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CONDITIONAL EXPECTATIONS GIVEN THE SUM OF INDEPENDENT RANDOM VARIABLES WITH REGULARLY VARYING DENSITIES

Michel Denuit
Institute of Statistics, Biostatistics and Actuarial Science - ISBA
Louvain Institute of Data Analysis and Modeling - LIDAM
UCLouvain
Louvain-la-Neuve, Belgium
michel.denuit@uclouvain.be

Patricia Ortega-Jimenez
Institute of Statistics, Biostatistics and Actuarial Science - ISBA
Louvain Institute of Data Analysis and Modeling - LIDAM
UCLouvain
Louvain-la-Neuve, Belgium
patricia.ortega@uclouvain.be

Christian Y. Robert
Laboratory of Actuarial and Financial Science - LSAF
Université Lyon 1
Lyon, France
christian.robert@univ-lyon1.fr

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Abstract

Stochastic monotonicity of two independent random variables X and Y given the value of their sum $S = X + Y$ has been linked to log-concave densities since Efron (1965). However, the log-concavity assumption is not realistic in some applications because it excludes heavy-tailed distributions. This paper considers random variables with regularly varying densities to illustrate how heavy tails can lead to a non-monotonic behavior for the conditional expectation $m_X(s) = \mathrm{E}[X|S = s]$, which turns out to be problematic in risk sharing or signal processing (including industry loss warranties or parametric insurance, for instance). This paper first aims to identify situations where a non-monotonic behavior appears according to the tail-heaviness of X and Y. Secondly the paper aims to study the asymptotic behavior of $m_X(s)$ as the value s of the sum gets large. The analysis is then extended to zero-augmented probability distributions, commonly encountered in applications to insurance and to sums of more than two random variables. Consequences for signal processing and risk sharing are discussed. Many numerical examples illustrate the results.

Keywords: log-concavity, asymptotic smoothness, size-bias transform, noisy signal, risk sharing, zero-augmented distributions.

JEL Classification: G22

1 Introduction and motivation

Given independent random variables X and Y the study of the stochastic monotonicity of each marginal given the value of their sum $S = X + Y$ is usually referred to as "Efron's monotonicity property" after Efron (1965). This author proved that a sufficient condition for X and Y to be stochastically increasing in S was that the summands X and Y possess log-concave densities. However, this assumption might be too strong in certain applications because log-concave densities are associated with light-tailed asymptotic behavior, as discussed in Asmussen and Lehtomaa (2017). Hence, this paper aims to study how heavy-tailedness affects the monotonicity of the conditional expectations of X and Y given the sum S.

Let $m_X(\cdot)$ denote the conditional expectation of X given the sum S, defined by $m_X(s) = \mathrm{E}[X|S = s]$. Similarly, we denote as $m_Y(\cdot)$ the conditional expectation of Y given the sum S, with $m_X(\cdot)$ and $m_Y(\cdot)$ summing to identity. The monotonicity of $m_X(\cdot)$ is known as regression dependence and appears to be weaker than Efron's monotonicity. It plays an important role in various contexts, as discussed next.

Consider for instance two economic agents, with respective insurance losses modeled as independent random variables X and Y, who decide to form a pool to share the total loss $S = X + Y$. As proposed in Denuit and Dhaene (2012), the conditional expectations $m_X(\cdot)$ and $m_Y(\cdot)$ may be used by participants to allocate the total loss among them. The conditional mean risk-sharing rule has been axiomatized by Jiao et al. (2022). If $m_X(\cdot)$ decreases over a range of values then the participant bringing loss X may be tempted to exaggerate the loss since this would decrease his or her contribution to the pool. Non-decreasingness of $m_X(\cdot)$ thus appears to be a reasonable and useful requirement in the context of risk sharing, where this property is referred to as the no-sabotage condition.

The results derived in this paper are also useful in signal processing. Following Mizuno (2006), Ernst et al. (2022) and Huang (2023), let us assume that a random variable X of interest is observed perturbed by an independent additive noise Y. The analyst then tries to recover information about X from the signal $S = X + Y$ with the help of the conditional expectation $m_X(S)$ of X given the signal S. In signal processing, it is often desirable that a larger signal corresponds to a larger value of the random variable under consideration. Hence, regression dependence is an important property for proper interpretation of the signal.

There are situations in insurance where signals are used to determine cash flows. This is for instance the case with industry loss warranties. The payout of these financial instruments is linked to the losses experienced at market level (taken as reference index, or signal). These insurance-linked securities have become increasingly attractive for investors and are bought by insurance companies as a substitute for reinsurance. Denoting as X the buyer's loss and as Y the total loss for the other companies on the market, the payout is tied to the industry loss index $S = X + Y$. The expected payout when $X + Y = s$ is thus $m_X(s)$ and the buyer is interested in its behavior in the adverse situation where s is large. This is precisely one of the topics investigated in this paper.

It turns out that $m_X(\cdot)$ may exhibit different behaviors when the log-concavity assumption is violated. For instance, Denuit and Robert (2021b) showed that regression dependence is not fulfilled when X and Y obey zero-augmented Gamma distributions, each of them highly spiked in a different mode. A second typical set of distributions which do not have log-concave densities are those with heavy tails, since log-concavity constrains the tails to be exponentially decreasing. In this paper, we consider variables with regularly varying densities to explore how the tail-heaviness affect the monotonicity of $m_X(\cdot)$.

Denuit and Robert (2020) showed that considering a set of random variables with regularly varying tails, when one of the random variables dominates, the conditional expectations of the others are asymptotically vanishingly small with respect to the dominating one. However, this does not necessarily have any implication on their monotonicity and neither mean that the other conditional expectations are bounded. Intuitively, we can expect that the more significant the difference in the tail-heaviness, the smaller the other conditional expectations. Hence, in this paper we aim to explore how the difference in the tail indices of the densities result in different behaviors of $m_X(\cdot)$ as the sum tends to infinity.

This paper innovates at both methodological and practical levels. First, the asymptotic behavior of $m_X(\cdot)$ is determined in the heavy-tailed case, considering the different situations that can be encountered depending on the difference in the tail indices of X and Y. Then, asymptotic approximations are derived for $m_X(\cdot)$ according to the respective values of these tail indices. Also, the results are extended to zero-augmented distributions describing insurance losses or accounting for a detection limit in signal processing or a threshold to be exceeded by the index in parametric insurance. In addition, a necessary condition for $m_X(\cdot)$ being decreasing in an interval is extended to the case of sums with more than two terms. Finally, these findings are applied in the closing discussion to gain understanding in the context of the examples considered in this paper.

The remainder of this paper is organized as follows. In Section 2, we recall some definitions and representations of $m_X(\cdot)$ and of it first derivative. We also establish a lower bound on the asymptotic value of $m_X(\cdot)$. Section 3 shows that $m_X(\cdot)$ may reach different asymptotic levels depending on the respective tail indices of X and Y. An expansion formula for $m_X(\cdot)$ is derived in Section 4. This expansion allows us to study the asymptotic behavior of $m_X(\cdot)$ according to the differences between the respective tail indices of X and Y. Extensions to zero-augmented distributions and to several terms are considered in Sections 5 and 6. Section 7 concludes the paper with a discussion of its main results. Technical material is given in appendix.

Let us say a few words about the notation adopted in this paper. For any positive functions $f(\cdot)$ and $g(\cdot)$, we write $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$, $g(x) = o(f(x))$ as $x \to \infty$ if $\lim_{x\to\infty} g(x)/f(x) = 0$ and $g(x) = O(f(x))$ as $x \to \infty$ if $\limsup_{x\to\infty} g(x)/f(x) < \infty$.

2 Background and definitions

2.1 Regularly varying and asymptotically smooth density functions

Regularly varying distribution functions have long been used in probability theory. For the study of the properties of $m_X(\cdot)$, it is however preferable to consider regularly varying probability density functions because $m_X(\cdot)$ possesses a useful representation in terms of density functions as explained in Section 2.2. Note that variables with regularly varying density functions have regularly varying distribution functions, but the reverse does not hold. Recall that a positive measurable function $L(\cdot)$ is said to be slowly varying if it is defined on some neighborhood (x_0, ∞) of infinity, with $x_0 \geq 0$, and $\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1$ for all $t > x_0$. Intuitively speaking, a slowly varying function is a function that grows/decays asymptotically slower than any polynomial. For example, $\log(\cdot)$ or any function converging to a constant are slowly varying functions. We are now in a position to recall the definition of a regularly varying probability density function.

Definition 2.1 (Regularly varying density). *A probability density function $f(\cdot)$ defined on $(0, \infty)$ is said to be regularly varying with index $\alpha > 1$, if there exists a slowly varying function $L(\cdot)$ such that $f(x) = x^{-\alpha} L(x)$.*

An important property of regularly varying densities of positive random variables is that they form a stable family by convolution. Precisely, the convolution of two regularly varying densities is still regularly varying with tail index equal to the minimum of the tail indices of the convoluted density functions. More precisely, we have the following property:

$$f_X \text{ and } f_Y \text{ regularly varying} \Rightarrow \lim_{s\to\infty} \frac{f_{X+Y}(s)}{f_X(s) + f_Y(s)} = 1, \tag{2.1}$$

where $f_{X+Y}(\cdot)$ is the convolution of $f_X(\cdot)$ and $f_Y(\cdot)$. See e.g. Theorem 1.1 in Bingham et al. (2006) for a proof.

In Table 2.1, we summarize some distributions with regularly varying densities, together with the index and slowly varying function $L(\cdot)$ appearing in the representation of the density $f(\cdot)$ as stated in Definition 2.1. The notation $\Gamma(\cdot)$ stands for the Gamma function and $\zeta(\cdot)$ for the Riemann Zeta function. Note that the parameters of the distribution and density functions have been chosen in order to have index α in all cases. To this end, we may depart from the standard parameter choices for these distributions. Type I and Type II Pareto, Log-Gamma and Dagun distributions are often defined with a parameter $\beta = \alpha - 1$ instead of the parameter α which appears in Table 2.1. Type III and Type VI Pareto distributions are often specified with parameter $\gamma = \frac{\lambda}{\alpha-1}$ ($\lambda = 1$ in the case of Type III) instead of α. Notice that the support is (ϑ, ∞) for certain distributions listed in Table 2.1. In the remainder of this paper, we assume that the supports of X and Y (and hence of their sum S) is $(0, \infty)$. If the support of the variables is on the form (ϑ, ∞), we can either extend the density function $f(\cdot)$ by assuming $f(x) = 0$ in $x \in (0, \vartheta]$ or shift the random variable under consideration by ϑ.

