FILTERED BICOLIMIT PRESENTATIONS OF LOCALLY PRESENTABLE LINEAR CATEGORIES, GROTHENDIECK CATEGORIES AND THEIR TENSOR PRODUCTS

JULIA RAMOS GONZÁLEZ

Abstract. We investigate two different ways of recovering a Grothendieck category as a filtered bicolimit of small categories and the compatibility of both with the tensor product of Grothendieck categories. Firstly, we show that any locally presentable linear category (and in particular any Grothendieck category) can be recovered as the filtered bicolimit of its subcategories of α-presentable objects, with α varying in the family of small regular cardinals. We then prove that the tensor product of locally presentable linear categories (and in particular the tensor product of Grothendieck categories) can be recovered as a filtered bicolimit of the Kelly tensor product of α-cocomplete linear categories of the corresponding subcategories of α-presentable objects. Secondly, we show that one can recover any Grothendieck category as a filtered bicolimit of its linear site presentations. We then prove that the tensor product of Grothendieck categories, in contrast with the first case, cannot be recovered in general as a filtered bicolimit of the tensor product of the corresponding linear sites. Finally, we show how the first presentation can be helpful when computing tensor products of cocontinuous linear functors between Grothendieck categories.

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1. Introduction

Grothendieck categories conform one of the most studied subfamilies within abelian categories, with the categories of modules and, more generally, the categories of quasi-coherent sheaves [34, Tag 077P] as the main and most well-known examples. They are essential objects in, among other fields, noncommutative algebraic geometry, where they play the role of (categorical models of) noncommutative schemes (see, for example [3], [35] or [33] among many others). For convenience of the reader, we recall their definition:

Definition 1.1. A *Grothendieck category* is a cocomplete abelian category with a generator and exact filtered colimits.

The combination of the Gabriel-Popescu theorem [30] with the theory of enriched sheaves of Borceux and Quinteiro [9] shows that Grothendieck categories are precisely the Ab-enriched topoi, that is, Grothendieck categories are the categories of Ab-enriched sheaves on small Ab-enriched sites [25], where Ab denotes the category of abelian groups.

On the other hand, it is well-known that Grothendieck categories are in particular locally presentable Ab-enriched categories [22, Cor 5.2]+[21, Prop 7.5]. Therefore, thanks to the Gabriel-Ulmer duality [15, Kor 7.11], or more accurately, to its enriched version [21, §9], we know that any Grothendieck category can be written as the Ind_α-completion of its full subcategory of α-presentable objects (which is an α-cocomplete Ab-enriched category) for a big enough regular cardinal α.

Both presentations, as categories of sheaves on an Ab-enriched site or as Ind_α-completions, provide a powerful tool when working with Grothendieck categories. This is the case, for example, when defining the tensor product of Grothendieck categories, which is our main object of interest in this paper.

The tensor product of Grothendieck categories $\boxtimes_{\mathsf{G}}$ [27] can be understood, in the framework of noncommutative algebraic geometry, as the correct notion of product between noncommutative schemes. Indeed, given two quasi-compact quasi-separated schemes X, Y we have that

$$\mathsf{Qcoh}(X) \boxtimes_{\mathsf{G}} \mathsf{Qcoh}(Y) \simeq \mathsf{Qcoh}(X \times Y) \tag{1}$$

where $\mathsf{Qcoh}(-)$ denotes the category of quasi-coherent sheaves of a scheme and $\times$ is the (fibred) product of schemes (over $\mathrm{Sp}(\mathbb{Z})$). This equivalence was first proven for projective schemes in [27, Thm 4.12], and then, more generally, shown for quasi-compact quasi-separated schemes in [10, Thm A].

The tensor product $\boxtimes_{\mathsf{G}}$ is defined in [27] in terms of the presentations of Grothendieck categories as categories of Ab-enriched sheaves. More concretely, a tensor product of Ab-enriched sites $\boxtimes_{\mathsf{S}}$ is defined, and given two Grothendieck categories $\mathcal{A}, \mathcal{B}$ with sheaf representations $\mathcal{A} = \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$, $\mathcal{B} = \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ for linear sites $(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$, $(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$, the tensor product $\mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B}$ is given by

$$\mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B} = \mathsf{Sh}((\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \boxtimes_{\mathsf{S}} (\mathfrak{b}, \mathcal{T}_\mathfrak{b})). \tag{2}$$

This is well-defined up to equivalence of categories, namely, the tensor product $\mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B}$ does not depend on the choice of Ab-enriched sites providing the sheaf presentations of $\mathcal{A}$ and $\mathcal{B}$.

On the other hand, we also have a tensor product of locally presentable categories [6] at our disposal, which is easily translated to the Ab-enriched setup, that is, to a tensor product of locally presentable Ab-enriched categories. It is shown in [27, Thm 5.4] that the tensor product of Grothendieck categories $\boxtimes_{\mathsf{G}}$ is an instance of the tensor product of locally presentable Ab-enriched categories, which we denote by $\boxtimes_{\mathsf{LP}}$.

The tensor product LP is defined in [6] by means of the presentations of locally presentable categories as Ind_α-completions of α-cocomplete small categories for suitable regular cardinal α, and this construction works both for the non-enriched and the enriched setups. More concretely, in the Ab-enriched case, we have that given $\mathcal{A}, \mathcal{B}$ two α-locally presentable Ab-enriched categories, the tensor product $\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$ is given by

$$\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} = \mathsf{Ind}_\alpha(\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha) \tag{3}$$

where $\mathcal{A}^\alpha, \mathcal{B}^\alpha$ denote the corresponding Ab-enriched subcategories of α-presentable objects and $\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha$ is the Kelly tensor product of α-cocomplete Ab-enriched categories [6], [20], [21].

The starting point of this paper is the fact that Ab-enriched site presentations and Ab-enriched categories of α-presentable objects provide us with two different ways to recover any Grothendieck category as a filtered bicolimit of small categories, which we will describe in detail using the theory of 2-filtered bicolimits of categories from [13, 14]. Our main goal is the understanding of the behaviour of these filtered bicolimits with respect to the tensor product of presentations. More concretely, given two locally presentable Ab-enriched categories $\mathcal{A}, \mathcal{B}$, we show that the filtered bicolimit over the regular cardinals α of the Kelly tensor products of α-cocomplete Ab-enriched categories of the α-presentable objects coincides with the tensor product $\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$. On the other hand, we show that the filtered bicolimit of the tensor product of Ab-enriched site presentations does not allow us to recover the tensor product of the corresponding Grothendieck categories in general.

The tensor product of locally presentable Ab-enriched categories induces a symmetric monoidal closed structure on the 2-category of locally presentable Ab-enriched categories (see, for example, [11, Cor 2.2.5]), while the Kelly tensor product induces a symmetric monoidal closed structure on the 2-category of α-cocomplete Ab-enriched categories (see [20, §6.5]). In the last part of the paper, as an application of the good behaviour of the filtered bicolimit presentations with respect to the Kelly tensor product of α-cocomplete Ab-enriched categories, we show that the functoriality, associativity and symmetry of the tensor product of locally presentable Ab-enriched categories can also be deduced from the same properties of the Kelly tensor product. In particular, this allows us to make use of the functoriality of the Kelly tensor product in order to compute tensor products of cocontinuous Ab-enriched functors between locally presentable Ab-enriched categories and in particular, between Grothendieck categories.

Grothendieck categories are our main object of interest in this paper due to their relevance in noncommutative algebraic geometry. For this reason, we focus exclusively on linear enrichments (over Ab or, more generally, over categories of modules over a commutative ring), although the results we obtain most likely hold for other enrichments of Grothendieck topoi and locally presentable categories. In order to get a well-behaved theory of enriched Grothendieck topoi [9] one needs to require that the category over which one enriches is regular in the sense of Barr [5, Chap I, §1, (1.3)], locally finitely presentable and symmetric monoidal closed, and moreover, that the full subcategory of finitely presentable objects is closed under the tensor product. We will not analyse here whether these assumptions on the enrichment are enough for our results to carry over.

Structure of the paper. We devote the first sections to the background and preliminary results required for our purposes:

In §2: We fix the notation and conventions that will be used along the paper.

In §3: We revise the main aspects of the theory of locally presentable linear categories and their presentations as Ind_α-completions. In particular, we review the relation between the Kelly tensor product of α-cocomplete linear categories and the tensor product of locally presentable linear categories and we study the behaviour of this relation when one raises the cardinality α.

In §4: We revise the basic background on the theory of linear sites and their tensor product and how they present, respectively, Grothendieck categories and their tensor product.

In §5: We revise briefly the general theory of 2-filtered bicolimits from [13, 14]. In particular, we show that the instance of 2-filtered bicolimits in which we will be interested, namely, the filtered bicolimits (i.e. 2-filtered bicolimits where the indexing category is just a filtered 1-category), is well behaved with respect to the linear enrichement (see Proposition 5.5).

Once the necessary preliminaries are settled, we present our main results:

In §6: We provide two different ways of representing a Grothendieck category as a filtered bicolimit of small linear categories. The first one, generalizable to any locally presentable linear category, states the following:

Proposition 1.2 (Corollary 6.5)**.** *Let C be a locally presentable linear category. Then C is the linear filtered bicolimit of its family of subcategories of α-presentable objects $(C^\alpha)_\alpha$, where α varies in the totally ordered class of small regular cardinals.*

The second one makes use of the topos theoretical nature of Grothendieck categories. Given a Grothendieck category C, we consider the category $\mathcal{J}_C$ of all site presentations of C, that is, all the LC morphisms $(\mathfrak{a}, \mathcal{T}) \to C$ from a linear site $(\mathfrak{a}, \mathcal{T})$ to C (see Definition 4.12), and we show it is filtered (see Proposition 6.7). We then consider the functor $G_C : \mathcal{J}_C \to \mathsf{Cat}$ assigning to each LC morphism its domain. We prove the following:

Theorem 1.3 (Theorem 6.8)**.** *Given a Grothendieck category C, we have that C is the linear filtered bicolimit of G_C.*

In §7: We analyse the behaviour of the Kelly tensor product of α-cocomplete linear categories and the tensor product of linear sites with respect to the corresponding filtered bicolimit presentations provided in §6. Given $C, \mathcal{D}$ two locally κ-presentable linear categories, in virtue of Proposition 1.2 above, one can show that the tensor product $C \boxtimes_{\mathsf{LP}} \mathcal{D}$ of locally presentable categories can be recovered as the filtered bicolimit of the family $((C \boxtimes_{\mathsf{LP}} \mathcal{D})^\alpha)_{\alpha \geq \kappa} = (C^\alpha \otimes_\alpha \mathcal{D}^\alpha)_{\alpha \geq \kappa}$ of subcategories of α-presentable objects, with the transition functors given by the natural embeddings $(C \boxtimes_{\mathsf{LP}} \mathcal{D})^\alpha \subseteq (C \boxtimes_{\mathsf{LP}} \mathcal{D})^\beta$. However, at first sight, it is not clear whether these transition functors are precisely those induced, through the universal property of the Kelly tensor product, by the natural embeddings $C^\alpha \subseteq C^\beta$ and $\mathcal{D}^\alpha \subseteq \mathcal{D}^\beta$. We show that this is indeed the case:

Theorem 1.4 (Theorem 7.3)**.** *Let $C, \mathcal{D}$ be two locally κ-presentable linear categories. The tensor product $C \boxtimes_{\mathsf{LP}} \mathcal{D}$ can be expressed as the k-linear filtered bicolimit of the tensor products $(C^\alpha \otimes_\alpha \mathcal{D}^\alpha)_\alpha$ of categories of α-presentable objects, where α takes values in the totally ordered set of small regular cardinals bigger or equal than κ, and the transition maps*

$$C^\alpha \otimes_\alpha \mathcal{D}^\alpha \to C^\beta \otimes_\beta \mathcal{D}^\beta$$

are those induced, through the universal property of the Kelly tensor product, by the canonical embeddings $\mathcal{A}^\alpha \subseteq \mathcal{A}^\beta$ and $\mathcal{B}^\alpha \subseteq \mathcal{B}^\beta$ for all $\alpha \leq \beta$.

While the Kelly tensor product of α-cocomplete linear categories is well-behaved in this sense, this is not the case for the tensor product of linear sites. More concretely, given two Grothendieck categories $\mathcal{A}, \mathcal{B}$, we show in Remark 7.4 that the filtered bicolimit of the tensor product of linear sites of the site presentations of $\mathcal{A}$ with the site presentations of $\mathcal{B}$ does not allow, in general, to recover $\mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B}$.

In §8: We describe the functoriality of the tensor product of locally presentable linear categories via the functoriality of the Kelly tensor product of α-cocomplete linear categories (see Proposition 8.8). This description provides an advantage when computing tensor products of cocontinuous linear functors in the cases where one has control of the subcategories of presentable objects. We illustrate this advantage by means of a geometrical example (see Example 8.9). To conclude, we describe the associativity and symmetry of the tensor product of locally presentable linear categories making use of the same properties of the Kelly tensor product of α-cocomplete linear categories (see Proposition 8.10 and Proposition 8.11).

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2. Conventions and notation

Throughout the paper we assume the axioms of the Tarski-Grothendieck (TG) set theory, that is, the axioms of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC) together with the universe axiom. We fix a universe $\mathcal{U}$.

Recall that a category is said to be *$\mathcal{U}$-small* if it has a $\mathcal{U}$-small set of objects and, between any two objects, a $\mathcal{U}$-small sets of morphisms. A category is said to be *essentially $\mathcal{U}$-small* if it is equivalent to a $\mathcal{U}$-small category.

We denote by $\mathcal{U}\text{-}\mathsf{Reg}$ the totally ordered class of regular $\mathcal{U}$-small cardinals, which we will envision as a category. This category is not $\mathcal{U}$-small [32, Cor 6.13].

We fix a $\mathcal{U}$-small commutative ring k for the rest of the article and we denote by $\mathcal{U}\text{-}\mathsf{Mod}(k)$ the category of $\mathcal{U}$-small k-modules. We will write "k-linear" to mean "enriched in $\mathcal{U}\text{-}\mathsf{Mod}(k)$".

The forgetful functor $\mathcal{U}\text{-}\mathsf{Mod}(k) \to \mathcal{U}\text{-}\mathsf{Set}$ is faithful, conservative (i.e. it reflects isomorphisms) and it preserves and reflects both $\mathcal{U}$-small limits and $\mathcal{U}$-small filtered colimits (see [7, Exa 2.9.10.b & 2.9.10.d, Prop 2.13.5] for the case $k = \mathbb{Z}$, the general case follows analogously). Moreover, k is a strong generator of $\mathcal{U}\text{-}\mathsf{Mod}(k)$ and, additionally, it is finitely presentable and dualizable with respect to $\otimes_k$, the tensor product of modules over k [23, Exa 4.7]. All these properties together ensure a very good behaviour of the theory of k-linear categories. For example, we have that:

- to give a k-natural transformation between k-linear functors amounts to give an ordinary natural transformation [20, §1.3];
- a k-linear functor has a k-linear adjoint if and only if its underlying non-enriched functor has an ordinary adjoint [20, §1.11];
- a k-linear category is a locally presentable k-linear category if and only if its underlying category is locally presentable [21, Prop 7.5];
- a k-linear category is $\mathcal{U}$-(co)complete in the $[\mathcal{U}\text{-}\mathsf{Mod}(k)]$-enriched sense (i.e. it admits all $\mathcal{U}$-small $[\mathcal{U}\text{-}\mathsf{Mod}(k)]$-indexed/weighted (co)limits) if and only if its underlying ordinary category is $\mathcal{U}$-(co)complete (i.e. it admits

all $\mathcal{U}$-small ordinary/conical (co)limits) (see [20, Prop 3.73 + Prop 3.46] and their duals);
- analogously, given $\alpha \in \mathcal{U}\text{-}\mathsf{Reg}$, a k-linear category is α-(co)complete in the $[\mathcal{U}\text{-}\mathsf{Mod}(k)]$-enriched sense if and only if its underlying ordinary category is α-(co)complete (see [21, Prop 4.3 + §7.4] together with the restriction of [20, Prop 3.46] to cotensors with α-presentable objects, and their duals);
- a k-linear functor is $\mathcal{U}$-(co)continuous in the $[\mathcal{U}\text{-}\mathsf{Mod}(k)]$-enriched sense (i.e. it preserves $\mathcal{U}$-small $[\mathcal{U}\text{-}\mathsf{Mod}(k)]$-indexed/weighted (co)limits) if and only if its underlying ordinary functor is $\mathcal{U}$-(co)continuous (i.e. it preserves $\mathcal{U}$-small ordinary/conical (co)limits) (see [20, Prop 3.73] and its dual statement together with [23, Prop 4.2]);
- analogously, given $\alpha \in \mathcal{U}\text{-}\mathsf{Reg}$, a k-linear functor is α-(co)continuous in the $[\mathcal{U}\text{-}\mathsf{Mod}(k)]$-enriched sense if and only if its underlying ordinary functor is α-(co)continuous (see [21, §4.6 + §7.4] and its dual statement together with [23, Prop 4.2]).

In particular, while working in the k-linear setup, we can safely restrict ourselves to work exclusively with ordinary (co)limits and the corresponding ordinary notions of (co)completeness and (co)continuity, and we will do so from this point on.

We denote by $\mathcal{U}\text{-}\mathsf{Cat}(k)$ the 2-category of $\mathcal{U}$-small k-linear categories with k-linear functors and k-linear natural transformations. This is actually not only a 2-category, but actually a $[\mathcal{U}\text{-}\mathsf{Cat}(k)]$-enriched category [20, §2.3]. In particular, given $\mathfrak{a}, \mathfrak{b} \in \mathcal{U}\text{-}\mathsf{Cat}(k)$, we denote by $\mathcal{U}\text{-}\mathsf{Cat}(k)(\mathfrak{a}, \mathfrak{b})$ the $\mathcal{U}$-small k-linear category of k-linear functors between $\mathfrak{a}$ and $\mathfrak{b}$ with morphisms the k-linear natural transformations.

We denote by $\otimes$ the *tensor product of k-linear categories*, that is, given $\mathfrak{a}, \mathfrak{b} \in \mathcal{U}\text{-}\mathsf{Cat}(k)$, $\mathfrak{a} \otimes \mathfrak{b}$ is the $\mathcal{U}$-small k-linear category with objects the pairs $(A, B) \in \mathrm{Obj}(\mathfrak{a}) \times \mathrm{Obj}(\mathfrak{b})$ and Hom-k-modules $\mathfrak{a} \otimes \mathfrak{b}((A, B), (A', B'))$ given by the tensor product of k-modules $\mathfrak{a}(A, A') \otimes_k \mathfrak{b}(B, B')$. The tensor product $\otimes$ endows the $[\mathcal{U}\text{-}\mathsf{Cat}(k)]$-enriched category $\mathcal{U}\text{-}\mathsf{Cat}(k)$ with a symmetric monoidal closed structure and the inner hom corresponds with the external hom [20, §2.3]. More concretely, given $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{U}\text{-}\mathsf{Cat}(k)$, we have the universal property

$$\mathcal{U}\text{-}\mathsf{Cat}(k)\left(\mathfrak{a} \otimes_k \mathfrak{b}, \mathfrak{c}\right) \cong \mathcal{U}\text{-}\mathsf{Cat}(k)\left(\mathfrak{a}, \mathcal{U}\text{-}\mathsf{Cat}(k)(\mathfrak{b}, \mathfrak{c})\right) \tag{4}$$

in $\mathcal{U}\text{-}\mathsf{Cat}(k)$.

Given $\mathfrak{a} \in \mathcal{U}\text{-}\mathsf{Cat}(k)$, we consider the *$k$-linear category of $\mathcal{U}$-small (right) $\mathfrak{a}$-modules* $\mathcal{U}\text{-}\mathsf{Cat}(k)(\mathfrak{a}^{\mathsf{op}}, \mathcal{U}\text{-}\mathsf{Mod}(k))$ and we denote it by $\mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{a})$. We denote by $Y_{\mathfrak{a}} : \mathfrak{a} \hookrightarrow \mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{a})$ the k-linear Yoneda embedding, which sends $A \in \mathfrak{a}$ to the k-linear functor $\mathfrak{a}(-, A) : \mathfrak{a}^{\mathsf{op}} \to \mathcal{U}\text{-}\mathsf{Mod}(k)$. Given $f : \mathfrak{a} \to \mathfrak{b}$ a morphism in $\mathcal{U}\text{-}\mathsf{Cat}(k)$, we denote the *restriction of scalars along f* by

$$f^* : \mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{b}) \to \mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{a}) : F \mapsto F \circ f^{\mathsf{op}}, \tag{5}$$

which is a k-linear functor. Its (k-linear) left adjoint, the *extension of scalars along f*, will be denoted by

$$f_! : \mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{a}) \to \mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{b}) \tag{6}$$

and its (k-linear) right adjoint, the *coextension of scalars along f*, by

$$f_* : \mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{a}) \to \mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{b}) \tag{7}$$

(see, for example, [36, Exp I, Prop 5.8.3]). In particular, $f_! \cong \mathsf{Lan}_{Y_{\mathfrak{a}}} Y_{\mathfrak{b}} f$ where $\mathsf{Lan}_{Y_{\mathfrak{a}}} Y_{\mathfrak{b}} f$ denotes the left Kan extension [20, §4] of the functor $Y_{\mathfrak{b}} f : \mathfrak{a} \to \mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{b})$ along the k-linear Yoneda embedding $Y_{\mathfrak{a}} : \mathfrak{a} \to \mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{a})$ (see [12, Prop A.6]).

Let α be a $\mathcal{U}$-small regular cardinal. We denote by $\mathcal{U}\text{-}\mathsf{Cat}_\alpha(k)$ the $[\mathcal{U}\text{-}\mathsf{Cat}(k)]$-enriched category of α-cocomplete (i.e. closed under α-small colimits) $\mathcal{U}$-small k-linear categories with α-cocontinuous (i.e. α-small colimit preserving) k-linear functors and k-linear natural transformations. In particular, given $\mathfrak{a}, \mathfrak{b} \in \mathcal{U}\text{-}\mathsf{Cat}_\alpha(k)$, we denote by $\mathsf{Rex}_\alpha(\mathfrak{a}, \mathfrak{b})$ the $\mathcal{U}$-small k-linear category of α-cocontinuous k-linear functors between $\mathfrak{a}$ and $\mathfrak{b}$.

Given $\mathfrak{a} \in \mathcal{U}\text{-}\mathsf{Cat}_\alpha(k)$, we consider the full k-linear subcategory $\mathcal{U}\text{-}\mathsf{Lex}_\alpha(\mathfrak{a}) \subseteq \mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{a})$ of α-continuous $\mathfrak{a}$-modules, that is the k-linear functors $\mathfrak{a}^{\mathsf{op}} \to \mathcal{U}\text{-}\mathsf{Mod}(k)$ preserving α-small limits.

Given a small k-linear category $\mathfrak{a}$, the k-linear categories $\mathcal{U}\text{-}\mathsf{Lex}_\alpha(\mathfrak{a})$ and $\mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{a})$ are no longer $\mathcal{U}$-small, as neither is the (ordinary) category $\mathcal{U}\text{-}\mathsf{Reg}$, as pointed out above. However, one can always consider a larger universe $\mathcal{V}$ containing $\mathcal{U}$, such that $\mathcal{U}\text{-}\mathsf{Reg}$, $\mathcal{U}\text{-}\mathsf{Lex}_\alpha(\mathfrak{a})$ and $\mathcal{U}\text{-}\mathsf{Mod}(\mathfrak{a})$ are $\mathcal{V}$-small for all $\mathcal{U}$-small k-linear categories $\mathfrak{a}$. From this point on and for the rest of the paper, we will omit the universes $\mathcal{U}$ and $\mathcal{V}$ from our notation and terminology, and will say something is small when it is $\mathcal{U}$-small and something is large when it is $\mathcal{V}$-small. In particular, we will replace the notation $\mathcal{V}\text{-}\mathsf{Cat}$ by CAT to refer to the 2-category of $\mathcal{V}$-small categories and the notation $\mathcal{V}\text{-}\mathsf{Cat}(k)$ by $\mathsf{CAT}(k)$ to refer to the 2-category of $\mathcal{V}$-small linear categories.

To conclude, let us fix some notations we will use when working with 1- and 2-categories.

Given a k-linear 1-category $\mathcal{C}$, we will denote the (k-module of) morphisms between two objects $A, B \in \mathcal{C}$ by $\mathcal{C}(A, B)$. Along the text, equivalences of categories will be denoted by the symbol $\simeq$ and isomorphisms (in any category) will be denoted by the symbol $\cong$.

Let us in addition fix the notation we will use when working with 2-morphisms in the various bicategories (and 2-categories) appearing in this paper. In any bicategory, we denote by $\bullet$ the vertical composition of 2-morphisms and by $\circ$ the horizontal composition of 2-morphisms, following the convention in [28]. In particular, given a diagram

$$
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow \alpha"] & C
\end{tikzcd}
\tag{8}
$$

in a bicategory $\mathcal{C}$, we denote by $\alpha \circ f$ to the horizontal composite $\alpha \circ 1_f$.

The reader can find a notation index at the end of the paper, where each symbol is accompanied by a short description of its meaning and the page where it is introduced.

3. Generalities on locally presentable linear categories

In this section we provide an overview on the theory of locally presentable k-linear categories and their tensor product, focused on the aspects that will be needed along the text. Due to the nice behaviour of the k-linear enrichment (see §2), the k-linear theory of locally presentable categories essentially amounts to a direct k-linear translation of the non-enriched theory of locally presentable categories. We refer the reader to [15] and [1] for the non-enriched setup, and to [21] for the enriched framework.

3.1. **Basic definitions and results.**

Definition 3.1. Let $\mathcal{C}$ be a k-linear category and α a small regular cardinal. We say that $C \in \mathcal{C}$ is *α-presentable* if the functor

$$\mathcal{C}(C, -) : \mathcal{C} \to \mathsf{Mod}(k)$$

preserves α-filtered colimits.

Notation 3.2. Given a k-linear category $\mathcal{C}$, we denote by $\mathcal{C}^\alpha$ its full subcategory of α-presentable objects. We denote by $\iota_\alpha : \mathcal{C}^\alpha \hookrightarrow \mathcal{C}$ the natural embedding. When necessary to avoid confusion, we will use a superscript to make explicit which category we are working with, e.g. $\iota_\alpha^{\mathcal{C}} : \mathcal{C}^\alpha \hookrightarrow \mathcal{C}$.

Proposition 3.3 ([1, Prop 1.16])**.** *Let $\mathcal{C}$ be a cocomplete k-linear category and α a small regular cardinal. Then $\mathcal{C}^\alpha$ is closed under α-small colimits in $\mathcal{C}$. In particular, $\mathcal{C}^\alpha$ is an α-cocomplete k-linear category and the k-linear embedding $\iota_\alpha : \mathcal{C}^\alpha \hookrightarrow \mathcal{C}$ is α-cocontinuous.*

Remark 3.4. Let $\mathcal{C}$ be k-linear category and α, β small regular cardinals. Assume that $\alpha < \beta$. Then, because β-filtered colimits are in particular α-filtered, every α-presentable object in $\mathcal{C}$ is also β-presentable and thus $\mathcal{C}^\alpha \subseteq \mathcal{C}^\beta$. Observe that, as a direct consequence of Proposition 3.3, we have that $\mathcal{C}^\alpha$ is closed under α-small colimits in $\mathcal{C}^\beta$.

Notation 3.5. In the hypothesis of Remark 3.4, we denote by $\iota_{\alpha,\beta} : \mathcal{C}^\alpha \hookrightarrow \mathcal{C}^\beta$ the α-cocontinuous natural embedding. When necessary to avoid confusion, we will use a superscript to make explicit which category we are working with, e.g. $\iota_{\alpha,\beta}^{\mathcal{C}} : \mathcal{C}^\alpha \hookrightarrow \mathcal{C}^\beta$.

