + \sum_{j=2}^m \frac{u_j^2}{2} \sigma^2(r(x_j - x_1)) + \sum_{2 \leq j < k \leq m} u_j u_k \Sigma(x_k - x_1, x_j - x_1) \right. \\ \left. + \sum_{j=2}^m \log G(1 + \frac{u_j}{2\pi i}) G(1 - \frac{u_j}{2\pi i}) + \mathcal{O} \left(\frac{\log r}{\sqrt{r}} \right) \right), \quad (1.28)$$

where the functions σ^2 and Σ are defined in (1.10), and $\tilde{\mu}_\alpha$ is given by

$$\tilde{\mu}_\alpha(r, x) = \frac{\sqrt{r(x - x_1)}}{\pi} - \frac{\alpha}{\pi} \arccos \left(\frac{\sqrt{x_1}}{\sqrt{x}} \right).$$

Furthermore, the error term is uniform in $u_2, \dots, u_m$ in compact subsets of $(-\infty, +\infty)$ and uniform in $x_1, \dots, x_m$ in compact subsets of $(0, +\infty)$, as long as there exists $\delta > 0$ such that

$$\min_{1 \leq j < k \leq m} x_k - x_j \geq \delta. \quad (1.29)$$

We can deduce from Theorem 1.4 the following new asymptotics for the expectation and variance of the counting function. The proof is similar to the one of Corollary 1.2, and we omit it.

Corollary 1.5. Let $x_3 > x_2 > x_1 > 0$ be fixed. As $r \rightarrow +\infty$, we have

$$\mathbb{E}_\alpha [N_{(rx_1, rx_2)} | N_{(0, rx_1)} = 0] = \tilde{\mu}(r, x_2) + \mathcal{O} \left(\frac{\log r}{\sqrt{r}} \right), \\ \text{Var}_\alpha [N_{(rx_1, rx_2)} | N_{(0, rx_1)} = 0] = \sigma^2(r(x_2 - x_1)) + \frac{1 + \gamma_E}{2\pi^2} + \mathcal{O} \left(\frac{\log r}{\sqrt{r}} \right), \\ \text{Cov}_\alpha [N_{(rx_1, rx_2)} | N_{(0, rx_1)} = 0, (N_{(rx_1, rx_3)} | N_{(0, rx_1)} = 0)] = \Sigma(x_3 - x_1, x_2 - x_1) + \mathcal{O} \left(\frac{\log r}{\sqrt{r}} \right).$$

We also obtain the following central limit theorem. The proof is similar to the one done in Corollary 1.3 and we omit it.

Corollary 1.6. *Consider the conditional random variables $\tilde{N}_j^{(r)}$ defined by*

$$\tilde{N}_j^{(r)} = \frac{(N_{(rx_1, rx_j)} | N_{(0, rx_1)} = 0) - \tilde{\mu}(r, x_j)}{\sqrt{\sigma^2(r(x_j - x_1))}}, \quad j = 2, \dots, m.$$

As $r \rightarrow +\infty$, we have

$$(\tilde{N}_2^{(r)}, \dots, \tilde{N}_m^{(r)}) \xrightarrow{d} \mathcal{N}(\vec{0}, I_{m-1}). \quad (1.30)$$

Outline. Section 2 is divided into two parts. In the first part, we recall a model RH problem (whose solution is denoted Φ) introduced in [17], which is of central importance in the present paper. In the second part, we obtain a differential identity which expresses $\partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s})$ (for an arbitrary $k \in \{1, \dots, m\}$) in terms of Φ . We obtain large r asymptotics for Φ with $s_1 \in (0, +\infty)$ in Section 3 via a Deift/Zhou steepest descent. In Section 4, we use the analysis of Section 3 to obtain large r asymptotics for $\partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s})$. We also proceed with successive integrations of these asymptotics in $s_1, \dots, s_m$, which proves Theorem 1.1. Section 5 and Section 6 are devoted to the proof of Theorem 1.4 (with $s_1 = 0$), and are organised similarly to Section 3 and Section 4.

Approach. In [21], the authors obtained the asymptotics (1.5) by expressing $F_\alpha(rx_1, 0)$ as a limit as $n \rightarrow +\infty$ of $n \times n$ Toeplitz determinants (whose symbol has an hard edge) and then performing a steepest descent on an RH problem for orthogonal polynomials on the unit circle. The parameter n is thus an extra parameter which disappears in the limit. It is a priori possible for us to generalize the same strategy by relating $F_\alpha(r\vec{x}, \vec{s})$ with Toeplitz determinants (with jump-type Fisher-Hartwig singularities accumulating near an hard-edge), but on a technical level this appears not obvious at all. Our approach takes advantage of the known result (1.5) (only needed to prove Theorem 1.4, but not Theorem 1.1), and is more direct in the sense that the parameter n does not appear in the analysis.

2 Model RH problem for Φ and a differential identity

As mentioned in the outline, the model RH problem introduced in [17] is of central importance in the present paper, and we recall its properties here. In order to have compact and uniform notations, it is convenient for us to define $x_0 = 0$ and $s_{m+1} = 1$, but they are not included in the notations for $\vec{x}$ and $\vec{s}$. To summarize, the parameters $x_0, s_{m+1}, \vec{x} = (x_1, \dots, x_m)$ and $\vec{s} = (s_1, \dots, s_m)$ are such that

$$0 = x_0 < x_1 < \dots < x_m < +\infty, \quad s_1, \dots, s_m \in [0, +\infty) \quad \text{and} \quad s_{m+1} = 1. \quad (2.1)$$

The model RH problem we consider depends on $\alpha, \vec{x}$ and $\vec{s}$, and its solution is denoted by $\Phi(z; \vec{x}, \vec{s})$, where the dependence in α is omitted. When there is no confusion, we will just denote it by $\Phi(z)$ where the dependence in $\vec{x}$ and $\vec{s}$ is also omitted. There is existence (if the parameters satisfy (2.1)) and uniqueness for Φ , and furthermore it satisfies $\det \Phi \equiv 1$. The RH problem for Φ is more easily stated in terms of the following matrices:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \quad (2.2)$$

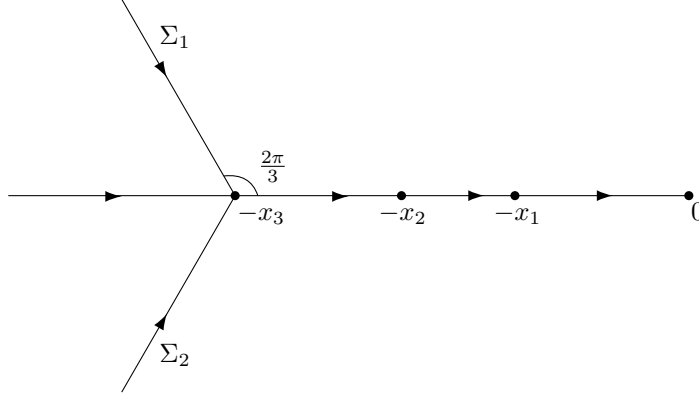


Figure 1: The jump contour for Φ with $m = 3$.

We also define for $y \in \mathbb{R}$ the following piecewise constant matrix:

$$H_y(z) = \begin{cases} I, & \text{for } -\frac{2\pi}{3} < \arg(z - y) < \frac{2\pi}{3}, \\ \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg(z - y) < \pi, \\ \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \text{for } -\pi < \arg(z - y) < -\frac{2\pi}{3}, \end{cases} \quad (2.3)$$

where the principal branch is chosen for the argument, such that $\arg(z - y) = 0$ for $z > y$.

RH problem for Φ

- (a) $\Phi : \mathbb{C} \setminus \Sigma_\Phi \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where the contour $\Sigma_\Phi = ((-\infty, 0] \cup \Sigma_1 \cup \Sigma_2)$ is oriented as shown in Figure 1 with

$$\Sigma_1 = -x_m + e^{\frac{2\pi i}{3}} \mathbb{R}^+, \quad \Sigma_2 = -x_m + e^{-\frac{2\pi i}{3}} \mathbb{R}^+.$$

- (b) The limits of $\Phi(z)$ as z approaches $\Sigma_\Phi \setminus \{0, -x_1, \dots, -x_m\}$ from the left (+ side) and from the right (− side) exist, are continuous on $\Sigma_\Phi \setminus \{0, -x_1, \dots, -x_m\}$ and are denoted by Φ_+ and Φ_- respectively. Furthermore they are related by:

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \quad z \in \Sigma_1, \quad (2.4)$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, -x_m), \quad (2.5)$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \quad z \in \Sigma_2, \quad (2.6)$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} e^{\pi i \alpha} & s_j \\ 0 & e^{-\pi i \alpha} \end{pmatrix}, \quad z \in (-x_j, -x_{j-1}), \quad (2.7)$$

where $j = 1, \dots, m$.

(c) As $z \rightarrow \infty$, we have

$$\Phi(z) = \left(I + \Phi_1 z^{-1} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} N e^{\sqrt{z}\sigma_3}, \quad (2.8)$$

where the principal branch is chosen for each root, and the matrix $\Phi_1 = \Phi_1(\vec{x}, \vec{s})$ is independent of z and traceless.

As z tends to $-x_j$, $j \in \{1, \dots, m\}$, Φ takes the form

$$\Phi(z) = G_j(z) \begin{pmatrix} 1 & \frac{s_{j+1}-s_j}{2\pi i} \log(z+x_j) \\ 0 & 1 \end{pmatrix} V_j(z) e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} H_{-x_m}(z), \quad (2.9)$$

where $G_j(z) = G_j(z; \vec{x}, \vec{s})$ is analytic in a neighborhood of $(-x_{j+1}, -x_{j-1})$, satisfies $\det G_j \equiv 1$, and $\theta(z)$, $V_j(z)$ are piecewise constant and defined by

$$\theta(z) = \begin{cases} +1, & \text{Im } z > 0, \\ -1, & \text{Im } z < 0, \end{cases} \quad V_j(z) = \begin{cases} I, & \text{Im } z > 0, \\ \begin{pmatrix} 1 & -s_j \\ 0 & 1 \end{pmatrix}, & \text{Im } z < 0. \end{cases} \quad (2.10)$$

As z tends to 0, Φ takes the form

$$\Phi(z) = G_0(z) z^{\frac{\alpha}{2} \sigma_3} \begin{pmatrix} 1 & s_1 h(z) \\ 0 & 1 \end{pmatrix}, \quad \alpha > -1, \quad (2.11)$$

where $G_0(z)$ is analytic in a neighborhood of $(-x_1, \infty)$, satisfies $\det G_0 \equiv 1$ and

$$h(z) = \begin{cases} \frac{1}{2i \sin(\pi \alpha)}, & \alpha \notin \mathbb{N}_{\geq 0}, \\ \frac{(-1)^\alpha}{2\pi i} \log z, & \alpha \in \mathbb{N}_{\geq 0}. \end{cases} \quad (2.12)$$

The quantities Φ_1 and G_j , $j = 0, \dots, m$ also depend on α , even though it is not indicated in the notation.

Differential identity

It is known [48, Theorem 2] that $F_\alpha(\vec{x}, \vec{s})$ can also be expressed as a Fredholm determinant with m discontinuities as follows

$$F_\alpha(\vec{x}, \vec{s}) = \det(1 - \mathcal{K}_{\vec{x}, \vec{s}}), \quad \mathcal{K}_{\vec{x}, \vec{s}} := \hat{\chi}_{(0, x_m)} \sum_{j=1}^m (1 - s_j) \mathcal{K}_\alpha^{\text{Be}} \hat{\chi}_{(x_{j-1}, x_j)}, \quad (2.13)$$

where $\mathcal{K}_\alpha^{\text{Be}}$ denotes the integral operator acting on $L^2(\mathbb{R}^+)$ whose kernel is K_α^{Be} (given by (1.1)), and where $\hat{\chi}_A$ is the projection operator onto $L^2(A)$. We consider the function $K_{\vec{x}, \vec{s}} : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ given by

$$K_{\vec{x}, \vec{s}}(u, v) = \chi_{(0, x_m)}(u) \sum_{j=1}^m (1 - s_j) K_\alpha^{\text{Be}}(u, v) \chi_{(x_{j-1}, x_j)}(v), \quad u, v > 0, \quad (2.14)$$

where for a given $A \subset \mathbb{R}$, χ_A denotes the characteristic function of A . For notational convenience, we omit the dependence of $K_{\vec{x}, \vec{s}}$ and $\mathcal{K}_{\vec{x}, \vec{s}}$ in α . Since $s_j \geq 0$ for all $j = 0, \dots, m$, we deduce from

(1.3) that $F_\alpha(\vec{x}, \vec{s}) = \det(1 - \mathcal{K}_{\vec{x}, \vec{s}}) > 0$. In particular, $1 - \mathcal{K}_{\vec{x}, \vec{s}}$ is invertible. Therefore, by standard properties of trace class operators, for any $k \in \{1, \dots, m\}$ we have

$$\begin{aligned} \partial_{s_k} \log \det(1 - \mathcal{K}_{\vec{x}, \vec{s}}) &= -\text{Tr} \left((1 - \mathcal{K}_{\vec{x}, \vec{s}})^{-1} \partial_{s_k} \mathcal{K}_{\vec{x}, \vec{s}} \right) \\ &= \frac{1}{1 - s_k} \text{Tr} \left((1 - \mathcal{K}_{\vec{x}, \vec{s}})^{-1} \mathcal{K}_{\vec{x}, \vec{s}} \chi_{(x_{k-1}, x_k)} \right) \\ &= \frac{1}{1 - s_k} \text{Tr} \left(\mathcal{R}_{\vec{x}, \vec{s}} \chi_{(x_{k-1}, x_k)} \right) = \frac{1}{1 - s_k} \int_{x_{k-1}}^{x_k} R_{\vec{x}, \vec{s}}(u, u) du, \end{aligned} \quad (2.15)$$

where $\mathcal{R}_{\vec{x}, \vec{s}}$ is the resolvent operator defined by

$$1 + \mathcal{R}_{\vec{x}, \vec{s}} = (1 - \mathcal{K}_{\vec{x}, \vec{s}})^{-1}, \quad (2.16)$$

and where $R_{\vec{x}, \vec{s}}$ is the associated kernel. From [17, equation (4.19)], for $u \in (x_{k-1}, x_k)$ we have

$$R_{\vec{x}, \vec{s}}(u, u) = \frac{-e^{\pi i \alpha}}{2\pi i} (1 - s_k) [\Phi_-^{-1}(-u; \vec{x}, \vec{s}) \partial_u (\Phi_-(-u; \vec{x}, \vec{s}))]_{21}. \quad (2.17)$$

Therefore, we obtain the following differential identity

$$\partial_{s_k} \log \det(1 - \mathcal{K}_{\vec{x}, \vec{s}}) = \frac{e^{\pi i \alpha}}{2\pi i} \int_{-x_k}^{-x_{k-1}} [\Phi_-^{-1}(u; \vec{x}, \vec{s}) \partial_u \Phi_-(u; \vec{x}, \vec{s})]_{21} du. \quad (2.18)$$

Note that we implicitly assumed $s_k \neq 1$ in (2.15), and thus (2.18) is a priori only true under this assumption. However, both sides of (2.18) are continuous as $s_k \rightarrow 1$, and therefore (2.18) also holds for $s_k = 1$ by continuity. (In fact both sides are analytic for s_k in a small complex neighborhood of 1. This follows from [48, Theorem 2] and the fact that $\det(1 - \mathcal{K}_{\vec{x}, \vec{s}})|_{s_k=1} > 0$ for the left-hand side, and from standard properties for RH problems for the right-hand side.)

Remark 7. It is quite remarkable that there are differential identities for $\log \det(1 - \mathcal{K}_{\vec{x}, \vec{s}})$ in terms of an RH problem. The reason behind this is that the kernel $\mathcal{K}_{\vec{x}, \vec{s}}$ is so-called *integrable* in the sense of Its, Izergin, Korepin and Slavnov [35]. This fact was also used extensively in [17] (even though the differential identities obtained in [17] are different from (2.18)).

In the rest of this section, we aim to simplify the integral on the right-hand side of (2.18), following ideas presented in [10, Section 3] and using some results of [17]. To prepare ourselves for that matter, we define for $r > 0$ the following quantities

$$\tilde{\Phi}(z; r) = \tilde{E}(r) \Phi(rz; r\vec{x}, \vec{s}), \quad \tilde{E}(r) = \begin{pmatrix} 1 & 0 \\ \frac{i}{\sqrt{r}} \Phi_{1,12}(r\vec{x}, \vec{s}) & 1 \end{pmatrix} e^{\frac{\pi i}{4} \sigma_3 r^{\frac{\sigma_3}{4}}}, \quad (2.19)$$

where we have omitted the dependence of $\tilde{\Phi}$ and $\tilde{E}$ in $\vec{x}$ and $\vec{s}$. It was shown in [17, equation (3.15)] that $\tilde{\Phi}$ satisfies a Lax pair, and in particular

$$\partial_z \tilde{\Phi}(z; r) = \tilde{A}(z; r) \tilde{\Phi}(z; r), \quad (2.20)$$

where $\tilde{A}$ is traceless, depends also on $\vec{x}$ and $\vec{s}$ and takes the form

$$\tilde{A}(z; r) = \begin{pmatrix} 0 & 0 \\ \frac{\sqrt{r}}{2} & 0 \end{pmatrix} + \sum_{j=0}^m \frac{\tilde{A}_j(r)}{z + x_j}, \quad (2.21)$$

for some traceless matrices $\tilde{A}_j(r) = \tilde{A}_j(r; \vec{x}, \vec{s})$. Therefore, we have

$$A(z; r) := \partial_z \left(\Phi(rz; r\vec{x}, \vec{s}) \right) \Phi^{-1}(rz; r\vec{x}, \vec{s}) = \tilde{E}(r)^{-1} \tilde{A}(z; r) \tilde{E}(r) = \begin{pmatrix} 0 & 0 \\ \frac{ir}{2} & 0 \end{pmatrix} + \sum_{j=0}^m \frac{A_j(r)}{z + x_j}, \quad (2.22)$$

where $A_j(r) = \tilde{E}(r)^{-1} \tilde{A}_j(r) \tilde{E}(r)$, $j = 0, 1, \dots, m$. We will use later the following relations between the matrices A_j and G_j . For $j = 1, \dots, m$, using (2.9) and $\det G_j \equiv 1$, we have

$$\begin{aligned} A_j(r) &= \frac{s_{j+1} - s_j}{2\pi i} (G_j \sigma_+ G_j^{-1})(-rx_j; r\vec{x}, \vec{s}) \\ &= \frac{s_{j+1} - s_j}{2\pi i} \begin{pmatrix} -G_{j,11} G_{j,21} & G_{j,11}^2 \\ -G_{j,21}^2 & G_{j,11} G_{j,21} \end{pmatrix}, \end{aligned} \quad (2.23)$$

and, from (2.11) and $\det G_0 \equiv 1$, we have

$$A_0(r) = \begin{cases} \frac{s_1}{2\pi i} (G_0 \sigma_+ G_0^{-1})(0; r\vec{x}, \vec{s}), & \text{if } \alpha = 0, \\ \frac{\alpha}{2} (G_0 \sigma_3 G_0^{-1})(0; r\vec{x}, \vec{s}), & \text{if } \alpha \neq 0. \end{cases} \quad (2.24)$$

Now, we rewrite the integrand on the right-hand side of (2.18) (with $\vec{x} \mapsto r\vec{x}$) using (2.22). Since A is traceless and $\det \Phi \equiv 1$, we have

$$\begin{aligned} [\Phi^{-1}(rz; r\vec{x}, \vec{s}) \partial_z (\Phi(rz; r\vec{x}, \vec{s}))]_{21} &= [\Phi^{-1}(rz; r\vec{x}, \vec{s}) A(z; r) \Phi(rz; r\vec{x}, \vec{s})]_{21} \\ &= \Phi_{11}^2 A_{21} - \Phi_{21}^2 A_{12} - 2\Phi_{11} \Phi_{21} A_{11}, \end{aligned} \quad (2.25)$$

which can be rewritten (again via (2.22)) as

$$\begin{aligned} [\Phi^{-1}(rz; r\vec{x}, \vec{s}) \partial_z (\Phi(rz; r\vec{x}, \vec{s}))]_{21} &= (\Phi \sigma_+ \Phi^{-1})_{12}(rz; r\vec{x}, \vec{s}) \left[\frac{ir}{2} + \sum_{j=0}^m \frac{A_{j,21}(r)}{z + x_j} \right] \\ &\quad + (\Phi \sigma_+ \Phi^{-1})_{21}(rz; r\vec{x}, \vec{s}) \sum_{j=0}^m \frac{A_{j,12}(r)}{z + x_j} + 2(\Phi \sigma_+ \Phi^{-1})_{11}(rz; r\vec{x}, \vec{s}) \sum_{j=0}^m \frac{A_{j,11}(r)}{z + x_j}. \end{aligned} \quad (2.26)$$

Let us define

$$\hat{F}(z) = \partial_{s_k} \Phi(rz; r\vec{x}, \vec{s}) \Phi(rz; r\vec{x}, \vec{s})^{-1}, \quad (2.27)$$

where $k \in \{1, \dots, m\}$ is fixed and omitted in the notation. From the RH problem for Φ , we deduce that $\hat{F}$ satisfies the following RH problem.