Distribution	**Parameters and support**	**Survival ($\overline{F}$) or density (f) function**	**Slowly varying function $L(x)$ associated to f**
Pareto (Type I) $P(I)(\theta,\alpha)$	$\alpha > 1,\ \theta > 0,$ $x > \theta$	$\overline{F}(x) = \theta^{\alpha-1}x^{-(\alpha-1)}$	$L(x) = \theta^{\alpha-1}(\alpha-1)$
Pareto (Type II) (Lomax dist. if $\vartheta = 0$) $P(II)(\theta,\alpha,\vartheta)$	$\alpha > 1,\ \theta > 0,$ $\vartheta \geq 0,\ x > \vartheta$	$\overline{F}(x) = \theta^{\alpha-1}\left(x+\theta-\vartheta\right)^{-(\alpha-1)}$	$L(x) = \theta^{\alpha-1}(\alpha-1)\left(\frac{x}{x+\theta-\vartheta}\right)^{\alpha}$
Pareto (Type III) $P(III)(\theta,\alpha,\vartheta)$	$\alpha > 1,\ \theta > 0,$ $\vartheta \geq 0,\ x > \vartheta$	$\overline{F}(x) = \left(1+\left(\frac{x-\vartheta}{\theta}\right)^{\alpha-1}\right)^{-1}$	$L(x) = (\alpha-1)\theta^{1-\alpha}\frac{x^{\alpha}(x-\vartheta)^{\alpha-2}}{\left(1+\left(\frac{x-\vartheta}{\theta}\right)^{\alpha-1}\right)^{2}}$
Pareto (Type IV) (Singh-Maddala dist.) $P(IV)(\theta,\alpha,\vartheta,\lambda)$	$\alpha > 1\ ,\theta,\lambda > 0,$ $\vartheta \geq 0,\ x > \vartheta$	$\overline{F}(x) = \left(1+\left(\frac{x-\vartheta}{\theta}\right)^{\frac{\alpha-1}{\lambda}}\right)^{-\lambda}$	$L(x) = (\alpha-1)\theta^{-\frac{\alpha-1}{\lambda}}\frac{x^{\alpha}(x-\vartheta)^{-1+\frac{\alpha-1}{\lambda}}}{\left(1+\left(\frac{x-\vartheta}{\theta}\right)^{\frac{\alpha-1}{\lambda}}\right)^{\lambda+1}}$
Log-Gamma $LG(\alpha,\lambda)$	$\alpha > 1,\ \lambda > 0,$ $x > 1$	$f(x) = \frac{(\alpha-1)^{\lambda}}{\Gamma(\lambda)}(\log x)^{\lambda-1}x^{-\alpha}$	$L(x) = \frac{(\alpha-1)^{\lambda}}{\Gamma(\lambda)}(\log x)^{\lambda-1}$
Dagun distribution (Burr III dist.) $D(\theta,\alpha,\lambda)$	$\alpha > 1\ ,\theta,\lambda > 0,$ $x > 0$	$\overline{F}(x) = 1-\left(1+\left(\frac{x}{\theta}\right)^{-(\alpha-1)}\right)^{-\lambda}$	$L(x) = \frac{(\alpha-1)\lambda\theta^{\alpha-1}}{\left(1+\left(\frac{x}{\theta}\right)^{-(\alpha-1)}\right)^{\lambda+1}}$
Davis distribution $Davis(\alpha,b,\vartheta)$	$\alpha > 1,\ b,\vartheta \geq 0,$ $x > \vartheta$	$f(x) = \frac{b^{\alpha}(x-\vartheta)^{-1-\alpha}}{\left(e^{\frac{b}{x-\vartheta}}-1\right)\Gamma(\alpha)\zeta(\alpha)}$	$L(x) = \frac{b^{\alpha}\left(\frac{x}{x-\vartheta}\right)^{\alpha+1}}{\left(e^{\frac{b}{x-\vartheta}}-1\right)\Gamma(\alpha)\zeta(\alpha)}$

Table 2.1: Families of distributions with regularly varying densities with index α.

To conduct our study, we need to impose a condition known in the literature as asymptotic smoothness after Barbe and McCormick (2005) and recalled next.

Definition 2.2. *A density function $f(\cdot)$ defined on $(0,\infty)$ is said to be asymptotically smooth with index $\alpha > 1$ if*

$$\lim_{\delta\to 0}\limsup_{t\to\infty}\sup_{0<|x|\leq\delta}\left|\frac{f(t(1-x))-f(t)}{xf(t)}-\alpha\right|=0.$$

Asymptotically smooth functions are related to regularly varying ones with the same index. When $f(\cdot)$ is asymptotically smooth and differentiable, then it is also regularly varying with the same index. Conversely, if $f(\cdot)$ is regularly varying and has an ultimately monotone derivative, then it is asymptotically smooth (see Proposition 2.1 in Barbe and McCormick (2005)). It must be remarked that all regularly varying densities considered in Table 2.1 have ultimately monotone derivatives and, therefore are asymptotically smooth. The proof of this statement is given in Appendix A.

2.2 Representation of the conditional expectation given the sum in terms of size-biasing

Let us consider a non-negative random variable X with distribution function F_X and finite and strictly positive expected value. If X is a random variable having a regularly varying density with index $\alpha > 1$, it is well-known that $k < \alpha - 1$ implies $\mathrm{E}[X^k] < \infty$. Hence, $\mathrm{E}[X] < \infty$ if $\alpha > 2$, and we retain this assumption for all variables considered throughout this paper. The distribution function of the size-biased version $\widetilde{X}$ of X is then given by

$$\mathrm{P}[\widetilde{X}\leq t]=\frac{1}{\mathrm{E}[X]}\int_0^t x\mathrm{d}F_X(x),\ \ t\geq 0.$$

We refer the reader to Arratia et al. (2019) for an introduction to the the size-biased transform.

In Denuit (2019) a representation for the regression function $m_X(\cdot)$ is established in terms of the size-biased transform of X. Assume that X and Y are independent with finite means, with respective probability density functions $f_X(\cdot)$ and $f_Y(\cdot)$. Since X possesses a density function $f_X(\cdot)$ and its expected value is finite, its size-biased version $\widetilde{X}$ also possesses a density given by $f_{\widetilde{X}}(x) = xf_X(x)/\mathrm{E}[X]$. If $\widetilde{X}$ is independent of Y then the representation

$$m_X(s)=\mathrm{E}[X]\frac{f_{\widetilde{X}+Y}(s)}{f_{X+Y}(s)} \tag{2.2}$$

holds true for any $s > 0$.

2.3 Minimum limit value of $m_X(\cdot)$

Understanding the asymptotic behavior of $m_X(s)$ as s gets large is important in practice. For example, in the case of risk sharing, $m_X(s)$ represents the contribution to be paid by the participant bringing X in the event of a large pool loss. If the supports of X and Y are not bounded, we might expect this contribution to tend towards infinity when s tends to infinity. On the one hand, we shall see that this is not always the case. On the other hand, there is a minimum value of the limit below which the contribution cannot fall which is $\mathrm{E}[X]$ as established below. This limit is reached in particular when X and Y have regularly varying densities with tail indices whose difference is larger than 1 as it will become clear in the next section.

Proposition 2.3. *Assume that X and Y have densities $f_X(\cdot)$ and $f_Y(\cdot)$ such that $f_Y(\cdot)$ is bounded and ultimately decreasing, i.e. there exists y_0 such that $f_Y(\cdot)$ is decreasing over (y_0,∞). Moreover assume that $\sup_{s\in(x-y_0,x)} f_X(s) = \mathrm{o}(f_{X+Y}(x))$ as $x\to\infty$, then*

$$\lim_{s\to\infty}\inf m_X(s) \geq \mathrm{E}[X].$$

Proof. Note that, for $s > y_0$,

$$m_X(s) = \frac{\int_0^s x f_X(x) f_Y(s-x)\,\mathrm{d}x}{\int_0^s f_X(x) f_Y(s-x)\,\mathrm{d}x} \geq \frac{\int_0^{s-y_0} x f_X(x) f_Y(s-x)\,\mathrm{d}x}{\int_0^{s-y_0} f_X(x) f_Y(s-x)\,dx + \int_{s-y_0}^{s} f_X(x) f_Y(s-x)\,\mathrm{d}x}.$$

Since $f_Y(s-\cdot)$ is increasing on $(0, s-y_0)$, we have

$$\mathrm{E}[X f_Y(s-X)\,|X\leq s-y_0] \geq \mathrm{E}[X|X\leq s-y_0]\mathrm{E}[f_Y(s-X)\,|X\leq s-y_0]$$

or equivalently

$$\frac{\int_0^{s-y_0} x f_X(x) f_Y(s-x)\,\mathrm{d}x}{\int_0^{s-y_0} f_X(x)\,\mathrm{d}x} \geq \frac{\int_0^{s-y_0} x f_X(x)\,\mathrm{d}x}{\int_0^{s-y_0} f_X(x)\,\mathrm{d}x}\frac{\int_0^{s-y_0} f_X(x) f_Y(s-x)\,\mathrm{d}x}{\int_0^{s-y_0} f_X(x)\,\mathrm{d}x}.$$

We deduce that

$$\begin{aligned}
&\frac{\int_0^{s-y_0} x f_X(x) f_Y(s-x)\,\mathrm{d}x}{\int_0^{s-y_0} f_X(x) f_Y(s-x)\,\mathrm{d}x + \int_{s-y_0}^{s} f_X(x) f_Y(s-x)\,\mathrm{d}x}\\
= \;& \frac{\int_0^{s-y_0} x f_X(x) f_Y(s-x)\,\mathrm{d}x}{\int_0^{s-y_0} f_X(x) f_Y(s-x)\,\mathrm{d}x}\frac{1}{1+\int_{s-y_0}^{s} f_X(x) f_Y(s-x)\,\mathrm{d}x/\int_0^{s-y_0} f_X(x) f_Y(s-x)\,\mathrm{d}x}\\
\geq \;& \mathrm{E}[X|X\leq s-y_0]\frac{1}{1+\int_{s-y_0}^{s} f_X(x) f_Y(s-x)\,\mathrm{d}x/[f_{X+Y}(x) - \int_{s-y_0}^{s} f_X(x) f_Y(s-x)\,\mathrm{d}x]}.
\end{aligned}$$

Moreover,

$$\int_{s-y_0}^{s} f_X(x) f_Y(s-x)\,\mathrm{d}x \leq \sup_{s\in(x-y_0,x)} f_X(s) \sup_{s>0} f_Y(s)$$

and it follows that

$$\lim_{s\to\infty}\frac{1}{1+\int_{s-y_0}^{s} f_X(x) f_Y(s-x)\,\mathrm{d}x/f_{X+Y}(s-y_0)} = 1$$

and $\lim_{s\to\infty}\inf m_X(s) \geq \mathrm{E}[X]$. □

2.4 Representation of the first derivative of the conditional expectation given the sum

Let us assume that the first derivative $f'_Y(\cdot)$ of the probability density function $f_Y(\cdot)$ exists and is well defined. Let (X_1, Y_1) and (X_2, Y_2) be independent copies of (X, Y), with respective sums $S_1 = X_1 + Y_1$ and $S_2 = X_2 + Y_2$. Then Denuit and Robert (2021a) demonstrated that the first derivative $m'_X(\cdot)$ of $m_X(\cdot)$ exists and can be represented as

$$m'_X(s) = \frac{1}{4}\mathrm{E}\left[(X_1 - X_2)\left(\frac{f'_Y(Y_1)}{f_Y(Y_1)} - \frac{f'_Y(Y_2)}{f_Y(Y_2)}\right)\mathrm{I}\left[X_1 > X_2\right]\middle| S_1 = s, S_2 = s\right]$$

where $\mathrm{I}[\cdot]$ denotes the indicator function (equal to 1 when the condition appearing within brackets is fulfilled and to 0 otherwise). In the latter expression, the inequality $Y_1 < Y_2$ must hold true because $X_1 > X_2$ and the sums $X_1 + Y_1$ and $X_2 + Y_2$ are constrained to be equal to some fixed value s. We deduce that if $f_Y(\cdot)$ is assumed to be log-concave, i.e. $\ln f_Y(\cdot)$ is concave, then the ratio $f'_Y(\cdot)/f_Y(\cdot)$ is decreasing and the second factor in the last conditional expectation must thus be positive, and therefore $m'_X(s) \geq 0$ for any $s > 0$. Log-concavity of $f_Y(\cdot)$ thus guarantees that $m_X(\cdot)$ is non-decreasing.

