

## A NOTE ON THE STABILITY FUNCTIONS OF THE MELLOR-YAMADA LEVEL $2\frac{1}{2}$ TURBULENCE CLOSURE

Eric DELEERSNIJDER

G. Lemaître Institute of Astronomy and Geophysics (ASTR)  
Catholic University of Louvain  
Chemin du Cyclotron, 2 - 1348 Louvain-la-Neuve, Belgium

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**Abstract.** It is shown that the occurrence of zones of high velocity shearing is an artefact that is probably due to an inappropriate behaviour of at least one of the stability functions of the Mellor-Yamada level  $2\frac{1}{2}$  turbulence closure. To avoid this problem, it is useful to introduce the assumption of local equilibrium in the stability functions.

### Introduction

Mellor and some of his co-workers have developed a hierarchy of turbulence closure models [1-4], which have been used in many geophysical fluid flow problems [5-11], including the study of the large-scale ocean circulation [12].

Throughout the present note, use will be made of the notations of Mellor and Yamada [4].

Among the various types of turbulence closures classified by Mellor and Yamada [3], the level  $2\frac{1}{2}$  associated with the boundary layer approximation has become the most popular. In this model, the vertical turbulent fluxes are computed by means of Fourier-Fick parameterizations requiring appropriate mixing coefficients. In particular, the eddy viscosity  $K_M$  and the eddy diffusivity  $K_H$  — used for most tracers — read

$$(K_M, K_H) = l q (S_M, S_H), \quad (1)$$

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where  $l$  and  $q^2/2$  are the turbulence macro-scale and the turbulent kinetic energy. The stability functions  $S_M$  and  $S_H$  satisfy the following system of algebraic equations:

$$S_M [6 A_1 A_2 G_M] + S_H [1 - 3 A_2 B_2 G_H - 12 A_1 A_2 G_H] = A_2, \quad (2)$$

$$S_M [1 + 6 A_1^2 G_M - 9 A_1 A_2 G_H] - S_H [12 A_1^2 G_H + 9 A_1 A_2 G_H] = A_1 (1 - 3 C_1), \quad (3)$$

with

$$(A_1, A_2, B_2, C_1) = (0.92, 0.74, 10.1, 0.08). \quad (4)$$

In the above equations,  $G_M$  and  $G_H$  are dimensionless quantities defined as

$$(G_M, G_H) = \frac{l^2}{q} \left( \left| \frac{\partial \mathbf{u}}{\partial z} \right|^2, - \frac{\partial b}{\partial z} \right), \quad (5)$$

where  $\mathbf{u}$ ,  $b$  and  $z$  denote the horizontal velocity vector, the buoyancy and the vertical coordinate — pointing upwards —, respectively. Notice that  $G_H$  is negative when the fluid column is stable.

One of the most prominent advantages of the Mellor-Yamada level  $2\frac{1}{2}$  turbulence closure is that  $q$  and  $l$  are obtained from the solution of partial differential equations that include numerous significant effects, namely time variation, advection, production by shear, inhibition by stratification, viscous dissipation and turbulent diffusion.

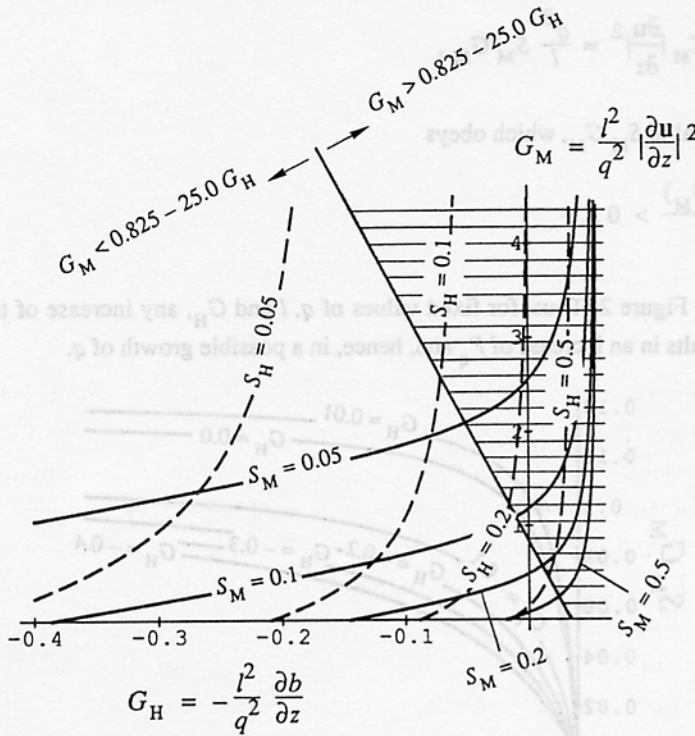
The geophysical fluid models that use the level  $2\frac{1}{2}$  closure generally achieve appreciable realism while keeping the complexities of the turbulence model at an acceptable level. In other words, the level  $2\frac{1}{2}$  model, as well as the  $k-\epsilon$  turbulence closure [13–16], turns out to be an excellent compromise between the accuracy of the results and the modelling/computational effort needed to compute the turbulent fluxes. Nevertheless, the results of the level  $2\frac{1}{2}$  closure have been repeatedly reported to be affected by a somewhat embarrassing shortcoming.