RH problem for $\hat{F}$

(a) $\hat{F} : \mathbb{C} \setminus [-x_k, -x_{k-1}] \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) $\hat{F}$ satisfies the jumps

$$\hat{F}_+(z) = \hat{F}_-(z) + e^{\pi i \alpha} (\Phi_- \sigma_+ \Phi_-^{-1})(rz; r\vec{x}, \vec{s}), \quad z \in (-x_k, -x_{k-1}). \quad (2.28)$$

(c) $\hat{F}$ satisfies the following asymptotic behaviors

$$\begin{aligned} \hat{F}(z) &= \frac{\partial_{s_k} \Phi_1(r\vec{x}, \vec{s})}{rz} + \mathcal{O}(z^{-2}), & \text{as } z \rightarrow \infty, \\ \hat{F}(z) &= \frac{\partial_{s_k} (s_{j+1} - s_j)}{s_{j+1} - s_j} A_j(r) \log(r(z + x_j)) + \hat{F}_j + o(1), & \text{as } z \rightarrow -x_j, j = 1, \dots, m, \end{aligned} \quad (2.29)$$

where $\widehat{F}_j = (\partial_{s_k} G_j G_j^{-1})(-rx_j; r\vec{x}, \vec{s})$. Furthermore, as $z \rightarrow 0$, we have

$$\widehat{F}(z) = \begin{cases} \widehat{F}_0 + o(1), & \text{if } \alpha > 0, \\ \frac{\partial_{s_k} s_1}{\partial_{s_k} s_1} A_0(r) \log(rz) + \widehat{F}_0 + o(1), & \text{if } \alpha = 0, \\ \frac{s_1}{2i \sin(\pi\alpha)} (G_0 \sigma_+ G_0^{-1})(0; r\vec{x}, \vec{s}) + \widehat{F}_0 + o(1), & \text{if } \alpha < 0, \end{cases} \quad (2.30)$$

where $\widehat{F}_0 = (\partial_{s_k} G_0 G_0^{-1})(0; r\vec{x}, \vec{s})$.

The RH problem for $\widehat{F}$ can be solved explicitly using Cauchy's formula, we have

$$\widehat{F}(z) = \frac{e^{\pi i \alpha}}{2\pi i} \int_{-x_k}^{-x_{k-1}} \frac{(\Phi_- \sigma_+ \Phi_-^{-1})(ru; r\vec{x}, \vec{s})}{u - z} du. \quad (2.31)$$

Expanding the above expression as $z \rightarrow \infty$ and comparing with (2.29), we obtain

$$-\frac{e^{\pi i \alpha}}{2\pi i} \int_{-x_k}^{-x_{k-1}} (\Phi_- \sigma_+ \Phi_-^{-1})(ru; r\vec{x}, \vec{s}) du = \frac{\partial_{s_k} \Phi_1(r\vec{x}, \vec{s})}{r}. \quad (2.32)$$

Substituting (2.26) into (2.18) (with $\vec{x} \mapsto r\vec{x}$), we can simplify the integral using the expansions of $\widehat{F}$ at ∞ and at $-x_j$, $j = 0, 1, \dots, m$ (given by (2.29)-(2.30)). Note that $\det A_j \equiv 0$ for $j = 1, \dots, m$. Therefore, the logarithmic part in the expansions of $\widehat{F}(z)$ as $z \rightarrow -x_j$ for $j = 1, \dots, m$ does not contribute in (2.18). One concludes the same for $j = 0$ if $\alpha = 0$. If $\alpha < 0$, the $\mathcal{O}(z^\alpha)$ term in the $z \rightarrow 0$ expansion of $\widehat{F}$ also does not contribute in (2.18), this follows from the relation

$$(G_0 \sigma_3 G_0^{-1})_{21} (G_0 \sigma_+ G_0^{-1})_{12} + (G_0 \sigma_3 G_0^{-1})_{12} (G_0 \sigma_+ G_0^{-1})_{21} + 2(G_0 \sigma_3 G_0^{-1})_{11} (G_0 \sigma_+ G_0^{-1})_{11} = 0, \quad (2.33)$$

where we have used $\det G_0 \equiv 1$. Therefore, for any $\alpha > -1$, we obtain

$$\partial_{s_k} \log \det(1 - \mathcal{K}_{r\vec{x}, \vec{s}}) = -\frac{i}{2} \partial_{s_k} \Phi_{1,12}(r\vec{x}, \vec{s}) + \sum_{j=0}^m [A_{j,21}(r) \widehat{F}_{j,12} + A_{j,12}(r) \widehat{F}_{j,21} + 2A_{j,11}(r) \widehat{F}_{j,11}].$$

Finally, substituting in the above equality the explicit forms for the A_j 's and $\widehat{F}_j$'s given by (2.23)-(2.24) and below (2.29)-(2.30), and simplifying the result with the identities $\det G_j \equiv 1$, we obtain

$$\partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s}) = K_\infty + \sum_{j=1}^m K_{-x_j} + K_0, \quad (2.34)$$

where

$$K_\infty = -\frac{i}{2} \partial_{s_k} \Phi_{1,12}(r\vec{x}, \vec{s}), \quad (2.35)$$

$$K_{-x_j} = \frac{s_{j+1} - s_j}{2\pi i} \left(G_{j,11} \partial_{s_k} G_{j,21} - G_{j,21} \partial_{s_k} G_{j,11} \right) (-rx_j; r\vec{x}, \vec{s}), \quad (2.36)$$

$$K_0 = \begin{cases} \frac{s_1}{2\pi i} \left(G_{0,11} \partial_{s_k} G_{0,21} - G_{0,21} \partial_{s_k} G_{0,11} \right) (0; r\vec{x}, \vec{s}) & \text{if } \alpha = 0, \\ \alpha \left(G_{0,21} \partial_{s_k} G_{0,12} - G_{0,11} \partial_{s_k} G_{0,22} \right) (0; r\vec{x}, \vec{s}) & \text{if } \alpha \neq 0. \end{cases} \quad (2.37)$$

3 Large r asymptotics for Φ with $s_1 \in (0, +\infty)$

In this section, we perform a Deift/Zhou steepest descent analysis to obtain large r asymptotics for $\Phi(rz; r\vec{x}, \vec{s})$ in different regions of the complex z -plane. On the level of the parameters, we assume that $s_1, \dots, s_m$ are in a compact subset of $(0, +\infty)$ and that $x_1, \dots, x_m$ are in a compact subset of $(0, +\infty)$ in such a way that there exists $\delta > 0$ independent of r such that

$$\min_{1 \leq j < k \leq m} x_k - x_j \geq \delta. \quad (3.1)$$

3.1 Normalization of the RH problem with a g -function

In the first transformation, we normalize the behavior at ∞ of the RH problem for $\Phi(rz; r\vec{x}, \vec{s})$ by removing the term that grows exponentially with z . This transformation is standard in the literature (see e.g. [22]) and uses a so-called g -function. In view of (2.8), we define our g -function by

$$g(z) = \sqrt{z}, \quad (3.2)$$

where the principal branch is taken. Define

$$T(z) = r^{\frac{\sigma_3}{4}} \Phi(rz; r\vec{x}, \vec{s}) e^{-\sqrt{r}g(z)\sigma_3}. \quad (3.3)$$

The asymptotics (2.8) of Φ then lead after a straightforward calculation to

$$T(z) = \left(I + \frac{T_1}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} N, \quad T_1 = r^{\frac{\sigma_3}{4}} \frac{\Phi_1(r\vec{x}, \vec{s})}{r} r^{-\frac{\sigma_3}{4}} \quad (3.4)$$

as $z \rightarrow \infty$. In particular, $T_{1,12} = \frac{\Phi_{1,12}}{\sqrt{r}}$. The jumps for T are obtained straightforwardly from those of Φ and the relation $g_+(z) + g_-(z) = 0$ for $z \in (-\infty, 0)$. Since $s_j \neq 0$, the jump matrix for T on $(-x_j, -x_{j-1})$ can be factorized as follows

$$\begin{pmatrix} e^{\pi i \alpha} e^{-2\sqrt{r}g_+(z)} & s_j \\ 0 & e^{-\pi i \alpha} e^{-2\sqrt{r}g_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{-\pi i \alpha} e^{-2\sqrt{r}g_-(z)} & 1 \end{pmatrix} \times \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{\pi i \alpha} e^{-2\sqrt{r}g_+(z)} & 1 \end{pmatrix}. \quad (3.5)$$

3.2 Opening of the lenses

Around each interval $(-x_j, -x_{j-1})$, $j = 1, \dots, m$, we open lenses $\gamma_{j,+}$ and $\gamma_{j,-}$, lying in the upper and lower half plane respectively, as shown in Figure 2. Let us also denote $\Omega_{j,+}$ (resp. $\Omega_{j,-}$) for the region inside the lenses around $(-x_j, -x_{j-1})$ in the upper half plane (resp. in the lower half plane). In view of (3.5), we define the next transformation by

$$S(z) = T(z) \prod_{j=1}^m \begin{cases} \begin{pmatrix} 1 & 0 \\ -s_j^{-1} e^{\pi i \alpha} e^{-2\sqrt{r}g(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_{j,+}, \\ \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{-\pi i \alpha} e^{-2\sqrt{r}g(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_{j,-}, \\ I, & \text{if } z \in \mathbb{C} \setminus (\Omega_{j,+} \cup \Omega_{j,-}). \end{cases} \quad (3.6)$$

It is straightforward to verify from the RH problem for Φ and from Section 3.1 that S satisfies the following RH problem.

RH problem for S

(a) $S : \mathbb{C} \setminus \Gamma_S \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, with

$$\Gamma_S = (-\infty, 0) \cup \gamma_+ \cup \gamma_-, \quad \gamma_{\pm} = \bigcup_{j=1}^{m+1} \gamma_{j,\pm}, \quad (3.7)$$

where $\gamma_{m+1,\pm} := -x_m + e^{\pm \frac{2\pi i}{3}}(0, +\infty)$, and Γ_S is oriented as shown in Figure 2.

(b) The jumps for S are given by

$$\begin{aligned} S_+(z) &= S_-(z) \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix}, & z \in (-x_j, -x_{j-1}), j = 1, \dots, m+1, \\ S_+(z) &= S_-(z) \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{\pm \pi i \alpha} e^{-2\sqrt{r}g(z)} & 1 \end{pmatrix}, & z \in \gamma_{j,\pm}, j = 1, \dots, m+1, \end{aligned}$$

where $x_{m+1} := +\infty$ (we recall that $x_0 = 0$ and $s_{m+1} = 1$).

(c) As $z \rightarrow \infty$, we have

$$S(z) = \left(I + \frac{T_1}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} N. \quad (3.8)$$

As $z \rightarrow -x_j$ from outside the lenses, $j = 1, \dots, m$, we have

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z + x_j)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z + x_j)) \end{pmatrix}. \quad (3.9)$$

As $z \rightarrow 0$ from outside the lenses, we have

$$S(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } \alpha = 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} z^{\frac{\alpha}{2}\sigma_3}, & \text{if } \alpha > 0, \\ \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \end{pmatrix}, & \text{if } \alpha < 0. \end{cases} \quad (3.10)$$

Since $\Re g(z) > 0$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\Re g_{\pm}(z) = 0$ for $z \in (-\infty, 0)$, the jump matrices for S tend to the identity matrix exponentially fast as $r \rightarrow +\infty$ on the lenses. This convergence is uniform for z outside of fixed neighborhoods of $-x_j$, $j \in \{0, 1, \dots, m\}$, but is not uniform as $r \rightarrow +\infty$ and simultaneously $z \rightarrow -x_j$, $j \in \{0, 1, \dots, m\}$.

3.3 Global parametrix

By ignoring the jumps for S that are pointwise exponentially close to the identity matrix as $r \rightarrow +\infty$, we are left with an RH problem which is independent of r , and whose solution is called the global parametrix and denoted $P^{(\infty)}$. It will appear later in Section 3.5 that $P^{(\infty)}$ is a good approximation for S away from neighborhoods of $-x_j$, $j = 0, 1, \dots, m$.

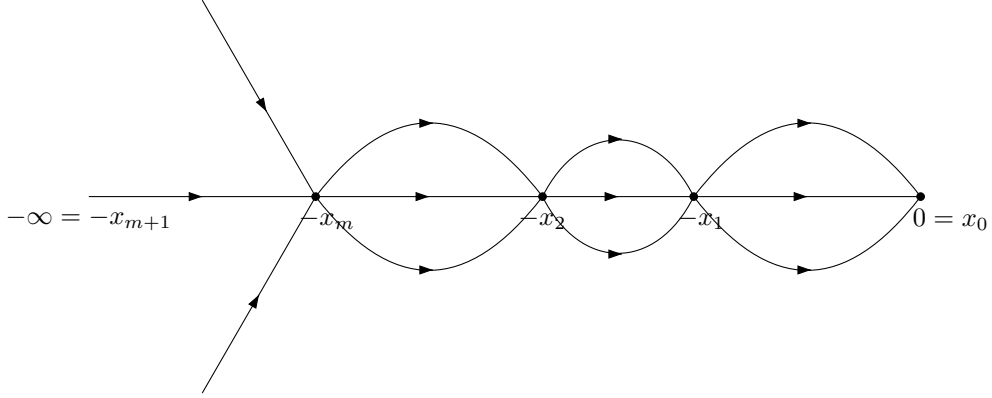


Figure 2: Jump contours Γ_S for the RH problem for S with $m = 3$ and $s_1 \neq 0$.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) The jumps for $P^{(\infty)}$ are given by

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix}, \quad z \in (-x_j, -x_{j-1}), j = 1, \dots, m+1.$$

(c) As $z \rightarrow \infty$, we have

$$P^{(\infty)}(z) = \left(I + \frac{P_1^{(\infty)}}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4} N}, \quad (3.11)$$

for a certain matrix $P_1^{(\infty)}$ independent of z .

(d) As $z \rightarrow -x_j$, $j \in \{1, \dots, m\}$, we have $P^{(\infty)}(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}$.

As $z \rightarrow 0$, we have $P^{(\infty)}(z) = \begin{pmatrix} \mathcal{O}(z^{-1/4}) & \mathcal{O}(z^{-1/4}) \\ \mathcal{O}(z^{-1/4}) & \mathcal{O}(z^{-1/4}) \end{pmatrix}$.