Definition 3.6. Let α be a small regular cardinal. A *locally α-presentable k-linear category* is a cocomplete k-linear category $\mathcal{C}$ with a small set of α-presentable objects $\mathfrak{c}$ such that every object in $\mathcal{C}$ is an α-filtered colimit of elements in $\mathfrak{c}$.

We say that a category $\mathcal{C}$ is a *locally presentable k-linear category* if there exists a small regular cardinal α such that $\mathcal{C}$ is a locally α-presentable k-linear category.

Remark 3.7 ([1, Remark under Thm 1.20]). If a k-linear category $\mathcal{C}$ is locally β-presentable, then it is locally γ-presentable for all regular cardinals $\gamma \geq \beta$, because β-presentable objects are in particular γ-presentable for all $\gamma \geq \beta$ (see Remark 3.4). Moreover, every object in a locally presentable k-linear category $\mathcal{C}$ is α-presentable for some regular cardinal α, and thus we have that

$$\mathcal{C} \simeq \bigcup_{\alpha \in \mathsf{Reg}} \mathcal{C}^\alpha.$$

This is a direct consequence of the first part of this remark together with the fact that, for any regular cardinal α, α-small colimits of α-presentable objects are α-presentable (see Proposition 3.3).

Notation 3.8.

- We denote by $\mathsf{LP}_\alpha(k)$ the 2-category of locally α-presentable k-linear categories with 1-morphisms the cocontinuous k-linear functors preserving α-filtered colimits and 2-morphisms the k-linear natural transformations. Note that $\mathsf{LP}_\alpha(k)$ actually constitutes a $\mathsf{Cat}(k)$-enriched category.
- We denote by $\mathsf{LP}(k)$ the 2-category of locally presentable k-linear categories with 1-morphisms the cocontinuous k-linear functors (i.e. the functors preserving all small colimits) and 2-morphisms the k-linear natural transformations. As in the case of $\mathsf{LP}_\alpha(k)$, we have that the categories of morphisms in $\mathsf{LP}(k)$ are also k-linear (although no longer small in general). In particular, given $\mathcal{A}, \mathcal{B}$ two locally presentable k-linear categories, we denote

by $\mathsf{Cocont}_k(\mathcal{A}, \mathcal{B})$ the k-linear category of cocontinuous k-linear functors between $\mathcal{A}$ and $\mathcal{B}$ and k-natural transformations between them.

Remark 3.9 ([1, Rem 1.19]). Given a locally α-presentable k-linear category C, its subcategory of α-presentable objects C^α is essentially small and hence we can consider it as an element in $\mathsf{Cat}_\alpha(k)$.

Gabriel-Ulmer duality (see [15, Kor 7.11] for the non-enriched setup and [21, §9] for the enriched one) is a key result in the theory of locally presentable categories and we will use it repeatedly along the text.

Theorem 3.10 (Gabriel-Ulmer duality)**.** *Let α be a small regular cardinal. There is a biequivalence of $\mathsf{Cat}(k)$-enriched categories*

$$(\mathsf{Cat}_\alpha(k))^{\mathsf{coop}} \to \mathsf{LP}_\alpha(k),$$

where $(-)^{\mathsf{coop}}$ denotes the conjugate-opposite 2-category.

Let's recall the explicit description of the duality at the level of objects.

- The biequivalence is given by the pseudofunctor
$$\mathsf{Ind}_\alpha : (\mathsf{Cat}_\alpha(k))^{\mathsf{coop}} \to \mathsf{LP}_\alpha(k)$$
sending an α-cocomplete small category $\mathfrak{a}$ to its Ind_α-completion $\mathsf{Ind}_\alpha(\mathfrak{a})$ (see [2, §3]), which is locally α-presentable [1, Thm 1.46]. In particular, we have that $\mathsf{Ind}_\alpha(\mathfrak{a}) \simeq \mathsf{Lex}_\alpha(\mathfrak{a})$ (see [2, Thm 2.4]) and the k-linear Yoneda embedding $Y_\mathfrak{a} : \mathfrak{a} \hookrightarrow \mathsf{Mod}(\mathfrak{a})$ factors as
$$\mathfrak{a} \xhookrightarrow{Y'_\alpha} \mathsf{Lex}_\alpha(\mathfrak{a}) \xhookrightarrow{j_\alpha} \mathsf{Mod}(\mathfrak{a})$$
where $j_\alpha : \mathsf{Lex}_\alpha(\mathfrak{a}) \hookrightarrow \mathsf{Mod}(\mathfrak{a})$ is the natural embedding. Moreover, the fully faithful functor $Y'_\alpha : \mathfrak{a} \hookrightarrow \mathsf{Lex}_\alpha(\mathfrak{a})$ exhibits $\mathsf{Lex}_\alpha(\mathfrak{a})$ as the α-free cocompletion of $\mathfrak{a}$ [1, Prop 1.45], that is, Y'_α preserves α-small colimits and for every cocomplete linear category $\mathcal{B}$, we have that precomposition with Y'_α induces an equivalence
$$\mathsf{Cocont}_k(\mathsf{Lex}_\alpha(\mathfrak{a}), \mathcal{B}) \xrightarrow{\simeq} \mathsf{Rex}_\alpha(\mathfrak{a}, \mathcal{B}) \tag{9}$$
whose left adjoint is given by taking the left Kan extension along Y'_α [21, Thm 9.9].
- The quasi-inverse pseudofunctor is given by
$$(-)^\alpha : \mathsf{LP}_\alpha(k) \to (\mathsf{Cat}_\alpha(k))^{\mathsf{coop}}$$
which sends a locally α-presentable category C to its subcategory of α-presentable objects C^α, which is an α-cocomplete (essentially) small category by Proposition 3.3 and Remark 3.9.

Remark 3.11. Observe that, given a locally α-presentable category C, we have an equivalence of categories

$$E_\alpha : C \xrightarrow{\simeq} \mathsf{Lex}_\alpha(C^\alpha).$$

This equivalence is given on objects as follows:

$$E_\alpha(C) \coloneqq C(\iota_\alpha(-), C)$$

where $\iota_\alpha : C^\alpha \hookrightarrow C$ is the natural embedding (see, e.g., the proof of [1, Thm 1.46]). If necessary to avoid confusion, we will write $E^{\mathcal{A}}_\alpha$ to make explicit the category we are working with.

Proposition 3.12 ([1, Thm 1.46 & Prop 1.35])**.** *Let $\mathfrak{a} \in \mathsf{Cat}_\alpha(k)$. Then, the natural k-linear embedding $j_\alpha : \mathsf{Lex}_\alpha(\mathfrak{a}) \hookrightarrow \mathsf{Mod}(\mathfrak{a})$ preserves α-filtered colimits and admits a (k-linear) right adjoint $R_\alpha : \mathsf{Mod}(\mathfrak{a}) \to \mathsf{Lex}_\alpha(\mathfrak{a})$. Moreover, we have that $R_\alpha Y_\mathfrak{a} \cong Y'_\alpha$.*

The following result will be useful for our purposes.

Proposition 3.13. *Let $\mathcal{C}$ be a locally κ-presentable k-linear category and consider small regular cardinals $\beta > \alpha \geq \kappa$. Let $\iota_{\alpha,\beta} : \mathcal{C}^\alpha \hookrightarrow \mathcal{C}^\beta$ be the natural embedding functor. Then, the k-linear functor $(\iota_{\alpha,\beta})^* : \mathsf{Mod}(\mathcal{C}^\beta) \to \mathsf{Mod}(\mathcal{C}^\alpha)$ restricts to a k-linear functor*

$$(\iota_{\alpha,\beta})_s : \mathsf{Lex}_\beta(\mathcal{C}^\beta) \to \mathsf{Lex}_\alpha(\mathcal{C}^\alpha).$$

Moreover, $(\iota_{\alpha,\beta})_s$ has a k-linear left adjoint $(\iota_{\alpha,\beta})^s : \mathsf{Lex}_\alpha(\mathcal{C}^\alpha) \to \mathsf{Lex}_\beta(\mathcal{C}^\beta)$ such that the diagram

$$\begin{tikzcd}
\mathcal{C}^\alpha \arrow[rr, hook, "\iota_{\alpha,\beta}"] \arrow[d, hook, "Y_{\mathcal{C}^\alpha}"] & & \mathcal{C}^\beta \arrow[d, hook, "Y_{\mathcal{C}^\beta}"] \\
\mathsf{Mod}(\mathcal{C}^\alpha) \arrow[rr, "(\iota_{\alpha,\beta})_!"] \arrow[d, "R_\alpha"] & & \mathsf{Mod}(\mathcal{C}^\beta) \arrow[d, "R_\beta"] \\
\mathsf{Lex}_\alpha(\mathcal{C}^\alpha) \arrow[rr, bend left, "(\iota_{\alpha,\beta})^s"] & \mathcal{C} \arrow[l, "\simeq", "E_\alpha"'] \arrow[r, "\simeq"', "E_\beta"] & \mathsf{Lex}_\beta(\mathcal{C}^\beta)
\end{tikzcd} \tag{10}$$

is commutative up to isomorphism. In particular, the adjunction $(\iota_{\alpha,\beta})^s \dashv (\iota_{\alpha,\beta})_s$ is an equivalence of k-linear categories and $(\iota_{\alpha,\beta})^s \cong \mathsf{Lan}_{Y'_\alpha} Y'_\beta \iota_{\alpha,\beta}$.

Proof. We first show that $(\iota_{\alpha,\beta})^*$ when restricted to the subcategory $\mathsf{Lex}_\beta(\mathcal{C}^\beta)$ takes values in $\mathsf{Lex}_\alpha(\mathcal{C}^\alpha)$. Indeed, if $X \in \mathsf{Lex}_\beta(\mathcal{C}^\beta)$ one easily sees that

$$(\iota_{\alpha,\beta})^*(X) = X((\iota_{\alpha,\beta})^{\mathsf{op}}(-))$$

preserves α-small limits because both $(\iota_{\alpha,\beta})^{\mathsf{op}}$ and X do. Therefore, $(\iota_{\alpha,\beta})^*$ does indeed restrict to a functor

$$(\iota_{\alpha,\beta})_s : \mathsf{Lex}_\beta(\mathcal{C}^\beta) \to \mathsf{Lex}_\alpha(\mathcal{C}^\alpha).$$

More concretely, we have that the diagram

$$\begin{tikzcd}
\mathsf{Mod}(\mathcal{C}^\alpha) & & \mathsf{Mod}(\mathcal{C}^\beta) \arrow[ll, "(\iota_{\alpha,\beta})^*"'] \\
\mathsf{Lex}_\alpha(\mathcal{C}^\alpha) \arrow[u, hook, "j_\alpha"] & & \mathsf{Lex}_\beta(\mathcal{C}^\beta) \arrow[u, hook, "j_\beta"'] \arrow[ll, "(\iota_{\alpha,\beta})_s"'] \\
& \mathcal{C} \arrow[ul, "E_\alpha", "\simeq"'] \arrow[ur, "E_\beta"', "\simeq"] &
\end{tikzcd}$$

is commutative, where the commutativity of the bottom triangle can be readily checked using the fact that $\iota_\beta \iota_{\alpha,\beta} = \iota_\alpha$. In particular, we have that $(\iota_{\alpha,\beta})_s$ is an equivalence.

Define now the functor $(\iota_{\alpha,\beta})^s := R_\beta (\iota_{\alpha,\beta})_! j_\alpha$. We show that this is the left adjoint to $(\iota_{\alpha,\beta})_s$. Indeed, we have that:

$$\begin{aligned}
\mathsf{Lex}_\beta(\mathcal{C}^\beta)(R_\beta(\iota_{\alpha,\beta})_! j_\alpha(M), X) &\cong \mathsf{Mod}(\mathcal{C}^\beta)((\iota_{\alpha,\beta})_! j_\alpha(M), j_\beta(X)) \\
&\cong \mathsf{Mod}(\mathcal{C}^\alpha)(j_\alpha(M), (\iota_{\alpha,\beta})^* j_\beta(X)) \\
&\cong \mathsf{Mod}(\mathcal{C}^\alpha)(j_\alpha(M), j_\alpha(\iota_{\alpha,\beta})_s(X)) \\
&\cong \mathsf{Lex}_\alpha(\mathcal{C}^\alpha)(M, (\iota_{\alpha,\beta})_s(X)).
\end{aligned} \tag{11}$$

Therefore, because $(\iota_{\alpha,\beta})_s$ was an equivalence, so is $(\iota_{\alpha,\beta})^s$, and it is in particular the quasi-inverse.

We now proceed to check the commutativity of the diagram (10). The commutativity up to isomorphism of the top square is a direct consequence of the fact that $(\iota_{\alpha,\beta})_! \cong \mathsf{Lan}_{Y_\mathfrak{a}} Y_\mathfrak{b}\iota_{\alpha,\beta}$ (see §2). It can also be directly checked using the (k-linear) Yoneda lemma. Indeed, given $C \in \mathcal{C}^\alpha$, we have that

$$\begin{aligned}
\mathsf{Mod}(\mathcal{C}^\beta)((\iota_{\alpha,\beta})_! Y_{\mathcal{C}^\alpha}(C), M) &\cong \mathsf{Mod}(\mathcal{C}^\alpha)(Y_{\mathcal{C}^\alpha}(C), (\iota_{\alpha,\beta})^*(M)) \\
&\cong (\iota_{\alpha,\beta})^*(M)(C) \\
&= M(\iota_{\alpha,\beta}(C)) \\
&\cong \mathsf{Mod}(\mathcal{C}^\beta)(Y_{\mathcal{C}^\beta}(\iota_{\alpha,\beta}(C)), M)
\end{aligned}$$

naturally in $M \in \mathsf{Mod}(\mathcal{C}^\beta)$ and thus $(\iota_{\alpha,\beta})_! Y_{\mathcal{C}^\alpha}(C) \cong Y_{\mathcal{C}^\beta}\iota_{\alpha,\beta}(C)$ as desired. We now show the commutativity up to isomorphism of the triangle in the bottom. Using the fact that $(\iota_{\alpha,\beta})_s E_\beta = E_\alpha$, we obtain that

$$E_\beta \cong (\iota_{\alpha,\beta})^s (\iota_{\alpha,\beta})_s E_\beta = (\iota_{\alpha,\beta})^s E_\alpha$$

as we wanted to prove. It remains to show the commutativity up to isomorphism of the bottom square. Because $(\iota_{\alpha,\beta})^s$ is an equivalence, it preserves small colimits, as so do $(\iota_{\alpha,\beta})_!, R_\alpha$ and R_β because they are left adjoints. Therefore, as $\mathsf{Mod}(\mathcal{C}^\alpha)$ is generated via small colimits by the elements $\{Y_{\mathcal{C}^\alpha}(C)\}_{C\in\mathcal{C}^\alpha}$, we have that the bottom square in the diagram (10) is commutative up to isomorphism if and only if

$$R_\beta(\iota_{\alpha,\beta})_! Y_{\mathcal{C}^\alpha} \cong (\iota_{\alpha,\beta})^s R_\alpha Y_{\mathcal{C}^\alpha}.$$

Now observe that:

$$\begin{aligned}
(\iota_{\alpha,\beta})^s R_\alpha Y_{\mathcal{C}^\alpha}(C) &= R_\beta(\iota_{\alpha,\beta})_! j_\alpha R_\alpha Y_{\mathcal{C}^\alpha}(C) \\
&\cong R_\beta(\iota_{\alpha,\beta})_! Y_{\mathcal{C}^\alpha}(C),
\end{aligned}$$

where the second isomorphism comes from the fact that $Y_{\mathcal{C}^\alpha} = j_\alpha Y'_{\mathcal{C}^\alpha}$ and $R_\alpha j_\alpha \cong 1_{\mathsf{Lex}_\alpha(\mathcal{C}^\alpha)}$. Therefore, we have an isomorphism

$$\lambda : Y'_\alpha \iota_{\alpha,\beta} \cong R_\beta Y_{\mathcal{C}^\beta}\iota_{\alpha,\beta} \Rightarrow (\iota_{\alpha,\beta})^s R_\alpha Y_{\mathcal{C}^\alpha} \cong (\iota_{\alpha,\beta})^s Y'_\beta.$$

It remains to check that $((\iota_{\alpha,\beta})^s, \lambda) \cong \mathsf{Lan}_{Y'_\alpha} Y'_\beta \iota_{\alpha,\beta}$, which exists because $\mathsf{Lex}_\beta(\mathcal{C}^\beta)$ is cocomplete [20, Prop 4.33]. Denote by

$$\delta : Y'_\beta \iota_{\alpha,\beta} \Rightarrow (\mathsf{Lan}_{Y'_\alpha} Y'_\beta \iota_{\alpha,\beta}) Y'_\alpha$$

the structure isomorphism of the left Kan extension. Now, by the universal property of the left Kan extension [20, Thm 4.43], there exists a unique natural transformation $\gamma : \mathsf{Lan}_{Y'_\alpha}\iota_{\alpha,\beta} Y'_\beta \Rightarrow (\iota_{\alpha,\beta})^s$ such that $\lambda = (\gamma \circ Y'_\alpha) \bullet \delta$. In particular, $\gamma \circ Y'_\alpha$ is an isomorphism. Now, because $(\iota_{\alpha,\beta})^s$ is cocontinuous, by the universal property of the α-free cocompletion (9) we can conclude that γ is an isomorphism. This concludes the proof. □

Remark 3.14. Proposition 3.13 can be compared to the topos-theoretic result [36, Exp.III, Prop 1.2] (see [31, Prop 2.3] for the k-linear counterpart). In fact, our choice of the notation $(-)^s$ and $(-)_s$ is borrowed from [36].

3.2. **The tensor product: α-cocomplete linear categories and locally presentable linear categories.** In this section we revise the monoidal structures in the 2-categories $\mathsf{Cat}_\alpha(k)$, $\mathsf{LP}_\alpha(k)$ and $\mathsf{LP}(k)$ and the relations among them.

The 2-category $\mathsf{Cat}_\alpha(k)$ of α-cocomplete k-linear categories can be endowed with a closed monoidal structure [20, §6.5] as follows:

Definition 3.15. Given $\mathfrak{a}, \mathfrak{b} \in \mathsf{Cat}_\alpha(k)$, there exists an $\mathfrak{a} \otimes_\alpha \mathfrak{b} \in \mathsf{Cat}_\alpha(k)$ and a k-linear functor $u^\alpha_{\mathfrak{a},\mathfrak{b}} : \mathfrak{a} \otimes \mathfrak{b} \to \mathfrak{a} \otimes_\alpha \mathfrak{b}$ which is α-cocontinuous in each variable, such that, for every $\mathfrak{c} \in \mathsf{Cat}_\alpha(k)$, composition with $u^\alpha_{\mathfrak{a},\mathfrak{b}}$ induces an equivalence

$$\mathsf{Rex}_\alpha(\mathfrak{a} \otimes_\alpha \mathfrak{b}, \mathfrak{c}) \simeq \mathsf{Rex}_{\alpha,\alpha}(\mathfrak{a} \otimes \mathfrak{b}, \mathfrak{c}) \simeq \mathsf{Rex}_\alpha(\mathfrak{a}, \mathsf{Rex}_\alpha(\mathfrak{b}, \mathfrak{c})) \tag{12}$$

in $\mathsf{Cat}_\alpha(k)$, where $\mathsf{Rex}_{\alpha,\alpha}(\mathfrak{a} \otimes \mathfrak{b}, \mathfrak{c})$ denotes the full subcategory of $\mathsf{Cat}(k)(\mathfrak{a} \otimes \mathfrak{b}, \mathfrak{c})$ consisting of the functors that are α-cocontinuous in each variable. We say that $\mathfrak{a} \otimes_\alpha \mathfrak{b}$ is the *α-Kelly tensor product* of $\mathfrak{a}$ and $\mathfrak{b}$ [20],[21].

Proposition 3.16 ([20, §6.5])**.** *The 2-category $\mathsf{Cat}_\alpha(k)$ together with the α-Kelly tensor product $\otimes_\alpha$ is a closed monoidal bicategory. The monoidal unit is given by the full subcategory $(\mathsf{Mod}(k))^\alpha \subseteq \mathsf{Mod}(k)$ of α-presentable k-modules and the inner hom between $\mathfrak{a}$ and $\mathfrak{b}$ is given by $\mathsf{Rex}_\alpha(\mathfrak{a}, \mathfrak{b})$.*

The α-Kelly tensor product can be explicitly constructed as follows:

Proposition 3.17 ([20, §6.5])**.** *Consider $\mathfrak{a}$, $\mathfrak{b} \in \mathsf{Cat}_\alpha(k)$. The category $\mathfrak{a} \otimes_\alpha \mathfrak{b}$ can be recovered as the closure under α-small colimits of the image of the k-linear composite*

$$\mathfrak{a} \otimes \mathfrak{b} \xhookrightarrow{Y_{\mathfrak{a}\otimes\mathfrak{b}}} \mathsf{Mod}(\mathfrak{a} \otimes \mathfrak{b}) \xrightarrow{R_{\alpha,\alpha}} \mathsf{Lex}_{\alpha,\alpha}(\mathfrak{a}, \mathfrak{b}) \tag{13}$$

where $\mathsf{Lex}_{\alpha,\alpha}(\mathfrak{a}, \mathfrak{b})$ denotes the full subcategory of $\mathsf{Mod}(\mathfrak{a} \otimes \mathfrak{b})$ consisting of the bi-modules $F : \mathfrak{a}^{\mathsf{op}} \otimes \mathfrak{b}^{\mathsf{op}} \to \mathsf{Mod}(k)$ that are α-continuous in each variable, and the k-linear functor $R_{\alpha,\alpha} : \mathsf{Mod}(\mathfrak{a} \otimes \mathfrak{b}) \to \mathsf{Lex}_{\alpha,\alpha}(\mathfrak{a}, \mathfrak{b})$ is the left adjoint to the natural embedding $j_{\alpha,\alpha} : \mathsf{Lex}_{\alpha,\alpha}(\mathfrak{a}, \mathfrak{b}) \hookrightarrow \mathsf{Mod}(\mathfrak{a} \otimes \mathfrak{b})$. Moreover, the universal functor $u^\alpha_{\mathfrak{a},\mathfrak{b}} : \mathfrak{a} \otimes \mathfrak{b} \to \mathfrak{a} \otimes_\alpha \mathfrak{b}$ is given by the corestriction of the functor (13) and the equivalence

$$(-) \circ u^\alpha_{\mathfrak{a},\mathfrak{b}} : \mathsf{Rex}_\alpha(\mathfrak{a} \otimes_\alpha \mathfrak{b}, \mathfrak{c}) \xrightarrow{\simeq} \mathsf{Rex}_{\alpha,\alpha}(\mathfrak{a} \otimes \mathfrak{b}, \mathfrak{c})$$

is the right adjoint of the functor

$$\mathsf{Lan}_{u^\alpha_{\mathfrak{a},\mathfrak{b}}}(-) : \mathsf{Rex}_{\alpha,\alpha}(\mathfrak{a} \otimes \mathfrak{b}, \mathfrak{c}) \xrightarrow{\simeq} \mathsf{Rex}_\alpha(\mathfrak{a} \otimes_\alpha \mathfrak{b}, \mathfrak{c})$$

given by taking the left Kan extension along $u^\alpha_{\mathfrak{a},\mathfrak{b}}$.

Remark 3.18. The adjunction $\left[\mathsf{Lan}_{u^\alpha_{\mathfrak{a},\mathfrak{b}}}(-)\right] \dashv \left[(-) \circ u^\alpha_{\mathfrak{a},\mathfrak{b}}\right]$ is also an adjoint equivalence when the α-cocomplete k-linear category of the codomain is not small (see [20, Thm 6.23]). More concretely, given $\mathfrak{a}, \mathfrak{b} \in \mathsf{Cat}_\alpha(k)$ and $\mathcal{C}$ a (non-necessarily small) α-cocomplete k-linear category, we have that precomposing with $u^\alpha_{\mathfrak{a},\mathfrak{b}}$ provides an equivalence

$$(-) \circ u^\alpha_{\mathfrak{a},\mathfrak{b}} : \mathsf{Rex}_\alpha(\mathfrak{a} \otimes_\alpha \mathfrak{b}, \mathcal{C}) \xrightarrow{\simeq} \mathsf{Rex}_{\alpha,\alpha}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{C})$$

whose left adjoint is given by

$$\mathsf{Lan}_{u^\alpha_{\mathfrak{a},\mathfrak{b}}}(-) : \mathsf{Rex}_{\alpha,\alpha}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{C}) \xrightarrow{\simeq} \mathsf{Rex}_\alpha(\mathfrak{a} \otimes_\alpha \mathfrak{b}, \mathcal{C}).$$

Notation 3.19. Let $\mathcal{A}$ and $\mathcal{B}$ be two locally α-presentable k-linear categories and consider $\mathcal{A}^\alpha, \mathcal{B}^\alpha \in \mathsf{Cat}_\alpha(k)$ their corresponding subcategories of α-presentable objects. We denote by

$$k_\alpha : \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \hookrightarrow \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha)$$

the natural embedding provided by Proposition 3.17 above. If necessary to avoid confusion, we will use a superindex to indicate which categories we are working with and write $k_\alpha^{\mathcal{A},\mathcal{B}}$.

Proposition 3.20 ([20, Prop 6.21]). *Let $\mathcal{A},\mathcal{B} \in \mathsf{LP}_\alpha(k)$. Then, the k-linear functor*

$$\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha) \to \mathsf{Mod}(\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha) : X \mapsto \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha)(k_\alpha(-),X)$$

factors via an equivalence

$$\begin{tikzcd}
\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha) \arrow[rr] \arrow[dr, "\simeq", "E_{\alpha,\alpha}"'] & & \mathsf{Mod}(\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha) \\
& \mathsf{Lex}_\alpha(\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha) \arrow[ur, hook, "j_\alpha"'] &
\end{tikzcd}$$

Remark 3.21. By definition, the diagram

$$\begin{tikzcd}
\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \arrow[r, "u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}"] \arrow[d, "R_{\alpha,\alpha}Y_{\mathcal{A}^\alpha\otimes\mathcal{B}^\alpha}"'] & \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[d, hook, "Y'_{\mathcal{A}^\alpha\otimes_\alpha\mathcal{B}^\alpha}"] \arrow[dl, hook, "k_\alpha"'] \\
\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha) \arrow[r, "E_{\alpha,\alpha}"'] & \mathsf{Lex}_\alpha(\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha)
\end{tikzcd}$$

is commutative.

The closed monoidal structure on $\mathsf{Cat}_\alpha(k)$ can be transported via the Gabriel-Ulmer duality (Theorem 3.10) to a closed monoidal structure in $\mathsf{LP}_\alpha(k)$ (see [6, §5.1]). More concretely, we have the following statement:

Proposition 3.22. *The 2-category $\mathsf{LP}_\alpha(k)$ is a closed monoidal bicategory with the k-linear bifunctor $\boxtimes_{\mathsf{LP}_\alpha} : \mathsf{LP}_\alpha(k) \otimes \mathsf{LP}_\alpha(k) \to \mathsf{LP}_\alpha(k)$ given by*

$$\mathcal{A} \boxtimes_{\mathsf{LP}_\alpha} \mathcal{B} := \mathsf{Lex}_\alpha(\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha).$$

The monoidal unit is given by $\mathsf{Mod}(k)$ and the inner hom between $\mathcal{A}$ and $\mathcal{B}$ is given by $\mathsf{Ind}_\alpha(\mathsf{Rex}_\alpha(\mathcal{A}^\alpha,\mathcal{B}^\alpha))$.

Remark 3.23. Given $\mathcal{A},\mathcal{B}$ locally α-presentable categories, we have that

$$(\mathcal{A} \boxtimes_{\mathsf{LP}_\alpha} \mathcal{B})^\alpha \simeq \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha.$$

The following remark will be useful further on.