This expression clarifies the role that log-concavity plays in the monotonicity of $m_X(\cdot)$. However, it must be remarked that, under the assumption of log-concavity, not only $m_X(\cdot)$ is non-decreasing, but the variable $\{X \mid S = s\}$ is non-decreasing in s in the usual stochastic order. This means that the expected value of any increasing transformation of the variable is non-decreasing in the conditioning value s of S. This result is a consequence of Section 2 in Efron (1965) and a proof computing $m'_X(s)$ is provided in Saumard and Wellner (2014) using symmetrization arguments for independent log-concave variables.

Consider a regularly varying density $f(\cdot)$ with tail index α such that $f'(\cdot)$ is ultimately monotonic (from Property A.1, it is the case for all distributions listed in Table 2.1). Then, we can derive from Karamata's theorem (Karamata, 1933) that

$$\frac{f'(x)}{f(x)} \sim -\alpha\frac{1}{x}$$

and therefore $f'(\cdot)/f(\cdot)$ cannot be ultimately decreasing. Hence it is natural to investigate whether $m_X(\cdot)$ can be decreasing for some tail indices of X and Y and for some values s of S. This is precisely one of the question investigated in the present paper.

3 Asymptotic level of the conditional expectation given the sum

When X and Y possess regularly varying density functions with respective indices α_X and α_Y, it turns out that the asymptotic level of $m_X(\cdot)$ depends on the difference in the tail indices α_X and α_Y. It is assumed that $\min\{\alpha_X, \alpha_Y\} > 2$ so that $\mathrm{E}[X] < \infty$ and $\mathrm{E}[Y] < \infty$. Throughout this section, we will also assume $\alpha_X \geq \alpha_Y$. If $\alpha_X < \alpha_Y$, it will be sufficient to interchange the indices α_X and α_Y in the results. Note that, under the assumption $\alpha_X \geq \alpha_Y$, the next result shows that $m_Y(\cdot)$ diverges, no matter on the size of the difference between the indices:

Property 3.1. *If X and Y possess regularly varying densities $f_X(\cdot)$ and $f_Y(\cdot)$ with respective indices α_X and α_Y such that $\alpha_Y \leq \alpha_X$, then $m_Y(s) \to \infty$ as $s \to \infty$.*

Proof. From (2.2), $m_Y(s) = \mathrm{E}[Y]\frac{f_{\widetilde{Y}+X}(s)}{f_{Y+X}(s)}$. Note that, since $f_Y(\cdot)$ is regularly varying with index $\alpha_Y > 2$, $f_{\widetilde{Y}}(\cdot)$ is regularly varying with index $1 < \alpha_{\widetilde{Y}} = \alpha_Y - 1 < \alpha_Y$. Therefore, $f_Y(s) = o\left(f_{\widetilde{Y}}(s)\right)$. As $f_X(\cdot)$ is a regularly varying density with index $\alpha_X \geq \alpha_Y$, it holds that $f_X(s) = o\left(f_{\widetilde{Y}}(s)\right)$ and:

$$m_Y(s) \sim \mathrm{E}[Y]\frac{f_{\widetilde{Y}}(s) + f_X(s)}{f_Y(s) + f_X(s)} = \mathrm{E}[Y]\frac{f_{\widetilde{Y}}(s) + o\left(f_{\widetilde{Y}}(s)\right)}{o\left(f_{\widetilde{Y}}(s)\right) + o\left(f_{\widetilde{Y}}(s)\right)}. \tag{3.1}$$

Hence, from this expression we conclude that $m_Y(s) \to \infty$ as $s \to \infty$. □

From Proposition 5.1 in Denuit and Robert (2020) and since variables with regularly varying densities have regularly varying tails, if $\alpha_X > \alpha_Y$, then, as s tends to infinity, $m_X(s) = o(s)$. Although, no information can be obtained with respect to the asymptotic level or the monotonicity of $m_X(\cdot)$. Therefore, in order to study the behavior of $m_X(\cdot)$, the following three cases can be distinguished.

3.1 Difference in tail indices larger than 1 $(\alpha_X - \alpha_Y > 1)$

The next result shows that when the difference between tail indices exceeds 1, the conditional expectation given the sum tends to the expected value for the term with larger index, as the sum tends to infinity.

Property 3.2. *If X and Y possess regularly varying densities $f_X(\cdot)$ and $f_Y(\cdot)$ with respective indices α_X and α_Y such that $\alpha_X > \alpha_Y + 1$, then*

$$\lim_{s\to\infty} m_X(s) = \mathrm{E}\left[X\right].$$

Proof. Since $f_X(\cdot)$ is regularly varying with index $\alpha_X > 2$, we have that $f_{\widetilde{X}}(\cdot)$ is regularly varying with index $\alpha_{\widetilde{X}} = \alpha_X - 1$. As $\alpha_X > \alpha_Y + 1$, then $\alpha_{\widetilde{X}} > \alpha_Y$ and, therefore, $f_{\widetilde{X}}(s) = o\left(f_Y(s)\right)$ and $f_X(s) = o\left(f_Y(s)\right)$, As a consequence of (2.1), then

$$\lim_{s\to\infty} m_X(s) = \mathrm{E}\left[X\right] \lim_{s\to\infty} \frac{f_{\widetilde{X}}(s) + f_Y(s)}{f_X(s) + f_Y(s)} = \mathrm{E}[X].$$

This ends the proof. □

Note that a direct implication of Property 3.2 for continuous distributions, since $m_X(0) < \infty$ is that $m_X(\cdot)$ is bounded. The following result shows that $m_X(\cdot)$ cannot be monotonic over $(0, \infty)$ in the case considered in Property 3.2.

Proposition 3.3. *Assume that X and Y are as described in Property 3.2, then there exists a non-empty interval in $(0, \infty)$ where $m_X(\cdot)$ is decreasing.*

Proof. Let us proceed by contradiction. Assume that $m_X(\cdot)$ is increasing on $(0, \infty)$. Notice that $m_X(\cdot)$ is a positive and continuous function such that $m_X(0) = 0$ and $\lim_{s\to\infty} m_X(s) = \mathrm{E}[X]$ by Property 3.2. Moreover, $\mathrm{E}[X] = \mathrm{E}[m_X(S)]$. If $m_X(\cdot)$ is increasing on $(0, \infty)$ and, since both variables are continuous, we do, however, deduce that $\mathrm{E}[m_X(S)] < \mathrm{E}[X]$, which is a contradiction. This ends the proof. □

The following example illustrates the behavior of $m_X(\cdot)$ in the case where the difference in the tail indices exceeds 1.

Example 3.4. *Let us consider two independent random variables $X \sim LG(\alpha_X, \lambda_X)$ and $Y \sim P(IV)(\theta, \alpha_Y, \vartheta, \lambda_Y)$ such that $\alpha_X > \alpha_Y + 1$. Figure 1 shows a numerical representation considering the parameter values $\vartheta = \theta = 1$, $\lambda_X = \lambda_Y = 2$, $\alpha_X = 5$ and $\alpha_Y = 2$. We can observe that $m_X(\cdot)$ is not monotonic over $(0, \infty)$, in accordance with Proposition 3.3. Here, $m_X(\cdot)$ reaches its maximum and starts decreasing beyond $F_S^{-1}(0.91)$.*

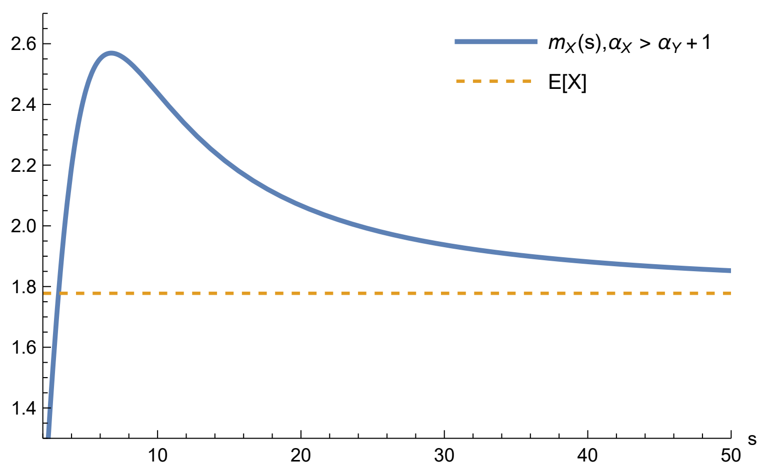


Figure 1: Conditional expectation $m_X(\cdot)$ (solid line) and horizontal line at $\mathrm{E}[X]$ (dashed line) when $X \sim LG(\alpha_X, \lambda_X)$ and $Y \sim P(IV)(\theta, \alpha_Y, \vartheta, \lambda_Y)$ with $\vartheta = \theta = 1$, $\lambda_X = \lambda_Y = 2$, $\alpha_X = 5$ and $\alpha_Y = 2$.

Taking into account that $m_X(s) + m_Y(s) = s$, both functions can not decrease simultaneously. Therefore, if either $m_X(\cdot)$ and $m_Y(\cdot)$ have a decreasing interval, as the other increases in such interval, it implies that, locally, $m_X(S)$ and $m_Y(S)$ are negatively dependent. This means that, given that S belongs to such interval, $m_X(S)$ tends to take larger values as $m_Y(S)$ takes smaller values and vice-versa. Although, in the scenario considered in this section, even if the conditional expectation given the sum must decrease over some values of the sum according to Proposition 3.3, it is interesting to point out that, globally, $m_X(S)$ and $m_Y(S)$ always remain positively correlated (i.e. their covariance is positive). This is precisely stated next.

Although this paper considers random variables X and Y possessing regularly varying density functions, the next result more generally holds for any positive variables with finite second-order moments.