Note that condition (d) for the RH problem for $P^{(\infty)}$ does not come from the RH problem for S . It is added to ensure uniqueness of the solution. The construction of $P^{(\infty)}$ relies on a so-called Szegő function D (see [41]). In our case, we need to define D as follows

$$D(z) = \exp \left(\frac{\sqrt{z}}{2\pi} \sum_{j=1}^m \log s_j \int_{x_{j-1}}^{x_j} \frac{du}{\sqrt{u}(z+u)} \right).$$

It satisfies the following jumps

$$D_+(z)D_-(z) = s_j, \quad \text{for } z \in (-x_j, -x_{j-1}), j = 1, \dots, m+1.$$

Furthermore, as $z \rightarrow \infty$, we have

$$D(z) = \exp \left(\sum_{\ell=1}^k \frac{d_\ell}{z^{\ell-\frac{1}{2}}} + \mathcal{O}(z^{-k-\frac{1}{2}}) \right), \quad (3.12)$$

where $k \in \mathbb{N}_{>0}$ is arbitrary and

$$d_\ell = \frac{(-1)^{\ell-1}}{2\pi} \sum_{j=1}^m \log s_j \int_{x_{j-1}}^{x_j} u^{\ell-\frac{3}{2}} du = \frac{(-1)^{\ell-1}}{\pi(2\ell-1)} \sum_{j=1}^m \log s_j \left(x_j^{\ell-\frac{1}{2}} - x_{j-1}^{\ell-\frac{1}{2}} \right).$$

Let us finally define

$$P^{(\infty)}(z) = \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} z^{-\frac{\sigma_3}{4}} N D(z)^{-\sigma_3}, \quad (3.13)$$

where the principal branch is taken for the root. From the above properties of D , one can check that $P^{(\infty)}$ satisfies criteria (a), (b) and (c) of the RH problem for $P^{(\infty)}$, with

$$P_{1,12}^{(\infty)} = id_1. \quad (3.14)$$

The rest of the current section is devoted to the computations of the first terms in the asymptotics of $D(z)$ as $z \rightarrow -x_j$, $j = 0, 1, \dots, m$. It will in particular prove that $P^{(\infty)}$ defined in (3.13) satisfies condition (d) of the RH problem for $P^{(\infty)}$. After integrations, we can rewrite D as follows

$$D(z) = \prod_{j=1}^m D_{s_j}(z), \quad (3.15)$$

where

$$D_{s_j}(z) = \left(\frac{(\sqrt{z} - i\sqrt{x_{j-1}})(\sqrt{z} + i\sqrt{x_j})}{(\sqrt{z} - i\sqrt{x_j})(\sqrt{z} + i\sqrt{x_{j-1}})} \right)^{\frac{\log s_j}{2\pi i}}. \quad (3.16)$$

As $z \rightarrow -x_j$, $j \in \{1, \dots, m\}$, $\Im z > 0$, we have

$$D_{s_j}(z) = \sqrt{s_j} T_{j,j}^{\frac{\log s_j}{2\pi i}} (z + x_j)^{-\frac{\log s_j}{2\pi i}} (1 + \mathcal{O}(z + x_j)), \quad T_{j,j} = 4x_j \frac{\sqrt{x_j} - \sqrt{x_{j-1}}}{\sqrt{x_j} + \sqrt{x_{j-1}}}. \quad (3.17)$$

As $z \rightarrow -x_{j-1}$, $j \in \{2, \dots, m\}$, $\Im z > 0$, we have

$$D_{s_j}(z) = T_{j,j-1}^{\frac{\log s_j}{2\pi i}} (z + x_{j-1})^{-\frac{\log s_j}{2\pi i}} (1 + \mathcal{O}(z + x_{j-1})), \quad T_{j,j-1} = \frac{1}{4x_{j-1}} \frac{\sqrt{x_j} + \sqrt{x_{j-1}}}{\sqrt{x_j} - \sqrt{x_{j-1}}}. \quad (3.18)$$

For $j \in \{1, \dots, m\}$, as $z \rightarrow -x_k$, $k \in \{1, \dots, m\}$, $k \neq j, j-1$, $\Im z > 0$, we have

$$D_{s_j}(z) = T_{j,k}^{\frac{\log s_j}{2\pi i}} (1 + \mathcal{O}(z + x_k)), \quad T_{j,k} = \frac{(\sqrt{x_k} - \sqrt{x_{j-1}})(\sqrt{x_k} + \sqrt{x_j})}{(\sqrt{x_k} - \sqrt{x_j})(\sqrt{x_k} + \sqrt{x_{j-1}})}. \quad (3.19)$$

From the above expansions, we obtain, as $z \rightarrow -x_j$, $j \in \{1, \dots, m\}$, $\Im z > 0$ that

$$D(z) = \sqrt{s_j} \left(\prod_{k=1}^m T_{k,j}^{\frac{\log s_k}{2\pi i}} \right) (z + x_j)^{\beta_j} (1 + \mathcal{O}(z + x_j)), \quad (3.20)$$

where β_j is given by that

$$\beta_j = \frac{1}{2\pi i} \log \frac{s_{j+1}}{s_j}, \quad \text{or equivalently} \quad e^{-2i\pi\beta_j} = \frac{s_j}{s_{j+1}}, \quad j = 1, \dots, m. \quad (3.21)$$

with $s_{m+1} := 1$. It will be more convenient to rewrite the product in (3.20) in terms of the β_k 's as follows

$$\prod_{k=1}^m T_{k,j}^{\frac{\log s_k}{2\pi i}} = (4x_j)^{-\beta_j} \prod_{\substack{k=1 \\ k \neq j}}^m \tilde{T}_{k,j}^{-\beta_k}, \quad \text{where} \quad \tilde{T}_{k,j} = \frac{\sqrt{x_j} + \sqrt{x_k}}{|\sqrt{x_j} - \sqrt{x_k}|}. \quad (3.22)$$

We will also need the first two terms of the asymptotics of D at the origin. From (3.15)-(3.16), we obtain

$$D(z) = \sqrt{s_1} \left(1 - d_0 \sqrt{z} + \mathcal{O}(z) \right), \quad \text{as } z \rightarrow 0, \quad (3.23)$$

where

$$d_0 = \frac{\log s_1}{\pi \sqrt{x_1}} - \sum_{j=2}^m \frac{\log s_j}{\pi} \left(\frac{1}{\sqrt{x_{j-1}}} - \frac{1}{\sqrt{x_j}} \right). \quad (3.24)$$

Note that for all $\ell \in \{0, 1, 2, \dots\}$, we can rewrite d_ℓ in terms of the β_j 's as follows

$$d_\ell = \frac{2i(-1)^\ell}{2\ell - 1} \sum_{j=1}^m \beta_j x_j^{\ell - \frac{1}{2}}. \quad (3.25)$$

3.4 Local parametrices

In this section, we aim to find approximations for S in small neighborhoods of $0, -x_1, \dots, -x_m$. This is the part of the RH analysis where we use the assumption that there exists $\delta > 0$ such that (1.12) holds. By (1.12), there exist small disks $\mathcal{D}_{-x_j}$ centred at $-x_j$, $j = 0, 1, \dots, m$, whose radii are fixed (independent of r), but sufficiently small such that they do not intersect. The local parametrix around $-x_j$, $j \in \{0, 1, \dots, m\}$, is defined in $\mathcal{D}_{-x_j}$ and is denoted by $P^{(-x_j)}$. It satisfies an RH problem with the same jumps as S (inside $\mathcal{D}_{-x_j}$) and a behavior near $-x_j$ “close” to S . Furthermore, on the boundary of the disk, $P^{(-x_j)}$ needs to “match” with $P^{(\infty)}$ (called the matching condition). More precisely, we require

$$S(z)P^{(-x_j)}(z)^{-1} = \mathcal{O}(1), \quad \text{as } z \rightarrow -x_j, \quad (3.26)$$

and

$$P^{(-x_j)}(z) = (I + o(1))P^{(\infty)}(z), \quad \text{as } r \rightarrow +\infty, \quad (3.27)$$

uniformly for $z \in \partial\mathcal{D}_{-x_j}$.

3.4.1 Local parametrices around $-x_j$, $j = 1, \dots, m$

For $j \in \{1, \dots, m\}$, $P^{(-x_j)}$ can be explicitly expressed in terms of confluent hypergeometric functions. This construction is standard (see e.g. [36, 31]) and involves a model RH problem whose solution is denoted Φ_{HG} (the details are presented in the appendix, Section 7.2). Let us first consider the function

$$f_{-x_j}(z) = -2 \begin{cases} g(z) - g_+(-x_j), & \text{if } \Im z > 0 \\ -(g(z) - g_-(-x_j)), & \text{if } \Im z < 0 \end{cases} = -2i(\sqrt{-z} - \sqrt{x_j}). \quad (3.28)$$

This is a conformal map from $\mathcal{D}_{-x_j}$ to a neighborhood of 0 and its expansion as $z \rightarrow -x_j$ is given by

$$f_{-x_j}(z) = ic_{-x_j}(z + x_j)(1 + \mathcal{O}(z + x_j)), \quad \text{with } c_{-x_j} = \frac{1}{\sqrt{x_j}} > 0. \quad (3.29)$$

Note also that $f_{-x_j}(\mathbb{R} \cap \mathcal{D}_{-x_j}) \subset i\mathbb{R}$. Now, we use the freedom we had in the choice of the lenses by requiring that f_{-x_j} maps the jump contour for $P^{(-x_j)}$ onto a subset of Σ_{HG} (see Figure 7):

$$f_{-x_j}((\gamma_{j+1,+} \cup \gamma_{j,+}) \cap \mathcal{D}_{-x_j}) \subset \Gamma_3 \cup \Gamma_2, \quad f_{-x_j}((\gamma_{j+1,-} \cup \gamma_{j,-}) \cap \mathcal{D}_{-x_j}) \subset \Gamma_5 \cup \Gamma_6, \quad (3.30)$$

where $\Gamma_3, \Gamma_2, \Gamma_5$ and Γ_6 are as shown in Figure 7. Let us define $P^{(-x_j)}$ by

$$P^{(-x_j)}(z) = E_{-x_j}(z) \Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} e^{-\sqrt{r}g(z)\sigma_3} e^{\frac{\pi i \alpha}{2} \theta(z)\sigma_3}, \quad (3.31)$$

where E_{-x_j} is analytic inside $\mathcal{D}_{-x_j}$ (and will be determined explicitly below) and where the parameter β_j for Φ_{HG} is given by (3.21). Since E_{-x_j} is analytic, it is straightforward from the jumps for Φ_{HG} (given by (7.7)) to verify that $P^{(-x_j)}$ given by (3.31) satisfies the same jumps as S inside $\mathcal{D}_{-x_j}$. In order to fulfil the matching condition (3.27), using (7.8), we need to choose

$$E_{-x_j}(z) = P^{(\infty)}(z) e^{-\frac{\pi i \alpha}{2} \theta(z) \sigma_3} (s_j s_{j+1})^{\frac{\sigma_3}{4}} \left\{ \begin{array}{ll} \sqrt{\frac{s_j}{s_{j+1}}}^{\sigma_3}, & \Im z > 0 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \Im z < 0 \end{array} \right\} \times e^{\sqrt{r} g_+(-x_j) \sigma_3} (\sqrt{r} f_{-x_j}(z))^{\beta_j \sigma_3}. \quad (3.32)$$

It can be verified from the jumps for $P^{(\infty)}$ that E_{-x_j} defined by (3.32) has no jump at all inside $\mathcal{D}_{-x_j}$. Furthermore, $E_{-x_j}(z)$ is bounded as $z \rightarrow -x_j$ and E_{-x_j} is then analytic in the full disk $\mathcal{D}_{-x_j}$, as desired. Since $P^{(-x_j)}$ and S have exactly the same jumps on $(\mathbb{R} \cup \gamma_+ \cup \gamma_-) \cap \mathcal{D}_{-x_j}$, $S(z) P^{(-x_j)}(z)^{-1}$ is analytic in $\mathcal{D}_{-x_j} \setminus \{-x_j\}$. As $z \rightarrow -x_j$ from outside the lenses, by condition (d) in the RH problem for S and by (7.10), $S(z) P^{(-x_j)}(z)^{-1}$ behaves as $\mathcal{O}(\log(z + x_j))$. This means that the singularity is removable and (3.26) holds. We will need later a more detailed knowledge than (3.27). Using (7.8), one shows that

$$P^{(-x_j)}(z) P^{(\infty)}(z)^{-1} = I + \frac{1}{\sqrt{r} f_{-x_j}(z)} E_{-x_j}(z) \Phi_{\text{HG},1}(\beta_j) E_{-x_j}(z)^{-1} + \mathcal{O}(r^{-1}), \quad (3.33)$$

as $r \rightarrow +\infty$, uniformly for $z \in \partial \mathcal{D}_{-x_j}$, where $\Phi_{\text{HG},1}(\beta_j)$ is given by (7.9) (with β_j given by (3.21)). Also, a direct computation using (3.13), (3.20)-(3.22) and (3.29) shows that

$$E_{-x_j}(-x_j) = \begin{pmatrix} 1 & 0 \\ i d_1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4} \sigma_3 x_j^{-\frac{\sigma_3}{4}}} N \Lambda_j^{\sigma_3}, \quad (3.34)$$

where

$$\Lambda_j = e^{-\frac{\pi i \alpha}{2}} (4x_j)^{\beta_j} \left(\prod_{\substack{k=1 \\ k \neq j}}^m \tilde{T}_{k,j}^{\beta_k} \right) e^{\sqrt{r} g_+(-x_j)} r^{\frac{\beta_j}{2}} c_{-x_j}^{\beta_j}. \quad (3.35)$$

3.4.2 Local parametrices around 0

The local parametrix $P^{(0)}$ can be constructed in terms of Bessel functions, and relies on the model RH problem for Φ_{Be} (this model RH problem is well-known, see e.g. [44], and is presented in the appendix, Section 7.1). Let us first consider the function

$$f_0(z) = \frac{g(z)^2}{4} = \frac{z}{4}. \quad (3.36)$$

This is a conformal map from $\mathcal{D}_0$ to a neighborhood of 0. Similarly to the previous local parametrices, we use the freedom in the choice of the lenses by requiring that

$$f_0(\gamma_{1,+}) \subset e^{\frac{2\pi i}{3}} \mathbb{R}^+, \quad f_0(\gamma_{1,-}) \subset e^{-\frac{2\pi i}{3}} \mathbb{R}^+. \quad (3.37)$$

Thus the jump contour for $P^{(0)}$ is mapped by f_0 onto a subset of Σ_{Be} (Σ_{Be} is the jump contour for Φ_{Be} , see Figure 6). We take $P^{(0)}$ in the form

$$P^{(0)}(z) = E_0(z) \Phi_{\text{Be}}(r f_0(z); \alpha) s_1^{-\frac{\sigma_3}{2}} e^{-\sqrt{r} g(z) \sigma_3}, \quad (3.38)$$

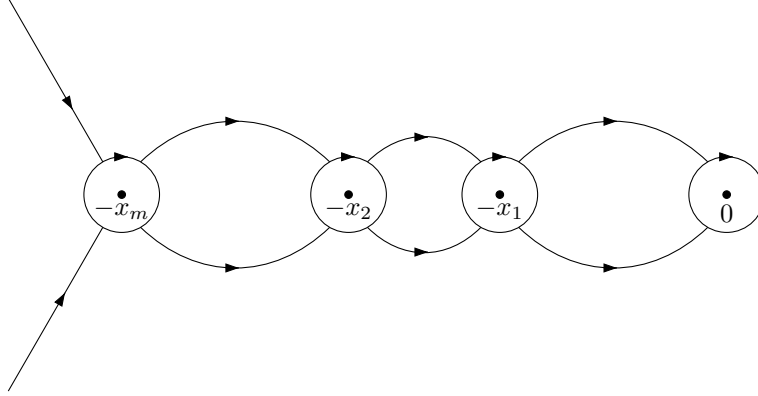


Figure 3: Jump contours Σ_R for the RH problem for R with $m = 3$ and $s_1 \neq 0$.

where E_0 is analytic inside $\mathcal{D}_0$ (and will be determined below). From (7.1), it is straightforward to verify that $P^{(0)}$ given by (3.38) has the same jumps as S inside $\mathcal{D}_0$. In order to satisfy the matching condition (3.27), by (7.2), we defined E_0 by

$$E_0(z) = P^{(\infty)}(z) s_1^{\frac{\sigma_3}{2}} N^{-1} \left(2\pi\sqrt{r} f_0(z)^{1/2} \right)^{\frac{\sigma_3}{2}}. \quad (3.39)$$

It can be verified from the jumps for $P^{(\infty)}$ that E_0 has no jumps in $\mathcal{D}_0$, and has a removable singularity at 0. Therefore, E_0 is analytic in $\mathcal{D}_0$. We will need later a more detailed knowledge of (3.27). Using (7.2), one shows that

$$P^{(0)}(z) P^{(\infty)}(z)^{-1} = I + \frac{1}{\sqrt{r} f_0(z)^{1/2}} P^{(\infty)}(z) s_1^{\frac{\sigma_3}{2}} \Phi_{\text{Be},1}(\alpha) s_1^{-\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1} + \mathcal{O}(r^{-1}), \quad (3.40)$$

as $r \rightarrow +\infty$ uniformly for $z \in \partial\mathcal{D}_0$, where $\Phi_{\text{Be},1}(\alpha)$ is given below (7.2). Furthermore, using (3.13), (3.23) and (3.36), we obtain

$$E_0(0) = \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -id_0 \\ 0 & 1 \end{pmatrix} (\pi\sqrt{r})^{\frac{\sigma_3}{2}}. \quad (3.41)$$

3.5 Small norm problem

The last transformation of the steepest descent is defined by

$$R(z) = \begin{cases} S(z) P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus \bigcup_{j=0}^m \mathcal{D}_{-x_j}, \\ S(z) P^{(-x_j)}(z)^{-1}, & \text{for } z \in \mathcal{D}_{-x_j}, j \in \{0, 1, \dots, m\}. \end{cases} \quad (3.42)$$

By definition of the local parametrices, R has no jumps and is bounded (by (3.26)) inside the $m+1$ disks. Therefore, R is analytic on $\mathbb{C} \setminus \Sigma_R$, where Σ_R consists of the boundaries of the disks, and the part of the lenses away from the disks, as shown in Figure 3. For $z \in \Sigma_R \cap (\gamma_+ \cup \gamma_-)$, from (3.13) and from the discussion at the end of Section 3.2, the jumps $J_R := R_-^{-1} R_+$ satisfy

$$J_R(z) = P^{(\infty)}(z) S_-(z)^{-1} S_+(z) P^{(\infty)}(z)^{-1} = I + \mathcal{O}(e^{-c\sqrt{r}\sqrt{z}}), \quad \text{as } r \rightarrow +\infty, \quad (3.43)$$

for a certain $c > 0$. Let us orient the boundaries of the disks in the clockwise direction (as in Figure 3). For $z \in \bigcup_{j=0}^m \partial\mathcal{D}_{-x_j}$, from (3.33) and (3.40), we have

$$J_R(z) = P^{(\infty)}(z) P^{(-x_j)}(z)^{-1} = I + \mathcal{O}\left(\frac{1}{\sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty. \quad (3.44)$$

Therefore, R satisfies a small norm RH problem. By standard theory for small norm RH problems [23, 24], R exists for sufficiently large r and satisfies

$$R(z) = I + \frac{R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}(r^{-1}), \quad R^{(1)}(z) = \mathcal{O}(1), \quad \text{as } r \rightarrow +\infty \quad (3.45)$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$. Also, the factors $\sqrt{r}^{\pm\beta_j}$ in the entries of E_{-x_j} (see (3.32)) induce factors of the form $\sqrt{r}^{\pm 2\beta_j}$ in the entries of J_R (see (3.33)). Thus, we have

$$\partial_{\beta_j} R(z) = \frac{\partial_{\beta_j} R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}\left(\frac{\log r}{r}\right), \quad \partial_{\beta_j} R^{(1)}(z) = \mathcal{O}(\log r), \quad \text{as } r \rightarrow +\infty. \quad (3.46)$$

Furthermore, since the asymptotics (3.43) and (3.44) hold uniformly for $\beta_1, \dots, \beta_m$ in compact subsets of $i\mathbb{R}$, and uniformly in $x_1, \dots, x_m$ in compact subsets of $(0, +\infty)$ as long as there exists $\delta > 0$ which satisfies (1.12), the asymptotics (3.45) and (3.46) also hold uniformly in $\beta_1, \dots, \beta_m, x_1, \dots, x_m$ in the same way.