Remark 3.24. Let α,β be two regular cardinals and $\mathcal{A},\mathcal{B}$ two γ-presentable k-linear categories where $\gamma = \min(\alpha,\beta)$. Then the tensor-inner hom adjunction of k-linear categories from (4) restricts to an isomorphism

$$\Phi_{\alpha,\beta} : \mathsf{Lex}_{\alpha,\beta}(\mathcal{A}^\alpha,\mathcal{B}^\beta) \to \mathsf{Lex}_\alpha((\mathcal{A}^\alpha)^{\mathsf{op}},\mathsf{Lex}_\beta(\mathcal{B}^\beta)). \tag{14}$$

More concretely, a functor $F \in \mathsf{Lex}_{\alpha,\beta}(\mathcal{A}^\alpha,\mathcal{B}^\beta)$ is sent to the functor $\Phi_{\alpha,\beta}(F) \in \mathsf{Lex}_\alpha((\mathcal{A}^\alpha)^{\mathsf{op}},\mathsf{Lex}_\beta(\mathcal{B}^\beta))$ defined by $\Phi_{\alpha,\beta}(F)(A)(B) := F(A,B)$ for all $A \in \mathcal{A}^\alpha, B \in \mathcal{B}^\beta$ and a natural transformation $\sigma : F \Rightarrow G$ in $\mathsf{Lex}_{\alpha,\beta}(\mathcal{A}^\alpha,\mathcal{B}^\beta)$ is sent to the natural transformation $\Phi_{\alpha,\beta}(\sigma) : \Phi_{\alpha,\beta}(F) \Rightarrow \Phi_{\alpha,\beta}(G)$ defined by $\left[\left[\Phi_{\alpha,\alpha}(\sigma)\right]_A\right]_B := \sigma_{(A,B)}$ for all $A \in \mathcal{A}^\alpha, B \in \mathcal{B}^\beta$. We will denote by

$$\Psi_{\alpha,\beta} : \mathsf{Lex}_\alpha((\mathcal{A}^\alpha)^{\mathsf{op}},\mathsf{Lex}_\beta(\mathcal{B}^\beta)) \to \mathsf{Lex}_{\alpha,\beta}(\mathcal{A}^\alpha,\mathcal{B}^\beta) \tag{15}$$

the inverse of this isomorphism. More concretely, any $F \in \mathsf{Lex}_\alpha((\mathcal{A}^\alpha)^{\mathsf{op}},\mathsf{Lex}_\beta(\mathcal{B}^\beta))$ is sent to the functor $\Psi_{\alpha,\beta}(F) \in \mathsf{Lex}_{\alpha,\beta}(\mathcal{A}^\alpha,\mathcal{B}^\beta)$ given by $\Psi_{\alpha,\beta}(F)(A,B) = F(A)(B)$ and a natural transformation $\eta : F \Rightarrow G$ is sent to the natural transformation $\Psi_{\alpha,\beta}(\eta) : \Psi_{\alpha,\beta}(F) \Rightarrow \Psi_{\alpha,\beta}(G)$ defined by $\left[\Psi_{\alpha,\beta}(\eta)\right]_{(A,B)} = [\eta_A]_B$.

In [6, §5.1] it is shown that the tensor product $\boxtimes_{\mathsf{LP}_\alpha}$ admits a nice description independent of the cardinal α.

Proposition 3.25. *Let $\alpha \in \mathsf{Reg}$. Given $\mathcal{A}, \mathcal{B} \in \mathsf{LP}_\alpha(k)$, there is an equivalence of k-linear categories*

$$H_\alpha : \mathsf{Cont}_k(\mathcal{A}^{\mathsf{op}}, \mathcal{B}) \to \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha)$$

where $\mathsf{Cont}_k(\mathcal{A}^{\mathsf{op}}, \mathcal{B})$ denotes the k-linear category of continuous k-linear functors between $\mathcal{A}^{\mathsf{op}}$ and $\mathcal{B}$ and k-natural transformations between them. More concretely, H_α is given by the composite of k-linear equivalences

$$\begin{aligned}
\mathsf{Cont}_k(\mathcal{A}^{\mathsf{op}}, \mathcal{B}) &\overset{- \circ (\iota_\alpha^{\mathcal{A}})^{\mathsf{op}}}{\simeq} \mathsf{Lex}_\alpha((\mathcal{A}^\alpha)^{\mathsf{op}}, \mathcal{B}) \\
&\overset{E_\alpha^{\mathcal{B}} \circ -}{\simeq} \mathsf{Lex}_\alpha((\mathcal{A}^\alpha)^{\mathsf{op}}, \mathsf{Lex}_\alpha(\mathcal{B}^\alpha)) \\
&\overset{\Psi_{\alpha,\alpha}}{\cong} \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha)
\end{aligned}$$

where $\iota_\alpha^{\mathcal{A}}$ is the natural embedding defined in Notation 3.2, $E_\alpha^{\mathcal{B}}$ is the equivalence from Remark 3.11 and $\Psi_{\alpha,\alpha}$ is the isomorphism defined in (15). If necessary to avoid confusion, we will use a superindex to indicate which categories we are working with and write $H_\alpha^{\mathcal{A},\mathcal{B}}$.

By virtue of Remark 3.7, Proposition 3.20 and Proposition 3.25, one can extend the tensor product in $\mathsf{LP}_\alpha(k)$ to a tensor product in $\mathsf{LP}(k)$ [6, §5.1].

Proposition 3.26. *The bifunctor $\boxtimes_{\mathsf{LP}} : \mathsf{LP}(k) \otimes \mathsf{LP}(k) \to \mathsf{LP}(k)$ given by*

$$\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} := \mathsf{Cont}_k(\mathcal{A}^{\mathsf{op}}, \mathcal{B})$$

induces a closed monoidal structure in $\mathsf{LP}(k)$, with monoidal unit given by $\mathsf{Mod}(k)$ and inner hom given by $\mathsf{Cocont}_k(\mathcal{B}, \mathcal{C})$. More concretely, given $\mathcal{A}, \mathcal{B} \in \mathsf{LP}(k)$, there exists a k-linear functor $s_{\mathcal{A},\mathcal{B}} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$ which is cocontinuous in each variable and such that precomposition with $s_{\mathcal{A},\mathcal{B}}$ provides an equivalence

(16)
$$\mathsf{Cocont}_k(\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}, \mathcal{C}) \simeq \mathsf{Cocont}_k, \mathsf{Cocont}_k(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathsf{Cocont}_k(\mathcal{A}, \mathsf{Cocont}_k(\mathcal{B}, \mathcal{C}))$$

for all $\mathcal{C} \in \mathsf{LP}(k)$, where $\mathsf{Cocont}_k, \mathsf{Cocont}_k(- \otimes -, -)$ denotes the k-linear functors that are cocontinuous in each variable, whose left adjoint is given by the equivalence

$$\mathsf{Lan}_{s_{\mathcal{A},\mathcal{B}}} : \mathsf{Cocont}_k, \mathsf{Cocont}_k(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \to \mathsf{Cocont}_k(\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}, \mathcal{C}).$$

Moreover, if both $\mathcal{A}$ and $\mathcal{B}$ are α-presentable one has that

$$(\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\alpha \simeq \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha.$$

Remark 3.27. The universal properties of $\otimes_\alpha$ and $\boxtimes_{\mathsf{LP}}$ are compatible in a nice way (see the proof of [6, Prop 5.3]). In other words, if $\mathcal{A}$ and $\mathcal{B}$ are locally α-presentable k-linear categories, there is an isomorphism

$$\begin{tikzcd}
\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \arrow[rr, hook, "i_\alpha^{\mathcal{A}} \otimes i_\alpha^{\mathcal{B}}"] \arrow[dd, "u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}"'] & & \mathcal{A} \otimes \mathcal{B} \arrow[d, "s_{\mathcal{A},\mathcal{B}}"] \\
& \Gamma_\alpha \swarrow & \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \\
\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[rr, hook, "k_\alpha"] & & \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha), \arrow[u, "H_\alpha^{-1}"']
\end{tikzcd}$$

where H_α^{-1} is a quasi-inverse of the equivalence H_α from Proposition 3.25.

3.2.1. *Raising the cardinality.* Let $\mathcal{A}$, $\mathcal{B}$ be two locally presentable k-linear categories and choose the smallest regular cardinal κ such that both are locally κ-presentable. Consider regular cardinals $\kappa \leq \alpha \leq \beta$ and denote by $\iota^{\mathcal{A}}_{\alpha,\beta} : \mathcal{A}^\alpha \hookrightarrow \mathcal{A}^\beta$, $\iota^{\mathcal{B}}_{\alpha,\beta} : \mathcal{B}^\alpha \hookrightarrow \mathcal{B}^\beta$ the natural embeddings, which in particular are α-cocontinuous. Observe that we have a canonical α-cocontinuous k-linear functor

$$f_{\alpha,\beta} : \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \to \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta \tag{17}$$

that makes the diagram

$$\begin{tikzcd} \mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \arrow[r, "\iota^{\mathcal{A}}_{\alpha,\beta}\otimes\iota^{\mathcal{B}}_{\alpha,\beta}"] \arrow[d, "u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}"'] & \mathcal{A}^\beta \otimes \mathcal{B}^\beta \arrow[d, "u_{\mathcal{A}^\beta,\mathcal{B}^\beta}"] \\ \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[r, "f_{\alpha,\beta}"] & \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta. \end{tikzcd} \tag{18}$$

commutative up to isomorphism. Indeed, $f_{\alpha,\beta}$ is defined as the image via the universal property (12) in $\mathsf{Rex}_\alpha(\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha, \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta)$ of the composite

$$\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \xrightarrow{\iota^{\mathcal{A}}_{\alpha,\beta}\otimes\iota^{\mathcal{B}}_{\alpha,\beta}} \mathcal{A}^\beta \otimes \mathcal{B}^\beta \xrightarrow{u_{\mathcal{A}^\beta,\mathcal{B}^\beta}} \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta,$$

which is α-cocontinuous in each variable. More concretely, we have that

$$f_{\alpha,\beta} = \mathsf{Lan}_{(u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha})}\left(u_{\mathcal{A}^\beta,\mathcal{B}^\beta}(\iota^{\mathcal{A}}_{\alpha,\beta} \otimes \iota^{\mathcal{B}}_{\alpha,\beta})\right) \tag{19}$$

(see Proposition 3.17).

The following is a 2-variables version of Proposition 3.13.

Proposition 3.28. *Let $\mathcal{A}, \mathcal{B}$ be locally κ-presentable k-linear categories and consider small regular cardinals $\beta \geq \alpha \geq \kappa$. Let $\iota^{\otimes}_{\alpha,\beta} := \iota^{\mathcal{A}}_{\alpha,\beta} \otimes \iota^{\mathcal{B}}_{\alpha,\beta} : \mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \to \mathcal{A}^\beta \otimes \mathcal{B}^\beta$ be the natural functor. Then, the k-linear functor $(\iota^{\otimes}_{\alpha,\beta})^* : \mathsf{Mod}(\mathcal{A}^\beta \otimes \mathcal{B}^\beta) \to \mathsf{Mod}(\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha)$ sends objects in $\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta)$ to objects in $\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha)$ and hence restricts to a k-linear functor*

$$(\iota^{\otimes}_{\alpha,\beta})_s : \mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta) \to \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha).$$

Moreover, $(\iota^{\otimes}_{\alpha,\beta})_s$ has a left adjoint $(\iota^{\otimes}_{\alpha,\beta})^s : \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha) \to \mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta)$ such that the diagram

$$\begin{tikzcd} \mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \arrow[rr, "\iota^{\otimes}_{\alpha,\beta}"] \arrow[d, hook] & & \mathcal{A}^\beta \otimes \mathcal{B}^\beta \arrow[d, hook] \\ \mathsf{Mod}(\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha) \arrow[rr, "(\iota^{\otimes}_{\alpha,\beta})_!"] \arrow[d, "R_{\alpha,\alpha}"] & & \mathsf{Mod}(\mathcal{A}^\beta \otimes \mathcal{B}^\beta) \arrow[d, "R_{\beta,\beta}"] \\ \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha) \arrow[rr, "(\iota^{\otimes}_{\alpha,\beta})^s"] & & \mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta) \\ & \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \arrow[ul, "H_\alpha"] \arrow[ur, "H_\beta"'] & \end{tikzcd} \tag{20}$$

is commutative up to isomorphism. Furthermore, the adjunction $(\iota^{\otimes}_{\alpha,\beta})^s \dashv (\iota^{\otimes}_{\alpha,\beta})_s$ is a k-linear equivalence of categories and $(\iota^{\otimes}_{\alpha,\beta})^s \cong \mathsf{Lan}_{(R_{\alpha,\alpha}Y_{\mathcal{A}^\alpha\otimes\mathcal{B}^\alpha})}\left(R_{\beta,\beta}Y_{\mathcal{A}^\beta\otimes\mathcal{B}^\beta}\iota^{\otimes}_{\alpha,\beta}\right)$.

Proof. We first prove that $(\iota^{\otimes}_{\alpha,\beta})^s$ when restricted to the subcategory $\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta,\mathcal{B}^\beta)$ takes values in $\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha)$. Indeed, given $X\in\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta,\mathcal{B}^\beta)$, we have that

$$(\iota^{\otimes}_{\alpha,\beta})^*(X)(-,-) = X((\iota^{\otimes}_{\alpha,\beta})^{\mathsf{op}}(-,-)) = X\left((\iota^{\mathcal{A}}_{\alpha,\beta})^{\mathsf{op}}(-),(\iota^{\mathcal{B}}_{\alpha,\beta})^{\mathsf{op}}(-)\right)$$

belongs to $\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha)$ because $(\iota^{\mathcal{A}}_{\alpha,\beta})^{\mathsf{op}}$ and $(\iota^{\mathcal{B}}_{\alpha,\beta})^{\mathsf{op}}$ preserve α-small limits and X preserves β-small (and in particular α-small) limits in each variable. We thus have a functor $(\iota^{\otimes}_{\alpha,\beta})_s:\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta,\mathcal{B}^\beta)\to\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha)$ such that the diagram

$$\begin{tikzcd}
\mathsf{Mod}(\mathcal{A}^\alpha\otimes\mathcal{B}^\alpha) & \mathsf{Mod}(\mathcal{A}^\beta\otimes\mathcal{B}^\beta) \arrow[l, "(\iota^{\otimes}_{\alpha,\beta})^*"'] \\
\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha) \arrow[u, hook, "j_{\alpha,\alpha}"] & \mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta,\mathcal{B}^\beta) \arrow[l, "(\iota^{\otimes}_{\alpha,\beta})_s"'] \arrow[u, hook, "j_{\beta,\beta}"']
\end{tikzcd}$$

is commutative. Observe now that, for any $X\in\mathsf{Cont}_k(\mathcal{A}^{\mathsf{op}},\mathcal{B})$, one has that

$$(\iota^{\mathcal{B}}_{\alpha,\beta})_sE^{\mathcal{B}}_\beta X(\iota^{\mathcal{A}}_\beta)^{\mathsf{op}}(\iota^{\mathcal{A}}_{\alpha,\beta})^{\mathsf{op}} = E^{\mathcal{B}}_\alpha X(\iota^{\mathcal{A}}_\alpha)^{\mathsf{op}},$$

because $(\iota^{\mathcal{B}}_{\alpha,\beta})_sE^{\mathcal{B}}_\beta=E^{\mathcal{B}}_\alpha$ (see the proof of Proposition 3.13) and $\iota^{\mathcal{A}}_\alpha=\iota^{\mathcal{A}}_\beta\iota^{\mathcal{A}}_{\alpha,\beta}$. Now applying $\Psi_{\alpha,\alpha}$ to both sides of the equality, one can conclude that

$$H_\alpha = (\iota^{\otimes}_{\alpha,\beta})_sH_\beta, \tag{21}$$

showing the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha) & & \mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta,\mathcal{B}^\beta). \arrow[ll, "(\iota^{\otimes}_{\alpha,\beta})_s"'] \\
& \mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B} \arrow[ul, "H_\alpha"] \arrow[ur, "H_\beta"'] &
\end{tikzcd}$$

In particular, because both H_α and H_β are both equivalences, we have that $\iota^{\otimes}_{\alpha,\beta}$ is also an equivalence.

Define now the functor $(\iota^{\otimes}_{\alpha,\beta})^s := R_{\beta,\beta}(\iota^{\otimes}_{\alpha,\beta})_!j_{\alpha,\alpha}$. We are going to show that $(\iota^{\otimes}_{\alpha,\beta})^s$ is a left adjoint of $(\iota^{\otimes}_{\alpha,\beta})_s$. Indeed, given $M\in\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha)$ and $N\in\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta,\mathcal{B}^\beta)$ we have that

$$\begin{aligned}
\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta,\mathcal{B}^\beta)\left((\iota^{\otimes}_{\alpha,\beta})^s(M),N\right) &\cong \mathsf{Mod}(\mathcal{A}^\beta\otimes\mathcal{B}^\beta)\left((\iota^{\otimes}_{\alpha,\beta})_!j_{\alpha,\alpha}(M),j_{\beta,\beta}(N)\right)\\
&\cong \mathsf{Mod}(\mathcal{A}^\alpha\otimes\mathcal{B}^\alpha)\left(j_{\alpha,\alpha}(M),(\iota^{\otimes}_{\alpha,\beta})^*j_{\beta,\beta}(N)\right)\\
&\cong \mathsf{Mod}(\mathcal{A}^\alpha\otimes\mathcal{B}^\alpha)\left(j_{\alpha,\alpha}(M),j_{\alpha,\alpha}(\iota^{\otimes}_{\alpha,\beta})_s(N)\right)\\
&\cong \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha)\left(M,(\iota^{\otimes}_{\alpha,\beta})_s(N)\right),
\end{aligned}$$

naturally in M and N. Therefore $(\iota^{\otimes}_{\alpha,\beta})^s\dashv(\iota^{\otimes}_{\alpha,\beta})_s$, and as $(\iota^{\otimes}_{\alpha,\beta})_s$ is an equivalence, this adjunction is an adjoint equivalence.

We now prove that the diagram (20) is commutative up to isomorphism. The commutativity up to isomorphism of the top square follows from the Yoneda lemma by the same argument as the one used in the proof of Proposition 3.13. Moreover, using the fact that $H_\alpha=(\iota^{\otimes}_{\alpha,\beta})_sH_\beta$, we can conclude that

$$(\iota^{\otimes}_{\alpha,\beta})^sH_\alpha = (\iota^{\otimes}_{\alpha,\beta})^s(\iota^{\otimes}_{\alpha,\beta})_sH_\beta\cong H_\beta$$

where we are using the counit of the equivalence adjunction $(\iota^{\otimes}_{\alpha,\beta})^s\dashv(\iota^{\otimes}_{\alpha,\beta})_s$. This shows the commutativity up to isomorphism of the lower triangle. We now prove that the bottom square of the diagram commutes up to isomorphism. Observe that, for all $X \in \mathsf{Mod}(\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha)$ and $M \in \mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta)$ we have that

$$\begin{aligned}\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta)\left((\iota^{\otimes}_{\alpha,\beta})^s R_{\alpha,\alpha}(X), M\right) &\cong \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha)\left(R_{\alpha,\alpha}(X), (\iota^{\otimes}_{\alpha,\beta})_s(M)\right)\\ &\cong \mathsf{Mod}(\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha)\left(X, j_{\alpha,\alpha}(\iota^{\otimes}_{\alpha,\beta})_s(M)\right)\\ &\cong \mathsf{Mod}(\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha)\left(X, (\iota^{\otimes}_{\alpha,\beta})^* j_{\beta,\beta}(M)\right)\\ &\cong \mathsf{Mod}(\mathcal{A}^\beta \otimes \mathcal{B}^\beta)\left((\iota^{\otimes}_{\alpha,\beta})_!(X), j_{\beta,\beta}(M)\right)\\ &\cong \mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta)\left(R_{\beta,\beta}(\iota^{\otimes}_{\alpha,\beta})_!(X), M\right),\end{aligned}$$

naturally in X and M and thus by Yoneda lemma we have that $R_{\beta,\beta}(\iota^{\otimes}_{\alpha,\beta})_! \cong (\iota^{\otimes}_{\alpha,\beta})^s R_{\alpha,\alpha}$ as desired. Therefore, we have an isomorphism

$$\lambda : R_{\beta,\beta} Y_{\mathcal{A}^\beta \otimes \mathcal{B}^\beta}(\iota^{\otimes}_{\alpha,\beta}) \Rightarrow (\iota^{\otimes}_{\alpha,\beta})^s R_{\alpha,\alpha} Y_{\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha}.$$

It only remains to show that $((\iota^{\otimes}_{\alpha,\beta})^s, \lambda) \cong \mathsf{Lan}_{(R_{\alpha,\alpha}Y_{\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha})}(R_{\beta,\beta} Y_{\mathcal{A}^\beta \otimes \mathcal{B}^\beta}(\iota^{\otimes}_{\alpha,\beta}))$, which exists because $\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta)$ is cocomplete [20, Prop 4.33]. Denote by

$$\delta : R_{\beta,\beta} Y_{\mathcal{A}^\beta \otimes \mathcal{B}^\beta}(\iota^{\otimes}_{\alpha,\beta}) \Rightarrow \left(\mathsf{Lan}_{(R_{\alpha,\alpha}Y_{\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha})}(R_{\beta,\beta} Y_{\mathcal{A}^\beta \otimes \mathcal{B}^\beta}(\iota^{\otimes}_{\alpha,\beta}))\right) R_{\alpha,\alpha} Y_{\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha}$$

the structure isomorphism of the left Kan extension. Now, by the universal property of the left Kan extension [20, Thm 4.43], there exists a unique natural transformation

$$\gamma : \mathsf{Lan}_{(R_{\alpha,\alpha}Y_{\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha})}(R_{\beta,\beta} Y_{\mathcal{A}^\beta \otimes \mathcal{B}^\beta}(\iota^{\otimes}_{\alpha,\beta})) \Rightarrow (\iota^{\otimes}_{\alpha,\beta})^s$$

such that $\lambda = (\gamma \circ R_{\alpha,\alpha} Y_{\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha}) \bullet \delta$. In particular, $\gamma \circ (R_{\alpha,\alpha} Y_{\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha}) = \gamma \circ (k_\alpha u_{\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha})$ is an isomorphism. As $\left(\mathsf{Lan}_{(R_{\alpha,\alpha}Y_{\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha})}(R_{\beta,\beta} Y_{\mathcal{A}^\beta \otimes \mathcal{B}^\beta}(\iota^{\otimes}_{\alpha,\beta}))\right) k_\alpha$ and $(\iota^{\otimes}_{\alpha,\beta})^s k_\alpha$ are α-cocontinuous, we can apply Remark 3.18 and conclude that $\gamma \circ k_\alpha$ is an isomorphism. Using now the fact that $k_\alpha \cong E^{-1}_{\alpha,\alpha} E_{\alpha,\alpha} k_\alpha = E^{-1}_{\alpha,\alpha} Y'_{\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha}$ (see Remark 3.21) and the fact that $\left(\mathsf{Lan}_{(R_{\alpha,\alpha}Y_{\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha})}(R_{\beta,\beta} Y_{\mathcal{A}^\beta \otimes \mathcal{B}^\beta}(\iota^{\otimes}_{\alpha,\beta}))\right)$ and $(\iota^{\otimes}_{\alpha,\beta})^s$ are cocontinuous, we can apply (9) to conclude that $\gamma \circ E^{-1}_{\alpha,\alpha}$, and thus γ, is an isomorphism. This finishes the proof. □

4. Generalities on linear sites

In this section we revise the basic notions and results concerning k-linear sites, as they will be an essential tool in the rest of the paper. For a more complete account we point the reader to [26, §2] and [31, §2].

4.1. **Basic definitions and results.** k-Linear sites and Grothendieck k-linear categories can be seen as the k-linear counterpart of the classical Grothendieck sites and Grothendieck topoi from [36]. We point the reader to [9] for more general enhancements of sites and topoi.

Let $\mathfrak{a}$ be a small k-linear category.

Definition 4.1. Given an object $A \in \mathfrak{a}$, a *k-linear sieve on A* is a subobject R of the representable module $\mathfrak{a}(-, A)$ in the category $\mathsf{Mod}(\mathfrak{a})$. Given a family $S = (f_i : A_i \to A)_{i \in I}$ of morphisms in $\mathfrak{a}$, the *k-linear sieve generated by S* is the smallest k-linear sieve R on A such that $f_i \in R(A_i)$ for all $i \in I$.

Definition 4.2. A *k-linear cover system* $\mathcal{R}$ on $\mathfrak{a}$ consists of providing for each $A \in \mathfrak{a}$ a family of k-linear sieves $\mathcal{R}(A)$ on A. The k-linear sieves in a k-linear cover system $\mathcal{R}$ are called *covering sieves* or *covers* (for $\mathcal{R}$). We will say that a family

$(f_i : A_i \to A)_{i \in I}$ is a *cover*, or a *covering family*, if the k-linear sieve it generates is a cover.

Definition 4.3. Given R a k-linear sieve on $A \in \mathfrak{a}$ and $g : A' \to A$ a morphism in $\mathfrak{a}$, the *pullback of R along g*, denoted $g^{-1}R$, is the k-linear sieve on A' obtained as the pullback

$$\begin{tikzcd} g^{-1}R \arrow[r, hook] \arrow[d] & \mathfrak{a}(-, A') \arrow[d, "g \circ -"] \\ R \arrow[r, hook] & \mathfrak{a}(-, A) \end{tikzcd}$$

in $\mathsf{Mod}(\mathfrak{a})$. In particular, we have that $g^{-1}R(A'') = \{f : A'' \to A' \mid g \circ f \in R(A'')\}$ for all $A'' \in \mathfrak{a}$.

Definition 4.4. A k-linear cover system $\mathcal{T}$ on $\mathfrak{a}$ is *localizing* if it satisfies the following:

(Id) Identity axiom: Given any object $A \in \mathfrak{a}$, the k-linear sieve generated by 1_A is a cover for $\mathcal{T}$, i.e. $\mathfrak{a}(-, A) \in \mathcal{T}(A)$ for all $A \in \mathfrak{a}$;

(Pb) Pullback axiom: Given a covering sieve $R \subseteq \mathcal{T}(A)$ and $g : A' \to A$ a morphism in $\mathfrak{a}$, the pullback sieve $g^{-1}R$ is also a covering sieve for $\mathcal{T}$.

If moreover $\mathcal{T}$ also satisfies the following:

(Glue) Glueing axiom: Let R be a k-linear sieve on A. If there exists a k-linear sieve S on A in $\mathcal{T}(A)$ such that for all morphism $g : A' \to A$ in S the pullback sieve $g^{-1}R \in \mathcal{T}(A')$, then $R \in \mathcal{T}(A)$;

we say $\mathcal{T}$ is a *k-linear Grothendieck topology*.

Definition 4.5. A *k-linear site* is a pair $(\mathfrak{a}, \mathcal{T})$ where $\mathfrak{a}$ is a k-linear category and $\mathcal{T}$ is a k-linear Grothendieck topology on $\mathfrak{a}$.

Given a k-linear site $(\mathfrak{a}, \mathcal{T})$ one can define linearised versions of presheaves and sheaves, in analogy with the classical notions.

Definition 4.6.

- A *k-linear presheaf* F on $(\mathfrak{a}, \mathcal{T})$ is an $\mathfrak{a}$-module, that is $F \in \mathsf{Mod}(\mathfrak{a})$.
- A *k-linear sheaf* F on $(\mathfrak{a}, \mathcal{T})$ is a k-linear presheaf such that the restriction functor
$$F(A) \cong \mathsf{Mod}(\mathfrak{a})(\mathfrak{a}(-, A), F) \to \mathsf{Mod}(\mathfrak{a})(R, F)$$
is an isomorphism for all $A \in \mathfrak{a}$ and all $R \in \mathcal{T}(A)$. We denote by $\mathsf{Sh}(\mathfrak{a}, \mathcal{T}) \subseteq \mathsf{Mod}(\mathfrak{a})$ the full subcategory of k-linear sheaves.

