

CLIFFORD DIVISION ALGEBRAS AND ANISOTROPIC QUADRATIC FORMS: TWO COUNTEREXAMPLES

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In a recent paper [3], D. W. Lewis proposed the following conjecture. (The notation is the same as that in [2] and [3].)

CONJECTURE. *Let F be a field of characteristic not 2 and let $a_1, b_1, \dots, a_n, b_n \in F^\times$. The tensor product of quaternion algebras*

$$\left(\begin{smallmatrix} a_1 & b_1 \\ & F \end{smallmatrix} \right) \otimes_F \dots \otimes_F \left(\begin{smallmatrix} a_n & b_n \\ & F \end{smallmatrix} \right)$$

is a division algebra if and only if the quadratic form over F

$$\perp_{i=1}^n (-1)^{i+1} \langle a_i, b_i, -a_i b_i \rangle$$

is anisotropic.

This equivalence indeed holds for $n=1$ as is well known [2, Theorem 2.7], and Albert [1] (see also [4, §15.7]) has shown that it also holds for $n=2$. The aim of this note is to provide counterexamples to both of the implications for $n \geq 3$.

Let k be a field of characteristic different from 2, and let $x_1, \dots, x_{n-1}, y_1, \dots, y_n$ be independent indeterminates over k (with $n \geq 3$). Let also $f(x_1, x_2) \in k(x_1, x_2)$ and

$$F = k(x_1, \dots, x_{n-1}, y_1, \dots, y_n).$$

THEOREM. (1) *The tensor product of quaternion algebras*

$$T = \left(\begin{smallmatrix} x_1 & y_1 \\ & F \end{smallmatrix} \right) \otimes \dots \otimes \left(\begin{smallmatrix} x_{n-1} & y_{n-1} \\ & F \end{smallmatrix} \right) \otimes \left(\begin{smallmatrix} f(x_1, x_2) & y_n \\ & F \end{smallmatrix} \right)$$

is a division algebra if and only if $f(x_1, x_2)$ is not a square in $k(\sqrt{x_1}, \sqrt{x_2})$.

(2) *The quadratic form over F*

$$Q = \langle x_1, y_1, -x_1 y_1 \rangle \perp \dots \perp (-1)^n \times \langle x_{n-1}, y_{n-1}, -x_{n-1} y_{n-1} \rangle \perp (-1)^{n+1} \langle f(x_1, x_2), y_n, -f(x_1, x_2) y_n \rangle$$

is anisotropic if and only if $(-1)^n f(x_1, x_2)$ is not represented by the quadratic form $\langle x_1, -x_2 \rangle$ over $k(x_1, x_2)$ and $f(x_1, x_2)$ is not a square in $k(x_1, x_2)$.

The proof will follow by repeated use of the following results.

LEMMA. *Let K be a field of characteristic different from 2 and let t be an indeterminate over K .*

(1) *If A is a central simple algebra over K and $c \in K^\times$, then $A \otimes_K ({}^c_{K(t)'})$ is a division algebra if and only if $A \otimes_K K(\sqrt{c})$ is a division algebra.*

(2) If q_1 and q_2 are quadratic forms over K , then $q_1 \perp \langle t \rangle q_2$ is anisotropic over $K(t)$ if and only if q_1 and q_2 are anisotropic over K .

Proof. (1) See [5, Proposition 2.4]; (2) see [2, p. 273].

Proof of the theorem. (1) We apply part (1) of the lemma $(n-1)$ times, taking successively $t = y_1, t = y_2, \dots, t = y_{n-1}$. It follows that T is a division algebra if and only if

$$(k(x_1, x_2, \dots, x_{n-1}, y_n)^{f(x_1, x_2)}) \otimes k(\sqrt{x_1}, \sqrt{x_2}, \dots, \sqrt{x_{n-1}}, y_n)$$

is a division algebra. This last condition is equivalent to the following: the quadratic form

$$\langle 1, -f(x_1, x_2) \rangle \perp -\langle y_n \rangle \langle 1, -f(x_1, x_2) \rangle$$

is anisotropic over $k(\sqrt{x_1}, \sqrt{x_2}, \dots, \sqrt{x_{n-1}}, y_n)$. Applying then the second part of the lemma with successively $t = y_n, \sqrt{x_{n-1}}, \dots, \sqrt{x_3}$, we see that this condition holds if and only if $\langle 1, -f(x_1, x_2) \rangle$ is anisotropic over $k(\sqrt{x_1}, \sqrt{x_2})$, i.e. $f(x_1, x_2)$ is not a square in $k(\sqrt{x_1}, \sqrt{x_2})$.

(2) readily follows from the second part of the lemma, applied successively with $t = y_1, y_2, \dots, y_n, x_3, x_4, \dots, x_{n-1}$.

Now, for $f(x_1, x_2) = (-1)^n(x_1 - x_2)$, the theorem shows that the tensor product T is a division algebra, while the corresponding quadratic form Q is isotropic.

Conversely, for $f(x_1, x_2) = x_1 x_2$, the tensor product T is not a division algebra, but the corresponding quadratic form Q is anisotropic, since part (2) of the lemma, with $t = x_2$, shows that $\langle x_1, -x_2, (-1)^n x_1 x_2 \rangle$ is anisotropic over $k(x_1, x_2)$.

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