The goal for the rest of this section is to obtain $R^{(1)}(z)$ for $z \in \mathbb{C} \setminus \bigcup_{j=0}^m \mathcal{D}_{-x_j}$ and for $z = 0$ explicitly. Since R satisfies the equation

$$R(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{R_-(s)(J_R(s) - I)}{s - z} ds \quad (3.47)$$

and since

$$J_R(z) = I + \frac{J_R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}(r^{-1}), \quad J_R^{(1)}(z) = \mathcal{O}(1), \quad (3.48)$$

as $r \rightarrow \infty$ uniformly for $z \in \bigcup_{j=0}^m \mathcal{D}_{-x_j}$, we obtain that $R^{(1)}$ is simply given by

$$R^{(1)}(z) = \frac{1}{2\pi i} \int_{\bigcup_{j=0}^m \partial \mathcal{D}_{-x_j}} \frac{J_R^{(1)}(s)}{s - z} ds. \quad (3.49)$$

We recall that the expressions for $J_R^{(1)}$ are given by (3.33) and (3.40). These expressions can be analytically continued on the interior of the disks, except at the centers where they have poles of order 1. Since the disks are oriented in the clockwise direction, by a direct residue calculation we have

$$R^{(1)}(z) = \sum_{j=0}^m \frac{1}{z + x_j} \text{Res}(J_R^{(1)}(s), s = -x_j), \quad \text{for } z \in \mathbb{C} \setminus \bigcup_{j=0}^m \mathcal{D}_{-x_j}, \quad (3.50)$$

and

$$R^{(1)}(0) = -\text{Res}\left(\frac{J_R^{(1)}(s)}{s}, s = 0\right) + \sum_{j=1}^m \frac{1}{x_j} \text{Res}(J_R^{(1)}(s), s = -x_j). \quad (3.51)$$

From (3.23), (3.36) and (3.40), we obtain

$$\text{Res}(J_R^{(1)}(s), s = 0) = \frac{d_1(1 - 4\alpha^2)}{8} \begin{pmatrix} -1 & -id_1^{-1} \\ -id_1 & 1 \end{pmatrix}, \quad (3.52)$$

and with increasing effort

$$\text{Res}\left(\frac{J_R^{(1)}(s)}{s}, s = 0\right) = \frac{1}{2} \begin{pmatrix} -(d_0 + d_1 d_0^2) & -id_0^2 \\ -i(\alpha^2 + d_0^2 d_1^2 + 2d_0 d_1 + \frac{3}{4}) & d_0 + d_1 d_0^2 \end{pmatrix}. \quad (3.53)$$

From (3.29) and (3.33)-(3.35), for $j \in \{1, \dots, m\}$, we have

$$\begin{aligned} \text{Res} \left(J_R^{(1)}(s), s = -x_j \right) &= \frac{\beta_j^2}{ic_{-x_j}} \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4} \sigma_3 x_j - \frac{\sigma_3}{4}} N \begin{pmatrix} -1 & \tilde{\Lambda}_{j,1} \\ -\tilde{\Lambda}_{j,2} & 1 \end{pmatrix} \\ &\quad \times N^{-1} x_j^{\frac{\sigma_3}{4}} e^{\frac{\pi i}{4} \sigma_3} \begin{pmatrix} 1 & 0 \\ -id_1 & 1 \end{pmatrix}, \end{aligned}$$

where

$$\tilde{\Lambda}_{j,1} = \tau(\beta_j) \Lambda_j^2 \quad \text{and} \quad \tilde{\Lambda}_{j,2} = \tau(-\beta_j) \Lambda_j^{-2}. \quad (3.54)$$

4 Proof of Theorem 1.1

This section is divided into two parts. In the first part, using the RH analysis done in Section 3, we find large r asymptotics for the differential identity

$$\partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s}) = K_\infty + \sum_{j=1}^m K_{-x_j} + K_0, \quad (4.1)$$

which was obtained in (2.34) with the quantities K_∞ , K_{-x_j} and K_0 defined in (2.35)-(2.37). In the second part, we integrate these asymptotics over the parameters $s_1, \dots, s_m$.

4.1 Large r asymptotics for the differential identity

Asymptotics for K_∞ . For z outside the disks and outside the lenses, by (3.42) we have

$$S(z) = R(z)P^{(\infty)}(z). \quad (4.2)$$

As $z \rightarrow \infty$, we can write

$$R(z) = I + \frac{R_1}{z} + \mathcal{O}(z^{-2}), \quad (4.3)$$

for a certain matrix R_1 independent of z . Thus, by (3.8) and (3.11), we have

$$T_1 = R_1 + P_1^{(\infty)}.$$

From (3.45) and the above expression, we infer that

$$T_1 = P_1^{(\infty)} + \frac{R_1^{(1)}}{\sqrt{r}} + \mathcal{O}(r^{-1}), \quad \text{as } r \rightarrow +\infty,$$

where $R_1^{(1)}$ is defined through the expansion

$$R^{(1)}(z) = \frac{R_1^{(1)}}{z} + \mathcal{O}(z^{-2}), \quad \text{as } z \rightarrow \infty. \quad (4.4)$$

Using (2.35), (3.4), (3.14), (3.46) and (3.50), the first part of the differential identity K_∞ is given by

$$\begin{aligned} K_\infty &= -\frac{i}{2} \sqrt{r} \partial_{s_k} T_{1,12} = -\frac{i}{2} \left(\partial_{s_k} P_{1,12}^{(\infty)} \sqrt{r} + \partial_{s_k} R_{1,12}^{(1)} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) \right) \\ &= \frac{1}{2} \partial_{s_k} d_1 \sqrt{r} - \sum_{j=1}^m \frac{\partial_{s_k} (\beta_j^2 (\tilde{\Lambda}_{j,1} - \tilde{\Lambda}_{j,2} + 2i))}{4ic_{-x_j} \sqrt{x_j}} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (4.5)$$

Asymptotics for K_{-x_j} with $j \in \{1, \dots, m\}$. By inverting the transformations (3.6) and (3.42), and using the expression for $P^{(-x_j)}$ given by (3.31), for z outside the lenses and inside $\mathcal{D}_{-x_j}$, we have

$$T(z) = R(z)E_{-x_j}(z)\Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} e^{-\sqrt{r}g(z) \sigma_3}. \quad (4.6)$$

If furthermore $\Im z > 0$, then by (3.29) and (7.13) we have

$$\Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j) = \widehat{\Phi}_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j). \quad (4.7)$$

Note from (3.21) and the connection formula for the Γ -function (see e.g. [45, equation 5.5.3]) that

$$\frac{\sin(\pi \beta_j)}{\pi} = \frac{1}{\Gamma(\beta_j)\Gamma(1-\beta_j)} = \frac{s_{j+1} - s_j}{2\pi i \sqrt{s_j s_{j+1}}}. \quad (4.8)$$

Therefore, using (3.29) and (7.14), as $z \rightarrow -x_j$ from the upper half plane and outside the lenses, we have

$$\Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} = \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix} (I + \mathcal{O}(z + x_j)) \begin{pmatrix} 1 & \frac{s_{j+1} - s_j}{2\pi i} \log(r(z + x_j)) \\ 0 & 1 \end{pmatrix}, \quad (4.9)$$

where the principal branch is taken for the log and

$$\begin{aligned} \Psi_{j,11} &= \frac{\Gamma(1 - \beta_j)}{(s_j s_{j+1})^{\frac{1}{4}}}, & \Psi_{j,12} &= \frac{(s_j s_{j+1})^{\frac{1}{4}}}{\Gamma(\beta_j)} \left(\log(c_{-x_j} r^{-1/2}) - \frac{i\pi}{2} + \frac{\Gamma'(1 - \beta_j)}{\Gamma(1 - \beta_j)} + 2\gamma_E \right), \\ \Psi_{j,21} &= \frac{\Gamma(1 + \beta_j)}{(s_j s_{j+1})^{\frac{1}{4}}}, & \Psi_{j,22} &= \frac{-(s_j s_{j+1})^{\frac{1}{4}}}{\Gamma(-\beta_j)} \left(\log(c_{-x_j} r^{-1/2}) - \frac{i\pi}{2} + \frac{\Gamma'(-\beta_j)}{\Gamma(-\beta_j)} + 2\gamma_E \right). \end{aligned} \quad (4.10)$$

From (2.9), (3.3), (4.6) and (4.9) we have

$$G_j(-rx_j; r\vec{x}, \vec{s}) = r^{-\frac{\sigma_3}{4}} R(-x_j)E_{-x_j}(-x_j) \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix}. \quad (4.11)$$

In fact K_{-x_j} does not depend on the pre-factor $r^{-\frac{\sigma_3}{4}}$ in (4.11). Let us define

$$H_j = r^{\frac{\sigma_3}{4}} G_j(-rx_j; r\vec{x}, \vec{s}) = R(-x_j)E_{-x_j}(-x_j) \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix}. \quad (4.12)$$

By a straightforward computation, we rewrite (2.36) as follows:

$$\sum_{j=1}^m K_{-x_j} = \sum_{j=1}^m \frac{s_{j+1} - s_j}{2\pi i} (H_{j,11} \partial_{s_k} H_{j,21} - H_{j,21} \partial_{s_k} H_{j,11}). \quad (4.13)$$

Using $\Gamma(1+z) = z\Gamma(z)$ (see e.g. [45, equation 5.5.1]) and (4.8), we note that

$$\Psi_{j,11} \Psi_{j,21} = \beta_j \frac{2\pi i}{s_{j+1} - s_j}, \quad j = 1, \dots, m. \quad (4.14)$$

Also, from (3.34), we have

$$\begin{aligned} \partial_{s_k} E_{-x_j,11}(-x_j) &= E_{-x_j,11}(-x_j) \partial_{s_k} \log \Lambda_j, & \partial_{s_k} E_{-x_j,12}(-x_j) &= -E_{-x_j,12}(-x_j) \partial_{s_k} \log \Lambda_j, \\ \partial_{s_k} E_{-x_j,21}(-x_j) &= E_{-x_j,21}(-x_j) \partial_{s_k} \log \Lambda_j + i E_{-x_j,11}(-x_j) \partial_{s_k} d_1, \\ \partial_{s_k} E_{-x_j,22}(-x_j) &= -E_{-x_j,22}(-x_j) \partial_{s_k} \log \Lambda_j + i E_{-x_j,12}(-x_j) \partial_{s_k} d_1. \end{aligned} \quad (4.15)$$

Therefore, using (3.45), (3.46), $\det E_{-x_j}(-x_j) = 1$ and (4.12)-(4.15), as $r \rightarrow +\infty$ we obtain

$$\begin{aligned} \sum_{j=1}^m K_{-x_j} &= \sum_{j=1}^m \frac{s_{j+1} - s_j}{2\pi i} \left(\Psi_{j,11} \partial_{s_k} \Psi_{j,21} - \Psi_{j,21} \partial_{s_k} \Psi_{j,11} \right) - \sum_{j=1}^m 2\beta_j \partial_{s_k} \log \Lambda_j \\ &\quad + i \partial_{s_k} d_1 \sum_{j=1}^m \frac{s_{j+1} - s_j}{2\pi i} (E_{-x_j,11}(-x_j) \Psi_{j,11} + E_{-x_j,12}(-x_j) \Psi_{j,21})^2 + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (4.16)$$

Again using (3.34) and (4.10), we can simplify (4.16) further by noting that

$$i \partial_{s_k} d_1 \sum_{j=1}^m \frac{s_{j+1} - s_j}{2\pi i} (E_{-x_j,11}(-x_j) \Psi_{j,11} + E_{-x_j,12}(-x_j) \Psi_{j,21})^2 = \sum_{j=1}^m \frac{\partial_{s_k} d_1}{2\sqrt{x_j}} \left(\beta_j^2 (\tilde{\Lambda}_{j,1} + \tilde{\Lambda}_{j,2}) + 2i\beta_j \right). \quad (4.17)$$

Asymptotics for K_0 . Note that we did not use the explicit expression for $R^{(1)}(-x_j)$ to compute the asymptotics for K_{-x_j} up to and including the constant term. The computations for K_0 are more involved and require explicitly $R^{(1)}(0)$ (given by (3.51)). We start by evaluating $G_0(0; r\vec{x}, \vec{s})$. For z outside the lenses and inside $\mathcal{D}_0$, by (3.6), (3.38) and (3.42) we have

$$T(z) = R(z) E_0(z) \Phi_{\text{Be}}(rf_0(z); \alpha) s_1^{-\frac{\sigma_3}{2}} e^{-\sqrt{r}g(z)\sigma_3}. \quad (4.18)$$

From (3.36), (4.18) and (7.5), as $z \rightarrow 0$ from outside the lenses, we have

$$T(z) = R(z) E_0(z) \Phi_{\text{Be},0}(rf_0(z); \alpha) 2^{-\alpha\sigma_3} s_1^{-\frac{\sigma_3}{2}} (rz)^{\frac{\alpha}{2}\sigma_3} \begin{pmatrix} 1 & s_1 h\left(\frac{rz}{4}\right) \\ 0 & 1 \end{pmatrix} e^{-\sqrt{r}g(z)\sigma_3}. \quad (4.19)$$

On the other hand, using (2.11) and (3.3), as $z \rightarrow 0$ we have

$$T(z) = r^{\frac{\sigma_3}{4}} G_0(rz; r\vec{x}, \vec{s}) (rz)^{\frac{\alpha}{2}\sigma_3} \begin{pmatrix} 1 & s_1 h(rz) \\ 0 & 1 \end{pmatrix} e^{-\sqrt{r}g(z)\sigma_3}. \quad (4.20)$$

Therefore, we obtain

$$G_0(0; r\vec{x}, \vec{s}) = r^{-\frac{\sigma_3}{4}} R(0) E_0(0) \Psi_0, \quad \Psi_0 := \begin{cases} \Phi_{\text{Be},0}(0; \alpha) 2^{-\alpha\sigma_3} s_1^{-\frac{\sigma_3}{2}}, & \text{if } \alpha \neq 0, \\ \Phi_{\text{Be},0}(0; 0) s_1^{-\frac{\sigma_3}{2}} \begin{pmatrix} 1 & -\frac{s_1}{\pi i} \log 2 \\ 0 & 1 \end{pmatrix}, & \text{if } \alpha = 0, \end{cases} \quad (4.21)$$

and $\Phi_{\text{Be}}(0; \alpha)$ is computed in the appendix, see (7.6). In the same way as for K_{-x_j} , we define

$$H_0 = r^{\frac{\sigma_3}{4}} G_0(0; r\vec{x}, \vec{s}) = R(0) E_0(0) \Psi_0, \quad (4.22)$$

and we simplify K_0 (given by (2.37)) as follows

$$K_0 = \begin{cases} \frac{s_1}{2\pi i} (H_{0,11} \partial_{s_k} H_{0,21} - H_{0,21} \partial_{s_k} H_{0,11}) & \text{if } \alpha = 0, \\ \alpha (H_{0,21} \partial_{s_k} H_{0,12} - H_{0,11} \partial_{s_k} H_{0,22}) & \text{if } \alpha \neq 0. \end{cases} \quad (4.23)$$

We start with the case $\alpha = 0$. Using (3.41), (3.45)-(3.46), (4.21)-(4.23), and the fact that $R^{(1)}$ is traceless, after a careful calculation, we obtain the following asymptotics as $r \rightarrow +\infty$:

$$\begin{aligned} K_0 &= \frac{s_1}{2\pi i} (H_{0,11} \partial_{s_k} H_{0,21} - H_{0,21} \partial_{s_k} H_{0,11}) = \frac{1}{2} \partial_{s_k} d_1 \sqrt{r} \\ &\quad - \frac{1}{2} \left(d_1 \partial_{s_k} (R_{11}^{(1)}(0) - R_{22}^{(1)}(0)) + i d_1^2 \partial_{s_k} R_{12}^{(1)}(0) + i \partial_{s_k} R_{21}^{(1)}(0) \right) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (4.24)$$

The subleading term in (4.24) can be evaluated using the explicit form for $R^{(1)}(0)$ given by (3.51):

$$-\frac{1}{2}\left(d_1\partial_{s_k}(R_{11}^{(1)}(0)-R_{22}^{(1)}(0))+id_1^2\partial_{s_k}R_{12}^{(1)}(0)+i\partial_{s_k}R_{21}^{(1)}(0)\right)=\frac{d_0\partial_{s_k}d_1}{2} \\ +\sum_{j=1}^m\frac{1}{4ic_{-x_j}\sqrt{x_j}}\partial_{s_k}(\beta_j^2(\tilde{\Lambda}_{j,1}-\tilde{\Lambda}_{j,2}-2i))-\partial_{s_k}d_1\sum_{j=1}^m\frac{\beta_j^2(\tilde{\Lambda}_{j,1}+\tilde{\Lambda}_{j,2})}{2c_{-x_j}x_j}. \quad (4.25)$$

Now, we evaluate K_0 for the case $\alpha \neq 0$. Using the formula $\alpha\Gamma(\alpha) = \Gamma(1+\alpha)$, (3.41), (3.45)-(3.46), (4.21)-(4.23), and the fact that $R^{(1)}(0)$ is traceless, after a lot of cancellations, we obtain

$$K_0 = \alpha\left(H_{0,21}\partial_{s_k}H_{0,12}-H_{0,11}\partial_{s_k}H_{0,22}\right) = \frac{1}{2}\partial_{s_k}d_1\sqrt{r} - \frac{\alpha}{2}\partial_{s_k}(\log s_1) \\ -\frac{1}{2}\left(d_1\partial_{s_k}(R_{11}^{(1)}(0)-R_{22}^{(1)}(0))+id_1^2\partial_{s_k}R_{12}^{(1)}(0)+i\partial_{s_k}R_{21}^{(1)}(0)\right) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \quad (4.26)$$

as $r \rightarrow +\infty$, which is the same formula as (4.24) for $\alpha = 0$, plus the extra factor $-\frac{\alpha}{2}\partial_{s_k}(\log s_1)$ which can be rewritten using (3.21) as follows:

$$-\frac{\alpha}{2}\partial_{s_k}(\log s_1) = \pi i \alpha \partial_{s_k}(\beta_1 + \dots + \beta_m).$$

Asymptotics for the differential identity (2.34). By summing the contributions K_0 , K_{-x_j} , $j = 1, \dots, m$ and K_∞ using (4.5), (4.16), (4.17), (4.24) and (4.25), and by substituting the expression for c_{-x_j} given by (3.29), and the expression for d_0 given by (3.25), a lot of terms cancel each other out and we obtain

$$\partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s}) = \partial_{s_k} d_1 \sqrt{r} + \pi i \alpha \sum_{j=1}^m \partial_{s_k} \beta_j - \sum_{j=1}^m \left(2\beta_j \partial_{s_k} \log \Lambda_j + \partial_{s_k}(\beta_j^2)\right) \\ + \sum_{j=1}^m \frac{s_{j+1} - s_j}{2\pi i} (\Psi_{j,11} \partial_{s_k} \Psi_{j,21} - \Psi_{j,21} \partial_{s_k} \Psi_{j,11}) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty. \quad (4.27)$$

Using the explicit expressions for $\Psi_{j,11}$ and $\Psi_{j,21}$ (see (4.10)) together with the relation (4.14), we have

$$\sum_{j=1}^m \frac{s_{j+1} - s_j}{2\pi i} (\Psi_{j,11} \partial_{s_k} \Psi_{j,21} - \Psi_{j,21} \partial_{s_k} \Psi_{j,11}) = \sum_{j=1}^m \beta_j \partial_{s_k} \log \frac{\Gamma(1+\beta_j)}{\Gamma(1-\beta_j)}. \quad (4.28)$$