Definition 4.7. Consider a small k-linear category $\mathfrak{a}$. A k-linear Grothendieck topology $\mathcal{T}$ on $\mathfrak{a}$ is called *subcanonical* if for every $A \in \mathfrak{a}$, the representable presheaf $\mathfrak{a}(-, A) \in \mathsf{Mod}(\mathfrak{a})$ is a sheaf, that is, it belongs to $\mathsf{Sh}(\mathfrak{a}, \mathcal{T})$. The largest Grothendieck subcanonical topology is called the *canonical topology*.

Remark 4.8. We will frequently consider Grothendieck k-linear categories C themselves as (large) k-linear sites, endowed with their canonical topology $\mathcal{T}_{C,\mathsf{can}}$. In this particular case, the covering families are the jointly epimorphic families (see [36, Exp I, §10.3 + Exp II, Def 2.5 + Exp II, Prop 4.3(2)]) and $\mathsf{Sh}(C, \mathcal{T}_{C,\mathsf{can}}) \simeq C$ (see [29, Lem 1.3.14]).

The following is a consequence of Gabriel-Popescu theorem [30] in combination with enriched topos theory [9].

Theorem 4.9. *The categories of k-linear sheaves over k-linear sites are precisely the Grothendieck k-linear categories.*

Definition 4.10. Let $(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ and $(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ be k-linear sites. We say that a k-linear functor $f : \mathfrak{a} \to \mathfrak{b}$ is a *continuous morphism* (for $\mathcal{T}_{\mathfrak{a}}$ and $\mathcal{T}_{\mathfrak{b}}$) if the restriction of scalars functor $f^* : \mathsf{Mod}(\mathfrak{b}) \to \mathsf{Mod}(\mathfrak{a})$ sends objects in $\mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ to objects in $\mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$.

We denote by $f_s : \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \to \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ the corresponding restriction functor, and by $f^s : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ its left adjoint.

Definition 4.11. Let $\mathfrak{a}$ be a small k-linear category, $(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ a k-linear site and $f : \mathfrak{a} \to \mathfrak{b}$ a k-linear functor. The *k-linear cover system induced by f* and denoted by $f^{-1}\mathcal{T}_{\mathfrak{b}}$ is the k-linear cover system on $\mathfrak{a}$ consisting of all the sieves R on $\mathfrak{a}$ such that the covering sieve on $\mathfrak{b}$ generated by the family $fR := (f(a) \,|\, a$ is a morphism in $R)$ belongs to $\mathcal{T}_{\mathfrak{b}}$.

The class of LC morphisms between sites, where LC stands for "Lemme de comparison" [25, §4], will be used in the following sections:

Definition 4.12 ([27, Def 3.4])**.** Consider a k-linear functor $f : \mathfrak{a} \to \mathfrak{c}$, where $\mathfrak{a}$ is small and $\mathfrak{c}$ may be large.

(1) Suppose $\mathfrak{c}$ is endowed with a k-linear cover system $\mathcal{T}_{\mathfrak{c}}$. We say that $f : \mathfrak{a} \to (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ satisfies
  (G) if for every $C \in \mathfrak{c}$ there is a covering family $(f(A_i) \to C)_{i\in I}$ for $\mathcal{T}_{\mathfrak{c}}$.

(2) Suppose $\mathfrak{a}$ is endowed with a k-linear cover system $\mathcal{T}_{\mathfrak{a}}$. We say that $f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to \mathfrak{c}$ satisfies
  (F) if for every morphism $c : f(A) \to f(A')$ in $\mathfrak{c}$ there exists a covering family $(a_i : A_i \to A)_{i\in I}$ for $\mathcal{T}_{\mathfrak{a}}$ and morphisms $f_i : A_i \to A'$ for each $i \in I$ such that $cf(a_i) = f(f_i)$ for all $i \in I$;
  (FF) if for every morphism $a : A \to A'$ in $\mathfrak{a}$ with $f(a) = 0$ there exists a covering family $(a_i : A_i \to A)_{i\in I}$ for $\mathcal{T}_{\mathfrak{a}}$ with $aa_i = 0$ for all $i \in I$.

(3) Suppose $\mathfrak{a}$ and $\mathfrak{c}$ are endowed with k-linear cover systems $\mathcal{T}_{\mathfrak{a}}$ and $\mathcal{T}_{\mathfrak{c}}$ respectively. We say that $f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ satisfies
  (LC) if f satisfies (G) with respect to $\mathcal{T}_{\mathfrak{c}}$, (F) and (FF) with respect to $\mathcal{T}_{\mathfrak{a}}$, and we further have $\mathcal{T}_{\mathfrak{a}} = f^{-1}\mathcal{T}_{\mathfrak{c}}$.

We denote the class of LC morphisms between k-linear sites by LC.

Remark 4.13. Observe that the identity morphism of a k-linear site $(\mathfrak{a}, \mathcal{T})$ belongs to LC. In addition, LC is closed under composition [25, Prop 4.4].

The importance of LC morphisms between linear sites relies in the fact that they are continuous morphisms inducing equivalences between the corresponding sheaf categories [25, Cor 4.5] together with the following key result:

Theorem 4.14 ([31, Thm 5])**.** *Let $(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ and $(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ be k-linear sites and consider a colimit preserving k-linear functor $F : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$. There exist a subcanonical site $(\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ and a diagram*

$$\begin{tikzcd} & \mathfrak{c} & \\ \mathfrak{a} \arrow[ur, "f"] & & \mathfrak{b}, \arrow[ul, "w"'] \end{tikzcd}$$

where f is a continuous morphism between k-linear sites and w is an LC morphism, such that

$$\begin{tikzcd} \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \arrow[rr, "F"] \arrow[dr, "f^s"'] & & \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \\ & \mathsf{Sh}(\mathfrak{c}, \mathcal{T}_{\mathfrak{c}}) \arrow[ur, "w_s"'] & \end{tikzcd}$$

is a commutative diagram up to isomorphism.

4.2. **The tensor product: linear sites and Grothendieck categories.** In [27] a tensor product $\boxtimes_{\mathsf{S}}$ of Grothendieck categories was introduced based on the definition of a tensor product of k-linear sites.

Definition 4.15. Given two k-linear sites $(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}), (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$, the *tensor product of k-linear sites* $(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \boxtimes_{\mathsf{S}} (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ is given by the k-linear site $(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes_{\mathsf{S}} \mathcal{T}_{\mathfrak{b}})$, where $\mathcal{T}_{\mathfrak{a}} \boxtimes_{\mathsf{S}} \mathcal{T}_{\mathfrak{b}}$ is the smallest Grothendieck topology on $\mathfrak{a} \otimes \mathfrak{b}$ such that, for each object $(A, B) \in \mathfrak{a} \otimes \mathfrak{b}$, $\mathcal{T}_{\mathfrak{a}} \boxtimes_{\mathsf{S}} \mathcal{T}_{\mathfrak{b}}(A, B)$ contains the images of all the canonical morphisms

$$R \otimes_k S \to \mathfrak{a}(-, A) \otimes_k \mathfrak{b}(-, B) = \mathfrak{a} \otimes \mathfrak{b}(-, (A, B))$$

where $R \in \mathcal{T}_{\mathfrak{a}}(A)$ and $S \in \mathcal{T}_{\mathfrak{b}}(B)$.

The tensor product of Grothendieck k-linear categories is defined in [27] as follows.

Definition 4.16. Given two Grothendieck k-linear categories $\mathcal{A} = \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ and $\mathcal{B} = \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$, the *tensor product of Grothendieck k-linear categories* $\mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B}$ is given by

$$\mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B} = \mathsf{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes_{\mathsf{S}} \mathcal{T}_{\mathfrak{b}})$$

Remark 4.17. The tensor product is well-defined, namely, it is independent of the k-linear site presentations of $\mathcal{A}$ and $\mathcal{B}$ chosen [27, Prop 4.1]. The proof of this relies on the following result regarding LC morphisms.

Proposition 4.18 ([27, Prop 3.14])**.** *Let $f : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ and $g : (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}}) \to (\mathfrak{d}, \mathcal{T}_{\mathfrak{d}})$ be k-linear functors between k-linear sites. If f and g are LC morphisms, so is the functor*

$$f \otimes g : (\mathfrak{a} \otimes \mathfrak{c}, \mathcal{T}_{\mathfrak{a}} \boxtimes_{\mathsf{S}} \mathcal{T}_{\mathfrak{c}}) \to (\mathfrak{b} \otimes \mathfrak{d}, \mathcal{T}_{\mathfrak{b}} \boxtimes_{\mathsf{S}} \mathcal{T}_{\mathfrak{d}}).$$

The following result provides a useful description of the tensor product of Grothendieck k-linear categories.

Proposition 4.19 ([27, Cor 2.16 + Cor 2.18])**.** *Consider Grothendieck k-linear categories $\mathcal{A} = \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ and $\mathcal{B} = \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$. Then, the tensor product $\mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B} = \mathsf{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes_{\mathsf{S}} \mathcal{T}_{\mathfrak{b}}) \subseteq \mathsf{Mod}(\mathfrak{a} \otimes \mathfrak{b})$ is given by the full subcategory of bimodules $F : \mathfrak{a}^{\mathsf{op}} \otimes \mathfrak{b}^{\mathsf{op}} \to \mathsf{Mod}(k)$ for which $F(A, -) \in \mathsf{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ for all $A \in \mathfrak{a}$ and $F(-, B) \in \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ for all $B \in \mathfrak{b}$.*

Recall that Grothendieck categories are in particular locally presentable categories [22, Cor 5.2]+[21, Prop 7.5]. The following result was proven in [27, Thm 5.4].

Theorem 4.20. *Given two Grothendieck categories $\mathcal{A}, \mathcal{B}$, we have that*

$$\mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B} \simeq \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}.$$

5. Generalities on the 2-filtered bicolimit of categories

Given a (pseudo)functor $F : \mathcal{A} \to \mathsf{Cat}$ where $\mathcal{A}$ is a category and Cat denotes the 2-category of small categories, we can consider Grothendieck's construction of the *weak* 2-*colimit* or *bicolimit* $\varinjlim_{\mathcal{A}}(F)$ from [37, Exposé VI][1]. In particular, the bicolimits of the pseudofunctors for which the indexing category $\mathcal{A}$ is filtered, are called *filtered bicolimits.* In this section, we will revise the construction of the filtered bicolimit of a pseudofunctor $F : \mathcal{A} \to \mathsf{Cat}$ as presented in [13, 14]. Actually, [13, 14] provides a construction of the more general 2-*filtered bicolimits*, i.e. the bicolimits

[1]Do not confuse this construction with the well-known *Grothendieck's construction* [36, Exposé VI], which given a pseudofunctor $F : \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ computes its oplax colimit [16, Thm 1.1], which has a different universal property than the bicolimit.

of pseudofunctors $F : \mathcal{A} \to \mathsf{Cat}$ where $\mathcal{A}$ is a 2-*filtered* 2-*category*. Because we only need to compute (1-)filtered bicolimits, we will just revise how this more general construction applies to this particular case. In addition, we show that in such case, if the pseudofunctor F factors through the 2-category of k-linear categories $\mathsf{Cat}(k)$, the filtered bicolimit is also k-linear.

We recall some important definitions from the theory of 2-categories. We point the reader to [19] for an extensive account on 2-categories and bicategories.

Definition 5.1. Let $F, G : \mathcal{A} \to \mathcal{B}$ be two pseudofunctors between 2-categories $\mathcal{A}, \mathcal{B}$.

(1) A *pseudonatural transformation* $\Phi : F \Rightarrow G$ is given by a family of 1-morphisms
$$(\Phi_A : F(A) \to G(A))_{A \in \mathrm{Obj}(\mathcal{A})}$$
and a family of invertible 2-morphisms
$$(\Phi_f : \Phi'_A F(f) \Rightarrow G(f)\Phi_A)_{(f:A \to A') \in \mathcal{A}}$$
such that for all $A \in \mathcal{A}$ one has that
$$\Phi_{1_A} \bullet (\Phi_A \circ F^0_A) = G^0_A \circ \Phi_A,$$
where $F^0_A : 1_{F(A)} \Rightarrow F(1_A)$ (resp. $G^0_A : 1_{G(A)} \Rightarrow G(1_A)$) is the invertible 2-cell in $\mathcal{B}$ providing the lax unity constraint of the pseudofunctor F (resp. G) at the object A, and for every pair of composable 1-morphisms $f : A \to A'$, $g : A' \to A''$ in $\mathcal{A}$ one has that
$$\Phi_{gf} \bullet \left(\Phi_{A''} \circ F^2_{g,f}\right) = \left(G^2_{g,f} \circ \Phi_A\right) \bullet \left(G(g) \circ \Phi_f\right) \bullet \left(\Phi_g \circ F(f)\right),$$
where $F^2_{f,g} : F(g)F(f) \Rightarrow F(gf)$ (resp. $G^2_{g,f} : G(g)G(f) \Rightarrow G(gf)$) is the invertible 2-cell in $\mathcal{B}$ providing the lax functoriality constraint of the pseudofunctor F (resp. G) at the pair of composable morphisms f, g.

(2) Given two pseudonatural transformations $\Phi, \Psi : F \Rightarrow G$, a *morphism of pseudonatural transformations* r between Φ and Ψ is a modification, that is a family of 2-morphisms
$$(r_A : \Phi_A \Rightarrow \Psi_A)_{A \in \mathcal{A}}$$
such that
$$(G(f) \circ r_A) \bullet \Phi_f = \Psi_f \bullet (r_{A'} \circ F(f))$$
for all $f : A \to A'$ in $\mathcal{A}$.

We denote by $\mathsf{Psnat}(F, G)$ the category of pseudonatural transformations between F and G, with morphisms given by the modifications (see [13] or [24]).

The notion of *2-filtered 2-category* is introduced in [13, §2] (see the corrected version of the definition in [14, Def 2.1b]) as a suitable generalisation in the 2-categorical realm of the classical notion of filtered category. In particular, as it is already mentioned in the introduction of [13], any (1-)category considered as a trivial 2-category is 2-filtered if and only if it is filtered as an ordinary category (and this remains true when using the corrected definition in [14]). Throughout this paper we will always use an indexing category which is of this latter type, hence we can safely avoid going through the technicalities of the construction of 2-filtered bicolimit for a general indexing 2-category.

Definition 5.2 ([13, Thm. 1.19])**.** Given a pseudofunctor $F : \mathcal{I} \to \mathsf{Cat}$, where $\mathcal{I}$ is a 2-filtered 2-category[2], the *2-filtered bicolimit of* F is a category $\mathcal{L}(F)$ together

[2]The bicolimit and its construction actually work when the indexing category is a *pre* 2-*filtered* 2-*category*, which is a weaker notion than that of 2-filtered 2-category, as pointed out in the introduction of [13].

with a pseudonatural transformation $\Lambda : F \Rightarrow \mathcal{L}(F)$ from F to the constant 2-functor $\mathcal{I} \to \mathsf{Cat}$ taking the value $\mathcal{L}(F)$ such that, for every category $\mathcal{C}$, it induces via composition an equivalence of categories

$$\mathsf{Cat}(\mathcal{L}(F), \mathcal{C}) \simeq \mathsf{Psnat}(F, \mathcal{C}) \tag{22}$$

between the category of functors $\mathcal{L}(F) \to \mathcal{C}$ and the category of pseudonatural transformations between the pseudofunctor F and the constant 2-functor taking the value $\mathcal{C}$.

Remark 5.3. Note that such a category is uniquely determined up to a unique equivalence.

Remark 5.4. Observe that the original definition (see [13, Thm. 1.19]) only considers 2-filtered bicolimits of F with F a strict 2-functor. For our purposes we need to consider the more general situation in which F is a pseudofunctor.

The main result of [13] provides, given a 2-functor $F : \mathcal{I} \to \mathsf{Cat}$, a construction of the bicolimit $\mathcal{L}(F)$ in an intrinsic way in terms of the 2-functor F, which in particular makes of the equivalence (22) an isomorphism of categories. One can observe that, when the indexing category is just an ordinary filtered (1-)category the construction is greatly simplified. For this choice of indexing category, one can easily extend the construction from [13] to the case in which F is a pseudofunctor by means of a slight generalization of the results explained in loc.cit.[3]

We flesh out below the construction of the bicolimit for our particular situation, i.e. when

- $F : \mathcal{I} \to \mathsf{Cat}$ is not necessarily a strict 2-functor but a pseudofunctor,
- the indexing category $\mathcal{I}$ is a filtered (1-)category.

Description of the objects:

Objects of $\mathcal{L}(F)$ are pairs (x, A) where $A \in \mathcal{I}$ and $x \in F(A)$.

Description of the morphisms:

First we describe the class of premorphisms:

- Consider two objects (x, A) and (y, B). A *premorphism* $(x, A) \to (y, B)$ consists of a triple (u, f, v), where $u : A \to C$, $v : B \to C$ and $f : F(u)(x) \to F(v)(y)$ in $F(C)$. In order to make the object C explicit in the notation, we will write $(u, f, v)_C : (x, A) \to (y, B)$.
- Two pre-morphisms $(u_1, f, v_1)_{C_1}, (u_2, g, v_2)_{C_2} : (x, A) \to (y, B)$ are said to be *homotopic* if there exists an object $C \in \mathcal{I}$ and morphisms $w_i : C_i \to C$, for $i = 1, 2$ such that $w_1 \circ u_1 = w_2 \circ u_2$, $w_1 \circ v_1 = w_2 \circ v_2$ and the following diagram commutes

$$\begin{tikzcd}
F(w_1) \circ F(u_1)(x) \arrow[r, "\cong"] \arrow[d, "F(w_1)(f_1)"'] & F(w_1 \circ u_1)(x) = F(w_2 \circ u_2)(x) \arrow[r, "\cong"] & F(w_2) \circ F(u_2)(x) \arrow[d, "F(w_2)(f_2)"] \\
F(w_1) \circ F(v_1)(y) \arrow[r, "\cong"] & F(w_1 \circ v_1)(y) = F(w_2 \circ v_2)(y) \arrow[r, "\cong"] & F(w_2) \circ F(v_2)(y).
\end{tikzcd}$$

  This relation is an equivalence relation and we denote the equivalence class of a premorphism $(u, f, v)_C : (x, A) \to (y, B)$ by $[(u, f, v)_C]$.

By means of the homotopy relation, we define the morphisms:

- *Morphisms* in $\mathcal{L}(F)$ between two objects are given by premorphisms between those two objects modulo homotopy.
- The identity morphism of an object (x, A) in $\mathcal{L}(F)$ is given by $[(1_A, 1_{F(x)}, 1_A)_A]$.

[3]The general construction for F a pseudofunctor should be possible in full generality, without restrictions of the indexing category, just by readjusting the notion of *homotopy* in [13, 1.5(iii)], as we have done in our particular case.

- Given two morphisms
$$[(u_1, f_1, v_1)_{C_1}] : (x, A) \to (y, B),$$
$$[(u_2, f_2, v_2)_{C_2}] : (y, B) \to (z, C),$$
the composite $[(u_2, f_2, v_2)_{C_2}] \circ [(u_1, f_1, v_1)_{C_1}]$ in $\mathcal{L}(F)$ is given by the morphism $[(s_1 \circ u_1, f, s_2 \circ v_2)_D]$ for $s_i : C_i \to D$ for $i = 1, 2$ such that $s_1 \circ v_1 = s_2 \circ u_2$ and where the morphism $f : F(s_1 \circ u_1)(x) \to F(s_2 \circ v_2)(z)$ is defined as the following composite:
$$F(s_1u_1)(x) \cong F(s_1)(F(u_1)(x)) \xrightarrow{F(s_1)(f_1)} F(s_1)(F(v_1)(y)) \cong F(s_1v_1)(y) =$$
$$F(s_2u_2)(y) \cong F(s_2)(F(u_2)(y)) \xrightarrow{F(s_2)(f_2)} F(s_2)(F(v_2)(z)) \cong F(s_2v_2)(z).$$
One can check that this is well-defined following the same reasoning as in [13, Prop 1.11] but replacing the homotopies with the new notion of homotopy we have introduced above to address the pseudofunctor case. We do not write down the details, but the argument carries over almost verbatim once one takes into account that the isomorphisms $F^2_{g,f} : F(g)F(f) \Rightarrow F(g \circ f)$ are natural in both f and g for f and g composable morphisms in $\mathcal{I}$.

The category $\mathcal{L}(F)$ fulfills the universal property (22) above together with the pseudonatural transformation $\Lambda : F \Rightarrow \mathcal{L}(F)$ given by:

- for each $A \in \mathcal{I}$, the functor $\Lambda_A : F(A) \to \mathcal{L}(F)$ sends an object $x \in F(A)$ to $(x, A) \in \mathcal{L}(F)$ and a morphism $f : x \to y$ in $F(A)$ to $[(1_A, F(1_A)(f), 1_A)_A]$;
- for each $w : A \to B$ in $\mathcal{I}$, the invertible 2-morphism $\Lambda_w : \Lambda_B \circ F(w) \Rightarrow \Lambda_A$ is given, for each $x \in F(A)$, by the isomorphism
$$[(1_B, \left[F^2_{1_B,w}\right]_x, w)_B] : (F(w)(x), B) \to (x, A)$$
where $\left[F^2_{1_B,w}\right]_x$ is just the evaluation at x of the natural isomorphism $F^2_{1_B,w} : F(1_B)F(w) \Rightarrow F(w)$ given by the lax functoriality constraint of the pseudofunctor F.

In particular, given a category C and a pseudonatural transformation $\Omega \in \mathsf{Psnat}(F, C)$, the unique functor $H \in \mathsf{Cat}(\mathcal{L}(F), C)$ such that $H \circ \Lambda = \Omega$ provided by the isomorphism (22) is given by

- on objects: $H((x, A)) = \Omega_A(x) \in C$ for all $(x, A) \in \mathcal{L}(F)$;
- on morphisms: given a morphism $[(u, f, v)]_C : (x, A) \to (y, B)$, we have that $H([(u, f, v)]_C) = [\Omega_v]_y \, \Omega_C(f) \, [(\Omega_u)^{-1}]_x$.

We are interested in k-linear categories and their filtered bicolimits. More explicitely, we are interested in filtered bicolimits of pseudofunctors $F : \mathcal{I} \to \mathsf{Cat}$ that take values in the 2-category $\mathsf{Cat}(k)$ of small k-linear categories with k-linear functors and k-linear natural transformations, that is, in pseudofunctors $F : \mathcal{I} \to \mathsf{Cat}$ that factor through the forgetful 2-functor $\mathsf{Cat}(k) \to \mathsf{Cat}$. In what follows, we show that the filtered bicolimit $\mathcal{L}(F)$ under these hypothesis will be again a k-linear category:

Proposition 5.5. *Let $\mathcal{I}$ be a filtered category. Take $F : \mathcal{I} \to \mathsf{Cat}$ a pseudofunctor that factors through the forgetful 2-functor $\mathsf{Cat}(k) \to \mathsf{Cat}$. Then, $\mathcal{L}(F)$ is a k-linear category. Moreover, the universal pseudonatural tranformation $\Lambda : F \Rightarrow \mathcal{L}(F)$ is k-linear, meaning that Λ_A is a k-linear functor for all $A \in \mathcal{I}$ and Λ_w is a k-linear natural isomorphism for all w morphism in $\mathcal{I}$.*

Proof. Consider $(x, A), (y, B) \in \mathcal{L}(F)$. The class $\mathcal{L}(F)((x, A), (y, B))$ has a natural structure of k-module induced from the k-linear structure of the values of F. Indeed, given two morphisms

$$[(u_1, f_1, v_1)_{C_1}], [(u_2, f_2, v_2)_{C_2}] : (x, A) \to (y, B)$$

and an element $\lambda \in k$, we define

$$[(u_1, f_1, v_1)_{C_1}] + \lambda[(u_2, f_2, v_2)_{C_2}] = [(w_1 \circ u_1, F(w_1)(f_1) + \lambda F(w_2)(f_2), w_2 \circ v_2)_C]$$

where $w_1 : C_1 \to C$, $w_2 : C_2 \to C$ and $w_1 \circ u_1 = w_2 \circ u_2$ and $w_1 \circ v_1 = w_2 \circ v_2$. Observe that such w_1, w_2 exist because $\mathcal{I}$ is a filtered category. An easy check shows that this is well-defined and does not depend on the choice of w_1 and w_2. The fact that it provides a k-module structure is directly deduced from the fact that, by hypothesis, for each object $C \in \mathcal{I}$, $F(C)$ is a k-linear category and for each morphism $D \to E$ in $\mathcal{I}$, the functor $F(D \to E) : F(D) \to F(E)$ is k-linear.

In addition, one can easily show that the composition is k-linear. To show this, consider morphisms

$$[(u_1, f_1, v_1)_{C_1}], [(u_2, f_2, v_2)_{C_2}] : (x, A) \to (y, B),$$
$$[(s_1, g_1, t_1)_{D_1}], [(s_2, g_2, t_2)_{D_2}] : (y, B) \to (z, C)$$

and elements $\lambda, \lambda' \in k$. Without loss of generality we can assume that

$$D = C_1 = C_2 = D_1 = D_2,$$
$$u = u_1 = u_2 : A \to D,$$
$$v = v_1 = v_2 = s_1 = s_2 : B \to D$$

and

$$t = t_1 = t_2 : C \to D.$$

We have that

$$\begin{aligned}
[(v, g_1, t)_D] \circ &([(u, f_1, v)_D] + \lambda[(u, f_2, v)_D]) \\
&= [(v, g_1, t)_D] \circ [(u, f_1 + \lambda f_2, v)_D] \\
&= [(u, g_1 \circ (f_1 + \lambda f_2), t)_D] \\
&= [(u, g_1 \circ f_1 + \lambda g_1 \circ f_2, t)_D] \\
&= [(v, g_1, t)_D] \circ [(u, f_1, v)_D] + \lambda[(v, g_1, t)_D] \circ [(u, f_2, v)_D].
\end{aligned}$$

Similarly, one proves that

$$([(v, g_1, t)_D] + \lambda'[(v, g_2, t]_D) \circ [(u, f_1, v)_D] = [(v, g_1, t)_D].$$

Hence, the composition is k-linear as desired.

The k-linearity of Λ follows now by direct inspection. Indeed, the k-linearity of the functor Λ_A for all $A \in \mathcal{I}$ is immediate from its definition. Moreover, for any $w : A \to B$ in $\mathcal{I}$, we have that $\Lambda_w : \Lambda_B \circ F(w) \Rightarrow \Lambda_A$ is a natural isomorphism between k-linear functors and therefore it is in particular a k-linear natural isomorphism (see §2). □

Observe that one could define a k-linear version of the 2-filtered bicolimit by suitably linearizing Definition 5.2 above.

Definition 5.6. Given a pseudofunctor $G : \mathcal{I} \to \mathsf{Cat}(k)$, where $\mathcal{I}$ is a 2-filtered 2-category, the *k-linear 2-filtered bicolimit* of G is a k-linear category $\mathcal{L}_k(G)$ together with a k-linear pseudonatural transformation $\Lambda : G \Rightarrow \mathcal{L}_k(G)$ from G to the constant 2-functor $\mathcal{I} \to \mathsf{Cat}(k)$ taking the value $\mathcal{L}_k(G)$ such that, for every k-linear category $\mathcal{C}$, it induces via composition an equivalence of categories

$$\mathsf{Cat}(k)(\mathcal{L}_k(G), \mathcal{C}) \simeq \mathsf{Psnat}_k(G, \mathcal{C}) \tag{23}$$

between the category of k-linear functors $\mathcal{L}_k(G) \to \mathcal{C}$ and the category of k-linear pseudonatural transformations between the pseudofunctor G and the constant 2-functor taking the value $\mathcal{C}$.

