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Hybrid Approaches for Life Insurance and Pension Schemes in an Uncertain Environment

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To my family,
parents and husband

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Chapter 1

Introduction

1.1 Financial and mortality challenges

In today's uncertain financial landscape, the world of life insurance and pension is facing enormous challenges, where the delicate balance between protecting against the uncertainties of life and managing the unpredictable events of the financial market becomes of essential importance. The current financial market, characterized by its persistent fluctuations, presents a tremendous set of risks for the insurance sector: the low level of interest rates that drops sometimes to even negative rates, the volatile stock market, and exceptionally high rates of inflation.

Another significant challenge is longer lifespans, which on the one hand are a sign of progress but on the other hand are a big financial issue that might encounter the life insurance and pension fund industries and even governments. Even though experts have less focused on longevity risk as much because it's a long-term concern, they're now showing greater interest in it. This risk can suddenly change due to unexpected and extreme events, like the recent COVID-19 pandemic.

In this evolving landscape, an interesting connection emerges between mortality and finance, deviating from the traditional separation in life insurance pricing. The pandemic, typically seen as a mortality risk, has demonstrated its influence on the financial market. This dynamic requires an exploration of the potential impact of this interdependence on the pricing of life insurance contracts. It is widely known that the current environment of low interest rates and market turbulence presents a significant challenge and threatens the viability of maintaining appealing long-term guarantees from life insurance. Moreover, the persistently higher trend in life expectancy makes it harder to offset mortality losses with financial gains, forcing novel strategies from insurers to deal with this complex relationship. Therefore, the twin pillars of life insurance—financial performance and mortality risk—have become complicated to manage. Added to this complexity are the solvency constraints that insurers must comply with. These constraints are not solely tied to profitability but also cover capital requirements, taking into account long-term guarantees. Additionally, a prevailing trend in modern finance highlights the fact that every guarantee has a cost in accordance with Solvency II. As a result, providing policyholders with long-term guarantees has grown into a serious burden for insurers, leading them to shift towards new risk-sharing-oriented offerings marked by a reduction in guarantees. For instance, there is a switch from traditional life insurance products with significant guarantees, even on future premiums, known as classical products, to universal life contracts where guarantees are only given on existing reserves. This product has become the new classic on the Belgian market. Another switch is to unit-linked products, where the guarantee component has nearly disappeared. Another notable transition is from life-long products with mortality to pure financial products. Currently, a considerable proportion of contracts proposed by life insurers lack a mortality component, concentrating instead on financial performance. A last example in occupational pension plans is the switch from defined benefit (DB) pension plans—entailing guarantees for affiliates—to defined contribution (DC) plans, which is observable even in certain national social security systems.

On the other side, amidst all the daily uncertainties, the trend of removing traditional guarantees needs a profound re-evaluation. While the trend has shifted towards the removal of guarantees, as evinced by empirical and experimental studies (Arkes et al. [2]; Gatzert et al. [20]; Knoller [25]; Ruβ & Schelling [29]), it remains clear that customers still want protection from risks and safety stays a legitimate expectation. The pursuit of downside protection emerges as a basis for consumer preference, a sentiment stressed by several elements proving the anxiety of consumers to fully embrace risk. For example, the remarkable popularity of pure savings banking accounts, which are seen as the illusion of a guarantee, as well as the remaining success of traditional products in life insurance and pension. Another example within group life insurance is the minimum compulsory guaranteed return for second pillar schemes in Belgium or in Switzerland. The very existence of such regulations reinforces the persisting demand for safeguards against excessive risk. This feeling is not just an individual preference; it is even supported by regulatory corridors. For instance, the European Commission is emphasizing the importance of providing some sort of guarantee by suggesting options for capital protection for customers in their recent regulation called the Pan European Pension Product ([18]).

These elements pose a threat not only to creating long-term guarantees from life insurers, but they also expose the plan member in pension insurance to investment risk, and they increase the requirement for some form of protection from the client, making the very act of offering, negotiating, and demanding a complex endeavor.

1.2 Diversification and hybridization

The challenges faced by life insurance and pension schemes, arising from the volatile financial market and unpredictable longevity risks, find a potential solution in the pioneering work of Markowitz [28] and his theory of diversification. Markowitz's innovative insight was that by diversifying investments—spreading investments across different types of assets—one could potentially reduce the overall risk in a portfolio while maintaining, or even enhancing, the potential for returns. This concept becomes particularly relevant when examining the inherent risks in the insurance industry, as it emerges as a potential solution to counter the downward trends in financial markets. Firstly, when it comes to the highly fluctuating financial market, the application of diversification aligns perfectly with Markowitz's theory. By investing in a mix of assets that react differently to market shifts, for instance, stocks and bonds, the potential negative impact of a downturn in one area can be offset by the positive performance of another. In principle, it is mitigating the vulnerability posed by sudden changes in interest rates, stock market fluctuations, and inflation spikes. Secondly, in addressing the longevity risk—the challenge posed by increasing lifespans—diversification once again shines as a strategic approach. By incorporating a variety of insurance products and investment vehicles that deal with different stages of life and various potential scenarios, life insurance and pension providers can more effectively manage the impact of unexpectedly extended lifespans and work toward more stable, resilient, and adaptable strategies in the face of the dynamic challenges presented by today's ever-changing financial landscape and the shifting demographics of human longevity. This diversification strategy helps distribute the risk associated with longer lives, ensuring that the potential stress on resources is less concentrated and more manageable.

A widely embraced strategy in actuarial science, both from an academic point of view and from a practical point of view, lies in the innovative approach known as hybridization. This approach seeks to mitigate the risks and offer good compromises justified by diversification principles. It is possible to generate diversification in two different ways: either by investing in many items, which is a normal diversification concept, or by creating a form of hybridization within the product itself, which can be an additional idea of diversification. Under the latter approach, a popular product has been developed in finance called structured products, which strives to ensure a minimum guarantee alongside participation in the returns of financial risky assets. These products were launched on the US market in 1987. Since then, many examples of structured products have been investigated; for instance, Chen & Kensinger [14] studied Market-Index Certificates of Deposit, which are characterized by paying a guaranteed minimum interest and a variable interest that is linked to the performance of the S&P 500, while Chen & Sears [15] analyzed the S&P 500 Index Note, which is similar to the latter product with a participation in the performance of the S&P 500 index by a predetermined ratio. A more recent study conducted by Bernard & Boyle [5] in which they investigated capped structured products. They offer a payoff at maturity that is equal to a guaranteed amount as a global floor with a bonus that is an accumulation of periodic returns with a local cap at some maximum level. Through their research, they talked about the dilemma of investors choosing between simple and sophisticated products. They discovered that despite the fact that pre-packaged solutions make structured products very expensive for regular investors, these instruments keep attracting them.

1.2.1 Hybridization in life insurance

In life insurance, the pursuit of maintaining certain forms of guarantees shines through hybrid products, which hold promise as a practical response to today's financial landscape. Notably in Germany, dynamic hybrid products, primarily expressed as deferred annuity contracts, have since emerged after static hybrid products. As stated by Kochanski & Karnarski [26], static hybrid products have their allocation of premiums determined at the beginning and stay unchanged, while the subsequent dynamic hybrid approach employs a dynamic rebalancing strategy that involves shifting between two to three investments on a periodic basis, driven by a constant proportion portfolio insurance-type strategy (see, Bohnert & Gatzert [11] and Hambardzumyan & Korn [22]). These dynamic products have acquired popularity among several life insurance firms since their introduction in 2007, rapidly increasing in importance. They seamlessly integrate the advantages of traditional participating life insurance and unit-linked products through periodic fund reallocations, including an insurer's policy reserve, a guarantee fund, and/or an equity fund. These products, tailored to the German market, aim to fulfill contemporary consumer needs by combining the stability inherent in traditional life insurance via conventional policy reserves with the growth potential characteristic of unit-linked policies, facilitated by investments in guarantee and/or equity funds. The landscape of dynamic hybrid insurance products has been carefully examined in the literature. They were initially introduced and explored in the scientific literature by Kochanski & Karnarski [26]. Their work encompassed the formulation of solvency capital requirements for both static and dynamic hybrids, utilizing a rules-based shifting mechanism. Their approach focused primarily on calculating the solvency capital needs for dynamic hybrid products within the framework of Solvency II, revealing its superiority over the conventional formula for innovative life insurance contracts. Although their model framework featured a 3-fund dynamic hybrid product and a monthly reallocation process, the examination primarily concentrated on solvency capital requirements and did not extend to the analysis of potential interaction effects with other products. Bohnert & Gatzert [11] has conducted extensive research on the latter, spotlighting the insurer's risk situation and policyholders' net present value and analyzing its impact on fair valuation and risk evaluation; however, Bohnert [10] offers a comprehensive overview of the German dynamic hybrid product market, showing the diverse contract options and the implications of varying shifting mechanisms through an exploration of industry risk-return profiles.

In addition, there are products developed in Germany that represent a form of hybridization themselves and bridge the gap between guaranteed stability and potential returns, such as the structured products in finance that are outlined above, catering to the evolving needs and preferences of policyholders. "Select Products" refer to a type of life insurance product that has gained prominence in various markets as well. These products are designed with innovative features that aim to strike a balance between reduced risk for the insurance company and attractiveness for policyholders, as explained by Alexandrova et al. [1]. Furthermore, most Select Products provide a money-back guarantee, assuring policyholders that they will receive at least the initial premium paid or the sum of premiums paid upon maturity.

Substantially, these hybrid products offer policyholders both security and the potential for higher returns compared to traditional options on the one hand, and they allow insurers to provide attractive guarantees instead of switching to products that are based only on investment risk without guarantee on the other. In this regard, recent years have seen the development of new products in life insurance in Belgium known as "products of Branch 44", a mixed investment that combines elements of Branch 21 (a participating life insurance contract with a guaranteed rate) and Branch 23 (a pure unit-linked product) within the same insurance contract, Van Maldegem [32]. Branch 44 allows you to achieve the balance best suited to your profile between the security offered by Branch 21 and the potentially higher performance that characterizes Branch 23. It typically offers a guaranteed minimum return on a portion of the invested premiums, while the remaining portion is invested in a selection of investment funds. As a result, policyholders get some level of security along with the potential for higher returns based on the performance of the underlying investments. In fact, Branch 21 insurance is often referred to as a participating life insurance product. It is a type of life insurance policy where the insurance company guarantees a minimum interest rate on the invested premiums for a specific period. This means that policyholders are assured that their initial investment, along with the guaranteed interest, will be paid out at the end of the contract term. Branch 21 policies are considered low-risk because of the guaranteed return, making them an attractive option for those seeking capital protection and stable, predictable growth. However, the potential for higher returns is limited compared to other investment options. Branch 23 insurance, on the other hand, is often referred to as a unit-linked insurance product. Unlike Branch 21, there is no guaranteed return in Branch 23 policies. Instead, the premiums paid by policyholders are invested in a range of investment funds, such as stocks, bonds, or mutual funds, based on the policyholder's risk tolerance and investment preferences. The value of the policy is linked to the performance of these underlying investments. This means that while Branch 23 policies offer the potential for higher returns than Branch 21, they also come with a higher level of risk, as the value can fluctuate based on market conditions.

Clearly, policyholders often exhibit a greater preference for insurance products falling under Branch 21 due to the inherent sense of security they provide. Branch 21 offerings come with guaranteed interest rates and capital protection, resonating with individuals seeking stability and assured returns for their investments. This appeal is particularly strong among risk-averse customers who prioritize safeguarding their financial resources and value the certainty offered by these policies. On the contrary, insurers tend to lean towards Branch 23 products within their portfolio. Branch 23 aligns with the interests of insurance companies seeking to balance risk and return. These investment-linked policies offer the potential for higher yields as they are linked to market performance and investment opportunities. Insurers, drawn by the prospect of greater returns, are motivated to include Branch 23 options to optimize their profitability and meet policyholders seeking potentially enhanced growth opportunities. Therefore, the combination of these divergent perspectives results in the hybrid product, Branch 44. The synthesis of guaranteed security and investment-linked potential within Branch 44 summarizes both policyholders' desire for stability and insurers' drive for growth. In fact, the diverse nature of Branch 44 emphasizes the complex interplay between policyholders' risk tolerance and insurers' financial objectives, shaping an innovative solution that guides the range of investor preferences.

1.2.2 Hybridization in occupational pension schemes

The same kind of hybridization can also be observed in the occupational pension market. Pension plans can generally be categorized into two main types: DB and DC plans. In a DB plan, employers commit to providing employees with a predetermined retirement benefit based on factors like salary history and length of service. This means that the employer bears the investment risk and is obligated to ensure that the promised benefits are met, even if the plan's investments underperform. This creates a financial burden on the employer, as they may need to allocate additional funds to meet these obligations in the current situation of volatile investment returns and increased life expectancies of retirees, as mentioned before. On the other hand, in a DC plan, employers and sometimes workers contribute to individual retirement accounts. The eventual retirement benefit is determined by the contributions made and the investment performance of those contributions. Here, the investment risk is shifted to the employees, as the value of their retirement fund directly depends on market fluctuations. Employers' obligations are typically limited to making the agreed-upon contributions, making DC plans more cost-predictable for employers compared to DB plans. However, both DB and DC plans have their limitations and risks. DB plans can strain employers' finances if investments don't perform as expected or if retirees live longer than anticipated. DC plans expose employees to market risks, potentially leading to inadequate retirement funds in the event of an investment collapse.

Fundamentally, pension insurance entails a collaborative arrangement involving two key stakeholders: the employer, who organizes the pension plan, and the worker, who is the plan's beneficiary. Each party is guided by distinct interests that shape their roles within the pension landscape. For the employer, the primary interest revolves around attracting and retaining a skilled workforce. Offering a robust and appealing pension plan is a strategic tool to achieve this goal. By providing employees with a reliable and comprehensive retirement benefit, the employer enhances their job satisfaction, loyalty, and long-term commitment. From a financial perspective, the employer also seeks to manage the costs associated with the pension plan, striving to strike a balance between providing valuable benefits to employees and maintaining the organization's financial stability. However, from the worker's standpoint, the central interest is securing a financially sound and stable retirement. Employees look to the pension plan as a means to ensure their post-work life is comfortable and free from financial uncertainty. In addition to financial security, the pension plan contributes to workers' peace of mind, enabling them to focus on their work and career growth without the constant worry of future financial challenges.

According to Blommestein et al. [9], hybrid plans—those between standard DB and individual DC—seem to be more effective and long-lasting ways of sharing risk than either of the other two. One of the hybrid products is the well-known "cash balance" (CB) plan. A CB plan is a retirement savings agreement that bridges the gap between traditional DB and DC plans. In this hybrid model, each participant's account receives annual contributions from the employer, typically expressed as a percentage of their salary, similar to a DC plan. Yet, what sets the CB plan apart is its feature of guaranteeing a specific annual contribution, similar to the guarantees offered by a DB plan. The innovation of CB plans emerges from their ability to mix the stability and predictable growth attributes of DB plans with the individual account transparency and flexibility seen in DC plans. This hybrid structure appeals to a diverse range of participants, meeting risk-averse individuals seeking security alongside those desiring the growth potential inherent in investment-linked products. Through this creative mixing of features, CB plans reflect the harmonization of risk mitigation and market exposure in the dynamic landscape of pension insurance.

1.3 Fair valuation for the provider

From the perspective of the provider, a fair valuation of life or pension insurance liabilities is an important regulation outlined in Solvency II, where market-consistent valuations are required. This technique involves pricing the contract consistently in an arbitrage-free market. Providers must account for both the financial guarantees extended to policyholders and the uncertainties associated with mortality and longevity risks. In this pursuit, sophisticated modeling techniques are often employed to simulate various scenarios and assess potential outcomes. This dynamic approach allows insurers to make informed decisions that protect the interests of both policyholders and stakeholders. Moreover, a fair valuation reinforces the insurer's accountability to policyholders by ensuring that pricing and reserves are adequately aligned to fulfill promises made in long-term contracts. The process of fair valuation necessitates ongoing review and adjustment, ensuring that the insurer maintains fairness even in the face of evolving market dynamics.

Within the framework of classical market assumptions such as completeness and the absence of arbitrage opportunities, the pricing of guarantees becomes analogous to pricing options. Under these assumptions, there exists only one unique equivalent martingale measure known as the risk-neutral measure. It is used for pricing by computing the expectation of discounted future payoffs under this measure. A lot of papers in the literature have been dedicated to computing the fair valuation of life insurance and pension products in a Black and Scholes [8] framework (Grosen & Jørgensen [21] and Bacinello [3]) as well as in interest rates stochastic frameworks (Briys & De Varenne [12], Bernard et al. [6], and Barbarin & Devolder [4]). For instance, Hansen & Miltersen [23] priced contracts with minimum rate of return guarantees using the principle of fair valuation under a constant interest rate environment and extended the model to include a stochastic model for interest rates as they also studied the effects of mortality risk. Bacinello [3] priced consistently the classical life insurance policy of an endowment type given a mortality table and a technical interest rate in order to obtain a fair relation between the parameters of the model; however, the pricing has been made under the assumption of independence between mortality and financial elements. Afterwards, the longevity risk has been taken into account in the literature after proposing models of stochastic longevity (Biffis [7] and Cairns et al. [13]), and accordingly, several papers have stochastically modeled the uncertainty related to longevity (Devolder & Azizieh [16]) and more recently studied its correlation with finance (Gao et al. [19], Jalen & Mamon [24], and Liu et al. [27]).

1.4 Optimal strategies for the client

Exploring the realm of hybridization in life insurance and pension schemes naturally gives rise to a crucial idea: the development of an optimal investment strategy. An important framework that offers valuable insights into optimal portfolio selection and is closely aligned with this pursuit is Markowitz's pioneering theory [28]. Markowitz's theory of optimal portfolio selection revolves around a fundamental principle: achieving an equilibrium between risk and return to optimize an investor's utility. The theory assumes that, by wisely allocating assets across various investment options, an investor can achieve a portfolio that maximizes their satisfaction or utility, considering both financial gains and risk tolerance. Markowitz's theory delves into the dynamics of diversification, asserting that a well-constructed portfolio should include a mix of assets with varying degrees of risk and return. The objective is to identify an optimal combination that capitalizes on potential gains while mitigating the adverse impacts of market fluctuations. Central to Markowitz's theory is the concept of the efficient frontier, a graphical representation of portfolios that offer the highest returns for a given level of risk or the lowest risk for a given level of return. Portfolios lying on the efficient frontier are considered optimal, as they provide the best risk-reward trade-off. Linking this optimality concept to life insurance and pension schemes, the pursuit of an optimal investment strategy assumes profound relevance. By tailoring investment allocations to the specific needs, risk preferences, and objectives of policyholders, insurance providers can enhance the overall utility or satisfaction derived by their clients. As well, because employers and workers have opposing interests in pension insurance, a mixed pension contract that meets the needs of both parties in terms of utility offers important insights for striking a balance between retirement benefits and cost control.

In this respect, many theories have been developed to describe the preferences of rational individuals in situations of uncertainty based on their risk aversion. For instance, the von Neumann and Morgenstern [33] framework of expected utility may be used to represent the preferences of customers with a specific utility function known as the Constant Relative Risk Aversion (CRRA) utility function. This function quantifies an individual's aversion to risk by incorporating a risk aversion coefficient that characterizes their tendency to tolerate risk in pursuit of potential rewards. The CRRA utility function elegantly captures the balance between risk tolerance and the desire for higher returns, providing a mathematical foundation to assess the optimal investment strategy for policyholders. Another theory that helps explain guarantees in life insurance and pension schemes is the Cumulative Prospect Theory (CPT), a behavioral model advanced by Tversky and Kahneman [31] in 1992. This framework delves into human decision-making under conditions of uncertainty and risk. CPT departs from traditional utility theory by acknowledging that individuals do not always make choices based solely on rational calculations of expected values and probabilities. Instead, CPT takes into account the psychological effects of gains and losses, acknowledging that people often deviate from strict rationality due to cognitive biases. Basically, these preference functions have been taken into account to compare various insurance products. For example, Døskeland & Nordahl [17] maximize customer welfare using typical life and pension contracts with a guaranteed rate of return and show that, for contracts exposed just to financial risk, CPT can explain demand for guarantees, whereas expected utility theory cannot.

1.5 Purpose and contribution of the thesis

The main purpose of this thesis is to answer the global question of how to design and evaluate financially and actuarially life insurance products as well as occupational pension schemes in an uncertain financial and demographic landscape. As stated above, amidst this struggle between the desire for guarantees and the appeal of higher returns, a need for compromise has arisen. Mixed or hybrid insurance products have then emerged as a popular option that represents a dynamic approach to closing the gap between guarantee-oriented and return-focused insurance offerings, enhancing the industry's capacity to meet evolving customer needs and regulatory expectations. Our research is a sort of generalization and adaptation of structured products to the world of life insurance and pension schemes. We intend to merge not assets, as commonly understood in finance, but liabilities, which relate to our domain of life insurance and pension schemes. This combination aims to provide clients with a distinctive mix of liabilities, not merely a superposition between them but jointly integrated into a single product. This integration rests on two fundamental concepts: firstly, an interplay between these two components through a dynamic strategy; and secondly, a transfer from Branch 23 to Branch 21, enhancing the guarantee within the latter.

This hybrid insurance contract is designed in **Chapter 2** of this thesis, combining features of participating and unit-linked elements to achieve a compromise between a large guarantee from the participating part and no guarantee yet good returns from the unit-linked part. As already mentioned, the two components of the contract are linked by permitting contracts with or without periodic rebalancing on one side and introducing a continuous fee on the unit-linked component on the other side. This fee acts as an internal subsidy, strengthening the investment guarantee in the participating segment. This technique implies cross-subsidizing between the two parts of the contract[1]. The primary contribution lies in demonstrating the appeal of this hybrid product compared to standalone unit-linked or participating contracts, particularly for risk-averse investors seeking greater guarantees.

Examining various guarantee forms within the participating contract, this chapter offers insights from insurer and customer perspectives and serves as the basis for the subsequent two chapters. In **Chapter 2**, we use a common standard model called the Black and Scholes model without considering the risk related to mortality. We focus on a pure financial contract and derive forms of fairness relations between the parameters of the two components of the mixed contract. This depends on the type of guarantee the customer selects and the constant allocation level between the participating and the unit-linked contracts considered in this chapter. Furthermore, this chapter examines utility theory and cumulative prospect theory as behavioral functions from the perspective of the consumer to determine the optimal constant investment strategy that optimizes his preferences. The outcomes demonstrate that the mixed product is appealing and that the risk/return characteristics of the unit-linked and participation funds have a significant impact on the optimal level of fees.

Instead of using constant allocation between the two components as in **Chapter 2**, we look in

[1]There could be some legal issues generated by that subsidization, but the purpose of this chapter is just actuarial.

Chapter 3 at fluctuating proportions over time based on age profile. Basically, the lifestyle investment strategy offers a flexible approach to pension and life insurance investments in the real world of finance by customizing long-term strategies and matching investment portfolios to policyholders' ages and risk tolerance. In this respect, **Chapter 3** delves into fair guaranteed rate pricing and optimal investment strategies, particularly under predefined deterministic investment styles. Its core objective is to check if the preferences of the customer, specifically the balance between Branch 21 and Branch 23, align with the lifestyle investment strategy concept when introducing a deterministic investment strategy, chosen at the inception of the contract, that guides the premium allocation to the participating life insurance contract over time. Similar to **Chapter 2**, the focus remains on financial aspects, utilizing the classical Black and Scholes model for fair valuation and employing the Expected Utility approach to put the contract in an optimal utility framework. We assume in this chapter that the type of guarantee is annual. The analysis incorporates various investment style families, including linear, step-wise, and piece-wise linear strategies. The findings emphasize constant allocation as an optimal strategy for single premium payments and constant risk attitudes, in line with Merton's stock allocation fraction [30]. Incorporating periodic premium payments highlights the dynamic nature of investment decisions, allowing policyholders to adjust their allocations in response to changing financial circumstances or risk tolerances. This adaptive approach provides a more tailored and responsive investment strategy, enhancing the overall suitability to lifestyle strategy. Alternatively, Recognizing potential variations in lifetime risk preferences, this study further incorporates age-dependent risk attitudes, capturing evolving risk preferences over time. This accommodates policyholders' changing attitudes and aligns with lifestyle investment strategies.

Traditional life insurance pricing methods often neglect correlations between financial and mortality elements, an approach challenged by contemporary factors like the COVID-19 pandemic. **Chapter 4** aims to price and assess different contracts within this stochastic environment while considering the interplay between financial and demographic risk factors. Building upon **Chapter 2**, this chapter shifts from pure financial to real life insurance contracts by integrating mortality components using stochastic models for both mortality and interest rates and introducing correlations between mortality and financial market trends to have a whole view of risks. More specifically, we expand the **Chapter 2** by, first, taking into account the mortality component and comparing different products, such as pure endowment for the entire contract or blends of pure endowment and pure investment. Second, we use the Hull and White model to simulate the fluctuation and evolution of interest rates by incorporating stochastic interest rates. Finally, the exploration delves into studying correlations among all risk factors, encompassing both financial and actuarial risks. This multi-dimensional approach examines the interdependence of financial and mortality risks within a continuous-time model encompassing interest rates, stock returns, and mortality, presenting a more realistic picture of the insurance landscape.

In this chapter, we employ the change of measure technique to yield simplified formulas for insurance contract pricing under correlation assumptions. The explicit fair valuation of maturity guarantee contracts is achieved, allowing for analysis of the correlation's influence on fair guaranteed rates. The introduction of correlations introduces diverse impacts on pricing and fair guarantee levels. Moreover, the correlation parameters' effects on fair guaranteed rates are connected with volatility levels.

Moving to the pension insurance landscape, the divergent interests and risks of employers and workers necessitate balanced solutions, including the risk of insufficient benefits for the worker compared to a target and the risk of insufficient funds for the employer to meet his liability. Namely, while the employer wants to lower the supplemental cost of his commitment to the worker, the latter seeks a higher level of benefit upon retirement. In this regard, the central objective of **Chapter 5** is to try to conciliate the distinct interests of employers and workers through the selection of pension plans. In particular, traditional defined contribution plans managed with participating life insurance (DC-PL) plans prioritize robust retirement benefits, while cash balance plans managed with unit-linked insurance (CB-UL) plans are favored by employers for their potential surplus generation during favorable market conditions. Within this context, the main contribution of this chapter lies in focusing on the intermediate hybrid solutions—the DC-PL and CB-UL pension plans—and striving to identify an optimal mix that reconciles both parties' objectives.

The innovation of this study is its multi-objective perspective, viewing the pension plan as a negotiated contract between employers and workers. This perspective integrates the cumulative prospect theory (CPT) model to capture the dynamics of decision-making under uncertainty. The analysis incorporates these risk dimensions through both additive utility and Nash equilibrium methods, providing insights into the optimal alignment of pension plans that simultaneously address cost management and retirement benefit optimization.

The study's methodology integrates stochastic interest rates through the Hull and White model, offering a robust framework for computing benefits, pricing participating contracts to ensure fair guaranteed rates, and determining the crediting interest rate. The initial analysis examines a single premium scenario, splitting the premium allocation between DC-PL and CB-UL investments. Sensitivity analyses emphasize the inclination of employers towards CB-UL plans and workers' preference for DC-PL plans. Notably, the study highlights that when both employer and worker considerations are accorded equal importance, the CB-UL plan emerges as the preferred choice due to its inherent advantages.

Furthermore, the scope extends to accommodate a periodic premium scenario and a stochastic crediting rate based on par yields. The chapter makes a significant contribution by addressing the complex interplay between employer and worker preferences within pension planning. In summary, this chapter offers an innovative approach to balancing risk-sharing and optimizing utility-based decision-making within the context of pension plan selection.

1.6 Published material

In this thesis, we have replicated the text of the previously submitted or published paper to facilitate an independent reading of the different chapters. **Chapter 2** has been published in Scandinavian Actuarial Journal (2022), **Chapter 3** has been submitted in Decisions in Economics and Finance Journal (2023), **Chapter 4** has been published in European Actuarial Journal (2023), and **Chapter 5** has been published in Risks (2023).

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Chapter 2

Mixed participating and unit-linked life insurance contracts: design, pricing and optimal strategy[1]

2.1 Introduction

In many countries, the industry of life insurance has traditionally focused on products with guarantees. The typical example is given by participating life insurance contracts (or life insurance with profit), where the insurer offers ex ante a return guarantee and afterwards provides the customer a participation in case the actual return exceeds this return guarantee. These participation schemes may have different forms such as reversionary bonus or terminal bonus (Bacinello [3]; Bohnert & Gatzert [11]; Hieber et al. [27]). The premiums are usually invested in the general fund of the insurer with no possible financial decision from the customer. In parallel to this activity, new forms of life insurance products have emerged for decades in many markets under the form of unit-linked contracts or equity-linked contracts. In these products, the financial risk is transferred to the investor allowing a more risky investment strategy. The investor can then realize his choices between several funds. In some countries, additional guarantees can be included in the product (Boyle & Schwartz [13]; Brennan & Schwartz [14]; Hansen & Miltersen [23]; Bauer et al. [6]; Bacinello et al. [4]; Mackay et al. [30]; Hieber [26]).

The recent evolution of the financial market, characterized by persistent low (or even negative) interest rates and volatile equity markets, threatens the existence of attractive guarantees from the life insurer and makes, therefore, the offer of participating contracts very challenging. Then, the unit-linked products could appear as the only future for life insurance. Removing completely the guarantees has been indeed an important trend for insurers in many markets these last decades. Nevertheless, it is confirmed by many empirical and experimental studies that the customers are risk-averse and specifically interested in a downside protection by guarantees (Arkes et al. [2]; Gatzert et al. [21]; Knoller [28]; Ruβ & Schelling [34]). The demand and interest for guarantees from the customers have not disappeared. The importance to provide some form of guarantees has been also emphasized by the European Commission in the context of the recent draft regulation on a new Pan European Pension Product (European Insurance and Occupational Pensions Authority [20]), where the providers should propose possibilities of capital protection for the customers. One solution to keep some form of guarantees in this challenging financial environment is hybrid or mixed products including partial or lighter guarantees. This is a promising area of development, in order to find intermediate solutions between pure participating products and pure equity-linked policies. As observed by Hambardzumyan & Korn [22], finding a good compromise between the search for potential returns and the possibility to generate guarantees is a major issue for the life insurance industry.

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The ability of the product to generate guarantees can also play an important role when legal guarantees are imposed by the regulator or incentivized by the state (for instance in Belgium, the minimum guaranteed rate to offer in occupational pensions, Devolder & De Valeriola [18], or in Germany the reduced taxation for products with a so-called money-back guarantee, Alexandrova et al. [1]).

Mixed or hybrid products have already been considered in the literature (Bohnert [9]; Bohnert et al. [10]; Bohnert & Gatzert [12]) and developed in practice (for instance "Select Products" in Germany, Alexandrova et al. [1]). "Select Products" constitute a static hybrid insurance product where the investor can choose between the investment in a unit-linked index participation and a traditional participating life insurance contract. The detailed product design varies from insurer to insurer but the contracts typically also include a money-back guarantee (i.e. a 0% maturity guarantee) and possibly caps or floors on the unit-linked fund return, similar to equity-indexed annuities in the US (see, e.g. Bernard & Boyle [7]). Apart from static hybrid products, so-called "3-fund dynamic hybrid products" have been introduced to the German insurance market (Kochanski & Karnarski [29]; Mahayni & Schneider [31]; Hambardzumyan & Korn [22]). The dynamic version includes a period rebalancing between the investment possibilities, typically a participating life insurance contract, a guarantee fund and a (unit-linked) equity fund. The dynamic rebalancing is used to finance and replicate additional guarantees, i.e. a money-back guarantee. The design allows to also adapt the product to the individual policyholder's needs (see also the related case of so-called flexibility riders in Chen et al. [16]).

In Belgium, new products have emerged these last years, called "products of Branch 44", mixing a part in Branch 21 (participating life insurance contract with guaranteed rate) and a part in Branch 23 (pure unit-linked product) (Van Maldegem [36]).

In this chapter, we introduce hybrid products based on a combination of a participating product and a pure unit-linked product. The latter is a product without any maturity guarantee, thereafter called "unit-linked product". Instead of considering just the superposition of these two parts, we create an automatic link between them, by a double mechanism. First, we allow contracts with or without periodical rebalancing between the two accounts. We also permit transfer from the unit-linked part to the guaranteed part by introducing a continuous fee on the unit-linked part. This technique is similar to the concept of guarantee fees in variable annuities (see, e.g. Bauer et al. [6]) and to well-known structured funds in finance offering only a part of the return of the market in exchange for a money-back guarantee. Our guarantee fee is chosen by the customer and works as a subsidy from the unit-linked part to increase the investment guarantee in the participating part. In the environment of low interest rates, this kind of internal subsidy inside the contract can boost the guaranteed part. In particular, we show that this technique permits the insurer to offer guarantees higher than the risk-free rate. This could be precious for two reasons. First, from a marketing point of view in a situation of negative returns, the insurer could propose some products with positive guaranteed rates (even if technically possible, life insurance contracts with a negative interest rate guarantee seem unrealistic). Second, we assume that the continuous fee in the unit-linked part is a fixed percentage that is due in both bad and good market scenarios. This fee is used to increase the downside protection of the investment boosting the guaranteed return in the participating contract. This leads to return distributions with a higher asymmetry, i.e. a lower return in good market scenarios but a higher downside protection. Such asymmetric return distributions are popular for many customers (Ruβ & Schelling [34]). For the participating part, we look at various forms of guarantees (annual or maturity guarantees, and intermediate guarantees on a predefined number of years).

We analyse in the chapter successively the point of view of the insurer and of the customer. Using a classical Black and Scholes market without mortality risk, we compute the fair value of various hybrid contracts. Generalizing the fairness relation between the parameters of a participating contract as developed by Bacinello [3], we obtain different forms of equivalent relations depending on the parameters of the two parts and on the kind of guarantee chosen by the customer. The numerical results show the ability of the product to offer a range of guaranteed levels, including guarantees higher than the risk-free rate. In a second part, we look at the value of these contracts for the customer, using successively a utility approach and the cumulative prospect theory. These two preference functions were also considered by Døskeland & Nordahl [19] and Chen et al. [15] comparing several insurance products and a pure unit-linked product according to their expected utility. The analysis of the paper by Chen et al. [15] shows that by incorporating realistic aspects like mortality, long time horizons and high risk-aversion, the demand for products with guarantee increases. Our results suggest that a hybrid product leads to a higher expected utility than the two special cases of a pure unit-linked and a participating product interpreted in Døskeland & Nordahl [19] and Chen et al. [15] for both expected utility and prospect theory. In particular, we obtain, for various risk aversion, optimal proportions between participating and unit-linked parts and optimal level of guarantee fees.

We organize the remainder of this chapter as follows. In Section 2.2, we present the main financial assumptions used throughout the chapter. Section 2.3 describes the mechanism of the hybrid product considered. In Section 2.4, we compute fair values and obtain various fairness relations between the parameters of the contract, depending on the kind of bonus. In Section 2.5, we look at optimal contracts for the customer in terms of mixing degree and level of fees. Section 2.6 concludes.

2.2 Basic financial and mortality assumptions

The insurance company has to take into account the mortality risk as well as the financial risk in order to fulfill their contractual obligations towards the customers. In this chapter, we focus on the financial risk and we ignore mortality risk assuming that these two risks are independent and that the mortality risk will not have an effect on the fairness relation of the contract (as illustrated, for example, in Bacinello [3]).

To follow the evolution of the wealth and the price dynamics, we assume a continuous-time model with Black-Scholes economy (Black & Scholes [8]) with the following assets :

1. A money market account which is a risk-free asset denoted by B earning a constant interest rate $r \in \mathbb{R}$ with dynamics:

$$\mathrm{d}B_t = r\, B_t\, \mathrm{d}t, \quad B_0 = 1.$$

2. Two risky funds G and H, where G is a general fund of the company (in general with a low volatility), not directly traded on the market and used as a reference for the participating contracts. However, H is an investment fund (with high volatility) traded on the market and used to compute the value of the unit-linked contract. These two funds are described as follows:

   2.a. The fund G is following a geometric Brownian motion with dynamics:

$$\mathrm{d}G_t = \mu_G\, G_t\, \mathrm{d}t + \sigma_G\, G_t\, \mathrm{d}W_t^G, \quad G_0 = 1,\ t \in [0, T], \tag{2.2.1}$$

   where W^G is a standard Brownian motion under the real probability measure $\mathbb{P}$ and $\sigma_G > 0$. In the following, we will discretize time in $n \in \mathbb{N}$ periods of equal length $\Delta s = \frac{T}{n}$. Figure (2.1) illustrates n periods with maturity T where each period is made up of $\Delta s = \frac{T}{n}$ years. These periods are counted as $t = 1, 2, \ldots, n$.
   The period return of G is defined as follows:

$$g_{t\cdot\Delta s} = \frac{G_{t\cdot\Delta s}}{G_{(t-1)\cdot\Delta s}} - 1, \quad t = 1, 2, \ldots, n. \tag{2.2.2}$$

   It is well known that the solution of the stochastic differential equation (2.2.1), under the real probability measure $\mathbb{P}$, is $G_{t\cdot\Delta s} = G_0 \exp\Big((\mu_G - \frac{\sigma_G^2}{2}) \cdot t \cdot \Delta s + \sigma_G \cdot W_{t\cdot\Delta s}^G\Big)$. Hence, $1+g_{t\cdot\Delta s} = \exp\Big((\mu_G - \frac{\sigma_G^2}{2}) \cdot \Delta s + \sigma_G(W_{t\cdot\Delta s}^G - W_{(t-1)\cdot\Delta s}^G)\Big)$ are independent and log-normally distributed for all t.

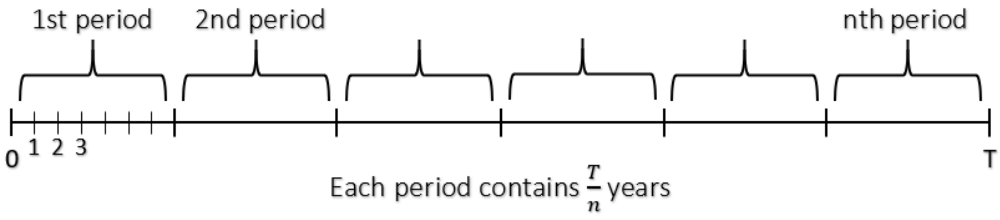

Figure 2.1: Representation of a $\frac{T}{n}$-years product

2.b. The fund H is following a geometric Brownian motion with dynamics:

$$\mathrm{d}H_t = \mu_H\, H_t\, \mathrm{d}t \;+\; \sigma_H\, H_t\, \mathrm{d}W_t^H, \quad H_0 = 1,\; t \in [0,T], \tag{2.2.3}$$

where σ_H is a positive constant and W^H is a standard Brownian motion under the real probability measure $\mathbb{P}$, that is correlated with W^G. We define $\rho \in]-1,1[$ as the parameter of correlation between the log-returns of the two funds H and G, such that $\mathbb{E}^{\mathbb{Q}}[\mathrm{d}W_t^G \mathrm{d}W_t^H] = \rho \cdot \mathrm{d}t$. This allows us to write the following differential equation using Cholesky decomposition:

$$\mathrm{d}H_t = \mu_H\, H_t\, \mathrm{d}t \;+\; \sigma_H\, \rho\, H_t\, \mathrm{d}W_t^G \;+\; \sigma_H \sqrt{1-\rho^2}\, H_t\, \mathrm{d}W_t \tag{2.2.4}$$

with

$$\mathrm{d}W_t^H = \rho \cdot \mathrm{d}W_t^G + \sqrt{1-\rho^2} \cdot \mathrm{d}W_t,$$

where W is a standard Brownian motion independent of W^G.
Similarly to the return of the fund G, the period return of the risky fund is defined as follows:

$$h_{t\cdot\Delta s} = \frac{H_{t\cdot\Delta s}}{H_{(t-1)\cdot\Delta s}} - 1, \quad t = 1,2,\ldots,n. \tag{2.2.5}$$

Noting that the solution of the stochastic differential equation (2.2.4), under the real probability measure $\mathbb{P}$, is $H_{t\cdot\Delta s} = H_0 \exp\Big((\mu_H - \frac{\sigma_H^2}{2}) \cdot t \cdot \Delta s + \sigma_H\, \rho\, W^G_{t\cdot\Delta s} + \sigma_H \sqrt{1-\rho^2}\, W_{t\cdot\Delta s}\Big)$ and then $1 + h_{t\cdot\Delta s} = \exp\Big((\mu_H - \frac{\sigma_H^2}{2}) \cdot \Delta s + \sigma_H \cdot \rho \cdot (W^G_{t\cdot\Delta s} - W^G_{(t-1)\cdot\Delta s})$
$+\sigma_H \cdot \sqrt{1-\rho^2} \cdot (W_{t\cdot\Delta s} - W_{(t-1)\cdot\Delta s})\Big)$ are also independent and log-normally distributed for all t.

Since we assume that the interest rate r is deterministic and constant, the uncertainty is limited to the stochastic evolution of the two funds and is reflected by the standard Brownian motions W^G and W under the real probability measure $\mathbb{P}$. These standard Brownian motions are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in the time interval $[0,T]$.

We further assume that the investment is riskier in the second fund than in the first fund, that is: $\mu_H > \mu_G > r$, μ_G and $\mu_H \in \mathbb{R}$ and $\sigma_H > \sigma_G$.

We consider the financial market to be complete and arbitrage-free. Let $\mathbb{Q}$ be the risk-neutral measure used for pricing in this market as per Harrison & Kreps [24]. Using Girsanov's theorem, we can write the change of measure between the real-world measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$ as

$\frac{d\mathbb{Q}}{d\mathbb{P}}\,|_{\mathbb{E}_t} = Z_t = e^{-\frac{1}{2}\|\theta\|^2 t - \theta \overline{W}_t}$, where $\theta = \begin{pmatrix} \frac{\mu_G - r}{\sigma_G} \\ \frac{\mu_H - r}{\sigma_H\sqrt{1-\rho^2}} - \frac{\rho(\mu_G - r)}{\sigma_G\sqrt{1-\rho^2}} \end{pmatrix}$ and $\overline{W}_t = \begin{pmatrix} W_t^G \\ W_t \end{pmatrix}$ is a two-dimensional $\mathbb{P}$-Brownian motion.

Then, $\begin{pmatrix} \widetilde{W}_t^G \\ \widetilde{W}_t \end{pmatrix} = \widetilde{W}_t = \overline{W}_t + \theta \cdot t$ is a two-dimensional $\mathbb{Q}$-Brownian motion.

2.3 Structure of the policy

We consider a life insurance contract with a single premium P paid at the inception of the contract. The contract pays a lump sum at maturity $T > 0$. A life insurance company decides on investing the customers' contributions such that a part is invested dynamically in a participating life insurance contract (for example the "Branch 21" insurance in Belgium) while the remaining is invested in a pure unit-linked contract (known as "Branch 23" in Belgium). In the following, we point out the difference between these two contracts:

A participating life insurance contract is a rather safe investment since it guarantees a minimum investment return known in advance, in addition to a possible bonus participating in the investment profits of the insurer's asset portfolio. More precisely, if the real rate of return exceeds the guaranteed rate, the difference between a part of the real return and the guaranteed rate is granted to the insured. In each period, the insured receives a surplus return of

$$\delta_{t\cdot\triangle s} = \max(\beta \cdot \textit{real rate of return}_{t\cdot\triangle s} - \textit{guaranteed rate of return},\ 0),$$

where $\beta \in [0, 1]$ is the participation level assumed to be constant over time, the guaranteed minimum rate of return, thereafter called $\tilde{i}$, is a technical interest rate that the insurance company determines at contract initiation while the real rate of return is a market rate acquired from the insurer's asset portfolio. We look at different guarantee periods ranging between one year (annual guarantee), several years (maturity guarantee) or simply by an arbitrary guarantee period $\triangle s$ (see Figure 2.1). We note that the rate of return $\tilde{i}$ is the total guarantee given on a period of $\frac{T}{n}$ and can be computed as $\tilde{i} = (1+i)^{\frac{T}{n}} - 1$, for an annual technical interest rate i.

The participating life insurance policy invests into the fund G defined in Section 2.2 and its value is computed by investing the premium with the guaranteed interest rate and adding the bonus rate in good years. So, the total return on the participating fund is the guaranteed interest rate $\tilde{i}$ in addition to a surplus return $\delta_{t\cdot\triangle s} = (\beta \cdot g_{t\cdot\triangle s} - \tilde{i})^+$.

A pure unit-linked contract is more attractive for investors who are likely to take more risks to pursue higher returns since the return depends on the investment funds to which this policy is linked to. Referring to Bacinello & Persson [5], this type of insurance offers advantages to the customers in terms of higher returns and the flexibility to choose or change the financial asset to which the policy is linked as well as it allows the insurance company to offer competitive products in the market. In contrast to classical life insurance products, the returns of a pure unit-linked contract are more volatile which makes the benefit at maturity uncertain and not guaranteed. There is no investment guarantee provided by the insurer.

We assume that the investment in the unit-linked part is subject to a continuous guarantee fee ε used to increase the guarantee of the participating part of the mixed product. The dynamics of this investment $F_t := e^{-\varepsilon t} H_t$ is given by:

$$\mathrm{d}F_t = (\mu_H - \varepsilon)\, F_t\, \mathrm{d}t \,+\, \sigma_H\, \rho\, F_t\, \mathrm{d}W_t^G \,+\, \sigma_H \sqrt{1-\rho^2}\, F_t\, \mathrm{d}W_t, \quad F_0 = 1,\ t \in [0, T], \tag{2.3.1}$$

where the period return of the unit-linked fund is

$$f_{t\cdot\triangle s} = \frac{F_{t\cdot\triangle s}}{F_{(t-1)\cdot\triangle s}} - 1, \quad t = 1, 2, \ldots, n. \tag{2.3.2}$$

Under the real probability measure $\mathbb{P}$, $1 + f_{t\cdot\triangle s} = \exp\Big((\mu_H - \varepsilon - \frac{\sigma_H^2}{2}) \cdot \triangle s + \sigma_H \cdot \rho \cdot (W^G_{t\cdot\triangle s} - W^G_{(t-1)\cdot\triangle s})$ $+\, \sigma_H \cdot \sqrt{1-\rho^2} \cdot (W_{t\cdot\triangle s} - W_{(t-1)\cdot\triangle s})\Big)$ are independent and log-normally distributed for all t.

The guarantee fee ε is a continuous-time dividend extracted from process H and credited to the guaranteed part.

Basically, the benefit granted to the customer is a function of the guaranteed interest rate, the surplus return and the rate of return from the investment in a risky fund.

More specifically, the contract is a combination of the participating life insurance and the unit-linked contract where

- a constant proportion $\pi \in]0,1]$ of the financial portfolio is invested dynamically in a participating life insurance contract with period return $\tilde{i} + \delta_{t\cdot\Delta s}$. The real return is modeled by a fund G.
- a constant proportion $1-\pi$ is invested in a risky fund F without guarantee.

With this investment, we analyse two different cases :

- *Rebalancing case :* After each time period, the investment share is rebalanced such that, again, a constant proportion $\pi \in]0,1]$ is invested in the participating contract. More precisely, the eventual rebalancing can occur only at the moment when the guarantee is consolidated. Here is the benefit at maturity for a mixed product:

$$V_T = V_0 \cdot \prod_{t=1}^{n} \left[\pi \cdot \left(1 + \tilde{i} + \left(\beta \cdot g_{t\cdot\frac{T}{n}} - \tilde{i}\right)^+\right) + (1-\pi) \cdot \left(1 + f_{t\cdot\frac{T}{n}}\right) \right]. \tag{2.3.3}$$

- *Non-rebalancing case :* The proportion π is set once at the beginning of the contract. No further action is taken, so the benefit at maturity is written as follows:

$$V_T = V_0 \cdot \left[\pi \cdot \prod_{t=1}^{n} \left(1 + \tilde{i} + \left(\beta \cdot g_{t\cdot\frac{T}{n}} - \tilde{i}\right)^+\right) + (1-\pi) \cdot \prod_{t=1}^{n} \left(1 + f_{t\cdot\frac{T}{n}}\right) \right]. \tag{2.3.4}$$

We note that the initial reserve V_0 is the premium invested at the inception of the contract P.

2.4 Valuation of the contract

Apart from the single premium, no payments occur until the maturity of the contract. Let us point out also that the guarantee can be given by the insurer on an annual basis ($n = T$), periodical basis or at maturity ($n = 1$). In order to price and evaluate the different forms of contract and define their parameters, we designate two main concepts:

a. *The retrospective reserve* : it corresponds to the initial premium capitalized each period, following the specifications of the contract.
We will denote this reserve by $V_{t\cdot\frac{T}{n}}$ $(t = 0, 1, \ldots, n)$ with initial condition : $V_0 = P$. We give the example of an annual guarantee contract ($n = T$):

  - For a pure participating life insurance contract ($\pi = 1$) with annual guarantee, this reserve is given by:

$$V_{t\cdot\frac{T}{n}} = V_t = V_0 \cdot \prod_{k=0}^{t-1} \left(1 + i + (\beta \cdot g_{k+1} - i)^+\right) = V_{t-1} \cdot \left(1 + i + (\beta \cdot g_t - i)^+\right).$$

  - For a pure unit-linked contract ($\pi = 0$) with annual guarantee, this reserve is given by:

$$V_{t\cdot\frac{T}{n}} = V_t = V_0 \cdot \prod_{k=0}^{t-1} (1 + f_{k+1}).$$

The last two formulas can be easily generalized for a mixed contract ($\pi \in]0, 1[$):

For a rebalanced portfolio:

$$V_{t\cdot\frac{T}{n}} = V_0 \cdot \prod_{k=0}^{t-1} \Bigg[\pi \cdot \left((1+i)^{\frac{T}{n}} + \left(\beta \cdot g_{(k+1)\cdot\frac{T}{n}} - ((1+i)^{\frac{T}{n}} - 1) \right)^+ \right) + (1-\pi) \cdot \left(1 + f_{(k+1)\cdot\frac{T}{n}} \right) \Bigg].$$

For a non-rebalanced portfolio:

$$V_{t\cdot\frac{T}{n}} = V_0 \cdot \Bigg[\pi \cdot \prod_{k=0}^{t-1} \left((1+i)^{\frac{T}{n}} + \left(\beta \cdot g_{(k+1)\cdot\frac{T}{n}} - ((1+i)^{\frac{T}{n}} - 1) \right)^+ \right) + (1-\pi) \cdot \prod_{k=0}^{t-1} \left(1 + f_{(k+1)\cdot\frac{T}{n}} \right) \Bigg].$$

We note that the final retrospective reserve V_T corresponds to the benefit paid at maturity to the customer. Particularly, we obtain the reserve of a pure participating contract and a pure unit-linked contract by, respectively, extracting $\pi = 1$ and $\pi = 0$ from the above mixed contract.

b. *The prospective reserve (or fair valuation)*: it corresponds to the discounted risk-neutral expectation of the final retrospective reserve.
We denote the fair value at time $t \cdot \frac{T}{n}$ by $FV_{t\cdot\frac{T}{n}}$ ($t = 0, 1, \ldots, n$), i.e.

$$FV_{t\cdot\frac{T}{n}} = e^{-r(T - t\cdot\frac{T}{n})} \cdot \mathbb{E}^{\mathbb{Q}}[V_T \mid \mathcal{F}_{t\cdot\frac{T}{n}}].$$

We say that the contract is fairly priced if and only if the initial fair value is equal to the initial retrospective reserve:

$$FV_0 = V_0 = P.$$

In the next sections, we develop this condition of fairness for different kinds of mixed contracts.

