

State estimation within a guaranteed interval

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Abstract—This paper presents a state estimator for systems linear in an unknown parameter. The estimate is given via a weighted mean of both an under- and an over-estimate provided by an interval observer. The weighting factor is computed in real-time from the difference between the output measurements and the interval bounds. The convergence of the estimate is first shown for a class of LTI systems. The generalization to a class of nonlinear systems is then presented. Both cases are illustrated with numerical simulations.

Index Terms—State Estimation, Observers for linear systems, Uncertain systems

I. INTRODUCTION

Implementing an efficient control strategy often requires to know the state variables of a dynamical system. However in most applications the whole state of the system cannot be measured and it has to be estimated from its output via a state observer.

The first state observers that have been developed several decades ago by Luenberger and Kalman are able to reconstruct the state of a dynamical system provided the model parameters are perfectly known. However in practice the model parameters can be uncertain or even unknown. These inaccuracies can become a serious limitation for the application of the Kalman or Luenberger observers since they converge toward a biased state estimate.

In most situations the parameter values are badly known but not totally unknown, and it would be interesting to take advantage of this partial knowledge to perform a reliable estimate. Two different lines of reasoning can be adopted in particular to deal with the knowledge of the parameter values: the multiple-model estimation and the interval observer.

The multiple-model estimation provides an estimate which is a combination of the estimates obtained for several systems with different parameter values [1]. This approach is particularly well suited for processes subject to multiple operating points [2] [3]. However the determination of the number of models to be considered is a difficult issue. On the one hand, with a limited number of models, the provided information about the process dynamics might reveal to be insufficient to obtain reliable estimates. On the other hand the estimate performance might be degrading with a too high number of

models due to the competition of "unnecessary" models as explained in [4].

The interval observer [5] [6] [7] does not consider several systems with different parameters value but it only provides upper and lower bounds for the actual state. Both estimates are computed from the parameters bounds to guarantee that they remain lower and greater than the state, respectively. This technique might be easy to implement but it has the disadvantage that it provides an interval and not a precise estimate value.

In this paper we present a way to estimate the state variables of a class of dynamical systems from the bounds provided by an interval observer. The estimate is computed as a weighted mean of the upper- and the lower-estimate for which the weighting factor is computed from the output measurements and the interval bounds. The underlying idea is based on the link between the position of the unmeasured variables within the interval and the position of the measured output within the interval.

The paper is organized as follows. The first section presents the weighted mean estimate for a class of linear systems. The estimation performances are illustrated with numerical simulations of a chemical reactor model. The second section presents the generalization to a class of nonlinear systems and the performances are illustrated with numerical simulations on an exothermic chemical reactor model.

II. ESTIMATE FOR A CLASS OF LTI SYSTEMS

Let us consider a stable linear time invariant dynamical system with the following structure:

$$\dot{x}(t) = Ax(t) + \theta Bu(t) + Dv(t) \quad (1)$$

$$y = Cx \quad (2)$$

with

$$x(t_0) = x_0 \quad (3)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{m_1}$, $v \in \mathbb{R}^{m_2}$, $y \in \mathbb{R}$ and where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m_1}$, $D \in \mathbb{R}^{n \times m_2}$ and $C \in \mathbb{R}^{n \times 1}$ are constant matrices. Let us also consider that the matrix A is cooperative i.e. its non-diagonal entries are non-negative and that the input u remains positive:

$$\forall t > t_0 : 0 < u_{\min} \leq u(t) \quad (4)$$

Let us assume that θ is an unknown constant parameter satisfying the following conditions:

$$\theta_w Bu < \theta Bu < \theta_z Bu \quad (5)$$

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Furthermore let us assume that there exists a gain matrix K such that the following couple of systems is an interval observer for the system (1):

$$\dot{w} = Aw + \theta_w Bu + Dv + K(y - Cw) \quad (6)$$

$$\dot{z} = Az + \theta_z Bu + Dv + K(y - Cz) \quad (7)$$

with for any time $t \geq t_0$:

$$w(t) < x(t) < z(t) \quad (8)$$

and in particular

$$w(t_0) < x(t_0) < z(t_0) \quad (9)$$

Then we propose to compute an estimate of the state x as a weighted mean of both under- and over-estimates as follows:

$$\hat{x} = \alpha w + (1 - \alpha) z \quad (10)$$

where the weighting factor α is computed in real time from the system output y and the under- and over-estimates w and z as follows :

$$\alpha = \frac{y - Cz}{C(w - z)} \quad (11)$$

It is worth noting that the above definition only holds for single output systems.