Also, using (3.35), we have

$$\sum_{j=1}^m -2\beta_j \partial_{s_k} \log \Lambda_j = -2 \sum_{j=1}^m \beta_j \partial_{s_k}(\beta_j) \log(4x_j c_{-x_j} \sqrt{r}) - 2 \sum_{j=1}^m \beta_j \sum_{\substack{\ell=1 \\ \ell \neq j}}^m \partial_{s_k}(\beta_\ell) \log(\tilde{T}_{\ell,j}). \quad (4.29)$$

It will more convenient to integrate with respect to $\beta_1, \dots, \beta_m$ instead of $s_1, \dots, s_m$. Therefore, we define

$$\tilde{F}_\alpha(r\vec{x}, \vec{\beta}) = F_\alpha(r\vec{x}, \vec{s}), \quad (4.30)$$

where $\vec{\beta} = (\beta_1, \dots, \beta_m)$ and $\vec{s} = (s_1, \dots, s_m)$ are related via the relations (3.21). By substituting (4.28) and (4.29) into (4.27), and by writing the derivative with respect to β_k instead of s_k , as $r \rightarrow +\infty$ we

obtain

$$\begin{aligned} \partial_{\beta_k} \log \tilde{F}_\alpha(r\vec{x}, \vec{\beta}) &= \partial_{\beta_k} d_1 \sqrt{r} - 2 \sum_{j=1}^m \beta_j \partial_{\beta_k}(\beta_j) \log(4x_j c_{-x_j} \sqrt{r}) + \pi i \alpha \\ &\quad - 2 \sum_{j=1}^m \beta_j \sum_{\substack{\ell=1 \\ \ell \neq j}}^m \partial_{\beta_k}(\beta_\ell) \log(\tilde{T}_{\ell,j}) - \sum_{j=1}^m \partial_{\beta_k}(\beta_j^2) + \sum_{j=1}^m \beta_j \partial_{\beta_k} \log \frac{\Gamma(1+\beta_j)}{\Gamma(1-\beta_j)} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (4.31)$$

Using the value of d_1 in (3.25) and the value of c_{-x_j} in (3.29), the above asymptotics can be rewritten more explicitly as follows

$$\begin{aligned} \partial_{\beta_k} \log \tilde{F}_\alpha(r\vec{x}, \vec{\beta}) &= -2i\sqrt{rx_k} - 2\beta_k \log(4\sqrt{rx_k}) + \pi i \alpha \\ &\quad - 2 \sum_{\substack{j=1 \\ j \neq k}}^m \beta_j \log(\tilde{T}_{k,j}) - 2\beta_k + \beta_k \partial_{\beta_k} \log \frac{\Gamma(1+\beta_k)}{\Gamma(1-\beta_k)} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (4.32)$$

4.2 Integration of the differential identity

By the steepest descent of Section 3 (see in particular the discussion in Section 3.5), the asymptotics (4.32) are valid uniformly for $\beta_1, \dots, \beta_m$ in compact subsets of $i\mathbb{R}$. First, we use (4.32) with $\beta_2 = 0 = \beta_3 = \dots = \beta_m$, and we integrate in β_1 from $\beta_1 = 0$ to an arbitrary $\beta_1 \in i\mathbb{R}$. It is important for us to note the following relation (see e.g. [36]):

$$\int_0^\beta x \partial_x \log \frac{\Gamma(1+x)}{\Gamma(1-x)} dx = \beta^2 + \log G(1+\beta)G(1-\beta), \quad (4.33)$$

where G is Barnes' G -function. Let us use the notation $\vec{\beta}_1 = (\beta_1, 0, \dots, 0)$. After integration of (4.32) (with $k = 1$) from $\vec{\beta} = \vec{0} = (0, \dots, 0)$ to $\vec{\beta} = \vec{\beta}_1$, we obtain

$$\log \frac{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_1)}{\tilde{F}_\alpha(r\vec{x}, \vec{0})} = -2i\beta_1 \sqrt{rx_1} - \beta_1^2 \log(4\sqrt{rx_1}) + \pi i \alpha \beta_1 + \log(G(1+\beta_1)G(1-\beta_1)) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right),$$

as $r \rightarrow +\infty$. Now, we use (4.32) with $k = 2$ and $\beta_3 = \dots = \beta_m = 0$, β_1 fixed but not necessarily 0, and we integrate in β_2 . With the notation $\vec{\beta}_2 = (\beta_1, \beta_2, 0, \dots, 0)$, as $r \rightarrow +\infty$ we obtain

$$\begin{aligned} \log \frac{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_2)}{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_1)} &= -2i\beta_2 \sqrt{rx_2} - \beta_2^2 \log(4\sqrt{rx_2}) + \pi i \alpha \beta_2 \\ &\quad - 2\beta_1 \beta_2 \log(\tilde{T}_{2,1}) + \log(G(1+\beta_2)G(1-\beta_2)) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (4.34)$$

By integrating successively in $\beta_3, \dots, \beta_m$, and then by summing the expressions, we obtain

$$\begin{aligned} \log \frac{\tilde{F}_\alpha(r\vec{x}, \vec{\beta})}{\tilde{F}_\alpha(r\vec{x}, \vec{0})} &= - \sum_{j=1}^m 2i\beta_j \sqrt{rx_j} - \sum_{j=1}^m \beta_j^2 \log(4\sqrt{rx_j}) + \pi i \alpha \sum_{j=1}^m \beta_j \\ &\quad - 2 \sum_{1 \leq j < k \leq m} \beta_j \beta_k \log(\tilde{T}_{j,k}) + \sum_{j=1}^m \log(G(1+\beta_j)G(1-\beta_j)) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (4.35)$$

By (4.30) and (2.13), we have $\tilde{F}_\alpha(r\vec{x}, \vec{0}) = F_\alpha(r\vec{x}, \vec{1}) = 1$. This finishes the proof of Theorem 1.1 (after identifying $u_j = -2\pi i \beta_j$).

5 Large r asymptotics for Φ with $s_1 = 0$

In this section, we perform an asymptotic analysis of $\Phi(z; r\vec{x}, \vec{s})$ as $r \rightarrow +\infty$ and $s_1 = 0$. This steepest descent differs from the one done in Section 3 in several aspects. In particular, we need a different g -function, the local parametrix at $-x_1$ is now built in terms of Bessel functions (instead of hypergeometric functions for $s_1 > 0$), and there is no need for a local parametrix at 0 (as opposed to Section 3). On the level of the parameters, we assume that $s_1 = 0$, that $s_2, \dots, s_m$ are in a compact subset of $(0, +\infty)$ and that $x_1, \dots, x_m$ are in a compact subset of $(0, +\infty)$ in such a way that there exists $\delta > 0$ independent of r such that

$$\min_{1 \leq j < k \leq m} x_k - x_j \geq \delta. \quad (5.1)$$

5.1 Normalization of the RH problem with a g -function

Since $s_1 = 0$, we need a different g -function than (3.2). We define

$$g(z) = \sqrt{z + x_1}, \quad (5.2)$$

where the principal branch is taken. It satisfies

$$g(z) = \sqrt{z} + \frac{x_1}{2} z^{-1/2} + \mathcal{O}(z^{-3/2}), \quad \text{as } z \rightarrow \infty. \quad (5.3)$$

We define the first transformation T similarly to (3.3) (however with an extra pre-factor matrix to compensate the asymptotic behavior (5.3) of the g -function)

$$T(z) = \begin{pmatrix} 1 & 0 \\ i \frac{x_1}{2} \sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} \Phi(rz; r\vec{x}, \vec{s}) e^{-\sqrt{r}g(z)\sigma_3}. \quad (5.4)$$

The asymptotics (2.8) of Φ then lead after some calculation to

$$T(z) = \left(I + \frac{T_1}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} N, \quad T_{1,12} = \frac{\Phi_{1,12}(r\vec{x}, \vec{s})}{\sqrt{r}} + i\sqrt{r} \frac{x_1}{2} \quad (5.5)$$

as $z \rightarrow \infty$. For $z \in (-\infty, -x_1)$, since $g_+(z) + g_-(z) = 0$, the jumps for T can be factorized in the same way as (3.5).

5.2 Opening of the lenses

Around each interval $(-x_j, -x_{j-1})$, $j = 2, \dots, m$, we open lenses $\gamma_{j,+}$ and $\gamma_{j,-}$, lying in the upper and lower half plane respectively, as shown in Figure 4. Let us also denote $\Omega_{j,+}$ (resp. $\Omega_{j,-}$) for the region inside the lenses around $(-x_j, -x_{j-1})$ in the upper half plane (resp. in the lower half plane). The next transformation is defined by

$$S(z) = T(z) \prod_{j=2}^m \begin{cases} \begin{pmatrix} 1 & 0 \\ -s_j^{-1} e^{\pi i \alpha} e^{-2\sqrt{r}g(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_{j,+}, \\ \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{-\pi i \alpha} e^{-2\sqrt{r}g(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_{j,-}, \\ I, & \text{if } z \in \mathbb{C} \setminus (\Omega_{j,+} \cup \Omega_{j,-}). \end{cases} \quad (5.6)$$

It is straightforward to verify from the RH problem for Φ and from Section 5.1 that S satisfies the following RH problem.

RH problem for S

(a) $S : \mathbb{C} \setminus \Gamma_S \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, with

$$\Gamma_S = (-\infty, 0) \cup \gamma_+ \cup \gamma_-, \quad \gamma_{\pm} = \bigcup_{j=2}^{m+1} \gamma_{j,\pm}, \quad (5.7)$$

where $\gamma_{m+1,\pm} := -x_m + e^{\pm \frac{2\pi i}{3}}(0, +\infty)$, and Γ_S is oriented as shown in Figure 4.

(b) The jumps for S are given by

$$\begin{aligned} S_+(z) &= S_-(z) \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix}, & z \in (-x_j, -x_{j-1}), j = 2, \dots, m+1, \\ S_+(z) &= S_-(z) e^{\pi i \alpha \sigma_3}, & z \in (-x_1, 0), \\ S_+(z) &= S_-(z) \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{\pm \pi i \alpha} e^{-2\sqrt{r}g(z)} & 1 \end{pmatrix}, & z \in \gamma_{j,\pm}, j = 2, \dots, m+1, \end{aligned}$$

where $x_{m+1} = +\infty$ (we recall that $x_0 = 0$ and $s_{m+1} = 1$).

(c) As $z \rightarrow \infty$, we have

$$S(z) = \left(I + \frac{T_1}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} N. \quad (5.8)$$

As $z \rightarrow -x_j$ from outside the lenses, $j = 1, \dots, m$, we have

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z + x_j)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z + x_j)) \end{pmatrix}. \quad (5.9)$$

As $z \rightarrow 0$, we have

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} z^{\frac{\sigma_3}{2}}. \quad (5.10)$$

Since $\Re g(z) > 0$ for all $z \in \mathbb{C} \setminus (-\infty, -x_1]$ and $\Re g_{\pm}(z) = 0$ for $z \in (-\infty, -x_1)$, the jump matrices for S tend to the identity matrix exponentially fast as $r \rightarrow +\infty$ on the lenses. This convergence is uniform for z outside of fixed neighborhoods of $-x_j$, $j \in \{1, \dots, m\}$, but is not uniform as $r \rightarrow +\infty$ and simultaneously $z \rightarrow -x_j$, $j \in \{1, \dots, m\}$.

5.3 Global parametrix

By ignoring the jumps for S that are pointwise exponentially close to the identity matrix as $r \rightarrow +\infty$, we are left with an RH problem for $P^{(\infty)}$ which is similar to the one done in Section 3.3. However, there are some important differences: the jumps along $(-x_1, 0)$ and the behavior near 0. It will appear later in Section 5.5 that $P^{(\infty)}$ is a good approximation for S away from neighborhoods of $-x_j$, $j = 1, \dots, m$. In particular, $P^{(\infty)}$ will be a good approximation for S in a neighborhood of 0, and thus we will not need a local parametrix near 0 in this steepest descent analysis.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

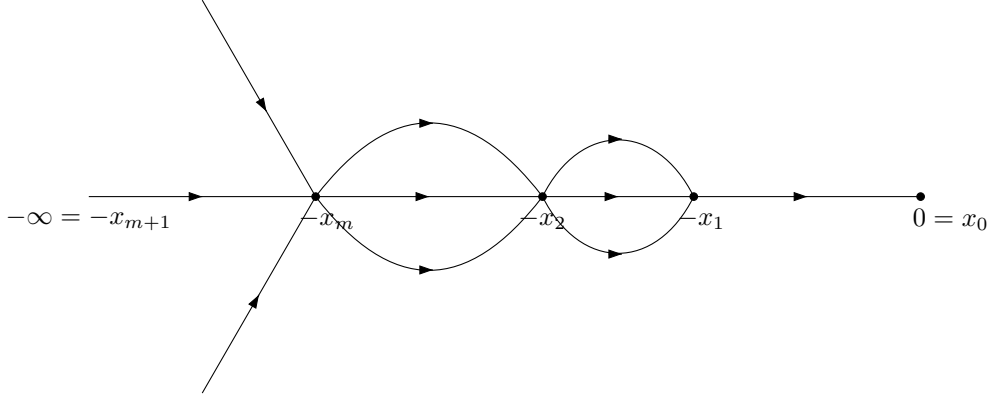


Figure 4: Jump contours Γ_S for the model RH problem for S with $m = 3$ and $s_1 = 0$.

(b) The jumps for $P^{(\infty)}$ are given by

$$\begin{aligned} P_+^{(\infty)}(z) &= P_-^{(\infty)}(z) \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix}, & z \in (-x_j, -x_{j-1}), j = 2, \dots, m+1, \\ P_+^{(\infty)}(z) &= P_-^{(\infty)}(z) e^{\pi i \alpha \sigma_3}, & z \in (-x_1, 0). \end{aligned}$$

(c) As $z \rightarrow \infty$, we have

$$P^{(\infty)}(z) = \left(I + \frac{P_1^{(\infty)}}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} N, \quad (5.11)$$

for a certain matrix $P_1^{(\infty)}$ independent of z .

(d) As $z \rightarrow -x_j$, $j \in \{2, \dots, m\}$, we have $P^{(\infty)}(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}$.

As $z \rightarrow -x_1$, we have $P^{(\infty)}(z) = \begin{pmatrix} \mathcal{O}((z+x_1)^{-\frac{1}{4}}) & \mathcal{O}((z+x_1)^{-\frac{1}{4}}) \\ \mathcal{O}((z+x_1)^{-\frac{1}{4}}) & \mathcal{O}((z+x_1)^{-\frac{1}{4}}) \end{pmatrix}$.

As $z \rightarrow 0$, we have $P^{(\infty)}(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} z^{\frac{\alpha}{2} \sigma_3}$.

Note that the condition (d) for the RH problem for $P^{(\infty)}$ does not come from the RH problem for S (with the exception of the behavior at 0). It is added to ensure uniqueness of the solution. The construction of $P^{(\infty)}$ relies on the following Szegő functions

$$\begin{aligned} D_\alpha(z) &= \exp \left(\frac{\alpha}{2} \sqrt{z+x_1} \int_0^{x_1} \frac{1}{\sqrt{x_1-u} (z+u)} du \right) = \left(\frac{\sqrt{z+x_1} + \sqrt{x_1}}{\sqrt{z+x_1} - \sqrt{x_1}} \right)^{\frac{\alpha}{2}}, \\ D_{\vec{s}}(z) &= \exp \left(\frac{\sqrt{z+x_1}}{2\pi} \sum_{j=2}^m \log s_j \int_{x_{j-1}}^{x_j} \frac{du}{\sqrt{u-x_1}(z+u)} \right). \end{aligned}$$

They satisfy the following jumps

$$\begin{aligned} D_{\alpha,+}(z) D_{\alpha,-}(z) &= 1, & \text{for } z \in (-\infty, -x_1), \\ D_{\alpha,+}(z) &= D_{\alpha,-}(z) e^{-\pi i \alpha}, & \text{for } z \in (-x_1, 0), \\ D_{\vec{s},+}(z) D_{\vec{s},-}(z) &= s_j, & \text{for } z \in (-x_j, -x_{j-1}), j = 2, \dots, m+1. \end{aligned}$$

Furthermore, as $z \rightarrow \infty$, we have

$$\begin{aligned} D_\alpha(z) &= \exp \left(\sum_{\ell=1}^k \frac{d_{\ell,\alpha}}{(z+x_1)^{\ell-\frac{1}{2}}} + \mathcal{O}(z^{-k-\frac{1}{2}}) \right), \\ D_{\bar{s}}(z) &= \exp \left(\sum_{\ell=1}^k \frac{d_{\ell,\bar{s}}}{(z+x_1)^{\ell-\frac{1}{2}}} + \mathcal{O}(z^{-k-\frac{1}{2}}) \right), \end{aligned} \quad (5.12)$$

where $k \in \mathbb{N}_{>0}$ is arbitrary and

$$\begin{aligned} d_{\ell,\alpha} &= \frac{\alpha}{2} \int_0^{x_1} (x_1 - u)^{\ell-\frac{3}{2}} du = \frac{\alpha x_1^{\ell-\frac{1}{2}}}{2\ell-1}, \\ d_{\ell,\bar{s}} &= \frac{(-1)^{\ell-1}}{2\pi} \sum_{j=2}^m \log s_j \int_{x_{j-1}}^{x_j} (u - x_1)^{\ell-\frac{3}{2}} du = \frac{(-1)^{\ell-1}}{\pi(2\ell-1)} \sum_{j=2}^m \log s_j \left((x_j - x_1)^{\ell-\frac{1}{2}} - (x_{j-1} - x_1)^{\ell-\frac{1}{2}} \right). \end{aligned} \quad (5.13)$$

For $\ell \geq 1$, we define $d_\ell = d_{\ell,\alpha} + d_{\ell,\bar{s}}$. Let us finally define

$$P^{(\infty)}(z) = \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} (z+x_1)^{-\frac{\sigma_3}{4}} N D(z)^{-\sigma_3}, \quad (5.14)$$

where the principal branch is taken for the root, and where $D(z) = D_\alpha(z)D_{\bar{s}}(z)$. From the above properties of D_α and $D_{\bar{s}}$, one can check that $P^{(\infty)}$ satisfies criteria (a), (b) and (c) of the RH problem for $P^{(\infty)}$, with

$$P_{1,12}^{(\infty)} = id_1. \quad (5.15)$$

The rest of the current section consists of computing of the first terms in the asymptotics of $D(z)$ as $z \rightarrow -x_j$, $j = 0, 1, \dots, m$. In particular, it will prove that $P^{(\infty)}$ defined in (5.14) satisfies condition (d) of the RH problem for $P^{(\infty)}$. After integrations, we can rewrite $D_{\bar{s}}$ as follows

$$D_{\bar{s}}(z) = \prod_{j=2}^m D_{s_j}(z), \quad (5.16)$$

where

$$D_{s_j}(z) = \left(\frac{(\sqrt{z+x_1} - i\sqrt{x_{j-1}-x_1})(\sqrt{z+x_1} + i\sqrt{x_j-x_1})}{(\sqrt{z+x_1} - i\sqrt{x_j-x_1})(\sqrt{z+x_1} + i\sqrt{x_{j-1}-x_1})} \right)^{\frac{\log s_j}{2\pi i}}. \quad (5.17)$$