Property 3.5. *Considering random variables X and Y, if their second order moments are finite, then*
$$\mathrm{Cov}[m_X(S), m_Y(S)] \geq 0. \tag{3.2}$$

Proof. First notice that for any s in the support of $S = X + Y$, we have

$$\mathrm{Var}\left[X|S=s\right] = \mathrm{Var}\left[X-s|S=s\right] = \mathrm{Var}\left[-Y|S=s\right] = \mathrm{Var}\left[Y|S=s\right].$$

This implies that
$$\mathrm{Var}[Y \mid S=s] = \mathrm{Var}[X \mid S=s]. \tag{3.3}$$

Now, we can write

$$0 = \mathrm{Var}[S|S=s] = \mathrm{Var}[X|S=s] + \mathrm{Var}[Y \mid S=s] + 2\mathrm{Cov}[X,Y|S=s].$$

It follows from (3.3) that

$$\mathrm{Cov}[X,Y|S=s] = -\mathrm{Var}[X|S=s]. \tag{3.4}$$

Then,

$$\begin{aligned}
\mathrm{Cov}[X,Y] &= \mathrm{E}\big[\mathrm{Cov}[X,Y \mid S]\big] + \mathrm{Cov}\big[\mathrm{E}[X \mid S], \mathrm{E}[Y \mid S]\big] \\
&= -\mathrm{E}\big[\mathrm{Var}[Y \mid S]\big] + \mathrm{Cov}\big[\mathrm{E}[X \mid S], \mathrm{E}[Y \mid S]\big] \\
&= \mathrm{Var}\big[\mathrm{E}[Y \mid S]\big] - \mathrm{Var}[Y] + \mathrm{Cov}\big[\mathrm{E}[X \mid S], \mathrm{E}[Y \mid S]\big].
\end{aligned}$$

so that we get

$$\mathrm{Cov}\big[\mathrm{E}[X \mid S], \mathrm{E}[Y \mid S]\big] = \mathrm{Cov}[X,Y] + \mathrm{Var}[Y] - \mathrm{Var}\big[\mathrm{E}[Y \mid S]\big]. \tag{3.5}$$

Considering (3.5), Jensen's inequality ensures that $\mathrm{Var}[Y] \geq \mathrm{Var}\big[\mathrm{E}[Y \mid S]\big]$. The announced inequality (3.2) finally follows from $\mathrm{Cov}[X,Y] = 0$. □

Let us remark that, if X and Y possess regularly varying densities with tail indices α_X and α_Y respectively, then the assumption of finite second moments is equivalent to $\min\{\alpha_X, \alpha_Y\} > 3$. We can now briefly comment the result stated in Property 3.5:

- As a direct consequence of (3.3), $\mathrm{Var}[X \mid S] = \mathrm{Var}[Y \mid S]$ a.s.

- We see from (3.4) that $\mathrm{Cov}[X,Y|S=s] \leq 0$ for all s, which could be expected since both variables are positive and the sum is fixed. Therefore, a greater value taken by one of the variables is negatively influencing the second variable. Again, (3.4) implies $\mathrm{Cov}[X,Y|S] = -\mathrm{Var}[X \mid S]$ a.s. Note that considering variables with log-concave densities, X and Y would not only have a negative covariance given S but would also be negatively associated (see Theorem 2.8 in Joag-Dev and Proschan (1983)).

In the signal framework, the positive covariance in (3.2) reflects that the variable X and the noise Y are likely to take large or small values together when the signal S is given. Regarding the risk-sharing scheme, positive dependence means that, wholesomely, participants have common interests as their contributions are likely to be large or small together.

This positive global relationship between $m_X(S)$ and $m_Y(S)$ holds true even if the functions $m_X(\cdot)$ and $m_Y(\cdot)$ are not everywhere increasing. When $m_X(\cdot)$ and $m_Y(\cdot)$ refer to the contributions to the pool, if one of them decreases in terms of the total loss, it will then be at the expense of the other contribution assuming a greater part of such total loss. Inequality (3.2) indicates that, if there are values for which the monotonicity of $m_X(\cdot)$ and $m_Y(\cdot)$ differ, those must be values with small probability of occurrence or with slight differences in the increasing/decreasing rate, as in overall terms the dependence remains positive. Nevertheless, in order to recover information about a variable from a signal or to build risk-sharing schemes, positive covariance is reasonable but it is more relevant to study the monotonicity of $m_X(\cdot)$ and $m_Y(\cdot)$.

3.2 Difference in tail indices less than 1 $(0 \leq \alpha_X - \alpha_Y < 1)$

When the difference between tail indices is less than 1, the conditional expectation given the sum is not bounded as the sum tends to infinity. This is in contrast with the preceding case.

Property 3.6. *If X and Y possess regularly varying densities $f_X(\cdot)$ and $f_Y(\cdot)$ with respective indices α_X and α_Y such that $\alpha_X > 2$ and $\alpha_Y \leq \alpha_X < \alpha_Y + 1$, then $m_X(s) \to \infty$ as $s \to \infty$.*

Proof. Since $f_X(\cdot)$ is regularly varying with index $\alpha_X > 2$, $f_{\widetilde{X}}(\cdot)$ is regularly varying with index $\alpha_{\widetilde{X}} = \alpha_X - 1 < \alpha_Y$. Because $f_Y(\cdot)$ is a regularly varying densities, it implies that $f_Y(s) = o\big(f_{\widetilde{X}}(s)\big)$. Since $\alpha_{\widetilde{X}} < \alpha_X$, it also follows that $f_X(s) = o\big(f_{\widetilde{X}}(s)\big)$. Therefore, by (2.1):

$$m_X(s) = \mathrm{E}[X]\frac{f_{\widetilde{X}+Y}(s)}{f_{X+Y}(s)} \sim \mathrm{E}[X]\frac{f_{\widetilde{X}}(s) + f_Y(s)}{f_X(s) + f_Y(s)} = \mathrm{E}[X]\frac{f_{\widetilde{X}}(s) + o\big(f_{\widetilde{X}}(s)\big)}{o\big(f_{\widetilde{X}}(s)\big) + o\big(f_{\widetilde{X}}(s)\big)}.$$

From this expression we can see that, as $s \to \infty$, $m_X(s) \to \infty$ and the assertion follows. □

From the previous result, we know that $m_X(\cdot)$ diverges as $s \to \infty$. Although, this is not sufficient to ensure that it is an increasing function. A variety of different behaviors may occur when the difference in tail indices is less than 1. Example 3.7 shows that $m_X(\cdot)$ can be increasing over $(0, \infty)$ while Example 3.8 shows that it may be decreasing over a range of central values.

Example 3.7. *Let us consider two independent random variables $X \sim P(II)(\theta, \alpha_X, \vartheta)$ and $Y \sim P(II)(\theta, \alpha_Y, \vartheta)$ such that $\alpha_X = 4.5$, $\alpha_Y = 4$, $\theta = 1$ and $\vartheta = 0$. Figure 2 shows that $m_X(\cdot)$ increases over $(0, \infty)$.*

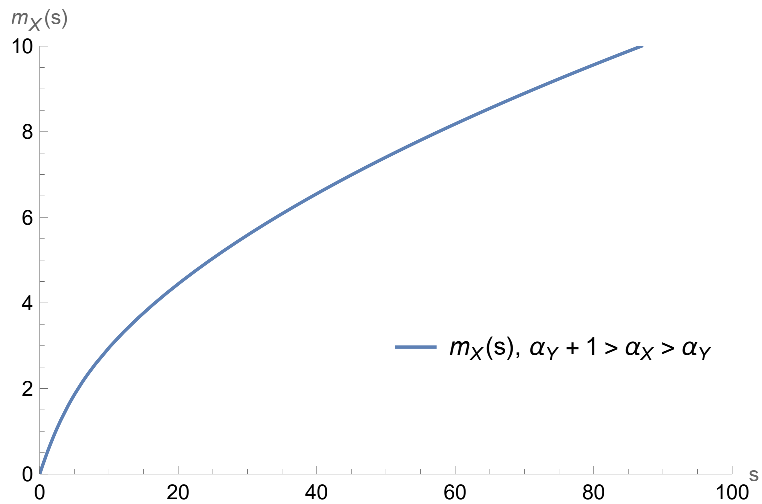


Figure 2: Conditional expectation $m_X(\cdot)$ when $X \sim P(II)(\theta, \alpha_X, \vartheta)$ and $Y \sim P(II)(\theta, \alpha_Y, \vartheta)$ with $\alpha_X = 4.5$, $\alpha_Y = 4$, $\theta = 1$ and $\vartheta = 0$.

Example 3.8. *Let us consider two independent random variables $X \sim P(I)(\alpha_X, \theta)$ and $Y \sim LG(\alpha_Y, \lambda)$ with $\alpha_Y + 1 > \alpha_X > \alpha_Y$. Figure 3 shows that the conditional expectation given the sum decreases over a range of central values and after it increases.*

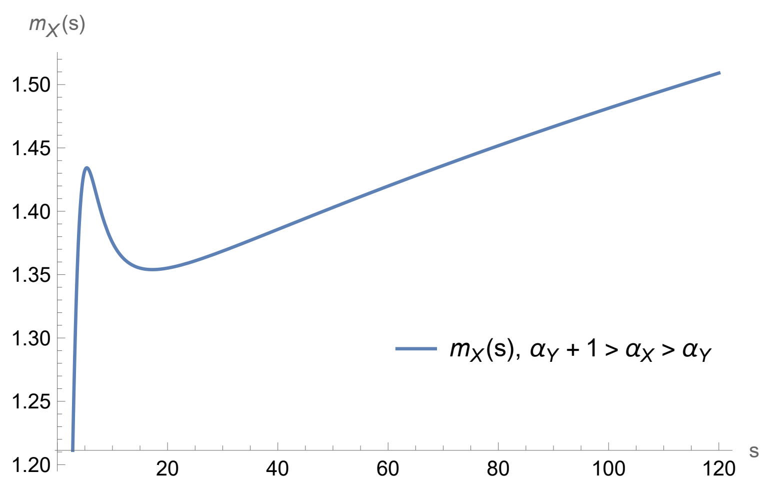


Figure 3: Conditional expectation $m_X(\cdot)$ when $X \sim P(I)(\alpha_X, \theta)$ and $Y \sim LG(\alpha_Y, \lambda)$ with $\alpha_X = 8$, $\alpha_Y = 7.8$, $\lambda = 2.5$ and $\theta = 1$.

3.3 Boundary case: Difference in tail indices equal to 1 ($\alpha_X - \alpha_Y = 1$)

The next result shows that when the difference between tail indices is exactly 1, the asymptotic behavior of the conditional expectation given the sum depends on the asymptotic behavior of the ratio between the slowly varying functions associated to the densities.