2.4.1 Maturity guarantee

As known, the timing of the allocation of the guarantee and the profit sharing can be annually, periodically or can be applied only at maturity. In the latter case where $n = 1$, the investment is developing throughout the years until maturity without assessing the performance of the funds which means that this is a product with a maturity guarantee (that is sometimes called terminal bonus). In practical terms, this implies that there is no need for periodic portfolio rebalancing. The guarantee and any surplus return are consolidated and applied only at the conclusion of the contract, when it matures. Therefore, adjustments to the portfolio composition throughout the investment years are not relevant, as the final outcome is determined solely at the contract's maturity. For this reason, we will price hereinafter the case of a non-rebalanced portfolio where the guarantee and the profit sharing are computed at maturity, while the rebalancing case will be hidden in such a sort of product.

Maturity guarantee without rebalancing

In this type of contract, the guarantee does not have to interfere during the years of investment and so there is somehow a compensation between good and bad years. We can conclude that this is a kind of static investment that leads to the following benefit at maturity:

$$V_T = V_0\left[\pi \cdot \left((1+i)^T + \left(\beta \cdot (\tfrac{G_T}{G_0} - 1) - ((1+i)^T - 1)\right)^+\right) + (1-\pi) \cdot \left(\tfrac{F_T}{F_0}\right)\right]$$

(see Equation (2.3.4) with $n = 1$ and $\tilde{i} = (1+i)^{\frac{T}{n}} - 1$).

The value of this contract is assumed by computing the time 0 price of V_T under the risk-neutral expectation : $FV_0(V_T) = \mathbb{E}^{\mathbb{Q}}[e^{-rT} \cdot V_T]$. This end gives, after replacing the retrospective reserve by its expression:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[e^{-rT} \cdot V_T] = \;& \pi \cdot V_0 \cdot \left[e^{-rT} \cdot (1+i)^T \; + \beta \cdot e^{-rT} \cdot \mathbb{E}^{\mathbb{Q}}\left[\left(\tfrac{G_T}{G_0} - \left(1 + \tfrac{(1+i)^T - 1}{\beta}\right)\right)^+\right]\right] \\ & + (1-\pi) \cdot V_0 \cdot e^{-rT} \cdot \mathbb{E}^{\mathbb{Q}}\left[\tfrac{F_T}{F_0}\right].\end{aligned}$$

The term $c_0^{\{T\}} = e^{-rT} \cdot \mathbb{E}^{\mathbb{Q}}\left[\left(\tfrac{G_T}{G_0} - \left(1 + \tfrac{(1+i)^T - 1}{\beta}\right)\right)^+\right]$ is the time 0 value of a European call option with a maturity T, an initial price of 1 of the underlying fund G and a strike price $1 + \frac{(1+i)^T - 1}{\beta}$. The value of this call option is given by Black-Scholes under the following formula:

$$c_0^{\{T\}} = \Phi(d_1) - \left(1 + \frac{(1+i)^T - 1}{\beta}\right) \cdot e^{-rT} \cdot \Phi(d_2),$$

where $d_1 = \dfrac{\left(r + \frac{\sigma_G^2}{2}\right)T - \ln\left(1 + \frac{(1+i)^T - 1}{\beta}\right)}{\sigma_G\sqrt{T}}, \quad d_2 = d_1 - \sigma_G\sqrt{T}$

and Φ is the cumulative distribution function of a standard normal variable.

In addition, $\frac{F_T}{F_0}$ expresses $1+$return of the fund F from the inception of the contract till maturity. Based on Equation (2.3.2), by taking $n = 1$, the $\mathbb{Q}$-expectation of $\frac{F_T}{F_0}$ gives $e^{(r-\varepsilon)T}$.

After these indications, the contract is fairly priced by equalizing the expected present value of benefits to the single premium paid by the insured ($FV_0(V_T) = P$):

$$\pi \cdot V_0 \cdot \left[e^{-rT} \cdot (1+i)^T \; + \beta \cdot c_0^{\{T\}}\right] + (1-\pi) \cdot V_0 \cdot e^{-rT} \cdot e^{(r-\varepsilon)T} = V_0.$$

Eventually, the premium equivalence is as follows :

$$\pi \cdot [\,((1+i)e^{-r})^T \; + \beta \cdot c_0^{\{T\}}] \; + \; (1-\pi) \cdot e^{-\varepsilon T} = 1. \tag{2.4.1}$$

This relation gives us a fair condition between all the parameters in order to generate a fair contract. It creates a link between the two components of the mixed product. In particular, we can interpret the relation as a feedback between the guaranteed rate i and the guarantee fee ε, depending on the allocation level π between these two parts.

2.4.2 $\frac{T}{n}$-years guarantee

We introduce a new product in which the bonus is incorporated not at maturity but each period, as represented in Figure 2.1. It is considered as a periodical guarantee contract since the guarantee and the profit sharing are investigated and computed each period Δs (or each $\frac{T}{n}$ years). Similarly, in this type of contract, we differentiate between periodically rebalancing or not rebalancing our portfolio.

$\frac{T}{n}$-years guarantee with rebalancing

Normally, the portfolio is rebalanced just after providing the guarantee and computing the profit sharing. Therefore, we rebalance the portfolio each period in order to maintain the original level of risk. To construct the benefit, we reinvest the capitalized amount each period with the same proportion, π in a participating contract and $1-\pi$ in a pure unit-linked contract :

$$\begin{aligned} V_T = V_0 \cdot \prod_{t=1}^{n} \Bigg[\quad & \pi \cdot \left((1+i)^{\frac{T}{n}} + \left(\beta \cdot \left(\frac{G_{t\frac{T}{n}}}{G_{(t-1)\frac{T}{n}}} - 1 \right) - ((1+i)^{\frac{T}{n}} - 1) \right)^+ \right) \\ & + (1-\pi) \cdot \left(\frac{F_{t\frac{T}{n}}}{F_{(t-1)\frac{T}{n}}} \right) \Bigg]. \end{aligned} \tag{2.4.2}$$

The time 0 price of V_T is computed by applying the martingale approach which is exploited by the following $\mathbb{Q}$-expectation : $\mathbb{E}^{\mathbb{Q}}[e^{-rT} \cdot V_T]$. By replacing V_T by its expression in Equation (2.4.2), taking into consideration the stochastic independence of the periodical rates $g_{t\cdot\triangle s}$ for all t, as well as for $f_{t\cdot\triangle s}$. After algebraic manipulations, we get

$$\begin{aligned} V_0 \cdot \prod_{t=1}^{n} \Bigg[\quad & \pi \cdot \left((e^{-r} \cdot (1+i))^{\frac{T}{n}} + \beta \cdot e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[\left(\frac{G_{t\frac{T}{n}}}{G_{(t-1)\frac{T}{n}}} - \left(1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta} \right) \right)^+ \right] \right) \\ & + (1-\pi) \cdot e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[\frac{F_{t\frac{T}{n}}}{F_{(t-1)\frac{T}{n}}} \right] \Bigg]. \end{aligned}$$

The term

$$c_0^{\{\frac{T}{n}\}} = e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[\left(\frac{G_{t\frac{T}{n}}}{G_{(t-1)\frac{T}{n}}} - \left(1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta} \right) \right)^+ \right]$$

is obviously the time 0 value of a European call option with maturity $\frac{T}{n}$ with the following formula:

$$c_0^{\{\frac{T}{n}\}} = \Phi(d_1) - \left(1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta} \right) \cdot e^{-r\frac{T}{n}} \cdot \Phi(d_2),$$

where

$$d_1 = \frac{(r + \frac{\sigma_G^2}{2})\frac{T}{n} - \ln\left(1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta}\right)}{\sigma_G \sqrt{\frac{T}{n}}}, \quad d_2 = d_1 - \sigma_G \sqrt{\frac{T}{n}}$$

and Φ is the cumulative distribution function of a standard normal variable.

Recalling Equation (2.3.2), the $\mathbb{Q}$-expectation of

$$\frac{F_{t\frac{T}{n}}}{F_{(t-1)\frac{T}{n}}}$$

gives $e^{\frac{T}{n}(r-\varepsilon)}$.

To this end, we can deduce the explicit formula of the premium equivalence:

$$\pi \cdot \left[((1+i) \cdot e^{-r})^{\frac{T}{n}} + \beta \cdot c_0^{\{\frac{T}{n}\}} \right] + (1-\pi) \cdot e^{-\varepsilon\frac{T}{n}} = 1. \tag{2.4.3}$$

It is worth mentioning that we have computed the reserve at time 0 to obtain the fairness relation between the parameters. As well, we can compute the reserve at any moment where the moment in this case is expressed periodically. For instance, we can express the time $t \cdot \frac{T}{n}$ price of V_T by solving

$$FV_{t\cdot\frac{T}{n}} = e^{-r(T - t\cdot\frac{T}{n})} \cdot \mathbb{E}^{\mathbb{Q}}[V_T \mid \mathcal{F}_{t\cdot\frac{T}{n}}], \quad t \leqslant n.$$

By replacing Equation (2.4.2) in the $\mathbb{Q}$-expectation and conditioning on the information available at time $t \cdot \frac{T}{n}$, we obtain the following:

$$FV_{t\cdot\frac{T}{n}} = V_{t\cdot\frac{T}{n}} \cdot \prod_{s=t}^{n-1} \Bigg[\pi \cdot \left((e^{-r} \cdot (1+i))^{\frac{T}{n}} + \beta \cdot e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[\left(\frac{G_{(s+1)\frac{T}{n}}}{G_{s\frac{T}{n}}} - \left(1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta} \right) \right)^+ \right] \right) + (1-\pi) \cdot e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[\frac{F_{(s+1)\frac{T}{n}}}{F_{s\frac{T}{n}}} \right] \Bigg].$$

Hence, the fair value will be $FV_{t\cdot\frac{T}{n}} = V_{t\cdot\frac{T}{n}} \cdot \left[\pi \cdot [((1+i)\cdot e^{-r})^{\frac{T}{n}} + \beta \cdot c_0^{\{\frac{T}{n}\}}] + (1-\pi) \cdot e^{-\varepsilon\frac{T}{n}} \right]^{n-t}$.

Consequently, if the contract is fairly priced at origin following condition (2.4.3), it remains fair in terms of the reserve at any time, by stating $FV_{t\cdot\frac{T}{n}} = V_{t\cdot\frac{T}{n}}$.

$\frac{T}{n}$-years guarantee without rebalancing

If the portfolio is not rebalanced each period, each contract is capitalized separately with the proportion invested in each asset at the beginning of the contract. At maturity, the structure of the retrospective reserve is in the following form:

$$V_T = V_0 \cdot \left[\pi \cdot \prod_{t=1}^{n} \left((1+i)^{\frac{T}{n}} + \left(\beta \cdot \left(\frac{G_{t\frac{T}{n}}}{G_{(t-1)\frac{T}{n}}} - 1 \right) - \left((1+i)^{\frac{T}{n}} - 1 \right) \right)^+ \right) + (1-\pi) \cdot \prod_{t=1}^{n} \left(\frac{F_{t\frac{T}{n}}}{F_{(t-1)\frac{T}{n}}} \right) \right].$$

To value this contract, we calculate the time 0 price of V_T under the risk-neutral measure and then deduce the premium equivalence in closed form. The fair value of the contract is written as follows:

$$\begin{aligned} FV_0(V_T) &= e^{-rT} \cdot \mathbb{E}^{\mathbb{Q}}[V_T] \\ &= V_0 \cdot \Bigg[\pi \cdot \prod_{t=1}^{n} \left((e^{-r} \cdot (1+i))^{\frac{T}{n}} + \beta \cdot e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[\left(\frac{G_{t\frac{T}{n}}}{G_{(t-1)\frac{T}{n}}} - \left(1 + \frac{(1+i)^{\frac{T}{n}} - 1}{\beta} \right) \right)^+ \right] \right) \\ &\quad + (1-\pi) \cdot \prod_{t=1}^{n} e^{-r\frac{T}{n}} \cdot \mathbb{E}^{\mathbb{Q}} \left[\frac{F_{t\frac{T}{n}}}{F_{(t-1)\frac{T}{n}}} \right] \Bigg]. \end{aligned}$$

To write the premium equivalence, we simply connect the last expression to the premium paid at inception after considering the $\mathbb{Q}$-expectations to obtain the following fairness relation:

$$\pi \cdot [((1+i)\cdot e^{-r})^{\frac{T}{n}} + \beta \cdot c_0^{\{\frac{T}{n}\}}]^n + (1-\pi) \cdot e^{-\varepsilon T} = 1. \tag{2.4.4}$$

Special case: annual guarantee (n=T)

We note that the annual guarantee (sometimes referred as reversionary bonus) is a special case from the $\frac{T}{n}$-years guarantee contract and is obtained by considering $n = T$ in Equations (2.4.3) and (2.4.4) for rebalancing and non-rebalancing contracts, respectively.

The premium equivalence of an annual guarantee with rebalancing is then

$$\pi \cdot ((1+i)\cdot e^{-r} + \beta \cdot c_1) + (1-\pi) \cdot e^{-\varepsilon} = 1,$$

and without rebalancing is

$$\pi \cdot ((1+i)\cdot e^{-r} + \beta \cdot c_1)^T + (1-\pi) \cdot e^{-\varepsilon T} = 1.$$

2.4.3 Numerical illustrations

In this section, we provide numerical solutions to the fairness relations obtained in each case. More precisely, we present some examples to Equations (2.4.1), (2.4.3) and (2.4.4) with respect to the technical interest rate i. The idea is to boost the minimum guaranteed rate by making a partial investment, which means, by investing a part in the unit-linked contract. This is because by increasing the guarantee fee ε, we are reinforcing the guaranteed rate i. For this reason, we will find thereafter solutions for the fairness relations with respect to i according to different values of the investment share π for each type of guarantee.

We consider a life insurance contract with a maturity $T = 20$ years. For the upcoming work, we suppose that the risk-free rate is $r = 1.5\%$ and the volatility of the guarantee fund is $\sigma_G = 3\%$. We note that the initial reserve V_0 does not need to be specified in the calculation of the fair guaranteed rate for the different forms of contracts.

Numerical insights: $\frac{T}{n}$-years guarantee

By considering the general form of a partial guarantee contract which is the $\frac{T}{n}$-years guarantee with a number of periods $n = 4$, we vary the share π invested in the participating contract and find the guaranteed rates i that satisfy fairness relations (2.4.3) and (2.4.4) obtained in the theoretical part of the rebalancing and non-rebalancing cases, respectively, supposing that the insurance company has fixed all other parameters. Furthermore, we analyze the impact of the participation level β and the guarantee fee ε on these findings.

Briefly, we assess the change of i with respect to different investment shares between the participating life insurance contract and the pure unit-linked contract. For this valuation, we consider a contract of 20 years divided in 4 periods with 5 successive years.

		With Rebalancing			***Without Rebalancing***		
π	$\beta = 70\%$ $\varepsilon = 0\%$	$\beta = 70\%$ $\varepsilon = 0.25\%$	$\beta = 50\%$ $\varepsilon = 0.25\%$	$\beta = 70\%$ $\varepsilon = 0.5\%$	$\beta = 70\%$ $\varepsilon = 0.25\%$	$\beta = 50\%$ $\varepsilon = 0.25\%$	$\beta = 70\%$ $\varepsilon = 0.5\%$
0.10	1.18	3.68	3.69	5.67	3.37	3.38	4.70
0.20	1.18	2.47	2.50	3.44	2.38	2.42	3.16
0.30	1.18	2.01	2.08	2.63	1.97	2.05	2.50
0.40	1.18	1.75	1.87	2.19	1.73	1.85	2.12
0.50	1.18	1.58	1.73	1.90	1.57	1.72	1.86
0.60	1.18	1.46	1.64	1.69	1.45	1.64	1.66
0.70	1.18	1.36	1.57	1.53	1.36	1.57	1.51
0.80	1.18	1.29	1.52	1.39	1.29	1.52	1.38
0.90	1.18	1.23	1.48	1.28	1.23	1.48	1.27
1.00	1.18	1.18	1.45	1.18	1.18	1.45	1.18

Table 2.1: Fair guaranteed rates i (expressed in %) with respect to the share in the participating contract π for different participation levels β and guarantee fees ε.

It is obvious from Table 2.1 the capacity of the insurance company to increase the guarantee by investing more in the unit-linked contract. In addition, we notice that a higher participation level leads to a lower guarantee; however, the more the company imposed fees the more it is able to give guarantees. This is confirmed in the first column of Table 2.1 where no fees are charged to the customer, which leads to a fixed guarantee regardless the asset allocation π.

Besides, we can observe that the difference in the values of the guaranteed rates between a contract with rebalancing and without rebalancing can be almost negligible especially when a higher proportion is invested in the participating life insurance contract.

We note that, by taking $n = T$ and $n = 1$, we come across the annual guarantee and the maturity guarantee types, respectively. Hence, we will thereafter analyze the above points for these specific types of contracts.

Numerical insights: annual guarantee

In the context of finding a solution to the fairness relation of an annual guarantee contract with respect to i, we assess the impact of the participation level β for given values of ε, r and σ_G as well as the impact of the guarantee fee ε for given values of β, r and σ_G on both rebalancing and non-rebalancing cases (Table 2.2).

		With Rebalancing			***Without Rebalancing***		
π	$\beta = 70\%$ $\varepsilon = 0\%$	$\beta = 70\%$ $\varepsilon = 0.25\%$	$\beta = 50\%$ $\varepsilon = 0.25\%$	$\beta = 70\%$ $\varepsilon = 0.5\%$	$\beta = 70\%$ $\varepsilon = 0.25\%$	$\beta = 50\%$ $\varepsilon = 0.25\%$	$\beta = 70\%$ $\varepsilon = 0.5\%$
0.10	0.11	3.67	3.78	6.06	3.19	3.34	4.65
0.20	0.11	2.09	2.41	3.38	1.93	2.29	2.93
0.30	0.11	1.42	1.90	2.32	1.34	1.84	2.10
0.40	0.11	1.02	1.62	1.70	0.98	1.59	1.57
0.50	0.11	0.76	1.43	1.27	0.73	1.42	1.18
0.60	0.11	0.56	1.31	0.94	0.55	1.30	0.89
0.70	0.11	0.41	1.21	0.67	0.40	1.20	0.64
0.80	0.11	0.29	1.14	0.46	0.29	1.13	0.44
0.90	0.11	0.19	1.08	0.27	0.19	1.08	0.26
1.00	0.11	0.11	1.03	0.11	0.11	1.03	0.11

Table 2.2: Fair guaranteed rates i (expressed in %) with respect to the share in the participating contract π for different participation levels β and guarantee fees ε.

We point out that, in both cases, the fair guaranteed rate i increases with the decrease of the share in the participating contract π. This is trivial because the more the insurance company invests in the unit-linked contract, the less is its commitment to the customer, and thus, it is capable to give a higher guarantee. Moreover, we observe that, for a fixed guarantee fee ε, a higher participation level produces a low fair technical rate. For instance in the rebalancing case, taking $\pi = 50\%$ and $\varepsilon = 0.25\%$, we can guarantee to $i = 1.43\%$ for a participation level $\beta = 50\%$ and to $i = 0.76\%$ for a $\beta = 70\%$. However, for a fixed β, it is obvious that taking a higher fee results in a higher guarantee. Take an example in the rebalancing case, for $\pi = 50\%$ and $\beta = 70\%$, we can give a guarantee of $i = 1.27\%$ for $\varepsilon = 0.50\%$ and $i = 0.76\%$ for $\varepsilon = 0.25\%$. Whilst we have to note that in the case of $\pi = 1$, the technical interest rate is constant for different values of ε and that is because we are going back to the classical life insurance product where the ε does not have any effect. Similarly, the technical interest rate is constant when ε is equal to 0 which confirms that ε is used to boost i. Furthermore, we notice that the guarantee is slightly better in the rebalanced case, for all different values of β and ε, since by rebalancing the portfolio we are avoiding somehow the exposure to undesirable risk. Overall, we can say that, in a partial guarantee product, we can get much higher guaranteed rate.

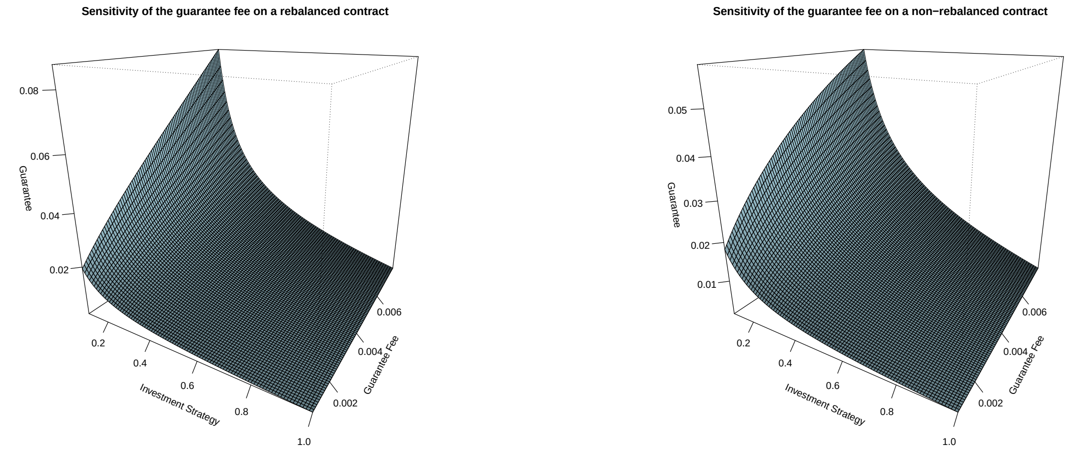

Figure 2.2: Sensitivity of the guarantee fee on the annual guarantee contract.

We present, in Figure 2.2, the effect of the guarantee fee on the guaranteed rate taking all other parameters fixed.

It is noticeable that, in both cases, the fee does not have any effect when investing in a participating life insurance contract and then the more we invest in the unit-linked contract, the more the guaranteed rate is increasing. In addition, it is increasing with the increase of the guarantee fee ε, and besides, it goes up faster in a rebalanced portfolio than in a non-rebalanced portfolio.

Consequently, this figure confirms that the guarantee fee is financing the participating insurance contract to boost the guaranteed rate.

We now examine the impact of the volatility coefficient σ_G on the technical interest rate i in order to satisfy the fairness relations of the rebalanced and non-rebalanced annual guarantee contract for given values of $\beta = 70\%$ and $\varepsilon = 0.25\%$ (Table 2.3).

	With Rebalancing		***Without Rebalancing***	
π	$\sigma_G = 3\%$	$\sigma_G = 1\%$	$\sigma_G = 3\%$	$\sigma_G = 1\%$
0.10	3.67	3.79	3.19	3.38
0.20	2.09	2.52	1.93	2.41
0.30	1.42	2.08	1.34	2.03
0.40	1.02	1.84	0.98	1.82
0.50	0.76	1.69	0.73	1.68
0.60	0.56	1.58	0.55	1.58
0.70	0.41	1.50	0.40	1.50
0.80	0.29	1.44	0.29	1.44
0.90	0.19	1.39	0.19	1.39
1.00	0.11	1.35	0.11	1.35

Table 2.3: Fair guaranteed rates i (expressed in %) with respect to π for different σ_G.

It is obvious that for a very low G-fund volatility, the fairness relation is satisfied with a higher guaranteed rate comparing to the results for a $\sigma_G = 3\%$. For instance, with a high participation level of $\beta = 70\%$ and a guarantee fee of $\varepsilon = 0.25\%$ and with an investment of $\pi = 70\%$ of the premium in participating life insurance product, a volatility coefficient of $\sigma_G = 3\%$ leads to a fair technical rate around $i = 0.4\%$ while a volatility coefficient of $\sigma_G = 1\%$ leads to a fair technical rate of $i = 1.50\%$

in the with and without rebalancing cases.

In addition, we notice the effect of the volatility in the classical insurance product which means when making a full investment in the participating life contract (π = 100%), the guaranteed rate is increased when we decrease σ_G.

Numerical insights: maturity guarantee

Once again, we are going to give some solutions with respect to the guaranteed rate to hold the fair relation of the maturity guarantee without rebalancing determined by Equation (2.4.1) in Section 2.4.1. Table 2.4 analyzes the effect of the participation level and the guarantee fee on the positive guaranteed rates.

		Without Rebalancing		
π	β = 70% ε = 0%	β = 70% ε = 0.25%	β = 50% ε = 0.25%	β = 70% ε = 0.5%
0.10	1.43	3.38	3.38	4.70
0.20	1.43	2.42	2.42	3.16
0.30	1.43	2.05	2.06	2.53
0.40	1.43	1.85	1.87	2.19
0.50	1.43	1.72	1.75	1.96
0.60	1.43	1.63	1.67	1.80
0.70	1.43	1.57	1.61	1.68
0.80	1.43	1.51	1.57	1.58
0.90	1.43	1.47	1.53	1.50
1.00	1.43	1.43	1.50	1.43

Table 2.4: Fair guaranteed rates i (expressed in %) with respect to the share in the participating contract π for different participation levels β and guarantee fees ε.

By fixing the guarantee fee ε and looking at the different technical rates with respect to the participation level β, we notice that the more the investor is sharing the profits with the insurance company the lower will be the guarantee. For instance, when ε is fixed at 0.25% and investing with a proportion of $\pi = 0.7$ of the premium, the fair guarantee is i = 1.61% for β = 50% against i = 1.57% for β = 70%. Whilst by fixing the participation level β and concentrating on the effect of the guarantee fee ε, the guarantee is absolutely increasing with the guarantee fee. We can check this result from Table 2.4 by taking β = 70% and $\pi = 0.7$, an ε of 0.25% gives a fair guarantee of 1.57% while an ε of 0.5% results in a fair guarantee of i = 1.68%. The results can be checked graphically.

The first chart in Figure 2.3 confirms the increase of the guaranteed rate with the increase of the guarantee fee ε as well as with a greater investment in the unit-linked part. On the other side, the impact of the participation level β on the guaranteed rate is negative that is decreasing the more the insurance company is sharing its profits with the customer.

Comparison between annual and maturity guarantee contracts

It is worth making a little comparison between the findings of the annual guarantee and the maturity guarantee. Two points can be noticeable:

1. The participation level β causes a small effect on the results of the technical interest rates in a fair maturity guarantee contract compared to the annual guarantee contract.

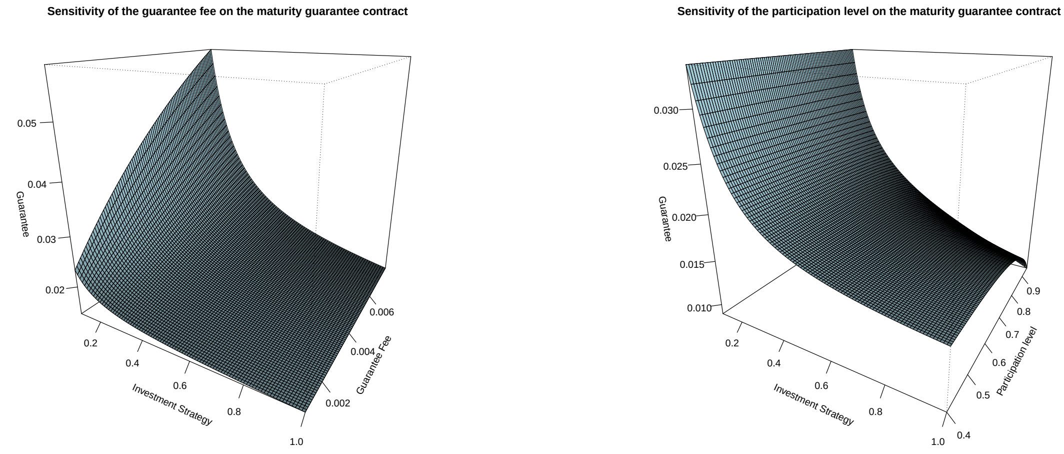

Figure 2.3: Sensitivity of the guarantee fee and the participation level on the maturity guarantee contract.

2. When considering a pure participating insurance product, the guarantee is much higher in the maturity guarantee contract than it is in the annual guarantee. Table 2.5 displays different values of T and σ_G and their impacts on the fair guaranteed rates derived from Equation (2.4.1) for $\beta = 70\%$, $\varepsilon = 0.25\%$ and $\pi = 1$. Table 2.5 shows that the guarantee decreases with the decrease of the maturity T for a given value of σ_G. Hence, the high value of the technical interest rate in the maturity guarantee is proved since the annual guarantee is a maturity guarantee with $T = 1$. In addition, the choice of a portfolio with low volatility coefficient on a long maturity leads to a higher value of the guaranteed rate.

Maturity	**Volatility (%)**	**Guarantee (%)**
20	10	0.60
	7	1.00
	3	1.43

Volatility (%)	**Maturity**	**Guarantee (%)**
3	20	1.43
	10	1.35
	1	0.11

Table 2.5: Solutions of equations (2.4.1) with respect to the guaranteed rate i for different maturities T and volatilities σ_G.

Sensitivity analysis

After analyzing the annual and maturity contracts, we can go back to the $\frac{T}{n}$-years guarantee contract to check the sensitivity of the volatility of the fund G, the maturity T and the number of periods n on the values of the technical interest rate with respect to π and fixing all other parameters.

Starting by the impact of the volatility, we consider a $\frac{T}{n}$-years guarantee contract with the same assumptions as before in order to look at the variation of the technical rate.

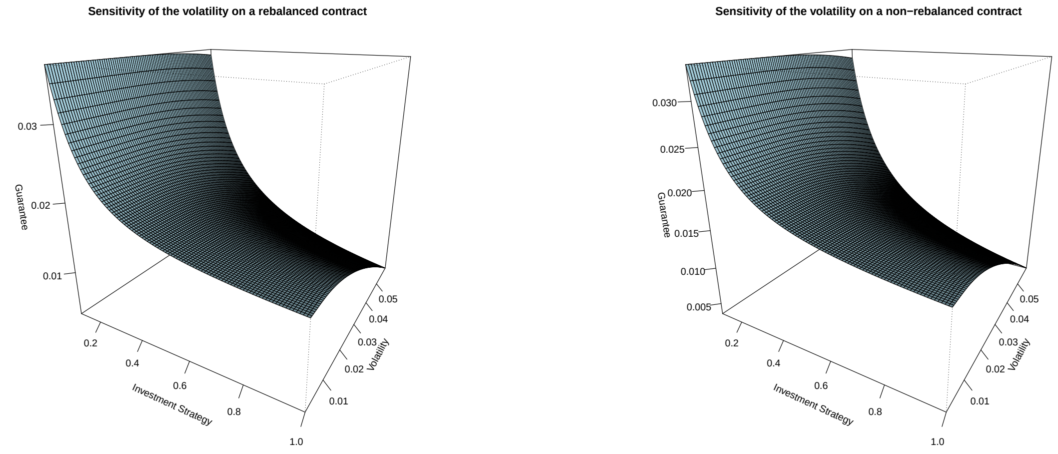

Figure 2.4: Sensitivity of the volatility σ_G on a $\frac{T}{n}$-years guarantee contract.

Figure 2.4 shows the variation of the guarantee for different values of the parameter σ_G and with respect to the investment strategy π. We notice that the volatility can have a significant effect when we invest more in a participating life insurance contract. Furthermore, the insurance company can give a better guarantee by investing in a portfolio less volatile in both contracts, with and without rebalancing.

Subsequently, we will examine the impact of the number of periods and maturity on the guaranteed interest rate. It is detected by fixing all the parameters and changing the interest rate according to the number of periods n for two different maturities T with respect to an investment in the participating life insurance contract and in the pure unit-linked contract.

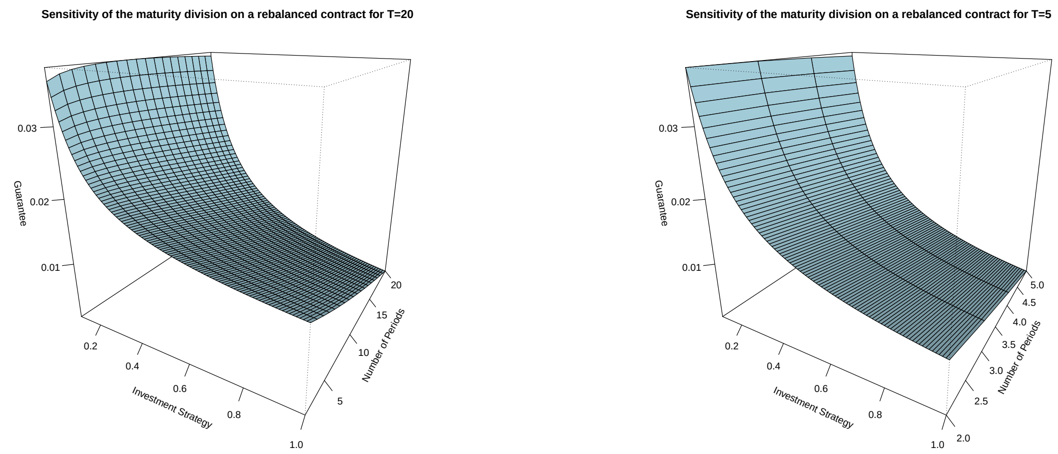

Figure 2.5: Sensitivity of the number of periods n and the maturity T on a $\frac{T}{n}$-years guarantee contract.

In Figure 2.5, we consider a $\frac{T}{n}$-years guarantee contract with rebalancing. First, we fix the time to maturity and all other parameters and then we look at the variation of the guaranteed interest rate i with respect to π and for different values of n. We notice that the guarantee is increasing when we take a fewer number of periods n. It is worth mentioning that in the rebalancing portfolio, the number of periods is fixed from $n > 1$, since for $n = 1$ we come across the maturity guarantee contract. On the other side, to check the effect of the maturity T on the guaranteed rate i for different number of periods, we observe that the guaranteed rate is increasing with the increase of T.

Short-term versus long-term contracts

Based on all the above different scenarios, we summarize with a comparison between a short- and a long-term contract where we consider, in each one, two investment cases in order to observe the variation in the guaranteed rate according to the different types of contracts that are supposed not to be rebalanced. In this comparison, we choose the same parameters β, σ_G, ε and n as 70%, 3%, 0.25% and 4, respectively. Whilst regarding the interest rate, we are going to construct the first table with the value assumed previously in the numerical illustrations $r = 1.5\%$ (Table 2.6) and in the second table we try to get closer to the current situation and take $r = 0.5\%$ (Table 2.7).

Short term contract $T = 5$, $r = 1.5\%$

	Pure participating product ($\pi = 1$)	**Mixed product** ($\pi = 0.5$)
Annual Guarantee	0.11	0.73
$\frac{5}{4}$-years Guarantee	0.34	0.94
Maturity Guarantee	1.18	1.58

Long term contract $T = 20$, $r = 1.5\%$

	Pure participating product ($\pi = 1$)	**Mixed product** ($\pi = 0.5$)
Annual Guarantee	0.11	0.73
$\frac{20}{4}$-years Guarantee	1.18	1.57
Maturity Guarantee	1.43	1.72

Table 2.6: Fair guaranteed rates i (expressed in %) for different maturities T when $r = 1.5\%$.

From a general look on the two above tables, it is obvious that the guarantee can be boosted in different ways :

- By considering a long-term contract because a higher time to maturity T leads to a higher guarantee, except for the annual guarantee contract where the maturity T does not have any effect.
- By combining a participating life insurance contract to a unit-linked contract.
- By turning into a maturity guarantee where the surplus does not occur until the end of the contract.

As a conclusion from the above suggested products and the numerical illustrations, we are able to mention some remarks and inferences. First of all, referring to the relations obtained, we can put the light on the exclusion of the parameters σ_H and ρ from the pricing part of each type of product. Hence, the changes in these parameters did not have an impact on the numerical solutions, particularly on the fair guaranteed interest rates. Second, we observe two dimensions that allow us to present a higher guarantee. The first dimension includes the concept of a partial guarantee which consists of investing a part of the premium in a pure unit-linked contract where a fee is charged to finance the participating life insurance contract in order to get a higher guarantee. While the second dimension consists of expanding the first dimension and turning it into a maturity guarantee product where a higher guarantee can be presented.

Short term contract $T = 5$, $r = 0.5\%$

	Pure participating product ($\pi = 1$)	**Mixed product** ($\pi = 0.5$)
Annual Guarantee	−1.88	−0.75
$\frac{5}{4}$-years Guarantee	−1.54	−0.50
Maturity Guarantee	−0.24	0.39

Long term contract $T = 20$, $r = 0.5\%$

	Pure participating product ($\pi = 1$)	**Mixed product** ($\pi = 0.5$)
Annual Guarantee	−1.88	−0.78
$\frac{20}{4}$-years Guarantee	−0.24	0.38
Maturity Guarantee	0.26	0.66

Table 2.7: Fair guaranteed rates i (expressed in %) for different maturities T when $r = 0.5\%$.

2.5 Customer utility and optimal investment

After pricing the different types of contracts, we aim to maximize the customers' utility based on individuals' preferences in order to put the above products in an optimal utility framework. Under this framework, the affiliate is the one responsible for his own investment strategy, and so decides to invest a part of his initial wealth in the participating fund and the remaining in the risky fund.

To measure the preferences in terms of risk and return, we maximize the expected utility of the terminal wealth over the decision variable which is the investment strategy chosen by the affiliate. More precisely, it is the proportion π of the premium to invest in the participating life insurance contract. Hence, for each of the above fair relations we introduce the following optimization problem:

$$\max_{\pi \in [0,1]} \quad \mathbb{E}\,[\,U(V_T)\,], \tag{2.5.1}$$

where U is the utility function of the affiliate assuming that $U' > 0$ and $U'' < 0$.

We mention that the previous maximization problem is solved subject to the following constraint:

$$\mathbb{E}^{\mathbb{Q}}\,[e^{-rT} \cdot V_T] = V_0 = \; P,$$

which means that the fair pricing relation is satisfied.

2.5.1 Power utility

We assume that the preferences of the customer are modeled by a constant relative risk aversion (CRRA) utility function with a risk aversion coefficient γ. The latter one is used to measure the level of risk of the affiliate and his behavior towards the uncertainty of the amount of money received at maturity. The utility function is described as a power utility function of the form

$$U(z) = \frac{z^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0, \gamma \neq 1.$$

We note that the higher the customer's unwillingness to take a risk, the greater the value of γ. Based on Davies [17], the best suggestion for γ may typically be between 3 and 5.

In order to solve numerically problem (2.5.1) for each type of contract, we refer to the same contract we introduced in Section 2.4.3 for which $T = 20$ years, $r = 1.5\%$, $\sigma_G = 3\%$, $\beta = 70\%$ and $\varepsilon = 0.25\%$. In addition, we suppose the following values for the asset parameters: $\mu_G = 3\%$, $\mu_H = 7\%$ and $\sigma_H = 15\%$, whereas the correlation is $\rho = 0.1$ and the initial value is $V_0 = 1000$.

The numerical results are based on Monte Carlo simulations. We refer in our computation to the certainty equivalent concept defined as $U(CEQ) = \mathbb{E}[U(V_T)]$. The investor is indifferent between receiving a safe amount CEQ at time T and the random payout V_T from the insurance contract. The certainty equivalent allows us to easily compare risky payoffs. Under this framework, we compare the CEQ to the different types of contracts according to various parameters as indicated below.

Optimality with respect to the risk aversion

With Rebalancing	$\gamma = 3$			$\gamma = 4$		
	Optimal π (%)	Guarantee i (%)	CEQ	Optimal π (%)	Guarantee i (%)	CEQ
Annual Guarantee	30	1.42	2160	48	0.80	1987
$\frac{20}{4}$-years Guarantee	28	2.07	2134	47	1.62	1957

Without Rebalancing	$\gamma = 3$			$\gamma = 4$		
	Optimal π (%)	Guarantee i (%)	CEQ	Optimal π (%)	Guarantee i (%)	CEQ
Annual Guarantee	27	1.49	2147	46	0.82	1968
$\frac{20}{4}$-years Guarantee	25	2.14	2126	46	1.62	1944
Maturity Guarantee	26	2.17	2115	48	1.75	1938

Table 2.8: CEQ and optimal investment in the different types of contracts with respect to the risk aversion γ.

Table 2.8 displays the optimal asset allocation given a fair guaranteed rate and based on the highest value of CEQ for the different types of contracts. The results show that the higher CEQ is obtained the lower the guarantee.

We note that the optimal asset allocation π obtained in Table 2.8, for instance, $\pi = 0.3$, confirms that the mixed product is more attractive than the pure participating ($\pi = 1$) and unit-linked ($\pi = 0$) products. In addition, we notice that the risk aversion has a strong effect on the asset allocation. If the customer is risk-averse, the larger share of the premium is invested in the participating fund and, equivalently, he is less guaranteed. Comparing both cases, with and without rebalancing, better results are obtained by rebalancing the portfolio yearly or periodically because we take the gains from high average return investments and reinvest them in assets with low average returns.

However, if we look at the table vertically, we remark that switching from one product to the other does not affect significantly the investment strategy of the investor but we can deduce that by increasing the number of periods n we get higher certainty equivalents. For instance, the table shows that the annual guarantee ($n = T$) performs better than the maturity guarantee ($n = 1$) given a fair guaranteed rate and an optimal asset allocation. The results obtained are confirmed by the fact that the guarantee and the surplus return are computed inside the contract on a shorter basis for a higher value of n.

We want to also connect our results to the literature on participating and unit-linked insurance contracts and optimal asset allocation, as, for example, Døskeland & Nordahl [19] and Chen et al. [15]. In Døskeland & Nordahl [19], the riskiness of the underlying risky funds can be controlled by the investment strategy of the insurance company. More specifically, it is possible to invest or borrow at a risk-free rate r; a constant share $\theta_G > 0$ of funds is invested in some risky asset. In our previous setting, this means that the fund G is replaced by the fund $\tilde{G}$:

$$\begin{aligned}\frac{\mathrm{d}\tilde{G}_t}{\tilde{G}_t} &= \theta_G \cdot \frac{\mathrm{d}G_t}{G_t} + (1-\theta_G)\cdot r\mathrm{d}t\\ &= (r+\theta_G(\mu_G - r))\mathrm{d}t + \theta_G \cdot \sigma_G \mathrm{d}W_t^G, \quad \tilde{G}_0 = 1,\ t \in [0,T].\end{aligned}$$

The parameter θ_G is a free parameter that may help to further increase the contract's expected utility; if $\theta_G = 1$ we are back to our setting. The same can also be done for the unit-linked fund H that is replaced by $\tilde{H}$:

$$\mathrm{d}\tilde{H}_t = (r+\theta_H(\mu_H - r))\,\tilde{H}_t\,\mathrm{d}t \;+\; \theta_H\sigma_H\,\rho\,\tilde{H}_t\,\mathrm{d}W_t^G \;+\; \theta_H\sigma_H\sqrt{1-\rho^2}\,\tilde{H}_t\,\mathrm{d}W_t, \quad \tilde{H}_0 = 1,\ t\in[0,T], \tag{2.5.2}$$

where $\theta_H > 0$ is the share invested in the risky fund. Following Døskeland & Nordahl [19], we choose θ_G^* such that the utility of a participating contract (i.e. the case $\pi = 1$ in our setting) is maximized. Secondly, we choose θ_H^* such that the utility of a unit-linked contract with a guarantee fee of $\varepsilon = 0$ is maximized (i.e. the case $\pi = 0$ in our setting). Given the two parameters θ_G^* and θ_H^*, we can now compute the utility of a mixed product for any share $\pi \in [0,1]$. As limiting cases $\pi = 0$ and $\pi = 1$, we obtain the same optimal contracts as considered in Døskeland & Nordahl [19]. As expected, the optimal participating contract leads to a higher certainty equivalent than the optimal unit-linked investment. However, we also see that a mixed product may lead to a higher certainty equivalent than these two extremes. Figure 2.6 gives the certainty equivalent as a function of the share π using the same parameter set as in Table 2.8 with a risk aversion $\gamma = 3$. We note that the large values of certainty equivalents observed in Figure 2.6 are due to the optimal choice of asset allocations θ_G^* and θ_H^*. The optimal mixing share is $\pi = 0.62$, confirming our previous results that mixed participating products can be more attractive than participating and unit-linked products, respectively.

Optimality with respect to the volatility of the risky fund

On the other side, we present the sensitivity of the parameters of the risky fund on the certainty equivalents. Table 2.9 shows that, for the same risk appetite, the optimal asset allocation is decreasing with the choice of a fund that is less volatile, which means that the part invested in the risky fund is higher when its volatility decreases. In addition, as shown above, the certainty equivalents increase for a greater investment in the unit-linked contract.

Consequently, it is worth noting that the risk aversion and the risky asset parameters are the most important parameters in measuring the preferences of the affiliate in terms of risk and return since they identify the profile of the customers.

Optimality with respect to the parameter of correlation

In addition to the sensitivity of the risk aversion and the volatility of the risky asset we check the impact of the correlation parameter ρ on the optimal investment by fixing the risk aversion γ to 4 and considering an annual guarantee contract. The first plot in Figure 2.7 shows that the correlation is not a significant measure of the preference of the customer; however, it affects the CEQ as shown in the second plot of Figure 2.7. The CEQ is decreasing with the increase of ρ and we mention that in the case of perfectly negative correlation the portfolio is diversified which yields a maximum risk reduction and as a result a higher certainty equivalent.

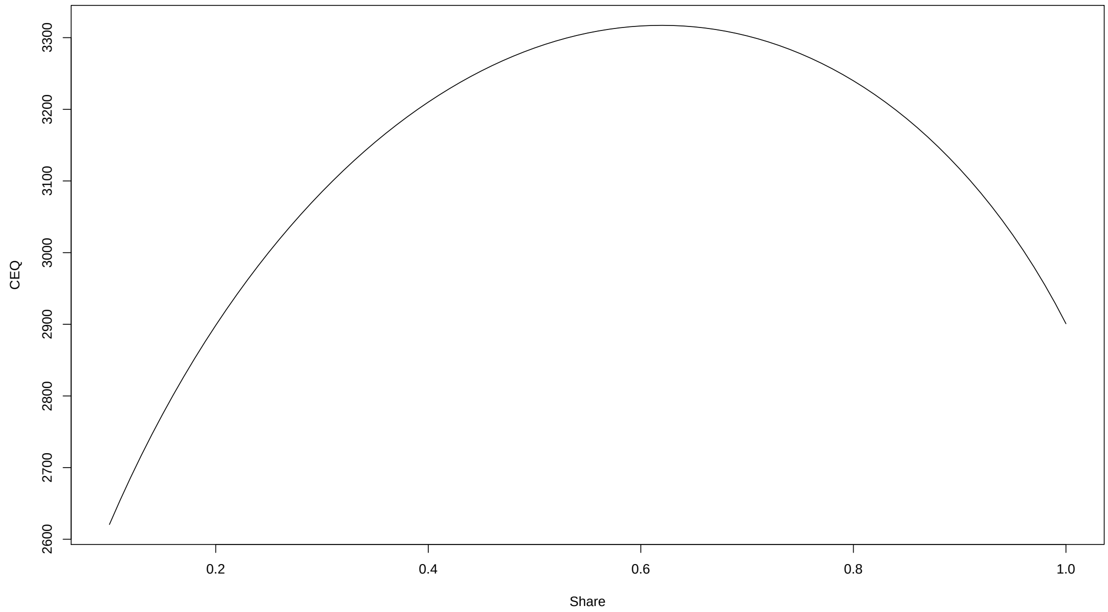

Figure 2.6: CEQ as a function of the share π - Maturity guarantee.

With Rebalancing, $\gamma = 4$	$\sigma_H = 15\%$		$\sigma_H = 20\%$		$\sigma_H = 25\%$	
	Optimal π (%)	CEQ	Optimal π (%)	CEQ	Optimal π (%)	CEQ
Annual Guarantee	48	1985	71	1784	82	1697
$\frac{20}{4}$-years Guarantee	47	1958	71	1749	83	1662

Without Rebalancing, $\gamma = 4$	$\sigma_H = 15\%$		$\sigma_H = 20\%$		$\sigma_H = 25\%$	
	Optimal π (%)	CEQ	Optimal π (%)	CEQ	Optimal π (%)	CEQ
Annual Guarantee	46	1970	72	1755	84	1665
$\frac{20}{4}$-years Guarantee	46	1943	72	1729	84	1640
Maturity Guarantee	48	1938	73	1730	85	1644

Table 2.9: CEQ and optimal investment in the different types of contracts with respect to the volatility of the risky fund σ_H.

Optimality with respect to the guarantee fee

In this section, we look at the effect of the guarantee fee ε on the optimal asset allocation decision and the corresponding CEQ for the customer. Opposite to the effect of the risk aversion parameter, Table 2.10 shows that the guarantee fee does hardly affect the optimal investment strategy of the affiliate. In terms of the CEQ of the optimal contract, we analyse different parameter sets and find situations where the introduction of the guarantee fee can increase the CEQ but also situations where the case $\varepsilon = 0\%$ leads to the highest CEQ (in Table 2.10, for example, the CEQ is decreasing in ε). Overall,

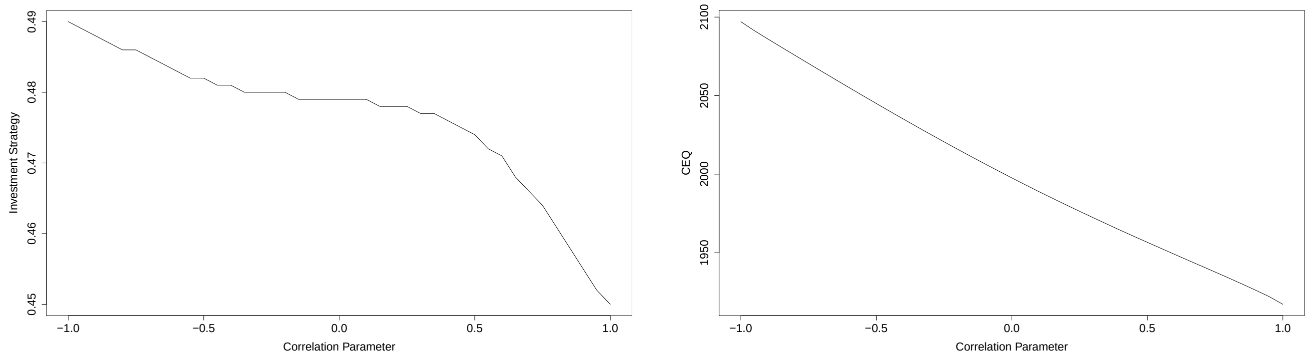

Figure 2.7: Sensitivity of the correlation parameter on the optimal investment in an annual guarantee contract.

we find three possible effects of ε on the CEQ of the optimal contract.