Theorem 2.1: Assuming K is chosen so that the matrix $F = A - KC$ is Hurwitz and cooperative, then $\hat{x}$ defined by (10) (11) converges asymptotically to the state of the dynamical system (1).

Proof

The proof proceeds in three steps. The first step shows that the weighting factor α is bounded so that the estimate $\hat{x}$ can be defined. The second step shows that there exists a linear combination of both systems (6) and (7) that converges toward x . Finally, the convergence of the estimate $\hat{x}$ to the state x is guaranteed by showing that the estimate converges to the state observer designed in the second step of the proof.

The existence of the weighting factor α is guaranteed by showing that the state variables w and z do not converge to each other. Let us start showing that both systems (6) and (7) define an interval within which the state is guaranteed to lie. In the following we only focus on the under-estimate w since the proof for the over-estimate z can be carried out in a similar way.

Let us define ε_{xw} as the reconstruction error related to (6)

$$\varepsilon_{xw}(t) = x(t) - w(t) \quad (12)$$

whose dynamics can be computed from (1) and (6):

$$\dot{\varepsilon}_{xw} = F\varepsilon_{xw} + (\theta - \theta_w)Bu \quad (13)$$

The above equation shows that the reconstruction error is stable since matrix F is Hurwitz. Now we shall show that the reconstruction error remains positive.

Let us note that by assumption, the initial error is positive

$$\varepsilon_{xw}(t_0) > 0 \quad (14)$$

Let us assume that the error can become negative and let $\varepsilon_{xw,i}$ be the first coordinates that becomes equals to zero. Then there exists a time instant τ such that

$$\varepsilon_{xw,i}(\tau) = 0 \quad (15)$$

The dynamics of $\varepsilon_{xw,i}$ is then given by

$$\dot{\varepsilon}_{xw,i}(\tau) = \sum_{j=1}^n F(i, j)\varepsilon_{xw,j}(\tau) + (\theta - \theta_w)Bu \quad (16)$$

where $F(i, j)$ denotes the entry of row i and column j of the matrix F . As the matrix F is cooperative by assumption, we have

$$F(i, j) \geq 0 \quad \forall i \neq j \quad (17)$$

and from (4) and (5), it follows that

$$\dot{\varepsilon}_{xw,i}(\tau) > 0 \quad (18)$$

so that $\varepsilon_{xw,i}$ cannot become negative. This shows that w is an under-estimate of x and that we have for any time $t \geq t_0$:

$$w(t) < x(t) \quad (19)$$

The same line of reasoning can be followed to show that z is an over-estimate of x . This finally allows to conclude that for any time $t \geq t_0$

$$w(t) < x(t) < z(t) \quad (20)$$

Furthermore, as the input u is bounded by assumption (4), (13) ensures that the deviation $w - z$ is bounded. The above result guarantees that the weighting factor α defined by (11) is bounded at any time instant $t \geq t_0$ and therefore the estimate $\hat{x}$ defined by (10) is bounded at any time instant t .

Let us now proceed to the second step of the proof by showing that a state observer for system (1) can be made up of both systems (6) and (7). Let us define the constant a as follows :

$$a = \frac{\theta - \theta_z}{\theta_w - \theta_z} \quad (21)$$

and let s be the following linear combination of w and z :

$$s = aw + (1 - a)z \quad (22)$$

As a is a constant, s is governed by the following dynamics:

$$\dot{s} = a\dot{w} + (1 - a)\dot{z} \quad (23)$$

Remembering that the definition of a (21) allows to express θ as follows

$$\theta = a\theta_w + (1 - a)\theta_z \quad (24)$$

and substituting $\dot{w}$ and $\dot{z}$ by their expressions ((6), (7)) into (23) leads to :

$$\dot{s} = As + \theta Bu + Dv + KC(x - s) \quad (25)$$

Then the reconstruction error defined as follows:

$$\varepsilon_{xs} = x - s \quad (26)$$

is governed by the following dynamics:

$$\dot{\epsilon}_{xs} = F \epsilon_{xs} \quad (27)$$

As F is Hurwitz, s is an exponentially converging state estimate for the system (1).