Property 3.9. *Let X and Y possess regularly varying densities $f_X(\cdot)$ and $f_Y(\cdot)$ with respective indices α_X and α_Y verifying $\alpha_X = \alpha_Y + 1$ and respective associated slowly varying functions $L_X(\cdot)$ and $L_Y(\cdot)$.*

- *If $\frac{L_X(s)}{L_Y(s)} \to \infty$ as s tends to infinity, then $m_X(s) \to \infty$.*
- *If $\lim_{s\to\infty} \frac{L_X(s)}{L_Y(s)} = k$ with $k \geq 0$, then $\lim_{s\to\infty} m_X(s) = \mathrm{E}[X] + k$.*

Proof. Since f_X is regularly varying with index $\alpha_X > 3$, then $f_{\widetilde{X}}(\cdot)$ is regularly varying with index $\alpha_{\widetilde{X}} = \alpha_X - 1 = \alpha_Y$. As a consequence of (2.1), taking into account that $f_X(\cdot)$ and $f_Y(\cdot)$ are regularly varying densities, $f_X(s) = o\left(f_Y(s)\right)$ and $f_{\widetilde{X}}(s) = \frac{s}{\mathrm{E}[X]} f_X(s)$. Then

$$m_X(s) = \mathrm{E}[X]\frac{f_{\widetilde{X}+Y}(s)}{f_{X+Y}(s)} \sim \mathrm{E}[X]\frac{f_{\widetilde{X}}(s) + f_Y(s)}{f_Y(s) + o(f_Y(s))} = \frac{L_X(s) + \mathrm{E}[X]L_Y(s)}{L_Y(s) + o(L_Y(s))} \sim \frac{L_X(s)}{L_Y(s)} + \mathrm{E}[X]. \tag{3.6}$$

Hence, if $\frac{L_X(s)}{L_Y(s)} \to \infty$ as s tends to infinity, it is direct that $m_X(s)$ diverges. On the other hand, if $\lim_{s\to\infty} \frac{L_X(s)}{L_Y(s)} = k$ with $k \geq 0$, from (3.6) then $\lim_{s\to\infty} m_X(s) = \mathrm{E}[X] + k$. □

The following example illustrates the different cases of Property 3.9.

Example 3.10 (Log-Gamma)**.** *Consider two independent random variables $X \sim LG(\alpha_X, \lambda_X)$ and $Y \sim LG(\alpha_Y, \lambda_Y)$ with $\alpha_X = \alpha_Y + 1$. Here,*

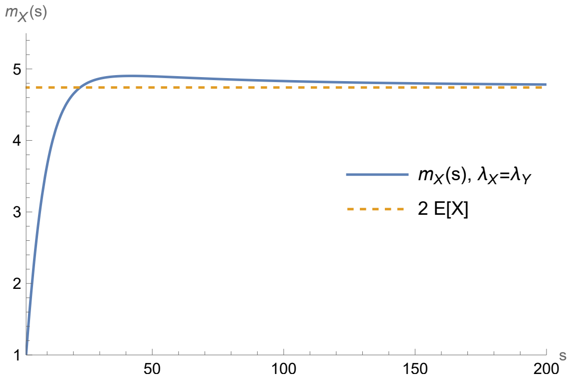
$$f_X(x) = \frac{(\alpha_X - 1)^{\lambda_X}}{\Gamma(\lambda_X)}(\log x)^{\lambda_X - 1} x^{-\alpha_X}, \quad x \geq 1.$$

Then $L_X(x) = \frac{(\alpha_X-1)^{\lambda_X}}{\Gamma(\lambda_X)}(\log x)^{\lambda_X-1}$ and,

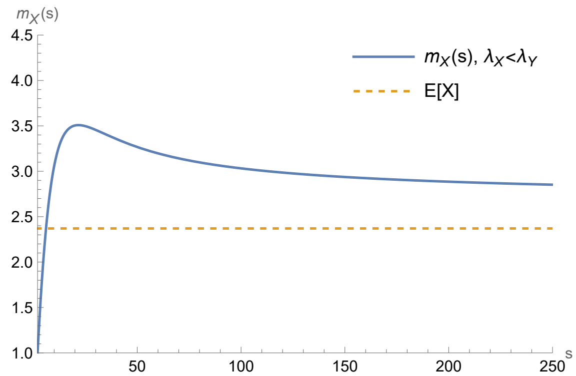
$$\frac{L_X(x)}{L_Y(x)} = \frac{(\alpha_X - 1)^{\lambda_X}}{(\alpha_X - 2)^{\lambda_Y}}\frac{\Gamma(\lambda_Y)}{\Gamma(\lambda_X)}(\log x)^{\lambda_X - \lambda_Y}.$$

Hence, Figure 4 illustrates that, in accordance with Property 3.9:

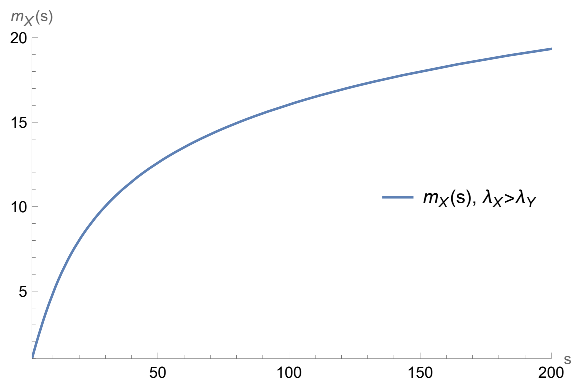
- *If $\lambda_X < \lambda_Y$, the latter ratio converges to zero and $\lim_{s\to\infty} m_X(s) = \mathrm{E}[X] = \left(\frac{\alpha_X-1}{\alpha_X-2}\right)^{\lambda_X}$;*
- *If $\lambda_X = \lambda_Y$, then $\frac{L_X(x)}{L_Y(x)} = \mathrm{E}[X]$ and, therefore $\lim_{s\to\infty} m_X(s) = 2\mathrm{E}[X]$;*
- *If $\lambda_X > \lambda_Y$, the ratio $\frac{L_X(x)}{L_Y(x)}$ diverges and $m_X(s) \to \infty$ as $s \to \infty$.*



(a) Case $\lambda_Y = 3$



(b) Case $\lambda_Y = 4$



(c) Case $\lambda_Y = 2$

Figure 4: Conditional expectation $m_X(\cdot)$ when $X \sim LG(\alpha_X, \lambda_X)$ and $Y \sim LG(\alpha_Y, \lambda_Y)$ with $\alpha_X = \alpha_Y + 1$ with $\alpha_X = 5$, $\alpha_Y = 4$, $\lambda_X = 3$ and considering different values of λ_Y in each case.

Note that the case where the difference between the tail indices is exactly 1 and the constant considered in Property 3.9 is $c = 0$, similarly as in Property 3.3, we can ensure that there exists a non-empty interval in $(0, \infty)$ where $m_X(\cdot)$ is decreasing. Although, contrarily to this case or to the case where the difference in tail indices exceeds 1, there is no guarantee that there exists a non-empty interval of $(0, \infty)$ where $m_X(\cdot)$ is decreasing when tail indices differ by 1 and $c > 0$. Nevertheless, when the ratio among the slowly varying functions associated to the densities is finite, since the limit is finite, $m_X(\cdot)$ is necessarily bounded. Under this framework, we can now face a variety of different behaviors for $m_X(\cdot)$, as illustrated in Example 3.11. In the first situation considered there, the conditional expectation given the sum is monotonically increasing over $(0, \infty)$. However, in the second situation, the conditional expectation given the sum peaks before decreasing to tend to its limit from above.

Example 3.11. *Let us consider two independent random variables $X \sim P(I)(\theta, \alpha_X)$ and $Y \sim P(I)(\theta, \alpha_Y)$ such that $\alpha_X = \alpha_Y + 1$. Then, $\widetilde{X} \sim P(I)(\theta, \alpha_X - 1)$ so that $\widetilde{X} \overset{\mathrm{d}}{=} Y$, where $\overset{\mathrm{d}}{=}$ stands for "equally distributed", and $c = 1$ in Property 3.9. Furthermore, we deduce from Property 3.9 that $\lim_{s\to\infty} m_X(s) = 2\mathrm{E}[X]$. However, the behavior of $m_X(\cdot)$ differs according to the value of α_Y, as shown in the two following situations:*

1. *Figure 5 shows that there are situations where* $m_X(\cdot)$ *increases over* $(0,\infty)$ *and tends to its limit from below.*

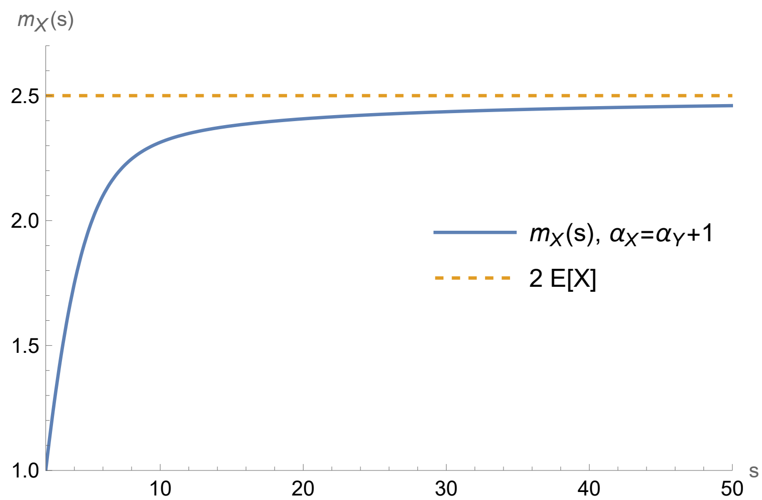


Figure 5: Conditional expectation $m_X(\cdot)$ (blue solid line) and horizontal line at $2\mathrm{E}[X]$ (orange dashed line) when $X \sim P(I)(\theta, \alpha_X)$ and $Y \sim P(I)(\theta, \alpha_Y)$ with $\theta = 1$, $\alpha_X = 6$ and $\alpha_Y = 5$.

2. *Figure 6 shows that there are situations where* $m_X(\cdot)$ *attains a maximum before decreasing to its limit. Specifically, we can see there that, considering the values* $\theta = 1$, $\alpha_X = 3.5$ *and* $\alpha_Y = 2.5$, $m_X(\cdot)$ *increases over* $(0, 64.46)$ *where* $64.46 \simeq F^{-1}(0.998)$ *and then decreases over* $(64.46, \infty)$, *tending to its limit from above.*

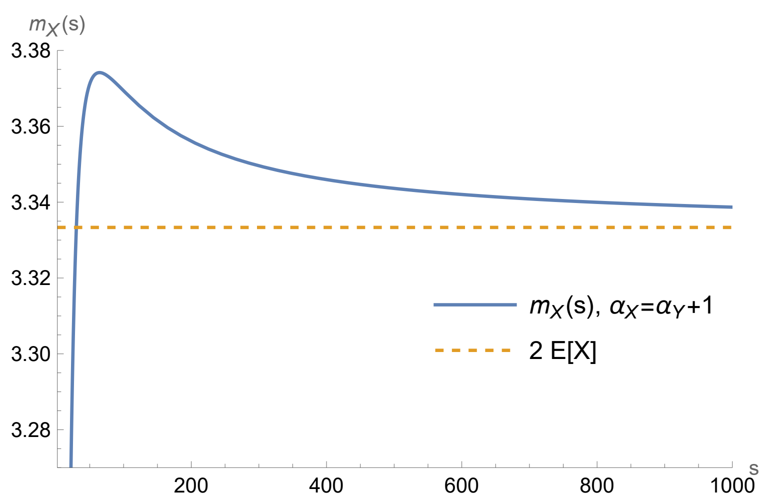


Figure 6: Conditional expectation $m_X(\cdot)$ (blue solid line) and horizontal line at $2\mathrm{E}[X]$ (orange dashed line) when $X \sim P(I)(\theta, \alpha_X)$ and $Y \sim P(I)(\theta, \alpha_Y)$ with $\theta = 1$, $\alpha_X = 3.5$ and $\alpha_Y = 2.5$.