With Rebalancing, $\gamma = 4$

	$\varepsilon = 0\%$		$\varepsilon = 0.25\%$		$\varepsilon = 0.5\%$	
	Optimal π (%)	CEQ	Optimal π (%)	CEQ	Optimal π (%)	CEQ
Annual Guarantee	47	2006	48	1987	48	1969
$\frac{20}{4}$-years Guarantee	46	1987	47	1957	47	1932

Without Rebalancing, $\gamma = 4$

	$\varepsilon = 0\%$		$\varepsilon = 0.25\%$		$\varepsilon = 0.5\%$	
	Optimal π (%)	CEQ	Optimal π (%)	CEQ	Optimal π (%)	CEQ
Annual Guarantee	47	1991	46	1968	45	1951
$\frac{20}{4}$-years Guarantee	46	1975	46	1944	45	1918
Maturity Guarantee	46	1981	48	1938	47	1904

Table 2.10: Optimal asset allocation parameter π and optimal certainty equivalents (CEQ) for different contracts with respect to the guarantee fee.

To demonstrate this, let us consider an annual guarantee contract with three different scenarios defined by a set of different parameters for the less risky asset G and the risky asset F illustrated in Table 2.11. All the other parameters are set as in the previous examples.

	Scenario 1	Scenario 2	Scenario 3
μ_G	2	2	2
σ_G	6	6	6
μ_H	6	6	6
σ_H	14	20	15.5

Table 2.11: Different sets of parameter values (expressed in %) for the less risky and the risky assets.

In Figure 2.8, we display graphically the impact of the guarantee fee ε on the CEQ for the three parameter choices from Table 2.11. For $\varepsilon \in [0, 0.01]$, we observe a decreasing, an increasing (first two figures, scenarios 1,2) or a hump-shaped effect (third figure, scenario 3) on the CEQ. The increase of CEQ with ε can be explained by the fact that the constant fee ε due in both good and bad years is used to reduce the risk in the part invested in the participating contract by increasing the investment guarantee.

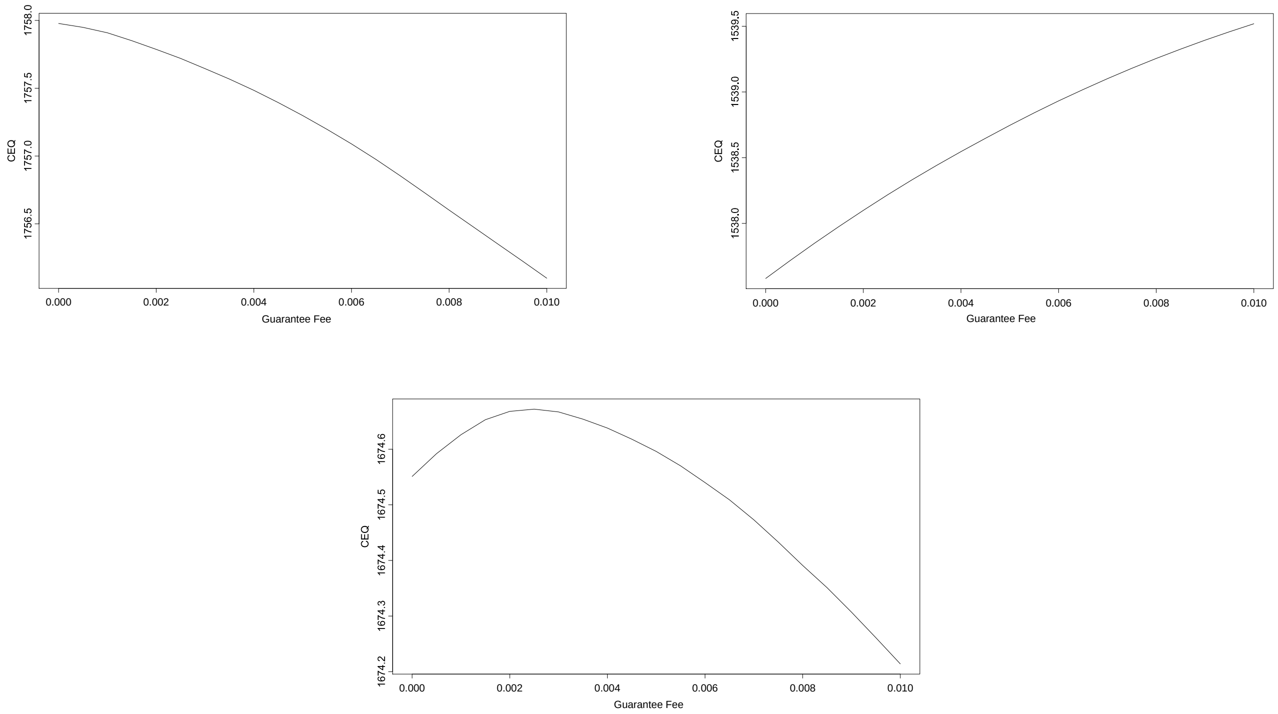

Figure 2.8: Different effects of the guarantee fee on the CEQ for the annual guarantee contract.

In conclusion, we point out that the main parameters for the optimal investment framework are the risk aversion that represents the individual risk and the risky asset parameters. Therefore, the product answers different profiles of customers from small to large risk aversion and allows for different unit-linked funds based on each profile. The mixed product turns out to be more attractive than a single investment in a participating or a unit-linked contract. Further, we observe that the introduction of a guarantee fee can sometimes (but not always) help to further improve the optimal product's CEQ for the customer. In the next section, we confirm these results using prospect theory as preference functions. Prospect theory takes into account risk preferences of customers who are more sensitive to losses but are (at the same time) risk-seeking with respect to gains. Such a behavior is observed in many empirical or experimental studies (see, e.g. Tversky & Kahnemann [35]).

2.5.2 Cumulative prospect theory

In this part, we describe an alternative behavioral model proposed by Tversky & Kahnemann [35] called the cumulative prospect theory (CPT) that describes the human behaviors under risk and uncertainty. Generally, people tend to value gains and losses differently. Experimental studies confirm that they tend to be risk-averse with respect to losses (e.g. in an insurance product) but risk-seeking with respect to gains (e.g. in a lottery) (see also Tversky & Kahnemann [35]). In prospect theory, this is accounted for by a reference point Γ that separates the portfolio values in a gain part ($V_T - \Gamma$ is positive) and a loss part ($V_T - \Gamma$ is negative). The different view towards gains and losses is accounted for by an

S-shaped preference function:

$$U(z) = \begin{cases} (z-\Gamma)^{\phi} & z \geq \Gamma, \\ -\lambda \cdot (\Gamma - z)^{\phi} & z < \Gamma, \end{cases}$$

where ϕ is the sensitivity parameter that measures gains or losses. z refers to the final retrospective reserve where some values fall above the reference point and the others fall below. λ represents the sensitivity to losses over gains and takes into account that losses "hurt more". Prospect theory helps to explain the high demand for guarantees in life and pension insurance – an observation that cannot be explained by a power utility preference function. The second aspect of human behavior according to prospect theory is a probability distortion: The likelihood of extreme events (e.g. a stock market crash) is overweighted while the likelihood of "normal" events is underweighted. This effect is modeled by a weighting function w: $[0,1] \longrightarrow [0,1]$ that is strictly increasing with $w(0) = 0$ and $w(1) = 1$. It holds that $w(p) > p$ for small values of p and $w(p) < p$ for high values of p. We rely on Prelec [32] to define the weighting function as $w(p) = e^{-(-\log p)^{\varphi}}$ where φ is a free parameter that controls the curvature of the function. Given the weighting function w and the function U, the prospect utility function is defined as[2]

$$CPT(V_T) := \int_{-\infty}^{0} U(x)\,\mathrm{d}(w(F(x))) + \int_{0}^{\infty} U(x)\,\mathrm{d}(-w(1-F(x)))$$

with distribution function of terminal wealth $F(x) = P(V_T \leq x)$. This is a natural generalization (cf. Hens & Rieger [25], Ruβ & Schelling [33]) of the discrete case introduced by Tversky & Kahnemann [35]. Without probability weighting (i.e. for $\varphi = 1$ and $w(x) = x$), the prospect utility function reduces to

$$CPT(V_T) = \int_{-\infty}^{\infty} U(x)\,\mathrm{d}F(x) = E[U(V_T)]\,.$$

To solve the problem numerically, we refer to the concept of certainty equivalent defined previously to a better comparison. We fix the same parameter values as in the power utility part, however, there are several studies that estimate the parameters of CPT and based on these studies and on the long time horizon $T = 20$ chosen in this chapter, we assign the following values to our parameters: $\Gamma = (1+r)^T \cdot V_0$ as a reference point, $\lambda = 2.25$ as estimated by Tversky & Kahnemann [35] and $\varphi = 0.78$ inspired by several studies that use Prelec [32] to estimate the probability weighting function. Under this framework, we compare the CEQ of the different types of contracts with respect to the sensitivity parameter ϕ.

Table 2.12 shows that, consistently with the results of the power utility, the share in the participating fund π is almost the same for different types of contracts for a given sensitivity parameter ϕ while the CEQ is increasing in the number of periods n. On the other hand, the increase of the sensitivity parameter ϕ, which indicates an increased sensitivity to gains than to losses, generates a higher CEQ as well as a greater investment in the unit-linked contract.

[2]To estimate the utility by Monte-Carlo simulation with N simulation runs, denote by $V_T^{(1)} \leq V_T^{(2)} \leq \ldots \leq V_T^{(N)}$ the ordered realizations of terminal wealth. If there is no distortion, expected utility is simply obtained as $\mathbb{E}[U(V_T)] = \frac{1}{N}\sum_{j=1}^{N} U(V_T^{(j)})$. If extreme scenarios are perceived more likely, the expected utility is estimated as

$$\mathbb{E}[CPT(V_T)] = w(p) \cdot U(V_T^{(1)}) + \sum_{j=1}^{N-1} \Big[w((j+1)\cdot p) - w(j \cdot p) \Big] \cdot U(V_T^{(j+1)}),$$

where $p = \frac{1}{N}$ is the true probability of each Monte-Carlo realization.

With Rebalancing	$\phi = 0.15$			$\phi = 0.18$		
	Optimal π (%)	Guarantee i (%)	CEQ	Optimal π (%)	Guarantee i (%)	CEQ
Annual Guarantee	85	0.24	1869	79	0.30	2259
$\frac{20}{4}$-years Guarantee	85	1.26	1814	82	1.28	2191

Without Rebalancing	$\phi = 0.15$			$\phi = 0.18$		
	Optimal π (%)	Guarantee i (%)	CEQ	Optimal π (%)	Guarantee i (%)	CEQ
Annual Guarantee	83	0.25	1896	80	0.29	2328
$\frac{20}{4}$-years Guarantee	86	1.25	1833	80	1.29	2227
Maturity Guarantee	82	1.50	1797	76	1.53	2200

Table 2.12: CEQ and optimal investment in the different types of contracts with respect to the sensitivity parameter ϕ.

Sensitivity analysis with respect to the sensitivity parameter and the guarantee fee

In the following, we analyze the effect of the guarantee fee ε for different values of the sensitivity parameter ϕ by considering an annual guarantee contract. We notice that the CEQ is decreasing with the guarantee fee while the asset allocation π is hardly affected (Table 2.13). This result depends on the careful choice of CPT parameters.

	$\phi = 0.12$			$\phi = 0.15$			$\phi = 0.18$		
ε	Optimal π (%)	Guarantee i (%)	CEQ	Optimal π (%)	Guarantee i (%)	CEQ	Optimal π (%)	Guarantee i (%)	CEQ
0.1%	87	0.22	1556	84	0.25	1877	76	0.34	2267
0.25%	86	0.23	1554	85	0.24	1869	79	0.30	2259
0.8%	89	0.20	1543	87	0.22	1851	83	0.26	2238

Table 2.13: CEQ and optimal investment with respect to the sensitivity parameter and the guarantee fee.

Under the power utility and CPT framework, we get higher CEQ by considering a mixed product under the annual guarantee contract. However, with the parameters we choose in this chapter, the investor has a preference to invest more in the participating contract under a CPT framework, while he prefers to invest equally or more in the unit-linked contract under a power utility framework.

2.6 Conclusion

In this chapter, we have designed a new life insurance product that combines features of participating and unit-linked life insurance contracts and is adapted to the needs in a low interest rate environment. The product is not just a superposition of participating and unit-linked contracts but includes a continuous guarantee fee on the unit-linked part that permits a transfer from the unit-linked part to the participating part.

Under this framework, we have considered different forms of contracts, varying the guarantee period (annual guarantee, maturity guarantee or periodical guarantee) and the type of investment (rebalancing or non-rebalancing). In each contract, a minimum return is guaranteed to the customer along with a participation in the profits of the insurer. We have developed fair parameter combinations under a given valuation measure, demonstrating that our product allows for fair guaranteed rates in the participating part that may even exceed the risk-free rate. This design leads to attractive guaranteed rates also in a low interest rate environment. The guaranteed rate is increasing with a greater investment in the unit-linked part, a higher value of the guarantee fee as well as a lower participation in the profits of the insurer. Furthermore, the model shows that the guaranteed rate can be boosted by considering a long-term contract or by turning it into a maturity guarantee contract. Eventually, the interference of the guarantee fee in the model improved the guarantee on the participating part.

Moreover, we have analyzed the set of fair products from the customer's viewpoint working with preference functions. We have discussed the optimal share in the different parts as well as the effect of the guarantee fee on the attractiveness of the mixed product. We use two behavioral models, the utility approach and the cumulative prospect theory. Our numerical results show that the mixed product is in almost all cases more attractive than a simple participating or unit-linked contract. In some cases, the introduction of the guarantee fee can further improve the attractiveness of the product. The numerical results further show that the annual guarantee contract typically leads to the highest CEQ and that the main parameters that affect the optimal investment strategy of the affiliate are the risk aversion and the risky asset parameters. In addition, the product proves that the optimal level of fees strongly depends on the risk/return profiles of the unit-linked and participating funds involved.

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Chapter 3

Deterministic lifestyle investment strategy in mixed life insurance contracts[1]

3.1 Introduction

In the realm of real-world finance, the lifestyle investment strategy offers a dynamic approach to pension and life insurance investments, tailoring long-term strategies based on individuals' concepts of lifestyle. This strategy aligns investment portfolios with policyholders' ages and risk tolerance, enabling them to benefit from equities' growth potential when young while transitioning to more stable assets as retirement or contract maturity approaches. Historically, equities have exhibited higher returns and risk. As policyholders age, the strategy shifts allocation towards bonds and cash, which are less volatile and more conservative assets. This empowers policyholders to adapt investment strategies through life stages and changing risk appetites. Typically, these investment strategies are deterministic in nature, with a predefined allocation stated at the inception of the contract. Policyholders agree upon a specific investment allocation for the entire duration of the contract. For instance, an investment plan might involve constant risky asset investments in the first half, followed by a linear decrease.

There is an important amount of research on the lifestyle subject that determines the optimal allocation in different types of contracts. For instance, Zhaoxun [14] introduces a new pension contract that has a transparent structure and clear rule for bonus distribution that smoothes the investment return and shows that its outcome is consistent with traditional pension investment advice, which is the lifestyle strategy. On the other hand, Blake et al. [4] argue that the optimal strategy is age-dependent and stochastic lifestyling, as opposed to the more traditional deterministic lifestyling. In this regard, Antolín et al. [1] point out that dynamic investment strategies can offer slightly higher replacement rates for a given degree of risk than more deterministic strategies. Nevertheless, they indicate that there is no standard approach to investing. In every scenario, none of the investment strategies is the clear winner. Additionally, they show that the deterministic life cycle strategy that maintains a steady exposure to risky assets throughout the majority of the accumulation phase while fast-moving to bonds in the final ten years before retirement seems to generate appropriate returns. Moreover, several studies highlight that lifestyle investment strategies are not always beneficial (see, e.g., Arts & Vigna [2], Booth & Yacoubov [5] and Cairns et al. [6]).

In the second chapter, we examine a mixed contract incorporating a risky asset (the pure unit-linked product—Branch 23) and a less risky asset (the participating life insurance with a guaranteed rate—Branch 21), and we look at the constant proportion invested in Branch 21 that maximizes the expected utility of the policyholder. So, this product has been designed and priced by considering a constant allocation between the two parts of the contract and studying the viewpoints of both the insurer and the policyholder. The purpose of this chapter lies in proposing a predefined deterministic

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investment strategy that determines the fraction of the premium invested in the participating life insurance contract at each time. Under this framework, we design, price, and optimize this innovative contract with an annual guarantee, the most popular type on the market, as highlighted by Hieber et al. [9]. Following the second chapter, we focus solely on the financial aspect and adopt the classical Black & Scholes model to achieve a fair valuation approach that ensures that the mixed contract is justly priced, satisfying both policyholders and investors, and adhering to principles of financial arbitrage-free pricing. More precisely, the objective of this chapter is to evaluate whether the balancing between Branch 21 and Branch 23 aligns with the lifestyle investment strategy, using the Expected Utility Theory developed by von Neumann & Morgenstern [15] to explain people's behavior under uncertainty, as it is frequently applied in designing insurance contracts when customers make ambiguous decisions. Hence, this study delves into pricing and the optimal investment strategy based on a predetermined deterministic investment style. The final goal is to fairly price the guaranteed rate within the lifestyle strategy and maximize the expected utility of the policyholder at the contract's maturity while ensuring the fair pricing relation of the contract is satisfied.

In this study, we explore various families of investment strategies under the lifestyle strategy framework. These include linear, step-wise, and piece-wise linear strategies, and we find that an optimal strategy is a constant allocation for a policyholder who pays a single premium and maintains the same attitude over the duration of the contract, which typically corresponds to a classical utility function. This outcome is consistent with Merton's optimal stock allocation fraction (Merton [11]). However, incorporating periodic premiums can interact with investment strategies on one side, and on the other side, several studies have discovered that a person's lifetime risk preferences might vary. They proved that by incorporating age-dependent risk preferences can greatly enhance strategies for life-cycle pension funds and pension insurance, optimizing investment behavior and aligning with evolving risk attitudes over time (see, for instance, Tang et al. [13], Lichtenstern et al. [10] and Steffensen [12]). Campbell & Viceira [7] were the first to raise concern about the uncertainty regarding the risk aversion coefficient. They highlighted that putting investing advice on a scientific basis has been one of the main goals of modern financial economics, and it has been accomplished for investors with short time horizons and constant investment strategies but not for those with long time horizons and facing time-varying expected returns on risky assets. Accordingly, we extend the analysis in two directions: the first is to consider periodic premium payments, and the second is to consider policyholders' risk preferences in a more dynamic manner by introducing time-varying risk aversion, and we explore how this coefficient changes over time to capture policyholders' evolving risk preferences as they age. This approach allows us to design optimal investment strategies that align with policyholders' changing risk attitudes and reflect lifestyle investment strategies.

The chapter is organized as follows. In Section 3.2, we present the financial assumptions and the structure of the policy, in addition to the families of investment strategies to consider in this study. On the other hand, we state the fairness relation for each investment strategy with some numerical illustrations. In Section 3.3, we look at the optimal allocation for each investment strategy by considering first a classical utility function and then extending to consider periodic premiums on one side and a time-varying risk aversion on the other. Numerical results are illustrated as well. Section 3.4 concludes.

3.2 Mixed contract and investment strategy

3.2.1 Structure and financial assumptions

Our analysis focuses purely on the financial aspects of these contracts, disregarding mortality risk and concentrating solely on the uncertainties associated with the evolution of financial markets.

We recall the financial assumptions from Hanna et al. [8], where they assume a Black & Scholes model with three assets in the market that take over the time horizon $[0, T]$. The risk-free asset B earning a constant interest rate $r \in \mathbb{R}$, and two risky funds G and H. The fund G is linked to the participating policy and has generally a low volatility, while the fund H is linked to the unit-linked policy with a high volatility.

The risk-free asset follows a deterministic dynamics:

$$\mathrm{d}B_t = r\,B_t\,\mathrm{d}t, \quad B_0 = 1.$$

The dynamics of fund G follow a geometric Brownian motion:

$$\mathrm{d}G_t = \mu_G\,G_t\,\mathrm{d}t + \sigma_G\,G_t\,\mathrm{d}W_t^G, \quad G_0 = 1,\ t \in [0,T], \tag{3.2.1}$$

where W^G is a standard Brownian motion under the real probability measure $\mathbb{P}$ and $\sigma_G > 0$. The annual returns of G denoted by g_t are defined as follows:

$$1 + g_t = \frac{G_t}{G_{t-1}} = \exp\left(\left(\mu_G - \frac{\sigma_G^2}{2}\right) + \sigma_G\left(W_t^G - W_{t-1}^G\right)\right), \quad t = 1,2,\ldots,T.$$

and are independent and log-normally distributed for all t.

The dynamics of fund H also follow a geometric Brownian motion:

$$\mathrm{d}H_t = \mu_H\,H_t\,\mathrm{d}t + \sigma_H\,\rho\,H_t\,\mathrm{d}W_t^G + \sigma_H\sqrt{1-\rho^2}\,H_t\,\mathrm{d}W_t, \quad H_0 = 1,\ t \in [0,T], \tag{3.2.2}$$

where W is a standard Brownian motion under the real probability measure $\mathbb{P}$, independent of W^G and $\sigma_H > 0$. We define $\rho \in]-1,1[$ as the correlation parameter between the log-returns of the two funds G and H such that $\mathrm{d}W_t^H = \rho \cdot \mathrm{d}W_t^G + \sqrt{1-\rho^2} \cdot \mathrm{d}W_t$. The annual returns of H denoted by h_t are defined as follows:

$$\begin{aligned}1 + h_t &= \frac{H_t}{H_{t-1}}\\ &= \exp\left(\left(\mu_H - \frac{\sigma_H^2}{2}\right) + \sigma_H \cdot \rho \cdot \left(W_t^G - W_{t-1}^G\right) + \sigma_H \cdot \sqrt{1-\rho^2} \cdot \left(W_t - W_{t-1}\right)\right), \ t = 1,2,\ldots,T.\end{aligned}$$

and are also independent and log-normally distributed.

The financial market is assumed to be complete and arbitrage-free. So, the pricing is done using the risk-neutral measure $\mathbb{Q}$, obtained through Girsanov's theorem.

In the framework of this chapter, we design the mixed life insurance contract initially proposed by Hanna et al. [8]. These contracts combine features of participating and pure unit-linked life insurance contracts, aiming to strike a balance between the security provided by the participating component and the absence of guarantees in the pure unit-linked component. To ensure the fulfillment of guaranteed rates within the participating contract, a constant fee is deducted from the pure unit-linked account value to increase the guaranteed return in the participating fund.

We assume that the contract pays a lump sum at the policy's maturity $T > 0$. The allocation of the single premium P, paid at the inception of the contract, between the two mentioned contracts follows a predefined strategy, where π_t is the proportion of the premium defined at time $t \in [0, T-1]$ and invested in the participating life insurance contract, and the remaining fraction of the premium $(1-\pi_t)$ is allocated to the pure unit-linked contract. We note that the allocation of the premium includes various predefined strategies, which could be constant over the contract's maturity, deterministic allocations, as well as dynamic or stochastic strategies. In this chapter, the allocation is no longer constant as in the second chapter, but we move to a deterministic strategy.

The investment strategy π_t invested in the participating life insurance contract is directed towards the insurer's general fund G, also known as the participating fund. This fund typically carries a lower level of risk compared to other investment options. One of the key features of this contract is the guarantee of asset protection by providing an annual minimum guaranteed interest rate i. Moreover, the participating fund entitles the policyholder to a surplus return in addition to the guaranteed benefit. The annual surplus return is derived from the financial profits generated by the investments made by the insurance company as follows:

$$\delta_t = \max(\beta \cdot g_t - i,\, 0),$$

where $\beta \in [0,1]$ is called the participation level and it is assumed to be constant over time. The total return on the participating fund comprises two components: the guaranteed interest rate and the surplus return.

The remaining strategy (1 - π_t) is allocated to the pure unit-linked contract. Unlike the participating life insurance contract, the pure unit-linked contract does not provide any guarantees regarding investment returns or asset protection. Instead, the policyholder's contributions are directed towards the investment fund H, commonly referred to as the risky fund. The risky fund presents higher investment risks compared to the participating fund. The returns generated by this fund are subject to market fluctuations and can vary significantly over time. Policyholders who choose the pure unit-linked contract bear the risk associated with the investment performance of the underlying assets. In addition, we introduce a continuous guarantee fee ε that is deducted from the investment fund H to enhance the guarantee of the participating part in the mixed product. The investment in the unit-linked part is represented by $F_t = e^{-\varepsilon t} H_t$, and its dynamics are given by:

$$\mathrm{d}F_t = (\mu_H - \varepsilon)\, F_t\, \mathrm{d}t \;+\; \sigma_H\, \rho\, F_t\, \mathrm{d}W_t^G \;+\; \sigma_H \sqrt{1-\rho^2}\, F_t\, \mathrm{d}W_t, \quad F_0 = 1,\; t \in [0,T]. \tag{3.2.3}$$

The guarantee fee ε is used to boost the guarantee inside the contract.

3.2.2 Investment strategy

In this chapter, we consider different families of investment strategies under the presented framework known as lifestyle strategies, including linear forms, step-wise, and piece-wise linear strategies, in order to check their impact on the guaranteed rate i and to determine, based on a predefined allocation form, the optimal investment strategy that maximizes the policyholder's expected utility:

- Linear strategy: it establishes a linear relationship between the initial share at the inception of the contract, π_0, and the final share, π_{T-1}, at the final allocation time, both invested in the participating contract, and it is expected to be a linear form as the time approaches maturity, such that

$$\pi_t \;=\; \frac{\pi_{T-1} - \pi_0}{T-1} \cdot t \;+\; \pi_0, \quad t \text{ is discrete, } t = 0, 1, \ldots, T-1. \tag{3.2.4}$$

- Step-wise linear strategy: The investment strategy varies according to a step function where the analysis is examined on two and three paths as follows

$$\textit{Two-step function:} \quad \pi_t \;=\; \begin{cases} \pi_0 & \text{for } t \in [0, \frac{T}{n} - 1], \quad n > 1, \\ \pi_{\frac{T}{n}} & \text{for } t \in [\frac{T}{n}, T-1], \end{cases} \tag{3.2.5}$$

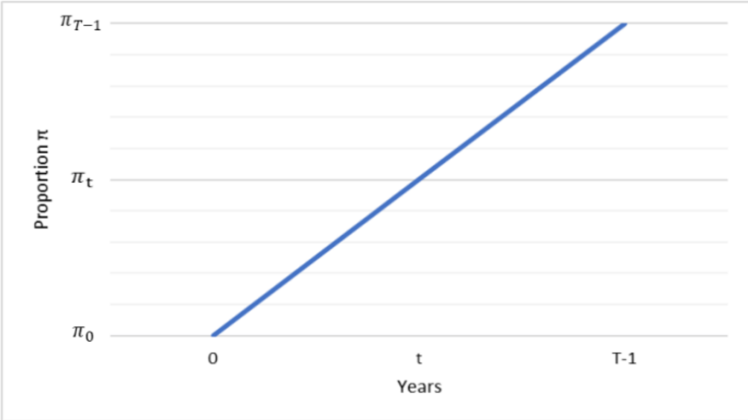

Figure 3.1: Shape of a linear investment strategy.

or

$$\textit{Three-step function:}\quad \pi_t = \begin{cases} \pi_0 & \text{for } t \in [0, \frac{T}{n} - 1], \quad n > 1, \\ \pi_{\frac{T}{n}} & \text{for } t \in [\frac{T}{n}, \frac{k \cdot T}{l} - 1], \quad 1 \leq k < l, \\ \pi_{\frac{k \cdot T}{l}} & \text{for } t \in [\frac{k \cdot T}{l}, T - 1], \quad \frac{k \cdot T}{l} + 1 \leq T - 1, \end{cases} \tag{3.2.6}$$

where t is a discrete natural number, and n, k, and l are natural numbers.

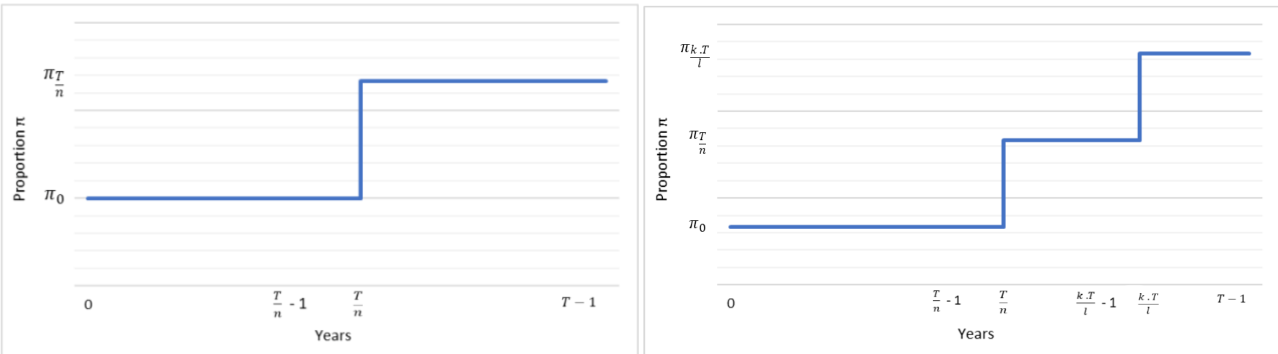

Figure 3.2: Shapes of two- and three-step investment strategies.

- Piece-wise linear strategy: This investment approach follows a segmented pattern, allowing for different allocation strategies over specific periods of time. The analysis considers both two- and three-path scenarios.
  In the two-path strategy, the investment allocation begins with a fixed percentage and remains constant for all time steps within the first segment. Subsequently, there is a transition to a linear increase or decrease in allocation, which is maintained until contract maturity.
  In the three-path strategy, three distinct segments are observed. The initial segment features a constant allocation. This is followed by a period of linear increase or decrease in allocation. Finally, the strategy reverts back to a constant allocation leading up to the contract's maturity.

This strategy is mathematically expressed as follows:

$$\pi_t = \begin{cases} \pi_0 & \text{for } t \in [0, \frac{T}{n} - 1], \quad n > 1, \\ \pi_{\frac{t}{\frac{T}{n}}} = \dfrac{\pi_{\frac{k\cdot T}{l}-1} - \pi_0}{\frac{k\cdot T}{l} - 1 - \frac{T}{n}} \cdot t + \dfrac{(\frac{k\cdot T}{l} - 1)\cdot \pi_0 - \frac{T}{n}\cdot \pi_{\frac{k\cdot T}{l}-1}}{\frac{k\cdot T}{l} - 1 - \frac{T}{n}} & \text{for } t \in [\frac{T}{n}, \frac{k\cdot T}{l} - 1], \quad 1 \leq k < l, \\ \pi_{\frac{k\cdot T}{l}} = \dfrac{(\frac{k\cdot T}{l} - \frac{T}{n})\cdot \pi_{\frac{k\cdot T}{l}-1} - \pi_0}{\frac{k\cdot T}{l} - 1 - \frac{T}{n}} & \text{for } t \in [\dfrac{k\cdot T}{l}, T-1], \\ & \dfrac{k\cdot T}{l} + 1 \leq T - 1, \end{cases} \tag{3.2.7}$$

where t is a discrete natural number, and n, k, and l are natural numbers. By taking $l = k = 1$, we examine the two-path strategy.

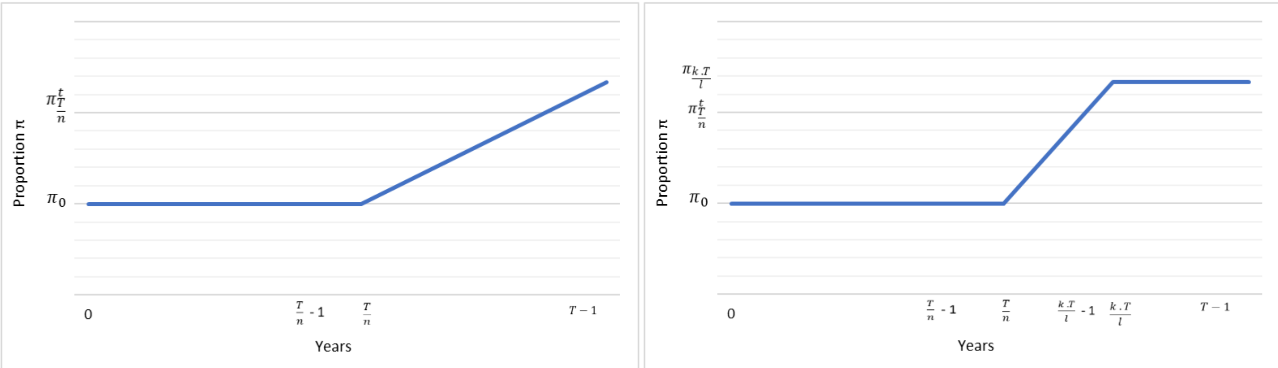

Figure 3.3: Shapes of two- and three-path investment strategies.

Figures 3.1, 3.2, and 3.3 are just examples of such behaviors and do not present the definitive form.

Since the policyholder is meant to invest more in the less risky fund over time, if the lifestyle investment is used in practice, we assume that the investment strategy π_t is a non-decreasing function.

It is worth noting that, in our analysis, the investment strategy π_t for the whole duration of the contract is predetermined at $t = 0$, and accordingly, the shares $\pi_0, \dots, \pi_{T-1}$ are real positive numbers, taking values in $]0, 1]$, chosen at the issuance of the contract.

3.2.3 Fair valuation of the mixed contract

In our investment framework, we adopt a strategy where the investment portfolio is rebalanced each year to maintain the predefined allocation in the mixed contract. Apart from the initial single premium paid at the contract's inception, no further payments occur until the maturity of the contract. As a result, the benefit at maturity for the mixed product, which is an annual guarantee contract with rebalancing, is determined based on the performance of the participating contract and the unit-linked fund, and it is expressed as:

$$V_T = P \cdot \prod_{t=0}^{T-1} \Big[\pi_t \cdot \big((1+i) + (\beta \cdot g_{t+1} - i)^+ \big) + (1 - \pi_t) \cdot (1 + f_{t+1}) \Big]. \tag{3.2.8}$$

It is important to highlight that the parameters of the mixed contract are chosen fairly, ensuring that the contract is fair. The fairness of the contract is achieved by equalizing the discounted risk-neutral expectation of the final benefit, which is the initial fair value of the contract denoted by FV_0, to the initial premium P:

$$FV_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} \cdot V_T\right] = P.$$

Solving this equation leads to the following fairness relation:

$$\prod_{t=0}^{T-1}\left(\pi_t \cdot ((1+i)\cdot e^{-r} + \beta \cdot c_1) + (1-\pi_t)\cdot e^{-\varepsilon}\right) = 1, \tag{3.2.9}$$

where c_1 is the time 0 value of the one-year European call option given by the classical Black & Scholes [3]:

$$c_1 = \Phi(d_1) - \left(1+\frac{i}{\beta}\right)\cdot e^{-r}\cdot \Phi(d_2),$$

$$d_1 = \frac{r + \frac{\sigma_G^2}{2} - \ln\left(1+\frac{i}{\beta}\right)}{\sigma_G},$$

$$d_2 = d_1 - \sigma_G,$$

where Φ is the cumulative distribution function of a standard normal variable.

For a comprehensive understanding of the process used to derive this fairness relation in Equation (3.2.9), detailed steps and explanations can be found in the work of Hanna et al. [8], where the premium equivalence of the annual guarantee contract under the same framework is treated. The only difference is the investment strategy, which is not constant but time-dependent in this study.
Equation (3.2.9) establishes a meaningful connection between various parameters, revealing a feedback mechanism between the guaranteed rate i and the guarantee fee ε, depending on the investment strategy π_t between the two parts of the contract. This relationship provides a valuable tool for policyholders and investors to comprehend how modifying one aspect affects other components, leading to a fair contract structure. In particular, Equation (3.2.9) helps the insurer assess the effect of the lifestyle strategy on boosting the fair guaranteed rate for a given guarantee fee.

In the following, we numerically solve the fairness relation of Equation (3.2.9) with respect to the guaranteed interest rate i. The aim is to explore how the minimum guaranteed rate can be enhanced through the mixed product, as by increasing the guarantee fee ε, we strengthen the guaranteed rate i, and to compare these results to the constant investment strategy case presented in Hanna et al. [8]. To achieve this, we will present various solutions and sensitivity analyses for the fairness relation based on different investment strategies.

We consider a life insurance contract with a maturity T of 20 years. We recall the parameter values of the contract introduced in Hanna et al. [8] for which the interest rate r is equal to 1.5%, and the volatility of the general asset G is $\sigma_G = 3\%$. Moreover, for the participating contract, we fix a participation level β of 70%, and for the unit-linked contract, we impose a guarantee fee ε of 0.25%.

Linear strategy

Under this strategy, the fair condition, using Equations (3.2.9) and (3.2.4), is expressed by the following equation:

$$\prod_{t=0}^{T-1}\left(\left(\frac{\pi_{T-1}-\pi_0}{T-1}\cdot t + \pi_0\right)\cdot((1+i)\cdot e^{-r} + \beta\cdot c_1) + \left(1 - \frac{\pi_{T-1}-\pi_0}{T-1}\cdot t - \pi_0\right)\cdot e^{-\varepsilon}\right) = 1. \tag{3.2.10}$$

In all the subsequent numerical illustrations, we impose that the initial and final shares (π_0 and π_{T-1}) vary between 5%[2] and 100%, equidistantly by 5%. The fair guaranteed rates are presented in Figure 3.4, and some examples are displayed in Table 3.1.

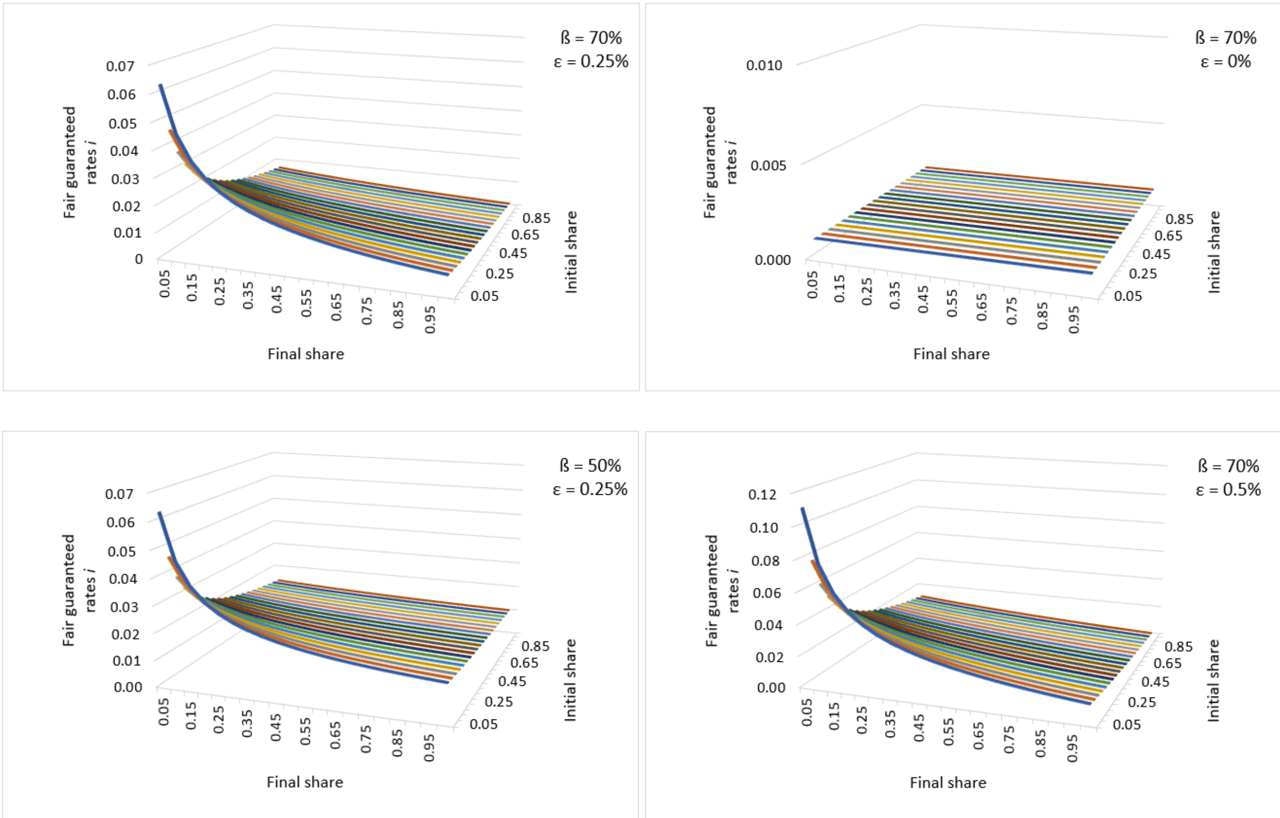

Figure 3.4: Fair guaranteed rates i with respect to a linear investment strategy for different participation levels β and guarantee fees ε.

π_0	π_{T-1}	$\beta = 70\%$ $\varepsilon = 0\%$	$\beta = 70\%$ $\varepsilon = 0.25\%$	$\beta = 50\%$ $\varepsilon = 0.25\%$	$\beta = 70\%$ $\varepsilon = 0.5\%$
0.05	0.05	0.11	6.32	6.33	11.13
0.15	0.65	0.11	1.02	1.62	1.70
0.20	0.70	0.11	0.88	1.52	1.47
0.30	0.55	0.11	0.95	1.56	1.58
0.50	0.05	0.11	1.55	2.00	2.53
0.50	0.50	0.11	0.76	1.43	1.27
0.65	0.65	0.11	0.48	1.25	0.80
0.75	0.40	0.11	0.61	1.33	1.01
0.85	0.50	0.11	0.45	1.23	0.74
1.00	1.00	0.11	0.11	0.10	0.11

Table 3.1: Fair guaranteed rates i (expressed in %) with respect to a linear investment strategy for different participation levels β and guarantee fees ε.

[2]We skip 0% in order to keep some form of guarantee inside the contract through the guaranteed interest rate of the participating contract.

Step-wise linear strategy

The step-wise strategy involves making gradual shifts in the investment portfolio at predefined time points. This strategy allows for more precise adjustments to the asset allocation as the policyholder ages or approaches contract maturity. We will provide the numerical results for both the two-step function and the three-step function.

In this part, we present two examples. The first example is the two-step strategy by taking $n = \frac{4}{3}$ in Equation (3.2.5). In this setting, we assume that the policyholder would like to invest a constant proportion in the participating contract for the first three quarters of the time, then invest the same or another constant proportion for the last quarter of the time. In this regard, the fair condition is expressed as follows:

$$\left(\pi_0 \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1-\pi_0) \cdot e^{-\varepsilon}\right)^{\frac{3}{4}} \cdot \left(\pi_{\frac{3 \cdot T}{4}} \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1-\pi_{\frac{3 \cdot T}{4}}) \cdot e^{-\varepsilon}\right)^{\frac{1}{4}} = 1. \quad (3.2.11)$$

Again, we impose that the shares π_0 and $\pi_{\frac{3 \cdot T}{4}}$ vary between 5% and 100% equidistantly by 5% in all the subsequent numerical figures. The results of the two-step function are illustrated in Figure 3.5, and some examples are displayed in Table 3.2.

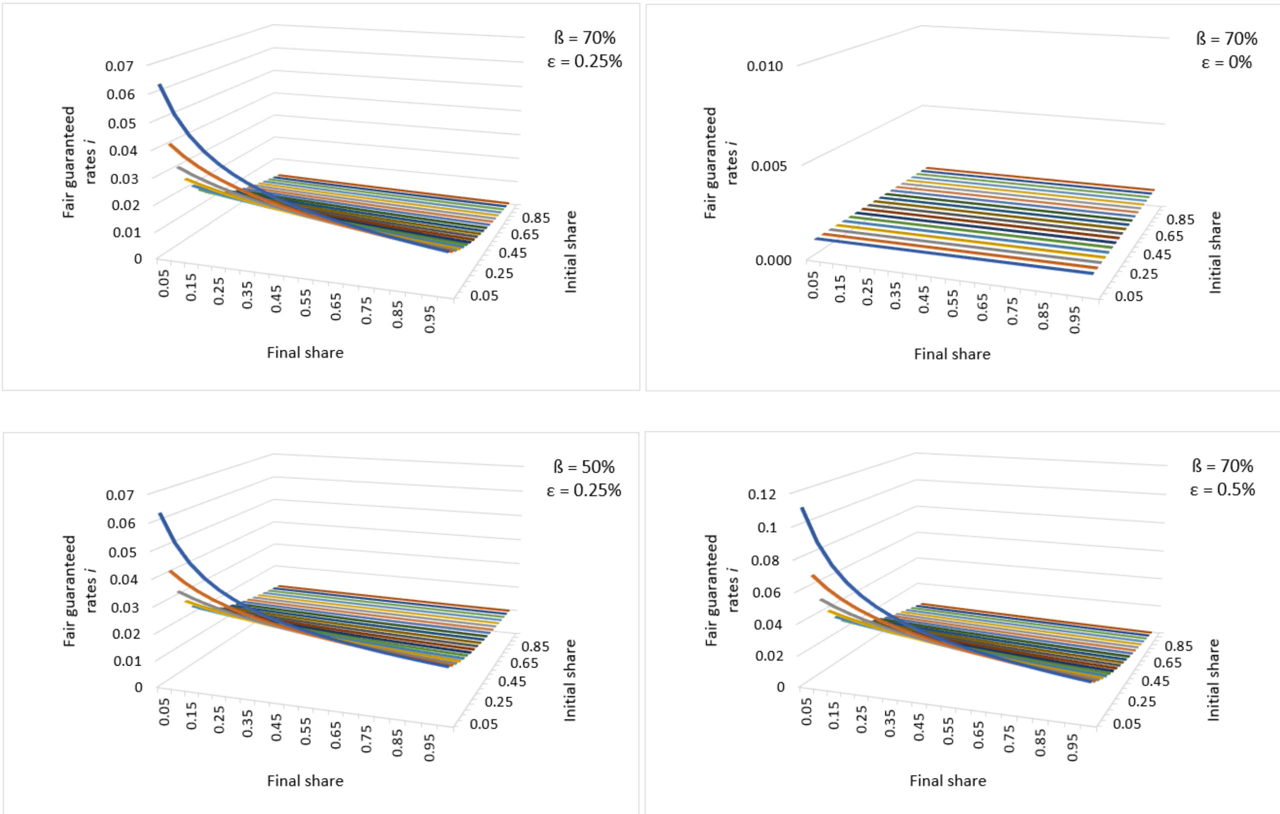

Figure 3.5: Fair guaranteed rates i with respect to a step-wise investment strategy (two-step function with $n = \frac{4}{3}$) for different participation levels β and guarantee fees ε.

Another example is the three-step strategy by taking $n = 2$, $k = 3$, and $l = 4$ in Equation (3.2.6). In this setting, we assume that the policyholder invests a constant proportion π_0 in the participating contract for the first half of the time, then invests the same or another constant proportion $\pi_{\frac{T}{2}}$ for the

π_0	$\pi_{\frac{3\cdot T}{4}}$	$\beta = 70\%$ $\varepsilon = 0\%$	$\beta = 70\%$ $\varepsilon = 0.25\%$	$\beta = 50\%$ $\varepsilon = 0.25\%$	$\beta = 70\%$ $\varepsilon = 0.5\%$
0.05	0.05	0.11	6.32	6.33	11.13
0.05	0.65	0.11	2.09	2.42	3.39
0.20	0.70	0.11	1.30	1.81	2.14
0.30	0.55	0.11	1.15	1.71	1.90
0.45	0.65	0.11	0.76	1.43	1.27
0.50	0.50	0.11	0.76	1.43	1.27
0.65	0.65	0.11	0.48	1.25	0.80
0.75	0.40	0.11	0.46	1.24	0.77
0.90	0.50	0.11	0.29	1.14	0.46
1.00	1.00	0.11	0.11	0.10	0.11

Table 3.2: Fair guaranteed rates i (expressed in %) with respect to a step-wise investment strategy (two-step function with $n = \frac{4}{3}$) for different participation levels β and guarantee fees ε.

remaining quarter of the time, and eventually invests the same or another constant proportion $\pi_{\frac{3\cdot T}{4}}$ in the last quarter of the time. In this regard, the fair condition is expressed as follows:

$$\left(\pi_0 \cdot ((1+i)\cdot e^{-r} + \beta \cdot c_1) + (1-\pi_0)\cdot e^{-\varepsilon}\right)^{\frac{1}{2}} \cdot \left(\pi_{\frac{T}{2}} \cdot ((1+i)\cdot e^{-r} + \beta \cdot c_1) + (1-\pi_{\frac{T}{2}})\cdot e^{-\varepsilon}\right)^{\frac{1}{4}} \cdot \left(\pi_{\frac{3\cdot T}{4}} \cdot ((1+i)\cdot e^{-r} + \beta \cdot c_1) + (1-\pi_{\frac{3\cdot T}{4}})\cdot e^{-\varepsilon}\right)^{\frac{1}{4}} = 1. \tag{3.2.12}$$

Also, we impose that the shares π_0, $\pi_{\frac{T}{2}}$ and $\pi_{\frac{3\cdot T}{4}}$ vary between 5% and 100% equidistantly by 5% in all the subsequent numerical figures. The results of the three-step function are displayed in Table 3.3.

π_0	$\pi_{\frac{T}{2}}$	$\pi_{\frac{3\cdot T}{4}}$	$\beta = 70\%$ $\varepsilon = 0\%$	$\beta = 70\%$ $\varepsilon = 0.25\%$	$\beta = 50\%$ $\varepsilon = 0.25\%$	$\beta = 70\%$ $\varepsilon = 0.5\%$
0.05	0.05	0.05	0.11	6.32	6.33	11.13
0.05	0.45	0.65	0.11	1.42	1.90	2.33
0.20	0.50	0.70	0.11	1.02	1.62	1.70
0.35	0.55	0.65	0.11	0.82	1.47	1.36
0.50	0.50	0.50	0.11	0.76	1.43	1.27
0.65	0.65	0.65	0.11	0.48	1.25	0.80
0.75	0.80	0.85	0.11	0.30	1.14	0.48
0.75	0.80	1.00	0.11	0.26	1.12	0.41
0.75	0.90	1.00	0.11	0.24	1.11	0.36
1.00	1.00	1.00	0.11	0.11	0.10	0.11

Table 3.3: Fair guaranteed rates i (expressed in %) with respect to a step-wise investment strategy (three-step function with $n = 2$, $k = 3$ and $l = 4$) for different participation levels β and guarantee fees ε.