We now show that not only $\mathcal{L}(F)$ is a k-linear category when $F : \mathcal{I} \to \mathsf{Cat}$ factors through the forgetful 2-functor $\mathsf{Cat}(k) \to \mathsf{Cat}$, it actually coincides with the k-linear filtered bicolimit $\mathcal{L}_k(F)$:

Proposition 5.7. *Let $\mathcal{I}$ be a filtered category. Take $F : \mathcal{I} \to \mathsf{Cat}$ a pseudofunctor that factors through the forgetful 2-functor $\mathsf{Cat}(k) \to \mathsf{Cat}$. Then, $\Lambda : F \Rightarrow \mathcal{L}(F)$ satisfies the universal property (23) of the k-linear filtered bicolimit of F.*

Proof. By Proposition 5.5 we know that $\mathcal{L}(F)$ is a k-linear category and $\Lambda : F \Rightarrow \mathcal{L}(F)$ is a k-linear pseudonatural transformation. Therefore, when we restrict the isomorphism $[-\circ\Lambda] : \mathsf{Cat}(\mathcal{L}(F), C) \to \mathsf{Psnat}(F, C)$ from (22) to the full subcategory $\mathsf{Cat}(k)(\mathcal{L}(F), C) \subseteq \mathsf{Cat}(\mathcal{L}(F), C)$, to obtain a functor

$$\mathsf{Cat}(k)(\mathcal{L}(F), C) \to \mathsf{Psnat}_k(F, C).$$

Moreover, given a k-linear pseudonatural transformation $\Omega : F \Rightarrow C$, the unique functor $H : \mathcal{L}(F) \to C$ such that $H \circ \Lambda = \Omega$ can be immediately checked to be k-linear using our assumption on F and the fact that Ω is k-linear. Consequently, we can conclude that $[-\circ\Lambda] : \mathsf{Cat}(\mathcal{L}(F), C) \to \mathsf{Psnat}(F, C)$ from (22) does restrict to an equivalence (actually an isomorphism)

$$\mathsf{Cat}(k)(\mathcal{L}(F), C) \simeq \mathsf{Psnat}_k(F, C),$$

showing that $\mathcal{L}_k(F) \simeq \mathcal{L}(F)$. □

Remark 5.8. Proposition 5.7 shows that the forgetful 2-functor $U : \mathsf{Cat}(k) \to \mathsf{Cat}$ preserves and reflects filtered bicolimits, which can be seen as a bicategorical parallel of the fact that the forgetful functor $\mathsf{Mod}(k) \to \mathsf{Set}$ preserves and reflects filtered colimits [7, Prop 2.13.5]. For this reason, from this point on, we will use the notation $\mathcal{L}(F)$ to refer to both the k-linear filtered bicolimit and to its non-enriched version.

6. Locally presentable linear categories and Grothendieck categories as filtered bicolimits of small categories

In this section we show, based on §5, that every locally presentable k-linear category can be written as a filtered bicolimit of small categories, and hence in particular this holds for any Grothendieck k-linear category. Furthermore, we show that Grothendieck k-linear categories can be written as filtered bicolimits of certain filtered categories of k-linear sites.

Remark 6.1. From this point on we will need to work with large filtered bicolimits, i.e. with filtered bicolimits of pseudofunctors $F : \mathcal{I} \to \mathsf{CAT}$, where $\mathcal{I}$ is a possibly large filtered 1-category. In other words, we will use the theory of filtered bicolimits inside the universe $\mathcal{V}$ we fixed in §2 with respect to which our large categories were small. Observe that even if such a pseudofunctor $F : \mathcal{I} \to \mathsf{CAT}$ takes values in the category of small categories Cat, the resulting filtered bicolimit may still be large because $\mathcal{I}$ is itself large.

Theorem 6.2. *Let C be a category which is a union of full small subcategories indexed by a (possibly large) directed poset. Then, C is the filtered bicolimit of that family of subcategories. More precisely, if C is a category such that $C \simeq \bigcup_{i\in I} C_i$, where $C_i \subseteq C$ are full small subcategories, I is a (possibly large) directed poset and $C_i \subseteq C_j$ if $i \leq j$, then C is a filtered bicolimit of the family $(C_i)_{i\in I}$.*

Proof. Denote by $\mathcal{I}$ the filtered category given by the directed poset I, and denote by $\iota_{i,j} : C_i \hookrightarrow C_j$ the natural embeddings for $i \leq j$. We define the 2-functor

$$F_C : \mathcal{I} \to \mathsf{Cat} \subset \mathsf{CAT} \tag{24}$$

to be given by $F_C(i) = C_i$ for every $i \in \mathcal{I}$ and $F_C(i \leq j) = \iota_{i,j} : C_i \hookrightarrow C_j$ for every morphism $i \leq j$ in $\mathcal{I}$.

We build a functor

$$\phi : \mathcal{L}(F_C) \to C$$

defined as follows:

- $\phi(x, i) = x \in C_i \subseteq C$ for every $(x, i) \in \mathcal{L}(F_C)$;
- $\phi\left([(i \leq k, f : \iota_{i,k}(x) \to \iota_{j,k}(y), j \leq k)]\right) = f \in C_k(\iota_{i,k}(x), \iota_{j,k}(y)) = C(x, y)$ for every morphism $[(i \leq k, f : \iota_{i,k}(x) \to \iota_{j,k}(y), j \leq k)] : (x, i) \to (y, j)$ in $\mathcal{L}(F_C)$.

One can readily check that this is well-defined and it defines a functor. As $C \simeq \bigcup_{i \in \mathcal{I}} C_i$, one trivially has that this functor is essentially surjective, and because C_i are full subcategories of C for all $i \in \mathcal{I}$, one easily deduces that the functor is fully faithful. □

Remark 6.3. Assume C is k-linear, and hence C_i is k-linear for all $i \in \mathcal{I}$ and so are the fully faithful functors $\iota_{i,j} : C_i \subseteq C_j$ for $i \leq j$ in $\mathcal{I}$. Then we have that both $\mathcal{I}$ and F_C are in the hypothesis of Proposition 5.5 above, hence $\mathcal{L}(F_C)$ is k-linear. Observe that the functor ϕ defined in the proof above is as well k-linear, and thus C and $\mathcal{L}(F_C)$ are equivalent as k-linear categories.

Remark 6.4. Theorem 6.2 is just a bicategorical parallel to the easy (1-categorical) statement that says that every object in a category that can be written as a union of a filtered system of subobjects is in particular the filtered colimit of these system of subobjects.

Corollary 6.5. *Let C be a locally presentable k-linear category. Then C is the k-linear filtered bicolimit of its family of subcategories of locally presentable objects $(C^\alpha)_\alpha$, where α varies in* Reg.

Proof. As $C \simeq \bigcup_{\alpha \in \mathsf{Reg}} C^\alpha$ (see Remark 3.7), the result follows from Theorem 6.2 and Remark 6.3, together with the fact that Reg is directed (because it is a totally ordered class). □

Remark 6.6. Because Grothendieck k-linear categories are in particular locally presentable k-linear categories, we can apply Corollary 6.5 to any Grothendieck k-linear category. More concretely, every Grothendieck k-linear category C can be recovered as the k-linear filtered bicolimit of its family of subcategories of locally presentable objects $(C^\alpha)_\alpha$, where α varies in Reg.

We now introduce another presentation of a Grothendieck k-linear category as a filtered bicolimit, where the indexing filtered category will be a certain category of k-linear sites.

Let C be a Grothendieck k-linear category. Consider the category $\mathcal{J}_C$ defined as follows:

- Objects of $\mathcal{J}_C$ are given by
$$\{u_{\mathfrak{a}} : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \to C \mid (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \text{ small } k\text{-linear site}, u_{\mathfrak{a}} \in \mathsf{LC}\}$$
where C is endowed with the canonical topology (see Remark 4.8). For readibility, we will frequently omit the topology from our notation and write $u_{\mathfrak{a}} : \mathfrak{a} \to C$.
- Morphisms between two objects $u_{\mathfrak{a}} : \mathfrak{a} \to C$ and $u_{\mathfrak{b}} : \mathfrak{b} \to C$ are given by the k-linear functors $f : \mathfrak{a} \to \mathfrak{b}$ which belong to LC and such that $u_{\mathfrak{b}} \circ f = u_{\mathfrak{a}}$. We write $f : u_{\mathfrak{a}} \to u_{\mathfrak{b}}$.

One can readily check this is a well-defined category as a direct consequence of Remark 4.13.

Proposition 6.7. *Given a Grothendieck category C, the category $\mathcal{J}_C$ constructed above is filtered.*

Proof. Observe that the category $\mathcal{J}_C$ is not empty. Given two objects $u_{\mathfrak{a}} : \mathfrak{a} \to C$ and $u_{\mathfrak{b}} : \mathfrak{b} \to C$, we want to find a third object $u_{\mathfrak{c}} : \mathfrak{c} \to C$ and morphisms $f : u_{\mathfrak{a}} \to u_{\mathfrak{c}}$ and $g : u_{\mathfrak{b}} \to u_{\mathfrak{c}}$. Let $\mathfrak{c}$ be the full subcategory of C with objects $\{u_{\mathfrak{a}}(A)\}_{A\in\mathfrak{a}} \cup \{u_{\mathfrak{b}}(B)\}_{B\in\mathfrak{b}}$ and endow it with the topology induced by the canonical topology in C. We hence have that the embedding $u_{\mathfrak{c}} : \mathfrak{c} \to C$ is an LC morphism. Now consider the corestrictions $u_{\mathfrak{a}} : \mathfrak{a} \to \mathfrak{c}$ and $u_{\mathfrak{b}} : \mathfrak{b} \to \mathfrak{c}$. We trivially have that these define morphisms $u_{\mathfrak{a}} \to u_{\mathfrak{c}}$ and $u_{\mathfrak{b}} \to u_{\mathfrak{c}}$.

Consider now two morphisms $f, g : u_{\mathfrak{a}} \to u_{\mathfrak{b}}$. We want to find an object $u_{\mathfrak{c}} : \mathfrak{c} \to C$ and a morphism $h : u_{\mathfrak{b}} \to u_{\mathfrak{c}}$ equalizing f and g. Take $\mathfrak{c}$ to be the full subcategory of C with objects $u_{\mathfrak{b}}(\mathfrak{b})$ and endow it with the (restriction of the) canonical topology. Take $u_{\mathfrak{c}} : \mathfrak{c} \to C$ to be the embedding (which is LC) and $h : \mathfrak{b} \to \mathfrak{c}$ the corestriction of $u_{\mathfrak{b}} : \mathfrak{b} \to C$ to $\mathfrak{c}$. Then, by definition, one has that $hu_{\mathfrak{c}} = u_{\mathfrak{b}}$. Furthermore, we have that $hf = hg$ as a direct consequence of the fact that $u_{\mathfrak{b}}f = u_{\mathfrak{a}} = u_{\mathfrak{b}}g$.

We can thus conclude that $\mathcal{J}_C$ is a filtered category. □

We now consider the functor $G_C : \mathcal{J}_C \to \mathsf{Cat}(k) \subset \mathsf{CAT}(k)$ given by forgetting the "slice structure", i.e. defined by sending each object $u_{\mathfrak{a}} : \mathfrak{a} \to C$ to the small k-linear category $\mathfrak{a}$ and each morphism $f : u_{\mathfrak{a}} \to u_{\mathfrak{b}}$ to itself seen as a k-linear morphism $f : \mathfrak{a} \to \mathfrak{b}$.

We proceed to describe $\mathcal{L}(G_C)$ using the construction from §5 above. Observe that in this case the description of the homotopies between premorphisms is that of [13, Def 1.5(iii)] because G_C is a strict functor, and it is therefore slightly simpler than in the pseudofunctor case:

- Objects are $\{(x, u_{\mathfrak{a}} : \mathfrak{a} \to C) \mid (u_{\mathfrak{a}} : \mathfrak{a} \to C) \in \mathcal{J}_C, x \in \mathfrak{a}\}$
- Morphisms $(x, u_{\mathfrak{a}} : \mathfrak{a} \to C) \to (y, u_{\mathfrak{b}} : \mathfrak{b} \to C)$ are given by homotopy classes of triples (u, f, v) where $u : u_{\mathfrak{a}} \to u_{\mathfrak{c}}$, $v : u_{\mathfrak{b}} \to u_{\mathfrak{c}}$ are morphisms in $\mathcal{J}_C$ and $f : u(x) \to v(y)$ is a morphism in $\mathfrak{c}$. As before, we use the notation $(u, f, v)_{u_{\mathfrak{c}}}$ to make explicit the codomain of u and v. Two morphisms $(u_1, f, v_1)_{u_{\mathfrak{c}_1}}$ and $(u_2, g, v_2)_{u_{\mathfrak{c}_2}}$ are homotopic if there exist morphisms $w_1 : u_{\mathfrak{c}_1} \to u_{\mathfrak{c}}$ and $w_2 : u_{\mathfrak{c}_2} \to u_{\mathfrak{c}}$ such that $w_1u_1 = w_2u_2$, $w_1v_1 = w_2v_2$ and $w_1(f) = w_2(g)$. As in §5, we denote the homotopy class of $(u_1, f, v_1)_{u_{\mathfrak{c}_1}}$ by $[(u_1, f, v_1)_{u_{\mathfrak{c}_1}}]$.

Theorem 6.8. *Given a Grothendieck category C, we have that C is the k-linear filtered bicolimit of G_C.*

Proof. Observe that G_C factors through $\mathsf{Cat}(k)$ and hence, by Proposition 5.7, we have that $\mathcal{L}(G_C)$ is the k-linear filtered bicolimit, so in particular a k-linear category. To conclude, it suffices to construct a k-linear equivalence $\psi_C : \mathcal{L}(G_C) \to C$. With the notation introduced in §5 for the objects and morphisms of $\mathcal{L}(G_C)$, we consider the following assignations:

- To every object $(x, u_{\mathfrak{a}} : \mathfrak{a} \to C) \in \mathcal{J}_C$ we assign the object
$$\psi_C(x, u_{\mathfrak{a}} : \mathfrak{a} \to C) := u_{\mathfrak{a}}(x) \in C.$$
- To every morphism $[(u, f, v)_{u_{\mathfrak{c}}}] : (x, u_{\mathfrak{a}} : \mathfrak{a} \to C) \to (y, u_{\mathfrak{b}} : \mathfrak{b} \to C)$ we assign the morphism
$$\psi_C([(u, f, v)_{u_{\mathfrak{c}}}]) := (u_{\mathfrak{c}}(f) : u_{\mathfrak{a}}(x) = u_{\mathfrak{c}}u(x) \to u_{\mathfrak{c}}v(y) = u_{\mathfrak{b}}(y)).$$

We have the following:

(1) The assignation on morphisms is well-defined. Consider two homotopic morphisms $(u_1, f_1, v_1)_{u_{\mathfrak{c}_1}}$, $(u_2, f_2, v_2)_{u_{\mathfrak{c}_2}} : (x, \mathfrak{a} \to C) \to (y, \mathfrak{b} \to C)$, that is, there exist morphisms $w_1 : u_{\mathfrak{c}_1} \to u_{\mathfrak{c}}$ and $w_2 : u_{\mathfrak{c}_2} \to u_{\mathfrak{c}}$ such that

$w_1u_1 = w_2u_2$, $w_1v_1 = w_2v_2$ and $w_1(f_1) = w_2(f_2)$. We want to show that $u_{\mathfrak{c}_1}(f_1) = u_{\mathfrak{c}_2}(f_2)$. Observe that $u_{\mathfrak{c}_1}(f_1) = u_{\mathfrak{c}}w_1(f_1) = u_{\mathfrak{c}}w_2(f_2) = u_{\mathfrak{c}_2}(f_2)$, as desired.

(2) The assignations define a functor $\psi_C : \mathcal{L}(G_C) \to C$. First observe that it preserves identities. Indeed, the identity morphism $[(1_{\mathfrak{a}}, 1_x, 1_{\mathfrak{a}})_{u_{\mathfrak{a}}}]$ of the object $(x, u_{\mathfrak{a}} : \mathfrak{a} \to C)$ gets sent to $u_{\mathfrak{a}}(1_x) = 1_{u_{\mathfrak{a}}(x)}$. We now check that it preserves compositions. Consider two composable morphisms $[(u_1, f_1, v_1)_{u_{\mathfrak{c}_1}}] : (\mathfrak{a}, u_{\mathfrak{a}}) \to (\mathfrak{b}, u_{\mathfrak{b}})$ and $[(u_2, f_2, v_2)_{u_{\mathfrak{c}_2}}] : (\mathfrak{b}, u_{\mathfrak{b}}) \to (\mathfrak{c}, u_{\mathfrak{c}})$ in $\mathcal{L}(G_C)$. Because $\mathcal{J}_C$ is filtered, we can assume that $u_{\mathfrak{d}} = u_{\mathfrak{c}_1} = u_{\mathfrak{c}_2}$, i.e. that the morphisms u_1, v_1, u_2, v_2 in $\mathcal{J}_C$ have the same codomain $u_{\mathfrak{d}} : \mathfrak{d} \to C$, and that $v_1 = u_2$. Their composite in $\mathcal{L}(G_C)$ is given by $[(u_1, f_2 \circ f_1, v_2)_{u_{\mathfrak{d}}}]$ and gets sent to $u_{\mathfrak{d}}(f_2f_1)$. On the other hand, we have that $[(u_1, f_1, v_1)_{u_{\mathfrak{d}}}]$ and $[(u_2, f_2, v_2)_{u_{\mathfrak{d}}}]$ get sent to $u_{\mathfrak{d}}(f_1)$ and $u_{\mathfrak{d}}(f_2)$ respectively, whose composite is $u_{\mathfrak{d}}(f_2)u_{\mathfrak{d}}(f_1)$. As $u_{\mathfrak{d}} : \mathfrak{d} \to C$ is a functor, we have that $u_{\mathfrak{d}}(f_2 \circ f_1) = u_{\mathfrak{d}}(f_2)u_{\mathfrak{d}}(f_1)$ as desired.

(3) The functor ψ_C is k-linear. Let $[(u_1, f_1, v_1)_{u_{\mathfrak{c}_1}}], [(u_2, f_2, v_2)_{u_{\mathfrak{c}_2}}] : (\mathfrak{a}, u_{\mathfrak{a}}) \to (\mathfrak{b}, u_{\mathfrak{b}})$ be two morphisms in $\mathcal{L}(G_C)$. Because $\mathcal{J}_C$ is filtered, we may assume that $u_{\mathfrak{c}} = u_{\mathfrak{c}_1} = u_{\mathfrak{c}_2}$ and that $u = u_1 = u_2$, $v = v_1 = v_2$. Then, given $\lambda \in k$, we have that

$$\psi_C([(u, f_1, v)_{u_{\mathfrak{d}}}] + \lambda[(u, f_2, v)_{u_{\mathfrak{d}}}]) = \psi_C([u, f_1 + \lambda f_2, v]_{u_{\mathfrak{d}}}) = u_{\mathfrak{d}}(f_1 + \lambda f_2).$$

On the other hand, we have that

$$\psi_C([(u, f_1, v)_{u_{\mathfrak{d}}}]) + \lambda\psi_C([(u, f_2, v)_{u_{\mathfrak{d}}}]) = u_{\mathfrak{d}}(f_1) + \lambda u_{\mathfrak{d}}(f_2).$$

As $u_{\mathfrak{d}}$ is a k-linear functor, we have that $u_{\mathfrak{d}}(f_1 + \lambda f_2) = u_{\mathfrak{d}}(f_1) + \lambda u_{\mathfrak{d}}(f_2)$ as desired.

(4) The functor ψ_C is essentially surjective. Let y be an object in C. Consider the small full subcategory $\mathfrak{a}$ of C with objects $\{y\} \cup \{g\}_{g \in G}$, where G is a small set of generators of C. We endow $\mathfrak{a}$ with the topology induced by the canonical topology in C. Then the embedding $\iota : \mathfrak{a} \to C$ is trivially an LC morphism and hence we have that $\psi_C(y, \iota : \mathfrak{a} \to C) = \iota(y) = y$, as desired.

(5) The functor ψ_C is faithful. Let $[(u_1, f_1, v_1)_{u_{\mathfrak{c}_1}}]$, $[(u_2, f_2, v_2)_{u_{\mathfrak{c}_2}}] : (x, u_{\mathfrak{a}}) \to (y, u_{\mathfrak{b}})$ be two morphisms, such that $u_{\mathfrak{c}_1}(f_1) = u_{\mathfrak{c}_2}(f_2)$. Consider the full subcategory $\mathfrak{d}$ of C with objects $\{u_{\mathfrak{c}_1}(\mathfrak{c}_1)\} \cup \{u_{\mathfrak{c}_2}(\mathfrak{c}_2)\}$ endowed with the topology induced by the canonical topology in C and the associated embedding $\iota : \mathfrak{d} \to C$. Then we have that $u_{\mathfrak{c}_1}$ and $u_{\mathfrak{c}_2}$ factor through ι:

$$\begin{tikzcd}
\mathfrak{c}_1 \arrow[dr, "\tilde{u}_{\mathfrak{c}_1}"] \arrow[drr, bend left, "u_{\mathfrak{c}_1}"] & & \\
& \mathfrak{d} \arrow[r, "\iota"] & C. \\
\mathfrak{c}_2 \arrow[ur, "\tilde{u}_{\mathfrak{c}_2}"'] \arrow[urr, bend right, "u_{\mathfrak{c}_2}"'] & &
\end{tikzcd}$$

Observe that, because $u_{\mathfrak{c}_1}(f_1) = u_{\mathfrak{c}_2}(f_2)$ and ι is fully faithful and injective on objects, we have that $\tilde{u}_{\mathfrak{c}_1}u_1 = \tilde{u}_{\mathfrak{c}_2}u_2$, $\tilde{u}_{\mathfrak{c}_1}v_1 = \tilde{u}_{\mathfrak{c}_2}v_2$ and $\tilde{u}_{\mathfrak{c}_1}(f_1) = \tilde{u}_{\mathfrak{c}_2}(f_2)$. This implies that $(u_1, f_1, v_1)_{u_{\mathfrak{c}_1}}$ and $(u_2, f_2, v_2)_{u_{\mathfrak{c}_2}}$ are homotopic, and hence $[(u_1, f_1, v_1)_{u_{\mathfrak{c}_1}}] = [(u_2, f_2, v_2)_{u_{\mathfrak{c}_2}}]$.

(6) The functor ψ_C is full. Consider $(x, u_{\mathfrak{a}} : \mathfrak{a} \to C), (y, u_{\mathfrak{b}} : \mathfrak{b} \to C) \in \mathcal{J}_C$ and a morphism $f : u_{\mathfrak{a}}(x) \to u_{\mathfrak{b}}(y)$ in C. As previously, consider the full subcategory $\mathfrak{c}$ of C with objects $\{u_{\mathfrak{a}}(\mathfrak{a})\} \cup \{u_{\mathfrak{b}}(\mathfrak{b})\}$ endowed with the topology induced by the canonical topology in C and the embedding $\iota : \mathfrak{c} \to C$, which

is an LC morphism. Then, we have that $u_{\mathfrak{a}}$ and $u_{\mathfrak{b}}$ factor through ι as above:

$$\begin{tikzcd}
\mathfrak{a} \arrow[dr, "\tilde{u}_{\mathfrak{a}}"] \arrow[drr, bend left, "u_{\mathfrak{a}}"] & & \\
& \mathfrak{c} \arrow[r, "\iota"] & C. \\
\mathfrak{b} \arrow[ur, "\tilde{u}_{\mathfrak{b}}"'] \arrow[urr, bend right, "u_{\mathfrak{b}}"'] & &
\end{tikzcd}$$

We can hence consider $\tilde{f} : \tilde{u}_{\mathfrak{a}}(x) \to \tilde{u}_{\mathfrak{b}}(y)$ the image of f via the isomorphism $C(\iota\tilde{u}_{\mathfrak{a}}(x), \iota\tilde{u}_{\mathfrak{b}}(y)) \cong \mathfrak{c}(\tilde{u}_{\mathfrak{a}}(x), \tilde{u}_{\mathfrak{b}}(y))$ induced by ι. Take the morphism $[(\tilde{u}_{\mathfrak{a}}, \tilde{f}, \tilde{u}_{\mathfrak{b}})_{\iota=u_{\mathfrak{c}}}] : (x, u_{\mathfrak{a}} : \mathfrak{a} \to C) \to (y, u_{\mathfrak{b}} : \mathfrak{b} \to C)$. By construction we have that ψ_C sends this morphism to $\iota(\tilde{f}) = f$, proving fulness.

We hence have proven that $\psi_C : \mathcal{L}(G_C) \to C$ is a k-linear equivalence of categories as desired. □

Remark 6.9. Observe that, in order to recover any locally presentable category C as a filtered bicolimit using the construction from Corollary 6.5, we can always use the same filtered category, namely the category Reg associated to the totally ordered class of small regular cardinals. Notice that this is not the case for this last presentation of Grothendieck categories provided by Theorem 6.8, as the filtered category $\mathcal{J}_C$ depends on the Grothendieck category C we want to recover.

7. The tensor product of locally presentable linear categories as a filtered bicolimit

In this section we analyse the tensor product of locally presentable k-linear categories in terms of the realization of locally presentable categories as filtered bicolimits provided in §6. In particular, this applies to the tensor product of Grothendieck k-linear categories.

Let $\mathcal{A}, \mathcal{B}$ be locally presentable k-linear categories and let $\kappa \in \mathsf{Reg}$ the smallest regular cardinal such that both $\mathcal{A}$ and $\mathcal{B}$ are locally κ-presentable. First observe that, as a direct consequence of Theorem 6.2, we have that $\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$ is the k-linear filtered bicolimit of the family $((\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\alpha)_{\alpha \in \mathsf{Reg}}$ and that from $\alpha \geq \kappa$ we have that $(\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\alpha \simeq \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha$. However, it is not directly obvious whether the fully faithful functors $\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \hookrightarrow \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta$ with $\beta \geq \alpha \geq \kappa$ provided by the inclusion $(\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\alpha \subseteq (\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\beta$ coincide with the canonical functors $f_{\alpha,\beta}$ from (17), and hence whether the filtered bicolimit is compatible with the α-cocomplete tensor products for α varying in Reg. This is precisely the content of the next theorem.

Theorem 7.1. *Let $\mathcal{A}$, $\mathcal{B}$ be two locally presentable k-linear categories and choose the smallest regular cardinal κ such that both $\mathcal{A}$ and $\mathcal{B}$ are locally κ-presentable. Then, for all $\alpha, \beta \in \mathsf{Reg}$ such that $\beta \geq \alpha \geq \kappa$ the canonical functor*

$$f_{\alpha,\beta} : \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \to \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta$$

defined in (17) is fully faithful. In particular, the functor $f_{\alpha,\beta}$ coincides up to isomorphism, modulo the equivalences $(\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\alpha \simeq \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha$ and $(\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\beta \simeq \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta$, with the canonical inclusion $\iota_{\alpha,\beta}^{\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B}} : (\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\alpha \subseteq (\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\beta$.