The following development shows that the estimate $\hat{x}$ converges to x by showing that $\|x - \hat{x}\|$ becomes smaller than an arbitrary small positive constant. Let us choose an arbitrary positive constant:

$$\epsilon > 0 \quad (28)$$

Equations (10) (22) allow to write the deviation between s and $\hat{x}$ as follows:

$$s - \hat{x} = (a - \alpha)(w - z) \quad (29)$$

Noting that the constant a can be written as follows from the definition of s (22) :

$$a = \frac{Cs - Cz}{Cw - Cz} \quad (30)$$

Then substituting a and α (11) into Equation (29) leads to:

$$s - \hat{x} = \frac{C(s - x)}{C(w - z)}(w - z) \quad (31)$$

By taking the norm of the above equation, we can write:

$$\|s - \hat{x}\| \leq \|C\epsilon_{xs}\| \left\| \frac{1}{C(w - z)}(w - z) \right\| \quad (32)$$

Let us focus on the deviation between w and z

$$\epsilon_{wz} = w - z \quad (33)$$

which is governed by the following dynamics:

$$\dot{\epsilon}_{wz} = F \epsilon_{wz} + (\theta_w - \theta_z)Bu \quad (34)$$

Since u does not vanish and does not change of sign, there exists a positive constant $\delta > 0$ such that for any time $t \geq t_0$, we have :

$$|Cw - Cz| > \delta \quad (35)$$

Furthermore as the matrix F is Hurwitz, there also exists a constant D such that for any time $t > t_0$, we have :

$$\|w - z\| < D \quad (36)$$

Therefore (32) can be bounded and we can write:

$$\|s - \hat{x}\| < \|Cs - Cx\| \frac{D}{\delta} \quad (37)$$

Let us now focus on the reconstruction error ϵ_{xs} which is governed by Equation (27). Let us define ϵ_1 as follows:

$$\epsilon_1 = \frac{\epsilon \delta}{2D} \quad (38)$$

then, as the reconstruction error converges to the origin with an exponential rate, there exists a time instant τ_1 such that

$$\forall t \geq \tau_1 : \|C\epsilon_{xs}\| < \epsilon_1 \quad (39)$$

so that (37) allows to write that for any time $t \geq \tau_1$ we have:

$$\|s - \hat{x}\| < \frac{\epsilon}{2} \quad (40)$$

As s is a state estimate for x we can choose ϵ_2 as follows

$$\epsilon_2 = \frac{\epsilon}{2} \quad (41)$$

and there exists a time instant τ_2 such that

$$\forall t \geq \tau_2 : \|s - x\| < \epsilon_2 \quad (42)$$

Let τ be the maximum of (τ_1, τ_2) . Both conditions (40) and (42) then hold for any time $t \geq \tau$:

$$\|s - \hat{x}\| < \frac{\epsilon}{2} \quad (43)$$

$$\|s - x\| < \frac{\epsilon}{2} \quad (44)$$

Finally, the norm of the difference between x and $\hat{x}$ is lower than the arbitrary small constant ϵ for any time instant $t > \tau$

$$\|x - \hat{x}\| < \epsilon$$

■

In this section we have shown that the state of a LTI dynamical system with the structure (1) can be expressed as an unknown linear combination of both biased observers. The proof assumes that both biased observers are respectively lower and greater than the state. However it would have been sufficient to assume that the reconstructed output of both biased observers are different at any time. In other words similar results can be obtained with two different over- or under-estimates but in practice it is more interesting to have two bounds of the state.

Implementing the estimator described here requires to tune a gain vector K so that the matrix $F = A - KC$ is Hurwitz and cooperative. It is worth noting that the gain vector $K = 0$ can be used since the matrix A is stable and cooperative by assumption. This explains why it has been assumed that system (1) is stable and cooperative since the proof can be achieved by assuming that only the matrix F (but not the matrix A) is Hurwitz and cooperative.