Piece-wise linear strategy

The piece-wise linear strategy combines elements of both the linear and step-wise strategies. This strategy segments the investment horizon into multiple stages, with each stage characterized by a distinct linear relationship between the initial and final investment shares. In this strategy, we distinguish two different strategies and present an example for each. For the two-path strategy, we assume an investment strategy that is a step function for the first three quarters, followed by a linear increase in the last quarter. This is obtained by replacing n by $\frac{4}{3}$ and taking $k = l = 1$ in Equation (3.2.7). In this respect, we obtain the following fairness relation:

$$\left(\pi_0 \cdot ((1+i)\cdot e^{-r} + \beta \cdot c_1) + (1-\pi_0)\cdot e^{-\varepsilon}\right)^{\frac{3\cdot T}{4}} \cdot \prod_{t=\frac{3\cdot T}{4}}^{T-1} \left(\pi^{t}_{\frac{3\cdot T}{4}} \cdot ((1+i)\cdot e^{-r} + \beta \cdot c_1) + (1-\pi^{t}_{\frac{3\cdot T}{4}})\cdot e^{-\varepsilon}\right) = 1. \tag{3.2.13}$$

The results of the two-path strategy are illustrated in Figure 3.6, and some examples are displayed in Table 3.4.

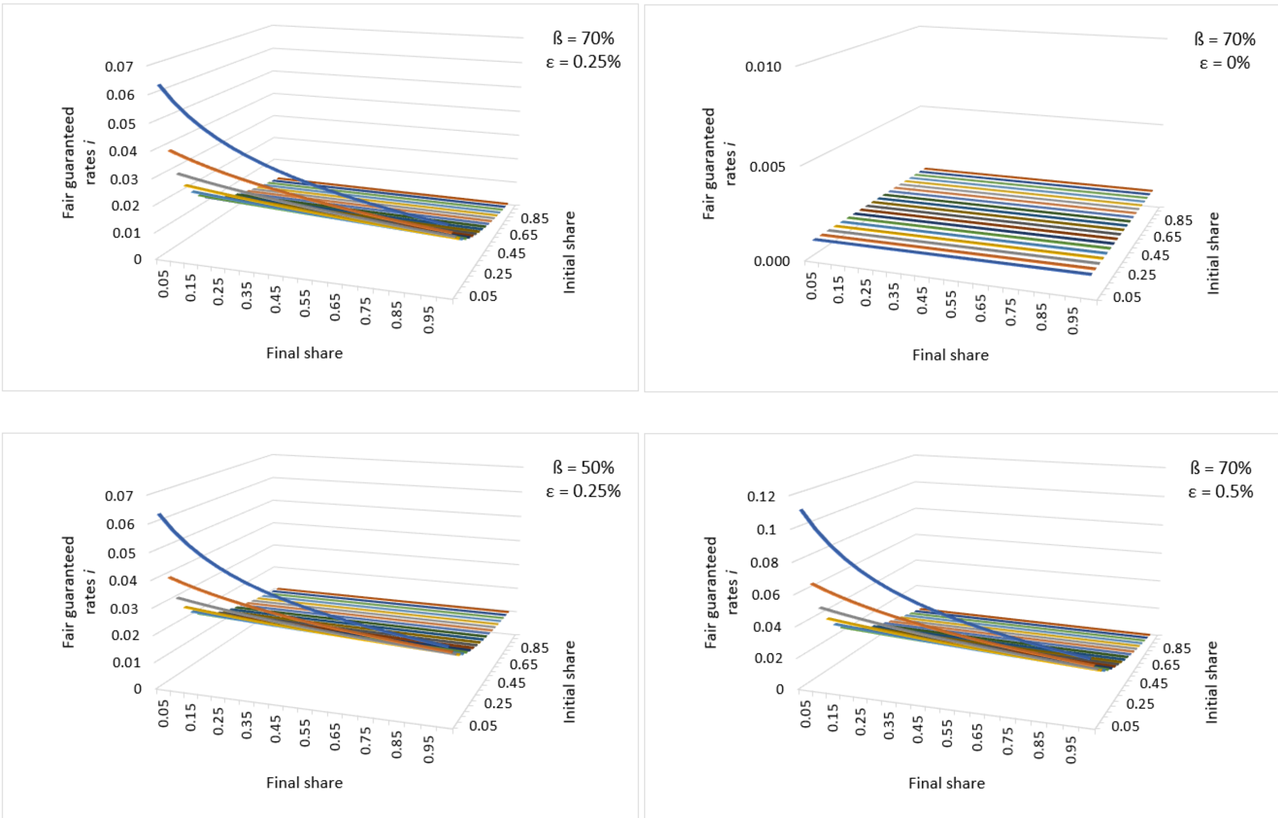

Figure 3.6: Fair guaranteed rates i with respect to a piece-wise investment strategy (two-path strategy with $n = \frac{4}{3}$) for different participation levels β and guarantee fees ε.

For the three-path strategy, we assume an investment strategy that is a step function for the first half, followed by a linear increase in the subsequent quarter, and then a step function for the last quarter of the time. This is obtained by replacing n by 2, k by 3, and l by 4 in Equation (3.2.7). In this respect,

π_0	π_{T-1}	$\beta = 70\%$ $\varepsilon = 0\%$	$\beta = 70\%$ $\varepsilon = 0.25\%$	$\beta = 50\%$ $\varepsilon = 0.25\%$	$\beta = 70\%$ $\varepsilon = 0.5\%$
0.05	0.05	0.11	6.32	6.33	11.13
0.05	0.65	0.11	3.09	3.25	5.04
0.20	0.70	0.11	1.62	2.05	2.65
0.35	0.75	0.11	1.02	1.65	1.70
0.50	0.50	0.11	0.76	1.43	1.27
0.50	0.70	0.11	0.70	1.40	1.18
0.65	0.65	0.11	0.48	1.25	0.80
0.75	0.40	0.11	0.40	1.21	0.66
0.90	0.50	0.11	0.24	1.11	0.36
1.00	1.00	0.11	0.11	0.10	0.11

Table 3.4: Fair guaranteed rates i (expressed in %) with respect to a piece-wise investment strategy (two-path strategy with $n = \frac{4}{3}$) for different participation levels β and guarantee fees ε.

we obtain the following fairness relation:

$$\left(\pi_0 \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1-\pi_0) \cdot e^{-\varepsilon}\right)^{\frac{T}{2}} \cdot \prod_{t=\frac{T}{2}}^{\frac{3\cdot T}{4}-1} \left(\pi_{\frac{t}{\frac{T}{2}}} \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1-\pi_{\frac{t}{\frac{T}{2}}}) \cdot e^{-\varepsilon}\right)$$
$$\cdot \left(\pi_{\frac{3\cdot T}{4}} \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1-\pi_{\frac{3\cdot T}{4}}) \cdot e^{-\varepsilon}\right)^{\frac{T}{4}} = 1. \tag{3.2.14}$$

The results of the three-path strategy are illustrated in Figure 3.7, and some examples are displayed in Table 3.5.

π_0	$\pi_{\frac{3\cdot T}{4}-1}$	$\beta = 70\%$ $\varepsilon = 0\%$	$\beta = 70\%$ $\varepsilon = 0.25\%$	$\beta = 50\%$ $\varepsilon = 0.25\%$	$\beta = 70\%$ $\varepsilon = 0.5\%$
0.05	0.05	0.11	6.32	6.33	11.13
0.05	0.55	0.11	1.59	2.02	2.59
0.20	0.70	0.11	0.97	1.58	1.61
0.40	0.60	0.11	0.79	1.45	1.31
0.40	0.65	0.11	0.74	1.42	1.23
0.50	0.50	0.11	0.76	1.43	1.27
0.65	0.65	0.11	0.48	1.25	0.80
0.75	0.40	0.11	0.57	1.31	0.95
0.90	0.50	0.11	0.38	1.19	0.62
1.00	1.00	0.11	0.11	0.10	0.11

Table 3.5: Fair guaranteed rates i (expressed in %) with respect to a piece-wise investment strategy (three-path strategy with $n = 2$, $k = 3$ and $l = 4$) for different participation levels β and guarantee fees ε.

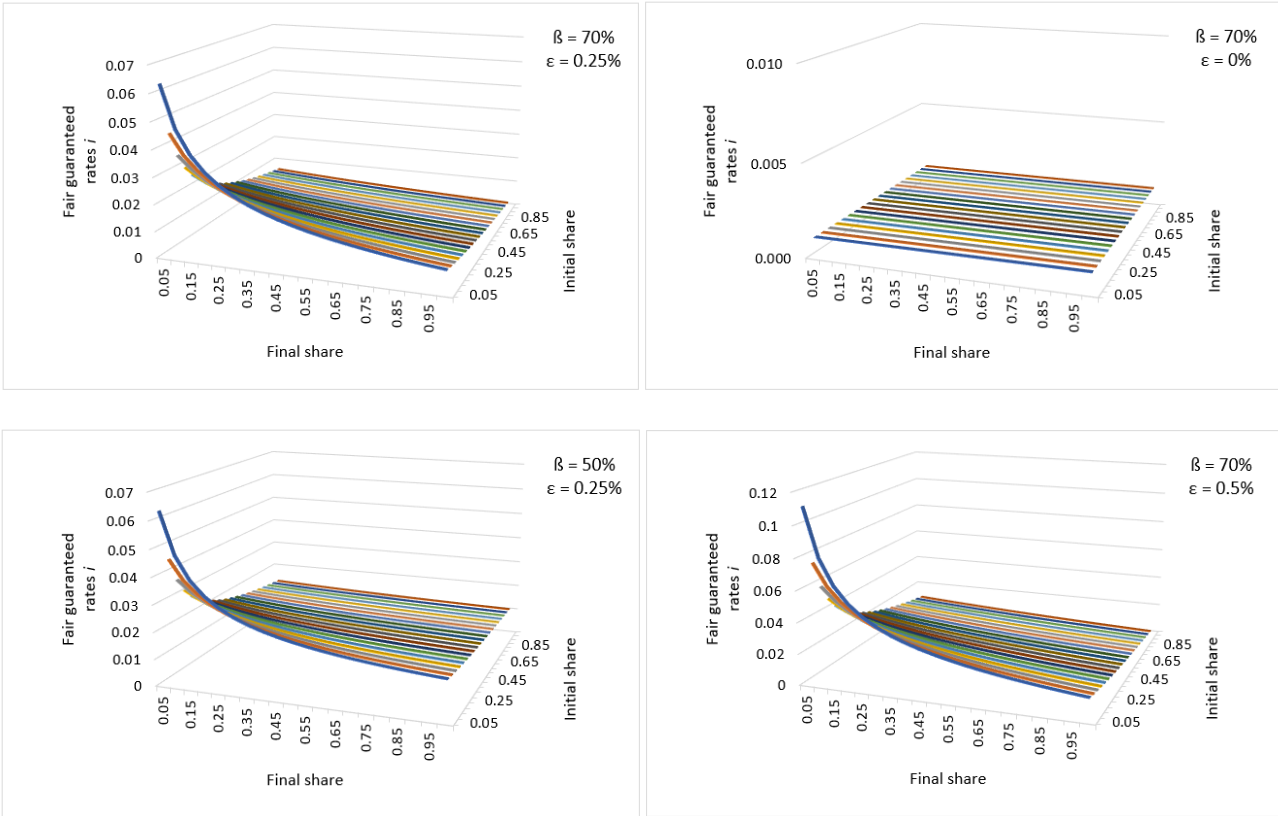

Figure 3.7: Fair guaranteed rates i with respect to a piece-wise investment strategy (three-path strategy with $n = 2$, $k = 3$ and $l = 4$) for different participation levels β and guarantee fees ε.

All the results exhibited in Figures 3.4, 3.5, 3.6, and 3.7, and in Tables 3.1, 3.2, 3.3, 3.4, and 3.5, for the different examples of investment strategies, draw attention to the fact that for a constant proportion invested in the participating contract throughout the duration of the contract, the fair guaranteed rates are in line with the results presented in the second chapter where the allocation level was assumed constant; as well, we notice in this framework that these rates increase the more we invest in the unit-linked part, either by investing initially a high percentage in the unit-linked part and increasing this percentage until maturity or by investing a lower initial share and increasing it gradually until maturity. In fact, the insurance company is less committed to the policyholder the more we invest in the unit-linked part, which allows it to offer a higher guarantee. For instance, Table 3.1 shows that if we initially invest 50% in the participating contract ($\pi_0 = 50\%$) and decrease this proportion to $\pi_{T-1} = 5\%$, the insurance company grants a higher guaranteed rate than if the investment share stays constant. Additionally, we point out that, for a fixed guarantee fee ε, the lower the participation in the profits of the insurer ($\beta \searrow$), the higher the guaranteed rates granted to the policyholder. This is also true if the policyholder accepts a larger guarantee fee ($\varepsilon \nearrow$) for a fixed β. Furthermore, we highlight the scenario of the zero guarantee fee ($\varepsilon = 0\%$), in which the guaranteed rates stay constant at any stage of the investment strategy. In this case, there is no transfer from the unit-linked to the participating part of the mixed contract, which results in the same guaranteed rate, regardless of the investment strategy. The insurer actually offers higher guaranteed rates in exchange for the guarantee fee. This instance demonstrates how the guarantee fee contributed to the mixed contract. Eventually, it is important to highlight the case of $\pi_t = 100\% \; \forall t$. In this case, the mixed contract is a traditional life insurance policy, and therefore, the guarantee fee has no impact at all, which leads to a low guaranteed rate that remains the same for all ε.

Consequently, the figures confirm that the guaranteed rate is boosted by an increase in the guarantee fee, a decrease in the participation level, and a higher allocation in the unit-linked contract over time. We can state that, indeed, the mixed contract increases the guaranteed rates significantly and that the shape of the strategy has an important impact on the fair guaranteed rates as well.

3.3 Lifestyle strategy and classical utility function

3.3.1 Optimal allocation for classical utility function and single premium

By analyzing the fair condition of Equation (3.2.9), investors can make informed decisions on the optimal allocation strategy and adjust the parameters to achieve the desired risk and return profiles for the mixed product. Their preferences and attitude towards risk are vital considerations in designing this optimal investment strategy. Hence, we model the preferences using a constant relative risk aversion (CRRA) utility function, which exhibits risk aversion. The level of risk aversion is determined by the risk aversion coefficient. Power utility functions are commonly employed to represent CRRA utility functions. The power utility function captures the trade-off between risk and return and enables us to quantify the policyholder's utility based on their risk preferences:

$$U(z) = \frac{z^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0, \gamma \neq 1, \tag{3.3.1}$$

where U is the utility function of the affiliate satisfying $U' > 0$ and $U'' < 0$, and γ is the risk aversion coefficient that is assumed to be constant over time, and its increase in value reflects the unwillingness of the policyholder to take a risk. The goal is to find the optimal deterministic investment strategy that maximizes the expected utility of the terminal benefit. Hence, the optimization problem is formulated as follows

$$\max_{\pi_t} \quad \mathbb{E}^{\mathbb{P}} \left[U(V_T) \right], \quad t = 0, 1, \ldots, T-1. \tag{3.3.2}$$

This optimization problem is subject to the constraint that ensures the fairness of the pricing relation of the mixed contract, which depends on π_t, as displayed:

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-rT} \cdot V_T \right] = P.$$

Under this framework, we present numerical illustrations of three investment strategies within the framework of the lifestyle investment strategy: the linear, the step-wise, and the piece-wise linear strategies. These illustrations provide insights into the optimal investment strategies based on a predetermined deterministic investment style that maximize the expected utility of the policyholder at the maturity of the contract.

We refer to the same contract introduced before, and we assume that the premium is paid in a single payment at inception ($P = 1000$). We suppose the following values for the asset parameters: $\mu_G = 3\%$, $\mu_H = 7\%$, and $\sigma_H = 15\%$, and the correlation between the funds, ρ, is supposed to be 10%. The results are based on Monte Carlo simulations.

In order to present different policyholder preferences, we assume three different values of the risk aversion coefficient ($\gamma = 2, 4, \text{and } 6$). To examine the linear strategy, we apply the form of the investment strategy as indicated in Equation (3.2.4). Next, to explore the step-wise strategy, we provide the numerical results for both the two-step function (Equation (3.2.5)) and the three-step function (Equation (3.2.6)). For these functions, we refer to the same examples presented in the previous section. Lastly, we examine the piece-wise linear strategy, in which we consider the two- and three-path strategies and refer to the same examples presented in the previous section. The results, illustrated in Figure 3.8, show the optimal investment strategy for each risk aversion coefficient. The numerical results for the three strategies under the classical utility function and single premium are given in Table 3.A.1 in Appendix 3.A. The results are presented in terms of certainty equivalent concept defined as $U(CEQ) = \mathbb{E}^{\mathbb{P}} \left[U(V_T) \right]$.

In the different investment strategies, the optimal investment strategy varies with the risk aversion coefficient, but it remains constant over time for each specific coefficient. Figure 3.8 shows that as the

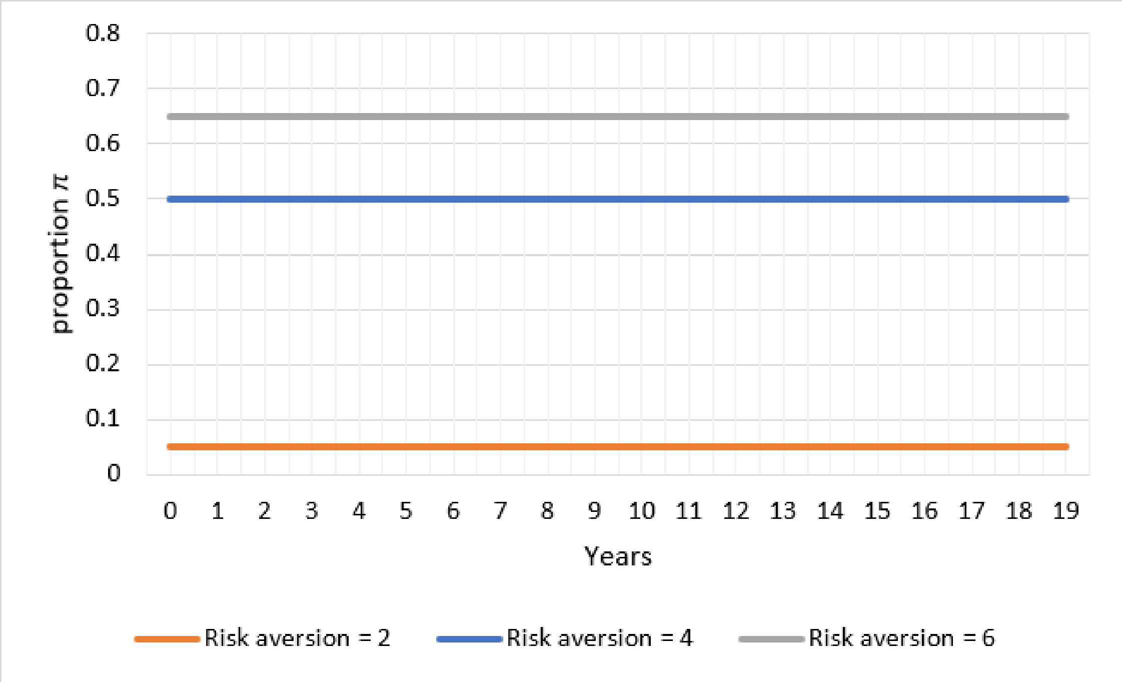

Figure 3.8: Optimal proportion in the participating contract using a classical utility function for different risk aversion values.

risk aversion increases, the policyholder becomes more risk-averse and allocates a higher proportion of the premium to the participating contract. They maintain the same allocation in the unit-linked contract at each moment if their risk aversion remains unchanged. Particularly, we see from Figure 3.8 that the optimal investment strategy for an investor with a risk aversion of $\gamma = 2$ is $\pi_t = 0.5\%$, for $\gamma = 4$ is $\pi_t = 50\%$, and for $\gamma = 6$ is $\pi_t = 65\% \, \forall t \in [0, T-1]$. Therefore, investors with the same risk appetite at all points in the contract prefer a constant investment strategy that aligns with their behavior throughout the contract.

In conclusion, the results obtained for the constant risk aversion case across different investment strategies demonstrate that the lifestyle investment strategy does not hold in this setting. Regardless of the strategy considered—linear, step-wise, or piece-wise linear—the optimal investment strategy remains constant for policyholders using a classical utility function. Therefore, the findings suggest that under the assumption of constant risk aversion, policyholders tend to choose an investment strategy that remains consistent throughout the contract, regardless of their age or time to maturity. If we want to justify and generate some form of lifestyle strategy, we can generalize this section in two directions, either considering periodic premiums or time-varying risk aversion. The latter suggests to consider a variable attitude towards risk with ageing.

3.3.2 Optimal allocation for classical utility function and periodic premiums

The presence of constant periodic premiums introduces an additional layer of complexity to the optimization problem. It requires a thorough examination of how premium payments interact with investment strategies and their impact on the policyholder's overall utility. When considering the impact of periodic premiums on the optimal allocation strategy, it is essential to account for the additional cash flows associated with these payments. Unlike the single premium scenario discussed in Section 3.3.1, periodic premiums introduce a time-varying cash inflow into the investment portfolio. This dynamic element necessitates adjustments in the optimization framework.

To incorporate periodic premiums, the benefit at maturity of Equation (3.2.8) becomes:

$$\tilde{V}_T = \tilde{P} \cdot \sum_{k=0}^{T-1} \left(\prod_{t=k}^{T-1} \left(\pi_t \cdot \left((1+i) + (\beta \cdot g_{t+1} - i)^+ \right) + (1-\pi_t) \cdot (1+f_{t+1}) \right) \right), \tag{3.3.3}$$

where $\tilde{P}$ is the periodic premium, and π_t is the part of the total reserve invested at time t in the participating contract. To determine the fair guaranteed interest rate, we ensured fair pricing of the contract by equating the fair value of Equation (3.3.3) with the fair value of the premiums in the following manner:

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-rT} \cdot \tilde{V}_T \right] = \mathbb{E}^{\mathbb{Q}} \left[\sum_{k=0}^{T-1} \tilde{P} \cdot e^{-rk} \right],$$

which gives the following fairness relation:

$$\tilde{P} \cdot \sum_{k=0}^{T-1} \left(e^{-rk} \cdot \prod_{t=k}^{T-1} \left(\pi_t \cdot ((1+i) \cdot e^{-r} + \beta \cdot c_1) + (1-\pi_t) \cdot e^{-\varepsilon} \right) \right) = \tilde{P} \cdot \sum_{k=0}^{T-1} e^{-rk}. \tag{3.3.4}$$

Formally, the modified optimization problem can be stated as:

$$\max_{\pi_t} \quad \mathbb{E}^{\mathbb{P}} \left[U(\tilde{V}_T) \right], \quad t = 0, 1, \ldots, T-1, \tag{3.3.5}$$

considering a fair guaranteed rate through the satisfaction of Equation (3.3.4). This formulation acknowledges the dual influence of investment decisions and premium payments on the policyholder's utility.

In the following, we will explore various investment strategies tailored for scenarios involving periodic premiums. Specifically, we will delve into numerical illustrations for strategies such as linear, step-wise, and piece-wise linear approaches. These illustrations will shed light on the optimal investment strategies that maximize the expected utility of the policyholder, considering periodic premium payments. We fix the risk aversion parameter γ to 4, and we divided the single premium P by the expected number of annuity payments over a 20-year period to determine the amount of each periodic premium.

Incorporating periodic premiums, we observe in Figure 3.9 that the optimal proportion invested in the participating contract varies over time for each of the three investment strategies considered: linear, step-wise, and piece-wise. The numerical results for the three strategies under the classical utility function and periodic premiums are given in Table 3.B.1 in Appendix 3.B.

For the linear strategy, the allocation to the participating contract starts at 5% and gradually increases to 40% over the 20-year period, reflecting a steady progression towards a more conservative investment approach.

The step-wise linear strategy employs two distinct phases: an initial 5% allocation to the participating contract for the first 15 years, followed by a substantial increase to 40% for the remaining period. Analogously, the three-step strategy follows a similar pattern, but with a transition at year 10 to an allocation of 45% in the participating contract, and this proportion stays the same for the remaining period of time.

In addition, the piece-wise linear strategy offers a more interesting approach. The two-path strategy maintains a 15% allocation for the first three quarters of time and increases linearly to 55%. Conversely, the three-path strategy undergoes gradual shifts in allocation, starting at 5% and progressively increasing to 43% by the contract's maturity.

Incorporating periodic premiums has revealed a notable shift in the application of the lifestyle investment strategy. Each investment strategy offers policyholders unique risk-return profiles, enabling

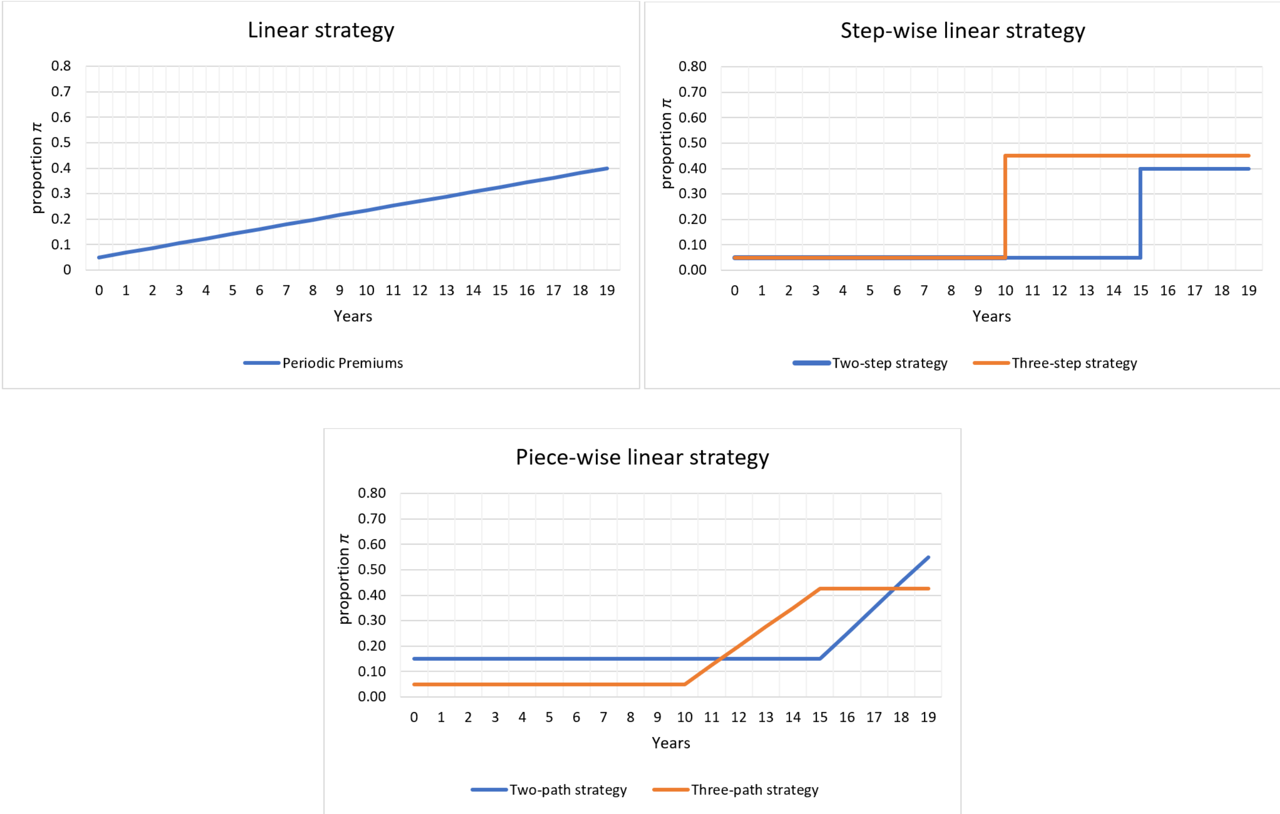

Figure 3.9: Optimal proportion in the participating contract for periodic premium payments.

them to customize their investments to align with their specific preferences and goals throughout the contract's duration. This dynamic approach ensures that the investment strategy reflects the lifestyle strategy concept, which is in contrast to the scenario with a single premium payment. In the next section, we move on to consider time-varying risk aversion with a single premium payment to check its impact on the optimal strategy for the different investment strategy shapes and, accordingly, check if it aligns with the lifestyle investment strategy.

3.3.3 Optimal allocation for time-varying risk aversion utility function

Tang et al. [13] conducted research that challenges the assumption that investors' risk aversion remains constant over time, highlighting the unrealistic nature of such an assumption. Consequently, it becomes crucial to incorporate time variation in the risk aversion coefficient to capture the dynamic nature of investors' risk preferences. Typically, investors tend to undergo a shift in their investment style, transitioning from a certain level of risk-loving preference in their youth to risk aversion as they age. In other words, changes in risk preferences over an investor's lifetime are reflected by a risk aversion that increases as age advances. Consequently, it is plausible to suggest that investors become more risk-averse as they grow older.

Returning to the single premium case, the classical utility framework defined in Equation (3.3.1) is the common way to define aggregate utility, which is the utility function based on the final wealth. However, the final wealth of Equation (3.2.8) can be written equivalently as

$$V_T = P \cdot \prod_{t=0}^{T-1} R_{t+1}, \tag{3.3.6}$$

where

$$R_{t+1} = \pi_t \cdot \left(1 + i + (\beta \cdot g_{t+1} - i)^+\right) + (1 - \pi_t) \cdot (1 + f_{t+1}).$$

Using Equation (3.3.6), the utility function can be seen as a function of successive return factors and can be written as

$$\begin{aligned} U(V_T) &= \frac{(P \cdot R_1 \cdot \ldots \cdot R_T)^{1-\gamma}}{1-\gamma} \\ &= \frac{P^{1-\gamma}}{1-\gamma} \cdot R_1^{1-\gamma} \cdot \ldots \cdot R_T^{1-\gamma}. \end{aligned}$$

Based on that, we could generalize this approach by introducing time-varying risk attitudes. This allows us to write a new aggregate utility function:

$$\tilde{U}(R_1, \ldots, R_T) = \frac{P^{1-\gamma_0}}{1-\gamma_0} \cdot R_1^{1-\gamma_1} \cdot \ldots \cdot R_T^{1-\gamma_T}, \qquad \gamma_0, \ldots, \gamma_T > 1, \tag{3.3.7}$$

where $\tilde{U}$ is the new utility function satisfying the following conditions for $\gamma_0, \ldots, \gamma_T > 1$ and $i = 1, \ldots, T$:

$$\begin{aligned} \frac{\partial \tilde{U}}{\partial R_i} &= \frac{P^{1-\gamma_0}}{1-\gamma_0} \cdot R_1^{1-\gamma_1} \cdot \ldots \cdot (1-\gamma_i) \cdot R_i^{-\gamma_i} \cdot \ldots \cdot R_T^{1-\gamma_T} > 0, \quad \text{and} \\ \frac{\partial^2 \tilde{U}}{\partial R_i^2} &= \frac{P^{1-\gamma_0}}{1-\gamma_0} \cdot R_1^{1-\gamma_1} \cdot \ldots \cdot (1-\gamma_i) \cdot (-\gamma_i) \cdot R_i^{-1-\gamma_i} \cdot \ldots \cdot R_T^{1-\gamma_T} < 0. \end{aligned}$$

$(\gamma_0, \ldots, \gamma_T)$ is a vector of successive time-varying risk aversions with $\gamma_0 < \ldots < \gamma_T$. Investors' risk appetites change each year as they become more conservative with age, and based on this, we have annual decisions reflected by the time-varying risk aversions in Equation (3.3.7).

As a result, the new optimization problem is defined as

$$\max_{\pi_t} \quad \mathbb{E}^{\mathbb{P}} \left[\tilde{U}(R_1, \ldots, R_T) \right], \quad t = 0, 1, \ldots, T-1. \tag{3.3.8}$$

In the following part, we will focus on the numerical results of the time-varying risk aversion cases for each of the three investment strategies: the linear, the step-wise, and the piece-wise linear strategies.

Linear strategy

In this subsection, we present numerical illustrations of the linear strategy in the time-varying risk aversion case. We consider the policyholder's behavior as risk aversion increases with age from $\gamma_0 = 2$ to $\gamma_T = 6$, equidistantly by 0.2 as follows

$$\gamma_t = 0.2 \cdot t + 2, \quad t = 0, 1, \ldots, T. \tag{3.3.9}$$

The results, illustrated in Figure 3.10, show the optimal investment strategy for each age-dependent risk aversion coefficient. The numerical results of the utility are displayed in Table 3.C.4 in Appendix 3.C.

Figure 3.10 highlights that by assuming a change in the preference of the policyholder at each point in the contract, the lifestyle strategy performs better than the case of constant risk aversion. More precisely, the policyholder becomes more risk-averse as he ages than when he is young. Therefore, the increase in the risk aversion parameter results in a lower investment in the unit-linked contract over time. A policyholder who starts with a risk aversion of $\gamma = 2$ at the inception of the contract invests 80% of the premium in the risky asset. This allocation decreases to 30% at contract maturity, which satisfies the lifestyle strategy concept.

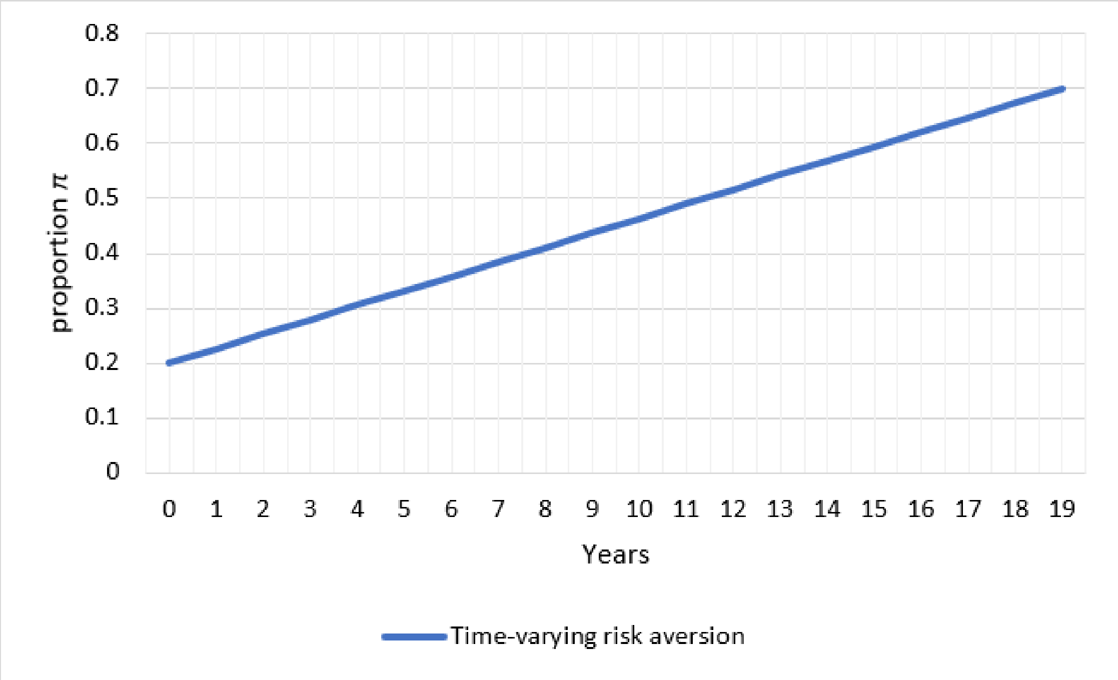

Figure 3.10: Optimal proportion in the participating contract based on a linear strategy for increasing values of risk aversion.

Step-wise linear strategy

Next, we explore the step-wise strategy in the time-varying risk aversion case. We will provide the numerical results for both the yearly-varying risk aversion example and the step-varying risk aversion example. For each example, we analyze the policyholder's behavior and their optimal investment strategy over time.

Yearly-varying risk aversion:

Recalling the two examples of two- and three-step strategies presented above, we assume that the behavior of the policyholder changes each year as risk aversion increases with age from $\gamma_0 = 2$ to $\gamma_T = 6$, equidistantly by 0.2, see Equation (3.3.9).

Figure 3.11 emphasizes that by increasing their risk aversion each year, the policyholder tends to invest less in the risky part of the contract for each next step. For instance, we notice that the investment in the unit-linked contract moves from 55% to 35% for the last quarter of the time in the two-step strategy, whereas it moves from 65% between $t = 0$ and $t = \frac{T}{2}$, to 45% for the next quarter of the time, then to 35% for the last quarter of the time in the three-step strategy.
We note that the results of the utility are exhibited in Table 3.C.4 in Appendix 3.C.

Step-varying risk aversion:

This part highlights a special case of time-varying risk aversion, that is, instead of varying the risk aversion on a yearly basis, increase it just when moving from one step to another. More precisely, risk aversion increases with the investment strategy of this approach. Recalling the two examples of two- and three-step strategies presented above, for the two-step function, the policyholder has a constant risk aversion $\gamma_0 = 2$ between $t = 0$ and $t = \frac{3 \cdot T}{4} - 1$, and another higher constant risk aversion $\gamma_{\frac{3 \cdot T}{4}} = 6$

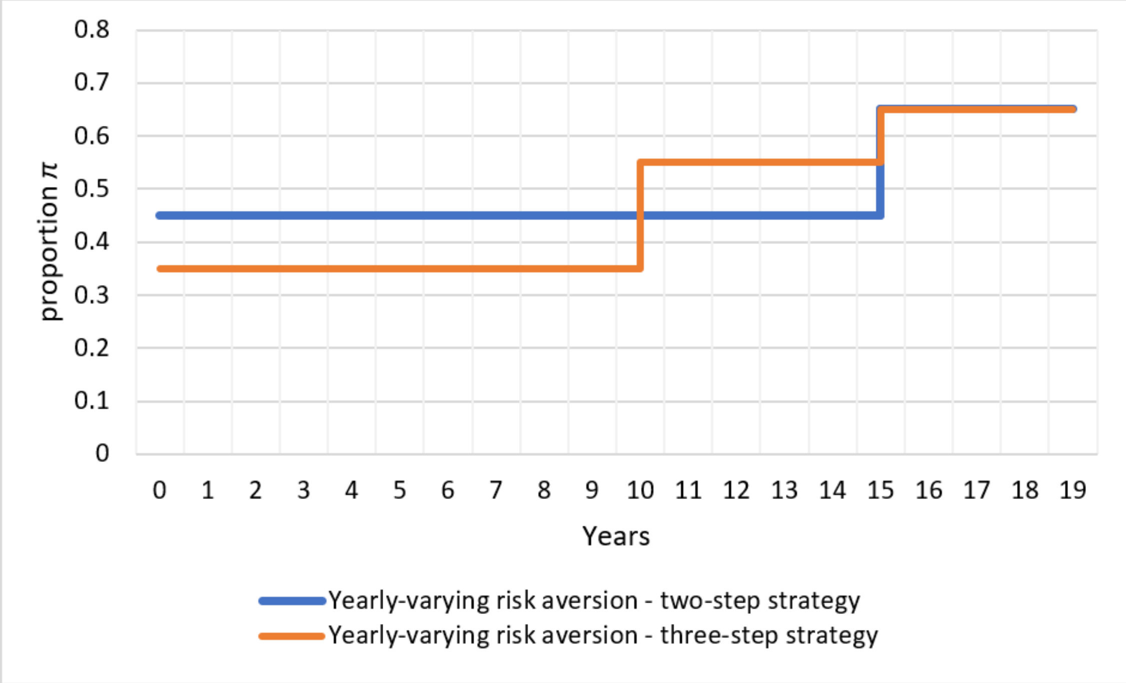

Figure 3.11: Optimal proportion in the participating contract based on two and three-step strategies for increasing values of risk aversion.

for the remaining time, displayed as follows

$$\gamma_t = \begin{cases} 2 & \text{for } t \in \left[0, \frac{3 \cdot T}{4} - 1\right], \\ 6 & \text{for } t \in \left[\frac{3 \cdot T}{4}, T\right]. \end{cases} \tag{3.3.10}$$

Analogously, for the three-step function, the policyholder has a constant risk aversion $\gamma_0 = 2$ between $t = 0$ and $t = \frac{T}{2} - 1$, another higher but also constant risk aversion $\gamma_{\frac{T}{2}} = 4$ between $t = \frac{T}{2}$ and $t = \frac{3 \cdot T}{4} - 1$, and eventually a higher constant risk aversion $\gamma_{\frac{3 \cdot T}{4}} = 6$ for the remaining time, represented as

$$\gamma_t = \begin{cases} 2 & \text{for } t \in \left[0, \frac{T}{2} - 1\right], \\ 4 & \text{for } t \in \left[\frac{T}{2}, \frac{3 \cdot T}{4} - 1\right], \\ 6 & \text{for } t \in \left[\frac{3 \cdot T}{4}, T\right]. \end{cases} \tag{3.3.11}$$

In Figure 3.12, we point out that the investment in the unit-linked contract is higher, in the first three quarters of the time, in the step-varying risk aversion case for both the two- and three-step strategies. This is intuitive, as in the yearly-varying risk aversion case, the policyholder becomes more conservative from one year to another. However, his attitude toward risk stays the same for a longer period in the step-varying risk aversion case. On the other side, the optimal constant investment strategy for the last quarter is the same because the risk aversion levels are close in both cases at this stage of time.
The results of the utilities for the two- and three-step functions are exhibited in Table 3.C.5 in the Appendix 3.C.

Piece-wise linear strategy

Lastly, we consider the piece-wise linear strategy under the time-varying risk aversion case. We will present the numerical results for both the yearly-varying risk aversion example and the piece-varying

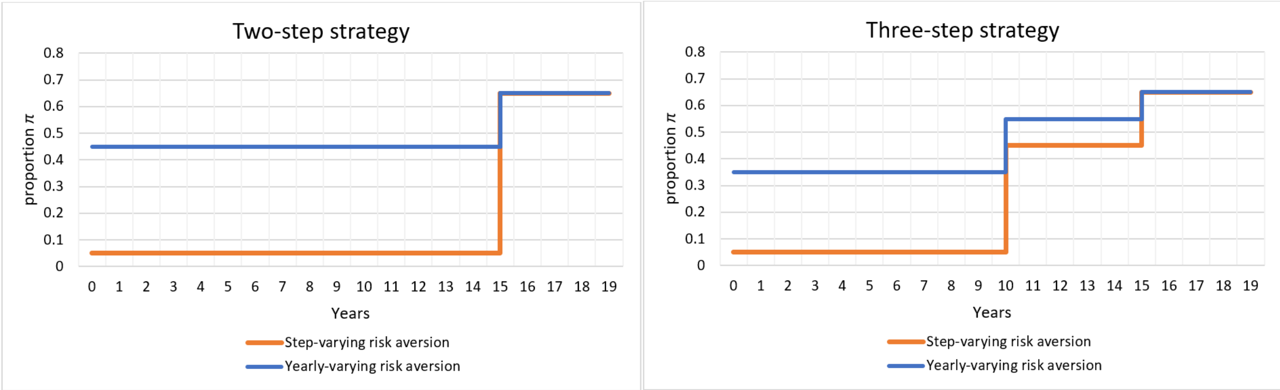

Figure 3.12: Optimal proportion in the participating contract based on two and three-step strategies for step-varying vs yearly-varying risk aversion.

risk aversion example. For each example, we analyze the policyholder's changing risk aversion and its impact on the optimal investment allocation over time.

Two-path strategy:

Recalling the two-step strategy defined above, we present the yearly-varying risk aversion case, which states that the risk aversion increases from $\gamma_0 = 2$ to $\gamma_T = 6$ equidistantly by 0.2 (see Equation (3.3.9)), and the piece-varying risk aversion case, which alters the risk aversion at $t = \frac{3 \cdot T}{4}$ only to be $\gamma_0 = 2$ before and $\gamma_{\frac{3 \cdot T}{4}} = 6$ after (see Equation (3.3.10)).
For the utility results, please refer to Tables 3.C.4 and 3.C.5 in Appendix 3.C for yearly-varying and step-varying risk aversions, respectively.

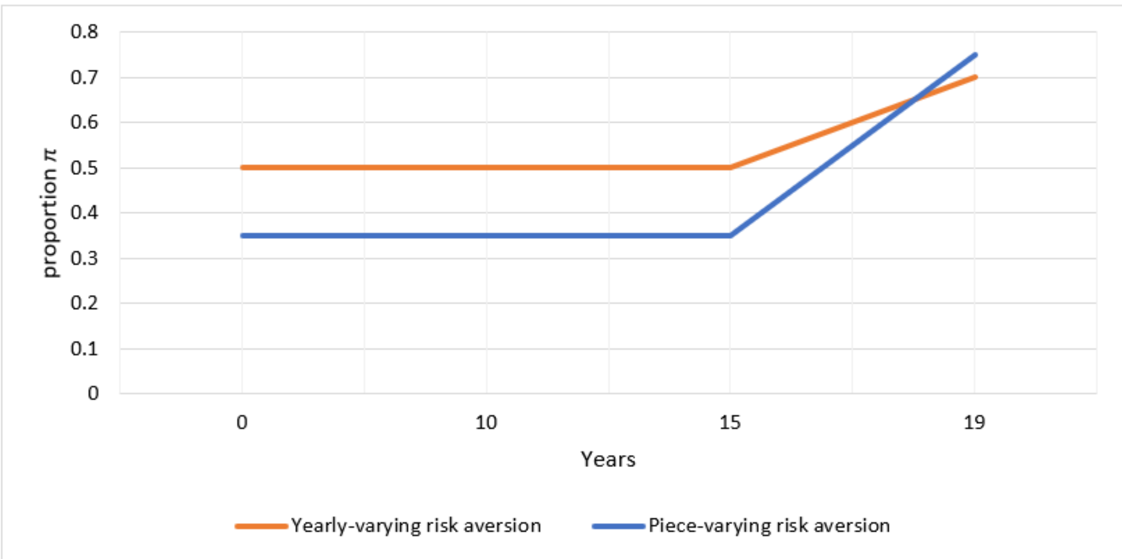

Figure 3.13: Optimal proportion in the participating contract based on a piece-wise linear strategy for different values of risk aversion - Two-path strategy.

Figure 3.13 points out that, by increasing the risk aversion, either on a yearly basis or just when moving from one piece to the next, the lifestyle investment strategy concept is fulfilled. It is intuitive to see that the policyholder who maintains a low level of risk aversion for the first three quarters of the time invests in the risky fund for a longer period compared to the one who becomes more risk averse from one year to the next. However, as the optimal investment strategy is a linear form in the last quarter of time, the strategy of a policyholder with a risk aversion of 6 (in the piece-varying risk aversion case) would intersect with another policyholder with a risk aversion between 5 and 6 in this interval of time (in the yearly-varying risk aversion case) since the latter is less risk averse.

Three-path strategy:

The last segment covers the three-path strategy stated in Equation (3.2.7) for $n = 2$, $k = 3$, and $l = 4$. We note that the risk aversion increases from $\gamma_0 = 2$ to $\gamma_T = 6$ equidistantly by 0.2 for the yearly-varying risk aversion case (see Equation (3.3.9)) and changes from $\gamma_0 = 2$, $\gamma_{\frac{T}{2}} = 4$, and $\gamma_{\frac{3 \cdot T}{4}} = 6$ for the piece-varying risk aversion case (see Equation (3.3.11)). The results are presented in Figure 3.14.
For the utility results, please refer to Tables 3.C.4 and 3.C.5 in Appendix 3.C for yearly-varying and step-varying risk aversions, respectively.

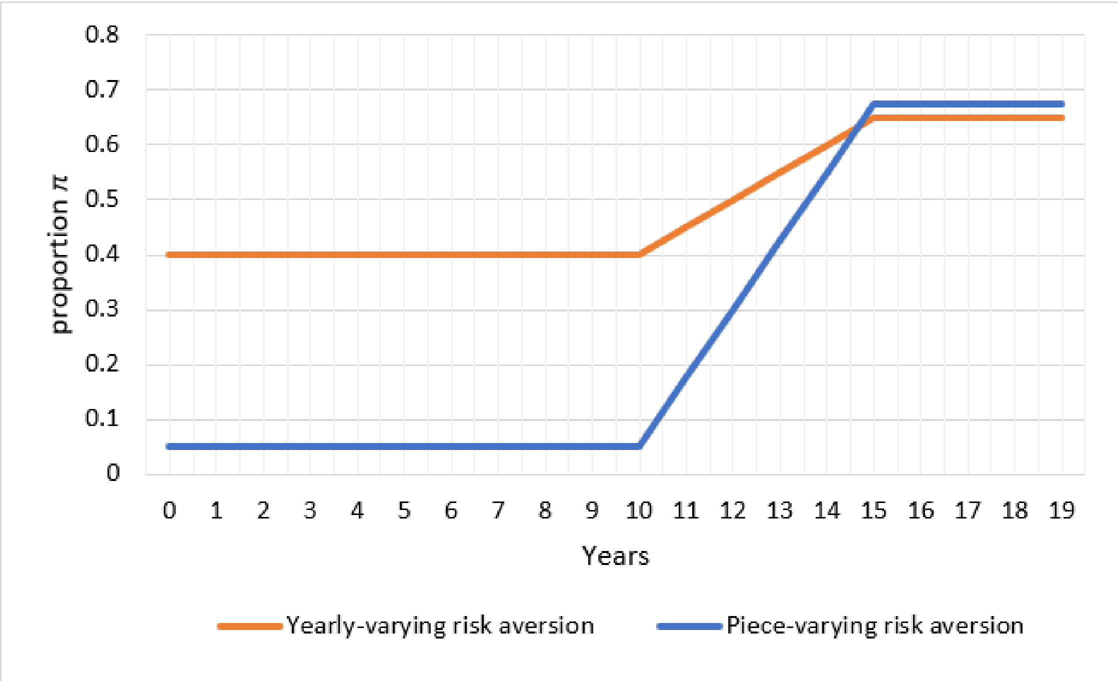

Figure 3.14: Optimal proportion in the participating contract based on a piece-wise linear strategy for different values of risk aversion - Three-path strategy.