As $z \rightarrow -x_j$, $j \in \{2, \dots, m\}$, $\Im z > 0$, we have

$$D_{s_j}(z) = \sqrt{s_j} T_{j,j}^{\frac{\log s_j}{2\pi i}} (z+x_j)^{-\frac{\log s_j}{2\pi i}} (1 + \mathcal{O}(z+x_j)), \quad T_{j,j} = 4(x_j-x_1) \frac{\sqrt{x_j-x_1} - \sqrt{x_{j-1}-x_1}}{\sqrt{x_j-x_1} + \sqrt{x_{j-1}-x_1}}. \quad (5.18)$$

As $z \rightarrow -x_{j-1}$, $j \in \{3, \dots, m\}$, $\Im z > 0$, we have

$$D_{s_j}(z) = T_{j,j-1}^{\frac{\log s_j}{2\pi i}} (z+x_{j-1})^{\frac{\log s_j}{2\pi i}} (1 + \mathcal{O}(z+x_{j-1})), \quad T_{j,j-1} = \frac{1}{4(x_{j-1}-x_1)} \frac{\sqrt{x_j-x_1} + \sqrt{x_{j-1}-x_1}}{\sqrt{x_j-x_1} - \sqrt{x_{j-1}-x_1}}. \quad (5.19)$$

For $j \in \{2, \dots, m\}$, as $z \rightarrow -x_k$, $k \in \{2, \dots, m\}$, $k \neq j, j-1$, $\Im z > 0$, we have

$$D_{s_j}(z) = T_{j,k}^{\frac{\log s_j}{2\pi i}} (1 + \mathcal{O}(z+x_k)), \quad T_{j,k} = \frac{(\sqrt{x_k-x_1} - \sqrt{x_{j-1}-x_1})(\sqrt{x_k-x_1} + \sqrt{x_j-x_1})}{(\sqrt{x_k-x_1} - \sqrt{x_j-x_1})(\sqrt{x_k-x_1} + \sqrt{x_{j-1}-x_1})}. \quad (5.20)$$

From the above expansion, we obtain, as $z \rightarrow -x_j$, $j \in \{2, \dots, m\}$, $\Im z > 0$ that

$$D(z) = \sqrt{s_j} \left(\prod_{k=2}^m T_{k,j}^{\frac{\log s_k}{2\pi i}} \right) D_{\alpha,+}(-x_j)(z+x_j)^{\beta_j} (1 + \mathcal{O}(z+x_j)), \quad (5.21)$$

where $\beta_2, \dots, \beta_m$ are given by

$$\beta_j = \frac{1}{2\pi i} \log \frac{s_{j+1}}{s_j}, \quad \text{or equivalently} \quad e^{-2i\pi\beta_j} = \frac{s_j}{s_{j+1}}, \quad j = 2, \dots, m, \quad (5.22)$$

with $s_{m+1} := 1$. Note that

$$\prod_{k=2}^m T_{k,j}^{\frac{\log s_k}{2\pi i}} = (4(x_j - x_1))^{-\beta_j} \prod_{\substack{k=2 \\ k \neq j}}^m \tilde{T}_{k,j}^{-\beta_k}, \quad \text{where} \quad \tilde{T}_{k,j} = \frac{\sqrt{x_j - x_1} + \sqrt{x_k - x_1}}{|\sqrt{x_j - x_1} - \sqrt{x_k - x_1}|}. \quad (5.23)$$

As $z \rightarrow -x_1$, we have

$$D_{\bar{s}}(z) = \sqrt{s_2} \left(1 - d_{0,\bar{s}} \sqrt{z+x_1} + \mathcal{O}(z+x_1) \right), \quad (5.24)$$

where

$$d_{0,\bar{s}} = \frac{\log s_2}{\pi \sqrt{x_2 - x_1}} - \sum_{j=3}^m \frac{\log s_j}{\pi} \left(\frac{1}{\sqrt{x_{j-1} - x_1}} - \frac{1}{\sqrt{x_j - x_1}} \right). \quad (5.25)$$

As $z \rightarrow -x_1$, $\Im z > 0$, we have

$$D_{\alpha}(z) = e^{-\frac{\pi i \alpha}{2}} \left(1 - d_{0,\alpha} \sqrt{z+x_1} + \mathcal{O}(z+x_1) \right), \quad d_{0,\alpha} = \frac{-\alpha}{\sqrt{x_1}}. \quad (5.26)$$

It follows that as $z \rightarrow -x_1$, $\Im z > 0$, we have

$$D(z) = \sqrt{s_2} e^{-\frac{\pi i \alpha}{2}} \left(1 - d_0 \sqrt{z+x_1} + \mathcal{O}(z+x_1) \right), \quad d_0 := d_{0,\alpha} + d_{0,\bar{s}}. \quad (5.27)$$

Note that for all $\ell \in \{0, 1, 2, \dots\}$, we can rewrite d_{ℓ} in terms of the β_j 's as follows

$$d_{\ell} = \frac{\alpha x_1^{\ell-\frac{1}{2}}}{2\ell-1} + \frac{2i(-1)^{\ell}}{2\ell-1} \sum_{j=2}^m \beta_j (x_j - x_1)^{\ell-\frac{1}{2}}. \quad (5.28)$$

As $z \rightarrow 0$, we have

$$D(z) = D_0 z^{-\frac{\alpha}{2}} (1 + \mathcal{O}(z)), \quad (5.29)$$

where

$$D_0 = \exp \left(\frac{\alpha}{2} \log(4x_1) - \sum_{j=2}^m 2i\beta_j \arccos \left(\frac{\sqrt{x_1}}{\sqrt{x_j}} \right) \right). \quad (5.30)$$

5.4 Local parametrixes

In this section, we aim to find approximations for S in small neighborhoods of $-x_1, \dots, -x_m$ (as already mentioned, there is no need for a local parametrix in a neighborhood of 0). By (1.29), there exist small disks $\mathcal{D}_{-x_j}$ centred at $-x_j$, $j = 1, \dots, m$, whose radii are fixed (independent of r), but sufficiently small such that they do not intersect. The local parametrix around $-x_j$, $j \in \{1, \dots, m\}$,

is defined in $\mathcal{D}_{-x_j}$ and is denoted by $P^{(-x_j)}$. It satisfies an RH problem with the same jumps as S (inside $\mathcal{D}_{-x_j}$) and in addition we require

$$S(z)P^{(-x_j)}(z)^{-1} = \mathcal{O}(1), \quad \text{as } z \rightarrow -x_j, \quad (5.31)$$

and

$$P^{(-x_j)}(z) = (I + o(1))P^{(\infty)}(z), \quad \text{as } r \rightarrow +\infty, \quad (5.32)$$

uniformly for $z \in \partial\mathcal{D}_{-x_j}$.

5.4.1 Local parametrices around $-x_j$, $j = 2, \dots, m$

For $j \in \{2, \dots, m\}$, $P^{(-x_j)}$ can be explicitly expressed in terms of the model RH problem for Φ_{HG} (see Section 7.2). This construction is very similar to the one done in Section 3.4.1, and we provide less details here. Let us first consider the function

$$f_{-x_j}(z) = -2 \begin{cases} g(z) - g_+(-x_j), & \text{if } \Im z > 0 \\ -(g(z) - g_-(-x_j)), & \text{if } \Im z < 0 \end{cases} = -2i(\sqrt{-z - x_1} - \sqrt{x_j - x_1}). \quad (5.33)$$

This is a conformal map from $\mathcal{D}_{-x_j}$ to a neighborhood of 0, and its expansion as $z \rightarrow -x_j$ is given by

$$f_{-x_j}(z) = ic_{-x_j}(z + x_j)(1 + \mathcal{O}(z + x_j)) \quad \text{with} \quad c_{-x_j} = \frac{1}{\sqrt{x_j - x_1}} > 0. \quad (5.34)$$

Note also that $f_{-x_j}(\mathbb{R} \cap \mathcal{D}_{-x_j}) \subset i\mathbb{R}$. Now, we deform the lenses in a similar way as in (3.30), that is, such that f_{-x_j} maps the jump contour for $P^{(-x_j)}$ onto a subset of Σ_{HG} (see Figure 7). It can be checked that the local parametrix is given by

$$P^{(-x_j)}(z) = E_{-x_j}(z)\Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} e^{-\sqrt{r}g(z)\sigma_3} e^{\frac{\pi i \alpha}{2}\theta(z)\sigma_3}, \quad (5.35)$$

where E_{-x_j} is analytic inside $\mathcal{D}_{-x_j}$ and given by

$$E_{-x_j}(z) = P^{(\infty)}(z) e^{-\frac{\pi i \alpha}{2}\theta(z)\sigma_3} (s_j s_{j+1})^{\frac{\sigma_3}{4}} \begin{cases} \sqrt{\frac{s_j}{s_{j+1}}}, & \Im z > 0 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \Im z < 0 \end{cases} \times e^{\sqrt{r}g_+(-x_j)\sigma_3} (\sqrt{r}f_{-x_j}(z))^{\beta_j \sigma_3}. \quad (5.36)$$

We will need later a more detailed knowledge than (5.32). Using (7.8), one shows that

$$P^{(-x_j)}(z)P^{(\infty)}(z)^{-1} = I + \frac{1}{\sqrt{r}f_{-x_j}(z)} E_{-x_j}(z)\Phi_{\text{HG},1}(\beta_j)E_{-x_j}(z)^{-1} + \mathcal{O}(r^{-1}), \quad (5.37)$$

as $r \rightarrow +\infty$, uniformly for $z \in \partial\mathcal{D}_{-x_j}$, where $\Phi_{\text{HG},1}(\beta_j)$ is given by (7.9) with the parameter β_j given by (5.22). Also, a direct computation shows that

$$E_{-x_j}(-x_j) = \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3} (x_j - x_1)^{-\frac{\sigma_3}{4}} N\Lambda_j^{\sigma_3}, \quad (5.38)$$

where

$$\Lambda_j = D_{\alpha,+}(-x_j)^{-1} e^{-\frac{\pi i \alpha}{2}} (4(x_j - x_1))^{\beta_j} \left(\prod_{\substack{k=2 \\ k \neq j}}^m \tilde{T}_{k,j}^{\beta_k} \right) e^{\sqrt{r}g_+(-x_j)} r^{\frac{\beta_j}{2}} c_{-x_j}^{\beta_j}. \quad (5.39)$$

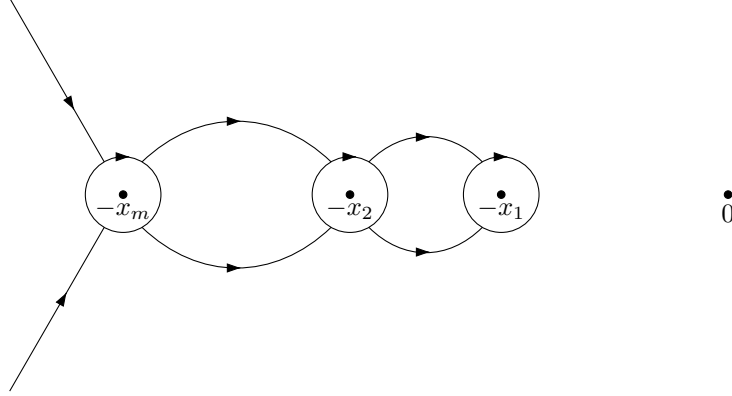


Figure 5: Jump contours Σ_R for the RH problem for R with $m = 3$ and $s_1 = 0$.

5.4.2 Local parametrix around $-x_1$

The local parametrix $P^{(-x_1)}$ can be expressed in terms of the model RH problem $\Phi_{\text{Be}}(z; 0)$ presented in Section 7.1. This construction is similar to the one done in Section 3.4.2 (note however that in Section 3.4.2 we needed $\Phi_{\text{Be}}(z; \alpha)$), and we provide less details here. Let us first consider the function

$$f_{-x_1}(z) = \frac{g(z)^2}{4} = \frac{z + x_1}{4}. \quad (5.40)$$

This is a conformal map from $\mathcal{D}_{-x_1}$ to a neighborhood of 0. Similarly to (3.37), we choose $\gamma_{2,\pm}$ such that the jump contour for $P^{(-x_1)}$ is mapped by f_{-x_1} onto a subset of Σ_{Be} (see Figure 6). It can be verified that $P^{(-x_1)}$ is given by

$$P^{(-x_1)}(z) = E_{-x_1}(z) \Phi_{\text{Be}}(r f_{-x_1}(z); 0) s_2^{-\frac{\sigma_3}{2}} e^{-\sqrt{r} g(z) \sigma_3} e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3}, \quad (5.41)$$

where E_{-x_1} is analytic inside $\mathcal{D}_{-x_1}$ and is given by

$$E_{-x_1}(z) = P^{(\infty)}(z) e^{-\frac{\pi i \alpha}{2} \theta(z) \sigma_3} s_2^{\frac{\sigma_3}{2}} N^{-1} \left(2\pi \sqrt{r} f_{-x_1}(z)^{1/2} \right)^{\frac{\sigma_3}{2}}. \quad (5.42)$$

We will need later a more detailed knowledge than (5.32). Using (7.2), one shows that

$$\begin{aligned} P^{(-x_1)}(z) P^{(\infty)}(z)^{-1} &= I \\ &+ \frac{1}{\sqrt{r} f_{-x_1}(z)^{1/2}} P^{(\infty)}(z) e^{-\frac{\pi i \alpha}{2} \theta(z) \sigma_3} s_2^{\frac{\sigma_3}{2}} \Phi_{\text{Be},1}(0) s_2^{-\frac{\sigma_3}{2}} e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} P^{(\infty)}(z)^{-1} + \mathcal{O}(r^{-1}), \end{aligned} \quad (5.43)$$

as $r \rightarrow +\infty$ uniformly for $z \in \partial \mathcal{D}_{-x_1}$, where $\Phi_{\text{Be},1}(0)$ is given below (7.2). Furthermore,

$$E_{-x_1}(-x_1) = \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -id_0 \\ 0 & 1 \end{pmatrix} (\pi \sqrt{r})^{\frac{\sigma_3}{2}}. \quad (5.44)$$

5.5 Small norm problem

The last transformation of the steepest descent is defined by

$$R(z) = \begin{cases} S(z) P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus \bigcup_{j=1}^m \mathcal{D}_{-x_j}, \\ S(z) P^{(-x_j)}(z)^{-1}, & \text{for } z \in \mathcal{D}_{-x_j}, j \in \{1, \dots, m\}. \end{cases} \quad (5.45)$$

The analysis of R is similar to the one done in Section 3.5, and we provide less details here. The main difference lies in the analysis of $R(z)$ for z in a neighborhood of 0. From the RH problems for S and $P^{(\infty)}$, it is straightforward to verify that R has no jumps along $(-x_1, 0)$ and is bounded as $z \rightarrow 0$. Thus R is analytic in a neighborhood of 0. Also, by definition of the local parametrices, R is analytic on $\mathbb{C} \setminus \Sigma_R$, where Σ_R consists of the boundaries of the disks, and the part of the lenses away from the disks, as shown in Figure 5. As in Section 3.5, the jumps for R on the lenses are uniformly exponentially close to I as $r \rightarrow +\infty$. On the boundary of the disks, the jumps are close to I by an error of order $\mathcal{O}(r^{-1/2})$. Therefore, R satisfies a small norm RH problem. By standard theory [23, 24] (see also Section 3.5), R exists for sufficiently large r and satisfies

$$R(z) = I + \frac{R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}(r^{-1}) \quad R^{(1)}(z) = \mathcal{O}(1), \quad (5.46)$$

$$\partial_{\beta_j} R(z) = \frac{\partial_{\beta_j} R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}\left(\frac{\log r}{r}\right), \quad \partial_{\beta_j} R^{(1)}(z) = \mathcal{O}(\log r) \quad (5.47)$$

as $r \rightarrow +\infty$, uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, uniformly for $\beta_2, \dots, \beta_m$ in compact subsets of $i\mathbb{R}$, and uniformly in $x_1, \dots, x_m$ in compact subsets of $(0, +\infty)$ as long as there exists $\delta > 0$ which satisfies (1.29).

The goal for the rest of this section is to obtain $R^{(1)}(z)$ for $z \in \mathbb{C} \setminus \bigcup_{j=1}^m \mathcal{D}_{-x_j}$ and for $z = -x_1$ explicitly. Let us take the clockwise orientation on the boundaries of the disks, and let us denote by $J_R(z)$ for the jumps of R . Since J_R admits a large r expansion of the form

$$J_R(z) = I + \frac{J_R^{(1)}(z)}{\sqrt{r}} + \mathcal{O}(r^{-1}), \quad (5.48)$$

as $r \rightarrow \infty$ uniformly for $z \in \bigcup_{j=1}^m \mathcal{D}_{-x_j}$, we obtain (in the same way as in Section 3.5) that $R^{(1)}$ is simply given by

$$R^{(1)}(z) = \frac{1}{2\pi i} \int_{\bigcup_{j=1}^m \partial \mathcal{D}_{-x_j}} \frac{J_R^{(1)}(s)}{s - z} ds. \quad (5.49)$$

By a direct residue calculation we have

$$R^{(1)}(z) = \sum_{j=1}^m \frac{1}{z + x_j} \text{Res}(J_R^{(1)}(s), s = -x_j), \quad \text{for } z \in \mathbb{C} \setminus \bigcup_{j=1}^m \mathcal{D}_{-x_j} \quad (5.50)$$

and

$$R^{(1)}(-x_1) = -\text{Res}\left(\frac{J_R^{(1)}(s)}{s + x_1}, s = -x_1\right) + \sum_{j=2}^m \frac{1}{x_j - x_1} \text{Res}(J_R^{(1)}(s), s = -x_j). \quad (5.51)$$

From (5.43), we have

$$\text{Res}(J_R^{(1)}(s), s = -x_1) = \frac{d_1}{8} \begin{pmatrix} -1 & -id_1^{-1} \\ -id_1 & 1 \end{pmatrix}, \quad (5.52)$$

and with increasing effort

$$\text{Res}\left(\frac{J_R^{(1)}(s)}{s + x_1}, s = -x_1\right) = \frac{1}{2} \begin{pmatrix} -(d_0 + d_1 d_0^2) & -id_0^2 \\ -i(d_0^2 d_1^2 + 2d_0 d_1 + \frac{3}{4}) & d_0 + d_1 d_0^2 \end{pmatrix}. \quad (5.53)$$

From (5.37)-(5.39), for $j \in \{2, \dots, m\}$, we have

$$\begin{aligned} \text{Res} \left(J_R^{(1)}(s), s = -x_j \right) &= \frac{\beta_j^2}{ic_{-x_j}} \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4} \sigma_3 (x_j - x_1) - \frac{\sigma_3}{4}} N \begin{pmatrix} -1 & \tilde{\Lambda}_{j,1} \\ -\tilde{\Lambda}_{j,2} & 1 \end{pmatrix} \\ &\quad \times N^{-1}(x_j - x_1)^{\frac{\sigma_3}{4}} e^{\frac{\pi i}{4} \sigma_3} \begin{pmatrix} 1 & 0 \\ -id_1 & 1 \end{pmatrix}, \end{aligned}$$

where

$$\tilde{\Lambda}_{j,1} = \tau(\beta_j) \Lambda_j^2 \quad \text{and} \quad \tilde{\Lambda}_{j,2} = \tau(-\beta_j) \Lambda_j^{-2}. \quad (5.54)$$

6 Proof of Theorem 1.4

This section is divided into two parts in the same way as in Section 4. In the first part, using the RH analysis done in Section 5, we find large r asymptotics for the differential identity

$$\partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s}) = K_\infty + \sum_{j=1}^m K_{-x_j} + K_0, \quad (6.1)$$

which was obtained in (2.34) with the quantities K_∞ , K_{-x_j} and K_0 defined in (2.35)-(2.37). In the second part, we integrate these asymptotics over the parameters $s_2, \dots, s_m$. Some parts of the computations in this section are close to those done in Section 4. However, it requires some adaptation and we provide the details for completeness.