The following example illustrates the estimation performances with a continuous stirred tank reactor model.

Example 1: Let us consider a continuous stirred tank reactor with the following consecutive chemical reactions :



The dynamical model of the isothermal reactor with first order kinetics is given by the following differential equations:

$$\dot{C}_A = \frac{Q}{V}(C_{A,in} - C_A) - k_1 C_A \quad (46)$$

$$\dot{C}_B = -\frac{Q}{V}C_B + k_1 C_A - k_2 C_B \quad (47)$$

$$\dot{C}_C = -\frac{Q}{V}C_C + k_2 C_B \quad (48)$$

where C_A , C_B , C_C , Q , V , $C_{A,in}$, k_1 and k_2 are the concentration of A (mol/m^3), the concentration of B (mol/m^3), the concentration of C (mol/m^3), the feed flowrate (m^3/s), the

reactor volume (m^3), the inlet concentration of A (mol/m^3), the kinetics constant associated to the transformation of A (s^{-1}) and of B (s^{-1}), respectively.

Let us consider that only the concentration C_C is measured and that the inlet concentration $C_{A,in}$ is uncertain and bounded as follows:

$$C_{A,in,min} < C_{A,in} < C_{A,in,max} \quad (49)$$

Let $\bar{Q}$, $\bar{V}$ be the nominal operating conditions and $\bar{C}_A$, $\bar{C}_B$ and $\bar{C}_C$ be the corresponding steady state values. The reactor model described by Equations (46) to (48) linearized around the equilibrium point is then given by the following differential equations:

$$\dot{x} = Ax + \theta Bu \quad (50)$$

$$y = Cx \quad (51)$$

with

$$A = \begin{pmatrix} -\frac{\bar{Q}}{\bar{V}} - k_1 & 0 & 0 \\ k_1 & -\frac{\bar{Q}}{\bar{V}} - k_2 & 0 \\ 0 & k_2 & -\frac{\bar{Q}}{\bar{V}} \end{pmatrix} \quad (52)$$

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (53)$$

$$C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \quad (54)$$

where x , u , θ are the concentrations vector, the dilution rate Q/V and the inlet concentration $C_{A,in}$, respectively.

Let θ_w and θ_z be $C_{A,in,min}$ and $C_{A,in,max}$ respectively, we now have to tune a gain vector K so that both following systems are an interval observer for (50):

$$\dot{w} = Aw + \theta_w Bu + K(y - Cw) \quad (55)$$

$$\dot{z} = Az + \theta_z Bu + K(y - Cz) \quad (56)$$

This can be achieved by imposing that the matrix F is stable and cooperative:

$$F = \begin{pmatrix} -\frac{\bar{Q}}{\bar{V}} - k_1 & 0 & -K_1 \\ k_1 & -\frac{\bar{Q}}{\bar{V}} - k_2 & -K_2 \\ 0 & k_2 & -\frac{\bar{Q}}{\bar{V}} - K_3 \end{pmatrix} \quad (57)$$

As mentioned above, the following gain vector can be used

$$K = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T \quad (58)$$

since the matrix A is stable and cooperative. The results of a numerical simulation with the numerical values listed in Table I is illustrated on Figure 1.

However the estimator convergence dynamics is governed by the eigenvalues of matrix F and can then be improved via the gain vector K . It is worth noting that these eigenvalues cannot be arbitrarily assigned even if the system is observable since the matrix F has to be cooperative. In this example, the cooperativity condition requires to set non-positive values for the gains K_1 and K_2 which limits the convergence rate. In particular the estimation performances can be improved by

TABLE I
NUMERICAL VALUES USED FOR THE SIMULATION

Parameter	Value	Variable	Initial condition
$\bar{Q}/\bar{V}$ (s^{-1})	2	$x(0)$ (mol/l)	$(0.83 \ 1.2 \ 3)^T$
k_2 (s^{-1})	5	$w(0)$ (mol/l)	$(0 \ 1 \ 0)^T$
k_1 (s^{-1})	10	$z(0)$ (mol/l)	$(1.1 \ 2 \ 5)^T$
$C_{A,in}$ (mol/l)	5		
$C_{A,in,min}$ (mol/l)	4		
$C_{A,in,max}$ (mol/l)	8		