It is interesting to point out that the lifestyle strategy concept holds for the time-varying risk aversion case. In fact, we observe in Figure 3.14 that an investor with a risk aversion that changes each year prefers to invest 60% of his initial wealth in the unit-linked contract for the first half of the time, then decreases this allocation linearly to 40% at the third quarter $t = \frac{3 \cdot T}{4} - 1$ and continue with a constant allocation of 35% for the last quarter of the time. On the other hand, an investor with a risk aversion that changes only at $t = \frac{T}{2}$ and $t = \frac{3 \cdot T}{4}$ is satisfied by investing 95% of his initial wealth in the unit-linked contract for the first half of the time, analogously decreasing this allocation linearly to 45% at $t = \frac{3 \cdot T}{4} - 1$, and continuing with a constant allocation of 32% till maturity. In fact, the risk aversion level in the piece-varying risk aversion case is below the risk aversion level in the yearly-varying risk aversion until $t = \frac{3 \cdot T}{4}$, and that's why the optimal strategy in the last quarter is to invest more in the safe contract in the piece-varying risk aversion case.

In practice, considering time-varying risk aversion in designing investment strategies for mixed participating and unit-linked life insurance contracts may better capture the dynamic nature of an individual's risk preferences and lead to more tailored and suitable investment strategies that align with the policyholder's changing needs and risk appetite over the contract duration.

In the linear strategy, as illustrated in Figure 3.10, policyholders with increasing risk aversion tend to adjust their investment strategy over time. Starting with a higher allocation to the risky asset when they are younger and less risk-averse, they gradually reduce their exposure to risk as they approach contract maturity. This behavior aligns with the lifestyle investment strategy concept, indicating that policyholders are willing to take on more risk when they have a longer investment horizon and gradually become more conservative as they age.

The step-wise and piece-wise linear strategies, as depicted in Figures 3.11, 3.13, and 3.14, further emphasize the impact of time-varying risk aversion in shaping the optimal investment strategy. Policyholders are observed to allocate more to the unit-linked contract in the early stages of the contract and gradually decrease this allocation over time. This dynamic risk management approach allows policyholders to tailor their investment strategies to their evolving risk preferences as they progress through different life stages.

In conclusion, the findings suggest that by taking time-varying risk aversion into account in mixed life insurance contracts, the lifestyle investment strategy may be reflected as it takes into account the policyholders' evolving risk preferences over time. Insurers can accommodate clients with changing risk attitudes by including time-varying risk aversion into the design of their investment strategies.

3.4 Conclusion

In this chapter, we have explored the optimal investment strategy within the framework of a mixed insurance contract, focusing on a predetermined deterministic investment style. By combining participating and unit-linked features, mixed contracts offer policyholders a balance between a guarantee and investment growth by deducting a continuous guarantee fee from the unit-linked part to increase, in exchange, the guarantee on the participating part. This work analyzes different families of investment strategies under the lifestyle strategy framework, including linear, step-wise, and piece-wise linear strategies.

In this context, we have considered an annual guarantee contract with rebalancing, striving to create a product that offers advantageous fair guaranteed rates in the participating part, and examining the impact of various investment strategies on the attractiveness of the fair guaranteed rates. We have found that the guaranteed rate increases as the investment in the unit-linked part increases over time, the guarantee fee's value rises, and the participation in the insurer's profits decreases. The incorporation of the guarantee fee into the model significantly enhances the guarantee on the participating part, making it more promising.

Moreover, we have maximized the policyholder's expected utility at the contract's maturity for the different investment strategies proposed while ensuring the contract's fair pricing relation is upheld. The results have clearly indicated that the lifestyle investment strategy does not hold under the classical utility function. Regardless of the chosen strategy—linear, step-wise, or piece-wise linear—policyholders with the same risk aversion coefficient tend to adopt a constant investment strategy throughout the contract's duration. Notably, the lifestyle investment strategy has been highlighted by considering either periodic premium payments or time-varying risk aversion in designing investment strategies for mixed life insurance contracts. As policyholders age and their needs change, the optimal investment strategy can be tailored accordingly, and so the lifestyle strategy is captured. These approaches lead to more personalized investment strategies that align with the policyholder's changing risk appetite over the contract duration. In summary, the way premiums are paid, whether in a single sum or periodically, along with the risk aversion coefficient, whether it is constant or changes over time, significantly influence the optimization of investment strategies for mixed participating and unit-linked life insurance contracts. They shape the policyholder's allocation decisions within various investment strategies.

Appendices

3.A. Classical utility function and single premium

Table 3.A.1 shows the CEQs of different possibilities of investment strategies under the three defined families: linear, step-wise, and piece-wise strategies, given a fair guaranteed rate. These results are based on risk preferences that remain the same throughout the contract period (the case of classical utility function).

		Linear strategy				*Step-wise linear strategy (two-step function with $n = \frac{4}{3}$)*			
			CEQ				*CEQ*		
π_0	π_{T-1}	i (%)	$\gamma = 2$	$\gamma = 4$	$\gamma = 6$	i (%)	$\gamma = 2$	$\gamma = 4$	$\gamma = 6$
0.05	0.05	6.32	2555	1679	1055	6.32	2555	1679	1055
0.10	0.55	1.30	2374	1907	1513	1.98	2429	1787	1219
0.35	0.70	0.70	2195	1961	1752	0.91	2272	1939	1641
0.50	0.50	0.76	2233	1983	1760	0.76	2233	1983	1760
0.50	0.70	0.56	2117	1954	1804	0.65	2171	1963	1773
0.65	0.65	0.48	2060	1936	1821	0.48	2060	1936	1821
0.85	0.35	0.56	2099	1922	1760	0.38	1923	1809	1704
0.90	1.00	0.15	1647	1634	1621	0.17	1682	1666	1650
1.00	1.00	0.11	1574	1564	1554	0.11	1574	1564	1554

		Piece-wise linear strategy			
			CEQ		
π_0	π_{T-1}	i (%)	$\gamma = 2$	$\gamma = 4$	$\gamma = 6$
0.05	0.05	6.32	2555	1679	1055
0.10	0.55	2.58	2484	1790	1301
0.35	0.70	1.04	2327	1957	1651
0.50	0.50	0.76	2233	1983	1760
0.50	0.70	0.70	2203	1977	1778
0.65	0.65	0.48	2060	1936	1821
0.85	0.35	0.30	1862	1798	1736
0.90	1.00	0.18	1702	1685	1668
1.00	1.00	0.11	1574	1564	1554

Table 3.A.1: CEQ with respect to constant risk aversion γ—single premium case.

3.B. Classical utility function and periodic premiums

Table 3.B.1 shows the CEQs of different possibilities of investment strategies under the three defined families: linear, step-wise, and piece-wise strategies, given a fair guaranteed rate. These results are based on a risk aversion γ of 4 that remains the same throughout the contract period (the case of the classical utility function).

		Linear strategy		*Step-wise linear strategy (two-step function with $n = \frac{4}{3}$)*	
π_0	π_{T-1}	i (%)	*CEQ*	i (%)	*CEQ*
0.05	0.05	6.32	1760	6.32	1760
0.05	0.40	1.52	1801	2.13	1798
0.10	0.55	1.04	1790	1.49	1793
0.35	0.70	0.60	1730	0.77	1754
0.50	0.50	0.76	1745	0.76	1745
0.50	0.70	0.51	1704	0.59	1724
0.65	0.65	0.48	1688	0.48	1688
0.85	0.35	0.71	1704	0.49	1623
0.90	1.00	0.13	1493	0.15	1511
1.00	1.00	0.11	1467	0.11	1467

		Piece-wise linear strategy (two-path function with $n = \frac{4}{3}$)	
π_0	π_{T-1}	i (%)	*CEQ*
0.05	0.05	6.32	1760
0.10	0.55	2.11	1791
0.15	0.55	1.79	1792
0.35	0.70	0.95	1770
0.50	0.50	0.76	1745
0.50	0.70	0.67	1736
0.65	0.65	0.48	1688
0.90	1.00	0.17	1526
1.00	1.00	0.11	1467

Table 3.B.1: CEQ with respect to constant risk aversion γ—periodic premiums case.

3.C. Time-varying risk aversion utility function

Table 3.C.4 displays the expected utilities of different possibilities of investment strategies under the three defined families: linear, step-wise, and piece-wise strategies, given a fair guaranteed rate. These results are based on the yearly increasing risk aversion parameters, as presented in Table 3.C.1 (the case of a yearly-varying risk aversion utility function).

Years (t)	0	1	2	...	10	...	18	19	20
Risk aversion (γ_t)	2	2.2	2.4	...	4	...	5.6	5.8	6

Table 3.C.1: Values of risk aversion parameters in a yearly-varying risk aversion case.

Table 3.C.5 displays the expected utilities of different possibilities of investment strategies under the two defined families: step-wise, and piece-wise strategies, given a fair guaranteed rate. These results are based on risk aversion parameters that increase with each step, as presented in Table 3.C.2 for two-step or two-path strategies and Table 3.C.3 for three-step or three-path strategies (the case of step- and piece-varying risk aversion utility function).

Years (t)	$[0, \frac{3 \cdot T}{4} - 1]$	$[\frac{3 \cdot T}{4}, T]$
Risk aversion (γ_t)	2	6

Table 3.C.2: Values of risk aversion parameters in a two-step/path varying risk aversion case.

Years (t)	$[0, \frac{T}{2} - 1]$	$[\frac{T}{2}, \frac{3 \cdot T}{4} - 1]$	$[\frac{3 \cdot T}{4}, T]$
Risk aversion (γ_t)	2	4	6

Table 3.C.3: Values of risk aversion parameters in a three-step/path varying risk aversion case.

Linear strategy			*Step-wise linear strategy*						
			(two-step function with $n = \frac{4}{3}$)			*(three-step function with $n = 2$, $k = 3$ and $l = 4$)*			
π_0	π_{T-1}	*Expected Utility*	π_0	$\pi_{\frac{3 \cdot T}{4}}$	*Expected Utility*	π_0	$\pi_{\frac{T}{2}}$	$\pi_{\frac{3 \cdot T}{4}}$	*Expected Utility*
0.05	0.55	-0.0001326	0.05	0.55	-0.0001695	0.05	0.45	0.65	-0.0001341
0.05	0.65	-0.0001266	0.05	0.65	-0.0001679	0.20	0.35	0.70	-0.0001328
0.20	0.70	-0.0001240	0.20	0.70	-0.0001435	0.35	0.55	0.65	-0.0001253
0.35	0.75	-0.0001269	0.35	0.75	-0.0001326	0.50	0.55	0.65	-0.0001277
0.40	0.60	-0.0001265	0.40	0.60	-0.0001288	0.60	0.60	0.95	-0.0001518
0.45	0.65	-0.0001273	0.45	0.65	-0.0001278	0.65	0.75	0.85	-0.0001483
0.50	0.70	-0.0001300	0.50	0.70	-0.0001288	0.75	0.75	0.50	-0.0001502
0.65	0.80	-0.0001458	0.65	0.80	-0.0001407	0.80	0.30	0.75	-0.0001581
0.80	0.95	-0.0001901	0.80	0.95	-0.0001778	0.95	0.05	0.55	-0.0002107
0.90	1.00	-0.0002256	0.90	1.00	-0.0002129	0.95	0.95	1.00	-0.0002303

Piece-wise linear strategy					
(two-path strategy with $n = \frac{4}{3}$)			*(three-path strategy with $n = 2$, $k = 3$ and $l = 4$)*		
π_0	π_{T-1}	*Expected Utility*	π_0	$\pi_{\frac{3 \cdot T}{4}-1}$	*Expected Utility*
0.05	0.55	-0.0001977	0.05	0.55	-0.0001423
0.05	0.65	-0.0001927	0.05	0.65	-0.0001461
0.20	0.70	-0.0001535	0.20	0.70	-0.0001370
0.35	0.75	-0.0001339	0.35	0.75	-0.0001352
0.40	0.60	-0.0001320	0.40	0.60	-0.0001261
0.45	0.65	-0.0001293	0.45	0.65	-0.0001273
0.50	0.70	-0.0001286	0.50	0.70	-0.0001305
0.65	0.80	-0.0001373	0.65	0.80	-0.0001463
0.80	0.95	-0.0001683	0.80	0.95	-0.0001926
0.90	1.00	-0.0002033	0.90	1.00	-0.0002274

Table 3.C.4: Power utility with respect to yearly-varying risk aversion.

Step-wise linear strategy							Piece-wise linear strategy					
(two-step function with $n = \frac{4}{3}$)			*(three-step function with $n = 2$, $k = 3$ and $l = 4$)*				*(two-path strategy with $n = \frac{4}{3}$)*			*(three-path strategy with $n = 2$, $k = 3$ and $l = 4$)*		
π_0	$\pi_{\frac{3\cdot T}{4}}$	*Expected Utility*	π_0	$\pi_{\frac{T}{2}}$	$\pi_{\frac{3\cdot T}{4}}$	*Expected Utility*	π_0	π_{T-1}	*Expected Utility*	π_0	$\pi_{\frac{3\cdot T}{4}-1}$	*Expected Utility*
0.05	0.55	-0.00007363	0.05	0.45	0.65	-0.0001695	0.05	0.55	-0.0002817	0.05	0.55	-0.0001761
0.05	0.65	-0.00007237	0.20	0.35	0.70	-0.0001770	0.05	0.65	-0.0002726	0.05	0.65	-0.0001820
0.20	0.70	-0.00007701	0.35	0.55	0.65	-0.0001812	0.20	0.70	-0.0002617	0.20	0.70	-0.0001866
0.35	0.75	-0.00008177	0.50	0.55	0.65	-0.0001884	0.35	0.75	-0.0002557	0.35	0.75	-0.0001947
0.40	0.60	-0.00008280	0.60	0.60	0.95	-0.0002244	0.40	0.60	-0.0002599	0.40	0.60	-0.0001836
0.45	0.65	-0.00008382	0.65	0.75	0.85	-0.0002184	0.45	0.65	-0.0002588	0.45	0.65	-0.0001869
0.50	0.70	-0.00008565	0.75	0.75	0.50	-0.0002224	0.50	0.70	-0.0002602	0.50	0.70	-0.0001926
0.65	0.80	-0.00009427	0.80	0.30	0.75	-0.0002158	0.65	0.80	-0.0002772	0.65	0.80	-0.0002154
0.80	0.95	-0.00011374	0.95	0.05	0.55	-0.0002576	0.80	0.95	-0.0003218	0.80	0.95	-0.0002753
0.90	1.00	-0.00012768	0.95	0.95	1.00	-0.0003104	0.90	1.00	-0.0003639	0.90	1.00	-0.0003131

Table 3.C.5: Power utility with respect to step- and piece-varying risk aversions.

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Chapter 4

Valuation of mixed life insurance contracts under stochastic correlated mortality and interest rates[1]

4.1 Introduction

It is well known that the life insurance sector is facing a major challenge in combining a reasonable balance between the desire for potential returns and the ability to generate long-term guarantees in this difficult financial environment. Various insurance markets tend to remove all forms of guarantees and move to new types of life insurance products with no investment guarantees, which effectively reduce the insurer's risk and transfer it to the policyholder. In general, the traditional form of insurance products, so-called participating life insurance contracts, in which the insurer guarantees a return with a possible bonus depending on the investment performance, have been replaced by so-called unit-linked life insurance contracts that allow the client to opt for investment funds with no guaranteed return, according to an investment strategy adapted to his risk profile. A main difference between the two products is the investment strategy, as no financial decision can be made by the customer in the traditional product because the participating contract fund is internal and controlled by the insurer, whereas the unit-linked fund is based on the investor's choice. Nonetheless, the customer's interest in downside protection is highly present, and removing guarantees entirely is not an appealing trend from the client's perspective, as many empirical and experimental studies have shown, see, e.g., Gatzert et al. [17] and Ruß & Schelling [27]. Based on behavioral models, products with guarantees are a preferred choice for investor (Døskeland & Nordahl [15]). Therefore, the development of attractive life and pension insurance product designs has become necessary but, at the same time, very difficult due to demographic changes and volatile equity and interest rate markets.

In this context, hybrid products can be a good solution for both the insurance industry and the customer. For the insurer, it is less risky than the participating product, for which the insurance company bears the entire risk, and for the customer, it is more attractive than the pure unit-linked product that does not offer any investment guarantees (Reuß et al. [26]). In particular, dynamic hybrid products are innovative life insurance products that provide guarantees through a periodic rebalancing process between the policy reserves, a guarantee fund, and an equity fund. Bohnert et al. [7] shows that in the case of dynamic hybrid contracts, a minimum guarantee level should be provided to ensure that the willingness-to-pay is greater than the minimum premium that the insurer must charge to sell the contract. Moreover, a study was conducted by Bohnert & Gatzert [8] to examine the fair valuation and risk condition of an insurer that offers both dynamic hybrid and traditional participating life insurance contracts. The results highlight the benefits and risks of offering dynamic hybrids by

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revealing significant interaction effects between the two contract types within the portfolio that are highly dependent on the portfolio composition. So, hybrid life insurance products are designed to fulfill customers' demands for guaranteed stability as well as adequate upside potential, and they have risen in popularity since their introduction to the German market by a number of life insurance companies. For example, there is the introduction of "Select Products" in the German market as a version of deferred equity-indexed annuity contracts that replaced the participating life insurance in 2013 (Alexandrova et al. [1]) or the development of participating products with alternative forms of guarantees (Reuß et al. [25]). Let us remark that even if these mixed contracts have appeared in many markets during a period of low interest rates, their design remains relevant in other circumstances as a natural way to incorporate sound principles of diversification inside a life insurance product.

In this chapter, we consider mixed life insurance contracts as proposed by Hanna et al. [18]. These products combine features of participating and pure unit-linked life insurance contracts to achieve a compromise between a guarantee from the participating part and no guarantee from the pure unit-linked part, where a constant fee is extracted from the pure unit-linked account value to finance the guaranteed rates in the participating contract with a maturity or annual guarantee. We emphasize that the pure unit-linked product considered in this chapter has no maturity guarantee and is referred to hereafter as a "unit-linked product". In Hanna et al. [18], these products were studied without mortality (pure financial products) and in a classical Black-Scholes environment with a constant risk-free rate. The purpose of this chapter is to extend this model in various directions: including the mortality component, and considering different products (pure endowment for the entire contract or combination of a pure endowment and a pure investment), modeling the movement and evolution of interest rates by allowing for stochastic interest rates using the Hull and White model, and finally studying the correlation between all the risk factors, including correlations between financial and actuarial risks.

Indeed, following the improvement in health sciences, human life expectancy has increased significantly, which puts both the life actuary and the elderly under considerable stress. Apart from the financial risk that notably affects life insurance contracts, mortality risk is an important factor that has been accounted for in the pricing and valuation of life and pension contracts for a long time now. For instance, Bacinello [4] prices a classical life insurance policy given a deterministic life table and a constant interest rate for a single premium and a sequence of periodical premium cases. Ultimately, deterministic mortality does not really change the structure of the fairness relation of the contract. So, mortality risk was typically modelled deterministically, but stochastic mortality models later showed up and are widely accepted as significant improvements and uncertainty have been detected in mortality trends. In addition, various papers admit that interest and mortality rates share a lot in common under the modeling framework (see, e.g., Milevsky & Promislow [24], Biffis [6] and Cairns et al. [10]). As a result, we can model the force of mortality for mortality risk in the same way that we model the short term as the interest rate term structure. In this regard, Zeddouk & Devolder [28] prove that considering a continuous-time mean-reverting model with a time-dependent target is suitable to model mortality. Accordingly, a paper held by Devolder & Azizieh [13] valued a life insurance product with a surplus return based on the insurer's overall profit, in which financial and longevity risks are modelled in a stochastic framework, assuming independence between these two components.

Generally, mortality and finance are separated in life insurance pricing, but this assumption of independence is no longer acceptable in the long term, and it was further strengthened after the recent COVID-19 pandemic that intuitively has an influence on the financial markets. A number of studies show that changes in demographic trends can affect the value of financial assets. For instance, Dacorogna & Cadena [11] study the relationship between mortality and financial risks in different countries. They show a weak effect on the financial markets by taking the full sample into account and a higher effect by taking the 10 worst years of the changes in mortality. Moreover, they highlight that extreme downturns in the mortality indices increase the size of this dependence. Analogously, Jalen & Mamon [20] cover the dependency between interest and mortality rates over the long term and note the impact of a catastrophic event on interest rates as well as population size in the short term. According to Maurer [23], most of the long-term time series variation in interest rates and stock market returns can be attributed to demographic shifts. He provides enough conditions for the interest rate to drop in the birth rate and rise in the death rate. In addition, Dhaene et al. [14] state that an independence assumption between financial and actuarial risks in the pricing world may be convenient but challenging to comprehend intuitively, and they show that it is often not maintained in the risk-neutral world.

In this chapter, we move from a pure financial contract in Hanna et al. [18] to a real life insurance contract with a mortality component. We consider stochastic models for mortality and interest rates, and we introduce some form of correlation between mortality evolution and the financial markets to obtain a whole view of the risks of an insurance contract. The aim of the chapter is to price and evaluate different contracts in this stochastic environment, assuming a dependence between financial and demographic risk factors. Some papers have already studied other products and have also tried to introduce the idea of correlation. Our study follows the classical method used by these papers, which is the change of measure technique (also known as the change of numéraire), as it leads to simple formulas in the pricing of insurance contracts under the dependence assumption (see for example, Jalen & Mamon [20], Dhaene et al. [14], Liu et al. [21], Gao et al. [16], Deelstra et al. [12], Zhao & Mamon [29] and Arık et al. [2]). Accordingly, we obtain an explicit form of the fair valuation for the maturity guarantee contract and analyze the impact of the correlation on the fair guaranteed rates. Our results emphasize the advantage of these mixed contracts, in particular their ability to offer better guaranteed rates. We also show that the introduction of correlations can have various impacts on the price and the level of a fair guarantee. Furthermore, we point out that the impact of the correlation parameters on the fair guaranteed rates is affected by the level of volatility in the model.

The remainder of this chapter is organized as follows: The financial market and the stochastic longevity model are introduced in Section 4.2. More precisely, the well-known Hull and White model is used to model the evolution of both mortality and interest rate processes. Furthermore, the correlation parameters between the interest rate, stock returns, and mortality are defined. The structure of the hybrid product is described in Section 4.3. In Section 4.4, we introduce the change of measure technique to compute fair values and obtain an explicit form of the fairness relation of a maturity guarantee contract. Section 4.5 deals with other kinds of contracts and particularly the annual guarantee contract. Section 4.6 illustrates a numerical example for the different products examined in Sections 4.4 and 4.5. Finally, we conclude in Section 4.7.

4.2 Financial and mortality assumptions

Life insurance policies, as long-term financial products, are particularly exposed to a variety of risks. The main risks that these contracts face are those linked to uncertain evolution in financial markets, such as market risk and interest rate risk, characterized recently by volatile stock and interest rate markets, as well as those associated with uncertain evolution in longevity, called mortality risk, which appears with the rapid ageing of the population. In order to model these risks, we allow ourselves to introduce the various stochastic assumptions of the financial market model and the longevity model. Furthermore, we attempt to represent the correlations between the assets defined in the market to explicitly study the dependence between financial and mortality risks.

We work on a finite time interval $[0, T]$, and we consider a global filtered probability space $(\Omega, \mathcal{M}, (\mathcal{M}_t)_t)$ to reflect all the possible financial and mortality events. We introduce two sub-filtrations of $(\mathcal{M}_t)_{t\in[0,T]}$: the financial filtration $(\mathcal{F}_t)_t$ generated by the information on the financial market and the actuarial filtration $(\mathcal{A}_t)_t$ generated by the survival information. Therefore, $\mathcal{M}_t := \mathcal{F}_t \vee \mathcal{A}_t$ is the sigma-algebra generated by $\mathcal{F}_t$ and $\mathcal{A}_t$ on the global space Ω. In this space, we introduce a probability measure $\mathbb{Q}$, which is our global valuation probability. Referring to Artzner et al. [3], this probability when restricted to the financial filtration $\mathcal{F}$, corresponds to the financial risk-neutral measure $\mathbb{Q}^f$. More precisely, $\mathbb{Q}^f$ is obtained when applying to events that are just generated by the financial filtration.

Therefore, the valuation in this chapter will be computed using this risk-neutral probability measure $\mathbb{Q}$.

4.2.1 The financial market

We assume a complete financial market (existence and uniqueness of $\mathbb{Q}^f$) on which the insurer trades three basic processes: the term structure of interest rates and two risky assets, one of which is more volatile than the other. In the following, we display the financial assumptions related to each process.

1. The term structure of interest rates evolves according to the one-factor Hull and White model [19]. Then, the risk-free rate r_t is a solution of the following stochastic equation:

$$\mathrm{d}r_t = \theta\,(\nu_t - r_t)\,\mathrm{d}t \,+\, \sigma_r\,\mathrm{d}W_t^r, \quad t \in [0,T] \text{ and } T \in \mathbb{N}, \tag{4.2.1}$$

where W^r is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^f$. θ and σ_r are positive constants, and ν_t is a deterministic function chosen to ensure the fit of the observed yields and expressed as $\nu_t = \frac{1}{\theta}\cdot\frac{\partial f(0,t)}{\partial t} + f(0,t) + \frac{\sigma_r^2}{2\theta^2}(1-e^{-2\theta t})$ with $f(0,t) = -\frac{\partial \ln P(0,t)}{\partial t}$, the instantaneous forward rate. ν_t is the mean reversion level toward which r_t converges, and θ is the speed of mean reversion.

In an arbitrage free environment, the price of a zero-coupon bond at time t with maturity T is defined by the following expression:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}^f}\left[e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t\right].$$

This price has an explicit form under Hull and White and is given by:

$$P(t,T) = \exp\left(-r_t\cdot B_r(t,T,\theta) - \theta\int_t^T \nu_s\cdot B_r(s,T,\theta)ds + \frac{\sigma_r^2}{2}\int_t^T B_r^2(s,T,\theta)ds\right), \tag{4.2.2}$$

where $B_r(t,T,\theta) = \frac{1-e^{-\theta(T-t)}}{\theta}$, and using the expression of ν_t, the explicit form of the future price $P(t,T)$ becomes:

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\cdot\exp\left(-r_t\cdot B_r(t,T,\theta) + B_r(t,T,\theta)\cdot f(0,t) - \frac{\sigma_r^2}{2}\cdot B_r(0,t,2\theta)\cdot B_r^2(t,T,\theta)\right).$$

The risk-neutral dynamic of $P(t,T)$ is given by:

$$\mathrm{d}P(t,T) = P(t,T)\,[r_t\,\mathrm{d}t - \sigma_r\,B_r(t,T,\theta)\,\mathrm{d}W_t^r].$$

2. Two risky funds, G and H, where G is a fund with low volatility and H is a fund with high volatility. Their dynamics are described as follows:

$$\mathrm{d}G_t = r_t\,G_t\,\mathrm{d}t \,+\, \sigma_G\,G_t\,\mathrm{d}W_t^G, \quad G_0 = 1,\, t\in[0,T] \tag{4.2.3}$$

$$\mathrm{d}H_t = r_t\,H_t\,\mathrm{d}t \,+\, \sigma_H\,H_t\,\mathrm{d}W_t^H, \quad H_0 = 1,\, t\in[0,T]. \tag{4.2.4}$$

The two funds follow a geometric Brownian motion where W^G and W^H are standard Brownian motions under the risk-neutral measure $\mathbb{Q}^f$. We state that the fund G is the insurer's general fund used as a reference for the participating contract, while the fund H is an investment fund used to compute the value of the unit-linked contract. We assume then that $\sigma_H > \sigma_G > 0$.

Recalling Equations (4.2.1), (4.2.3) and (4.2.4), we note that W^r, W^G and W^H are three correlated Brownian motions. We define $\rho_{r,G}$ as the correlation parameter that links the asset G with the interest rate, $\rho_{r,H}$ as the correlation parameter that links the asset H with the interest rate, and $\rho_{G,H}$ as the correlation parameter between the two assets, such that

$\mathbb{E}^{\mathbb{Q}^f}[\mathrm{d}W_t^r \mathrm{d}W_t^G] = \rho_{r,G} \cdot \mathrm{d}t$, $\mathbb{E}^{\mathbb{Q}^f}[\mathrm{d}W_t^r \mathrm{d}W_t^H] = \rho_{r,H} \cdot \mathrm{d}t$ and $\mathbb{E}^{\mathbb{Q}^f}[\mathrm{d}W_t^G \mathrm{d}W_t^H] = \rho_{G,H} \cdot \mathrm{d}t$.

The aforementioned stochastic differential equations (SDEs) are written as follows using the Cholesky decomposition[2] for correlated Brownian motions:

$$\mathrm{d}G_t = r_t\, G_t\, \mathrm{d}t \;+\; \sigma_G\, \rho_{r,G}\, G_t\, \mathrm{d}W_t^r \;+\; \sigma_G \sqrt{1-\rho_{r,G}^2}\, G_t\, \mathrm{d}\overline{W}_t^G, \quad G_0 = 1,\; t \in [0,T] \tag{4.2.5}$$

$$\begin{aligned} \mathrm{d}H_t = r_t\, H_t\, \mathrm{d}t \;+\; \sigma_H\, \rho_{r,H}\, H_t\, \mathrm{d}W_t^r \;+\; \sigma_H\, \overline{\rho}_{G,H} \sqrt{1-\rho_{r,H}^2}\, H_t\, \mathrm{d}\overline{W}_t^G \\ +\; \sigma_H \sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)}\, H_t\, \mathrm{d}\overline{\overline{W}}_t^H, \quad H_0 = 1,\; t \in [0,T], \end{aligned} \tag{4.2.6}$$

with

$$W_t^G = \rho_{r,G} \cdot W_t^r + \sqrt{1-\rho_{r,G}^2} \cdot \overline{W}_t^G, \tag{4.2.7}$$

$$W_t^H = \rho_{r,H} \cdot W_t^r + \overline{\rho}_{G,H} \cdot \sqrt{1-\rho_{r,H}^2} \cdot \overline{W}_t^G + \sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)} \cdot \overline{\overline{W}}_t^H \tag{4.2.8}$$

and

$$\overline{\rho}_{G,H} = \frac{\rho_{G,H} - \rho_{r,G} \cdot \rho_{r,H}}{\sqrt{(1-\rho_{r,G}^2)(1-\rho_{r,H}^2)}},$$

where W^r, $\overline{W}^G$ and $\overline{\overline{W}}^H$ are three uncorrelated Brownian motions.
We further exhibit the annual returns of G and H, respectively:

$$g_t = \frac{G_t}{G_{t-1}} - 1 \quad \text{and} \quad h_t = \frac{H_t}{H_{t-1}} - 1, \qquad t = 1, 2, \ldots, T, \tag{4.2.9}$$

where the solutions of the SDEs (4.2.5) and (4.2.6), under the risk-neutral measure, are given by:

$$G_t = G_0 \exp\left(\int_0^t r_s ds - \frac{\sigma_G^2}{2} \cdot t \;+\; \sigma_G\, \rho_{r,G} \int_0^t \mathrm{d}W_s^r + \sigma_G \sqrt{1-\rho_{r,G}^2} \int_0^t \mathrm{d}\overline{W}_s^G \right)$$

$$\begin{aligned} H_t = H_0 \exp\Bigg(\int_0^t r_s ds - \frac{\sigma_H^2}{2} \cdot t \;+\; \sigma_H\, \rho_{r,H} \int_0^t \mathrm{d}W_s^r + \sigma_H\, \overline{\rho}_{G,H} \sqrt{1-\rho_{r,H}^2} \int_0^t \mathrm{d}\overline{W}_s^G \\ +\; \sigma_H \sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)} \int_0^t \mathrm{d}\overline{\overline{W}}_s^H \Bigg). \end{aligned}$$

[2]We assume that the correlation matrix is positive-definite. Moreover, other decomposition methods can be applied, but we used Cholesky for computational purposes and because it is efficient when dealing with Gaussian random variables.

4.2.2 The stochastic longevity model

In addition to the financial risk, we include in this chapter the mortality risk, which is still a significant concern for life actuaries. According to Devolder & Azizieh [13], there are numerous criteria to consider when selecting a stochastic mortality model. We will use the Hull and White model, which is an appropriate mortality model. This is a continuous-time model using mean-reverting with a floating target, as suggested by Zeddouk & Devolder [28]. The main benefit is that this model provides for a complete and elegant equivalence between interest rate and mortality rate.

As a result, we introduce a new process μ_x known as the stochastic mortality intensity for an individual aged x at time 0 (from a cohort perspective). This process is assumed to be a solution of an affine equation under the risk-neutral measure $\mathbb{Q}$. So, the intensity of mortality $\mu_x(t)$ follows a stochastic process solution of the following equation:

$$\mathrm{d}\mu_x(t) = b\,(\xi_t - \mu_x(t))\,\mathrm{d}t \,+\, \sigma_\mu\,\mathrm{d}W_t^\mu, \quad t \in [0,T], \tag{4.2.10}$$

where W^μ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. b and σ_μ are positive constants, and ξ_t is a deterministic function. x is the initial age of the policyholder.

This stochastic intensity of mortality $\mu_x(t)$ generates a stochastic probability of survival process, denoted $p_x(t)$, that is the solution of a stochastic differential equation of the form:

$$\mathrm{d}p_x(t,T) = p_x(t,T)\,[\mu_x(t)\,\mathrm{d}t - \sigma_\mu\,B_\mu(t,T,b)\,\mathrm{d}W_t^\mu],$$

where $B_\mu(t,T,b) = \frac{1-e^{-b(T-t)}}{b}$.

As well, the survival probability at time t, under the risk-neutral measure $\mathbb{Q}$, is written as

$$p_x(t,T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T \mu_x(s)ds}\,|\,\mathcal{A}_t], \tag{4.2.11}$$

with an explicit formula under Hull and White:

$$p_x(t,T) = \exp\left(-\,\mu_x(t)\cdot B_\mu(t,T,b) - b\int_t^T \xi_s\cdot B_\mu(s,T,b)ds + \frac{\sigma_\mu^2}{2}\int_t^T B_\mu^2(s,T,b)ds\right). \tag{4.2.12}$$

As pointed out by Luciano & Vigna [22], the probability of negative values of μ_x turns out to be negligible with the calibrated parameters.

In recent decades, following many natural disasters, particularly the recent COVID-19 pandemic, it is illogical to ignore how their occurrence affects demographic tables as well as financial markets and, accordingly, the correlation between changes in demographic trends and the value of financial assets. This dependence is captured by the correlation parameters $\rho_{r,\mu}$, $\rho_{G,\mu}$ and $\rho_{H,\mu}$ such that

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{d}W_t^r\mathrm{d}W_t^\mu] = \rho_{r,\mu}\cdot\mathrm{d}t, \quad \mathbb{E}^{\mathbb{Q}}[\mathrm{d}W_t^G\mathrm{d}W_t^\mu] = \rho_{G,\mu}\cdot\mathrm{d}t \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}}[\mathrm{d}W_t^H\mathrm{d}W_t^\mu] = \rho_{H,\mu}\cdot\mathrm{d}t.$$

Using Cholesky decomposition[3], one can write

$$W_t^\mu = \rho_{r,\mu}\cdot W_t^r + \sqrt{1-\rho_{r,\mu}^2}\cdot\overline{W}_t^\mu, \tag{4.2.13}$$

Recalling Equations (4.2.7) and (4.2.13), the correlation between $\overline{W}_t^G$ and $\overline{W}_t^\mu$ is determined by $\overline{\rho}_{G,\mu}$ as follows

$$\overline{\rho}_{G,\mu} = \frac{\rho_{G,\mu} - \rho_{r,G}\cdot\rho_{r,\mu}}{\sqrt{(1-\rho_{r,G}^2)(1-\rho_{r,\mu}^2)}}.$$

[3]We assume that the correlation matrix is positive-definite.

So, Equation (4.2.13) is expressed as

$$W_t^\mu = \rho_{r,\mu} \cdot W_t^r + \overline{\rho}_{G,\mu} \cdot \sqrt{1-\rho_{r,\mu}^2} \cdot \overline{W}_t^G + \sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)} \cdot \overline{\overline{W}}_t^\mu, \tag{4.2.14}$$

where W_t^r, $\overline{W}_t^G$ and $\overline{\overline{W}}_t^\mu$ are uncorrelated Brownian motions.

Again, recalling Equations (4.2.8) and (4.2.14), the correlation between $\overline{\overline{W}}_t^H$ and $\overline{\overline{W}}_t^\mu$ is determined by $\overline{\overline{\rho}}_{H,\mu}$ as follows

$$\overline{\overline{\rho}}_{H,\mu} = \frac{\rho_{H,\mu} - \rho_{r,H} \cdot \rho_{r,\mu} - \overline{\rho}_{G,H} \cdot \overline{\rho}_{G,\mu}\sqrt{(1-\rho_{r,H}^2)(1-\rho_{r,\mu}^2)}}{\sqrt{(1-\rho_{r,H}^2)(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,H}^2)(1-\overline{\rho}_{G,\mu}^2)}}.$$

Consequently, we write Equation (4.2.14) as follows

$$\begin{aligned} W_t^\mu = \rho_{r,\mu} \cdot W_t^r &+ \overline{\rho}_{G,\mu} \cdot \sqrt{1-\rho_{r,\mu}^2} \cdot \overline{W}_t^G + \overline{\overline{\rho}}_{H,\mu} \cdot \sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)} \cdot \overline{\overline{W}}_t^H \\ &+ \sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)} \cdot \overline{\overline{W}}_t^\mu . \end{aligned} \tag{4.2.15}$$

Eventually, we exhibit the dynamic of the intensity of the mortality $\mu_x(t)$, under $\mathbb{Q}$, with independent Brownian motions:

$$\begin{aligned} \mathrm{d}\mu_x(t) = b\,(\xi_t - \mu_x(t))\,\mathrm{d}t &+ \sigma_\mu\,\rho_{r,\mu}\,\mathrm{d}W_t^r + \sigma_\mu\,\overline{\rho}_{G,\mu}\sqrt{1-\rho_{r,\mu}^2}\,\mathrm{d}\overline{W}_t^G \\ &+ \sigma_\mu\,\overline{\overline{\rho}}_{H,\mu} \cdot \sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)}\,\mathrm{d}\overline{\overline{W}}_t^H + \sigma_\mu\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)}\,\mathrm{d}\overline{\overline{W}}_t^\mu, \end{aligned}$$

where W_t^r is the noise associated with the interest rate, $\overline{W}_t^G$ is the specific risk associated with the insurer's general fund that is independent of the interest rate, $\overline{\overline{W}}_t^H$ is the risk of the investment fund that is not correlated with the interest rate and with the general fund, and finally, $\overline{\overline{W}}_t^\mu$ is the specific noise associated with mortality that is not correlated with the finance.

4.3 Design of the basic contract

We extend the model of Hanna et al. [18] to include stochastic interest rates and mortality risks. As a result, we have chosen a pure endowment life insurance contract with a single premium P paid at inception as the basic contract in order to examine the effect of mortality on the entire contract; However, we will look at other alternative forms of contracts in Section 4.5.

In a pure endowment type of policy, the insurer commits to pay the policyholder the sum assured if he survives to the end of the policy term. The sum assured is paid in a single instalment as a lump sum at the policy's maturity $T > 0$.

The insurance company invests the customers' contributions in a mixed contract such that a part $\pi \in]0,1]$ of the premium is invested in a participating life insurance contract with participation in the financial profits on the investments of the company, and the remaining part $1-\pi$ is invested in a pure unit-linked contract. In the participating part, we look first at the guarantee at maturity, so the basic contract that we study in Section 4.4 is a maturity guarantee contract. An annual guarantee contract will be considered as a generalization in Section 4.5.

The structure of the mixed contract as well as the mentioned sub-contracts are discussed in detail in Hanna et al. [18]. Shortly, the participating contract invests in the fund G defined in Section 4.2.1, and as is known, this type of contract guarantees the protection of the asset and provides the insured with a surplus return on top of the guaranteed benefit. So, the total return on the participating fund is the guaranteed interest rate in addition to a surplus return.

On the other side, the pure unit-linked contract invests in the investment fund H without any guarantee and is subject to a continuous guarantee fee $\varepsilon \in \mathbb{R}^+$ to form a new process F such that $F_t := e^{-\varepsilon t} \cdot H_t$ and given by:

$$\begin{aligned} \mathrm{d}F_t = (r_t - \varepsilon)\, F_t\, \mathrm{d}t \,+\, \sigma_H\, \rho_{r,H}\, F_t\, \mathrm{d}W_t^r \,+\, \sigma_H\, \overline{\rho}_{G,H} \sqrt{1-\rho_{r,H}^2}\, F_t\, \mathrm{d}\overline{W}_t^G \\ +\, \sigma_H \sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)}\, F_t\, \mathrm{d}\overline{\overline{W}}_t^H, \quad F_0 = 1,\ \varepsilon \geq 0,\ t \in [0,T]. \end{aligned}$$

We focus in the following on the fair valuation of the mixed contract, where the guarantee is given only at maturity and so the financial profit, under a stochastic interest rate and a stochastic mortality.

In order to price and evaluate the contract, the insurance company relies on two basis. The first-order basis is the basis for the pricing and computation of the premium. More precisely, this is what the insurer guarantees to the client: an interest rate i and a life table ${}_Tp_x$. However, for the provision, the insurer uses a second-order basis to compute the real valuation and check if the contract is fair across all the parameters. These two basis are discussed further below, where we link the first-order and second-order basis to the retrospective and prospective reserves, respectively.

1. The retrospective reserve that corresponds to the initial reserve (which is the initial premium) is capitalized to the maturity of the policy. The capitalization is based on the total returns of the participating and risky funds.
   As a result, the initial retrospective reserve is equal to the premium, so $V_0 = P$, and the final retrospective reserve is equal to the benefit at maturity and is written as follows:

$$V_T = \frac{P}{{}_Tp_x} \cdot \left[\pi \cdot \left((1+i)^T + \beta \cdot \left(\frac{G_T}{G_0} - \left(1 + \frac{(1+i)^T - 1}{\beta} \right) \right)^+ \right) + (1-\pi) \cdot \frac{F_T}{F_0} \right], \tag{4.3.1}$$

   where i is the annual guaranteed interest rate, $\beta \in [0,1]$ is the participation level, assumed to be constant over time, and ${}_Tp_x$ is the probability, knowing that one is alive at age x, of still being alive after T years. It is a tariff rate that arises from a deterministic life table (first-order basis). This probability is used only for pricing, to somehow share the part of the people dying with the others surviving until maturity[4]. We point out that no payments, apart from the premium, occur until the maturity of the contract.
   In summary, Equation (4.3.1) represents the final reserve, which is the initial premium invested in the different financial instruments, taking into account the initial guaranteed probability of survival.

2. The prospective reserve, which is the fair valuation price, is equal to the risk-neutral expectation of the discounted final retrospective reserve, taking into account the real mortality. So, the fair value at t is written as:

$$FV_t \;=\; \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_s \mathrm{d}s} \cdot e^{-\int_t^T \mu_x(s) \mathrm{d}s} \cdot V_T \,|\, \mathcal{M}_t \right].$$

So, the contract is fairly priced if

$$FV_0 \;=\; P \;=\; V_0.$$

[4] ${}_Tp_x$ is a tariff basis that does not necessarily correspond to the real best estimate of mortality defined by $p_x(0,T)$ in Equation (4.2.11) for $t = 0$.

4.4 Valuation of the basic contract

As previously mentioned, the dependence between the financial markets and demographic trends is a very important aspect to examine nowadays. According to several papers (see, e.g., Devolder & Azizieh [13] and Bernard et al. [5]), the intuitive way to price life insurance contracts in a stochastic environment is to use the change of measure technique, as it is consistent with finance. The advantage of this technique in our framework is that different changes of numéraire can be applied.

To price the basic contract, we split Equation (4.3.1) into three components as follows:

$$V_T = \frac{P}{{}_T p_x} \cdot \Big[C_i(T) + C_G(T) + C_F(T) \Big],$$

where,

$$\begin{aligned} C_i(T) &= \pi \cdot (1+i)^T \text{ is the guaranteed part,} \\ C_G(T) &= \pi \cdot \beta \cdot \left(\frac{G_T}{G_0} - K \right)^+ \text{ is the profit sharing part,} \\ C_F(T) &= (1-\pi) \cdot \frac{F_T}{F_0} \text{ is the unit-linked part;} \end{aligned}$$

with $K = 1 + \dfrac{(1+i)^T - 1}{\beta}$.

The valuation of these three components is given by these propositions:

Proposition 4.1. *The time 0 price of $C_i(T)$ is derived as follows*

$$C_i(0) = \pi \cdot (1+i)^T \cdot P(0,T) \cdot p_x(0,T) \cdot \rho(0,T),$$

where,

$$\rho(0,T) = \exp\left(\frac{\sigma_r\, \sigma_\mu\, \rho_{r,\mu}}{\theta \cdot b} \left[T + B_{r,\mu}(0,T,\theta+b) - B_r(0,T,\theta) - B_\mu(0,T,b) \right] \right) \tag{4.4.1}$$

is the term of correlation that arises only from the dependence between the interest rate and the mortality rate $\rho_{r,\mu}$,

with $B_{r,\mu}(t,T,\theta+b) = \dfrac{1 - e^{-(\theta+b)(T-t)}}{\theta+b}$.

Proposition 4.2. *The time 0 price of $C_G(T)$ is given by the following form*

$$C_G(0) = \pi \cdot \beta \cdot p_x(0,T) \cdot \rho(0,T) \cdot c_0^T,$$

where,

$$c_0^T = P(0,T) \cdot \left[e^{\hat{m}_G + \frac{\hat{v}_G}{2}} \cdot \Phi\Big(\frac{\hat{m}_G + \hat{v}_G - \ln K}{\sqrt{\hat{v}_G}} \Big) - K \cdot \Phi\Big(\frac{\hat{m}_G - \ln K}{\sqrt{\hat{v}_G}} \Big) \right].$$

The expectation $\hat{m}_G$ and the variance $\hat{v}_G$ are given by:

$$\begin{aligned} \hat{m}_G = E\left[\ln\left(\frac{G_T}{G_0} \right) \right] = \ln\left(\frac{1}{P(0,T)} \right) &- \frac{\sigma_G^2}{2} \cdot T - \frac{\sigma_G\, \sigma_r\, \rho_{r,G}}{\theta} \left[T - B_r(0,T,\theta) \right] \\ &- \frac{\sigma_G\, \sigma_\mu\, \rho_{G,\mu}}{b} \left[T - B_\mu(0,T,b) \right] \\ &- \frac{\sigma_r\, \sigma_\mu\, \rho_{r,\mu}}{\theta \cdot b} \left[T + B_{r,\mu}(0,T,\theta+b) - B_r(0,T,\theta) - B_\mu(0,T,b) \right] \\ &- \frac{\sigma_r^2}{2\theta^2} \left[T + B_r(0,T,2\theta) - 2\, B_r(0,T,\theta) \right] \end{aligned} \tag{4.4.2}$$

$$\hat{v}_G = V\left[\ln\left(\frac{G_T}{G_0}\right)\right] = \sigma_G^2 \cdot T + 2 \cdot \frac{\sigma_G\,\sigma_r\,\rho_{r,G}}{\theta}\,[T - B_r(0,T,\theta)] + \frac{\sigma_r^2}{\theta^2}\,[T + B_r(0,T,2\theta) - 2\,B_r(0,T,\theta)]. \tag{4.4.3}$$

We note that c_0^T is implicitly affected by $\rho_{r,\mu}$ and $\rho_{G,\mu}$, the correlations between the interest and the mortality rates and between the fund G and the mortality rate, through the expectation term $\hat{m}_G$.

Proposition 4.3. *The time 0 price of $C_F(T)$ is obtained as*

$$C_F(0) = (1-\pi)\cdot p_x(0,T)\cdot \rho^{'}(0,T)\cdot e^{-\varepsilon\cdot T},$$

where,

$$\rho^{'}(0,T) = \exp\left(-\frac{\sigma_H\,\sigma_\mu\,\rho_{H,\mu}}{b}\,[T - B_\mu(0,T,b)]\right) \tag{4.4.4}$$

is the term of correlation that arises from the dependence between the investment fund and the mortality rate.

Therefore, the total valuation of the contract is given by

$$FV_0 = P\cdot\frac{p_x(0,T)}{{}_Tp_x}\cdot\left(\pi\cdot\rho(0,T)\cdot((1+i)^T\cdot P(0,T)+\beta\cdot c_0^T) + (1-\pi)\cdot\rho^{'}(0,T)\cdot e^{-\varepsilon\cdot T}\right). \tag{4.4.5}$$

In the following, we display the proofs for these three propositions explicitly:

Proof 4.1. We start by estimating the time 0 price of $C_i(T)$, so we write:

$$C_i(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}\cdot e^{-\int_0^T \mu_x(s)ds}\cdot C_i(T)\right].$$

To solve this expectation, we choose the zero-coupon bond as the numéraire for the T-forward measure defined from the risk-neutral measure using Girsanov's theorem as follows:

$$\begin{aligned}\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} &= \frac{M_0}{M_T}\cdot\frac{P(T,T)}{P(0,T)} = \frac{e^{-\int_0^T r_s ds}}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}\right]}\\ &= \exp\left(-\sigma_r\int_0^T B_r(s,T,\theta)\,\mathrm{d}W_s^r - \frac{1}{2}\sigma_r^2\int_0^T B_r^2(s,T,\theta)\,\mathrm{d}s\right),\end{aligned}$$

where $M_t = e^{\int_0^t r_s ds}$ is the risk-free asset known as the money market account.

Therefore, the fair value of $C_i(T)$, under the T-forward measure, is

$$\begin{aligned}C_i(0) &= P(0,T)\cdot\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\cdot C_i(T)\right]\\ &= \pi\cdot(1+i)^T\cdot P(0,T)\cdot\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\right].\end{aligned}$$

We can prove that $\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\right] = p_x(0,T)\cdot\rho(0,T)$, see Appendix 4.A.

Consequently,

$$C_i(0) = \pi\cdot(1+i)^T\cdot P(0,T)\cdot p_x(0,T)\cdot\rho(0,T). \tag{4.4.6}$$

Proof 4.2. We move to the second term of the contract, $C_G(T)$, and we also compute its value at 0 under the T-forward measure as

$$\begin{aligned} C_G(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s)ds} \cdot C_G(T)\right] \\ &= P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds} \cdot C_G(T)\right] \\ &= \pi \cdot \beta \cdot P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[\left(\frac{G_T}{G_0} - K\right)^+ \cdot e^{-\int_0^T \mu_x(s)ds}\right]. \end{aligned}$$

Due to the dependence between the value of financial assets and the mortality rate, we define a new measure from the T-forward measure called the forward-mortality measure, denoted by $\mathbb{Q}^\mu$. Using Girsanov's theorem, the Radon-Nikodym derivative is, in this case

$$\frac{\mathrm{d}\mathbb{Q}^\mu}{\mathrm{d}\mathbb{Q}^T} = \frac{e^{-\int_0^T \mu_x(s)ds}}{\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\right]}.$$

Under this new measure, the expectation in the valuation of $C_G(T)$ becomes

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^T}\left[\left(\frac{G_T}{G_0} - K\right)^+ \cdot e^{-\int_0^T \mu_x(s)ds}\right] &= \mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\right] \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\left(\frac{G_T}{G_0} - K\right)^+\right] \\ &= p_x(0,T) \cdot \rho(0,T) \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\left(\frac{G_T}{G_0} - K\right)^+\right]. \end{aligned}$$

We show in Appendix 4.B that

$$\mathbb{E}^{\mathbb{Q}^\mu}\left[\left(\frac{G_T}{G_0} - K\right)^+\right] = e^{\hat{m}_G + \frac{\hat{v}_G}{2}} \cdot \Phi\left(\frac{\hat{m}_G + \hat{v}_G - \ln K}{\sqrt{\hat{v}_G}}\right) - K \cdot \Phi\left(\frac{\hat{m}_G - \ln K}{\sqrt{\hat{v}_G}}\right),$$

where Φ is the cumulative distribution function of a standard normal variable and $\hat{m}_G$ and $\hat{v}_G$ are respectively the expectation and the variance of $\ln\left(\frac{G_T}{G_0}\right)$ under the forward-mortality measure, defined in Equations (4.4.2) and (4.4.3).