Proof. Let $\iota^{\otimes}_{\alpha,\beta} \coloneqq \iota^{\mathcal{A}}_{\alpha,\beta} \otimes \iota^{\mathcal{B}}_{\alpha,\beta}$ as in Proposition 3.28. First consider the diagram

$$\begin{tikzcd}
\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \arrow[rrr, "\iota^{\otimes}_{\alpha,\beta}"] \arrow[dr, "u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}"] \arrow[dd, "R_{\alpha,\alpha}Y_{\alpha,\alpha}"'] & & & \mathcal{A}^\beta \otimes \mathcal{B}^\beta \arrow[dl, "u_{\mathcal{A}^\beta,\mathcal{B}^\beta}"'] \arrow[dd, "R_{\beta,\beta}Y_{\beta,\beta}"] \\
& \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[r, "f_{\alpha,\beta}"] \arrow[dl, hook, "k_\alpha"] & \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta \arrow[dr, hook, "k_\beta"] & \\
\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha) \arrow[rrr, "(\iota^{\otimes}_{\alpha,\beta})^s"] & & & \mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta).
\end{tikzcd}$$

By Proposition 3.28, there exists an isomorphism

$$\Delta_{\alpha,\beta} : R_{\beta,\beta}Y_{\beta,\beta}\iota^{\otimes}_{\alpha,\beta} \Rightarrow (\iota^{\otimes}_{\alpha,\beta})^s R_{\alpha,\alpha}Y_{\alpha,\alpha} \tag{25}$$

witnessing the fact that $(\iota^{\otimes}_{\alpha,\beta})^s \cong \mathsf{Lan}_{(R_{\alpha,\alpha}Y_{\alpha,\alpha})}(R_{\beta,\beta}Y_{\beta,\beta}\iota_{\alpha\otimes\beta})$, and providing the commutativity up to isomorphism of the outer square. Moreover, by (19) there exists an isomorphism

$$\Pi_{\alpha,\beta} : u_{\mathcal{A}^\beta,\mathcal{B}^\beta}\iota^{\otimes}_{\alpha,\beta} \Rightarrow f_{\alpha,\beta}u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha} \tag{26}$$

witnessing the fact that $f_{\alpha,\beta} = \mathsf{Lan}_{u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}} u_{\mathcal{A}^\beta,\mathcal{B}^\beta}\iota_{\alpha\otimes\beta}$ and providing the commutativity up to isomorphism of the upper square. In addition, we have by Remark 3.21 that the left and right triangles are strictly commutative. Therefore, we obtain an isomorphism

$$\Theta_{\alpha,\beta} \coloneqq \left[\Delta_{\alpha,\beta}\right] \bullet \left[k_\beta \circ (\Pi_{\alpha,\beta})^{-1}\right] : k_\beta f_{\alpha,\beta}u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha} \Rightarrow (\iota^{\otimes}_{\alpha,\beta})^s k_\alpha u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}. \tag{27}$$

As $k_\beta, f_{\alpha,\beta}, k_\alpha, (\iota^{\otimes}_{\alpha,\beta})^s$ all preserve α-small colimits, by applying Remark 3.18, we conclude that the lower square of the diagram also commutes up to an isomorphism

$$\Xi_{\alpha,\beta} : k_\beta f_{\alpha,\beta} \Rightarrow (\iota^{\otimes}_{\alpha,\beta})^s k_\alpha, \tag{28}$$

such that $\Xi_{\alpha,\beta} \circ u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha} = \Theta_{\alpha,\beta}$. Thus, because k_α, k_β and $(\iota^{\otimes}_{\alpha,\beta})^s$ are fully faithful, we can conclude that $f_{\alpha,\beta}$ is also fully faithful as desired.

Now, for each $\alpha \geq \kappa$ we consider the equivalence $H_\alpha : \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \to \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha)$ from Proposition 3.25. Recall from Proposition 3.28 that there is an isomorphism

$$\epsilon_{\alpha,\beta} : (\iota_{\alpha\otimes\beta})^s H_\alpha = (\iota^{\otimes}_{\alpha,\beta})^s(\iota^{\otimes}_{\alpha,\beta})_s H_\beta \Rightarrow H_\beta, \tag{29}$$

where $\epsilon_{\alpha,\beta} : (\iota^{\otimes}_{\alpha,\beta})^s(\iota_{\alpha\otimes\beta})_s \Rightarrow 1_{\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta,\mathcal{B}^\beta)}$ is the unit of the adjunction $(\iota_{\alpha\otimes\beta})^s \dashv (\iota^{\otimes}_{\alpha,\beta})_s$. Observe now that, because the α-presentable (resp. β-presentable) objects of $\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha)$ (resp. of $\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta)$) are given by the essential image of k_α (resp. of k_β), we have that there exists an equivalence $H'_\alpha : (\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\alpha \to \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha$ (resp. $H'_\beta : (\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\beta \to \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta$) such that $k_\alpha H'_\alpha = H_\alpha(\iota^{\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B}}_\alpha)$

(resp. $H'_\beta k_\beta = H_\beta(\iota_\beta^{\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B}})$). Consider now the diagram

$$\begin{tikzcd}
 & \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[rr, hook, "f_{\alpha,\beta}" description] \arrow[d, hook, "k_\alpha" description] \arrow[rr, phantom, shift right=6ex, "\Xi_{\alpha,\beta}\Swarrow"] & & \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta \arrow[d, hook, "k_\beta" description] & \\
 & \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha,\mathcal{B}^\alpha) \arrow[rr, "(\iota^{\otimes}_{\alpha,\beta})^s" description] \arrow[rr, phantom, shift right=6ex, "\Searrow\epsilon_{\alpha,\beta}"] & & \mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta,\mathcal{B}^\beta) & \\
(\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B})^\alpha \arrow[rr, hook, "\iota_\alpha^{\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B}}" description] \arrow[ur, bend left, "H'_\alpha" description] \arrow[ur, phantom, "\circlearrowleft"] \arrow[rrrr, hook, bend right, "\iota_{\alpha,\beta}^{\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B}}" description] \arrow[rrrr, phantom, shift right=5ex, "\circlearrowleft"] & & \mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B} \arrow[ul, "H_\alpha" description] \arrow[ur, "H_\beta" description] & & (\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B})^\beta . \arrow[ll, hook', "\iota_\beta^{\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B}}" description] \arrow[ul, bend right, "H'_\beta" description] \arrow[ul, phantom, "\circlearrowleft"]
\end{tikzcd}$$

Consequently, the composite $\left[\epsilon_{\alpha,\beta} \circ \iota_\alpha^{\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B}}\right] \bullet \left[\Xi_{\alpha,\beta} \circ H'_\alpha\right]$ provides an isomorphism

$$k_\beta f_{\alpha,\beta} H'_\alpha \cong k_\beta H'_\beta \iota_{\alpha,\beta}^{\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B}},$$

and as k_β is injective on objects and fully faithful, we can conclude that there is an isomorphism

$$f_{\alpha,\beta} H'_\alpha \cong H'_\beta \iota_{\alpha,\beta}^{\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B}},$$

and in particular we have that

$$f_{\alpha,\beta} \cong H'_\beta \iota_{\alpha,\beta}^{\mathcal{A}\boxtimes_{\mathsf{LP}}\mathcal{B}} (H'_\alpha)^{-1},$$

as desired. □

Let $\mathcal{A}, \mathcal{B}$ be two locally presentable k-linear categories and let κ be the smallest regular cardinal such that $\mathcal{A}$ and $\mathcal{B}$ are locally κ-presentable. Let $\mathsf{Reg}_{\geq\kappa}$ be the category provided by the totally ordered class of regular cardinals that are bigger or equal than κ. We can now define

$$F_{\mathcal{A},\mathcal{B}} : \mathsf{Reg}_{\geq\kappa} \to \mathsf{Cat}(k) \subset \mathsf{CAT}(k) \tag{30}$$

the pseudofunctor given by

- $F_{\mathcal{A},\mathcal{B}}(\alpha) = F_{\mathcal{A}}(\alpha) \otimes_\alpha F_{\mathcal{B}}(\alpha) = \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha$ for every $\alpha \in \mathsf{Reg}_{\geq\kappa}$;
- $F_{\mathcal{A},\mathcal{B}}(\alpha \leq \beta) = f_{\alpha,\beta}$ for every morphism $\alpha \leq \beta$ in $\mathsf{Reg}_{\geq\kappa}$;

with the notation from §6 above.

Remark 7.2. The fact that $F_{\mathcal{A},\mathcal{B}}$ is a pseudofunctor follows easily from the fact that $f_{\alpha,\beta} = \mathsf{Lan}_{u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}}(u_{\mathcal{A}^\beta,\mathcal{B}^\beta}(\iota^{\mathcal{A}}_{\alpha,\beta} \otimes \iota^{\mathcal{B}}_{\alpha,\beta}))$ using the universal property of the left Kan extension.

Theorem 7.3. *Let $\mathcal{A}, \mathcal{B}$ be two locally presentable k-linear categories and let κ be the smallest regular cardinal such that $\mathcal{A}$ and β are locally κ-presentable. Then, $\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$ is the k-linear filtered bicolimit of $F_{\mathcal{A},\mathcal{B}} : \mathsf{Reg}_{\geq\kappa} \to \mathsf{Cat}(k)$.*

Proof. First, consider an equivalence adjoint $H_\alpha^{-1} \dashv H_\alpha$ for every $\alpha \in \mathsf{Reg}_{\geq\kappa}$ with unit $\eta_\alpha : 1 \Rightarrow H_\alpha H_\alpha^{-1}$ and counit $\mu_\alpha : H_\alpha^{-1} H_\alpha \Rightarrow 1$. We build a k-linear pseudonatural transformation

$$\Omega^{\mathcal{A},\mathcal{B}} : F_{\mathcal{A},\mathcal{B}} \Rightarrow \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \tag{31}$$

as follows. Given $\alpha \in \mathsf{Reg}_{\geq\kappa}$, we take $\Omega_\alpha^{\mathcal{A},\mathcal{B}} : F_{\mathcal{A},\mathcal{B}}(\alpha) \to \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$ to be given by the k-linear functor:

$$\Omega_\alpha^{\mathcal{A},\mathcal{B}} := H_\alpha^{-1} k_\alpha : \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \to \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}. \tag{32}$$

Given $\alpha \leq \beta$ in $\mathsf{Reg}_{\geq\kappa}$, we take $\Omega^{\mathcal{A},\mathcal{B}}_{\alpha,\beta} : \Omega^{\mathcal{A},\mathcal{B}}_{\beta} F_{\mathcal{A},\mathcal{B}}(\alpha \leq \beta) \Rightarrow \Omega^{\mathcal{A},\mathcal{B}}_{\alpha}$ to be given by the k-linear natural isomorphism

$$\Omega^{\mathcal{A},\mathcal{B}}_{\alpha\leq\beta} \coloneqq \left[\Sigma_{\alpha,\beta} \circ k_\alpha\right] \bullet \left[H^{-1}_{\beta} \circ \Xi_{\alpha,\beta}\right] : H^{-1}_{\beta} k_\beta f_{\alpha,\beta} \Rightarrow H^{-1}_{\alpha} k_\alpha, \tag{33}$$

where $\Sigma_{\alpha,\beta}$ is obtained as the composite
(34)

$$\Sigma_{\alpha,\beta} \coloneqq \left[\mu_\alpha \circ H^{-1}_{\alpha}\right] \bullet \left[H^{-1}_{\beta} \circ \epsilon_{\alpha,\beta} \circ H^{-1}_{\alpha}\right] \bullet \left[(H^{-1}_{\beta}(\iota^{\otimes}_{\alpha,\beta})^s) \circ \eta_\alpha\right] : H^{-1}_{\beta}(\iota^{\otimes}_{\alpha,\beta})^s \Rightarrow H^{-1}_{\alpha}$$

and $\Xi_{\alpha,\beta}, \epsilon_{\alpha,\beta}$ are as in the proof Theorem 7.1 above. One now needs to check that $\Omega^{\mathcal{A},\mathcal{B}}$ so defined satisfies the coherence constraints from Definition 5.1. We do not spell out the argument in full detail, but the proof boils down to the fact that, by definition, $\Xi_{\alpha,\beta} \circ u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha} = \left[\Delta_{\alpha,\beta}\right] \bullet \left[k_\beta \circ \Pi^{-1}_{\alpha,\beta}\right]$ where $\Delta_{\alpha,\beta}$ and $\Pi_{\alpha,\beta}$ are both witnesses of left Kan extensions, together with the fact that $\Sigma_{\alpha,\beta} = \left[\mu_\alpha \circ H^{-1}_{\alpha}\right] \bullet \left[H^{-1}_{\beta} \circ \epsilon_{\alpha,\beta} \circ H^{-1}_{\alpha}\right] \bullet \left[(H^{-1}_{\beta}(\iota^{\otimes}_{\alpha,\beta})^s) \circ \eta_\alpha\right]$ where $\mu_\alpha, \epsilon_{\alpha,\beta}, \eta_\alpha$ are respectively a counit and two units of equivalence adjunctions.

Now, by the universal property of the k-filtered bicolimit (23), there is a unique k-linear functor $\omega : \mathcal{L}(F_{\mathcal{A},\mathcal{B}}) \to \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$ such that $\Omega^{\mathcal{A},\mathcal{B}} = \omega \circ \Lambda$, which is given by

- Given $(x,\alpha) \in \mathcal{L}(F_{\mathcal{A},\mathcal{B}})$, we have
$$\omega(x,\alpha) = \Omega^{\mathcal{A},\mathcal{B}}_{\alpha}(x) = H^{-1}_{\alpha} k_\alpha(x) \in \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B};$$
- Given $[(\alpha \leq \gamma, f, \beta \leq \gamma)] : (x,\alpha) \to (y,\beta)$, we have
$$\omega([(\alpha \leq \gamma, f, \beta \leq \gamma)]) = \left[\Omega^{\mathcal{A},\mathcal{B}}_{\beta\leq\gamma}\right]_y \Omega^{\mathcal{A},\mathcal{B}}_{\gamma}(f) \left[(\Omega^{\mathcal{A},\mathcal{B}}_{\alpha\leq\gamma})^{-1}\right]_x$$

(see §5). To conclude, it is enough to show that ω is an equivalence:

(1) ω is essentially surjective. Given $x \in \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$, we know there exits a regular cardinal $\alpha \geq \kappa$ such that $x \in (\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B})^\alpha$. Therefore,
$$x \cong H^{-1}_{\alpha} k_\alpha H'_{\alpha}(x) = \omega((H'_{\alpha}(x), \alpha))$$
as desired;

(2) ω is faithful. Let $[(\alpha \leq \gamma, f, \beta \leq \gamma)], [(\alpha \leq \delta, g, \beta \leq \delta)] : (x,\alpha) \to (y,\beta)$ be two morphisms in $\mathcal{L}(F_{\mathcal{A},\mathcal{B}})$ such that
$$\omega([(\alpha \leq \gamma, f, \beta \leq \gamma)]) = \omega([(\alpha \leq \delta, g, \beta \leq \delta)]).$$
Without loss of generality, we may assume that $\delta = \gamma$. We thus have that
$$\left[\Omega^{\mathcal{A},\mathcal{B}}_{\beta\leq\gamma}\right]_y \Omega^{\mathcal{A},\mathcal{B}}_{\gamma}(f) \left[(\Omega^{\mathcal{A},\mathcal{B}}_{\alpha\leq\gamma})^{-1}\right]_x = \left[\Omega^{\mathcal{A},\mathcal{B}}_{\beta\leq\gamma}\right]_y \Omega^{\mathcal{A},\mathcal{B}}_{\gamma}(g) \left[(\Omega^{\mathcal{A},\mathcal{B}}_{\alpha\leq\gamma})^{-1}\right]_x.$$
Because $\left[\Omega^{\mathcal{A},\mathcal{B}}_{\beta\leq\gamma}\right]_y$ and $\left[(\Omega^{\mathcal{A},\mathcal{B}}_{\alpha\leq\gamma})^{-1}\right]_x$ are isomorphisms, we can conclude that
$$\Omega^{\mathcal{A},\mathcal{B}}_{\gamma}(f) = \Omega^{\mathcal{A},\mathcal{B}}_{\gamma}(g).$$
Now, because $\Omega^{\mathcal{A},\mathcal{B}}_{\gamma} = H^{-1}_{\gamma} k_\gamma$ is fully faithful, we can conclude that $f = g$ which shows that
$$[(\alpha \leq \gamma, f, \beta \leq \gamma)] = [(\alpha \leq \gamma, g, \beta \leq \gamma)]$$
as desired.

(3) ω is full. Let $s : \omega(x,\alpha) = \Omega^{\mathcal{A},\mathcal{B}}_{\alpha}(x) \to \Omega^{\mathcal{A},\mathcal{B}}_{\beta}(y) = \omega(y,\beta)$ be a morphism in $\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$. Consider the composite

$$\Omega^{\mathcal{A},\mathcal{B}}_{\gamma} f_{\alpha,\gamma}(x) \xrightarrow{\left[\Omega^{\mathcal{A},\mathcal{B}}_{\alpha\leq\gamma}\right]_x} \Omega^{\mathcal{A},\mathcal{B}}_{\alpha}(x) \xrightarrow{\quad s \quad} \Omega^{\mathcal{A},\mathcal{B}}_{\beta}(y) \xrightarrow{\left[(\Omega^{\mathcal{A},\mathcal{B}}_{\beta\leq\gamma})^{-1}\right]_y} \Omega^{\mathcal{A},\mathcal{B}}_{\gamma} f_{\beta,\gamma}(y).$$

Then, because $\Omega_{\gamma}^{\mathcal{A},\mathcal{B}}$ is fully faithful, there exists a (unique) $f : f_{\alpha,\gamma}(x) \to f_{\beta,\gamma}(y)$ such that $\Omega_{\gamma}^{\mathcal{A},\mathcal{B}}(f) = \left[(\Omega_{\beta\leq\gamma}^{\mathcal{A},\mathcal{B}})^{-1}\right]_y s \left[\Omega_{\alpha\leq\gamma}^{\mathcal{A},\mathcal{B}}\right]_x$, from which we can readily conclude that $s = \omega([(\alpha \leq \gamma, f, \beta \leq \gamma)])$, as desired.

□

Remark 7.4. One may wonder if an analogous approach would allow to obtain a realization of the tensor product of Grothendieck categories as a filtered bicolimit by using, instead of the α-Kelly tensor product, the tensor product $\boxtimes_{\mathsf{S}}$ of linear sites and LC morphisms from §4 and, instead of the realization of Grothendieck categories as filtered bicolimits of α-presentable objects from Corollary 6.5, the realization of Grothendieck categories as filtered bicolimits of linear sites from Theorem 6.8. We will explain why this is not the case. Roughly, the argument goes as follows:

Let $\mathcal{A}, \mathcal{B}$ be Grothendieck k-linear categories. We use the notation introduced in §6 for the rest of the remark. Consider the filtered categories $\mathcal{J}_{\mathcal{A}}$ (resp. $\mathcal{J}_{\mathcal{B}}$) with objects the LC morphisms $u_{\mathfrak{c}} : (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}}) \to (\mathcal{A}, \mathcal{T}_{\mathcal{A},\mathsf{can}})$, (resp. the LC morphisms $v_{\mathfrak{d}} : (\mathfrak{d}, \mathcal{T}_{\mathfrak{d}}) \to (\mathcal{B}, \mathcal{T}_{\mathcal{B},\mathsf{can}})$). Denote by $\mathcal{J}_{\mathcal{A},\mathcal{B}}$ the category with objects given by tensor products $u_{\mathfrak{c}} \otimes v_{\mathfrak{d}} : (\mathfrak{c} \otimes \mathfrak{d}, \mathcal{T}_{\mathfrak{c}} \boxtimes_{\mathsf{S}} \mathcal{T}_{\mathfrak{d}}) \to (\mathcal{A} \otimes \mathcal{B}, \mathcal{T}_{\mathcal{A},\mathsf{can}} \boxtimes_{\mathsf{S}} \mathcal{T}_{\mathcal{B},\mathsf{can}})$ of objects in $\mathcal{J}_{\mathcal{A}}$ with objects of $\mathcal{J}_{\mathcal{B}}$, and with morphisms $f \otimes g : u_{\mathfrak{c}_1} \otimes v_{\mathfrak{d}_1} \to r_{\mathfrak{c}_2} \otimes s_{\mathfrak{d}_2}$ the LC morphisms given by the tensor product $f \otimes g$ of LC morphisms in $\mathcal{J}_{\mathcal{A}}$ with morphisms in $\mathcal{J}_{\mathcal{B}}$. In particular, we have that $\mathcal{J}_{\mathcal{A},\mathcal{B}}$ is filtered and we can define a functor $G_{\mathcal{A},\mathcal{B}} : \mathcal{J}_{\mathcal{A},\mathcal{B}} \to \mathsf{CAT}(k)$ consisting on forgetting the slice structure, parallelly to the definition of $G_{\mathcal{C}}$ in §6. Moreover, as LC is closed under composition (see Remark 4.13) and tensor products (see Proposition 4.18), composition with the LC morphism

$$s_{\mathcal{A},\mathcal{B}} : (\mathcal{A} \otimes \mathcal{B}, \mathcal{T}_{\mathcal{A},\mathsf{can}} \boxtimes_{\mathsf{S}} \mathcal{T}_{\mathcal{B},\mathsf{can}}) \to (\mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B}, \mathcal{T}_{\mathcal{A}\boxtimes_{\mathsf{G}}\mathcal{B},\mathsf{can}})$$

permits us to define a pseudonatural transformation $T : G_{\mathcal{A},\mathcal{B}} \Rightarrow \mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B}$ with components LC morphisms. More concretely,

- Given $u_{\mathfrak{a}} \otimes v_{\mathfrak{b}} : \mathfrak{a} \otimes \mathfrak{b} \to \mathcal{A} \otimes \mathcal{B}$, we define
$$T_{u\otimes v} \coloneqq s_{\mathcal{A},\mathcal{B}}(u_{\mathfrak{a}} \otimes v_{\mathfrak{b}}) : \mathfrak{a} \otimes \mathfrak{b} \to \mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B};$$
- Given a morphism $(f \otimes g) : u_{\mathfrak{a}} \otimes v_{\mathfrak{b}} \to r_{\mathfrak{a}'} \otimes s_{\mathfrak{b}'}$ in $\mathcal{J}_{\mathcal{A},\mathcal{B}}$, we define
$$T_{f\otimes g} : s_{\mathcal{A},\mathcal{B}}(r_{\mathfrak{a}'} \otimes s_{\mathfrak{b}'})(f \otimes g) \Rightarrow s_{\mathcal{A},\mathcal{B}}(u_{\mathfrak{a}} \otimes v_{\mathfrak{b}})$$
to be equal to the identity $s_{\mathcal{A},\mathcal{B}}(r_{\mathfrak{a}'} \otimes s_{\mathfrak{b}'})(f \otimes g) = s_{\mathcal{A},\mathcal{B}}(u_{\mathfrak{a}} \otimes v_{\mathfrak{b}})$.

As in Theorem 7.3, we can consider the unique k-linear functor

$$\tau : \mathcal{L}(G_{\mathcal{A},\mathcal{B}}) \to \mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B}$$

such that $\tau \circ \Lambda = T$, provided by the universal property of the k-linear filtered bicolimit (23). However, in this case, contrary to the case above, τ is not an equivalence. More concretely, τ fails to be essentially surjective in general. Indeed, it is straightforward to check that τ factors as $\tau = s_{\mathcal{A},\mathcal{B}} \circ \tau'$ for some $\tau' : \mathcal{L}(G_{\mathcal{A},\mathcal{B}}) \to \mathcal{A} \otimes \mathcal{B}$, and the functor $s_{\mathcal{A},\mathcal{B}} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A} \boxtimes_{\mathsf{G}} \mathcal{B}$ is not essentially surjective in general, thus τ also fails to be essentially surjective. Let us provide an example[4] to illustrate the failure to be essentially surjective of $s_{\mathcal{A},\mathcal{B}}$. Assume that our base ring k is a field and consider

$$A \coloneqq \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$$

[4]The example we describe belongs to the family of examples provided in the following entry of MathOverflow: https://mathoverflow.net/questions/354306/tensor-indecomposable-modules

the lower triangular matrix k-algebra. In particular, we know (see, for example, [4, Ex II.1.3(c)]) that A is the path algebra of the A_2 quiver

$$1 \xrightarrow{\alpha} 2,$$

where

- $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ corresponds to the trivial path from 1 to 1;
- $e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ corresponds to the path α;
- $e_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to the trivial path from 2 to 2.

Consider the Grothendieck k-linear categories $\mathcal{A} := \mathsf{Mod}(A^{\mathsf{op}})$ of left A-modules and $\mathcal{B} := \mathsf{Mod}(A)$ of right A-modules and their tensor product $\mathsf{Mod}(A^{\mathsf{op}}) \boxtimes_{\mathsf{G}} \mathsf{Mod}(A) \simeq \mathsf{Mod}(A^{\mathsf{op}} \otimes_k A)$. Recall that $\mathsf{Mod}(A^{\mathsf{op}})$ is equivalent to the category of representations of the quiver A_2, while $\mathsf{Mod}(A)$ is equivalent to the category of representations of the quiver $A_2^{\mathsf{op}} \cong A_2$ [4, Thm III.1.6]. Moreover, the k-algebra $A \otimes_k A^{\mathsf{op}}$ is the path algebra of the quiver $A_2 \times A_2$

$$\begin{tikzcd} (1,1) \arrow[r, "{(1,\alpha)}"] \arrow[d, "{(\alpha,1)}"'] & (1,2) \arrow[d, "{(\alpha,2)}"] \\ (2,1) \arrow[r, "{(2,\alpha)}"] & (2,2) \end{tikzcd}$$

with relation $\sigma := (\alpha,2)(1,\alpha) - (2,\alpha)(\alpha,1)$ [18, Prop 3]. Therefore, $\mathsf{Mod}(A^{\mathsf{op}} \otimes_k A)$ is equivalent to the category of representations of the quiver with relations $(A_2 \times A_2, \sigma)$ [4, Thm III.1.6]. Let us now consider the universal functor

$$s_{\mathsf{Mod}(A^{\mathsf{op}}),\mathsf{Mod}(A)} : \mathsf{Mod}(A^{\mathsf{op}}) \otimes \mathsf{Mod}(A) \to \mathsf{Mod}(A^{\mathsf{op}} \otimes_k A),$$

which in particular sends a pair (M,N) of a left A-module M and a right A-module N to the right $A^{\mathsf{op}} \otimes_k A$-module $M \otimes_k N$. We claim that A itself, when considered as a right $A^{\mathsf{op}} \otimes_k A$-module in the standard way, is not in the essential image of this functor. The representation associated to A is given by

$$\begin{tikzcd} e_1Ae_1 \arrow[r, "(-)\cdot e_2"] \arrow[d, "e_2\cdot(-)"'] & e_1Ae_3 \arrow[d, "e_2\cdot(-)"] \\ e_3Ae_1 \arrow[r, "(-)\cdot e_2"] & e_3Ae_3 \end{tikzcd} \quad \cong \quad \begin{tikzcd} k \arrow[r, "0"] \arrow[d, "1_k"'] & 0 \arrow[d, "0"] \\ k \arrow[r, "0"] & k. \end{tikzcd}$$

On the other hand, given $M \in \mathsf{Mod}(A^{\mathsf{op}})$ with associated representation

$$e_1M \xrightarrow{e_2\cdot(-)} e_3M \qquad \cong \qquad k^{m_1} \xrightarrow{f} k^{m_2}$$

and $N \in \mathsf{Mod}(A)$ with associated representation

$$Ne_1 \xrightarrow{(-)\cdot e_2} Ne_3 \qquad \cong \qquad k^{n_1} \xrightarrow{g} k^{n_2},$$

the associated presentation of $M \otimes_k N$ is of the form

$$\begin{tikzcd} k^{m_1} \otimes_k k^{n_1} \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"'] & k^{m_1} \otimes_k k^{n_2} \arrow[d, "f\otimes 1"] \\ k^{m_2} \otimes_k k^{n_1} \arrow[r, "1\otimes g"] & k^{m_2} \otimes_k k^{n_2}. \end{tikzcd}$$

Therefore, for A to be of the form $M \otimes_k N$, we would need that $m_1 = n_1 = m_2 = n_2 = 1$ but at the same time that either $m_1 = 0$ or $n_2 = 0$, which cannot happen. We can thus conclude that A does not belong to the essential image of $s_{\mathsf{Mod}(A^{\mathsf{op}}),\mathsf{Mod}(A)}$, as desired.