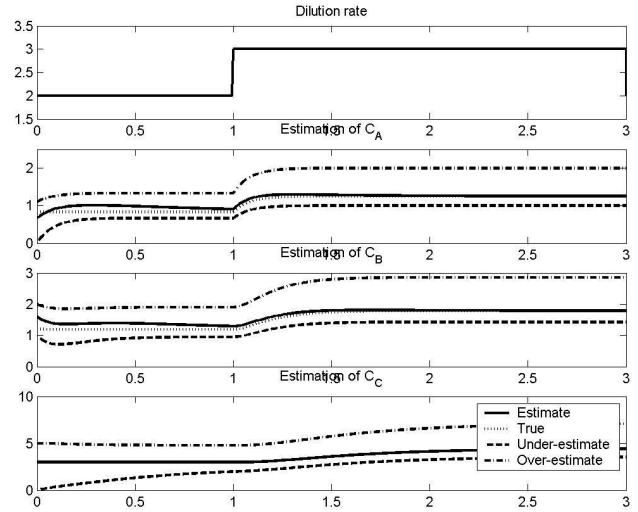


Fig. 1. Estimation of an LTI system with $K = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$

setting K_1 and K_2 equal to zero and increasing K_3 . Simulation results with the following gain vector:

$$K = \begin{pmatrix} 0 & 0 & 10 \end{pmatrix}^T \quad (59)$$

are shown on Figure 2 where it can be seen that the convergence rate is improved, as expected.

III. ESTIMATE FOR A CLASS OF NONLINEAR SYSTEMS

The results obtained for the LTI system (1) can be generalized for nonlinear systems with the following structure:

$$\dot{x} = f(y, t, u)x + \theta g(y, t, u) + h(y, t, u) \quad (60)$$

$$y = Cx \quad (61)$$

with the state $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}$, the input $u \in \mathbb{R}^p$ and where f , g , h are nonlinear bounded functions and where θ is an uncertain bounded parameter satisfying:

$$\theta_w g(y, t, u) < \theta g(y, t, u) < \theta_z g(y, t, u) \quad (62)$$

Note that the above system structure is very close to the linear time varying system corresponding to (1) however the function g does not need to be linear with respect to u .

Let us assume that there exists a vector $K \in \mathbb{R}^n$ such that both following systems are an interval observer for

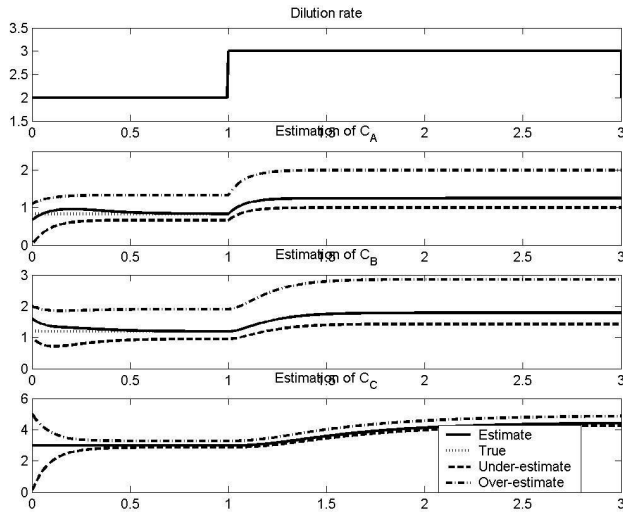


Fig. 2. Estimation of an LTI system with $K = (0 \ 0 \ 10)^T$

system (60):

$$\dot{w} = f(y, t, u)w + g(y, t, u)\theta_w + h(y, t, u) + K(y - Cw) \quad (63)$$

$$\dot{z} = f(y, t, u)z + g(y, t, u)\theta_z + h(y, t, u) + K(y - Cz) \quad (64)$$

with for any time $t \geq t_0$:

$$w(t) < x(t) < z(t) \quad (65)$$

The weighted mean estimate is defined similarly as for LTI systems by:

$$\hat{x} = \alpha w + (1 - \alpha)z \quad (66)$$

with

$$\alpha = \frac{y - Cz}{Cw - Cz} \quad (67)$$

Theorem 3.1: Assuming systems (63) and (64) are an interval observer for (60), the variable $\hat{x}$ given by (66) is a weighted mean estimate for (60).