From this section we can conclude that, if the difference between the indices exceeds one, $m_X(\cdot)$ will decrease in an interval; if the difference is less than one, $m_X(\cdot)$ diverges; and in the boundary case, if the difference is exactly one, it can be either bounded or not, depending on the asymptotic behavior of the ratio between the associated slowly varying functions. In the following section, in addition to the asymptotic levels, we aim to delve into the rate at which $m_X(\cdot)$ is either converging or diverging as the argument gets larger.

4 Asymptotic behavior of the conditional mean given the sum

The asymptotic levels of the conditional expectation given the sum were discussed in the preceding section. In this section, we supplement these results with the asymptotic behavior of $m_X(\cdot)$ and illustrate the accuracy of the resulting approximations with the help of various examples.

4.1 Asymptotic expansion of the conditional mean given the sum

Define the truncated mean functions as

$$\mu_X(t) = \int_0^t x f_X(x)\,\mathrm{d}x \text{ and } \mu_Y(t) = \int_0^t y f_Y(y)\,\mathrm{d}y. \tag{4.1}$$

If the expectation is finite (i.e. $\alpha_X > 2$ and $\alpha_Y > 2$), the truncated mean tends to the expected value:

$$\lim_{t\to\infty} \mu_X(t) = \mu_X = \mathrm{E}[X] < \infty \text{ and } \lim_{t\to\infty} \mu_Y(t) = \mu_Y = \mathrm{E}[Y].$$

Henceforth, we will denote as μ_Z the expected value of a variable Z. The following result provides us with an asymptotic expansion of the density function of the sum $X+Y$ when X and Y both possess asymptotically smooth densities. This is an extension of Theorem 1.1 in Bingham et al. (2006).

Proposition 4.1. *Let X and Y be positive random variables with asymptotically smooth density functions $f_X(\cdot)$ and $f_Y(\cdot)$ with tail indices α_X and α_Y such that* $\min\{\alpha_X, \alpha_Y\} \geq 2$. *Then, as s tends to infinity,*

$$f_{X+Y}(s) = f_X(s)\left(1 + \alpha_X \frac{\mu_Y(s)}{s}(1 + o(1))\right) + f_Y(s)\left(1 + \alpha_Y \frac{\mu_X(s)}{s}(1 + o(1))\right).$$

In particular, if $\min\{\alpha_X, \alpha_Y\} > 2$*,*

$$f_{X+Y}(s) = f_X(s)\left(1 + \alpha_X \frac{\mu_Y}{s}(1 + o(1))\right) + f_Y(s)\left(1 + \alpha_Y \frac{\mu_X}{s}(1 + o(1))\right).$$

Proof. The density function of $X+Y$ can be written as

$$f_{X+Y}(t) = \int_0^t f_X(t-x) f_Y(x)\,\mathrm{d}x = T_Y f_X(t) + T_X f_Y(t)$$

where
$$T_Y f_X\left(t\right) = \int_0^{t/2} f_X\left(t-x\right) f_Y\left(x\right) \mathrm{d}x.$$
The announced result then follows from the expressions for $T_Y f_X\left(t\right)$ and $T_X f_Y\left(t\right)$ derived in Property B.1 in Appendix B. □

An asymptotic expansion for the conditional mean given the sum is then deduced from this result.

Proposition 4.2. *Let X and Y be positive random variables with asymptotically smooth density functions $f_X(\cdot)$ and $f_Y(\cdot)$ with tail indices α_X and α_Y such that $\min\{\alpha_X - 1, \alpha_Y\} > 2$. Then, as s tends to infinity,*
$$m_X(s) = \frac{f_X\left(s\right)\left(s + (\alpha_X - 1)\mu_Y\left(1+o\left(1\right)\right)\right) + \mu_X f_Y\left(s\right)\left(1 + \frac{\alpha_Y \mu_{\widetilde{X}}}{s}\left(1+o\left(1\right)\right)\right)}{f_X\left(s\right)\left(1+\frac{\alpha_X\mu_Y}{s}\left(1+o\left(1\right)\right)\right) + f_Y\left(s\right)\left(1+\frac{\alpha_Y\mu_X}{s}\left(1+o\left(1\right)\right)\right)}. \tag{4.2}$$

Proof. Representation (2.2) shows that $m_X(\cdot)$ can be expressed with the help of the ratio of the densities of $\widetilde{X} + Y$ of $X + Y$. Applying Proposition 4.1 to these densities, and taking into account that $\alpha_{\widetilde{X}} = \alpha_X - 1 > 2$, we get
$$\begin{aligned} f_{\widetilde{X}+Y}\left(s\right) &= f_{\widetilde{X}}\left(s\right) + f_Y\left(s\right) + \left(\alpha_{\widetilde{X}}\frac{f_{\widetilde{X}}\left(s\right)}{s}\mu_Y + \alpha_Y\frac{f_Y\left(s\right)}{s}\mu_{\widetilde{X}}\right)\left(1+o\left(1\right)\right) \\ &= \frac{sf_X\left(s\right)}{\mu_X} + f_Y\left(s\right) + \left((\alpha_X - 1)\frac{f_X\left(s\right)}{\mu_X}\mu_Y + \alpha_Y\frac{f_Y\left(s\right)}{s}\mu_{\widetilde{X}}\right)\left(1+o\left(1\right)\right) \end{aligned}$$
and
$$f_{X+Y}\left(s\right) = f_X\left(s\right) + f_Y\left(s\right) + \left(\alpha_X\frac{f_X\left(s\right)}{s}\mu_Y + \alpha_Y\frac{f_Y\left(s\right)}{s}\mu_X\right)\left(1+o\left(1\right)\right)$$
as s tends to infinity. The announced result then follows from (2.2). □

Proposition 4.2 allows us to study the asymptotic behavior of the conditional expectation given the sum according to the values of the tail indices. In the following sections, we will study it under different frameworks and we assume again that $\alpha_X \geq \alpha_Y$. Note that, if $\alpha_Y > \alpha_X$, the asymptotic expression of $m_X(\cdot)$ can be obtained by computing the asymptotic expression of $m_Y(\cdot)$ and noting that $m_X(s) = s - m_Y(s)$. Although, it must be remarked that, in order to obtain the asymptotic expression of $m_X\left(\cdot\right)$, it is required that $\alpha_X > 3$ and $\alpha_Y > 2$. Hence, if computing the asymptotic expression of $m_Y\left(\cdot\right)$, we need to assume $\alpha_Y > 3$ and $\alpha_X > 2$.

Based on Proposition 4.2, we derive different types of asymptotic behaviors of the conditional mean of X given the sum, as detailed in the next sections. We consider two cases where $m_X\left(\cdot\right)$ converges to the unconditional expected value: when the difference of the indices is greater than two and when it is in between one and two. In addition, we compare the asymptotic behavior of $m_X\left(\cdot\right)$ as it diverges, by deriving the asymptotic expansion when indices are "near" (their difference is positive but smaller than one) and when both variables have the same index. Figure 7 illustrates the different cases considered in the next sections within the plane (α_X, α_Y). As it is common to the different cases, throughout the next sections, we will assume $\alpha_X > 3$ and $\alpha_Y > 2$. In the following sections, we will denote by $\widehat{m}_X(\cdot)$ an approximation of $m_X(\cdot)$ at infinity.

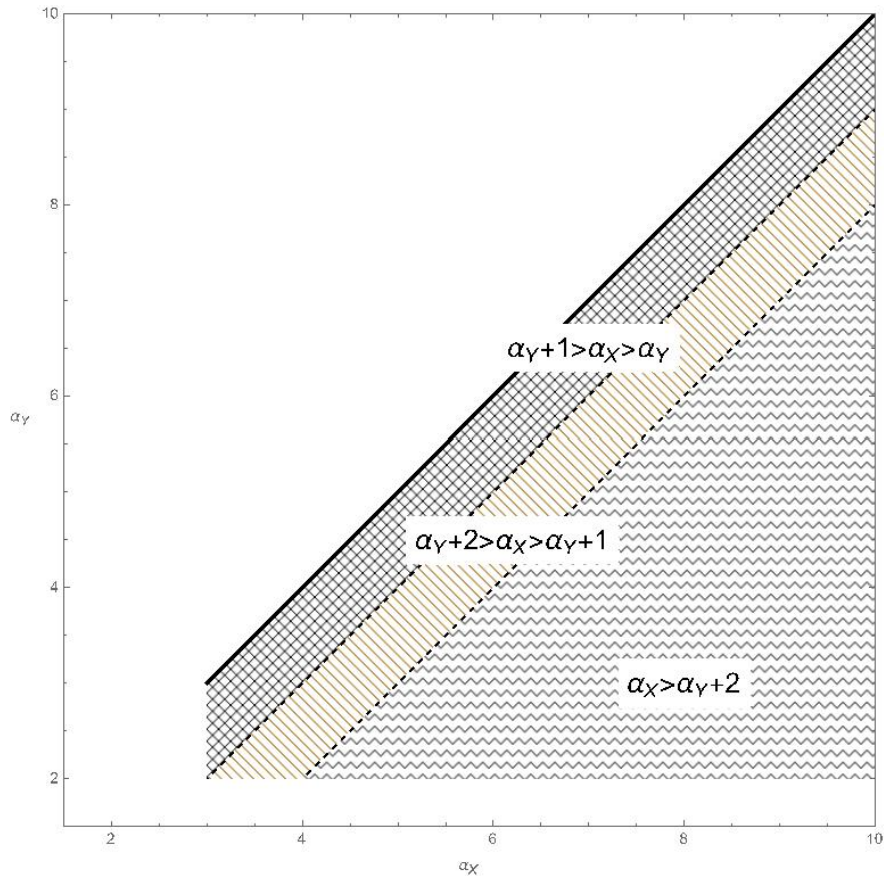


Figure 7: Discussion according to the position of (α_X, α_Y) in $(3, \infty) \times (2, \infty)$ with $\alpha_X \geq \alpha_Y$.