The purpose of the present note is to briefly describe that shortcoming and to show that it is certainly due to an inadequate behaviour of the stability function  $S_M$ . Finally, it will be stressed that the modified closure derived by Galperin *et al.* [10] is probably better than the classical level  $2\frac{1}{2}$  closure as regards the problem dealt with here.

## Odd behaviour of the stability functions

The author of the present note, as well as several other scientists [4, 17–18] have found that, on the vertical, “for some model simulations a discontinuity in velocity could develop and persist” [4]. Of course, no true discontinuity arises, but a region where  $|\partial u/\partial z|^2$  is exceedingly high develops. In such a region, high velocity shearing is usually associated with reduced values of the eddy viscosity, which obviously explains why it can persist. Needless to say that this phenomenon is a mere artefact.

Hereafter, a mechanism that could cause this artefact is put forward.



**Figure 1.** Contours of the stability functions  $S_M$  (solid lines) and  $S_H$  (dashed lines) in the  $G_H$ - $G_M$  space. This figure partly replaces Figure 3 of Mellor and Yamada [4] where the contours of  $S_M$  are erroneous. The hatched portion is where the value of  $G_M$  is too high to be used in the stability functions, according to the constraint suggested by Mellor and Yamada [4] (see (8)).

First, imagine that, at a given point in the flow, the vertical shearing of the horizontal velocity increases. Thus,  $G_M$  becomes higher, which, surprisingly, leads to a decrease of

$K_m$ , if  $l$ ,  $q$  and  $G_H$  are kept constant. Indeed, as illustrated in Figure 1,  $S_M$  is a decreasing function of  $G_M$ .

Next, because  $K_M$  is decreasing, the horizontal velocity shearing may continue to grow. In other words, a positive feedback between  $G_M$  and  $K_M$  develops, eventually leading to a "discontinuity in velocity", as reported by Mellor and Yamada [4].

One might object that when the velocity shearing increases the production of turbulent kinetic energy should enhance, which could allow  $q$  to grow so that the eddy viscosity might be increasing, instead of decreasing [19]. This argument is supported by the fact that the rate of production of turbulence kinetic energy,

$$P_q \equiv K_M \left| \frac{\partial u}{\partial z} \right|^2 = \frac{q^3}{l} S_M G_M, \quad (6)$$

is proportional to  $S_M G_M$ , which obeys

$$\frac{\partial(S_M G_M)}{\partial G_M} > 0, \quad (7)$$

as shown on Figure 2. Thus, for fixed values of  $q$ ,  $l$  and  $G_H$ , any increase of the velocity shearing results in an increase of  $P_q$  and, hence, in a possible growth of  $q$ .

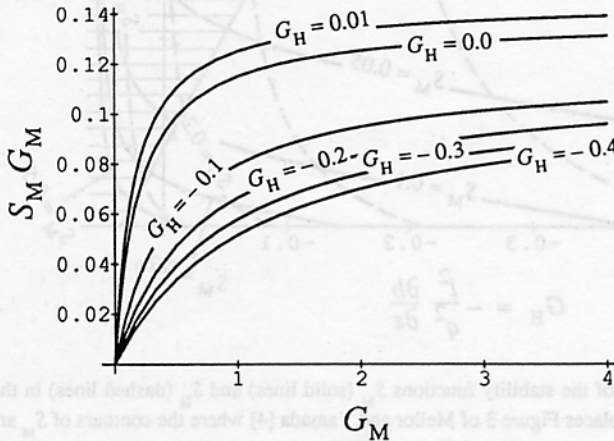


Figure 2. The product  $S_M G_M$  plotted as a function of  $G_M$ .

The above reasoning is probably correct. However, when an explicit single step numerical scheme is used, the positive feedback previously described is more likely to

prevail in certain circumstances. For the increase of  $P_q$  to counterbalance this process, a "sufficiently implicit" numerical scheme must be utilized, as confirmed by the numerical experiments of Luyten [17].

The present discussion is fully in agreement with the statement of Mellor and Yamada [4] according to which the occurrence of the phenomenon examined here "depends on the specifics of finite differencing".