Accordingly, we obtain the fair value of $C_G(T)$ at 0 as follows:

$$C_G(0) = \pi \cdot \beta \cdot p_x(0,T) \cdot \rho(0,T) \cdot c_0^T, \tag{4.4.7}$$

where $c_0^T = P(0,T) \cdot \left[e^{\hat{m}_G + \frac{\hat{v}_G}{2}} \cdot \Phi\left(\frac{\hat{m}_G + \hat{v}_G - \ln K}{\sqrt{\hat{v}_G}}\right) - K \cdot \Phi\left(\frac{\hat{m}_G - \ln K}{\sqrt{\hat{v}_G}}\right)\right]$.

We observe that $\rho_{G,\mu}$, the correlation between the fund G and the mortality rate, and $\rho_{r,\mu}$, the correlation between the interest and mortality rates, directly influence the expectation term $\hat{m}_G$, in turn implicitly impacting c_0^T.

Proof 4.3. Ultimately, we attempt to price the third term of the contract, $C_F(T)$. For this as well, we move first to the T-forward measure, then to the forward-mortality measure:

$$\begin{aligned} C_F(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s)ds} \cdot C_F(T)\right] \\ &= P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds} \cdot C_F(T)\right] \\ &= (1-\pi) \cdot P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[\frac{F_T}{F_0} \cdot e^{-\int_0^T \mu_x(s)ds}\right] \\ &= (1-\pi) \cdot P(0,T) \cdot p_x(0,T) \cdot \rho(0,T) \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\frac{F_T}{F_0}\right]. \end{aligned}$$

We prove that $P(0,T) \cdot \rho(0,T) \cdot \mathbb{E}^{\mathbb{Q}^{\mu}}\left[\frac{F_T}{F_0}\right] = \rho^{'}(0,T) \cdot e^{-\varepsilon \cdot T}$, see Appendix 4.C.

As a result,

$$C_F(0) \;=\; (1-\pi) \cdot p_x(0,T) \cdot \rho^{'}(0,T) \cdot e^{-\varepsilon \cdot T}, \tag{4.4.8}$$

where $\rho^{'}(0,T)$ is the term of correlation that arises from the dependence between the investment fund and the mortality rate, defined in Equation (4.4.4).

Corollary 4.1. *The premium equivalence of the maturity guarantee contract follows from Equation (4.4.5) under the condition that $FV_0 \;=\; P$:*

$$p_x(0,T) \cdot \left[\pi \cdot \rho(0,T) \cdot \left(P(0,T) \cdot (1+i)^T + \beta \cdot c_0^T\right) + (1-\pi) \cdot \rho^{'}(0,T) \cdot e^{-\varepsilon \cdot T}\right] = {}_T p_x, \tag{4.4.9}$$

where $\rho(0,T)$ and $\rho^{'}(0,T)$ are the terms of correlation exhibited in Equations (4.4.1) and (4.4.4), respectively.

This relation establishes a fair connection between all of the parameters, allowing us to create a fair contract.

It is interesting to shed light, in Equation (4.4.9), on the terms of correlation and their impact on the guaranteed interest rate and the value of the contract. The two terms of correlation, $\rho(0,T)$ and $\rho^{'}(0,T)$, are shown explicitly in Equation (4.4.9). Nevertheless, the term c_0^T is impacted by the term of correlation that reflects the dependence between the general fund G and the mortality rate. For instance, an increase in the correlation between interest and mortality rates is followed by an increase in the term of the correlation $\rho(0,T)$ that results in an increase in the price of the contract and a decrease in the guaranteed interest rate. On the other hand, the increase in the correlation between the risky fund H and the mortality rate leads to a decrease in the term of the correlation $\rho^{'}(0,T)$ that lowers the price of the contract and increases the guaranteed rate i. This impact of the terms of correlations on the fair value of contracts could be explained by the fact that moving the interest rate r and the mortality rate μ in the same direction affects positively the fair value of the product, whereas moving the fund H and the mortality rate μ in the same direction has a negative effect. Let's consider a scenario where both $\rho_{r,\mu}$ and $\rho_{H,\mu}$ are positive. In this situation: when the mortality rate μ decreases, it leads to an increase in longevity, subsequently raising the cost of the product. On the other hand, if the interest rate r decreases, the discount rate also decreases. This causes the price of zero-coupon bonds to increase, which in turn raises the cost of the product. This happens because the insurer needs to allocate more funds to ensure future payouts. Therefore, when both r and μ move in the same direction, they have a similar effect on the cost of the product. This implies that the positive sign within the exponential term in $\rho(0,T)$ is justified. Now, if the value of the risky fund H decreases, it directly impacts the value of the investment. This results in a decrease in the cost of the product. A lower value in the risky fund means fewer funds are required to cover potential payoffs. As a result, when both μ and H move in the same direction, they have opposing effects on the cost of the product. This implies that the negative sign within the exponential term in $\rho^{'}(0,T)$ is justified.

4.5 Other forms of pension products

In Sections 4.3 and 4.4, we have seen a product with longevity conditions on the two parts, which is a combination of two pure endowments, one for the participating product and the other for the unit-linked product. In this section, we consider other kinds of products and derive explicitly the premium equivalence of each product.

1. *Pure financial product*:

   This form of product is a direct generalization of Hanna et al. [18] in which no mortality is introduced; however, a stochastic interest rate is considered. So, this case can be simply extracted from Equation (4.4.9) by ignoring the presence of the mortality element in the life insurance contract. This leads to the following fair relation:

$$\pi \cdot \left(P(0,T) \cdot (1+i)^T + \beta \cdot C_0^{'T}\right) + (1-\pi) \cdot e^{-\varepsilon \cdot T} = 1, \tag{4.5.1}$$

   where $C_0^{'T} = \Phi\left(\frac{-\ln(K) - \ln(P(0,T)) + \frac{\hat{v}_G}{2}}{\sqrt{\hat{v}_G}}\right) - K \cdot P(0,T) \cdot \Phi\left(\frac{-\ln(K) - \ln(P(0,T)) - \frac{\hat{v}_G}{2}}{\sqrt{\hat{v}_G}}\right)$ is the time 0 value of a T-year European call option under stochastic interest rate.

   Relation (4.5.1) is similar to the result obtained in Hanna et al. [18]. The only difference lies in the price of a zero coupon and the call option, which are now valued with a stochastic interest rate instead of a constant interest rate.

2. *Combination of pure endowment and pure investment product:*

   This kind of product entails paying the sum assured on the participating part of the contract if the policyholder survives to the end of the policy term, in addition to the return of the investment fund on the risky part (minus the guarantee fee) at the end of the year of death or at maturity if he survives to that point. In this kind of contract, the terminal cash flow is written as follows:

$$V_T = \frac{P}{{}_Tp_x} \cdot \left[\pi \cdot \left((1+i)^T + \beta \cdot \left(\frac{G_T}{G_0} - \left(1 + \frac{(1+i)^T - 1}{\beta}\right)\right)^+\right)\right] + (1-\pi) \cdot P \cdot \frac{F_T}{F_0}. \tag{4.5.2}$$

   Compared to the cash flow of Equation (4.3.1), the risky part of Relation (4.5.2) is not affected by survival probabilities.

   Proposition 4.4. *The total valuation of this contract is given by:*

$$\begin{aligned} FV_0 = {} & \pi \cdot P \cdot \frac{p_x(0,T)}{{}_Tp_x} \cdot \rho(0,T) \cdot \left(P(0,T) \cdot (1+i)^T + \beta \cdot c_0^T\right) \\ & + (1-\pi) \cdot P \cdot \Big(p_x(0,T) \cdot \rho'(0,T) \cdot e^{-\varepsilon \cdot T} \\ & \qquad + \sum_{j=0}^{T-1} \left[p_x(0,j) \cdot \rho'(0,j) - p_x(0,j+1) \cdot \rho'(0,j+1)\right] \cdot e^{-\varepsilon(j+1)}\Big). \end{aligned} \tag{4.5.3}$$

   Proof 4.4. The fair valuation of V_T is expressed as follows:

$$\begin{aligned} FV_0 = {} & \mathbb{E}^{\mathbb{Q}}\left[V_T \cdot e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s) ds} \mid \mathcal{M}_0\right] \\ & + \mathbb{E}^{\mathbb{Q}}\left[(1-\pi) \cdot P \cdot \sum_{j=0}^{T-1} \frac{F_{j+1}}{F_0} \cdot e^{-\int_0^{j+1} r_s ds} \left(e^{-\int_0^j \mu_x(s) ds} - e^{-\int_0^{j+1} \mu_x(s) ds}\right) \mid \mathcal{M}_0\right], \end{aligned}$$

   where the second expectation takes into account the payment of the investment fund in case of death between the start of the contract and the maturity date. We divide this expectation into two halves and write down the fair value:

$$\begin{aligned} FV_0 &= \mathbb{E}^{\mathbb{Q}}\Big[V_T \cdot e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s)ds} \mid \mathcal{M}_0\Big] \\ &+ (1-\pi)\cdot P \cdot \sum_{j=0}^{T-1} \mathbb{E}^{\mathbb{Q}}\left[\frac{F_{j+1}}{F_0} \cdot e^{-\int_0^{j+1} r_s ds} \cdot e^{-\int_0^{j} \mu_x(s)ds} \mid \mathcal{M}_0\right] \\ &- (1-\pi)\cdot P \cdot \sum_{j=0}^{T-1} \mathbb{E}^{\mathbb{Q}}\left[\frac{F_{j+1}}{F_0} \cdot e^{-\int_0^{j+1} r_s ds} \cdot e^{-\int_0^{j+1} \mu_x(s)ds} \mid \mathcal{M}_0\right]. \end{aligned}$$

The computations of the three expectations are detailed in Appendix 4.D, which allows us to obtain the following result:

$$\begin{aligned} FV_0 = \pi \cdot P \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot \rho(0,T) \cdot \Big(P(0,T)\cdot(1+i)^T + \beta \cdot c_0^T\Big) \\ + (1-\pi)\cdot P \cdot \Big(p_x(0,T)\cdot \rho^{'}(0,T)\cdot e^{-\varepsilon\cdot T} \\ + \sum_{j=0}^{T-1}\Big[p_x(0,j)\cdot\rho^{'}(0,j) - p_x(0,j+1)\cdot\rho^{'}(0,j+1)\Big]\cdot e^{-\varepsilon(j+1)}\Big). \end{aligned}$$

Corollary 4.2. *The premium equivalence of a maturity guarantee contract under this special combination of products is given as follows:*

$$\begin{aligned} \pi \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot \rho(0,T) \cdot \Big(P(0,T)\cdot(1+i)^T + \beta \cdot c_0^T\Big) \\ + (1-\pi)\cdot \Bigg(p_x(0,T)\cdot \rho^{'}(0,T)\cdot e^{-\varepsilon\cdot T} \\ + \sum_{j=0}^{T-1}\Big[p_x(0,j)\cdot\rho^{'}(0,j) - p_x(0,j+1)\cdot\rho^{'}(0,j+1)\Big]\cdot e^{-\varepsilon(j+1)}\Bigg) = 1. \end{aligned} \tag{4.5.4}$$

Relation (4.5.4) gives a fair condition between all the parameters and differs from Equation (4.4.9) in the fact that the insured has the right to a return on the risky part if he dies before the maturity of the contract. We note that even if the final benefit is independent of mortality (Relation (4.5.2)), we can observe here that the valuation of the unit-linked part depends on the survival probabilities, and this is induced by the effect of the guarantee fee ε because by setting $\varepsilon = 0$, we obtain 1 as the value of the unit-linked fund.

3. *Annual guarantee*:

We recall our basic contract in which we combine two pure endowments; but in this form of contract, we look at the guarantee and the financial profit each year and not only at maturity, so we can distinguish between a rebalanced and a non-rebalanced portfolio. This means that we either create an automatic link between the participating and unit-linked accounts or that the proportion π invested in the participating part stays constant throughout the duration of the contract.
The terminal cash flow of a non-rebalanced contract (in which the proportion π is invested in

the participating part at the beginning of the contract and no further action is taken) is written as follows:

$$V_T = \frac{P}{{}_T p_x} \cdot \left[\pi \cdot \prod_{t=1}^{T} \left(1 + i + (\beta \cdot g_t - i)^+\right) + (1-\pi) \cdot \prod_{t=1}^{T} \frac{F_t}{F_{t-1}} \right], \tag{4.5.5}$$

however, the terminal cash flow of a rebalanced contract (in which the investment share is rebalanced so that, once more, the constant proportion π is invested in the participating contract each year) is written as follows:

$$V_T = \frac{P}{{}_T p_x} \cdot \prod_{t=1}^{T} \left[\pi \cdot \left(1 + i + (\beta \cdot g_t - i)^+\right) + (1-\pi) \cdot \frac{F_t}{F_{t-1}} \right]. \tag{4.5.6}$$

Following the same logic as Section 4.4, we use the change of measure technique to compute the real valuation of Equations (4.5.5) and (4.5.6) at the inception of the contract.

Proposition 4.5. *The total valuations of annual guarantee contracts without and with rebalancing are given, respectively, by Equations (4.5.7) and (4.5.8):*

$$FV_0 = P \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot \left[\pi \cdot P(0,T) \cdot \rho(0,T) \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\prod_{t=1}^{T} \left(1+i+(\beta \cdot g_t - i)^+\right)\right] + (1-\pi) \cdot \rho^{'}(0,T) \cdot e^{-\varepsilon T} \right] \tag{4.5.7}$$

$$FV_0 = P \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot P(0,T) \cdot \rho(0,T) \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\prod_{t=1}^{T} \left[\pi \cdot \left(1+i+(\beta \cdot g_t - i)^+\right) + (1-\pi) \cdot \frac{F_t}{F_{t-1}} \right]\right]. \tag{4.5.8}$$

Proof 4.5. The fair valuation of Equations (4.5.5) and (4.5.6) is stated as follows:

$$\begin{aligned} FV_0 &= \mathbb{E}^{\mathbb{Q}}\left[V_T \cdot e^{-\int_0^T r_s ds} \cdot e^{-\int_0^T \mu_x(s) ds} \mid \mathcal{M}_0 \right] \\ &= P(0,T) \cdot p_x(0,T) \cdot \rho(0,T) \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[V_T \right]. \end{aligned} \tag{4.5.9}$$

This is obtained by applying the change of measures defined in Section 4.4; from the risk-neutral measure to a T-forward measure, then to a forward-mortality measure.

For the rebalanced contract, it is not possible to go further because of the dependence between the annual returns of the funds G and F. Thus, the fair value of the non-rebalanced contract (4.5.5) follows from Equation (4.5.9):

$$\begin{aligned} FV_0 = V_0 \cdot P(0,T) \cdot \frac{p_x(0,T)}{{}_T p_x} \cdot \rho(0,T) \cdot &\left[\pi \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\prod_{t=1}^{T} \left(1 + i + (\beta \cdot g_t - i)^+\right)\right] \right. \\ &\left. + (1-\pi) \cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\underbrace{\prod_{t=1}^{T} \frac{F_t}{F_{t-1}}}_{=\frac{F_T}{F_0}} \right]\right]. \end{aligned}$$

Due to the dependence between the annual returns of the fund G, the term $\mathbb{E}^{\mathbb{Q}^\mu}\left[\prod_{t=1}^{T} \left(1 + i + (\beta \cdot g_t - i)^+\right)\right]$ could not be solved explicitly. However, the second term $\mathbb{E}^{\mathbb{Q}^\mu}\left[\frac{F_T}{F_0}\right]$ gives, referring to the computations in Section 4.4, $\dfrac{\rho^{'}(0,T)\cdot e^{-\varepsilon T}}{\rho(0,T)\cdot P(0,T)}$.

Corollary 4.3. *The fairness relations of annual guarantee contracts without and with rebalancing are given, respectively, in Equations (4.5.10) and (4.5.11):*

$$p_x(0,T)\Bigg[\pi\cdot P(0,T)\cdot\rho(0,T)\cdot\mathbb{E}^{\mathbb{Q}^\mu}\Bigg[\prod_{t=1}^{T}\Big(1+i+(\beta\cdot g_t-i)^+\Big)\Bigg] + (1-\pi)\cdot\rho^{'}(0,T)\cdot e^{-\varepsilon\cdot T}\Bigg] = {}_Tp_x \tag{4.5.10}$$

$$p_x(0,T)\cdot P(0,T)\cdot\rho(0,T)\cdot\mathbb{E}^{\mathbb{Q}^\mu}\Bigg[\prod_{t=1}^{T}\Bigg[\pi\cdot\Big(1+i+(\beta\cdot g_t-i)^+\Big)+(1-\pi)\cdot\frac{F_t}{F_{t-1}}\Bigg]\Bigg] = {}_Tp_x. \tag{4.5.11}$$

We state in the following some special cases deducted directly from Equation (4.4.9) of our basic contract:

Remark 1. (*Independence finance-mortality*). We assume independence between the financial and mortality elements. In many studies, this case is considered when pricing a life insurance contract for ease of computation on the one hand and assuming that a correlation between the demographic trend and the financial market is almost negligible in the absence of an actual pandemic on the other.

Practically, the independence leads to $\rho(0,T) = \rho^{'}(0,T) = 1$, resulting in a fair relation of the form

$$p_x(0,T)\cdot\Big[\pi\cdot\Big(P(0,T)\cdot(1+i)^T+\beta\cdot C_0^{'T}\Big)+(1-\pi)\cdot e^{-\varepsilon\cdot T}\Big] = {}_Tp_x, \tag{4.5.12}$$

where $C_0^{'T} = \Phi\Big(\frac{-\ln P(0,T)+\frac{\hat{v}_G}{2}-\ln K}{\sqrt{\hat{v}_G}}\Big) - K\cdot P(0,T)\cdot\Phi\Big(\frac{-\ln P(0,T)-\frac{\hat{v}_G}{2}-\ln K}{\sqrt{\hat{v}_G}}\Big)$.

Remark 2. (*Pure endowment contract without participation*). Another special case is presented as a comparison of the type of pure endowment for a classic life insurance contract, between a classical valuation and a stochastic framework.

The classical actuarial valuation of a pure endowment without profit sharing ($\pi = 1$, $\beta = 0$) is:

$${}_TE_x = \frac{1}{(1+i)^T}\cdot{}_Tp_x.$$

However, under the framework of this chapter, the valuation of the classical product becomes:

$$FV = P(0,T)\cdot p_x(0,T)\cdot\rho(0,T),$$

in which case we need three factors instead of two to represent the correlation between financial and longevity risks.

4.6 Numerical illustrations

In this section, we present some numerical applications for Equations (4.4.9), (4.5.4), (4.5.12) and (4.5.1) to compute the fair guaranteed interest rate i, and we attempt to compare these results to Hanna et al. [18] after accounting for stochastic interest rates and mortality rates. In this framework, we will find solutions for fairness relations with respect to i based on different values of the investment share π.

So, we consider a pure endowment contract, and we display in Table 4.1 the parameter values chosen to be used in this section.

Parameters	Symbols	Contract Parameters	Parameters	Symbols	Market Parameters
Maturity	T	15	G-volatility	σ_G	3%
Initial age	x	50	H-volatility	σ_H	15%
Survival probability	${}_Tp_x$	0.957453645[5]			

Parameters	Symbols	Interest Rates Parameters	Parameters	Symbols	Mortality Parameters
Interest Rate volatility	σ_r	0.8%	Mortality volatility	σ_μ	0.0519610%
Speed of mean reversion	θ	15%	Speed of mean reversion	b	13.8550587%
Initial interest rate	r_0	0.5%	Initial mortality force	$\mu_x(0)$	0.2600332%

Table 4.1: Model parameters' values.

We point out that we chose low and high volatility levels for the two funds G and H, respectively, using values from the paper by Hanna et al. [18]. Moreover, based on the same paper, we use as an example a participation level of $\beta = 70\%$ and a guarantee fee of $\varepsilon = 0.25\%$ for the valuation of the contract.

In order to simulate the price of a zero-coupon bond $P(0,T)$ in Equation (4.2.2), the calibration of the instantaneous forward rate $f(0,T)$ is given by the restricted exponential model defined by Cairns [9]:

$$f(0,t) = \beta_0 + \sum_{j=1}^{n} \beta_j \cdot e^{-c_j t}.$$

We fitted this model to the curve of Belgian zero-coupon rates on January 6, 2021, by using non-linear regression through the non-linear least squares function; we use a virtual adjustment with $n = 3$ and the following values for the parameters β_j and c_j:

$\beta_0 = -0.0039$, $\beta_1 = 0.001$, $\beta_2 = -0.0165$, $\beta_3 = 0.012$, $c_1 = 0.1385$, $c_2 = 0.0385$ and $c_3 = -0.02$.

Regarding the simulation of the survival probability $p_x(0,T)$ in Equation (4.2.12), we refer to Zeddouk & Devolder [28], which states that the target is given by the Gompertz law with the following expression:

$$\xi_t = \frac{A}{b} e^{Bt},$$

where A and B are two strictly positive constants and refer, respectively, to the baseline mortality at age x and the senescent component. Zeddouk & Devolder [28] show that using a mean-reverting model with an exponential target consistent with Gompertz law is appropriate for describing mortality

[5]The numbers of survivals at the corresponding ages are based on the Belgian official XR table, for 1 million of births.

over time, as they confirm its adequacy.
We note that the optimal parameters for the mortality model chosen in Table 4.1 are derived from the calibration performed by Zeddouk & Devolder [28], with an initial age of $x = 50$ and considering generation 1970. We add to these parameters the values of A and B, which are, respectively, 0.000307570 and 0.100627916.

We note that in the first three subsections, we analyze the results of Equation (4.4.9).

4.6.1 The effect of the correlation parameters

To study the correlation parameters clearly, we display graphically the effect of each parameter on the fair guaranteed rates. We assume that the insurer invests equal parts in the participating and the unit-linked contracts ($\pi = 50\%$). In this setting, we study each parameter separately by putting the two other correlation parameters at zero and varying the one in question in the range of values $[-20\%, 20\%]$. We note that, in the case where $\rho_{r,\mu}$ is different from zero, we have an indirect correlation between the funds G and F and the mortality through the interest rate by imposing $\rho_{G,\mu} = \rho_{r,\mu} \cdot \rho_{r,G}$ and $\rho_{H,\mu} = \rho_{r,\mu} \cdot \rho_{r,H}$.

The first graph in Figure 4.1 shows a decreasing effect on the guaranteed rate since for $\rho_{r,\mu} \in [-20\%, 20\%]$, the guaranteed interest rate i takes values between $[0.473\%, 0.479\%]$. So, a negative correlation between interest and mortality rates boosts the guaranteed rate.

In contrast to the effect of $\rho_{r,\mu}$, the guaranteed interest rate i remains relatively stable for the different values of $\rho_{G,\mu}$ and increases with the increase of $\rho_{H,\mu}$, as shown in the second and third graphs in Figure 4.1. Clearly, $\rho_{G,\mu}$ has no significant impact on i that takes values between 0.476% and 0.477%, whereas $\rho_{H,\mu}$ has a greater impact since the values of i emerge between 0.467% and 0.485%. As a result, stronger and positive correlations between the fund H and the mortality rate improve the guarantee on the participating part.

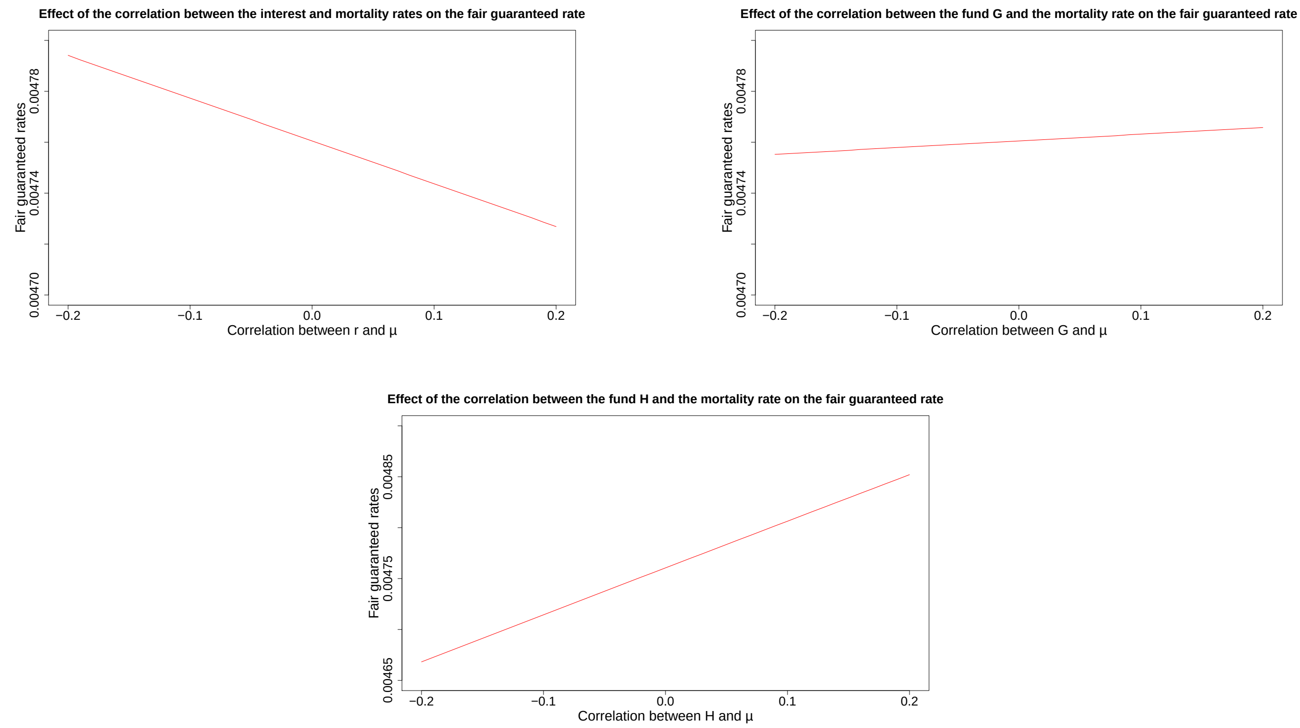

Figure 4.1: Effect of the different correlation parameters.

The empirical evidence provided by Dacorogna & Cadena [11] to explain the change in the financial market due to the mortality factor shows that the influence is quite weak in a normal situation and that the correlation between interest rates and mortality rates is higher in most countries than the correlation between stocks and the mortality index. For this reason, it is worth noting that the values of the correlation parameters, $\rho_{r,\mu}$, $\rho_{G,\mu}$, and $\rho_{H,\mu}$, are chosen relatively small, with $\rho_{r,\mu} > \rho_{G,\mu}$ and $\rho_{H,\mu}$. Therefore, we state in the following subsections that $|\rho_{r,\mu}| = 7\%$ and $|\rho_{G,\mu}| = |\rho_{H,\mu}| = 5\%$.

4.6.2 Sign sensitivity

Table 4.2 shows the sensitivity of the signs of the correlation parameters to the fair guaranteed rates i. We assume that the allocation level π is equal to 50% and that $\rho_{r,\mu}$, $\rho_{G,\mu}$ and $\rho_{H,\mu}$ are equal, in absolute value, to 7%, 5% and 5%, respectively, where each row in the table indicates the sign of these parameters. For instance, the sign "+" under $\rho_{r,\mu}$ means that $\rho_{r,\mu} = +7\%$, while the sign "−" means that $\rho_{r,\mu} = -7\%$. Moreover, the relations $\rho_{G,\mu} = \rho_{r,\mu} \cdot \rho_{r,G}$ and $\rho_{H,\mu} = \rho_{r,\mu} \cdot \rho_{r,H}$ in Table 4.2 imply that the correlation between the funds and the mortality rate arises only from their correlations with the interest rate.

We observe an increase in the fair guaranteed rate by switching to a negative correlation between the interest and mortality rates and a positive correlation between the funds and the mortality rate. In fact, a negative correlation between the two risk factors r and μ operates as a hedging mechanism against the overall unpredictability of the contract, which reduces its price and increases the guaranteed rate.

From this analysis, we present two central scenarios for the correlation parameters, and we find the combination of signs for the three parameters that gives the lowest as well as the highest values for the guaranteed rates i that maintain the fair relation of the maturity guarantee determined by Equation (4.4.9). The results are illustrated in Table 4.3.

$|\rho_{r,\mu}| = 7\%$, $|\rho_{G,\mu}| = 5\%$, $|\rho_{H,\mu}| = 5\%$,
$\rho_{r,G} = -20\%$, $\rho_{r,H} = -15\%$, $\pi = 50\%$

$\rho_{r,\mu}$	$\rho_{G,\mu}$	$\rho_{H,\mu}$	$i(\%)$
+	$\rho_{r,\mu} \cdot \rho_{r,G}$	$\rho_{r,\mu} \cdot \rho_{r,H}$	0.475
+	+	+	0.478
+	−	−	0.473
−	$\rho_{r,\mu} \cdot \rho_{r,G}$	$\rho_{r,\mu} \cdot \rho_{r,H}$	0.477
−	+	+	0.479
−	−	−	0.474

Table 4.2: Fair guaranteed rates i (expressed in %) depending on the sign of the correlation parameters.

The first and third columns of Table 4.3 are two central scenarios since they indicate the signs of the set of correlations $(\rho_{r,\mu}, \rho_{G,\mu}, \rho_{H,\mu})$ that give the lowest and highest fair guaranteed rates. However, the middle column refers to the case of independence between mortality and financial elements, for which we recall Equation (4.5.12). We notice that the highest values of guaranteed rates are obtained by considering a negative correlation between interest and mortality rates together with positive values

	$\rho_{r,G} = -20\%$		
π	$\rho_{r,\mu} = 7\%$ $\rho_{G,\mu} = -5\%$ $\rho_{H,\mu} = -5\%$	$\rho_{r,\mu} = 0\%$ $\rho_{G,\mu} = 0\%$ $\rho_{H,\mu} = 0\%$	$\rho_{r,\mu} = -7\%$ $\rho_{G,\mu} = 5\%$ $\rho_{H,\mu} = 5\%$
0.10	2.700	2.711	2.722
0.30	0.978	0.983	0.987
0.50	0.473	0.476	0.479
0.70	0.175	0.178	0.180
1.00	-0.144	-0.142	-0.141

Table 4.3: Fair guaranteed rates i (expressed in %) for three extreme scenarios of the correlations.

between equities and mortality rates, which is in line with the graphs in Figure 4.1. However, the impact of the correlation parameters is not significant with respect to the case where no correlation is considered, especially with a higher investment in the participating part.

In addition, we observe in Table 4.3 that the guaranteed rates increase the more we invest in the risky fund. So, the results show that a mixed product is preferred over a participating life insurance product in terms of the guaranteed interest rate, regardless of the values of the correlation parameters.

4.6.3 Sensitivity analysis with respect to the volatility of mortality σ_μ

The weak effect of the correlation parameters is due to the choice of very small values for the volatilities. Hence, we are interested in showing how the volatility σ_μ affects a product with a dependency between financial and mortality elements and another where no dependency is taken into account. A higher choice of volatility has an influence on the terms of correlation that, in turn, make all the difference.

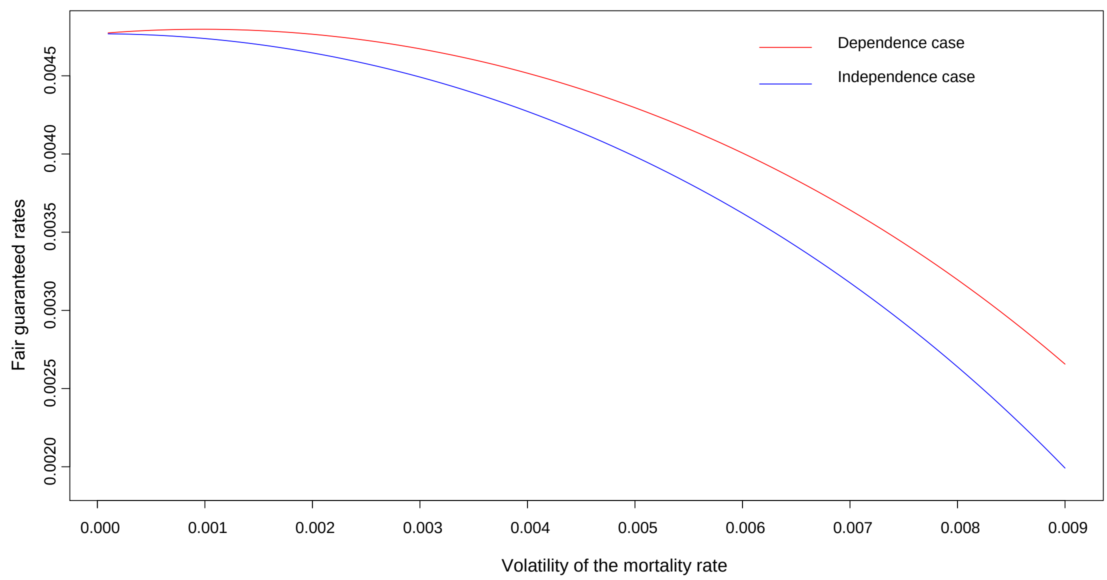

Figure 4.2: Sensitivity of the volatility σ_μ.

Figure 4.2 displays the fair guaranteed rates of a maturity guarantee product with and without dependence between financial and mortality elements for different values of the volatility σ_μ by investing equal parts in the participating and the unit-linked contracts ($\pi = 50\%$). The values of the correlation parameters used in the dependence case are as follows: $\rho_{r,\mu} = -7\%$, $\rho_{G,\mu} = 5\%$ and $\rho_{H,\mu} = 5\%$. The graph proves that by increasing the volatility, the impact on the fair guaranteed rates becomes more and more important, and consequently, taking the dependency into account in the model is more prudent. In Table 4.4, we compare the fair guaranteed rates for the three extreme correlation scenarios with two different volatility levels, $\sigma_\mu = 0.05\%$ and $\sigma_\mu = 0.5\%$.

	$\rho_{r,G} = -20\%$					
	$\sigma_\mu = 0.05\%$			$\sigma_\mu = 0.5\%$		
π	$\rho_{r,\mu} = 7\%$, $\rho_{G,\mu} = -5\%$, $\rho_{H,\mu} = -5\%$	$\rho_{r,\mu} = 0\%$, $\rho_{G,\mu} = 0\%$, $\rho_{H,\mu} = 0\%$	$\rho_{r,\mu} = -7\%$, $\rho_{G,\mu} = 5\%$, $\rho_{H,\mu} = 5\%$	$\rho_{r,\mu} = 7\%$, $\rho_{G,\mu} = -5\%$, $\rho_{H,\mu} = -5\%$	$\rho_{r,\mu} = 0\%$, $\rho_{G,\mu} = 0\%$, $\rho_{H,\mu} = 0\%$	$\rho_{r,\mu} = -7\%$, $\rho_{G,\mu} = 5\%$, $\rho_{H,\mu} = 5\%$
0.10	2.700	2.711	2.722	2.417	2.524	2.629
0.30	0.978	0.983	0.987	0.844	0.890	0.935
0.50	0.473	0.476	0.479	0.366	0.398	0.430
0.70	0.175	0.178	0.180	0.078	0.102	0.126
1.00	-0.144	-0.142	-0.141	-0.241	-0.226	-0.211

Table 4.4: Fair guaranteed rates i (expressed in %) for three extreme scenarios of the correlations with $\sigma_\mu = 0.5\%$.

Table 4.4 shows that with a higher value of σ_μ, the impact on the fair guaranteed rates is more significant between these three extreme scenarios; however, the fair guaranteed rates decrease with increasing volatility, and that's because with a higher risk, the cost of the guarantee increases, which decreases the guaranteed rate. In addition, we notice that a positive value of $\rho_{r,\mu}$ along with negative values of $\rho_{G,\mu}$ and $\rho_{H,\mu}$ deliver guaranteed rates that even fall below the case where no correlation is considered. As a result, the contribution of the correlations inside the model is prudent in terms of their signs since they can be optimistic as well as pessimistic.

In the following, we would like to present the consequences of taking these two parameters with the wrong sign or size when pricing the contract. More precisely, we will show the size of the loss in percentage of the premium in the worst-case scenario.

The first step is to remove any correlation between financial and mortality factors and to look at the fair guaranteed rate obtained by investing equal shares in both parts of the contract, which is 0.398%, referring to Table 4.4 (for $\sigma_\mu = 0.5\%$). The second step is to include the correlation parameters in order to measure the size of the loss as a percentage of the premium, which is supposed to be 1000. Hence, we evaluate the contract again with a guaranteed rate of $i = 0.398\%$, but we incorporate the correlation parameters to look at the level of the provision.

Table 4.5 shows the results for including a correlation between interest and mortality rates and a correlation between the risky fund H and the mortality rate. We note that the integration of $\rho_{r,\mu}$ and $\rho_{H,\mu}$ into the contract is significant because the insurer incurs a loss when the correlation between finance and mortality is at its worst. In addition, the impact of $\rho_{H,\mu}$ is greater as the percentage of loss on the level of the provision is higher (see first column, Table 4.5).

$\pi = 0.50,\quad i = 0.398\%,\quad \rho_{G,\mu} = \rho_{r,\mu} \cdot \rho_{r,G},\quad \rho_{r,G} = -20\%,\quad \rho_{r,H} = -15\%$

$\rho_{H,\mu}$ \ $\rho_{r,\mu}$	0%	5%	7%	10%	15%	20%	50%
$\rho_{r,\mu} \cdot \rho_{r,H}$	0%	0.042%	0.058%	0.084%	0.125%	0.167%	0.420%
−3%	0.070%	0.094%	0.104%	0.119%	0.143%	0.167%	0.314%
−5%	0.117%	0.141%	0.151%	0.166%	0.190%	0.214%	0.361%
−7%	0.164%	0.188%	0.198%	0.213%	0.237%	0.261%	0.408%
−10%	0.235%	0.259%	0.268%	0.283%	0.307%	0.332%	0.478%
−15%	0.352%	0.376%	0.386%	0.401%	0.425%	0.449%	0.596%
−20%	0.470%	0.494%	0.504%	0.519%	0.543%	0.567%	0.714%
−50%	1.184%	1.209%	1.218%	1.233%	1.257%	1.281%	1.428%

Table 4.5: Size of loss in terms of the premium.

4.6.4 Numerical comparison for other products

In this section, we look at the impact of each of the forms of product represented in Section 4.5 on the fair guaranteed rates i for the different values of the investment share π.

The third column of Table 4.6 displays the results of Equation (4.5.4), in which we consider a combination between a pure endowment and a pure financial contract. However, in the last column, we provide solutions for the guaranteed rates in order to maintain the fair relation determined in Equation (4.5.1). This allows us to look at our model without mortality. We compare these cases to the dependence case that is reflected by Equation (4.4.9). The results are displayed in Table 4.6 by choosing $\rho_{r,G} = -20\%$, $\rho_{r,\mu} = -7\%$, $\rho_{G,\mu} = 5\%$, $\rho_{H,\mu} = 5\%$ and $\sigma_\mu = 0.05\%$.

π	**Basic contract**	**Pure endowment and pure investment product**	**Pure financial product**
0.10	2.722	2.254	2.221
0.30	0.987	0.807	0.735
0.50	0.479	0.372	0.264
0.70	0.180	0.119	-0.032
1.00	-0.141	-0.141	-0.385

Table 4.6: Fair guaranteed rates i (expressed in %) with respect to the share in the participating contract π for different forms of products.

We notice in Table 4.6 that the case of a pure financial product delivers the worst values of i, which means that mortality has to be taken into account in life insurance pricing. In addition, we have to consider a correlation between financial and mortality elements since the best results with respect to the guaranteed rates are obtained in the dependence case. Eventually, the cost of the guarantee is more expensive in the contract in which we consider a combination of a pure endowment and a pure financial return (column 3) than in a pure endowment contract (column 2).

Ultimately, the results in Table 4.6 confirm that introducing a correlation between financial and mortality elements inside the model can be important in the case where the signs of the correlation parameters increase the guaranteed interest rate, as described in Section 4.6.2.

4.6.5 Annual guarantee

We close the numerical part with results from the fairness relations of Equations (4.5.10) and (4.5.11) with respect to the guaranteed interest rate for an annual guarantee contract with and without rebalancing. The results are based on Monte Carlo simulations with 1,000,000 iterations. In order to find the guaranteed interest rate in each equation, we simulate the Monte Carlo estimates for the expectations in Equations (4.5.10) and (4.5.11), respectively.

Tables 4.7 and 4.8 present the effect of the correlation parameters on the fair guaranteed rates, which is coherent with the results obtained in Table 4.4 for a $\sigma_\mu = 0.5\%$; a negative value of $\rho_{r,\mu}$ and positive values of $\rho_{G,\mu}$ and $\rho_{H,\mu}$ lead to the highest guaranteed rates. Moreover, we notice that the guaranteed rates are higher in the maturity guarantee case (Table 4.4), and this makes sense because in a maturity guarantee contract, the guarantee and the financial profit are computed only at maturity, so the good years of investment returns finance the bad years.

Furthermore, we also highlight here the increase in the guaranteed interest rate the more we invest in the unit-linked part, which makes the mixed product more attractive than the pure participating product.

Eventually, we perform the same analysis held in Section 4.6.3, which is the size of the loss in percentage of the premium in case the correlation parameters were not taken into account or considered with the wrong sign or size. The results are displayed in Appendix 4.E (Tables 4.E.1 and 4.E.2), and they are in the same direction as the base case.

	$\rho_{r,G} = -20\%$, $\rho_{r,H} = -15\%$, $\sigma_\mu = 0.5\%$		
π	$\rho_{r,\mu} = 7\%$ $\rho_{G,\mu} = -5\%$ $\rho_{H,\mu} = -5\%$	$\rho_{r,\mu} = 0\%$ $\rho_{G,\mu} = 0\%$ $\rho_{H,\mu} = 0\%$	$\rho_{r,\mu} = -7\%$ $\rho_{G,\mu} = 5\%$ $\rho_{H,\mu} = 5\%$
0.10	2.167	2.301	2.431
0.30	-0.125	-0.040	0.042
0.50	-1.049	-0.981	-0.916
0.70	-1.679	-1.623	-1.569
1.00	-2.457	-2.418	-2.380

Table 4.7: Fair guaranteed rates i (expressed in %) for three extreme scenarios of the correlations–Annual guarantee without rebalancing.

	$\rho_{r,G} = -20\%$, $\rho_{r,H} = -15\%$, $\rho_{G,H} = 10\%$, $\sigma_\mu = 0.5\%$		
π	$\rho_{r,\mu} = 7\%$ $\rho_{G,\mu} = -5\%$ $\rho_{H,\mu} = -5\%$	$\rho_{r,\mu} = 0\%$ $\rho_{G,\mu} = 0\%$ $\rho_{H,\mu} = 0\%$	$\rho_{r,\mu} = -7\%$ $\rho_{G,\mu} = 5\%$ $\rho_{H,\mu} = 5\%$
0.10	1.714	1.911	2.102
0.30	-0.494	-0.384	-0.279
0.50	-1.321	-1.237	-1.157
0.70	-1.855	-1.790	-1.727
1.00	-2.457	-2.418	-2.380

Table 4.8: Fair guaranteed rates i (expressed in %) for three extreme scenarios of the correlations−Annual guarantee with rebalancing.

4.7 Conclusion

In this chapter, we have priced a mixed life insurance product in stochastic models of mortality, interest rates, and stock returns, and we have presented a multi-dimensional approach to explicitly examine the dependence between financial and mortality elements. The mixed product is a combination of a participating contract with a minimum guaranteed rate and a unit-linked life insurance contract with no guarantee, where a guarantee fee is extracted continuously to boost the guarantee on the participating part. Different forms of products have been considered, on the one hand by varying the guarantee period (maturity guarantee or annual guarantee contracts) and on the other hand by differing between pure endowment contracts or mixing pure endowment and pure investment. Using the change of numéraire technique, we obtain a closed form of the premium equivalence for the different forms of products except for the annual guarantee contract due to the dependence between the annual returns.

It is clear from the numerical results that the mixed insurance contract is a more attractive product for the insurer than a pure investment in the participating contract in this complete stochastic environment. Indeed, the concept of a partial guarantee can be an alternative trend for the insurance industry to provide higher investment guarantees.

In this stochastic framework, including the dependence assumption, we have focused on the maturity guarantee contract, analyzed the effect of the correlation parameters, and developed fair combinations, in terms of the parameter's signs, that boost the fair guaranteed rates. The model shows that a negative correlation between interest and mortality rates and positive correlations between the funds and mortality increase the fair guaranteed rates. Hence, we present two central scenarios, based on the signs of the correlation parameters, that give the lowest as well as the highest values of fair guaranteed rates. We show that the impact of the correlation parameters is more pronounced with a higher value of the volatility of mortality. Basically, higher volatility increases the terms of correlations, allowing them to have a greater impact on the fairness relation. In addition, we have studied the percentage of loss on the level of the provision for different values of the correlation parameters, and we see that by considering the wrong sign of the correlation, the loss increases with the size of this parameter.

In future work, we will consider other forms of mortality models as well as calibrate the correlation parameters. We also plan to develop this kind of methodology for other forms of life insurance products (for instance, annuities, endowment insurance, and whole life insurance).

Appendices

4.A. Proof of Proposition 4.1

After defining the T-forward measure, $\tilde{W}_t^r = W_t^r + \int_0^t \sigma_r \, B_r(s,T,\theta)ds$ is a $\mathbb{Q}^T$-Brownian motion. We need then to write the dynamic of the intensity of mortality $\mu_x(t)$ under the T-forward measure, that is,

$$
\begin{aligned}
\mathrm{d}\mu_x(t) = & \, b\,(\tilde{\xi}_t - \mu_x(t))\,\mathrm{d}t \; + \; \sigma_\mu\,\rho_{r,\mu}\,\mathrm{d}\tilde{W}_t^r \; + \; \sigma_\mu\,\overline{\rho}_{G,\mu}\,\sqrt{1-\rho_{r,\mu}^2}\,\mathrm{d}\tilde{W}_t^G \\
& + \sigma_\mu\,\overline{\overline{\rho}}_{H,\mu}\,\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)}\,\mathrm{d}\tilde{W}_t^H \; + \; \sigma_\mu\,\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)}\,\mathrm{d}\tilde{W}_t^\mu; \\
\tilde{\xi}_t \; = & \; \xi_t - \sigma_\mu\,\sigma_r\,\rho_{r,\mu}\frac{B_r(t,T,\theta)}{b},
\end{aligned}
$$

where $\tilde{W}_t^G = W_t^G$, $\tilde{W}_t^H = W_t^H$ and $\tilde{W}_t^\mu = W_t^\mu$ are $\mathbb{Q}^T$-Brownian motions.

As a result,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}^T}\left[e^{-\int_0^T \mu_x(s)ds}\right] & = \exp\left(-\mu_x(0)\cdot B_\mu(0,T,b) - b\int_0^T \tilde{\xi}_s \cdot B_\mu(s,T,b)ds + \frac{\sigma_\mu^2}{2}\int_0^T B_\mu^2(s,T,b)ds\right) \\
& = \exp\Bigg(- \mu_x(0)\cdot B_\mu(0,T,b) - b\int_0^T \xi_s \cdot B_\mu(s,T,b)ds \\
& \quad + \sigma_r\,\sigma_\mu\,\rho_{r,\mu}\int_0^T B_r(s,T,\theta)\cdot B_\mu(s,T,b)ds \; + \; \frac{\sigma_\mu^2}{2}\int_0^T B_\mu^2(s,T,b)ds\Bigg) \\
& = \exp\left(-\mu_x(0)\cdot B_\mu(0,T,b) - b\int_0^T \xi_s \cdot B_\mu(s,T,b)ds + \frac{\sigma_\mu^2}{2}\int_0^T B_\mu^2(s,T,b)ds\right) \\
& \quad \cdot \exp\left(\sigma_r\,\sigma_\mu\,\rho_{r,\mu}\int_0^T B_r(s,T,\theta)\cdot B_\mu(s,T,b)ds\right) \\
& = p_x(0,T)\cdot\rho(0,T),
\end{aligned}
$$

where $\rho(0,T)$ is the term of correlation that arises from the dependence only between the interest rate and the mortality rate, defined in Equation (4.4.1).

4.B. Proof of Proposition 4.2

By computing the Radon-Nikodym derivative, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}\mathbb{Q}^{\mu}}{\mathrm{d}\mathbb{Q}^{T}} = \exp\Bigg(& -\sigma_\mu\,\rho_{r,\mu}\int_0^T B_\mu(s,T,b)\,\mathrm{d}\tilde{W}_s^r - \sigma_\mu\,\overline{\rho}_{G,\mu}\sqrt{1-\rho_{r,\mu}^2}\int_0^T B_\mu(s,T,b)\,\mathrm{d}\tilde{W}_s^G \\
& - \sigma_\mu\,\overline{\overline{\rho}}_{H,\mu}\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)}\int_0^T B_\mu(s,T,b)\,\mathrm{d}\tilde{W}_s^H \\
& - \sigma_\mu\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)}\int_0^T B_\mu(s,T,b)\,\mathrm{d}\tilde{W}_s^\mu \\
& - \frac{1}{2}\sigma_\mu^2\,\rho_{r,\mu}^2\int_0^T B_\mu^2(s,T,b)\,\mathrm{d}s - \frac{1}{2}\sigma_\mu^2\,\overline{\rho}_{G,\mu}^2(1-\rho_{r,\mu}^2)\int_0^T B_\mu^2(s,T,b)\,\mathrm{d}s \\
& - \frac{1}{2}\sigma_\mu^2\,\overline{\overline{\rho}}_{H,\mu}^2(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)\int_0^T B_\mu^2(s,T,b)\,\mathrm{d}s \\
& - \frac{1}{2}\sigma_\mu^2\,(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)\int_0^T B_\mu^2(s,T,b)\,\mathrm{d}s\Bigg).
\end{aligned}
$$

Then,
$$
\begin{cases}
\hat{W}_t^r = \tilde{W}_t^r + \sigma_\mu\,\rho_{r,\mu}\int_0^t B_\mu(s,T,b)\mathrm{d}s \\
\hat{W}_t^G = \tilde{W}_t^G + \sigma_\mu\,\overline{\rho}_{G,\mu}\sqrt{1-\rho_{r,\mu}^2}\int_0^t B_\mu(s,T,b)\mathrm{d}s \\
\hat{W}_t^H = \tilde{W}_t^H + \sigma_\mu\,\overline{\overline{\rho}}_{H,\mu}\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)}\int_0^t B_\mu(s,T,b)\mathrm{d}s \\
\hat{W}_t^\mu = \tilde{W}_t^\mu + \sigma_\mu\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)}\int_0^t B_\mu(s,T,b)\mathrm{d}s
\end{cases}
$$
are $\mathbb{Q}^\mu$-Brownian motions.