4.2 Difference in the tail indices larger than 2 ($\alpha_X - \alpha_Y > 2$)

From Proposition 4.2, we can write $m_X(\cdot)$ as the ratio of two asymptotic expansions, each considering four terms. Note that, assuming $\alpha_X - \alpha_Y > 1$, the terms of the denominator are regularly varying with arranged indices $\{\alpha_Y, \alpha_Y + 1, \alpha_X, \alpha_X + 1\}$. Therefore, α_Y and $\alpha_Y + 1$ are the two smallest indices (i.e. the two greatest orders). With respect to the numerator, the (non-arranged) indices of the terms are $\{\alpha_X - 1, \alpha_X, \alpha_Y, \alpha_Y + 1\}$. Under the assumption $\alpha_X - \alpha_Y > 1$, the smallest index is α_Y. However, if $\alpha_X - \alpha_Y > 2$, the second smallest index is $\alpha_Y + 1$ and, if $\alpha_X - \alpha_Y < 2$, it is $\alpha_X - 1$. This motivates to study the behavior of $m_X(\cdot)$ separately if the difference between indices trespasses two.

Property 4.3. *Let X and Y be positive random variables possessing asymptotically smooth density functions $f_X(\cdot)$ and $f_Y(\cdot)$ with respective tail indices α_X and α_Y. If $\alpha_X > \alpha_Y + 2$, then*

$$m_X(s) = \mu_X + \alpha_Y \frac{1}{s} \mathrm{Var}[X] \left(1 + o\left(1\right)\right)$$

as s tends to infinity.

Proof. Notice that
$$\mathrm{Var}[X] = \mu_{\widetilde{X}}\mu_X - \mu_X^2.$$
Since $\min\{\alpha_X - 1, \alpha_Y\} > 2$, (4.2) allows us to write

$$m_X(s) = \frac{sf_X(s) + \mu_X f_Y(s) + \left((\alpha_X - 1)f_X(s)\,\mu_Y + \alpha_Y \frac{f_Y(s)}{s}\left(\mathrm{Var}[X] + \mu_X^2\right)\right)(1 + o(1))}{f_X(s) + f_Y(s) + \left(\alpha_X \frac{f_X(s)}{s}\mu_Y + \alpha_Y \frac{f_Y(s)}{s}\mu_X\right)(1 + o(1))}$$

as s tends to infinity. As $\alpha_Y > 2$ and $\alpha_X > \alpha_Y + 2$, we have that $f_X(s) = o(\frac{f_Y(s)}{s^2})$ and we can write

$$\begin{aligned} m_X(s) &= \mu_X \frac{f_Y(s) + \alpha_Y \frac{f_Y(s)}{s}(\mu_X + \frac{\mathrm{Var}[X]}{\mu_X})(1 + o(1))}{f_Y(s) + \alpha_Y \frac{f_Y(s)}{s}\mu_X(1 + o(1))} \\ &= \mu_X \frac{1 + \alpha_Y \frac{1}{s}(\mu_X + \frac{\mathrm{Var}[X]}{\mu_X})(1 + o(1))}{1 + \alpha_Y \frac{1}{s}\mu_X(1 + o(1))} \\ &= \mu_X \left(1 + \alpha_Y \frac{1}{s}\frac{\mathrm{Var}[X]}{\mu_X}(1 + o(1))\right). \end{aligned}$$

This ends the proof. □

The following example illustrates Property 4.3.

Example 4.4. *Let us consider two independent random variables $X \sim P(II)(\theta, \alpha_X, \vartheta)$ and $Y \sim P(II)(\theta, \alpha_Y, \vartheta)$ such that $\alpha_X = 10$, $\alpha_Y = 5$, $\theta = 1$ and $\vartheta = 0$. Figure 8 shows the same asymptotic behavior of $m_X(\cdot)$ and $\widehat{m}_X(s) = \mu_X\left(1 + \alpha_Y \frac{1}{s}\frac{\mathrm{Var}[X]}{\mu_X}\right)$.*

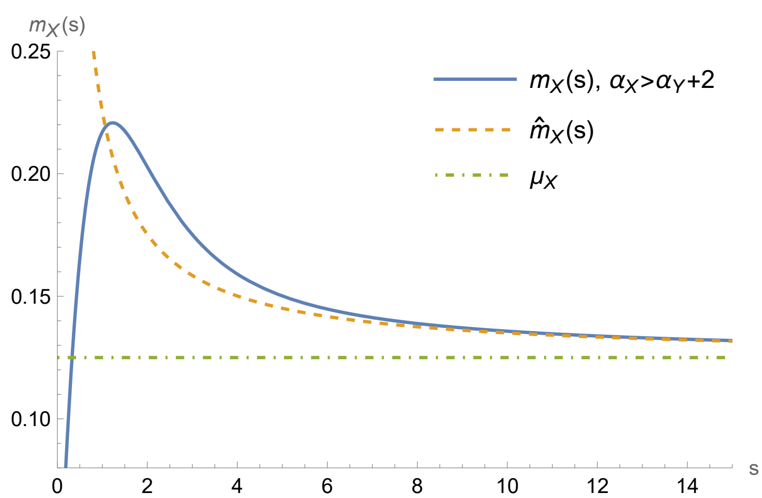


Figure 8: Conditional expectation $m_X(\cdot)$ when $X \sim P(II)(\theta, \alpha_X, \vartheta)$ and $Y \sim P(II)(\theta, \alpha_Y, \vartheta)$ with $\alpha_X = 10$, $\alpha_Y = 5$, $\theta = 1$ and $\vartheta = 0$.

4.3 Difference in the tail indices between 1 and 2 ($1 < \alpha_X - \alpha_Y < 2$)

Property 4.5. *Let X and Y be positive random variables possessing asymptotically smooth density functions $f_X(\cdot)$ and $f_Y(\cdot)$ with respective tail indices α_X and α_Y. If $\alpha_Y + 2 > \alpha_X > \alpha_Y + 1$, then*

$$m_X(s) = \mu_X + s\frac{f_X(s)}{f_Y(s)}(1 + o(1))$$

as s tends to infinity.

Proof. Since $\alpha_Y > 2$ and $\alpha_Y + 2 > \alpha_X > \alpha_Y + 1$, we have that $f_X(s) = o\left(\frac{f_Y(s)}{s}\right)$ and $\frac{f_Y(s)}{s} = o(sf_X(s))$. Hence, we can write

$$\begin{aligned} m_X(s) &= \frac{\mu_X f_Y(s) + sf_X(s)(1 + o(1))}{f_Y(s) + \alpha_Y \frac{f_Y(s)}{s}\mu_X(1 + o(1))} \\ &= \mu_X + s\frac{f_X(s)}{f_Y(s)}(1 + o(1)). \end{aligned}$$

This ends the proof. □

The following example illustrates Property 4.5

Example 4.6. *Let us consider two independent random variables $X \sim Davis(\alpha_X, b, \vartheta)$ and $Y \sim Davis(\alpha_Y, b, \vartheta)$ such that $\alpha_X = 6$, $\alpha_Y = 4.5$, $b = 2$ and $\vartheta = 0$. Figure 9 shows the same asymptotic behavior of $m_X(\cdot)$ and $\widehat{m}_X(s) = \mu_X + s\frac{f_X(s)}{f_Y(s)}$.*

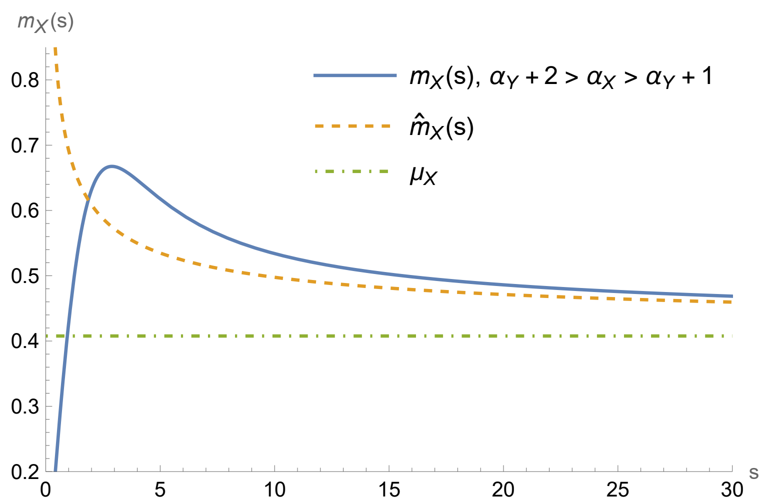


Figure 9: Conditional expectation $m_X(\cdot)$ when $X \sim Davis(\alpha_X, b, \vartheta)$ and $Y \sim Davis(\alpha_Y, b, \vartheta)$ with $\alpha_X = 6$, $\alpha_Y = 4.5$, $b = 2$ and $\vartheta = 0$.

From Proposition 4.3, if $\alpha_X - \alpha_Y > 2$ the asymptotic behavior of $m_X(\cdot)$ can be approximated by $\widehat{m}_X(s) = \mu_X + \alpha_Y \frac{1}{s}\mathrm{Var}[X]$, where the second term behaves proportionally to $1/s$. However, if $1 < \alpha_X - \alpha_Y < 2$, by Proposition 4.5, asymptotically, $m_X(\cdot)$ can be approximated by $\widehat{m}_X(s) = \mu_X + s\frac{f_X(s)}{f_Y(s)}$, whose second term is $s^{-\alpha_X+1+\alpha_Y}\frac{L_X(s)}{L_Y(s)}$. Note that $-\alpha_X + 1 + \alpha_Y > -1$. Hence, as we could expect, if $\alpha_X - \alpha_Y > 2$, asymptotically, $m_X(\cdot)$ decreases faster as it does if $1 < \alpha_X - \alpha_Y < 2$.

4.4 Difference in the tail indices less than 1 ($0 < \alpha_X - \alpha_Y < 1$)

When the indices are similar, and by similar we mean here that they do not differ by more than one unit, the following result provides the asymptotic expression of $m_X(\cdot)$, where the second and third terms vary depending on how similar are the indices. Note that the expressions are arranged in terms of their order, so, even if the terms considered are the same in the two first cases, they are given separately to emphasize that the second-order term differs in the two cases.