To prevent the development of velocity discontinuities, Mellor and Yamada [4] recommended to constrain the possible growth of the velocity shearing by requiring that the value of  $G_M$  to be used in the stability functions satisfy the following inequality (Figure 1):

$$G_M \leq 0.825 - 25.0 G_H, \quad (8)$$

with  $G_H \leq 0.033$ . Nevertheless, it seems that the implementation of the above constraint should be accompanied by the introduction of appropriate "implicitness" in the numerical scheme. And, even so, some problems may arise, depending on the nature of the flow being studied.

One is obviously authorized to conclude that, due to the fact that  $S_M$  is a decreasing function of  $G_M$ , the level  $2\frac{1}{2}$  closure is not sufficiently robust, no matter what precautions are taken.

It is worth adding that  $S_H$  is a decreasing function of  $G_M$  (Figure 1), which is also somewhat counterintuitive. On the other hand,  $S_M$  and  $S_H$  decrease as  $G_H$  becomes smaller, reflecting in a quite natural way the stabilizing influence of the stratification.

## Quasi-equilibrium closure

In the light of this discussion, one may find it desirable to modify the stability functions in order to dispense with  $G_M$  while retaining the supposedly favorable behaviour related to the stratification. This has been achieved, yet with different aims, by Galperin *et al.* [10]. They modified some of the closure assumptions underlying Mellor's hierarchy of turbulence models, leading to the following stability functions:

$$G_M = A_1 \times \frac{1 - 3C_1 - (6A_1/B_1) - 3A_2 G_H \{ (B_2 - 3A_2)[1 - (6A_1/B_1)] - 3C_1(B_2 + 6A_1) \}}{[1 - 3A_2 G_H (6A_1 + B_2)](1 - 9A_1 A_2 G_H)}, \quad (9)$$

$$G_H = A_2 \frac{1 - (6A_1/B_1)}{1 - 3A_2 G_H (6A_1 + B_2)}, \quad (10)$$

with  $B_1 = 16.6$ . The above stability functions are free of  $G_M$ . Furthermore, they are increasing functions of  $G_H$ , as expected (Figure 3).

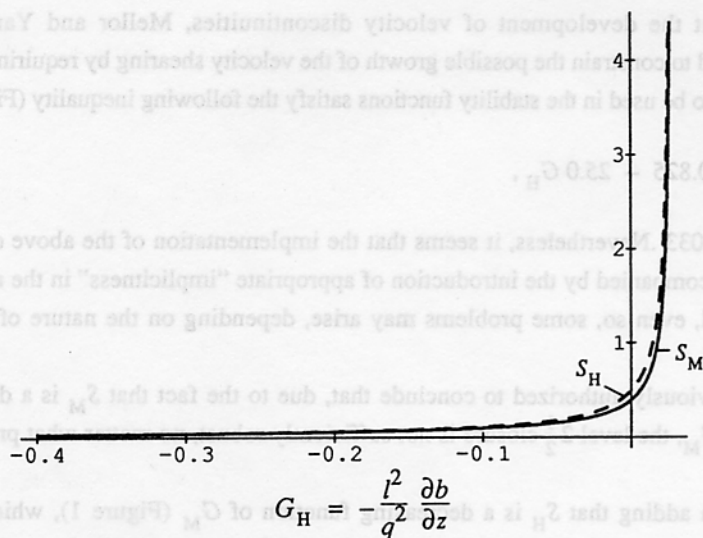


Figure 3. The stability functions  $S_M$  (solid line) and  $S_H$  (dashed line) of the quasi-equilibrium model suggested by Galperin *et al.* [10].

With this modified closure,  $q$  and  $l$  are computed by means of the same evolution equations as those used in the classical level  $2\frac{1}{2}$  closure, thus “preserving the principal advantage of the level  $2\frac{1}{2}$  model” [10].

Galperin *et al.* [10] pointed out that (9) and (10) could be obtained by assuming that, for the stability functions only, the production of turbulent kinetic energy by shear is balanced by the inhibition due to the stratification and by the viscous dissipation. This modification of the original level  $2\frac{1}{2}$  closure thus resorts to the local equilibrium hypothesis, which is primarily associated with the level 2 type of closure. Therefore, the model of Galperin *et al.* [10] lies between the level 2 and the level  $2\frac{1}{2}$ — and might be called level  $2\frac{1}{4}$ .

## Conclusion

Flaws due to the inappropriate behaviour of the stability functions of level  $2\frac{1}{2}$  turbulence closure models may be avoided by introducing the assumption of local equilibrium in the parameterizations of the stability functions. The resulting model, suggested by Galperin *et al.* [10], has proved to be more reliable and more robust. In particular, undesirable zones of high velocity shearing no longer arise [17].

The same kind of approximation may also be utilized to build robust buoyancy-extended  $k$ - $\epsilon$  closure of level  $2\frac{1}{4}$ . This is being tested [20].

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