The objective is to compute the expectation of $\left(\frac{G_T}{G_0} - K\right)^+$ under the new measure. So, we will express this expression as follows

$$
\left(\frac{G_T}{G_0} - K\right)^+ = \frac{G_T}{G_0}\cdot \mathbb{1}_{\frac{G_T}{G_0}>K} - K\cdot \mathbb{1}_{\frac{G_T}{G_0}>K}.
$$

Hence,

$$
\mathbb{E}^{\mathbb{Q}^\mu}\left[\left(\frac{G_T}{G_0} - K\right)^+\right] = \mathbb{E}^{\mathbb{Q}^\mu}\left[\frac{G_T}{G_0}\cdot \mathbb{1}_{\frac{G_T}{G_0}>K}\right] - K\cdot \mathbb{E}^{\mathbb{Q}^\mu}\left[\mathbb{1}_{\frac{G_T}{G_0}>K}\right]. \tag{4.B.1}
$$

Let's express the dynamic of the fund G under $\mathbb{Q}^\mu$ in order to get the solution of its differential equation. We move first from the risk-neutral measure to the T-forward measure, then to the forward-mortality measure.
Under the forward measure:

$$
\mathrm{d}G_t = (r_t - \sigma_G\,\sigma_r\,\rho_{r,G}\,B_r(t,T,\theta))\,G_t\,\mathrm{d}t + \sigma_G\,\rho_{r,G}\,G_t\,\mathrm{d}\tilde{W}_t^r + \sigma_G\sqrt{1-\rho_{r,G}^2}\,G_t\,\mathrm{d}\tilde{W}_t^G.
$$

Under the forward-mortality measure:

$$
\begin{aligned}
\mathrm{d}G_t = & \Big(r_t - \sigma_G\,\sigma_r\,\rho_{r,G}\,B_r(t,T,\theta) - \sigma_G\,\sigma_\mu\,\rho_{G,\mu}\,B_\mu(t,T,b)\Big)\,G_t\,\mathrm{d}t \\
& + \sigma_G\,\rho_{r,G}\,G_t\,\mathrm{d}\hat{W}_t^r + \sigma_G\sqrt{1-\rho_{r,G}^2}\,G_t\,\mathrm{d}\hat{W}_t^G.
\end{aligned}
$$

After integration, we obtain the SDE of the latter expression as follows:

$$
\begin{aligned}
G_t = G_0 \cdot \exp\Bigg(& \int_0^t r_s \mathrm{d}s \, - \, \sigma_G \, \sigma_r \, \rho_{r,G} \int_0^t B_r(s,T,\theta)\mathrm{d}s \, - \, \sigma_G \, \sigma_\mu \, \rho_{G,\mu} \int_0^t B_\mu(s,T,b)\mathrm{d}s \\
& - \frac{\sigma_G^2}{2} \cdot t \, + \, \sigma_G \, \rho_{r,G} \int_0^t \mathrm{d}\hat{W}_s^r \, + \, \sigma_G \sqrt{1-\rho_{r,G}^2} \int_0^t \mathrm{d}\hat{W}_s^G \Bigg).
\end{aligned}
$$

This can also be written as

$$
\begin{aligned}
\frac{G_T}{G_0} = \frac{1}{P(0,T)} \cdot \exp\Bigg(& - \, \sigma_G \, \sigma_r \, \rho_{r,G} \int_0^T B_r(s,T,\theta)\mathrm{d}s \, - \, \sigma_G \, \sigma_\mu \, \rho_{G,\mu} \int_0^T B_\mu(s,T,b)\mathrm{d}s \\
& - \frac{\sigma_r^2}{2} \int_0^T B_r^2(s,T,\theta)\mathrm{d}s \, - \, \sigma_r \, \sigma_\mu \, \rho_{r,\mu} \int_0^T B_r(s,T,\theta) \cdot B_\mu(s,T,b)\mathrm{d}s \\
& - \frac{\sigma_G^2}{2} \cdot T \, + \int_0^T (\sigma_G \, \rho_{r,G} \, + \, \sigma_r \, B_r(s,T,\theta))\mathrm{d}\hat{W}_s^r \, + \, \sigma_G \sqrt{1-\rho_{r,G}^2} \int_0^T \mathrm{d}\hat{W}_s^G \Bigg).
\end{aligned}
$$

Under $\mathbb{Q}^\mu$, $\frac{G_T}{G_0}$ is log-normally distributed which allows us to write Equation (4.B.1) as

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}^\mu}\left[\left(\frac{G_T}{G_0} - K\right)^+\right] &= \mathbb{E}^{\mathbb{Q}^\mu}\left[\frac{G_T}{G_0}\middle|\frac{G_T}{G_0} > K\right] \cdot \mathbb{Q}^\mu\left[\frac{G_T}{G_0} > K\right] \, - \, K \cdot \mathbb{Q}^\mu\left[\frac{G_T}{G_0} > K\right] \\
&= e^{\hat{m}_G + \frac{\hat{v}_G}{2}} \cdot \Phi\left(\frac{\hat{m}_G + \hat{v}_G - \ln K}{\sqrt{\hat{v}_G}}\right) \, - \, K \cdot \Phi\left(\frac{\hat{m}_G - \ln K}{\sqrt{\hat{v}_G}}\right),
\end{aligned}
$$

where Φ is the cumulative distribution function of a standard normal variable and $\hat{m}_G$ and $\hat{v}_G$ are, respectively, the expectation and the variance of $\ln\left(\frac{G_T}{G_0}\right)$ under the forward-mortality measure, defined in Equations (4.4.2) and (4.4.3).

4.C. Proof of Proposition 4.3

We start by writing the dynamic of the fund F under the forward-mortality measure

$$
\begin{aligned}
\mathrm{d}F_t \, &= (r_t \, - \varepsilon \, - \, \sigma_H \, \sigma_r \, \rho_{r,H} \, B_r(t,T,\theta) \, - \, \sigma_H \, \sigma_\mu \, \rho_{H,\mu} \, B_\mu(t,T,b)) \, F_t \, \mathrm{d}t \\
& \quad + \, \sigma_H \, \rho_{r,H} \, F_t \, \mathrm{d}\hat{W}_t^r \, + \, \sigma_H \, \overline{\rho}_{G,H} \sqrt{1-\rho_{r,H}^2} \, F_t \, \mathrm{d}\hat{W}_t^G \, + \, \sigma_H \sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)} \, F_t \, \mathrm{d}\hat{W}_t^H.
\end{aligned}
$$

After integration,

$$
\begin{aligned}
F_t = F_0 \cdot \exp\Bigg(& \int_0^t r_s \mathrm{d}s \, - \varepsilon \cdot t \, - \, \sigma_H \, \sigma_r \, \rho_{r,H} \int_0^t B_r(s,T,\theta)\mathrm{d}s \, - \, \sigma_H \, \sigma_\mu \, \rho_{H,\mu} \int_0^t B_\mu(s,T,b)\mathrm{d}s \\
& - \frac{\sigma_H^2}{2} \cdot t \, + \, \sigma_H \, \rho_{r,H} \int_0^t \mathrm{d}\hat{W}_s^r \, + \, \sigma_H \, \overline{\rho}_{G,H} \sqrt{1-\rho_{r,H}^2} \int_0^t \mathrm{d}\hat{W}_s^G \\
& + \, \sigma_H \sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)} \int_0^t \mathrm{d}\hat{W}_s^H \Bigg),
\end{aligned}
$$

hence, the expression of $\frac{F_T}{F_0}$ can be written as

$$\frac{F_T}{F_0} = \frac{1}{P(0,T)} \cdot \exp\Bigg(-\varepsilon \cdot T - \sigma_H\,\sigma_r\,\rho_{r,H} \int_0^T B_r(s,T,\theta)\mathrm{d}s - \sigma_H\,\sigma_\mu\,\rho_{H,\mu} \int_0^T B_\mu(s,T,b)\mathrm{d}s \\ - \frac{\sigma_r^2}{2} \int_0^T B_r^2(s,T,\theta)\mathrm{d}s - \sigma_r\,\sigma_\mu\,\rho_{r,\mu} \int_0^T B_r(s,T,\theta)\cdot B_\mu(s,T,b)\mathrm{d}s - \frac{\sigma_H^2}{2}\cdot T \\ + \int_0^T (\sigma_H\,\rho_{r,H} + \sigma_r\,B_r(s,T,\theta))\mathrm{d}\hat{W}_s^r + \sigma_H\,\overline{\rho}_{G,H}\sqrt{1-\rho_{r,H}^2}\int_0^T \mathrm{d}\hat{W}_s^G \\ + \sigma_H\sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)}\int_0^T \mathrm{d}\hat{W}_s^H\Bigg).$$

From this end, we can compute $\mathbb{E}^{\mathbb{Q}^\mu}\left[\frac{F_T}{F_0}\right] = e^{\hat{m}_F + \frac{\hat{v}_F}{2}}$ with

$$\hat{m}_F = E\left[\ln\left(\frac{F_T}{F_0}\right)\right] = \ln\left(\frac{1}{P(0,T)}\right) - \varepsilon\cdot T - \frac{\sigma_H^2}{2}\cdot T - \frac{\sigma_H\,\sigma_r\,\rho_{r,H}}{\theta}\left[T - B_r(0,T,\theta)\right] \\ - \frac{\sigma_H\,\sigma_\mu\,\rho_{H,\mu}}{b}\left[T - B_\mu(0,T,b)\right] \\ - \frac{\sigma_r\,\sigma_\mu\,\rho_{r,\mu}}{\theta\cdot b}\left[T + B_{r,\mu}(0,T,\theta+b) - B_r(0,T,\theta) - B_\mu(0,T,b)\right] \\ - \frac{\sigma_r^2}{2\theta^2}\left[T + B_r(0,T,2\theta) - 2\,B_r(0,T,\theta)\right]$$

$$\hat{v}_F = V\left[\ln\left(\frac{F_T}{F_0}\right)\right] = \sigma_H^2\cdot T + 2\cdot\frac{\sigma_H\,\sigma_r\,\rho_{r,H}}{\theta}\left[T - B_r(0,T,\theta)\right] \\ + \frac{\sigma_r^2}{\theta^2}\left[T + B_r(0,T,2\theta) - 2\,B_r(0,T,\theta)\right],$$

where $\hat{m}_F$ and $\hat{v}_F$ are, respectively, the expectation and the variance of $\ln\left(\frac{F_T}{F_0}\right)$ under the forward-mortality measure.

4.D. Proof of Proposition 4.4

Referring to Section 4.4, the first and the third expectations can be solved explicitly as follows:

$$\mathbb{E}^{\mathbb{Q}}\left[V_T\cdot e^{-\int_0^T r_s ds}\cdot e^{-\int_0^T \mu_x(s)ds}\right] = \pi\cdot P\cdot\frac{p_x(0,T)}{{}_T p_x}\cdot\rho(0,T)\cdot\left(P(0,T)\cdot(1+i)^T + \beta\cdot c_0^T\right) \\ + (1-\pi)\cdot P\cdot p_x(0,T)\cdot\rho'(0,T)\cdot e^{-\varepsilon\cdot T},$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{F_{j+1}}{F_0}\cdot e^{-\int_0^{j+1} r_s ds}\cdot e^{-\int_0^{j+1}\mu_x(s)ds}\right] = p_x(0,j+1)\cdot\rho'(0,j+1)\cdot e^{-\varepsilon\cdot(j+1)}.$$

Whereas the expectation in the middle can be written as

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{F_{j+1}}{F_0}\cdot e^{-\int_0^{j+1} r_s ds}\cdot e^{-\int_0^{j}\mu_x(s)ds}\right] = \mathbb{E}^{\mathbb{Q}}\Bigg[\underbrace{\frac{F_j}{F_0}\cdot e^{-\int_0^{j} r_s ds}\cdot e^{-\int_0^{j}\mu_x(s)ds}}_{\mathbf{I}}\cdot\underbrace{\frac{F_{j+1}}{F_j}\cdot e^{-\int_j^{j+1} r_s ds}}_{\mathbf{II}}\Bigg]. \quad (4.D.1)$$

We then develop the two processes I and II:

$$
\begin{aligned}
\mathrm{I} &= \frac{F_j}{F_0} \cdot e^{-\int_0^j r_s ds} \cdot e^{-\int_0^j \mu_x(s)ds} \\
&= \exp\Bigg(-\varepsilon \cdot j - \frac{\sigma_H^2}{2} \cdot j - \mu_x(0) \cdot B_\mu(0,j,b) \\
&\qquad - b\int_0^j \xi_s \cdot B_\mu(s,j,b)\mathrm{d}s + \int_0^j (\sigma_H \rho_{r,H} - \sigma_\mu \rho_{r,\mu} B_\mu(s,j,b))\mathrm{d}W_s^r \\
&\qquad + \int_0^j (\sigma_H \overline{\rho}_{G,H}\sqrt{1-\rho_{r,H}^2} - \sigma_\mu \overline{\rho}_{G,\mu}\sqrt{1-\rho_{r,\mu}^2} B_\mu(s,j,b))\mathrm{d}\overline{W}_s^G \\
&\qquad + \int_0^j (\sigma_H \sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)} - \sigma_\mu \overline{\overline{\rho}}_{H,\mu}\sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)} B_\mu(s,j,b))\mathrm{d}\overline{\overline{W}}_s^H \\
&\qquad - \sigma_\mu \sqrt{(1-\rho_{r,\mu}^2)(1-\overline{\rho}_{G,\mu}^2)(1-\overline{\overline{\rho}}_{H,\mu}^2)} \int_0^j B_\mu(s,j,b)\mathrm{d}\overline{\overline{\overline{W}}}_s^\mu \Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{II} = \frac{F_{j+1}}{F_j} \cdot e^{-\int_j^{j+1} r_s ds} = \exp\Bigg(&-\varepsilon - \frac{\sigma_H^2}{2} + \sigma_H \rho_{r,H} \int_j^{j+1} \mathrm{d}W_s^r + \sigma_H \overline{\rho}_{G,H}\sqrt{1-\rho_{r,H}^2} \int_j^{j+1} \mathrm{d}\overline{W}_s^G \\
&+ \sigma_H \sqrt{(1-\rho_{r,H}^2)(1-\overline{\rho}_{G,H}^2)} \int_j^{j+1} \mathrm{d}\overline{\overline{W}}_s^H \Bigg).
\end{aligned}
$$

The independence between these two processes allows us to split Equation (4.D.1) to

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\frac{F_j}{F_0} \cdot e^{-\int_0^j r_s ds} \cdot e^{-\int_0^j \mu_x(s)ds} \right] \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{F_{j+1}}{F_j} \cdot e^{-\int_j^{j+1} r_s ds} \right] \\
&= p_x(0,j) \cdot \rho^{'}(0,j) \cdot e^{-\varepsilon(j+1)}.
\end{aligned}
$$

4.E. Size of the loss in percentage of the premium - annual guarantee contract

See Tables 4.E.1 and 4.E.2.

$\pi = 0.50, \quad i = -0.981\%, \quad \rho_{G,\mu} = \rho_{r,\mu} \cdot \rho_{r,G}, \quad \rho_{r,G} = -20\%, \quad \rho_{r,H} = -15\%$

$\rho_{H,\mu}$ \ $\rho_{r,\mu}$	0%	5%	7%	10%	15%	20%	50%
$\rho_{r,\mu} \cdot \rho_{r,H}$	0%	0.036%	0.051%	0.073%	0.109%	0.146%	0.365%
−3%	0.070%	0.089%	0.097%	0.108%	0.127%	0.146%	0.259%
−5%	0.117%	0.136%	0.143%	0.155%	0.174%	0.196%	0.306%
−7%	0.164%	0.183%	0.190%	0.202%	0.221%	0.239%	0.353%
−10%	0.235%	0.253%	0.261%	0.272%	0.291%	0.310%	0.424%
−15%	0.352%	0.371%	0.379%	0.390%	0.409%	0.428%	0.541%
−20%	0.470%	0.489%	0.497%	0.508%	0.527%	0.546%	0.659%
−50%	1.184%	1.203%	1.211%	1.222%	1.241%	1.260%	1.373%

Table 4.E.1: Size of loss in terms of the premium−Annual guarantee without rebalancing.

$\pi = 0.50, \quad i = -1.237\%, \quad \rho_{G,\mu} = \rho_{r,\mu} \cdot \rho_{r,G}, \quad \rho_{r,G} = -20\%, \quad \rho_{r,H} = -15\%$

$\rho_{H,\mu}$ \ $\rho_{r,\mu}$	0%	5%	7%	10%	15%	20%	50%
$\rho_{r,\mu} \cdot \rho_{r,H}$	0%	0.036%	0.050%	0.071%	0.107%	0.143%	0.357%
−3%	0.073%	0.091%	0.097%	0.108%	0.125%	0.143%	0.247%
−5%	0.122%	0.139%	0.146%	0.157%	0.174%	0.192%	0.296%
−7%	0.171%	0.188%	0.195%	0.206%	0.223%	0.240%	0.345%
−10%	0.244%	0.262%	0.268%	0.279%	0.296%	0.314%	0.419%
−15%	0.366%	0.384%	0.391%	0.401%	0.419%	0.436%	0.541%
−20%	0.489%	0.506%	0.513%	0.524%	0.541%	0.559%	0.664%
−50%	1.227%	1.245%	1.252%	1.262%	1.280%	1.297%	1.403%

Table 4.E.2: Size of loss in terms of the premium−Annual guarantee with rebalancing.

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Chapter 5

Optimal choice between defined contribution and cash balance pension schemes: balancing interests of employers and workers[1]

5.1 Introduction

Many traditional pension plans were defined in the past as DB plans (defined benefit); in this kind of scheme, the employer bears the investment risk and is responsible for providing a specified level of retirement benefits to workers. As life expectancies and inflation have increased and investment returns have become less predictable, the costs and risks associated with defined benefit plans have become more and more challenging for employers to manage (Munnell et al. [16], Turner & Hughes [24], Perlman & Reddick [20]). Even if these plans offer the best guarantee to the workers, this solution has become too risky for many employers. Therefore, not surprisingly, DC (defined contribution) plans have emerged as a popular alternative to DB plans, with the goal of shifting investment risk from the employer to the worker (Munnell & Sundén [17], Poterba et al. [21]). In a pure DC scheme, the employer contributes a set percentage of the worker's salary into the worker's account without any further liability, and the worker bears the investment risk as the benefit is directly tied to the performance of the plan's investments. However, the reliance on worker investment decisions and potential market volatility can expose workers to substantial financial risks. In particular, if these contributions are invested in risky assets or in unit-linked insurance products, the worker can suffer a big loss at retirement. Consequently, the financial literacy of the worker can have profound implications for their overall retirement well-being, directly impacting their capacity to save and invest effectively, which reinforces their preference for DC plans. Other factors favoring the DC plans are that they facilitate labor market mobility and are more transparent (Brown & Weisbenner [7], Broadbent et al. [4], Munnell & Sundén [17]). Another solution could be the target-date funds that were suggested as a safe investment option to provide as the default in their DC plans. This strategy can be an improvement for people with relatively high risk aversion, as shown by Bodie & Treussard [2]. As a result of the aforementioned challenges, we can observe that pure DB and pure DC pension schemes can be seen as extreme solutions where the risks are concentrated on a single party (the employer or the worker).

Hybrid solutions have emerged in order to realize better risk sharing between employer and worker. A first way is to introduce a minimum guarantee in a DC plan (see, e.g., Boulier et al. [3]). For instance, the sponsor can manage the scheme through an insurance company that offers participating life insurance contracts. A participating life insurance contract is a type of life insurance policy that provides the policyholder with both insurance coverage and the opportunity to participate in the profits of the insurance company. Under this type of contract, the insurer pools the premiums paid

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by policyholders and invests them in a general fund to offer a guaranteed return with an additional bonus in the form of profit sharing depending on the investment's performance. This type of plan will be called DC-PL (defined contribution managed with participating life insurance). Depending on the level of the parameters, this kind of product can offer the worker a good compromise between guarantee and return while giving safety to the employer through a constant contribution rate. On the other hand, the employer will not take advantage of good financial returns, with all the eventual profit sharing being paid to the workers. Another way to implement safety within a pure DC philosophy is to switch from DC to CB (cash balance); this can be seen as a perfect example of a hybrid product between DB and DC (Brown et al. [6]). Cash balance plans offer the investment risk management of a pure defined contribution plan with the predictable retirement benefit of a defined benefit plan. The main difference between a cash balance plan and a defined contribution plan is the way employer contributions and returns are defined (Coronado & Copeland [9]).

In a cash balance plan, the employer bears the investment risk, and the worker is guaranteed a well-defined rate of return on their account balance, which is not, in general, equal to the real financial return on the assets. The employer attributes a set amount of money to an individual account for each worker. The account earns a predetermined interest rate, and the worker is guaranteed a rate of return on their account balance. In some sense, we could say that the contributions allocated to the account and the return given are virtual. They can be (and are, in general) different from the real contributions paid by the employer and the real investment returns of the assets (see, Hardy et al. [12]). These contributions are allocated to individual accounts for each worker and capitalized at predetermined times, according to the rules of the pension plan. The capitalization process involves calculating the amount that has accumulated in each individual account based on the tariff rules. The tariff rules determine a fixed interest rate or a yield defined by reference to a specified index. This guaranteed interest rate is referred to as the crediting interest rate. This kind of product can offer the worker great safety through pre-determined returns while giving the employer an expected cost close to a DC formula but with the possibility of making profit from additional investment returns. Indeed, in a DC scheme, all the returns are assigned to the workers; in a CB scheme, all the extra return above the crediting interest rate goes back to the employer and can decrease future costs to the pension plan. These returns could be generated, for instance, through a pension fund or through unit-linked insurance products without guarantee. This type of plan will be called CB-UL (cash balance managed with unit-linked insurance).

Cash balance plans have received significant attention in the financial literature as well as in actuarial research. Kopp & Sher [15] examined the benefits offered by DB and CB plans and suggested that CB plans were more advantageous for younger participants with shorter service, while traditional DB plans provided better benefits for older members. Coronado & Copeland [9] focused on the implications of transitioning and its practical aspects and concluded that the conversion from DB to DC plans is driven by labor market conditions rather than reducing benefit generosity. Moreover, Gold [10] approached CB design from the perspective of maximizing shareholder value, focusing on equity yields and optimal asset allocation. He showed that shareholders benefit by defining liabilities in terms of equities. The actuarial valuation of CB plans posed challenges, and different conclusions were drawn by researchers. Applying valuation techniques designed for traditional DB plans to the CB design was found to cause issues. For example, Murphy [18] shows that traditional actuarial methods sometimes generate a loss on termination where the account value exceeds the actuarial valuation. Broeders et al. [5] explore a model for valuing hybrid pension liabilities, where the return on these liabilities is a combination of the pension fund's asset return and the risk-free return. They distinguish between two types of pension schemes and find that the valuation of hybrid pension liabilities is influenced by contract nature, investment strategy, and asset return volatility. They conclude that the impact of these factors on valuation differs between the two schemes, which should be taken into account in labor compensation negotiations between employers and workers. Furthermore, Hardy et al. [12] approach CB benefits as financial liabilities of the employer and apply financial economics and risk management models to analyze and value these liabilities in a market-consistent manner. They argue that market-consistent valuation, based on objective market data rather than subjective judgement, provides a better measure of cost and risk compared to traditional actuarial valuation techniques. They conclude that the investment risk associated with CB plans is significant, especially when crediting rates are based on long bond rates or fixed. Stable costs can be achieved by using shorter rates for crediting, and the claim that CB plans cost less than notional contributions is shown to be untrue based on market value and corporate bond rate valuations. Also, when employing financial theory methods to calculate benefits, their analysis reveals dramatic variations in contributions and discounted benefits compared to an equivalent DC plan. Under the same approach, Brown et al. [6] examine the funding aspects of CB pension liabilities, emphasizing the use of techniques for valuing and hedging floating-rate bonds to determine the present-value cost and effective duration of these liabilities. They find that the present-value cost of funding a cash balance liability and its effective duration vary significantly depending on the chosen crediting rate.

As mentioned before, DC pension plans managed with participating life insurance have been very popular and more favorable to the worker as it provides a higher benefit in the long-term; however, CB pension plans managed with unit-linked insurance are more beneficial to the employer because, in years of good market performance, they end up with a surplus in their assets. The main contribution of this chapter is to compare these two intermediate hybrid solutions, namely DC-PL and CB-UL, and determine an optimal mix that balances the interests of both employers and workers. We recognize that the employer and the worker have different objectives and risk profiles when it comes to retirement benefits. The worker seeks higher benefits upon retirement, while the employer aims to minimize the supplementary costs of fulfilling their obligations. In fact, several papers focus on valuing CB plans and others on determining fair valuations for DC plans with participating contracts. However, the novelty of this study lies in its multi-objective perspective, which considers the pension plan as a contractual agreement between employers and workers, trying to model the equilibrium between these two parties. In order to analyze and evaluate the preferences and behaviors of the individuals involved in our study, we use a descriptive behavior model known as the cumulative prospect theory (CPT), and we formulate the maximization problem based on additive utility and Nash equilibrium approaches, seeking a fair agreement that maximizes the product of the two players' utilities. Indeed, by conducting a multi-criteria analysis, this study contributes to the existing literature on pension plan design by providing insights for employers and workers, particularly in the current landscape where numerous countries are considering the implementation of mandatory second-pillar pension systems. The decision between different pension systems becomes crucial in this context, emphasizing the importance of studies like ours.

Under this approach, this chapter incorporates stochastic interest rates using the Hull and White model to compute the benefits, price the participating contract to generate a fair minimum guaranteed rate, and determine the crediting interest rate. In the first part of the study, we focus on a single premium scenario where a portion of the premium is invested in the DC-PL plan and the remaining portion is invested in the CB-UL plan. In this part, the crediting interest rate is fixed throughout the contract period. Sensitivity analyses are performed to examine the impact of different model parameters on the optimal choices for both employers and workers. We show that the employer is, in all cases, attracted to the CB-UL plan, and the worker is, in most cases, attracted to the DC-PL plan. Interestingly, when considering equal importance between the employer and the worker, it becomes evident that the CB-UL plan emerges as the more favorable choice. This preference is based on the advantages and benefits associated with this plan. Additionally, two extensions are considered: a periodic premium scenario and a stochastic crediting rate based on par yields.

The remainder of the chapter is structured as follows: Section 5.2 develops our model; we describe the financial assumptions of both the DC-PL and CB-UL plans, and we highlight the utility model used in our framework. Moreover, we describe our optimization problem by stating the risk factors and maximizing the utilities of the two players in order to find the optimal mix depending on the weighting coefficient between them. In Section 5.3, we illustrate a numerical example stated as a basic scenario and carry out sensitivity analyses of various parameters. The study is extended in Section 5.4 to account for a periodic premium scenario as well as a stochastic crediting interest rate.

5.2 Model and assumptions

5.2.1 Purpose

In a pension scheme, the interests of the worker and the employer are clearly different. Both parties are exposed to different risks at the retirement date: the risk of insufficient benefits for the worker compared to a target, and the risk of insufficient fund for the employer to meet his liability. In other words, the worker aims for a higher level of benefit on retirement, while the employer aims to reduce the supplemental cost of his obligation toward the worker. In this regard, DC-PL and CB-UL plans seem to be suitable intermediate choices to answer the needs of the worker and the employer, so mixing these two plans based on the players' behaviors is a kind of compromise between these two parties and allows us to model the financial risk sharing between the two parties and eventually find the optimal choice depending on the difference of interests between them.

As a result, we have a market where two kinds of people, the employer and the worker, are interested in two different pension plans. The aim is to maximize their utilities based on their preferences by weighting the risk between the two parties, where a part is computed based on a DC-PL contract and the remaining part based on a CB-UL contract. In order to measure the utility, we use a behavior model called the cumulative prospect theory (CPT), a descriptive theory based on experimental evidence that aids in understanding how people make decisions in uncertain situations. This framework is detailed in Section 5.2.2.

5.2.2 Assumptions

In this chapter, we are interested in modeling the financial risks the employer and the worker are exposed to in order to find an optimal choice between their varying interests, so we examine the contracts from a pure financial point of view, ignoring mortality risk and assuming only the risks associated with an uncertain evolution in financial markets.

Defined contribution

The first system taken into account in our model is the DC plan, whose benefit at maturity is the value of the participant's account balance at the time of retirement. The account balance is made up of contributions made by the worker and employer, as well as credited returns on those contributions. We assume here that the plan is managed with a participating life insurance contract (DC-PL) so the participant receives a minimum guaranteed rate of return in addition to a bonus in years of good investment performance.

In our study, we use two important concepts in finance as the foundation for our analysis. The first concept is the valuation, as we aim to value the participating contract at zero to fix the control parameters (the parameters chosen by the insurance company), for which we need to define the processes under the risk-neutral measure $\mathbb{Q}$ that is used for the valuation in a complete arbitrage-free market. The second concept is risk management, as we want to compute the utility at maturity, and for this we look at the real situation under the real probability measure $\mathbb{P}$. Therefore, we need to define the different processes in both measures.

In our model, the insurer trades the term structure of interest rates on a complete financial market assuming that it evolves according to the one-factor Hull and White model [14], which is a stochastic model that describes the evolution of interest rates over time. The model assumes that the short-term interest rate r_t follows, under the risk-neutral measure $\mathbb{Q}$, a mean-reverting process that can be described by the following stochastic differential equation:

$$\mathrm{d}r_t = \theta\,(\nu_t - r_t)\,\mathrm{d}t \,+\, \sigma_r\,\mathrm{d}W_t^r, \quad t \in [0,T] \text{ and } T \in \mathbb{N}, \tag{5.2.1}$$

where T is the time of retirement, ν_t is the time-varying long-term mean of the interest rate, θ is the speed of mean reversion, and σ_r is the volatility parameter (θ and σ_r are positive constants). W^r is a Brownian motion under the risk-neutral measure $\mathbb{Q}$ defined on a financial filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{Q})$ for all t in the time interval [0,T].

We define under this measure the price of a zero-coupon bond, with no arbitrage, as follows:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t\Big],$$

where $P(t,T)$ is the price of the zero-coupon bond at time t that matures at time T. It is well known that the explicit form under Hull and White is given by:

$$P(t,T) = \exp\left(-r_t \cdot B_r(t,T,\theta) - \theta \int_t^T \nu_s \cdot B_r(s,T,\theta)ds + \frac{\sigma_r^2}{2}\int_t^T B_r^2(s,T,\theta)ds \right), \tag{5.2.2}$$

with $B_r(t,T,\theta) = \frac{1-e^{-\theta(T-t)}}{\theta}$. The price of the zero-coupon bond depends on the current short-term interest rate r_t, as well as the parameters of the model: the time-varying long-term mean of the interest rate ν_t, the speed of mean reversion θ, and the volatility parameter σ_r.

On the other hand, under the real probability measure $\mathbb{P}$, the risk-free rate r_t is a solution of the modified stochastic differential equation:

$$\mathrm{d}r_t = \theta\,(\overline{\nu}_t - r_t)\,\mathrm{d}t \,+\, \sigma_r\,\mathrm{d}\overline{W}_t^r, \quad t \in [0,T], \tag{5.2.3}$$

where $\overline{W}^r$ is the Brownian motion defined under $\mathbb{P}$ as $\mathrm{d}\overline{W}_t^r = \mathrm{d}W_t^r - \zeta \cdot \mathrm{d}t$; ζ stands for the market price of interest rate risk, and accordingly, $\overline{\nu}_t = \nu_t + \frac{\sigma_r \cdot \zeta}{\theta}$.

In addition, we assume a fund called G, defined as the insurer's general fund. It generally has a low level of volatility, is not traded directly on the market, and is used as a benchmark for the participating contracts. The fund G is assumed to follow, under the real probability measure, a geometric Brownian motion model, which is a popular model used to describe the dynamics of a fund price:

$$\mathrm{d}G_t = \mu_G\,G_t\,\mathrm{d}t \,+\, \sigma_G\,G_t\,\mathrm{d}\overline{W}_t^G, \quad G_0 = 1,\ t \in [0,T] \tag{5.2.4}$$

where G_t is the price of the fund at time t, μ_G is the drift or expected rate of return of the fund, σ_G is the volatility of the fund assumed to be positive and constant, and $\overline{W}^G$ is a standard Brownian motion under the real probability measure $\mathbb{P}$ that models the dynamics of the fund G.
It is trivial that the assets on the financial market are correlated, so we define $\rho_{r,G}$ as the correlation parameter between the Brownian motions $\overline{W}^r$ and $\overline{W}^G$ such that $\mathbb{E}^{\mathbb{P}}[\mathrm{d}\overline{W}_t^r\mathrm{d}\overline{W}_t^G] = \rho_{r,G} \cdot \mathrm{d}t$, which leads to the following stochastic differential equation for the fund G using the Cholesky decomposition for correlated Brownian motions:

$$\mathrm{d}G_t = \mu_G\,G_t\,\mathrm{d}t \,+\, \sigma_G\,\rho_{r,G}\,G_t\,\mathrm{d}\overline{W}_t^r + \sigma_G\sqrt{1-\rho_{r,G}^2}\,G_t\,\mathrm{d}\overline{\overline{W}}_t^G, \quad G_0 = 1,\ t \in [0,T] \tag{5.2.5}$$

with

$$\overline{W}_t^G = \rho_{r,G} \cdot \overline{W}_t^r + \sqrt{1-\rho_{r,G}^2} \cdot \overline{\overline{W}}_t^G, \tag{5.2.6}$$

where $\overline{\overline{W}}^G$ is a new $\mathbb{P}$-Brownian motion generated by Cholesky decomposition. It is the specific risk associated with the general fund and independent from $\overline{W}^r$.
Subsequently, we write down the annual return of G as follows:

$$g_t = \frac{G_t}{G_{t-1}} - 1, \qquad t = 1, 2, \ldots, T, \tag{5.2.7}$$

where G_t is the solution of Equation (5.2.5) under the real probability measure:

$$G_t = G_0 \, \exp\left(\left(\mu_G - \frac{\sigma_G^2}{2}\right) \cdot t \,+\, \sigma_G \, \rho_{r,G} \int_0^t \mathrm{d}\overline{W}_s^r + \sigma_G \sqrt{1-\rho_{r,G}^2} \int_0^t \mathrm{d}\overline{\overline{W}}_s^G\right).$$

After presenting the financial assumption of the DC-PL insurance, we start by considering a single contribution P paid at the inception of the contract and invested in the participating fund. The contract guarantees an annual minimum guaranteed interest rate i^{PL} in addition to a surplus return δ_t as a participation in the profits of the insurer if a part of the real rate of return, which is a market rate obtained from the investment in the general fund G, exceeds the guaranteed interest rate i^{PL}. The surplus return is computed annually inside the contract and translated into the following formula:

$$\delta_t = \max(\beta \cdot g_t - i^{PL}, \, 0), \quad t \in [0, T]$$

where $\beta \in [0, 1]$ is called the participation level that is assumed to be constant for the whole period of the contract. Consequently, the benefit at the time of retirement T, denoted V_T^{PL}, is equal to the contribution capitalized each year in the guaranteed interest rate in addition to the possible bonus as follows:

$$V_T^{PL} = P \cdot \prod_{t=1}^{T} \left(1 + i^{PL} + \delta_t\right). \tag{5.2.8}$$

It is worth noting that we assume the couple of parameters (i^{PL}, β) to be fair because the participating contract has to be priced fairly in order not to have a distortion of decision-making based on an arbitrary choice for these parameters that can be too generous as well as too miserly.
In this regard, we use the risk-neutral probability measure $\mathbb{Q}$ for the valuation in a complete and arbitrage-free market, and we write the dynamics of the fund G under this measure, as follows:

$$\mathrm{d}G_t = r_t \, G_t \, \mathrm{d}t \,+\, \sigma_G \, \rho_{r,G} \, G_t \, \mathrm{d}W_t^r \,+\, \sigma_G \sqrt{1-\rho_{r,G}^2} \, G_t \, \mathrm{d}W_t^G, \quad G_0 = 1, \, t \in [0, T] \tag{5.2.9}$$

where W^G is a $\mathbb{Q}$-Brownian motion, independent from W_t^r, and defined by

$$\mathrm{d}W_t^G \;=\; \mathrm{d}\overline{\overline{W}}_t^G \;+\; \frac{\mu_G - r_t - \sigma_G \cdot \rho_{r,G} \cdot \zeta}{\sigma_G \sqrt{1-\rho_{r,G}^2}} \, \mathrm{d}t.$$

Consequently, the financial fair valuation price of the benefit, defined in Equation (5.2.8), is obtained by computing the risk-neutral expectation of the terminal cash flow as follows:

$$FV_0^{PL} \;=\; \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s \mathrm{d}s} \cdot V_T^{PL}\right]. \tag{5.2.10}$$

In order to solve this expectation, we define the T-forward measure from the risk-neutral measure $\mathbb{Q}$, choosing as a numéraire the zero-coupon bond and using Girsanov's theorem for the change of measure:

$$\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} \;=\; \exp\left(-\sigma_r \int_0^T B_r(s, T, \theta) \, \mathrm{d}W_s^r - \frac{1}{2}\sigma_r^2 \int_0^T B_r^2(s, T, \theta) \, \mathrm{d}s\right).$$

Then, $\tilde{W}_t^r = W_t^r + \int_0^t \sigma_r B_r(s,T,\theta)ds$ is a $\mathbb{Q}^T$-Brownian motion.
Under this new measure, we write the fair valuation price in Equation (5.2.10) as follows:

$$\begin{aligned} FV_0^{PL} &= P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[V_T^{PL}\right] \\ &= P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[P \cdot \prod_{t=1}^{T}\left(1 + i^{PL} + \delta_t\right)\right] \\ &= P \cdot P(0,T) \cdot \mathbb{E}^{\mathbb{Q}^T}\left[\prod_{t=1}^{T}\left(1 + i^{PL} + \max(\beta \cdot g_t - i^{PL},\, 0)\right)\right], \end{aligned}$$

where g_k is the annual return of the fund G.

Then, we equalize the fair value to the initial contribution P in order to obtain the fairness relation. We note that the result is not explicit due to the dependency between the successive annual returns of the fund G; however, it will be simulated to find the fair couple (i^{PL}, β) to use, thereafter, as a reference for our model.

Cash balance

The second system accounted for in our model is the CB plan, whose benefit at maturity is expressed based on the notional cash balance account that grows over time with interest credits and is paid out as a lump sum at retirement. At the end of the plan's term or when the worker retires, the benefit at maturity is the accumulated balance in the worker's cash balance account, which includes contributions made by the employer and the guaranteed interest credits earned over the years. We note that the contribution is paid once at the inception of the contract. Let F_0 be the initial account value and i_{t-1}^c be the annual crediting interest rate declared at $t-1$ for the year $t-1$ to t:

$$F_T = F_0 \cdot \prod_{t=1}^{T}(1 + i_{t-1}^c), \tag{5.2.11}$$

where F_T is the benefit of the worker at the time of retirement T and $F_0 = P$ the contribution made at the inception of the contract. The crediting rates can be fixed over the years or stochastic based on Treasury bonds. Both cases will be treated in this chapter.

Apart from the worker's guaranteed retirement benefit, we assume that the CB plan is managed with a unit-linked life insurance contract with no maturity guarantee (CB-UL). Hence, the employer receives the return of the investment funds based on the performance of the underlying investments. The value of the unit-linked contract is determined by the fund S, an investment fund chosen by the employer. The fund follows a geometric Brownian motion with the following dynamic:

$$dS_t = \mu_S\, S_t\, dt + \sigma_S\, S_t\, d\overline{W}_t^S, \quad S_0 = 1,\ t \in [0,T] \tag{5.2.12}$$

where S_t is the price of the fund at time t, μ_S is the drift, σ_S is the volatility of the fund assumed to be positive, constant and higher than σ_G, and $\overline{W}^S$ is a standard Brownian motion under the real probability measure $\mathbb{P}$ that models the dynamics of the fund S. We define $\rho_{r,S}$ the correlation parameter between the risk factors $\overline{W}^r$ and $\overline{W}^S$ such that $\mathbb{E}^{\mathbb{P}}[d\overline{W}_t^r d\overline{W}_t^S] = \rho_{r,S} \cdot dt$. Analogously to the fund G, using Cholesky decomposition, the differential equation for the fund S is given by:

$$dS_t = \mu_S\, S_t\, dt + \sigma_S\, \rho_{r,S}\, S_t\, d\overline{W}_t^r + \sigma_S \sqrt{1 - \rho_{r,S}^2}\, S_t\, d\overline{\overline{W}}_t^S, \quad S_0 = 1,\ t \in [0,T] \tag{5.2.13}$$

with

$$\overline{W}_t^S = \rho_{r,S} \cdot \overline{W}_t^r + \sqrt{1 - \rho_{r,S}^2} \cdot \overline{\overline{W}}_t^S, \tag{5.2.14}$$

where $\overline{\overline{W}}^S$ is a new $\mathbb{P}$-Brownian motion generated by Cholesky decomposition. It is the risk of the investment fund independent of $\overline{W}^r$.
Note that the annual return of S is

$$s_t = \frac{S_t}{S_{t-1}} - 1, \quad t = 1, 2, \dots, T, \tag{5.2.15}$$

where S_t is the solution of Equation (5.2.13) under the real probability measure:

$$S_t = S_0 \exp\left(\left(\mu_S - \frac{\sigma_S^2}{2}\right) \cdot t + \sigma_S \rho_{r,S} \int_0^t \mathrm{d}\overline{W}_s^r + \sigma_S \sqrt{1-\rho_{r,S}^2} \int_0^t \mathrm{d}\overline{\overline{W}}_s^S\right).$$

Likewise, we consider a single contribution P paid at the inception of the contract and invested in the unit-linked fund. The returns on this fund are, in general, more volatile than those of the general fund, making the benefit at the time of retirement uncertain, not guaranteed, and expressed as follows:

$$V_T^{UL} = P \cdot \prod_{t=1}^{T} (1 + s_t) = P \cdot \frac{S_T}{S_0},$$

where V_T^{UL} is the value of the unit-linked contract given to the employer at maturity T.

CPT framework

To solve the optimization problem, we use a psychological theory of decision-making that provides a descriptive account of how people make decisions in situations where the outcomes are uncertain. According to CPT (Cumulative Prospect Theory), people evaluate potential outcomes, denoted by X, based on two main factors: the probability of the outcome occurring and the value or utility of the outcome. The model assumes that people display loss aversion as they tend to value potential losses more than equivalent gains and suggests that gains and losses are relative to some reference point where the utility function is S-shaped and concave to the right of the reference point (the gain part) and convex to the left (the loss part). Based on that, psychologists Tversky & Kahnemann [25] developed the following subjective utility function:

$$U_J(z) = \begin{cases} (z-\Gamma)^{\gamma_J} & z \geq \Gamma_J, \qquad J = E \text{ (Employer)} \\ -\lambda_J \cdot (\Gamma - z)^{\gamma_J} & z < \Gamma_J, \qquad \text{or } W \text{ (Worker)} \end{cases}$$

where γ_J is the risk aversion parameter that measures gains or losses. z refers to the outcome where some values fall above the reference point Γ_J and the others fall below. λ_J is the loss aversion parameter that represents the sensitivity to losses over gains.

However, CPT assumes that people do not evaluate these factors in a straightforward, rational manner. For example, people tend to overestimate the probability of rare events and underestimate the probability of common events. This is modeled by a probability weighting function proposed by Prelec [22]:

$$w_J(p) = e^{-(-\log p)^{\varphi_J}}, \qquad J = E \text{ (Employer) or } W \text{ (Worker)}$$

where $0 < \varphi_J < 1$ is a free parameter that controls the curvature of the function, w_J is strictly increasing with $w_J(0) = 0$, and $w_J(1) = 1$.

With the utility and weighting functions, we can define the cumulative prospect expected utility as follows:

$$CPT_J(X) := \int_{-\infty}^{0} U_J(q)\, \mathrm{d}(w_J(F(q))) + \int_0^{\infty} U_J(q)\, \mathrm{d}(-w_J(1-F(q))),$$

with $F(q) = \mathbb{P}(X \leq q) = \int_{-\infty}^{q} \mathrm{d}\mu_X$ is the distribution function and μ_X is the probability measure given by the random variable X.

5.2.3 The optimization problem

The employer's role in the pension plan is to make contributions on behalf of the workers, either as a single payment at the beginning of the contract or through periodic payments. The employer's objective is to manage the cost of the pension plan effectively. The cost is primarily determined by the contribution amount, which remains the same for both the DC-PL and CB-UL systems. However, the employer may also be responsible for additional payments at the time of retirement to fulfill the promised liabilities to the worker or to recover any surplus assets, particularly in the CB-UL plan. Therefore, the perception of risk is about this final surplus or final loss.
On the other hand, the worker does not pay contributions. In the DC-PL plan, he receives the value of the participating account, which includes a minimum guaranteed interest rate, whereas in the CB-UL pension plan, the worker receives the guaranteed benefit as specified in the cash balance contract, irrespective of the value of the unit-linked account, which could be higher due to favorable market conditions. At the same time, this provides the worker with protection in case of a market downturn. Essentially, the worker's cash flow is determined by the quality of their final liability, and the choice between the two systems influences the perceived risk associated with that cash flow.

As a result, we recall that the employer wants to lower the supplemental cost of his commitment to the worker, who, in turn, wants a higher level of benefit upon retirement. The criteria used to carry out this analysis are:

- The funding status of the employer, which is the difference between the assets according to the type of management operated by the insurance company and the liabilities, which are the obligations toward the worker. This amount can be positive (i.e., a surplus) or negative (i.e., a deficit), depending on whether the assets in the plan exceed or fall short of the promised benefits; it depends on the plan system along with the type of management as follows:

  - DC-PL contract: 0 as there is no obligation on the employer in the DC plan that does not offer any guarantee to the worker at retirement[2]. In fact, the risk is transferred from the employer to the insurer, who has an obligation to the worker through the management of the participating contract in a DC plan.
  - CB-UL contract: $P \cdot \left(\frac{S_T}{S_0} - \prod_{t=1}^{T} \left(1 + i_{t-1}^c\right) \right)$ as the employer receives on his asset's portfolio the value of the UL account and has to pay the guaranteed benefit, of the CB side, to the worker.

- The difference between the benefit at maturity and the premium capitalized at a worker target interest rate, called l, is set by the worker in a way that reflects his appreciation for what he would like to receive at maturity:

  - DC-PL contract: $P \cdot \left(\prod_{t=1}^{T} \left(1 + i^{PL} + \delta_t\right) - (1+l)^T \right)$ as the worker receives the value of the PL account at the time of retirement.
  - CB-UL contract: $P \cdot \left(\prod_{t=1}^{T} \left(1 + i_{t-1}^c\right) - (1+l)^T \right)$ as the worker receives the cash balance guaranteed benefit.

  We note that the worker can aim, in particular, for capital protection, for which the target l is set to zero.

[2]In some countries, there could be supplemental legal obligations (cf. the Law on Occupational Pensions (LPC) in Belgium).

Before measuring the utility of each party, we combine the two pension plans, where a fixed proportion $x \in [0,1]$ of the initial contribution is invested in the DC-PL plan and the remaining proportion $1-x$ is invested in the CB-UL plan. The investment of the contribution is based on a non-rebalancing case, which means that no additional action is performed after the first setting of the proportion x at the beginning of the contract. So, we see at the time of retirement:

For the funding status of the employer:

$$GE(T) = P \cdot \left[x \cdot 0 + (1-x) \cdot \left(\frac{S_T}{S_0} - \prod_{t=1}^{T} \left(1 + i_{t-1}^{c}\right) \right) \right]; \tag{5.2.16}$$

For the benefit-target of the worker:

$$GW(T) = P \cdot \left[x \cdot \prod_{t=1}^{T} \left(1 + i^{PL} + \delta_t\right) + (1-x) \cdot \prod_{t=1}^{T} \left(1 + i_{t-1}^{c}\right) \right] - P \cdot (1+l)^T. \tag{5.2.17}$$

After setting the mixed risk factor for each player, we measure the utility of the employer and the worker toward those mixed products by computing the expected utility of Equations (5.2.16) and (5.2.17), respectively, using the CPT framework. We note that the utility function of the employer could be different from the utility function of the worker because the latter, as an individual person seeking a better pension for his retirement, is, in general, supposed to be more risk averse than an employer managing the pension plan, so their risk appetites could be different.
As stated in Section 5.2.1, the objective is to maximize the utility of both parties under the described framework and indicated assumptions. Therefore, the next step is to weight the risk measures of the worker and the employer, eventually maximizing their utilities in order to find the optimal mix between DC-PL and CB-UL pension plans depending on the weighting coefficient between the two players. In this regard, we approach the problem of bargaining between the two parties based on an additive utility as well as the Nash equilibrium.

Hence, we first model the employer's decision-making process using CPT_E, the cumulative prospect function with the risk aversion parameter γ_E, the loss aversion parameter λ_E, and the free parameter φ_E that reflects the employer's risk appetite. The worker's decision-making process is modeled using CPT_W, another cumulative prospect function with the risk aversion parameter γ_W, loss aversion parameter λ_W, and free parameter φ_W adapted to the worker's risk profile.

To capture the multi-criteria nature of the decision problem, we introduce a weighted average of the utilities of the employer and worker, where the weight coefficients $\varrho \in [0,1]$ and $1-\varrho$ reflect the importance or power of each player in the decision-making process. In particular, when $\varrho = 0$, the worker's utility is the only consideration in the decision-making process, and the decision is made based on his preferences alone; however, when $\varrho = 1$, the employer's utility is the only consideration in the decision-making process, and the decision is made based on his preferences alone. For values of ϱ between zero and one, the decision takes into account both the employer's and worker's utilities, with the weight given to each depending on their respective power or importance.