Proof

The proof is similar to those for LTI systems. Let us start defining the constant a as follows:

$$a = \frac{\theta - \theta_z}{\theta_w - \theta_z} \quad (68)$$

and let us define the variable s by:

$$s = aw + (1 - a)z \quad (69)$$

whose dynamics is given by:

$$\dot{s} = f(y, t, u)s + g(y, t, u)\theta \quad (70)$$

so that the difference between s and x is governed by the following dynamics:

$$\dot{x} - \dot{s} = (f(y, t, u) - KC)(x - s) \quad (71)$$

It can be shown that if for any time t , output y and input u , $f - KC$ is Hurwitz and its norm is bounded, the above system converges to zero so that s is a state observer for x . Next we can compute the difference between $\hat{x}$ and s , which is equal to:

$$s - \hat{x} = (a - \alpha)(w - z) \quad (72)$$

which can be rewritten as follows:

$$s - \hat{x} = \frac{C(s - x)}{Cw - Cz}(w - z) \quad (73)$$

And the rest of the proof is similar to the proof for LTI systems. ■

The following example illustrates the performance of the estimation with the model of a continuous stirred tank reactor with nonlinear kinetics.

Example 2: Let us consider the model of a continuous stirred tank reactor with the following exothermic reaction :



whose kinetics follows the Arrhenius law :

$$r = k_0 e^{-E/RT} C \quad (75)$$

where k_0 , E , R , T and C are the kinetics constant (s^{-1}), the activation energy (J/mol), the gas constant ($J/mol/K$), the temperature (K) and the concentration of the reactant A (mol/m^3), respectively.

The dynamical reactor model obtained from mass and energy balances is given by the following differential equations:

$$\dot{T} = \frac{Q}{V}(T_{in} - T) - \frac{\Delta H}{\rho c_p} r + \frac{UA}{\rho c_p V}(T_j - T) \quad (76)$$

$$\dot{C} = \frac{Q}{V}(C_{in} - C) - r \quad (77)$$

where T , C , Q , V , T_{in} , C_{in} , ρ , c_p , ΔH , k , T_j , U and A are the temperature (K), the concentration of the limiting reactant (mol/m^3), the feed flowrate (m^3/s), the volume (m^3), the inlet temperature (K), the inlet concentration (mol/m^3), the density (kg/m^3), the heat capacity (J/kg), the reaction heat (J/mol), the kinetics constant (s^{-1}), the cooling fluid temperature (K), the heat exchange coefficient ($W/m^2/K$) and the heat exchange area (m^2), respectively.

Let us consider that the reactor temperature is measured on-line, that the cooling fluid temperature is the input variable and that the global heat exchange coefficient is badly known but satisfies the following inequalities:

$$UA_{min} < UA < UA_{max} \quad (78)$$

The reactor model described by (76) and (77) can be written

as (60) where

$$f(y, t, u) = \begin{pmatrix} -\frac{Q}{V} \\ 0 \\ -\frac{Q}{V} - k_0 e^{-E/RT} \end{pmatrix} \quad (79)$$

$$g(y, t, u) = \begin{pmatrix} -\frac{1}{\rho c_p V} (y - u) \\ 0 \end{pmatrix} \quad (80)$$

$$h(y, t, u) = \begin{pmatrix} \frac{Q}{V} T_{in} \\ \frac{Q}{V} C_{in} \end{pmatrix} \quad (81)$$

$$y = x_1 \quad (82)$$

where x_1 , x_2 , u and θ are the temperature T , the concentration C , the cooling fluid temperature T_j and the global heat exchange coefficient UA , respectively.