6.1 Large r asymptotics for the differential identity

Asymptotics for K_∞ . For z outside the disks and outside the lenses, by (5.45) we have

$$S(z) = R(z)P^{(\infty)}(z). \quad (6.2)$$

As $z \rightarrow \infty$, we can write

$$R(z) = I + \frac{R_1}{z} + \mathcal{O}(z^{-2}), \quad (6.3)$$

for a certain matrix R_1 independent of z . Thus, by (5.8) and (5.11), we have

$$T_1 = R_1 + P_1^{(\infty)}.$$

Using (5.46) and the above expressions, as $r \rightarrow +\infty$ we have

$$T_1 = P_1^{(\infty)} + \frac{R_1^{(1)}}{\sqrt{r}} + \mathcal{O}(r^{-1}),$$

where $R_1^{(1)}$ is defined through the expansion

$$R^{(1)}(z) = \frac{R_1^{(1)}}{z} + \mathcal{O}(z^{-2}), \quad \text{as } z \rightarrow \infty. \quad (6.4)$$

By (2.35), (5.5), (5.15), (5.47) and (5.50), the large r asymptotics for K_∞ are given by

$$\begin{aligned} K_\infty &= -\frac{i}{2} \sqrt{r} \partial_{s_k} T_{1,12} = -\frac{i}{2} \left(\partial_{s_k} P_{1,12}^{(\infty)} \sqrt{r} + \partial_{s_k} R_{1,12}^{(1)} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) \right) \\ &= \frac{1}{2} \partial_{s_k} d_1 \sqrt{r} - \sum_{j=2}^m \frac{\partial_{s_k} (\beta_j^2 (\tilde{\Lambda}_{j,1} - \tilde{\Lambda}_{j,2} + 2i))}{4ic_{-x_j} \sqrt{x_j - x_1}} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (6.5)$$

Asymptotics for K_{-x_j} with $j \in \{2, \dots, m\}$. By inverting the transformations (5.6) and (5.45), and using the expression for $P^{(-x_j)}$ given by (5.35), for z outside the lenses and inside $\mathcal{D}_{-x_j}$, we have

$$T(z) = R(z)E_{-x_j}(z)\Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} e^{-\sqrt{r}g(z) \sigma_3}. \quad (6.6)$$

If furthermore $\Im z > 0$, then by (5.33) and (7.13) we have

$$\Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j) = \widehat{\Phi}_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j). \quad (6.7)$$

Note from (5.22) and the connection formula for the Γ -function that

$$\frac{\sin(\pi \beta_j)}{\pi} = \frac{1}{\Gamma(\beta_j)\Gamma(1-\beta_j)} = \frac{s_{j+1} - s_j}{2\pi i \sqrt{s_j s_{j+1}}}. \quad (6.8)$$

Therefore, using (5.33) and (7.14), as $z \rightarrow -x_j$ from the upper half plane and outside the lenses, we have

$$\Phi_{\text{HG}}(\sqrt{r}f_{-x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} = \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix} (I + \mathcal{O}(z + x_j)) \begin{pmatrix} 1 & \frac{s_{j+1} - s_j}{2\pi i} \log(r(z + x_j)) \\ 0 & 1 \end{pmatrix}, \quad (6.9)$$

where the principal branch is taken for the log and

$$\begin{aligned} \Psi_{j,11} &= \frac{\Gamma(1-\beta_j)}{(s_j s_{j+1})^{\frac{1}{4}}}, & \Psi_{j,12} &= \frac{(s_j s_{j+1})^{\frac{1}{4}}}{\Gamma(\beta_j)} \left(\log(c_{-x_j} r^{-1/2}) - \frac{i\pi}{2} + \frac{\Gamma'(1-\beta_j)}{\Gamma(1-\beta_j)} + 2\gamma_E \right), \\ \Psi_{j,21} &= \frac{\Gamma(1+\beta_j)}{(s_j s_{j+1})^{\frac{1}{4}}}, & \Psi_{j,22} &= \frac{-(s_j s_{j+1})^{\frac{1}{4}}}{\Gamma(-\beta_j)} \left(\log(c_{-x_j} r^{-1/2}) - \frac{i\pi}{2} + \frac{\Gamma'(-\beta_j)}{\Gamma(-\beta_j)} + 2\gamma_E \right). \end{aligned} \quad (6.10)$$

From (2.9), (5.4), (6.6) and (6.9) we have

$$G_j(-rx_j; r\vec{x}, \vec{s}) = r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ -i\frac{x_1}{2}\sqrt{r} & 1 \end{pmatrix} R(-x_j)E_{-x_j}(-x_j) \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix}. \quad (6.11)$$

In fact K_{-x_j} does not depend on the first two pre-factors in (6.11). Let us define

$$H_j = \begin{pmatrix} 1 & 0 \\ i\frac{x_1}{2}\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} G_j(-rx_j; r\vec{x}, \vec{s}) = R(-x_j)E_{-x_j}(-x_j) \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix}. \quad (6.12)$$

By a straightforward computation, we rewrite (2.36) as

$$\sum_{j=2}^m K_{-x_j} = \sum_{j=2}^m \frac{s_{j+1} - s_j}{2\pi i} (H_{j,11} \partial_{s_k} H_{j,21} - H_{j,21} \partial_{s_k} H_{j,11}). \quad (6.13)$$

Using the connection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we note that

$$\Psi_{j,11} \Psi_{j,21} = \beta_j \frac{2\pi i}{s_{j+1} - s_j}, \quad j = 2, \dots, m. \quad (6.14)$$

Also, from (5.38), E_{-x_j} satisfies (4.15). Therefore, using (5.46), (5.47), $\det E_{-x_j}(-x_j) = 1$, (6.12)-(6.14) and (4.15), as $r \rightarrow +\infty$ we obtain

$$\begin{aligned} \sum_{j=2}^m K_{-x_j} &= \sum_{j=2}^m \frac{s_{j+1} - s_j}{2\pi i} (\Psi_{j,11} \partial_{s_k} \Psi_{j,21} - \Psi_{j,21} \partial_{s_k} \Psi_{j,11}) - \sum_{j=2}^m 2\beta_j \partial_{s_k} \log \Lambda_j \\ &\quad + i \partial_{s_k} d_1 \sum_{j=2}^m \frac{s_{j+1} - s_j}{2\pi i} (E_{-x_j,11}(-x_j) \Psi_{j,11} + E_{-x_j,12}(-x_j) \Psi_{j,21})^2 + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \end{aligned} \quad (6.15)$$

where, by (5.38) and (6.10), one has

$$i\partial_{s_k} d_1 \sum_{j=2}^m \frac{s_{j+1} - s_j}{2\pi i} (E_{-x_j, 11}(-x_j) \Psi_{j, 11} + E_{-x_j, 12}(-x_j) \Psi_{j, 21})^2 = \sum_{j=2}^m \frac{\partial_{s_k} d_1}{2\sqrt{x_j - x_1}} \left(\beta_j^2 (\tilde{\Lambda}_{j,1} + \tilde{\Lambda}_{j,2}) + 2i\beta_j \right). \quad (6.16)$$

Asymptotics for K_{-x_1} . Note that we did not use the explicit expression for $R^{(1)}(-x_j)$ to compute the asymptotics for K_{-x_j} up to and including the constant term for $j = 2, \dots, m$. The computations for K_{-x_1} are more involved and require explicitly $R^{(1)}(-x_1)$ (given by (5.51)). We start by evaluating $G_1(-rx_1; r\vec{x}, \vec{s})$. For z outside the lenses and inside $\mathcal{D}_{-x_1}$, by (5.6), (5.41) and (5.45), we have that

$$T(z) = R(z)E_{-x_1}(z)\Phi_{\text{Be}}(rf_{-x_1}(z); 0) s_2^{-\frac{\sigma_3}{2}} e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} e^{-\sqrt{r}g(z) \sigma_3}. \quad (6.17)$$

From (5.40), (6.17) and (7.5), as $z \rightarrow -x_1$ from outside the lenses, we have

$$T(z) = R(z)E_{-x_1}(z)\Phi_{\text{Be},0}(rf_{-x_1}(z); 0) s_2^{-\frac{\sigma_3}{2}} \begin{pmatrix} 1 & \frac{s_2}{2\pi i} \log \frac{r(z+x_1)}{4} \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} e^{-\sqrt{r}g(z) \sigma_3}. \quad (6.18)$$

On the other hand, using (2.9) and (5.4), as $z \rightarrow -x_1$, $\Im z > 0$, we have

$$T(z) = \begin{pmatrix} 1 & 0 \\ i\frac{x_1}{2}\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} G_1(rz; r\vec{x}, \vec{s}) \begin{pmatrix} 1 & \frac{s_2}{2\pi i} \log(r(z+x_1)) \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i \alpha}{2} \sigma_3} e^{-\sqrt{r}g(z) \sigma_3}. \quad (6.19)$$

Therefore, using also (7.6), we obtain

$$G_1(-rx_1; r\vec{x}, \vec{s}) = r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ -i\frac{x_1}{2}\sqrt{r} & 1 \end{pmatrix} R(-x_1)E_{-x_1}(-x_1) \begin{pmatrix} \Psi_{1,11} & \Psi_{1,12} \\ \Psi_{1,21} & \Psi_{1,22} \end{pmatrix}, \quad (6.20)$$

where

$$\begin{aligned} \Psi_{1,11} &= s_2^{-1/2}, & \Psi_{1,12} &= s_2^{1/2} \frac{\gamma_E - \log 2}{\pi i}, \\ \Psi_{1,21} &= 0, & \Psi_{1,22} &= s_2^{1/2}. \end{aligned}$$

In the same way as for K_{-x_j} with $j = 2, \dots, m$, we define

$$H_1 = \begin{pmatrix} 1 & 0 \\ i\frac{x_1}{2}\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} G_1(-rx_1; r\vec{x}, \vec{s}) = R(-x_1)E_{-x_1}(-x_1) \begin{pmatrix} \Psi_{1,11} & \Psi_{1,12} \\ \Psi_{1,21} & \Psi_{1,22} \end{pmatrix}, \quad (6.21)$$

and we simplify K_{-x_1} (given by (2.36)) as follows

$$K_{-x_1} = \frac{s_2}{2\pi i} (H_{1,11} \partial_{s_k} H_{1,21} - H_{1,21} \partial_{s_k} H_{1,11}). \quad (6.22)$$

Using (5.44), (5.46)-(5.47), (6.20)-(6.22), and the fact that $R^{(1)}$ is traceless, after a careful calculation we obtain

$$K_{-x_1} = \frac{1}{2} \partial_{s_k} d_1 \sqrt{r} - \frac{1}{2} \left(d_1 \partial_{s_k} (R_{11}^{(1)}(-x_1) - R_{22}^{(1)}(-x_1)) + i d_1^2 \partial_{s_k} R_{12}^{(1)}(-x_1) + i \partial_{s_k} R_{21}^{(1)}(-x_1) \right) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right) \quad (6.23)$$

as $r \rightarrow +\infty$. The subleading term in (6.23) can be computed more explicit using the expression for $R^{(1)}(-x_1)$ given by (5.51):

$$\begin{aligned} -\frac{1}{2} \left(d_1 \partial_{s_k} (R_{11}^{(1)}(-x_1) - R_{22}^{(1)}(-x_1)) + i d_1^2 \partial_{s_k} R_{12}^{(1)}(-x_1) + i \partial_{s_k} R_{21}^{(1)}(-x_1) \right) &= \frac{d_0 \partial_{s_k} d_1}{2} \\ &+ \sum_{j=2}^m \frac{1}{4ic_{-x_j} \sqrt{x_j - x_1}} \partial_{s_k} (\beta_j^2 (\tilde{\Lambda}_{j,1} - \tilde{\Lambda}_{j,2} - 2i)) - \partial_{s_k} d_1 \sum_{j=2}^m \frac{\beta_j^2 (\tilde{\Lambda}_{j,1} + \tilde{\Lambda}_{j,2})}{2c_{-x_j} (x_j - x_1)}. \end{aligned} \quad (6.24)$$

Asymptotics for K_0 . From (5.6), (5.14) and (5.45), for z in a neighborhood of 0, we have

$$T(z) = R(z) \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} (z + x_1)^{-\frac{\sigma_3}{4}} N D(z)^{-\sigma_3}. \quad (6.25)$$

On the other hand, from (2.11) and (5.4), as $z \rightarrow 0$ we have

$$T(z) = \begin{pmatrix} 1 & 0 \\ i\frac{x_1}{2}\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} G_0(rz; r\vec{x}, \vec{s})(rz)^{\frac{\alpha}{2}\sigma_3} e^{-\sqrt{r}g(z)\sigma_3}. \quad (6.26)$$

Therefore, using (5.29) and (6.25)-(6.26), we obtain

$$G_0(0; r\vec{x}, \vec{s}) = r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ -i\frac{x_1}{2}\sqrt{r} & 1 \end{pmatrix} R(0) \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} x_1^{-\frac{\sigma_3}{4}} N \widehat{c}^{-\sigma_3}, \quad (6.27)$$

for a certain $\widehat{c} \in \mathbb{C}$ whose exact value is unimportant for us. Let us define

$$H_0 = \begin{pmatrix} 1 & 0 \\ i\frac{x_1}{2}\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} G_0(0; r\vec{x}, \vec{s}) \widehat{c}^{\sigma_3} = R(0) \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} x_1^{-\frac{\sigma_3}{4}} N. \quad (6.28)$$

Since $s_1 = 0$, by (2.37), we have $K_0 = 0$ if $\alpha = 0$. If $\alpha \neq 0$, by (2.37), (5.46)-(5.47) and (6.27)-(6.28), we have

$$\begin{aligned} K_0 &= \alpha \left(H_{0,21} \partial_{s_k} H_{0,12} - H_{0,11} \partial_{s_k} H_{0,22} + (H_{0,12} H_{0,21} - H_{0,11} H_{0,22}) \partial_{s_k} \log D_0 \right) \\ &= \frac{\alpha \partial_{s_k} d_1}{2\sqrt{x_1}} - \alpha \partial_{s_k} \log D_0 + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \quad \text{as } r \rightarrow +\infty. \end{aligned} \quad (6.29)$$

Asymptotics for the differential identity (2.34). Summing the contribution K_0 , K_{-x_j} , $j = 1, \dots, m$ and K_∞ using (6.5), (6.15), (6.16), (6.23), (6.24) and (6.29), and substituting the expression for c_{-x_j} given by (5.34), and the expression for d_0 given by (5.28), after some calculations, we obtain

$$\begin{aligned} \partial_{s_k} \log F_\alpha(r\vec{x}, \vec{s}) &= \partial_{s_k} d_1 \sqrt{r} - \sum_{j=2}^m \left(2\beta_j \partial_{s_k} \log \Lambda_j + \partial_{s_k} (\beta_j^2) - 2i\alpha \arccos\left(\frac{\sqrt{x_1}}{\sqrt{x_j}}\right) \partial_{s_k} \beta_j \right) \\ &\quad + \sum_{j=2}^m \frac{s_{j+1} - s_j}{2\pi i} (\Psi_{j,11} \partial_{s_k} \Psi_{j,21} - \Psi_{j,21} \partial_{s_k} \Psi_{j,11}) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \end{aligned} \quad (6.30)$$

as $r \rightarrow +\infty$. Using the explicit expressions for $\Psi_{j,11}$ and $\Psi_{j,21}$ (see (6.10)) together with the relation (6.14), we have

$$\sum_{j=2}^m \frac{s_{j+1} - s_j}{2\pi i} (\Psi_{j,11} \partial_{s_k} \Psi_{j,21} - \Psi_{j,21} \partial_{s_k} \Psi_{j,11}) = \sum_{j=2}^m \beta_j \partial_{s_k} \log \frac{\Gamma(1 + \beta_j)}{\Gamma(1 - \beta_j)}. \quad (6.31)$$

Also, using (5.39), we have

$$\sum_{j=2}^m -2\beta_j \partial_{s_k} \log \Lambda_j = -2 \sum_{j=2}^m \beta_j \partial_{s_k} (\beta_j) \log(4\sqrt{r(x_j - x_1)}) - 2 \sum_{j=2}^m \beta_j \sum_{\substack{\ell=2 \\ \ell \neq j}}^m \partial_{s_k} (\beta_\ell) \log(\widetilde{T}_{\ell,j}). \quad (6.32)$$