Starting by the additive utility, the maximization problem is written as follows:

$$\max_{x\in[0,1]} \left[\varrho \cdot CPT_E(\,GE(T)\,) + (1-\varrho) \cdot CPT_W(\,GW(T)\,) \right]. \tag{5.2.18}$$

It is also possible to solve the same problem using another goal, which is based not on an additive but on a Nash equilibrium point of view, given by:

$$\max_{x\in[0,1]} \left[\left(CPT_E(\,GE(T)\,) \right)^{\varrho} \cdot \left(CPT_W(\,GW(T)\,) \right)^{1-\varrho} \right]. \tag{5.2.19}$$

In fact, the Nash bargaining problem was introduced by Nash [19] and is based on the idea that in a bargaining situation, the two players will negotiate over different products. The solution seeks to find a way that is fair to both players and that will lead to an agreement being reached by maximizing the product of the two players' utilities.

5.3 Numerical illustrations

To solve numerically our optimization problem described in Equations (5.2.18) and (5.2.19), we fix a number of parameters to find the optimal mix between the two products, DC-PL and CB-UL, based on the preferences of the employer and the worker and depending on the bargaining power between them. In this regard, we present hereafter the chosen values for the different parameters introduced in Section 5.2.2 to represent the basic scenario in this chapter. Afterwards, we carry out sensitivity analyses for a wide range of parameters in order to study their impact on the optimal result obtained in the basic scenario.

5.3.1 Basic scenario

We consider a pension contract that is funded by a single contribution P paid at the inception of the contract with $T = 15$ years to retirement. We assume the following parameters to illustrate our model:

- For the interest rates parameters, we fit the restricted exponential model (used to calibrate the instantaneous forward rate and developed by Cairns [8]) to the curve of Belgian zero-coupon rates on November 28, 2022, using a virtual adjustment through non-linear regression. The exponential model is given in the following form:

$$f(0,t) = \beta_0 + \sum_{j=1}^{n} \beta_j \cdot e^{-c_j t},$$

  where $f(0,t)$ is the instantaneous forward rate at 0 between t and an infinitesimal time forward. Assuming $n = 3$, a constant that determines the number of parameters in the model. The parameters β_j and c_j are estimated using non-linear least squares analysis, which generates the following values: $\beta_0 = 0.0049$, $\beta_1 = 0.005$, $\beta_2 = -0.0035$, $\beta_3 = 0.02$, $c_1 = 0.1385$, $c_2 = 0.0385$, and $c_3 = -0.02$.
  By defining the instantaneous forward rate, we can easily compute the mean reversion level. On the other hand, we assume that the interest rate volatility $\sigma_r = 0.8\%$, the initial rate $r_0 = 0.5\%$ and the speed of mean reversion $\theta = 15\%$.
  This model generates an initial curve of spot rates displayed in Table 5.1. These rates are obtained from the following formula: $P(0,t) = \dfrac{1}{(1+R(0,t))^t}$, where $P(0,t)$ is defined from Equation (5.2.2).

R(0,1)	**R(0,5)**	**R(0,10)**	**R(0,15)**	**R(0,20)**	**R(0,25)**
0.6486%	1.1367%	1.5883%	1.9347%	2.2207%	2.4728%

Table 5.1: Spot rates for different durations observed at zero.

- For the PL and UL insurance parameters, we recall the values from Hanna et al. [11], where low and high volatility levels are chosen for the general fund G and the investment fund S, respectively, $\sigma_G = 3\%$ and $\sigma_S = 15\%$, in addition to the following values for the drifts of the two funds, $\mu_G = 3\%$ and $\mu_S = 7\%$. Moreover, we choose as an example a participation level of $\beta = 70\%$ in the profits of the insurer. Based on that and on the risk-free rate, we compute the fair guaranteed interest rate by equalizing the result of Equation (5.2.10) to P, and we obtain $i^{PL} = 0.7357\%$.

- For the correlations between the general and the investment funds with the interest rate, we fix the following values, respectively: $\rho_{r,G} = -20\%$ and $\rho_{r,S} = -15\%$.

- For the cash balance parameter, we start by assuming that the crediting interest rate is constant for the whole period of the contract ($i^c_{t-1} = i^c \,\forall\, t$) and is chosen to be in line with the risk-free rate such that $i^c = R(0, 15) = 1.9347\%$.

- For the target rate parameter, we assume that the worker appreciates a capital protection so we fix $l = 0\%$ as the minimum desired level of return.

- For the CPT parameters, we choose a reference point of $\Gamma_E = \Gamma_W = 0$ because we aim to measure the utility of the employer based on his funding status and the utility of the worker based on the difference between the benefit and a target, so the utility is assessed relative to zero. Moreover, we fix the loss aversion parameter $\lambda_E = \lambda_W = 2.25$, as estimated by Tversky & Kahnemann [25]. Several empirical studies estimate, based on experimental data, the choice of the risk aversion value for the utility function and the free parameter for the weighting function as given by Prelec [22]. The range of values for the risk aversion γ_J is between 0.19 and 0.88, and for the free parameter φ_J, it is between 0.533 and 0.94. In our study, we chose an equal value of 0.5 for φ_E and φ_W as well as an equal value of 0.2 for γ_E and γ_W. The choice of a risk aversion of 0.2 could be influenced by the size of the payoffs, as pointed out by Stott [23]. On the other hand, the choice of the same value of risk aversion for the employer and the worker is chosen as a basic scenario; a sensitivity analysis will be conducted later on different values of γ_E and γ_W. In fact, if the two players have the same utility function, it is nevertheless possible for them to have distinct negotiating positions.

We note that the previous parameters' values are fixed for the numerical part unless otherwise noted. The results are based on 100,000 simulations in order to estimate the value of the participating fund, the unit-linked fund, and the value of the CB account at maturity. Relying on these estimations, we compute the funding status of the employer as well as the benefit-target of the worker in order to eventually simulate the prospect's expected utility depending on the weighting function for each player. In this setting, we present the numerical solution for Equations (5.2.18) and (5.2.19) as a basic scenario. So, Figure 5.1 shows graphically the optimal choice between the DC-PL and CB-UL plan schemes for both players, the employer and the worker.

In Figure 5.1, we see that the worker ($\varrho = 0$) is interested in a DC-PL plan, whereas the interest of the employer ($\varrho = 1$) is in a CB-UL plan. This strong result is rational because when the DC plan is managed with participating life insurance it motivates the worker by giving him the chance to participate in the profits of the insurer on top of the guaranteed return. On the other hand, from the employer's point of view, a CB-UL plan is more attractive because he benefits from good financial returns and makes profits after paying the guaranteed benefit to the worker. Furthermore, when equal importance is given to the employer and the worker, we notice that the CB-UL seems to be the preferable choice, as we see more advantage for this plan. Additionally, expanding the range and looking at ϱ between 0.4 and 0.6, we see the trend towards the CB-UL plan as an optimal choice for both parties. Furthermore, we point out that the results obtained through Nash equilibrium converge faster to the CB-UL plan than the additive utility model.

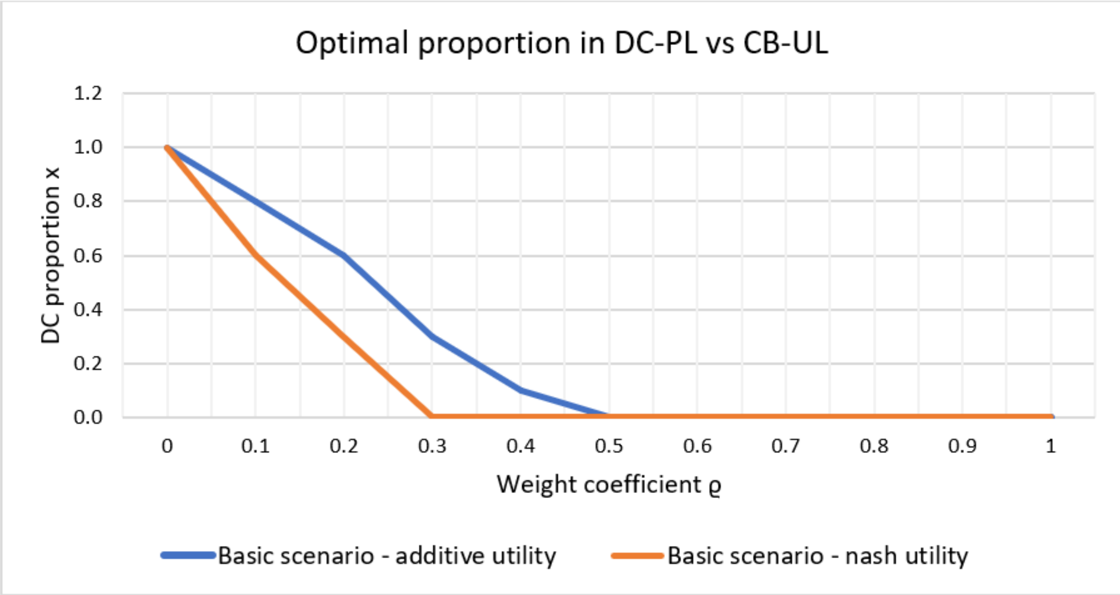

Figure 5.1: Optimal proportion in the DC (defined contribution) pension plan for the different values of the weight coefficient between employer and worker—basic scenario.

In the following, we perform sensitivity analyses for the different parameters and compare the results to the basic scenario in order to assess their impact on the optimal results.

5.3.2 Sensitivity analysis of the crediting interest rate

To check the sensitivity of the crediting interest rate i^c on the optimal choice of the plan and depending on the bargaining power, we increase and decrease i^c by 0.5%. The results are presented in Figure 5.2.

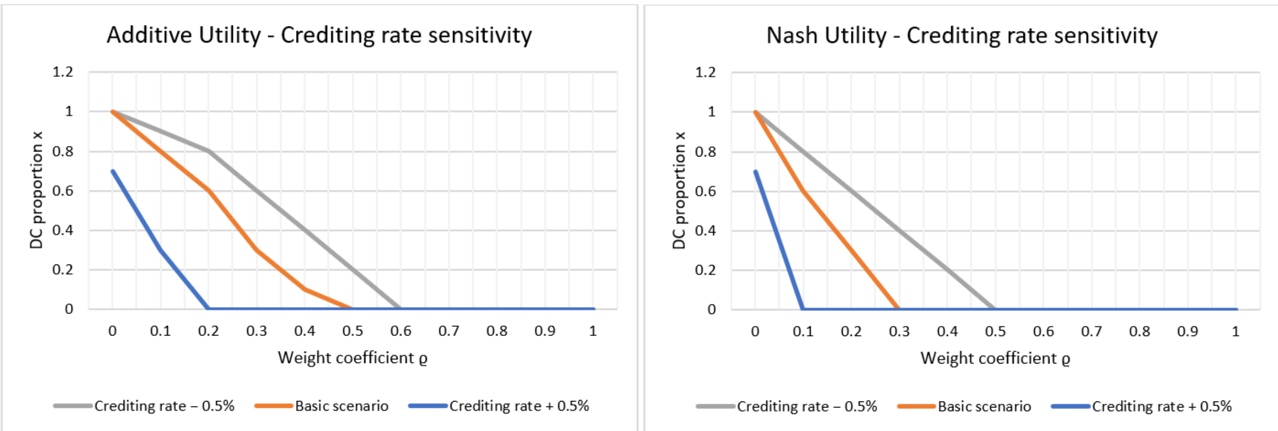

Figure 5.2: Impact of the crediting interest rate on the optimal results.

Figure 5.2 shows that by decreasing the crediting interest rate, there is still an advantage for the DC-PL plan. This advantage seems to be more interesting by sharing the power between the worker and the employer compared to the basic scenario. In other terms, if we have equal importance between the employer and the worker ($\varrho = 0.5$) in the additive utility model, the proportion x represents 20% instead of 0% in the DC-PL plan, and it needs a weight coefficient of 0.6 for the employer to reach 100% as an optimal choice in the CB-UL plan. However, by increasing the crediting interest rate by the same percentage, it is clear that the CB-UL plan becomes much more interesting than the DC-PL plan for both parties. Ultimately, we highlight that the impact of the crediting rate is similar for the two graphs in Figure 5.2, but the Nash equilibrium model tends to move the graph lines faster to a CB-UL plan.

5.3.3 Sensitivity analysis of the worker target interest rate

In this part, we study the impact of the target rate on the optimal results, so we change the value of the target rate by ± 1% and illustrate the results in Figure 5.3.

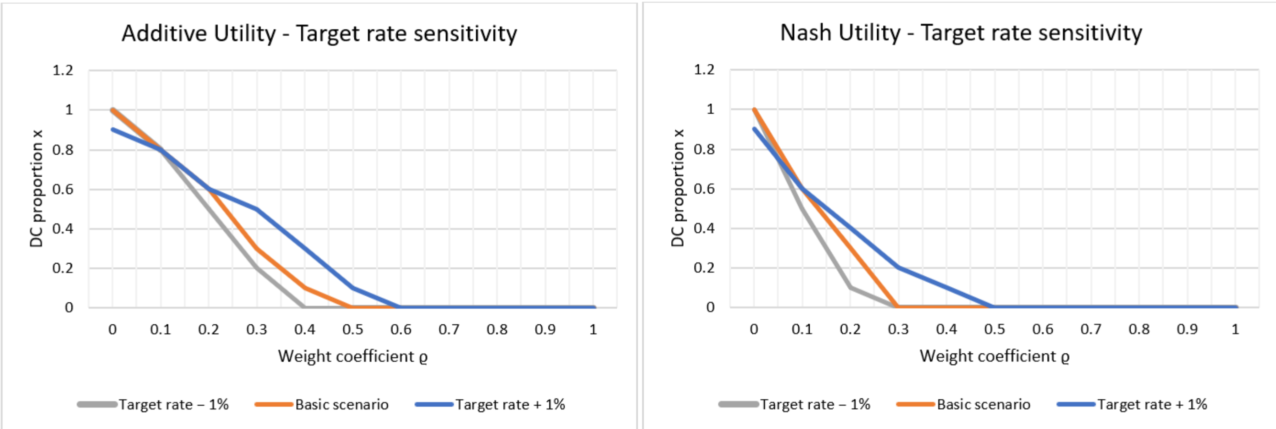

Figure 5.3: Impact of the target interest rate on the optimal results.

We point out from Figure 5.3 that with a higher target, the DC-PL plan meets less the needs of the worker, which leads to a higher preference for the CB-UL plan; however, the worker is attracted to a DC-PL plan with a smaller target. It is worth noting that for a weight coefficient between 0.2 and 0.6, the increasing effect of the target rate keeps the optimal choice between the two parties more in the DC-PL compared to the basic scenario before it converges to a 100% preference for CB-UL, yet it moves very quickly to the CB-UL plan with a decreasing effect.

5.3.4 Sensitivity analysis of the risk aversion

In this section, we focus on the impact of the risk aversion parameters γ_E and γ_W on the optimal choice of pension plans. In this respect, we once increase the risk aversion of the employer, γ_E, from 0.2 to 0.3 to compare its effect to the basic scenario, and then, analogously, we increase the risk aversion of the worker, γ_W, from 0.2 to 0.3 by setting again $\gamma_E = 0.2$, which makes the worker more risk-averse than the employer.

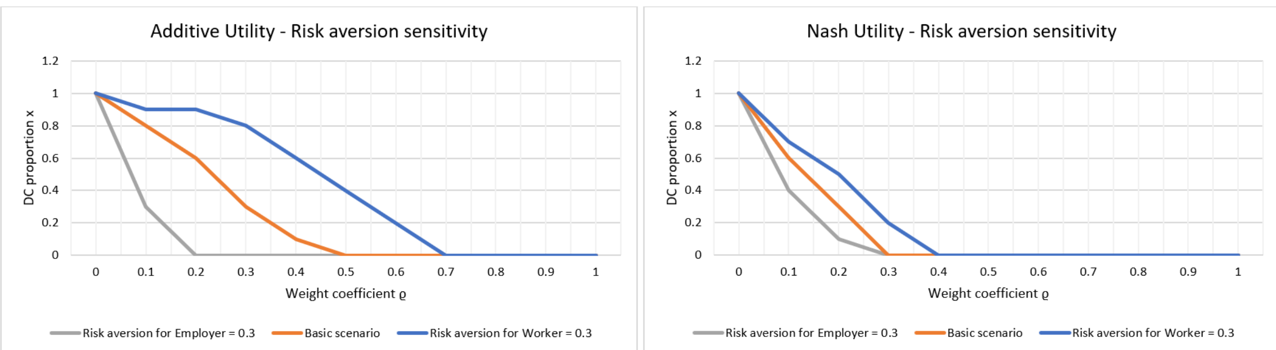

Figure 5.4: Impact of the risk aversion parameter on the optimal results.

Figure 5.4 proves that the increase of γ_W leads to the worker's unwillingness to take risks and therefore increases their preference for the DC-PL plan. Moreover, we observe in the additive utility model that for an equal weight between the two parties, the optimal decision is to invest 40% in the DC-PL plan. Similarly, the increase in the employer's risk aversion speeds up the convergence to the CB-UL plan. Interestingly, the impact is more pronounced in the additive utility than in the Nash utility model.

5.3.5 Sensitivity analysis of the participation level

In this section, we study the impact of the participation level β on the optimal investment decision for a DC pension plan managed with a participating life insurance contract versus a CB pension plan managed with a unit-linked insurance contract. We note that the participating contract must be fairly priced, which means that the change in the parameter β leads to a change in the minimum guaranteed interest rate i^{PL}. To conduct this study, we first decrease β from 70% to 60%, which gives a rate i^{PL} of 1.2285%, and then increase it to 80% and obtain a rate i^{PL} of 0.0240%.

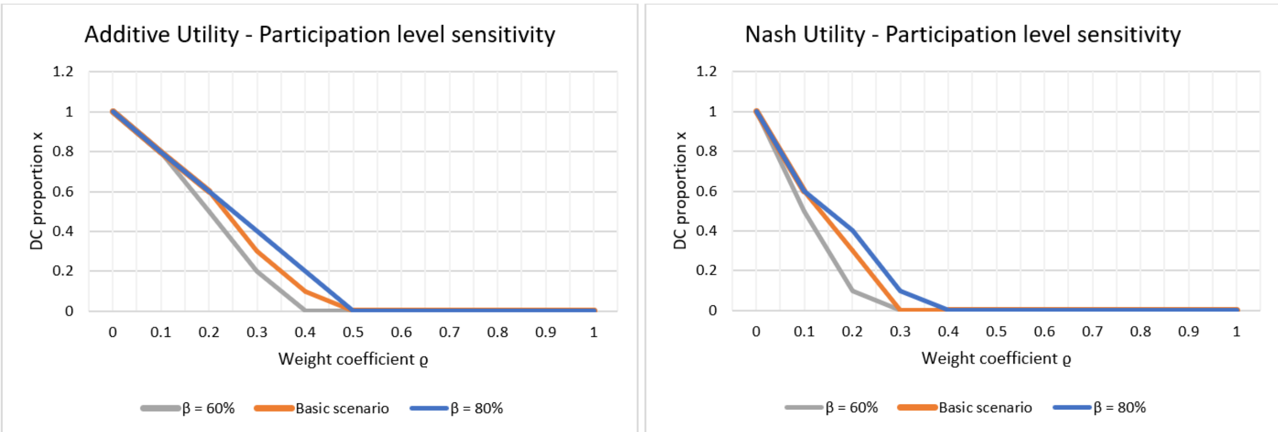

Figure 5.5: Impact of the participation level parameter on the optimal results.

We observe that as the participation level β increases (decreases), the minimum guaranteed interest rate granted by the insurer decreases (increases). Consequently, when the contract is priced fairly, the return on the participating fund remains relatively consistent. This leads to the worker's continued attraction towards the DC-PL plan, resulting in the same optimal outcome for ϱ values between 0 and 0.2 in the additive utility model, compared to the basic scenario (see Figure 5.5). However, as the bargaining power increases, there is a corresponding increase in interest towards the DC-PL plan when there is higher participation in the insurer's profits, and vice versa. It is worth noting that the impact remains largely unchanged whether we utilize additive or Nash utility models in our analysis.

5.3.6 Sensitivity analysis of the funds' volatilities

To check the sensitivity of the volatilities of the funds G and S on the optimal utility results under different bargaining power levels, we increase the G-volatility from 3% to 5% and the S-volatility from 15% to 20%. The increase in the G-volatility leads to a decrease in the fair guaranteed interest rate i^{PL}, which is now -0.7861%, and consequently an increase in the surplus return, which intuitively makes the DC-PL plan less attractive but also more preferable to the worker than the CB-UL plan, as proved in Figure 5.6. On the other hand, the increase in the return of the risky asset and in the surplus return in the participating contract keeps the optimal investment decision in the DC-PL plan longer between $\varrho = 0.2$ and $\varrho = 0.7$ in the additive utility model before it converges to the CB-UL plan. Conversely, in the Nash equilibrium model, the change in the volatilities has no impact on the optimal results for $\varrho \geq 0.4$.

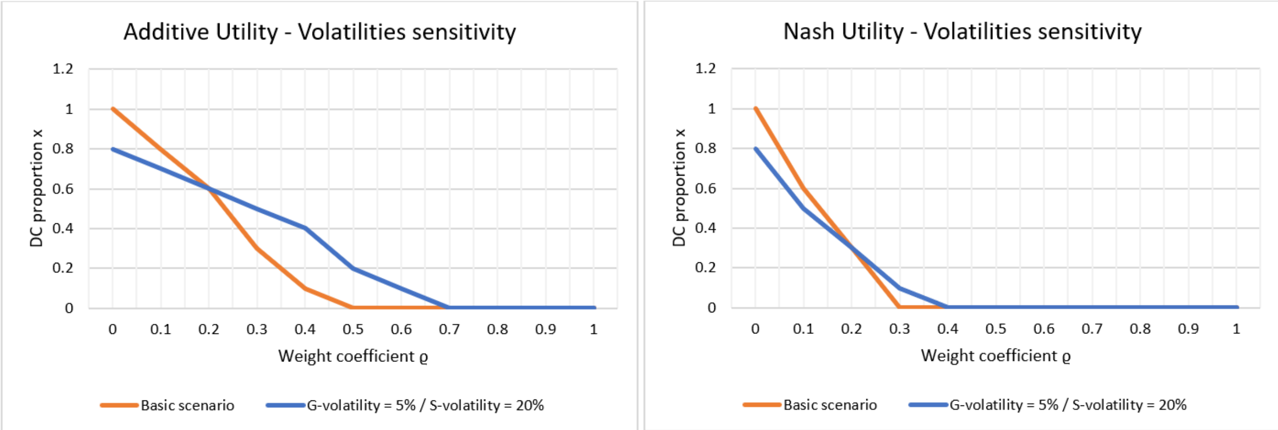

Figure 5.6: Impact of the funds volatilities parameters on the optimal results.

5.3.7 Sensitivity analysis of the time to maturity

To study the sensitivity of the maturity T on the optimal investment decision, we compare a contract with a maturity of 10 years and another with a maturity of 25 years to the basic scenario, whose maturity is 15 years, as illustrated in Figure 5.7. We note that the change in the time to maturity changes the crediting interest rate (which is chosen to be in line with the risk-free rate in this chapter) and the minimum guaranteed interest rate. A contract with maturity $T = 10$ yields a rate i^c of 1.5883% (see Table 5.1) and a rate i^{PL} of = 0.1447%, whereas a contract with maturity $T = 25$ yields a rate i^c of 2.4728% (see Table 5.1) and a rate i^{PL} of 1.3600%. We note that the crediting interest rate are chosen to be in line with the risk-free rate, as highlighted in Section 5.3.1.

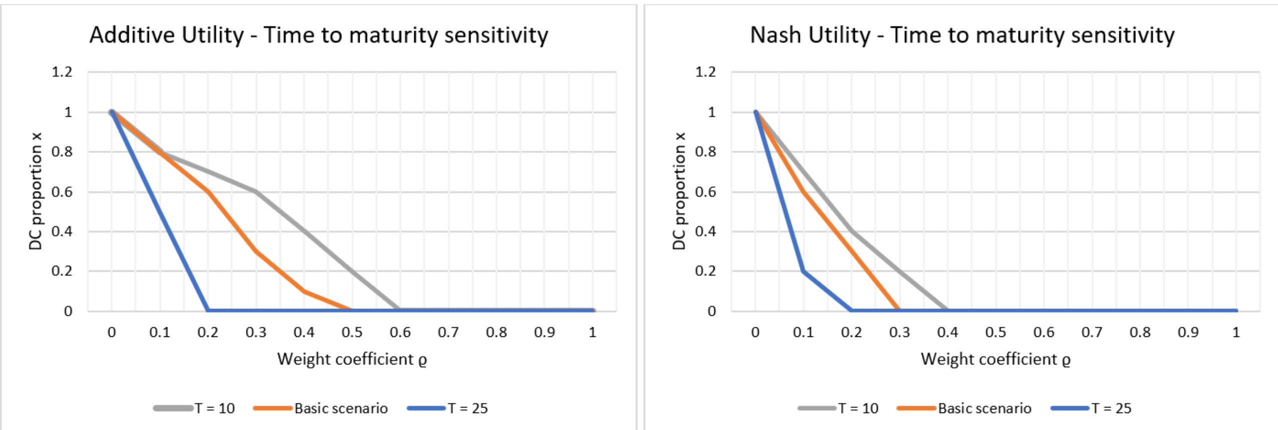

Figure 5.7: Impact of the time to maturity on the optimal results.

We point out that when the maturity of the contract decreases, the minimum guaranteed interest rate in the participating life insurance contract decreases. This should make the DC-PL plan less attractive to the worker, as they can earn a higher return with the CB plan. Interestingly, with the participation in the insurer's profits, the DC-PL plan stays more attractive to the worker for all maturity T, as eventually the return on the participating fund is higher than the return on the CB fund. At the same time, with a lower time to maturity, the return on the risky fund S decreases, which makes the CB-UL plan less attractive to the employer. Therefore, we see in Figure 5.7 that the DC-PL plan is more appreciated for a lower T, and vice versa. On the other hand, with a sharing of power between the worker and the employer, both parties' optimal decisions are more focused on the CB-UL plan. This is because, from one side, the crediting interest rate is higher and more interesting for the worker, and from the other side, the return from the unit-linked contract is higher and potentially more profitable for the employer.

5.4 Extensions

In this section, we study the impact of premium payment methods and variable crediting interest rates on the optimal choice of the employer and the worker and emphasize the importance of the stochastic crediting rate in the decision-making process.

5.4.1 Periodic premiums

In this section, we consider a pension contract that is funded by a periodic premium with a constant contribution paid annually at the beginning of the year. In this context, the single premium is divided by the expected number of annuity payments over a 15-year period to determine the amount of each periodic premium. Similar to the single premium case, a constant proportion x of the periodic premium is invested in the DC-PL plan, while $1 - x$ is allocated to the CB-UL plan. Therefore, the value of the participating contract is determined by the investment performance and contributions made over time, so the benefit at retirement for the worker in a DC-PL plan is the following:

$$\tilde{V}_T^{PL} = \tilde{P} \cdot \sum_{k=1}^{T} \left(\prod_{t=k}^{T} \left(1 + \tilde{i}^{PL} + \delta_t \right) \right), \tag{5.4.1}$$

where $\tilde{V}_T^{PL}$ is the benefit at the time of retirement, $\tilde{P}$ is the periodic premium, and $\tilde{i}^{PL}$ is the minimum interest rate guaranteed by the insurer for the whole period of the contract. In order to find the fair guaranteed interest rate, we priced the contract fairly by equalizing the fair value of Equation (5.4.1) to the fair value of the premiums as follows:

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T r_s \mathrm{d}s} \cdot \tilde{V}_T^{PL} \right] = \mathbb{E}^{\mathbb{Q}} \left[\sum_{t=0}^{T-1} \tilde{P} \cdot e^{-\int_0^t r_s \mathrm{d}s} \right]. \tag{5.4.2}$$

After some manipulations, we obtain the following fairness relation:

$$P(0,T) \cdot \sum_{k=1}^{T} \left(\mathbb{E}^{\mathbb{Q}^T} \left[\prod_{t=k}^{T} \left(1 + \tilde{i}^{PL} + \delta_t \right) \right] \right) = \sum_{k=0}^{T-1} P(0,k). \tag{5.4.3}$$

On the other hand, the second system is the CB-UL plan, where, under the periodic premium context, the worker is guaranteed a benefit at retirement based on the accumulated cash balance over the years:

$$F_T = \tilde{P} \cdot \sum_{k=1}^{T} \left(\prod_{t=k}^{T} (1 + i_{t-1}^c) \right), \tag{5.4.4}$$

whereas the employer receives the accumulation of the returns of the investment fund S and the contributions over time:

$$\tilde{V}_T^{UL} = \tilde{P} \cdot \sum_{k=1}^{T} \left(\prod_{t=k}^{T} \frac{S_t}{S_{t-1}} \right). \tag{5.4.5}$$

Eventually, the benefit of the worker, in both pension plan schemes, is compared to the accumulation of the contribution by the target rate up to the retirement date as follows:

$$\tilde{P} \cdot \sum_{k=1}^{T} (1+l)^k = \tilde{P} \cdot \frac{1+l}{l} \cdot \left((1+l)^T - 1 \right).$$

To compare numerically the periodic to the single premium case, we hold the same parameter values assumed in Section 5.3.1, and we solve Equations (5.2.18) and (5.2.19) in order to find the

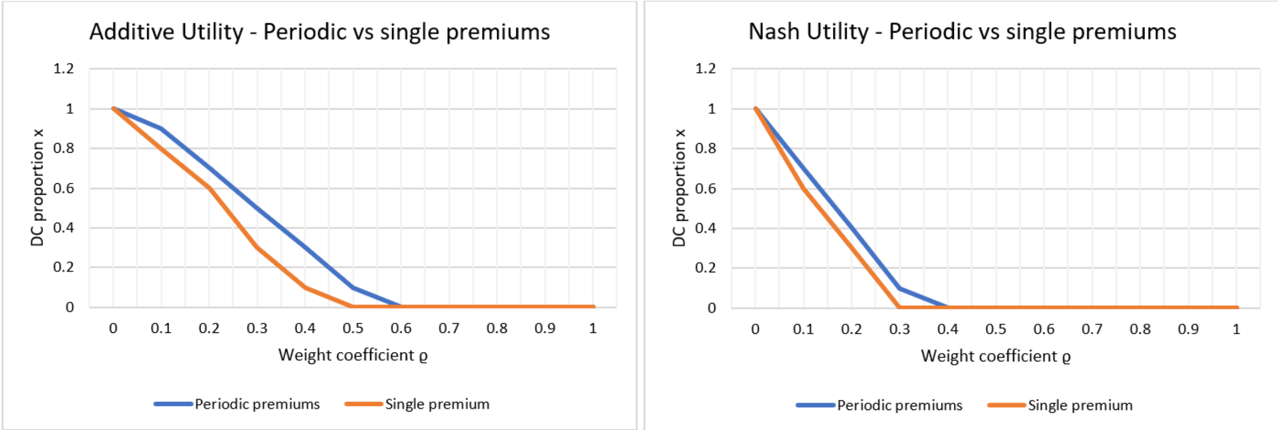

Figure 5.8: Optimal proportion in the DC pension plan for the different values of the weight coefficient between employer and worker—comparing periodic with single premiums.

optimal mix between the DC-PL and CB-UL plans, where the terms $GE(T)$ and $GW(T)$ are computed according to the risk factors described in this section above.

Comparing the results in Figure 5.8, we observe that the optimal mix favors the DC-PL plan when paying periodic premiums. This preference can be attributed to the higher minimum guaranteed interest rate offered in the participating contract for policies with periodic premiums. Specifically, the results show that the minimum guaranteed rate is 1.2547% for periodic premiums, while it is lower at 0.7357% for single premium policies. This significant difference in the guaranteed rates encourages workers to choose the DC-PL plan, as it provides the combined benefits of the minimum guaranteed rate and participation in the insurer's profits.

The results highlight the importance of considering the premium payment scenario when designing pension plans. Insurers offer a higher minimum guaranteed rate for periodic premium payments to compensate for the steady and predictable cash flow they receive. This allows them to better manage their investments and liabilities, and consequently, workers are motivated to choose the DC-PL plan to benefit from the attractive guaranteed rate and potential participation in insurer profits.

5.4.2 Stochastic crediting interest rate

Ultimately, we would like to consider the CB pension plan with a stochastic crediting interest rate, which introduces an element of uncertainty in the liabilities. As per the basic scenario, the premium is paid in a single payment at the beginning of the year, and the worker's retirement benefit is determined by the accumulated cash balance over time. Unlike traditional cash balance plans with fixed crediting rates, the stochastic crediting rate introduces variability in the notional investment performance. The crediting rate is subject to market fluctuations and can vary from year to year, influencing the growth of the cash balance. We recall Equation (5.2.11), the formula for the accumulated cash balance at retirement, assuming annual crediting interest rates:

$$F_T = F_0 \cdot \prod_{t=1}^{T} (1 + i_{t-1}^c), \tag{5.4.6}$$

where i_{t-1}^c represents the stochastic crediting rates in each year of contributions,which is a market-dependent variable. According to Hill et al. [13], most CB plans offer a crediting rate based on par yields on k-year Treasury bonds with an additional margin m, so we impose

$$i_{t-1}^c = c_k(t-1) + m, \quad t = 1, \ldots, T, \tag{5.4.7}$$

with $c_k(t-1)$ is the k-year par yield, with yearly coupons defined as follows:

$$c_k(t-1) = \frac{1 - P(t-1, t-1+k)}{\sum_{n=1}^{k} P(t-1, t-1+n)}, \quad t = 1, \ldots, T, \tag{5.4.8}$$

where $P(t-1, t-1+n)$ is the price of a zero-coupon bond defined in Equation (5.2.2) under $\mathbb{Q}$. In order to compute these prices, we simulate the short rates r_{t-1} under the real-world measure $\mathbb{P}$ using Monte Carlo simulation.

To compare numerically the stochastic to the constant crediting rate case, which is the basic scenario, we hold the same parameter values and simulate the short-term interest rates under the real probability measure, choosing a market price of risk $\zeta = -23.04\%$ as suggested in the paper by Barbarin & Devolder [1], and we choose a maturity reference of one year ($k = 1$) and a margin $m = 1\%$, as it is one of the most popular interest rate used in CB plans (see, Hill et al. [13]).

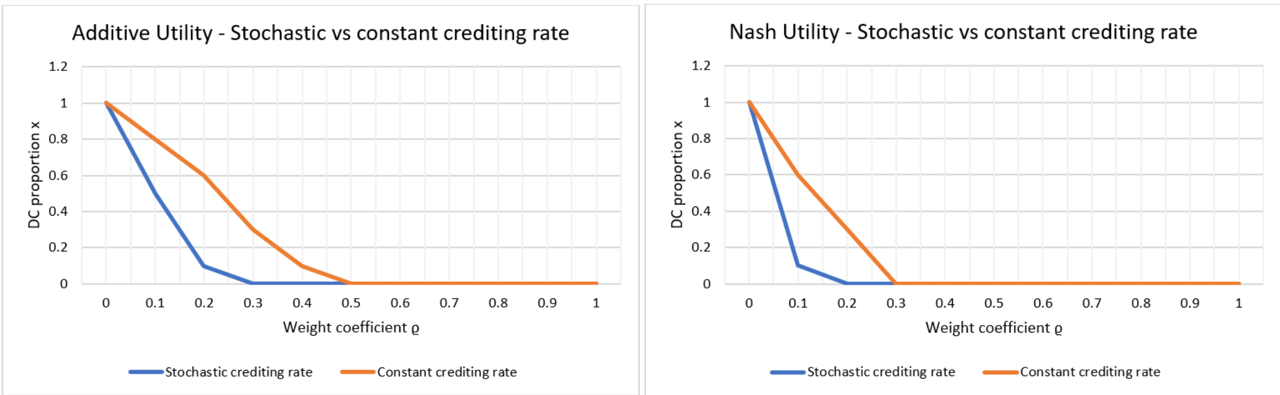

Figure 5.9: Optimal proportion in the DC pension plan for the different values of the weight coefficient between employer and worker—comparing stochastic with constant crediting rates.

We observe that the optimal utility solutions for the basic scenario are different when assuming a stochastic crediting interest rate for the CB plan, as illustrated in Figure 5.9. This result suggests that the choice between the CB and DC plans is influenced by the uncertainty in the credited interest rate. In fact, by incorporating a stochastic crediting rate, the CB plan aims to provide a more realistic representation of the potential gains or losses experienced by the underlying investments, leading to a dynamic retirement benefit estimation for workers. In our scenario, this stochastic framework generates a higher return on the CB plan, which leads to a higher preference for the CB-UL plan since the worker becomes less interested in the DC-PL plan.

Eventually, we would like to check the effect of the market price of risk parameter ζ on the optimal allocation between DC-PL and CB-UL pension plans. For this, we choose values of -30%, -23.04%, and -15%, and we derive the par yields. The results are illustrated in Figure 5.10.

The market price of risk seems to be a significant factor that determines the choice between the two plans. In fact, the increase in absolute value of the market price of risk leads to smaller values of the par yield and consequently decreases the return on the CB fund. Hence, we observe that the worker is attracted to the DC-PL plan for a high value of $|\zeta|$ as the return on the participating fund is higher.

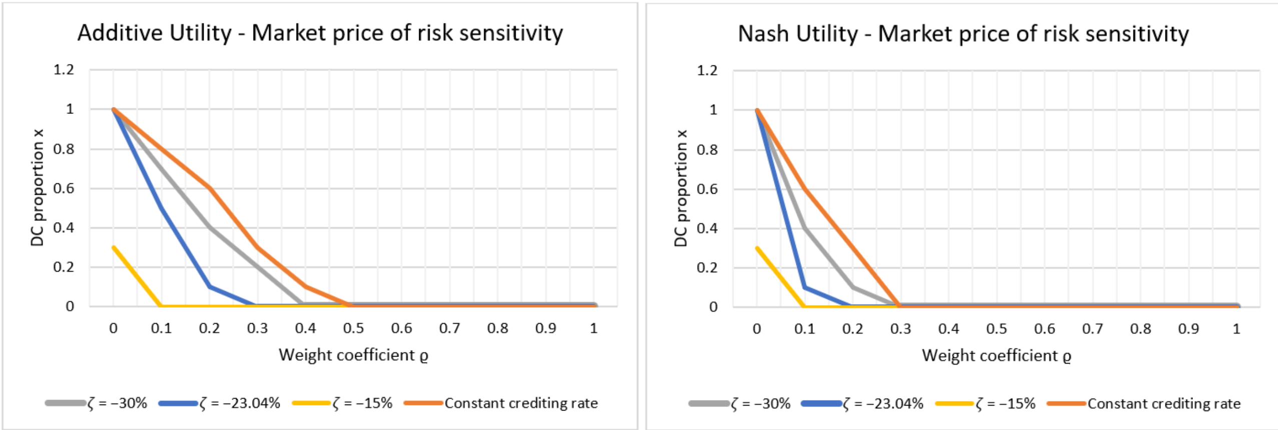

Figure 5.10: Impact of the market price of risk on the optimal results.

5.5 Discussion

The results of this study shed light on the preferences and decision-making dynamics between employers and workers in the context of pension plan design. Under this framework, we have presented a basic scenario based on a single premium with a fixed crediting interest rate throughout the contract period.

The findings demonstrate that the interests of the employer and the worker are indeed divergent, with the employer generally leaning towards the CB-UL plan and the worker being more attracted to the DC-PL plan. This aligns with the respective advantages of each plan, as the DC-PL plan offers profit-sharing potential to the worker, while the CB-UL plan provides the employer with the opportunity to benefit from favorable market performance. Particularly when equal importance is assigned to both parties, the CB-UL plan emerges as the more favorable choice, emphasizing its advantages in terms of utility maximization for both employers and workers. This is a strong result, as it holds for all the sensitivity analyses conducted in this research.

In order to examine the impact of various model parameters on the optimal choices for both employers and workers, we carried out sensitivity analyses. Some results indicate that the worker's preference may shift towards the CB-UL plan if the minimum guaranteed rate becomes less attractive, highlighting the influence of guarantee levels on decision-making. Furthermore, it is observed that when the worker desires a level of return at the retirement date higher than the single premium initially paid, there is an inclination towards a higher appreciation of the CB-UL plan, potentially driven by the worker's desire for more certainty to fulfill his aspiration. The convergent results obtained through the Nash equilibrium approach further support the dominance of the CB-UL plan in terms of utility maximization.

These findings provide valuable insights for decision-makers in selecting and designing pension plans that effectively balance the interests of both employers and workers while managing associated risks.

5.6 Conclusion

In this chapter, we looked at two intermediate hybrid pension solutions, DC-PL and CB-UL, with the aim of finding an optimal mix between the interests of both employers and workers, considering the differing interests and risk profiles of employers and workers. The analysis employs a multi-criteria approach, utilizing CPT as a behavior model and maximizing utilities through both the additive utility and Nash equilibrium approaches. The study examines various scenarios, including sensitivity analyses, periodic premiums, and stochastic crediting interest rates, to assess their impact on the optimal mix between the two pension plans based on the weighting coefficient. We have seen that the optimal choice between the two pension systems depends on market parameters as well as the risk attitudes of both parties.

This chapter contributes to the understanding of pension plan design by offering insights into the optimal mix of DC-PL and CB-UL plans, considering the preferences and objectives of both employers and workers. The results provide valuable guidance for decision-making in the design and selection of pension plans, highlighting the distinct preferences and considerations of employers and workers in achieving a balance between retirement benefits and cost management.

The issues addressed in this chapter could be very important in the context of occupational pension plans, particularly in systems where there exist mandatory second-pillar pensions. In future research, it could be useful to have a better design for the project of mandatory pension systems. In addition, it would be valuable to expand the scope of our analysis by incorporating additional risk factors, such as mortality risk. Furthermore, considering the impact of inflation on retirement benefits and extending the analysis to include the linkage of periodic premiums with salary would provide a more comprehensive understanding of the impact on optimal choices. By considering these additional factors, the generalizability of the obtained results in this broader setting can be explored.

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Chapter 6

General conclusion and perspectives

Amidst this uncertain landscape, the life insurance industry confronts a significant challenge: reconciling the policyholder's need for certain guarantees on one side and the insurer's endeavor to sustain enduring guarantees on the other. Likewise, pension schemes witness a divergence of interests between employers aiming to reduce supplementary costs associated with commitments toward workers, who in turn seek a minimum target retirement benefit. In response, this thesis has embarked on a comprehensive exploration of hybrid products in life insurance and pension schemes. They meet the needs of insurers and policyholders by reducing the gap between insurance policies that place a focus on guarantees and those that are return-oriented. Furthermore, these hybrid solutions provide a means to harmonize retirement benefits and cost management within pension plans, satisfying the distinctive preferences of employers and workers.

6.1 Summary of the main results

Chapter 2 focused on the design of a new life insurance product that combines elements from participating and unit-linked life insurance contracts. After careful analysis, it became evident that this hybrid offering provides an attractive alternative in an uncertain environment. To make the product more appealing, we have allowed for some kind of subsidy between the two components of the mixed contract by continuously deducting a fee from the unit-linked account value, and in exchange for this fee, a higher guaranteed rate is offered on the participating part. In this context, we have explored various contract structures with differing guarantee periods and investment types. Our product offered fair guaranteed rates in the participating contracts, sometimes exceeding risk-free rates, which could be particularly valuable in low interest rate environments. The model showed that a greater investment in the unit-linked contract, a higher guarantee fee, and reduced profit participation resulted in an increased guaranteed rate. So, the incorporation of a guarantee fee enhances guarantees within the model. Moreover, the model suggested that long-term or maturity guarantee contracts can boost these rates.

The study also included a customer-focused analysis employing preference functions. The study looked at the optimal investment share between the two components of the hybrid product using the utility approach and cumulative prospect theory, highlighting the superiority of the mixed product over conventional contracts. The numerical findings proved that the inclusion of a guarantee fee could enhance its attractiveness in certain scenarios, depending on the risk and return profiles of the involved unit-linked and participating funds. Intuitively, factors like risk aversion and risky asset parameters significantly influenced the optimal investment allocation.

As the contract approaches maturity, the policyholders are subject to certain risks that may ultimately affect their benefits, which has led to the creation of a lifestyle investment strategy in this respect. **Chapter 3** delved into testing the viability of this strategy within our designed mixed insurance contract. More precisely, the chapter adopted deterministic lifestyle strategies, predefined at the inception of the contract as generally proposed in practice, to examine their impact on the fair guaranteed rates offered by the insurer on one side and to find the optimal strategy from the customer's perspective by maximizing his expected utility on the other. The results in terms of guaranteed rates are in line with **Chapter 2**, while in this chapter we have also seen the impact of the shape of the investment on the fair guaranteed rates. On the other hand, regardless of the shape of the investment, the analysis revealed that the lifestyle investment strategy does not hold under the classical utility function. However, the incorporation of either periodic premiums or age-varying risk aversion led to the lifestyle investment strategy, allowing for personalized investment approaches that align with the evolving risk appetite of policyholders over the contract duration.

Incorporating more risk factors, **Chapter 4** examined the pricing dynamics of mixed life insurance products with a constant allocation between the two parts of the contract as in **Chapter 2**, but based on longevity, and proposed a multi-dimensional approach to explicitly studying the dependence between financial and mortality risks in a joint stochastic continuous time model of interest rates, stock returns, and mortality. The results confirmed that the idea of a partial guarantee, which is realized through the hybrid product, could represent an alternative trend for the insurance sector to offer better investment guarantees. Furthermore, we examined the impact of the correlation parameter combinations and generated fair combinations that increase the fair guaranteed rates according to the parameter's signs. According to the model, fair guaranteed rates rise when there is a positive correlation between funds and mortality and a negative correlation between interest and mortality rates.

Finally, the study was expanded to include pension plans in **Chapter 5**, which carefully examined the interactions between employer and worker interests, highlighting the conflict between retirement benefits and cost management since the worker seeks higher retirement benefits while the employer aims to minimize the cost of fulfilling his obligations. To address these diverse needs, we look at the two most commonly used pension schemes nowadays in practice: the defined contribution plan managed with participating life insurance (DC-PL) and the cash balance plan managed with unit-linked insurance (CB-UL). The multi-criteria analysis is conducted using the cumulative prospect theory model to measure the utility of the parties involved toward a mixed product combining these two pension plans. By assigning weights to risk measures and maximizing utilities, the study employs both additive utility and Nash equilibrium approaches. The findings revealed that the CB-UL plan aligns with employers' interests, offering potential financial gains, while the DC-PL plan attracts workers due to its profit-sharing aspect. Significantly, when equal importance is given to both parties, the CB-UL plan emerges as the prevailing choice. The results in this chapter offer a decision-making framework that successfully balances the interests of both employers and workers while managing associated risks.

6.2 Further research and perspectives

Several research subjects have been addressed throughout this dissertation, although many avenues have not been fully investigated. Here, we suggest several possible lines of research for future work to improve understanding as well as application of the ideas presented:

- In lifestyle strategies, our study has predominantly focused on deterministic investment strategies that align asset allocation with the age of policyholders, mirroring prevalent practices on the market. However, a powerful avenue emerges from a theoretical standpoint, which is the exploration of dynamic strategies such as stochastic optimal control methods. For instance, we can use the dynamic programming tool, a robust process for solving complex optimization problems, and compare these adaptable strategies with the more fixed approaches we discussed earlier in **Chapter 3**. As we navigate the terrain of stochastic optimal control, the potential result extends beyond just improved investment strategies. It could reshape how we plan for retirement by considering the true dynamics of financial markets and policyholders' evolving needs, especially when considering age-varying risk aversion.

- Instead of focusing only on financial models, another avenue of research expands upon considering mortality risk within both the lifestyle investment and optimal pension schemes chapters, offering distinct advantages and notable effects when incorporated into optimal investment decisions from the policyholder's as well as the pension parties viewpoints. Indeed, insurers and pension providers can tailor their offerings to better reflect the actual risks faced by policyholders, leading to fairer pricing and more sustainable products.

  In **Chapter 3**, accounting for mortality risk introduces a pivotal advantage because the model can design investment strategies that align more closely with the individual's risk tolerance and financial goals. This integration not only enhances the accuracy of investment decisions but also safeguards against the uncertainties of life expectancy, promoting a more resilient financial plan.

  Similarly, in **Chapter 5**, considering mortality risk facilitates the creation of pension plans that are tailored to the unique mortality profiles of policyholders. This customization optimizes the distribution of pension benefits over the course of retirement, reflecting a deep awareness of each individual's expected lifetime. Moreover, by incorporating mortality risk, policyholders can align their investment choices with their life expectancy, maximizing potential gains while mitigating risks associated with uncertain lifespans, which leads to choices that are more conducive to their long-term well-being.

  Furthermore, exploring the correlation between mortality risk and finance, as undertaken in **Chapter 4**, adds an additional layer of complexity and insight. Analyzing how fluctuations in financial markets might interact with changes in mortality rates can yield profound implications for investment strategies and pension design alike. Understanding these dynamics can lead to the development of strategies that are not only financially prudent but also responsive to shifts in mortality expectations.

- As conventional actuarial models, we have developed theory using Brownian environments, which are essentially characterized by Brownian motion processes without jumps. This approach has clearly produced valuable insights, but more study is still needed in this new territory. The proposition is to extend our existing interest rate and mortality models into a more inclusive framework capable of accommodating sudden and unexpected occurrences through the incorporation of jump processes or Levy processes. This ambitious journey is driven by the curiosity to uncover whether the conclusions derived from the traditional models still hold when confronted with the complexities of jump-induced dynamics.

  The incorporation of jump processes or Levy processes introduces an element of realism that has the potential to elevate our understanding of interest rate fluctuations and mortality dynamics. Additionally, this extended framework, incorporating jump processes or Levy processes, provides a more thorough evaluation of actuarial models, especially in periods of crisis. This enhancement considers elements that address sudden and significant market disruptions. The introduction of a Poisson process or a broader Levy process opens the door to capturing sudden shifts that have a marked impact on financial and life events. The resulting models summarize the possibility of rare but pivotal events, inherently enhancing the predictive power and applicability of the framework.

  Moreover, within this avenue of exploration lies the captivating prospect of delving into time-dependent volatility. As financial and demographic landscapes evolve over time, volatility becomes a dynamic force that shapes outcomes. Integrating time-dependent volatility into the extended models offers a sophisticated viewpoint through which we can observe the changing nature of risk and opportunity.

- Finally, in our framework of **Chapter 5**, we used a constant target interest rate that is mainly zero in our example, as the worker on one side and the European regulation on the other side appreciate capital protection. Nevertheless, it could be interesting to shed light on a crucial yet often overlooked factor that directly impacts retirement benefits. So, a final promising area is linking this target to a stochastic inflation model. A guiding principle for workers in the decision-making framework is the careful evaluation of how different inflation rates can safeguard the value of accumulated benefits. For policyholders, retirees, and financial planners alike, it aims to offer actionable insights by examining the relationship between inflation and retirement benefits.