8. The functoriality, associativity and symmetry of the tensor product of locally presentable linear categories in terms of filtered bicolimits

The tensor product of locally presentable linear categories endows the 2-category of locally presentable linear categories with a symmetric monoidal closed structure [11, Cor 2.2.5]. In this section we describe the functoriality with respect to cocontinuous k-linear functors, the associativity and the symmetry of the tensor product of locally presentable linear categories (and thus in particular that of Grothendieck categories) in terms of the functoriality, associativity and symmetry of the α-Kelly tensor products, making use of Theorem 7.3 above. In particular, this provides an advantage when computing the tensor product of cocontinuous functors when we have control of the subcategories of presentable objects, as illustrated by Example 8.9 below.

Definition 8.1. Consider locally presentable k-linear categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ and cocontinuous k-linear functors $F : \mathcal{A} \to \mathcal{C}$ and $F' : \mathcal{B} \to \mathcal{D}$. We define

$$F \boxtimes_{\mathsf{LP}} F' : \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \to \mathcal{C} \boxtimes_{\mathsf{LP}} \mathcal{D}$$

to be the cocontinuous k-linear functor obtained from applying the universal property of $\boxtimes_{\mathsf{LC}}$ from (16) to the (cococontinuous in each variable) k-linear functor

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{F \otimes F'} \mathcal{C} \otimes \mathcal{D} \xrightarrow{s_{\mathcal{C},\mathcal{D}}} \mathcal{C} \boxtimes_{\mathsf{LP}} \mathcal{D}.$$

More concretely, we have that $F \boxtimes_{\mathsf{LC}} F' = \mathsf{Lan}_{(s_{\mathcal{A},\mathcal{B}})} \left(s_{\mathcal{C},\mathcal{D}}(F \otimes F')\right)$.

In what follows, we are going to provide a description of the tensor product $\boxtimes_{\mathsf{LP}}$ of cocontinuous linear functors making use of the filtered bicolimit presentation of $\boxtimes_{\mathsf{LP}}$ from Theorem 7.3. In particular, we will use the functoriality of the α-Kelly tensor product:

Definition 8.2. Consider α-cocomplete k-linear categories $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and $\mathfrak{d}$ and α-cocontinuous k-linear functors $f : \mathfrak{a} \to \mathfrak{c}$ and $f' : \mathfrak{b} \to \mathfrak{d}$. We define

$$f \otimes_\alpha f' : \mathfrak{a} \otimes_\alpha \mathfrak{b} \to \mathfrak{c} \otimes_\alpha \mathfrak{d}$$

to be the α-cocontinuous k-linear functor obtained from applying the universal property of $\otimes_\alpha$ from (12) to the (α-cococontinuous in each variable) functor

$$\mathfrak{a} \otimes \mathfrak{b} \xrightarrow{f \otimes f'} \mathfrak{c} \otimes \mathfrak{d} \xrightarrow{u^\alpha_{\mathfrak{c},\mathfrak{d}}} \mathfrak{c} \otimes_\alpha \mathfrak{d}.$$

More concretely,

$$f \otimes_\alpha f' = \mathsf{Lan}_{(u^\alpha_{\mathfrak{a},\mathfrak{b}})} \left(u^\alpha_{\mathfrak{c},\mathfrak{d}}(f \otimes f')\right).$$

Consider $\mathcal{A}$, $\mathcal{B}$ two locally presentable k-linear categories and a regular cardinal α. Recall that a k-linear functor $F : \mathcal{A} \to \mathcal{B}$ is said to *have rank* α if it preserves α-filtered colimits [8, §5.5]. It is trivial to see that if a k-linear functor F has rank α, then it has rank β for every $\beta \geq \alpha$. We say a k-linear functor *has rank* if there exists a regular cardinal α such that it has rank α. We have the following useful proposition.

Proposition 8.3 ([8, Prop. 5.5.6])**.** *Let $G : \mathcal{A} \to \mathcal{B}$ a k-linear functor between locally presentable k-linear categories. If G has a left adjoint, then G has rank.*

The following is easy to show, but we provide a proof for the convenience of the reader.

Proposition 8.4. *Let $F : \mathcal{A} \to \mathcal{B}$ a cocontinuous k-linear functor between locally presentable k-linear categories. Then, there exists a regular cardinal α such that $F(\mathcal{A}^\beta) \subseteq \mathcal{B}^\beta$ for every $\beta \geq \alpha$.*

Proof. By the dual of the Special Adjoint Functor Theorem [7, Thm. 3.3.4], we have that F has a right adjoint G. In particular, by Proposition 8.3, G has rank. Fix the smallest α such that G has rank α. Then, given an element $C \in \mathcal{A}^\beta$, where $\beta \geq \alpha$, we have that

$$\begin{aligned}\mathcal{B}(F(C), \mathrm{colim}_i D_i) &= \mathcal{A}(C, G(\mathrm{colim}_i D_i)) \\ &= \mathcal{A}(C, \mathrm{colim}_i G(D_i)) \\ &= \mathrm{colim}_i \mathcal{A}(C, G(D_i)) \\ &= \mathrm{colim}_i (F(C), D_i)\end{aligned}$$

where $\mathrm{colim}_i D_i$ is any β-filtered colimit in $\mathcal{B}$. Hence $F(C) \in \mathcal{B}^\beta$ as desired. □

Remark 8.5. Given $F : \mathcal{A} \to \mathcal{B}$ as in the proposition, note that the restriction-corestriction $F_\beta : \mathcal{A}^\beta \to \mathcal{B}^\beta$ of F is β-cocontinuous for all $\beta \geq \alpha$.

We first show that the functoriality of the tensor product $\boxtimes_{\mathsf{LP}}$ and that of the tensor products $\otimes_\alpha$ are nicely compatible in the following sense:

Proposition 8.6. *Consider locally presentable k-linear categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ and cocontinuous k-linear functors $F : \mathcal{A} \to \mathcal{C}$ and $F' : \mathcal{B} \to \mathcal{D}$. Let $\bar{\kappa}$ be the smallest regular cardinal for which both F and F' preserve α-presentable objects for every $\alpha \geq \bar{\kappa}$ and let κ be the smallest regular cardinal such that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are locally κ-presentable and $\kappa \geq \bar{\kappa}$. For every $\gamma \geq \beta \geq \bar{\kappa}$, the diagram*

(35)
$$\begin{tikzcd} \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta \arrow[r, "F_\beta \otimes_\beta F'_\beta"] \arrow[d, "f_{\beta,\gamma}"'] & \mathcal{C}^\beta \otimes_\beta \mathcal{B}^\beta \arrow[d, "f_{\beta,\gamma}"] \\ \mathcal{A}^\gamma \otimes_\gamma \mathcal{B}^\gamma \arrow[r, "F_\gamma \otimes_\gamma F'_\gamma"] & \mathcal{C}^\gamma \otimes_\gamma \mathcal{D}^\gamma \end{tikzcd}$$

is commutative up to isomorphism, while for every $\alpha \geq \kappa$, the diagram

(36)
$$\begin{tikzcd} \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[r, "F_\alpha \otimes_\alpha F'_\alpha"] \arrow[d, hook] & \mathcal{C}^\alpha \otimes_\alpha \mathcal{D}^\alpha \arrow[d, hook] \\ \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \arrow[r, "F \boxtimes_{\mathsf{LP}} F'"] & \mathcal{C} \boxtimes_{\mathsf{LP}} \mathcal{D} \end{tikzcd}$$

is commutative up to isomorphism.

Proof. Consider the diagram

$$\begin{tikzcd} \mathcal{A}^\beta \otimes \mathcal{B}^\beta \arrow[rrr, "F_\beta \otimes F_\beta"] \arrow[dr, "u_{\mathcal{A}^\beta, \mathcal{B}^\beta}"] \arrow[ddd, "\iota^{\mathcal{A}}_{\beta,\gamma} \otimes \iota^{\mathcal{B}}_{\beta,\gamma}"'] & & & \mathcal{C}^\beta \otimes \mathcal{D}^\beta \arrow[dl, "u_{\mathcal{C}^\beta, \mathcal{D}^\beta}"'] \arrow[ddd, "\iota^{\mathcal{C}}_{\beta,\gamma} \otimes \iota^{\mathcal{D}}_{\beta,\gamma}"] \\ & \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta \arrow[r, "F_\beta \otimes_\beta F'_\beta"] \arrow[d, "f^{\mathcal{A},\mathcal{B}}_{\beta,\gamma}"'] \arrow[ur, phantom, "\swarrow M_\beta"] \arrow[dl, phantom, "\Pi^{\mathcal{A},\mathcal{B}}_{\beta,\gamma} \nearrow"] & \mathcal{C}^\beta \otimes_\beta \mathcal{D}^\beta \arrow[d, "f^{\mathcal{C},\mathcal{D}}_{\beta,\gamma}"] \arrow[dr, phantom, "\nwarrow \Pi^{\mathcal{C},\mathcal{D}}_{\beta,\gamma}"] & \\ & \mathcal{A}^\gamma \otimes_\gamma \mathcal{B}^\gamma \arrow[r, "F_\gamma \otimes_\gamma F'_\gamma"] \arrow[dr, phantom, "M_\gamma \nearrow"] & \mathcal{C}^\gamma \otimes_\gamma \mathcal{D}^\gamma & \\ \mathcal{A}^\gamma \otimes \mathcal{B}^\gamma \arrow[rrr, "F_\gamma \otimes F'_\gamma"] \arrow[ur, "u_{\mathcal{A}^\gamma, \mathcal{B}^\gamma}"] & & & \mathcal{C}^\gamma \otimes \mathcal{D}^\gamma, \arrow[ul, "u_{\mathcal{C}^\gamma, \mathcal{D}^\gamma}"'] \end{tikzcd}$$

where:

- The natural isomorphisms M_β and M_γ witness the fact that

$$F_\beta \otimes_\beta F'_\beta = \mathsf{Lan}_{(u_{\mathcal{A}^\beta, \mathcal{B}^\beta})} \left(u_{\mathcal{C}^\beta, \mathcal{B}^\beta} (F_\beta \otimes F'_\beta) \right)$$

  and

$$F_\gamma \otimes_\gamma F'_\gamma = \mathsf{Lan}_{(u_{\mathcal{A}^\gamma, \mathcal{B}^\gamma})} \left(u_{\mathcal{C}^\gamma, \mathcal{B}^\gamma} (F_\gamma \otimes F'_\gamma) \right),$$

providing the commutativity up to isomorphism of the upper and lower squares;

- The natural isomorphisms $\Pi^{\mathcal{A},\mathcal{B}}_{\beta,\gamma}$ and $\Pi^{\mathcal{C},\mathcal{D}}_{\beta,\gamma}$ witness the fact that
$$f^{\mathcal{A},\mathcal{B}}_{\beta,\gamma} = \mathsf{Lan}_{(u_{\mathcal{A}^\beta,\mathcal{B}^\beta})}\left(u_{\mathcal{C}^\gamma,\mathcal{B}^\gamma}(\iota^{\mathcal{A}}_{\beta,\gamma} \otimes \iota^{\mathcal{B}}_{\beta,\gamma})\right)$$
and
$$f^{\mathcal{C},\mathcal{D}}_{\beta,\gamma} = \mathsf{Lan}_{(u_{\mathcal{C}^\beta,\mathcal{D}^\beta})}\left(u_{\mathcal{C}^\gamma,\mathcal{B}^\gamma}(\iota^{\mathcal{C}}_{\beta,\gamma} \otimes \iota^{\mathcal{D}}_{\beta,\gamma})\right),$$
providing the commutativity up to isomorphism of the left and right squares;
- The outer square is strictly commutative by definition.

Therefore, the composite
$$\left[f^{\mathcal{C},\mathcal{D}}_{\beta,\gamma} \circ M_\beta\right] \bullet \left[\Pi^{\mathcal{C},\mathcal{D}}_{\beta,\gamma} \circ (F_\beta \otimes F'_\beta)\right] \bullet \left[M^{-1}_\gamma \circ (\iota^{\mathcal{A}}_{\beta,\gamma} \otimes \iota^{\mathcal{B}}_{\beta,\gamma})\right] \bullet \left[(F_\gamma \otimes_\beta F'_\gamma) \circ (\Pi^{\mathcal{A},\mathcal{B}}_{\beta,\gamma})^{-1}\right] \tag{37}$$
provides an isomorphism $(F_\gamma \otimes_\beta F'_\gamma) f^{\mathcal{A},\mathcal{B}}_{\beta,\gamma} u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha} \Rightarrow f^{\mathcal{C},\mathcal{D}}_{\beta,\gamma}(F_\beta \otimes_\beta F'_\beta) u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}$. Now, because $(F_\gamma \otimes_\beta F'_\gamma), f^{\mathcal{A},\mathcal{B}}_{\beta,\gamma}, f^{\mathcal{C},\mathcal{D}}_{\beta,\gamma}$ and $(F_\beta \otimes_\beta F'_\beta)$ are all β-cocontinuous, we can conclude, by applying Remark 3.18, that there exists a unique isomorphism
$$N_{\beta,\gamma} : (F_\gamma \otimes_\beta F'_\gamma) f^{\mathcal{A},\mathcal{B}}_{\beta,\gamma} \Rightarrow f^{\mathcal{C},\mathcal{D}}_{\beta,\gamma}(F_\beta \otimes_\beta F'_\beta) \tag{38}$$
such that $N_{\beta,\gamma} \circ u_{\mathcal{A}^\beta,\mathcal{B}^\beta}$ equals the composite from (37). In particular, $N_{\beta,\gamma}$ provides the desired commutativity up to isomorphism of (35).

Let us now proceed to prove the commutativity up to isomorphism of (36). First, consider the following diagram

$$\begin{tikzcd}
\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \arrow[rrr, "u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}"] \arrow[dd, "\iota^{\mathcal{A}}_\alpha \otimes \iota^{\mathcal{B}}_\alpha"'] \arrow[ddr, phantom, "\circlearrowleft"] & & & \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[ddd, hook, "k_\alpha"] \\
& \mathcal{A}^\kappa \otimes \mathcal{B}^\kappa \arrow[ul, "\iota^{\mathcal{A}}_{\kappa,\alpha} \otimes \iota^{\mathcal{B}}_{\kappa,\alpha}"'] \arrow[r, "u_{\mathcal{A}^\kappa,\mathcal{B}^\kappa}"] \arrow[dl, "\iota^{\mathcal{A}}_\kappa \otimes \iota^{\mathcal{B}}_\kappa"] \arrow[u, phantom, "\Downarrow\Pi^{\mathcal{A},\mathcal{B}}_{\kappa,\alpha}"] \arrow[dr, phantom, "\Nearrow\Gamma_\kappa"] & \mathcal{A}^\kappa \otimes_\kappa \mathcal{B}^\kappa \arrow[ur, "f^{\mathcal{A},\mathcal{B}}_{\kappa,\alpha}"'] \arrow[d, hook, "k_\kappa"] & \\
\mathcal{A} \otimes \mathcal{B} \arrow[r, "s_{\mathcal{A},\mathcal{B}}"] & \mathcal{A} \boxtimes \mathcal{B} & \mathrm{Lex}_{\kappa,\kappa}(\mathcal{A}^\kappa, \mathcal{B}^\kappa) \arrow[l, "H^{-1}_\kappa"'] \arrow[dr, phantom, "\Nwarrow\Omega^{\mathcal{A},\mathcal{B}}_{\kappa\leq\alpha}"] & \\
& & & \mathrm{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha), \arrow[llu, bend left, "H^{-1}_\alpha"]
\end{tikzcd}$$

where

- $\Pi^{\mathcal{A},\mathcal{B}}_{\kappa,\alpha}$ is the isomorphism witnessing the fact that
$$f^{\mathcal{A},\mathcal{B}}_{\kappa,\alpha} = \mathsf{Lan}_{(u_{\mathcal{A}^\kappa,\mathcal{B}^\kappa})}\left(u_{\mathcal{C}^\alpha,\mathcal{B}^\alpha}(\iota^{\mathcal{A}}_{\kappa,\alpha} \otimes \iota^{\mathcal{B}}_{\kappa,\alpha})\right);$$
- $\Omega^{\mathcal{A},\mathcal{B}}_{\kappa\leq\alpha}$ is the isomorphism provided by the evaluation at $\alpha \leq \beta$ of the pseudo-natural transformation $\Omega^{\mathcal{A},\mathcal{B}}$ from the proof of Theorem 7.3;
- Γ_κ is the isomorphism from Remark 3.27.

Therefore, there is an isomorphism
$$s_{\mathcal{A},\mathcal{B}}(\iota^{\mathcal{A}}_\alpha \otimes \iota^{\mathcal{B}}_\alpha)(\iota^{\mathcal{A}}_{\kappa,\alpha} \otimes \iota^{\mathcal{B}}_{\kappa,\alpha}) \Rightarrow H^{-1}_\alpha k_\alpha u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}(\iota^{\mathcal{A}}_{\kappa,\alpha} \otimes \iota^{\mathcal{B}}_{\kappa,\alpha}). \tag{39}$$
Because $H^{-1}_\alpha k_\alpha u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}$ is α-cocontinuous in each variable, as so is $s_{\mathcal{A},\mathcal{B}}(\iota^{\mathcal{A}}_\alpha \otimes \iota^{\mathcal{B}}_\alpha)$, and because $\mathcal{A}, \mathcal{B}$ are both locally κ-presentable, we can conclude that there exists a unique isomorphism
$$W^{\mathcal{A},\mathcal{B}}_\alpha : s_{\mathcal{A},\mathcal{B}}(\iota^{\mathcal{A}}_\alpha \otimes \iota^{\mathcal{B}}_\alpha) \Rightarrow H^{-1}_\alpha k_\alpha u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}$$

such that $W^{\mathcal{A},\mathcal{B}}_\alpha \circ (\iota^{\mathcal{A}}_{\kappa,\alpha} \otimes \iota^{\mathcal{B}}_{\kappa,\alpha})$ recovers (39). In a completely analogous way, we build an isomorphism

$$W^{\mathcal{C},\mathcal{D}}_\alpha : s_{\mathcal{A},\mathcal{B}}(\iota^{\mathcal{C}}_\alpha \otimes \iota^{\mathcal{D}}_\alpha) \Rightarrow k_\alpha u_{\mathcal{C}^\alpha,\mathcal{D}^\alpha}$$

for the case of $\mathcal{C}, \mathcal{D}$, satisfying the same parallel properties.

Let's now consider the diagram

$$\begin{tikzcd}
\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \arrow[rrr, "F_\alpha \otimes F'_\alpha"] \arrow[dr, "u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}"] \arrow[ddd, "\iota^{\mathcal{A}}_\alpha \otimes \iota^{\mathcal{B}}_\alpha"'] & \arrow[dr, phantom, "\Searrow M_\alpha"] & & \mathcal{C}^\alpha \otimes \mathcal{D}^\alpha \arrow[dl, "u_{\mathcal{C}^\alpha,\mathcal{D}^\alpha}"'] \arrow[ddd, "\iota^{\mathcal{C}}_\alpha \otimes \iota^{\mathcal{D}}_\alpha"] \\
\arrow[dr, phantom, "\Nearrow W^{\mathcal{A},\mathcal{B}}_\alpha"] & \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[r, "F_\alpha \otimes_\alpha F'_\alpha"] \arrow[d, hook] & \mathcal{C}^\alpha \otimes_\alpha \mathcal{D}^\alpha \arrow[d, hook] & \arrow[dl, phantom, "W^{\mathcal{C},\mathcal{D}}_\alpha \Nwarrow"] \\
& \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \arrow[r, "F \boxtimes_{\mathsf{LP}} F'"] \arrow[dr, phantom, "\Nwarrow R"] & \mathcal{C} \boxtimes_{\mathsf{LP}} \mathcal{D} & \\
\mathcal{A} \otimes \mathcal{B} \arrow[ur, "s_{\mathcal{A},\mathcal{B}}"] \arrow[rrr, "F \otimes F'"] & & & \mathcal{C} \otimes \mathcal{D}. \arrow[ul, "s_{\mathcal{C},\mathcal{D}}"']
\end{tikzcd}$$

where

- $W^{\mathcal{A},\mathcal{B}}_\alpha$, $W^{\mathcal{C},\mathcal{D}}_\alpha$ are the isomorphisms we just introduced witnessing the commutativity up to isomorphism of the left and right squares;
- M_α, as above, is the isomorphism witnessing the fact that
$$F_\alpha \otimes_\alpha F'_\alpha = \mathsf{Lan}_{(u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha})}\left(u_{\mathcal{C}^\alpha,\mathcal{B}^\alpha}(F_\alpha \otimes F'_\alpha)\right),$$
providing the commutativity up to isomorphism of the upper square;
- R is the isomorphism witnessing that
$$F \boxtimes_{\mathsf{LP}} F' = \mathsf{Lan}_{(s_{\mathcal{A},\mathcal{B}})}\left(s_{\mathcal{C},\mathcal{D}}(F \otimes F')\right),$$
providing the commutativity up to isomorphism of the lower square;
- The outer square is strictly commutative.

Therefore, the composite

$$\left[\left(H^{\mathcal{C},\mathcal{D}}_\alpha\right)^{-1} k^{\mathcal{C},\mathcal{D}}_\alpha \circ M_\alpha\right] \bullet \left[W^{\mathcal{C},\mathcal{D}}_\alpha \circ (F_\alpha \otimes F'_\alpha)\right] \bullet \left[R^{-1} \circ \left(\iota^{\mathcal{A}}_\alpha \otimes \iota^{\mathcal{B}}_\alpha\right)\right] \bullet \left[(F \boxtimes_{\mathsf{LP}} F') \circ \left(W^{\mathcal{A},\mathcal{B}}_\alpha\right)^{-1}\right] \tag{40}$$

is an isomorphism

$$(F \boxtimes_{\mathsf{LP}} F')(H^{\mathcal{A},\mathcal{B}}_\alpha)^{-1} k^{\mathcal{A},\mathcal{B}}_\alpha u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha} \Rightarrow (H^{\mathcal{C},\mathcal{D}}_\alpha)^{-1} k^{\mathcal{C},\mathcal{D}}_\alpha (F_\alpha \otimes_\alpha F'_\alpha) u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}.$$

Now, because $(F \boxtimes_{\mathsf{LP}} F')(H^{\mathcal{A},\mathcal{B}}_\alpha)^{-1} k^{\mathcal{A},\mathcal{B}}_\alpha$ and $(H^{\mathcal{C},\mathcal{D}}_\alpha)^{-1} k^{\mathcal{C},\mathcal{D}}_\alpha (F_\alpha \otimes_\alpha F'_\alpha)$ are α-cocomplete, we know by Remark 3.18 that there exists a unique isomorphism

$$Z_\alpha : (F \boxtimes_{\mathsf{LP}} F')(H^{\mathcal{A},\mathcal{B}}_\alpha)^{-1} k^{\mathcal{A},\mathcal{B}}_\alpha \Rightarrow (H^{\mathcal{C},\mathcal{D}}_\alpha)^{-1} k^{\mathcal{C},\mathcal{D}}_\alpha (F_\alpha \otimes_\alpha F'_\alpha) \tag{41}$$

such that $Z_\alpha \circ u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}$ coincides with the isomorphism (40). In particular, Z_α provides the desired commutativity up to isomorphism of (36), which concludes the proof. □

Remark 8.7. The reader may wonder why we have introduced the isomorphisms $W^{\mathcal{A},\mathcal{B}}_\alpha$ (resp. $W^{\mathcal{C},\mathcal{D}}_\alpha$) to witness the commutativity of the left (resp. right) square, when one could just have used the isomorphism Γ_α from Remark 3.27 which was already given to us. The reason behind this choice is to guarantee a good behaviour when we raise the cardinality. In particular, this choice will allow us to show in Proposition 8.8 below that the family $(Z_\alpha)_{\alpha \in \mathsf{Reg}_{\geq \kappa}}$ amounts to a modification.

Let $F : \mathcal{A} \to \mathcal{C}$ and $F' : \mathcal{B} \to \mathcal{D}$ be as in Proposition 8.6. We define a k-linear pseudonatural transformation

$$\Phi' : F_{\mathcal{A},\mathcal{B}} \Rightarrow F_{\mathcal{C},\mathcal{D}},$$

where $F_{\mathcal{A},\mathcal{B}}, F_{C,\mathcal{D}}$ are defined as in (30), as follows. For each $\alpha \in \mathsf{Reg}_{\geq\kappa}$, we set

$$\Phi'_\alpha \coloneqq F_\alpha \otimes_\alpha F'_\alpha : \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \to C^\alpha \otimes_\alpha \mathcal{D}^\alpha.$$

For each morphism $\alpha \leq \beta$ in $\mathsf{Reg}_{\geq\kappa}$, we set

$$\Phi'_{\alpha\leq\beta} \coloneqq N_{\alpha,\beta} : \Phi'_\beta F_{\mathcal{A},\mathcal{B}}(\alpha \leq \beta) \Rightarrow F_{C,\mathcal{D}}(\alpha \leq \beta)\Phi'_\alpha$$

where $N_{\alpha,\beta}$ is the isomorphism from (38). We do not provide the details here, but the fact that these data amount to a pseudonatural transformation is a direct consequence of the fact that $N_{\alpha,\beta}$ is defined in terms of left Kan extensions (see the proof of Proposition 8.6).

Let's now denote by

$$\Phi : F_{\mathcal{A},\mathcal{B}} \Rightarrow C \boxtimes_{\mathsf{LP}} \mathcal{D} \tag{42}$$

the k-linear pseudonatural transformation given by postcomposing Φ' with the pseudonatural transformation $\Omega^{C,\mathcal{D}} : F_{C,\mathcal{D}} \to C \boxtimes \mathcal{D}$ as defined in the proof of Theorem 7.3. More concretely,

$$\Phi(\alpha) \coloneqq (H^{C,\mathcal{D}}_\alpha)^{-1} k^{C,\mathcal{D}}_\alpha (F_\alpha \otimes_\alpha F'_\alpha)$$

for every $\alpha \in \mathsf{Reg}_{\geq\kappa}$ and

$$\Phi(\alpha \leq \beta) \coloneqq \left[\Omega^{C,\mathcal{D}}(\alpha \leq \beta) \circ (F_\alpha \otimes_\alpha F'_\alpha)\right] \bullet \left[(H^{C,\mathcal{D}}_\alpha)^{-1} k^{C,\mathcal{D}}_\alpha \circ N_{\alpha,\beta}\right]$$

for every $\alpha \leq \beta$ in $\mathsf{Reg}_{\geq\kappa}$.