It is worth noting that the under-estimate (resp., over-) uses the maximum (resp., minimum) value for the heat exchange coefficient

$$\theta_w = \theta_{max} \quad (83)$$

$$\theta_z = \theta_{min} \quad (84)$$

since the reaction is exothermic and we have at any time

$$T_j < T \quad (85)$$

It remains to tune the gain vector K so that the systems (63) and (64) are an interval observer for (60). It can be shown that any constant gain vector K satisfying the following conditions can be used

$$K_1 \geq 0 \quad (86)$$

$$K_2 \leq 0 \quad (87)$$

This can be achieved by considering the reconstruction errors

$$\varepsilon_w(t) = x(t) - w(t) \quad (88)$$

$$\varepsilon_z(t) = x(t) - z(t) \quad (89)$$

and noting that

$$\dot{\varepsilon}_w(0) > 0 \quad (90)$$

$$\dot{\varepsilon}_z(0) < 0 \quad (91)$$

since the kinetics law (75) is an increasing function of the concentration and since the reaction is exothermic.

As shown in the first example, an appropriate tuning of the gain vector K can improve the estimation dynamics. On the one hand, increasing the gains increases the convergence rate; and on the other hand, too high gains degrade the estimation performances in case of noisy measurements. For this example, we have chosen to set

$$K_2 = 0 \quad (92)$$

since a greater value does not guarantee that systems (63) and (64) are under- and over-estimates for x however we have chosen to assign a positive value to K_1 so that the estimates (63) and (64) are coupled to the actual systems. If K were set to zero then the over- and under-estimate could largely deviate from the actual state and lose any physical signification. We finally have chosen

$$K = \begin{pmatrix} 0.001 & 0 \end{pmatrix}^T \quad (93)$$

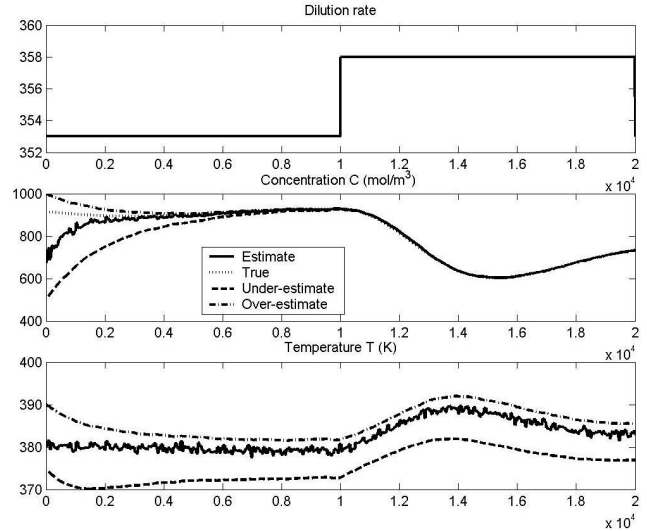


Fig. 3. Simulation results for an exothermic reactor

Parameter	Value	Variable	Initial Condition
Q (m^3/s)	0.00033	$T(0)$ (K)	380
V (m^3)	10	$C(0)$ (mol/m^3)	915
T_{in} (K)	283	$w_1(0)$ (K)	375
C_{in} (mol/m^3)	14000	$w_2(0)$ (mol/m^3)	500
ΔH (J/mol)	-76000	$z_1(0)$ (K)	390
c_p (J/kg)	2000	$z_2(0)$ (mol/m^3)	1000
ρ (Kg/m^3)	1000		
k_0 (s^{-1})	4.610^5		
E (J/mol)	65300		
UA (W/K)	10000		
UA_{min} (W/K)	8000		
UA_{max} (W/K)	15000		

TABLE II

NUMERICAL VALUES USED FOR THE SIMULATION

The performance of the estimation are illustrated on Figure 3 by a numerical simulation carried out with the values listed in Table 2.

IV. CONCLUSION

In this paper we have presented a way to compute a state estimate from an over-estimate and an under-estimate provided by an interval observer for systems that are linear with respect to the unknown parameter. First we have shown that the estimate converges exactly to the state for linear time invariant systems. The performance of the estimator has been illustrated with the linearized model of a continuous stirred tank reactor with consecutive reactions. Then we have extended the field of application to a class of nonlinear systems that are linear with respect to the unknown parameter. The performance of the estimator have been illustrated with numerical simulations of a continuous stirred tank reactor with an exothermic reaction whose kinetics follows the Arrhenius law.

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