It will more convenient to integrate with respect to $\beta_2, \dots, \beta_m$ instead of $s_2, \dots, s_m$. Therefore, we define

$$\widetilde{F}_\alpha(r\vec{x}, \vec{\beta}) = F_\alpha(r\vec{x}, \vec{s}), \quad (6.33)$$

where $\vec{\beta} = (\beta_2, \dots, \beta_m)$ and $\vec{s} = (s_2, \dots, s_m)$ are related via the relations (5.22). By substituting (6.31) and (6.32) into (6.30), and by writing the derivative with respect to β_k instead of s_k , we obtain

$$\begin{aligned} \partial_{\beta_k} \log \tilde{F}_\alpha(r\vec{x}, \vec{\beta}) &= \partial_{\beta_k} d_1 \sqrt{r} - 2 \sum_{j=2}^m \beta_j \partial_{\beta_k}(\beta_j) \log(4\sqrt{r(x_j - x_1)}) + \sum_{j=2}^m 2i\alpha \arccos\left(\frac{\sqrt{x_1}}{\sqrt{x_j}}\right) \partial_{\beta_k}(\beta_j) \\ &\quad - 2 \sum_{j=2}^m \beta_j \sum_{\substack{\ell=2 \\ \ell \neq j}}^m \partial_{\beta_k}(\beta_\ell) \log(\tilde{T}_{\ell,j}) - \sum_{j=2}^m \partial_{\beta_k}(\beta_j^2) + \sum_{j=2}^m \beta_j \partial_{\beta_k} \log \frac{\Gamma(1 + \beta_j)}{\Gamma(1 - \beta_j)} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \end{aligned} \quad (6.34)$$

as $r \rightarrow +\infty$. Using the value of d_1 in (5.28) and the value of c_{-x_j} in (5.34), the above asymptotics can be rewritten more explicitly as follows

$$\begin{aligned} \partial_{\beta_k} \log \tilde{F}_\alpha(r\vec{x}, \vec{\beta}) &= -2i\sqrt{r(x_k - x_1)} - 2\beta_k \log(4\sqrt{r(x_k - x_1)}) + 2i\alpha \arccos\left(\frac{\sqrt{x_1}}{\sqrt{x_k}}\right) \\ &\quad - 2 \sum_{\substack{j=2 \\ j \neq k}}^m \beta_j \log(\tilde{T}_{k,j}) - 2\beta_k + \beta_k \partial_{\beta_k} \log \frac{\Gamma(1 + \beta_k)}{\Gamma(1 - \beta_k)} + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (6.35)$$

6.2 Integration of the differential identity

By the steepest descent of Section 5 (see in particular the discussion in Section 5.5), the asymptotics (6.35) are valid uniformly for $\beta_2, \dots, \beta_m$ in compact subsets of $i\mathbb{R}$. First, we use (6.35) with $\beta_3 = 0 = \beta_4 = \dots = \beta_m$, and we integrate in β_2 from $\beta_2 = 0$ to an arbitrary $\beta_2 \in i\mathbb{R}$. Let us use the notations $\vec{\beta}_2 = (\beta_2, 0, \dots, 0)$ and $\vec{0} = (0, 0, \dots, 0)$. After integration (using (4.33)), we obtain

$$\begin{aligned} \log \frac{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_2)}{\tilde{F}_\alpha(r\vec{x}, \vec{0})} &= -2i\beta_2 \sqrt{r(x_2 - x_1)} - \beta_2^2 \log(4\sqrt{r(x_2 - x_1)}) + 2i\alpha\beta_2 \arccos\left(\frac{\sqrt{x_1}}{\sqrt{x_2}}\right) \\ &\quad + \log(G(1 + \beta_2)G(1 - \beta_2)) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \end{aligned} \quad (6.36)$$

as $r \rightarrow +\infty$. Now, we use (6.35) with $\beta_4 = \dots = \beta_m = 0$, β_2 fixed but not necessarily 0, and we integrate in β_3 . With the notation $\vec{\beta}_3 = (\beta_2, \beta_3, 0, \dots, 0)$, as $r \rightarrow +\infty$ we obtain

$$\begin{aligned} \log \frac{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_3)}{\tilde{F}_\alpha(r\vec{x}, \vec{\beta}_2)} &= -2i\beta_3 \sqrt{r(x_3 - x_1)} - \beta_3^2 \log(4\sqrt{r(x_3 - x_1)}) + 2i\alpha\beta_3 \arccos\left(\frac{\sqrt{x_1}}{\sqrt{x_3}}\right) \\ &\quad - 2\beta_2\beta_3 \log(\tilde{T}_{3,2}) + \log(G(1 + \beta_3)G(1 - \beta_3)) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right). \end{aligned} \quad (6.37)$$

By integrating successively in $\beta_4, \dots, \beta_m$, and then by summing the expressions, we obtain

$$\begin{aligned} \log \frac{\tilde{F}_\alpha(r\vec{x}, \vec{\beta})}{\tilde{F}_\alpha(r\vec{x}, \vec{0})} &= - \sum_{j=2}^m 2i\beta_j \sqrt{r(x_j - x_1)} - \sum_{j=2}^m \beta_j^2 \log(4\sqrt{r(x_j - x_1)}) + \sum_{j=2}^m 2i\alpha\beta_j \arccos\left(\frac{\sqrt{x_1}}{\sqrt{x_j}}\right) \\ &\quad - 2 \sum_{2 \leq j < k \leq m} \beta_j \beta_k \log(\tilde{T}_{j,k}) + \sum_{j=2}^m \log(G(1 + \beta_j)G(1 - \beta_j)) + \mathcal{O}\left(\frac{\log r}{\sqrt{r}}\right), \end{aligned} \quad (6.38)$$

as $r \rightarrow +\infty$. By adding the above asymptotics to (1.5), this finishes the proof of Theorem 1.4 (after identifying $u_j = -2\pi i\beta_j$).

6.3 Heuristic discussion of the asymptotics as $r \rightarrow +\infty$ and $s_1 \rightarrow 0$

Uniform asymptotics for $F_\alpha(r\vec{x}, \vec{s})$ as $r \rightarrow +\infty$ and simultaneously $s_1 \rightarrow 0$ should describe the transition between the asymptotics of Theorems 1.1 and 1.4. As can be seen from (1.11), (1.27) and (1.28) (and also from (1.5) and (1.6)), the leading term of $\log F_\alpha(r\vec{x}, \vec{s})$ is of order $\mathcal{O}(\sqrt{r})$ if s_1 is bounded away from 0, while it is of order $\mathcal{O}(r)$ if $s_1 = 0$. On the level of the RH analysis, this indicates that a new g -function is needed (which should interpolate between (3.2) and (5.2)). A similar transition has been studied for the sine point process in the series of papers [7, 8, 9], and (with less depth) for the Circular Unitary Ensemble in [13, 14]. In these works, the global parametrix $P^{(\infty)}$ is constructed in terms of elliptic θ -functions and the asymptotics that describe the transition are oscillatory. By analogy, we expect that asymptotics for $F_\alpha(r\vec{x}, \vec{s})$ as $r \rightarrow +\infty$ and simultaneously $s_1 \rightarrow 0$ will also be described in terms of elliptic θ -functions. A rigorous analysis is however very delicate, and is not addressed in this paper.

7 Appendix

In this section, we recall two well-known RH problems: 1) the Bessel model RH problem, which depends on a parameter $\alpha > -1$ and whose solution is denoted by $\Phi_{\text{Be}}(\cdot) = \Phi_{\text{Be}}(\cdot; \alpha)$, and 2) the confluent hypergeometric model RH problem, which depends on a parameter $\beta \in i\mathbb{R}$ and whose solution is denoted by $\Phi_{\text{HG}}(\cdot) = \Phi_{\text{HG}}(\cdot; \beta)$.

7.1 Bessel model RH problem

- (a) $\Phi_{\text{Be}} : \mathbb{C} \setminus \Sigma_{\text{Be}} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where Σ_{Be} is shown in Figure 6.
- (b) Φ_{Be} satisfies the jump conditions

$$\begin{aligned} \Phi_{\text{Be},+}(z) &= \Phi_{\text{Be},-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \mathbb{R}^-, \\ \Phi_{\text{Be},+}(z) &= \Phi_{\text{Be},-}(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, & z \in e^{\frac{2\pi i}{3}} \mathbb{R}^+, \\ \Phi_{\text{Be},+}(z) &= \Phi_{\text{Be},-}(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & z \in e^{-\frac{2\pi i}{3}} \mathbb{R}^+. \end{aligned} \tag{7.1}$$

- (c) As $z \rightarrow \infty$, $z \notin \Sigma_{\text{Be}}$, we have

$$\Phi_{\text{Be}}(z) = (2\pi z^{\frac{1}{2}})^{-\frac{\sigma_3}{2}} N \left(I + \frac{\Phi_{\text{Be},1}(\alpha)}{z^{\frac{1}{2}}} + \mathcal{O}(z^{-1}) \right) e^{2z^{\frac{1}{2}} \sigma_3}, \tag{7.2}$$

$$\text{where } \Phi_{\text{Be},1}(\alpha) = \frac{1}{16} \begin{pmatrix} -(1+4\alpha^2) & -2i \\ -2i & 1+4\alpha^2 \end{pmatrix}.$$

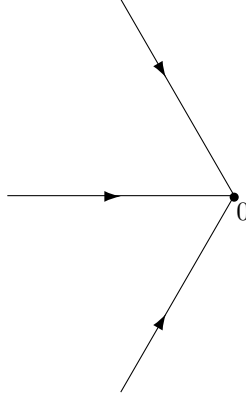


Figure 6: The jump contour Σ_{Be} for Φ_{Be} .

(d) As z tends to 0, the behavior of $\Phi_{\text{Be}}(z)$ is

$$\begin{aligned} \Phi_{\text{Be}}(z) &= \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases} & \text{if } \alpha = 0, \\ \Phi_{\text{Be}}(z) &= \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} z^{\frac{\alpha}{2}\sigma_3}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \\ \mathcal{O}(z^{-\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases} & \text{if } \alpha > 0, \\ \Phi_{\text{Be}}(z) &= \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \end{pmatrix}, & \text{if } \alpha < 0. \end{aligned} \quad (7.3)$$

This RH problem was introduced and solved in [44]. Its unique solution is given by

$$\Phi_{\text{Be}}(z) = \begin{cases} \begin{pmatrix} I_\alpha(2z^{\frac{1}{2}}) & \frac{i}{\pi} K_\alpha(2z^{\frac{1}{2}}) \\ 2\pi i z^{\frac{1}{2}} I'_\alpha(2z^{\frac{1}{2}}) & -2z^{\frac{1}{2}} K'_\alpha(2z^{\frac{1}{2}}) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \frac{1}{2} H_\alpha^{(1)}(2(-z)^{\frac{1}{2}}) & \frac{1}{2} H_\alpha^{(2)}(2(-z)^{\frac{1}{2}}) \\ \pi z^{\frac{1}{2}} (H_\alpha^{(1)})'(2(-z)^{\frac{1}{2}}) & \pi z^{\frac{1}{2}} (H_\alpha^{(2)})'(2(-z)^{\frac{1}{2}}) \end{pmatrix} e^{\frac{\pi i \alpha}{2} \sigma_3}, & \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \frac{1}{2} H_\alpha^{(2)}(2(-z)^{\frac{1}{2}}) & -\frac{1}{2} H_\alpha^{(1)}(2(-z)^{\frac{1}{2}}) \\ -\pi z^{\frac{1}{2}} (H_\alpha^{(2)})'(2(-z)^{\frac{1}{2}}) & \pi z^{\frac{1}{2}} (H_\alpha^{(1)})'(2(-z)^{\frac{1}{2}}) \end{pmatrix} e^{-\frac{\pi i \alpha}{2} \sigma_3}, & -\pi < \arg z < -\frac{2\pi}{3}, \end{cases} \quad (7.4)$$

where $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are the Hankel functions of the first and second kind, and I_α and K_α are the modified Bessel functions of the first and second kind.

A direct analysis of the RH problem for Φ_{Be} shows that in a neighborhood of z we have

$$\Phi_{\text{Be}}(z; \alpha) = \Phi_{\text{Be},0}(z; \alpha) z^{\frac{\alpha}{2}\sigma_3} \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_0(z), \quad (7.5)$$

where H_0 is given by (2.3), h by (2.12), and $\Phi_{\text{Be},0}$ is analytic in a neighborhood of 0. After some computation using asymptotics of Bessel functions near the origin (see [45, Chapter 10.30(i)]), we

obtain

$$\Phi_{\text{Be},0}(0; \alpha) = \begin{cases} \begin{pmatrix} \frac{1}{\Gamma(1+\alpha)} & \frac{i\Gamma(\alpha)}{2\pi} \\ \frac{i\pi}{\Gamma(\alpha)} & \frac{\Gamma(1+\alpha)}{2} \end{pmatrix}, & \text{if } \alpha \neq 0, \\ \begin{pmatrix} 1 & \frac{\gamma_E}{\pi i} \\ 0 & 1 \end{pmatrix}, & \text{if } \alpha = 0, \end{cases} \quad (7.6)$$

where γ_E is Euler's gamma constant.

7.2 Confluent hypergeometric model RH problem

- (a) $\Phi_{\text{HG}} : \mathbb{C} \setminus \Sigma_{\text{HG}} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where Σ_{HG} is shown in Figure 7.
- (b) For $z \in \Gamma_k$ (see Figure 7), $k = 1, \dots, 6$, Φ_{HG} has the jump relations

$$\Phi_{\text{HG},+}(z) = \Phi_{\text{HG},-}(z)J_k, \quad (7.7)$$

where

$$\begin{aligned} J_1 &= \begin{pmatrix} 0 & e^{-i\pi\beta} \\ -e^{i\pi\beta} & 0 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 0 & e^{i\pi\beta} \\ -e^{-i\pi\beta} & 0 \end{pmatrix}, \\ J_2 &= \begin{pmatrix} 1 & 0 \\ e^{i\pi\beta} & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\beta} & 1 \end{pmatrix}, \quad J_5 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\beta} & 1 \end{pmatrix}, \quad J_6 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\beta} & 1 \end{pmatrix}. \end{aligned}$$

- (c) As $z \rightarrow \infty$, $z \notin \Sigma_{\text{HG}}$, we have

$$\Phi_{\text{HG}}(z) = \left(I + \frac{\Phi_{\text{HG},1}(\beta)}{z} + \mathcal{O}(z^{-2}) \right) z^{-\beta\sigma_3} e^{-\frac{z}{2}\sigma_3} \begin{cases} e^{i\pi\beta\sigma_3}, & \frac{\pi}{2} < \arg z < \frac{3\pi}{2}, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & -\frac{\pi}{2} < \arg z < \frac{\pi}{2}, \end{cases} \quad (7.8)$$

where

$$\Phi_{\text{HG},1}(\beta) = \beta^2 \begin{pmatrix} -1 & \tau(\beta) \\ -\tau(-\beta) & 1 \end{pmatrix}, \quad \tau(\beta) = \frac{-\Gamma(-\beta)}{\Gamma(\beta+1)}. \quad (7.9)$$

In (7.8), the root is defined by $z^\beta = |z|^\beta e^{i\beta \arg z}$ with $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$.

As $z \rightarrow 0$, we have

$$\Phi_{\text{HG}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } z \in II \cup V, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } z \in I \cup III \cup IV \cup VI. \end{cases} \quad (7.10)$$

This model RH problem was first introduced and solved explicitly in [36]. Consider the matrix

$$\widehat{\Phi}_{\text{HG}}(z) = \begin{pmatrix} \Gamma(1-\beta)G(\beta; z) & -\frac{\Gamma(1-\beta)}{\Gamma(\beta)}H(1-\beta; ze^{-i\pi}) \\ \Gamma(1+\beta)G(1+\beta; z) & H(-\beta; ze^{-i\pi}) \end{pmatrix}, \quad (7.11)$$

where G and H are related to the Whittaker functions:

$$G(a; z) = \frac{M_{\kappa, \mu}(z)}{\sqrt{z}}, \quad H(a; z) = \frac{W_{\kappa, \mu}(z)}{\sqrt{z}}, \quad \mu = 0, \quad \kappa = \frac{1}{2} - a. \quad (7.12)$$

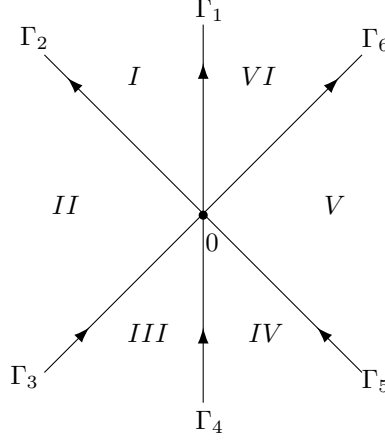


Figure 7: The jump contour Σ_{HG} for Φ_{HG} . The ray Γ_k is oriented from 0 to ∞ , and forms an angle with $\mathbb{R}^+$ which is a multiple of $\frac{\pi}{4}$.

The solution Φ_{HG} is given by

$$\Phi_{\text{HG}}(z) = \begin{cases} \widehat{\Phi}_{\text{HG}}(z)J_2^{-1}, & \text{for } z \in I, \\ \widehat{\Phi}_{\text{HG}}(z), & \text{for } z \in II, \\ \widehat{\Phi}_{\text{HG}}(z)J_3^{-1}, & \text{for } z \in III, \\ \widehat{\Phi}_{\text{HG}}(z)J_2^{-1}J_1^{-1}J_6^{-1}J_5, & \text{for } z \in IV, \\ \widehat{\Phi}_{\text{HG}}(z)J_2^{-1}J_1^{-1}J_6^{-1}, & \text{for } z \in V, \\ \widehat{\Phi}_{\text{HG}}(z)J_2^{-1}J_1^{-1}, & \text{for } z \in VI. \end{cases} \quad (7.13)$$

We need in the present paper a better knowledge than (7.10). From [45, Section 13.14 (iii)], as $z \rightarrow 0$ we have

$$\begin{aligned} G(\beta; z) &= 1 + \mathcal{O}(z), & G(1 + \beta; z) &= 1 + \mathcal{O}(z), \\ H(1 - \beta; z) &= \frac{-1}{\Gamma(1 - \beta)} \left(\log z + \frac{\Gamma'(1 - \beta)}{\Gamma(1 - \beta)} + 2\gamma_{\text{E}} \right) + \mathcal{O}(z \log z), \\ H(-\beta; z) &= \frac{-1}{\Gamma(-\beta)} \left(\log z + \frac{\Gamma'(-\beta)}{\Gamma(-\beta)} + 2\gamma_{\text{E}} \right) + \mathcal{O}(z \log z), \end{aligned}$$

where γ_{E} is Euler's gamma constant. Using the connection formula $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} = -\Gamma(-z)\Gamma(1 + z)$, as $z \rightarrow 0$, $z \in II$, we have

$$\widehat{\Phi}_{\text{HG}}(z) = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} (I + \mathcal{O}(z)) \begin{pmatrix} 1 & \frac{\sin(\pi\beta)}{\pi} \log z \\ 0 & 1 \end{pmatrix}, \quad (7.14)$$

where in the above expression

$$\log z = \log |z| + i \arg z, \quad \arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2} \right), \quad (7.15)$$

and

$$\begin{aligned} \Psi_{11} &= \Gamma(1 - \beta), & \Psi_{12} &= \frac{1}{\Gamma(\beta)} \left(\frac{\Gamma'(1 - \beta)}{\Gamma(1 - \beta)} + 2\gamma_{\text{E}} - i\pi \right), \\ \Psi_{21} &= \Gamma(1 + \beta), & \Psi_{22} &= \frac{-1}{\Gamma(-\beta)} \left(\frac{\Gamma'(-\beta)}{\Gamma(-\beta)} + 2\gamma_{\text{E}} - i\pi \right). \end{aligned}$$

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