This construction provides a way to recover the tensor product $F \boxtimes_{\mathsf{LP}} F'$ up to isomorphism. Indeed, we have the following:

Proposition 8.8. *Let $\mathcal{A}, \mathcal{B}, C$ and $\mathcal{D}$ be locally presentable k-linear categories and $F : \mathcal{A} \to C$ and $F' : \mathcal{B} \to \mathcal{D}$ cocontinuous k-linear functors. Let κ be the smallest regular cardinal such that $\mathcal{A}, \mathcal{B}, C$ and $\mathcal{D}$ are locally κ-presentable and F, F' preserve κ-presentable objects. The functor associated to the pseudonatural transformation $\Phi : F_{\mathcal{A},\mathcal{B}} \Rightarrow C \boxtimes_{\mathsf{LP}} \mathcal{D}$ above via the universal property of $\mathcal{L}(F_{\mathcal{A},\mathcal{B}})$ (23) is isomorphic to $F \boxtimes_{\mathsf{LP}} F'$.*

Proof. By virtue of (22), in order to conclude it is enough to show that the pseudonatural transformation $\Phi : F_{\mathcal{A},\mathcal{B}} \Rightarrow C \boxtimes_{\mathsf{LP}} \mathcal{D}$ is isomorphic to the pseudonatural transformation

$$(F \boxtimes_{\mathsf{LP}} F')\Omega^{\mathcal{A},\mathcal{B}} : F_{\mathcal{A},\mathcal{B}} \xrightarrow{\Omega^{\mathcal{A},\mathcal{B}}} \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \xRightarrow{F \boxtimes_{\mathsf{LP}} F'} C \boxtimes_{\mathsf{LP}} \mathcal{D} \tag{43}$$

where $\Omega^{\mathcal{A},\mathcal{B}}$ is the k-linear pseudonatural transformation introduced in the proof of theorem 7.3 giving rise to the equivalence $\mathcal{L}(F_{\mathcal{A},\mathcal{B}}) \simeq \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$ from Theorem 7.3 and where we are considering the categories $\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$ and $C \boxtimes_{\mathsf{LP}} \mathcal{D}$ as constant 2-functors from $\mathsf{Reg}_{\geq\kappa}$ to $\mathsf{CAT}(k)$ and $F \boxtimes_{\mathsf{LP}} F'$ as the corresponding constant k-linear pseudonatural transformation. We proceed to build a modification $Z : (F \boxtimes_{\mathsf{LP}} F')\Omega^{\mathcal{A},\mathcal{B}} \Rrightarrow \Phi$ as follows. For $\alpha \in \mathsf{Reg}_{\geq\kappa}$, we set

$$Z_\alpha : (F \boxtimes_{\mathsf{LP}} F')\Omega^{\mathcal{A},\mathcal{B}} \Rightarrow \Phi(\alpha)$$

to be given precisely by the natural isomorphism

$$\begin{tikzcd}
\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[r, "F_\alpha \otimes_\alpha F'_\alpha"] \arrow[d, "k^{\mathcal{A},\mathcal{B}}_\alpha"'] & C^\alpha \otimes_\alpha \mathcal{D}^\alpha \arrow[d, "k^{C,\mathcal{D}}_\alpha"] \\
\mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha) \arrow[r, phantom, "Z_\alpha \Nearrow"] \arrow[d, "(H^{\mathcal{A},\mathcal{B}}_\alpha)^{-1}"'] & \mathsf{Lex}_{\alpha,\alpha}(C^\alpha, \mathcal{D}^\alpha) \arrow[d, "(H^{C,\mathcal{D}}_\alpha)^{-1}"] \\
\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \arrow[r, "F \boxtimes_{\mathsf{LP}} F'"] & C \boxtimes_{\mathsf{LP}} \mathcal{D}
\end{tikzcd}$$

from Proposition 8.6 above. To show that Z is a modification, we need to prove that for every $\alpha \le \beta$ in $\mathsf{Reg}_{\geq\kappa}$ we have that

$$Z_\alpha \bullet \left[(F \boxtimes_{\mathsf{LP}} F') \circ \Omega^{\mathcal{A},\mathcal{B}}_{\alpha\le\beta} \right] = \Phi_{\alpha\le\beta} \bullet \left[Z_\beta \circ F_{\mathcal{A},\mathcal{B}}(\alpha \le \beta) \right],$$

that is, we need to show the following equality of pasting diagrams:

$$\begin{tikzcd}
\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[r, "F_\alpha \otimes_\alpha F'_\alpha"] \arrow[d, "f^{\mathcal{A},\mathcal{B}}_{\alpha,\beta}"'] & \mathcal{C}^\alpha \otimes_\alpha \mathcal{D}^\alpha \arrow[d, "f^{\mathcal{C},\mathcal{D}}_{\alpha,\beta}"] \arrow[ddr, bend left, "k^{\mathcal{C},\mathcal{D}}_\alpha"] & \\
\mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta \arrow[r, "F_\beta \otimes_\beta F'_\beta"'] \arrow[d, "k^{\mathcal{A},\mathcal{B}}_\beta"'] \arrow[ur, phantom, "N_{\alpha,\beta}\Nearrow"] & \mathcal{C}^\beta \otimes_\beta \mathcal{D}^\beta \arrow[d, "k^{\mathcal{C},\mathcal{D}}_\beta"] \arrow[r, phantom, "\Omega^{\mathcal{C},\mathcal{D}}_{\alpha,\beta}\Nearrow"] & {} \\
\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta) \arrow[d, "(H^{\mathcal{A},\mathcal{B}}_\beta)^{-1}"'] \arrow[r, phantom, "Z_\beta\Nearrow"] & \mathsf{Lex}_{\beta,\beta}(\mathcal{C}^\beta, \mathcal{D}^\beta) \arrow[d, "(H^{\mathcal{C},\mathcal{D}}_\beta)^{-1}"] & \mathsf{Lex}_{\alpha,\alpha}(\mathcal{C}^\alpha, \mathcal{D}^\alpha) \arrow[dl, bend left, "(H^{\mathcal{C},\mathcal{D}}_\alpha)^{-1}"] \\
\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \arrow[r, "F \boxtimes_{\mathsf{LP}} F'"] & \mathcal{C} \boxtimes_{\mathsf{LP}} \mathcal{D} &
\end{tikzcd} =$$

$$\begin{tikzcd}
\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \arrow[rr, "F_\alpha \otimes_\alpha F'_\alpha"] \arrow[d, "f^{\mathcal{A},\mathcal{B}}_{\alpha,\beta}"'] \arrow[ddr, "k^{\mathcal{A},\mathcal{B}}_\alpha"] & & \mathcal{C}^\alpha \otimes_\alpha \mathcal{D}^\alpha \arrow[dd, "k^{\mathcal{C},\mathcal{D}}_\alpha"] \\
\mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta \arrow[d, "k^{\mathcal{A},\mathcal{B}}_\beta"'] \arrow[r, phantom, "\Omega^{\mathcal{A},\mathcal{B}}_{\alpha,\beta}\Nearrow"] & {} \arrow[r, phantom, "Z_\alpha\Nearrow"] & {} \\
\mathsf{Lex}_{\beta,\beta}(\mathcal{A}^\beta, \mathcal{B}^\beta) \arrow[d, "(H^{\mathcal{A},\mathcal{B}}_\beta)^{-1}"'] & \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha) \arrow[dl, "(H^{\mathcal{A},\mathcal{B}}_\alpha)^{-1}"] & \mathsf{Lex}_{\alpha,\alpha}(\mathcal{C}^\alpha, \mathcal{D}^\alpha) \arrow[d, "(H^{\mathcal{C},\mathcal{D}}_\alpha)^{-1}"] \\
\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \arrow[rr, "F \boxtimes_{\mathsf{LP}} F'"] & & \mathcal{C} \boxtimes_{\mathsf{LP}} \mathcal{D}.
\end{tikzcd}$$

We do not reproduce the proof of this equality in full detail here, but we sketch the argument. First, one observes that proving this equality is equivalent to proving the equality of the pasting diagrams when precomposed with

$$u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha} \iota^{\otimes}_{\kappa,\alpha} : \mathcal{A}^\kappa \otimes \mathcal{B}^\kappa \to \mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha$$

because all the morphisms involved are α-cocontinuous, and $\alpha \geq \kappa$. Then, using the explicit definitions of Z_α and $N_{\alpha,\beta}$ from Proposition 8.6 above, together with the fact that $\Omega^{\mathcal{A},\mathcal{B}}$ and $\Omega^{\mathcal{C},\mathcal{D}}$ are pseudonatural transformations, allows us to conclude.

Therefore, we have find a modification $Z : (F \boxtimes_{\mathsf{LP}} F')\Omega^{\mathcal{A},\mathcal{B}} \Rrightarrow \Phi$ such that Z_α is invertible for every $\alpha \in \mathsf{Reg}_{\geq\kappa}$. Therefore, Z is an invertible modification and thus defines an isomorphism between $(F \boxtimes_{\mathsf{LP}} F')\Omega^{\mathcal{A},\mathcal{B}}$ and Φ (see [19, Lem 4.4.9]), which concludes the proof. ◻

The description of the tensor product of cocontinuous k-linear functors in terms of the Kelly tensor product of presentable objects provides a useful tool in order to perform computations when one has control of the presentable objects. Let us illustrate this with the following geometrical example.

Example 8.9. Let $f_1 : (X_1, \mathcal{O}_{X_1}) \to (Y_1, \mathcal{O}_{Y_1})$ be a morphism of k-schemes where X_1 and Y_1 are quasi-compact and quasi-separated. In particular, we know that:

- $\mathsf{Qcoh}(X_1)$ and $\mathsf{Qcoh}(Y_1)$ are Grothendieck k-linear categories [34, Tag 077P];
- $\mathsf{Qcoh}(X_1)$ and $\mathsf{Qcoh}(Y_1)$ are locally finitely presentable k-linear categories [17, Chap I, Cor 6.9.12];
- the pullback functor $f_1^* : \mathsf{Qcoh}(Y_1) \to \mathsf{Qcoh}(X_1)$, which is cocontinuous, preserves finitely presentable objects [17, Chap 0, §5.2.5].

Let $f_2 : (X_2, \mathcal{O}_{X_2}) \to (Y_2, \mathcal{O}_{Y_2})$ be another morphism between quasi-compact quasi-separated k-schemes and consider the *external tensor product* $\boxtimes$ of quasi-coherent sheaves [17, Chap I, Def 3.3.1], which is given by

$$\begin{aligned} \boxtimes : \mathsf{Qcoh}(X_1) \otimes \mathsf{Qcoh}(X_2) &\to \mathsf{Qcoh}(X_1 \times_k X_2) \\ (F, G) &\mapsto F \boxtimes G \coloneqq p_{X_1}^*(F) \otimes_{\mathcal{O}_{X_1 \times_k X_2}} p_{X_2}^*(G), \end{aligned}$$

where $p_{X_i} : X_1 \times_k X_2 \to X_i$ denote the corresponding projections for $i = 1, 2$. From [10, Thm 6.1] we have that the external tensor product $\boxtimes$ exhibits the Grothendieck category $\mathsf{Qcoh}(X_1 \times_k X_2)$ as the tensor product of Grothendieck k-linear categories $\mathsf{Qcoh}(X_1) \boxtimes_{\mathsf{G}} \mathsf{Qcoh}(X_2)$. In particular, by virtue of Remark 3.27, we have that the restriction-corestriction of the exterior tensor product

$$\boxtimes : \mathsf{Qcoh}(X_1)^{\aleph_0} \otimes \mathsf{Qcoh}(X_2)^{\aleph_0} \to \mathsf{Qcoh}(X_1 \times_k X_2)^{\aleph_0}$$

also exhibits the finitely cocomplete linear category $\mathsf{Qcoh}(X_1 \times_k X_2)^{\aleph_0}$ as the $\aleph_0$-Kelly tensor product $\mathsf{Qcoh}(X_1)^{\aleph_0} \otimes_{\aleph_0} \mathsf{Qcoh}(X_2)^{\aleph_0}$.

As a consequence of Proposition 8.6, we have the following diagram that commutes up to isomorphism:

$$\begin{tikzcd}
\mathsf{Qcoh}(Y_1)^{\aleph_0} \otimes \mathsf{Qcoh}(Y_2)^{\aleph_0} \arrow[rr, "(f_1^*)_{\aleph_0} \otimes (f_2^*)_{\aleph_0}"] \arrow[d, "\boxtimes"] \arrow[dd, bend right=60, "\boxtimes"'] & & \mathsf{Qcoh}(X_1)^{\aleph_0} \otimes \mathsf{Qcoh}(X_2)^{\aleph_0} \arrow[d, "\boxtimes"'] \arrow[dd, bend left=60, "\boxtimes"] \\
\mathsf{Qcoh}(Y_1 \times_k Y_2)^{\aleph_0} \arrow[rr, "(f_1^*)_{\aleph_0} \otimes_{\aleph_0} (f_2^*)_{\aleph_0}"] \arrow[d, hook] & & \mathsf{Qcoh}(X_1 \times_k X_2)^{\aleph_0} \arrow[d, hook] \\
\mathsf{Qcoh}(Y_1 \times_k Y_2) \arrow[rr, "f_1^* \boxtimes_{\mathsf{G}} f_2^*"] & & \mathsf{Qcoh}(X_1 \times_k X_2)
\end{tikzcd}$$

and thus, for any $F \in \mathsf{Qcoh}(Y_1)^{\aleph_0}$ and $G \in \mathsf{Qcoh}(Y_2)^{\aleph_0}$, we have that

$$(f_1^* \boxtimes_{\mathsf{G}} f_2^*)(F \boxtimes G) \cong f_1^*(F) \boxtimes f_2^*(G).$$

Moreover, notice that this fully determines the cocontinuous functor $f_1^* \boxtimes_{\mathsf{G}} f_2^*$ up to isomorphism. Indeed, every sheaf in $\mathsf{Qcoh}(Y_1 \times_k Y_2)$ is written as a filtered colimit of sheaves in $\mathsf{Qcoh}(Y_1 \times_k Y_2)^{\aleph_0}$ and furthermore, every sheaf in $\mathsf{Qcoh}(Y_1 \times_k Y_2)^{\aleph_0}$ is a finite colimit of sheaves of the form $F \boxtimes G$ where $F \in \mathsf{Qcoh}(Y_1)^{\aleph_0}$ and $G \in \mathsf{Qcoh}(Y_2)^{\aleph_0}$.

This example is particularly interesting in the case in which Y_1 and Y_2 are locally noetherian, as in that case, for $i = 1, 2$ we have that $\mathsf{Qcoh}(Y_i)^{\aleph_0}$ is precisely $\mathsf{Coh}(Y_i)$, the full subcategory of coherent sheaves [34, Tag 01XZ].

Now, we show how one can recover the associativity and the symmetry of the tensor product of locally presentable linear categories from the same properties of the α-Kelly tensor products.

Consider locally presentable k-linear categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. Define

$$F_{(\mathcal{A},\mathcal{B}),\mathcal{C}} : \mathsf{Reg} \to \mathsf{Cat}$$

by $F_{(\mathcal{A},\mathcal{B}),\mathcal{C}}(\alpha) = (\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha) \otimes_\alpha \mathcal{C}^\alpha$ with the natural transition functors

$$F_{(\mathcal{A},\mathcal{B}),\mathcal{C}}(\alpha \leq \beta) : (\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha) \otimes_\alpha \mathcal{C}^\alpha \to (\mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta) \otimes_\beta \mathcal{C}^\beta$$

induced by the universal property of $\otimes_\alpha$. Analogously, we put

$$F_{\mathcal{A},(\mathcal{B},\mathcal{C})} : \mathsf{Reg} \to \mathsf{Cat}$$

with $F_{\mathcal{A},(\mathcal{B},\mathcal{C})}(\alpha) = \mathcal{A}^\alpha \otimes_\alpha (\mathcal{B}^\alpha \otimes_\alpha \mathcal{C}^\alpha)$ with the natural transition functors

$$F_{\mathcal{A},(\mathcal{B},\mathcal{C})}(\alpha \leq \beta) : \mathcal{A}^\alpha \otimes_\alpha (\mathcal{B}^\alpha \otimes_\alpha \mathcal{C}^\alpha) \to \mathcal{A}^\beta \otimes_\beta (\mathcal{B}^\beta \otimes_\beta \mathcal{C}^\beta).$$

In a similar fashion to Theorem 7.3, one can show that

$$\mathcal{L}(F_{(\mathcal{A},\mathcal{B}),\mathcal{C}}) \simeq (\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}) \boxtimes_{\mathsf{LP}} \mathcal{C},$$

and analogously

$$\mathcal{L}(F_{\mathcal{A},(\mathcal{B},C)}) \simeq \mathcal{A} \boxtimes_{\mathsf{LP}} (\mathcal{B} \boxtimes_{\mathsf{LP}} C).$$

We know that, for any regular cardinal α, the category $\mathsf{Cat}_\alpha(k)$ of α-cocomplete small categories endowed with $\otimes_\alpha$ is a closed monoidal symmetric bicategory. In particular, for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathsf{Cat}_\alpha(k)$, we have an equivalence

$$(\mathfrak{a} \otimes_\alpha \mathfrak{b}) \otimes_\alpha \mathfrak{c} \simeq \mathfrak{a} \otimes_\alpha (\mathfrak{b} \otimes_\alpha \mathfrak{c})$$

natural in all three components. Moreover, these equivalences are inherited from the associativity of the monoidal structure $\otimes$ in $\mathsf{Cat}(k)$ [20, Eq (6.25)], namely, they make the diagram

$$\begin{tikzcd}
(\mathfrak{a} \otimes \mathfrak{b}) \otimes \mathfrak{c} \arrow[r, "\cong"] \arrow[d, "(u_{\mathfrak{a},\mathfrak{b}} \otimes_\alpha 1_{\mathfrak{c}})(u_{\mathfrak{a}\otimes\mathfrak{b},\mathfrak{c}})"'] & \mathfrak{a} \otimes (\mathfrak{b} \otimes \mathfrak{c}) \arrow[d, "(1_{\mathfrak{a}} \otimes_\alpha u_{\mathfrak{b},\mathfrak{c}})(u_{\mathfrak{a},\mathfrak{b}\otimes\mathfrak{c}})"] \\
(\mathfrak{a} \otimes_\alpha \mathfrak{b}) \otimes_\alpha \mathfrak{c} \arrow[r, "\simeq"] & \mathfrak{a} \otimes_\alpha (\mathfrak{b} \otimes_\alpha \mathfrak{c})
\end{tikzcd} \tag{44}$$

commutative for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathsf{Cat}_\alpha(k)$.

Consequently, there is a canonical equivalence $F_{(\mathcal{A},\mathcal{B}),C}(\alpha) \simeq F_{\mathcal{A},(\mathcal{B},C)}(\alpha)$ for each α, and (44) guarantees that it behaves naturally on $\alpha \in \mathsf{Reg}$. In other words, we have built a pseudonatural equivalence

$$F_{(\mathcal{A},\mathcal{B}),C} \simeq F_{\mathcal{A},(\mathcal{B},C)}, \tag{45}$$

i.e. a pseudonatural transformation that is point-wise an equivalence.

Proposition 8.10. *Let $\mathcal{A}, \mathcal{B}$ and C be locally presentable k-linear categories, then, there exists an equivalence*

$$(\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}) \boxtimes_{\mathsf{LP}} C \simeq \mathcal{A} \boxtimes_{\mathsf{LP}} (\mathcal{B} \boxtimes_{\mathsf{LP}} C).$$

Proof. Applying [19, Prop 6.2.3 + Prop 6.2.4] to (45), we obtain a pseudonatural equivalence

$$\mathsf{Psnat}(F_{(\mathcal{A},\mathcal{B}),C}, -) \simeq \mathsf{Psnat}(F_{\mathcal{A},(\mathcal{B},C)}, -)$$

and thus, by the universal property of the filtered bicolimit, a pseudonatural equivalence

$$\mathsf{Cat}((\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}) \boxtimes_{\mathsf{LP}} C, -) \simeq \mathsf{Cat}(\mathcal{A} \boxtimes_{\mathsf{LP}} (\mathcal{B} \boxtimes_{\mathsf{LP}} C), -).$$

We conclude by applying the bicategorical Yoneda lemma [19, Lem 8.3.12]. □

The argument to deduce the symmetry of the tensor product of locally presentable linear categories from the Kelly tensor products is analogous. Consider two locally presentable k-linear categories $\mathcal{A}$ and $\mathcal{B}$. As the monoidal bicategory $(\mathsf{Cat}_\alpha(k), \otimes_\alpha)$ is symmetric, we have

$$\mathfrak{a} \otimes_\alpha \mathfrak{b} \simeq \mathfrak{b} \otimes_\alpha \mathfrak{a}$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathsf{Cat}_\alpha(k)$, naturally in both components. Moreover, these equivalences are inherited from the symmetry of the monoidal structure $\otimes$ in $\mathsf{Cat}(k)$ [20, Eq (6.26)], namely, they make the diagram

$$\begin{tikzcd}
\mathfrak{a} \otimes \mathfrak{b} \arrow[r, "\simeq"] \arrow[d, "u_{\mathfrak{a},\mathfrak{b}}"'] & \mathfrak{b} \otimes \mathfrak{a} \arrow[d, "u_{\mathfrak{b},\mathfrak{a}}"] \\
\mathfrak{a} \otimes_\alpha \mathfrak{b} \arrow[r, "\simeq"] & \mathfrak{b} \otimes_\alpha \mathfrak{a}
\end{tikzcd}$$

commutative for all $\mathfrak{a}, \mathfrak{b} \in \mathsf{Cat}_\alpha(k)$.

Thus, reasoning as above, we have a canonical pseudonatural equivalence

$$F_{\mathcal{A},\mathcal{B}} \simeq F_{\mathcal{B},\mathcal{A}}, \tag{46}$$

where $F_{\mathcal{A},\mathcal{B}}$ and $F_{\mathcal{B},\mathcal{A}}$ are defined as in (30).

Proposition 8.11. *Let $\mathcal{A}, \mathcal{B}$ be locally presentable k-linear categories. Then, there exists an equivalence*

$$\mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B} \simeq \mathcal{B} \boxtimes_{\mathsf{LP}} \mathcal{A}.$$

Proof. It follows by applying the same argument of the proof of Proposition 8.10 to the pseudonatural equivalence (46). □

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Notation index

Symbol	Description	Page
R_α	k-linear right adjoint of j_α	9
$(\iota_{\alpha,\beta})_s, (\iota^C_{\alpha,\beta})_s$	restriction-corestriction $(\iota^C_{\alpha,\beta})_s : \mathsf{Lex}_\beta(C^\beta) \to \mathsf{Lex}_\alpha(C^\alpha)$ of the k-linear functor $(\iota^C_{\alpha,\beta})^* : \mathsf{Mod}(C^\beta) \to \mathsf{Mod}(C^\alpha)$ for $\alpha \le \beta$	10
$(\iota_{\alpha,\beta})^s, (\iota^C_{\alpha,\beta})^s$	k-linear left adjoint of $(\iota^C_{\alpha,\beta})_s$	10
$u^\alpha_{\mathfrak{a},\mathfrak{b}}$	universal k-linear functor $\mathfrak{a} \otimes \mathfrak{b} \to \mathfrak{a} \otimes_\alpha \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b}$ α-cocontinuous small k-linear categories	12
$\mathsf{Rex}_{\alpha,\beta}(- \otimes -, -)$	k-linear category of k-linear functors that are α-cocontinuous in the first variable and β-cocontinuous in the second variable	12
$\mathsf{Lex}_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b})$	k-linear subcategory of $\mathsf{Mod}(\mathfrak{a} \otimes \mathfrak{b})$ consisting of the right $\mathfrak{a} \otimes \mathfrak{b}$-modules that are α-continuous in the first variable and β-continuous on the second variable	12
$R_{\alpha,\alpha}$	k-linear right adjont of $j_{\alpha,\alpha}$	12
$j_{\alpha,\alpha}$	natural embedding $j_{\alpha,\alpha} : \mathsf{Lex}_{\alpha,\alpha}(\mathfrak{a}, \mathfrak{b}) \hookrightarrow \mathsf{Mod}(\mathfrak{a} \otimes \mathfrak{b})$	12
$k_\alpha, k^{\mathcal{A},\mathcal{B}}_\alpha$	the natural embedding $\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \hookrightarrow \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha)$	12
$E_{\alpha,\alpha}$	equivalence $E_{\alpha,\alpha} : \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha) \to \mathsf{Lex}_\alpha(\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha)$ for $\mathcal{A}, \mathcal{B}$ locally α-presentable k-linear categories	13
$\boxtimes_{\mathsf{LP}_\alpha}$	tensor product of locally α-presentable linear categories	13
$\Phi_{\alpha,\beta}$	k-linear functor $\mathsf{Lex}_{\alpha,\beta}(\mathcal{A}^\alpha, \mathcal{B}^\beta) \to \mathsf{Lex}_\alpha((\mathcal{A}^\alpha)^{\mathsf{op}}, \mathsf{Lex}_\beta(\mathcal{B}^\beta))$ obtained as restriction-corestriction of the tensor-hom adjunction	13
$\Psi_{\alpha,\beta}$	k-linear inverse functor of $\Phi_{\alpha,\beta}$	13
$H_\alpha, H^{\mathcal{A},\mathcal{B}}_\alpha$	the equivalence $H^{\mathcal{A},\mathcal{B}}_\alpha : \mathsf{Cont}_k(\mathcal{A}^{\mathsf{op}}, \mathcal{B}) \to \mathsf{Lex}_{\alpha,\alpha}(\mathcal{A}^\alpha, \mathcal{B}^\alpha)$	14
$\mathsf{Cont}_k(-,-)$	k-linear category of continuous k-linear functors between complete k-linear categories	14
$s_{\mathcal{A},\mathcal{B}}$	universal k-linear functor $\mathcal{A} \otimes \mathcal{B} \to \mathcal{A} \boxtimes_{\mathsf{LP}} \mathcal{B}$ for $\mathcal{A}, \mathcal{B}$ locally presentable k-linear categories	14
Γ_α	the isomorphism $\Gamma_\alpha : s_{\mathcal{A},\mathcal{B}}(\iota^{\mathcal{A}}_\alpha \otimes \iota^{\mathcal{B}}_\alpha) \Rightarrow H^{-1}_\alpha k_\alpha u_{\mathcal{A}^\alpha,\mathcal{B}^\alpha}$	14
$f_{\alpha,\beta}, f^{\mathcal{A},\mathcal{B}}_{\alpha,\beta}$	the α-cocontinuous functor $\mathcal{A}^\alpha \otimes_\alpha \mathcal{B}^\alpha \hookrightarrow \mathcal{A}^\beta \otimes_\beta \mathcal{B}^\beta$ induced via the universal property of $\otimes_\alpha$ by $u_{\mathcal{A}^\beta,\mathcal{B}^\beta}(\iota^C_{\alpha,\beta} \otimes \iota^{\mathcal{B}}_{\alpha,\beta})$, for $\mathcal{A}, \mathcal{B}$ locally presentable k-linear categories and $\alpha \le \beta$	15
$g^{-1}R$	pullback of a sieve R along a morphism g	18
$\mathcal{T}_{C,\mathsf{can}}$	canonical topology on a k-linear Grothendieck category C	18
$f^{-1}\mathcal{T}_{\mathfrak{b}}$	the cover system induced on a small category $\mathfrak{a}$ through a functor $f : \mathfrak{a} \to \mathfrak{b}$ by a cover system $\mathcal{T}_{\mathfrak{b}}$ on a small category $\mathfrak{b}$	19
LC	the class of LC morphisms between k-linear sites	19
F^0_A	the lax unity constraint $F^0_A : 1_{F(A)} \Rightarrow F(1_A)$ of a pseudofunctor $F : \mathcal{A} \to \mathcal{B}$ between bicategories on the object $A \in \mathcal{A}$	21
$F^2_{f,g}$	the lax functoriality constraint $F^2_{f,g} : F(g)F(f) \Rightarrow F(gf)$ of a pseudofunctor $F : \mathcal{A} \to \mathcal{B}$ between bicategories on the pair of composable arrows f, g in $\mathcal{A}$	21
$\mathsf{Psnat}(-,-)$	category of pseudonatural transformations between pseudofunctors	21
$\mathcal{L}(F)$	2-filtered bicolimit of the pseudofunctor F	21
$\Lambda : F \Rightarrow \mathcal{L}(F)$	universal pseudonatural transformation of the 2-filtered bicolimit of a pseudofunctor F	23
$\mathcal{L}_k(G)$	k-linear filtered bicolimit of a pseudofunctor G	24
$\mathcal{J}_C$	category of LC morphisms into the Grothendieck k-linear category C	26
G_C	functor $G_C : \mathcal{J}_C \to \mathsf{Cat}(k)$ forgetting the "slice structure"	27

(Julia Ramos González) Université Catholique de Louvain, Institut de Recherche en Mathématique et Physique, Chemin du Cyclotron 2/L7.01.02, 1348 Louvain-la-Neuve, Belgium.

Email address: julia.ramos@uclouvain.be