Property 4.7. *Let X and Y be positive random variables possessing asymptotically smooth density functions $f_X(\cdot)$ and $f_Y(\cdot)$ with respective tail indices α_X and α_Y. If $\alpha_Y+1 > \alpha_X > \alpha_Y$ then, as s tends to infinity:*

(a) if $1 > \alpha_X - \alpha_Y > \frac{1}{2}$, then:

$$m_X(s) = s\frac{f_X(s)}{f_Y(s)}\left(1 + \mu_X\frac{f_Y(s)}{sf_X(s)} - \frac{f_X(s)}{f_Y(s)}(1+o(1))\right);$$

(b) if $\frac{1}{2} > \alpha_X - \alpha_Y > \frac{1}{3}$, then:

$$m_X(s) = s\frac{f_X(s)}{f_Y(s)}\left(1 - \frac{f_X(s)}{f_Y(s)} + \mu_X\frac{f_Y(s)}{sf_X(s)}(1+o(1))\right);$$

(c) if $\frac{1}{3} > \alpha_X - \alpha_Y > 0$, then:

$$m_X(s) = s\frac{f_X(s)}{f_Y(s)}\left(1 - \frac{f_X(s)}{f_Y(s)} + \left(\frac{f_X(s)}{f_Y(s)}\right)^2(1+o(1))\right)$$

Proof. If $\alpha_Y + 1 > \alpha_X > \alpha_Y$, since $\alpha_X > 3$ and $\alpha_Y > 2$, then, as s tends to infinity:

$$\begin{aligned} m_X(s) &= \frac{f_X(s)\big(s+(\alpha_X-1)\mu_Y(1+o(1))\big)+\mu_X f_Y(s)\big(1+\alpha_Y\mu_{\widetilde{X}}s^{-1}(1+o(1))\big)}{f_X(s)\big(1+\alpha_X\mu_Y s^{-1}(1+o(1))\big)+f_Y(s)\big(1+\alpha_Y\mu_X s^{-1}(1+o(1))\big)} \\ &= s\frac{f_X(s)}{f_Y(s)}\frac{1+\mu_X\frac{f_Y(s)}{sf_X(s)}+\frac{1}{s}(\alpha_X-1)\mu_Y(1+o(1))+\alpha_Y\mu_{\widetilde{X}}\mu_X\frac{f_Y(s)}{s^2f_X(s)}(1+o(1))}{1+\frac{f_X(s)}{f_Y(s)}+\alpha_Y\mu_X\frac{1}{s}(1+o(1))+\alpha_X\mu_Y\frac{f_X(s)}{sf_Y(s)}(1+o(1))} \\ &= s\frac{f_X(s)}{f_Y(s)}\frac{1+\mu_X\frac{f_Y(s)}{sf_X(s)}+\frac{1}{s}(\alpha_X-1)\mu_Y(1+o(1))}{1+\frac{f_X(s)}{f_Y(s)}+\alpha_Y\mu_X\frac{1}{s}(1+o(1))}. \end{aligned}$$

where the last step holds since we can write $\alpha_Y\mu_{\widetilde{X}}\mu_X\frac{f_Y(s)}{s^2f_X(s)} = \frac{1}{s}o(1)$ and $\alpha_X\mu_Y\frac{f_X(s)}{sf_Y(s)} = \frac{1}{s}o(1)$. The term $\frac{f_X(s)}{f_Y(s)} + \alpha_Y\mu_X\frac{1}{s}$ tends to zero as s tends to infinity and therefore we can write:

$$\frac{1}{1+\frac{f_X(s)}{f_Y(s)}+\alpha_Y\mu_X\frac{1}{s}(1+o(1))} = 1 - \frac{f_X(s)}{f_Y(s)} - \alpha_Y\mu_X\frac{1}{s}(1+o(1)) + \left(\frac{f_X(s)}{f_Y(s)}\right)^2(1+o(1)).$$

Therefore, as s tends to infinity, $m_X(s) = s\frac{f_X(s)}{f_Y(s)}g(s)$ with

$$g(s) = 1-\frac{f_X\left(s\right)}{f_Y\left(s\right)}+\mu_X\frac{f_Y\left(s\right)}{sf_X\left(s\right)}+\left(\frac{f_X\left(s\right)}{f_Y\left(s\right)}\right)^2\left(1+o\left(1\right)\right)+\frac{(\alpha_X-1)\mu_Y-(\alpha_Y-1)\mu_X}{s}\left(1+o\left(1\right)\right).$$

We can now consider the different cases.

(a) If $\alpha_X - \alpha_Y > \frac{1}{2}$, then $\frac{f_X(s)}{f_Y(s)} = o(\mu_X\frac{f_Y(s)}{sf_X(s)})$ and, by noticing that $\left(\frac{f_X(s)}{f_Y(s)}\right)^2 = o(\frac{f_X(s)}{f_Y(s)})$ and $\frac{1}{s} = o(\frac{f_X(s)}{f_Y(s)})$, as s tends to infinity:

$$m_X(s) = s\frac{f_X(s)}{f_Y(s)}\left(1+\mu_X\frac{f_Y(s)}{sf_X(s)}-\frac{f_X(s)}{f_Y(s)}(1+o(1))\right).$$

(b) If $\frac{1}{2} > \alpha_X - \alpha_Y > \frac{1}{3}$, then $\mu_X\frac{f_Y(s)}{sf_X(s)} = o(\frac{f_X(s)}{f_Y(s)})$ and $\left(\frac{f_X(s)}{f_Y(s)}\right)^2 = o(\frac{f_Y(s)}{sf_X(s)})$. Therefore, because $\frac{1}{s} = o(\frac{f_Y(s)}{sf_X(s)})$, as s tends to infinity:

$$m_X(s) = s\frac{f_X(s)}{f_Y(s)}\left(1-\frac{f_X(s)}{f_Y(s)}+\mu_X\frac{f_Y(s)}{sf_X(s)}(1+o(1))\right).$$

(c) If $\frac{1}{3} > \alpha_X - \alpha_Y > 0$, then $\mu_X\frac{f_Y(s)}{sf_X(s)} = o(\frac{f_X(s)}{f_Y(s)})$ and $\frac{f_Y(s)}{sf_X(s)} = o(\left(\frac{f_X(s)}{f_Y(s)}\right)^2)$. Therefore, because $\frac{1}{s} = o(\left(\frac{f_X(s)}{f_Y(s)}\right)^2)$, as s tends to infinity:

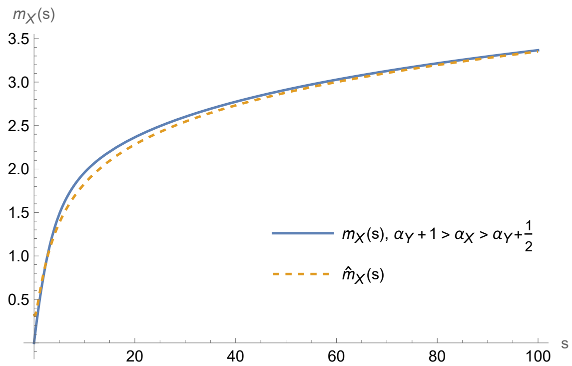
$$m_X(s) = s\frac{f_X(s)}{f_Y(s)}\left(1-\frac{f_X(s)}{f_Y(s)}+\left(\frac{f_X\left(s\right)}{f_Y\left(s\right)}\right)^2(1+o(1))\right).$$

This ends the proof. □

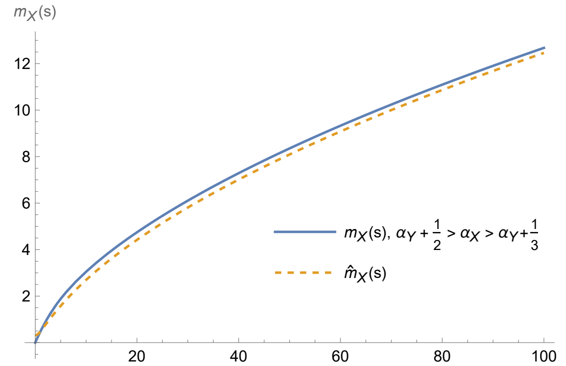
The following example illustrates Property 4.7.

Example 4.8. *Let us consider two independent random variables $X \sim P(II)(\theta, \alpha_X, \vartheta)$ and $Y \sim P(II)(\theta, \alpha_Y, \vartheta)$. Figure 10 shows the same asymptotic behavior of $m_X(\cdot)$ and:*

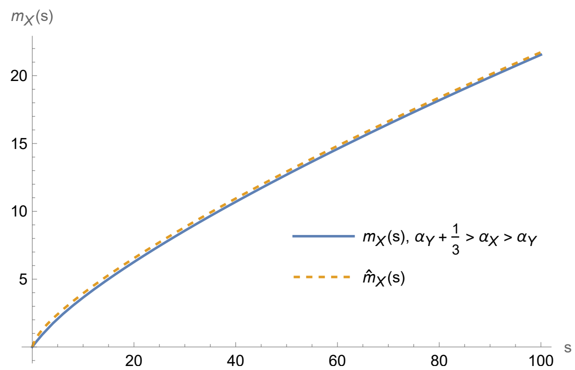
- $\widehat{m}_X(s) = s\frac{f_X(s)}{f_Y(s)}\left(1+\mu_X\frac{f_Y(s)}{sf_X(s)}-\frac{f_X(s)}{f_Y(s)}\right)$ *in cases (a) and (b).*
- $\widehat{m}_X(s) = s\frac{f_X(s)}{f_Y(s)}\left(1-\frac{f_X(s)}{f_Y(s)}+\left(\frac{f_X(s)}{f_Y(s)}\right)^2\right)$ *in case (c).*



(a) Case $\alpha_Y = 4.2$



(b) Case $\alpha_Y = 4.55$



(c) Case $\alpha_Y = 4.7$

Figure 10: Conditional expectation $m_X(\cdot)$ when $X \sim P(II)(\theta, \alpha_X, \vartheta)$ and $Y \sim P(II)(\theta, \alpha_Y, \vartheta)$ with $\alpha_X = 5$, $\theta = 1$ and $\vartheta = 0$ and considering different values of α_Y in each case.

4.5 Equal indices ($\alpha_X = \alpha_Y$)

The following results describe the asymptotic behavior of $m_X(\cdot)$ when both variables have the same index, and under the assumption of asymptotic proportionality between the densities. Let us note that if $X \stackrel{\mathrm{d}}{=} Y$, then $m_X(s) = m_Y(s)$ for all s in the support of S. Hence, as $m_X(s) + m_Y(s) = s$, it is direct to see that, for equally distributed variables, $m_X(s) = \frac{s}{2}$. The following result provides the asymptotic expression of $m_X(\cdot)$ when both densities have the same index and are asymptotically proportional, but not necessarily equal.

Property 4.9. *Let X and Y be positive random variables possessing asymptotically smooth density functions $f_X(\cdot)$ and $f_Y(\cdot)$ with respective tail indices α_X and α_Y such that $\alpha_Y = \alpha_X$ and $f_X(s) = c f_Y(s)\,(1 + o\,(1))$ with $c > 0$. Then*

$$m_X(s) = \frac{c}{1+c} s \left(1 + \frac{1}{s}\left(\frac{\alpha}{1+c}(\mu_Y - \mu_X) + \frac{\mu_X}{c} - \mu_Y\right)(1 + o\,(1))\right) \tag{4.3}$$

as s tends to infinity.

Proof. We consider $\alpha = \alpha_Y = \alpha_X$ and we can write:

$$\begin{aligned}
m_X(s) &= \frac{sf_X\left(s\right) + \mu_X f_Y\left(s\right) + \left(\left(\alpha_X - 1\right)f_X\left(s\right)\mu_Y + \alpha_Y \frac{f_Y(s)}{s}\left(\mathrm{Var}[X] + \mu_X^2\right)\right)\left(1 + o\left(1\right)\right)}{f_X\left(s\right) + f_Y\left(s\right) + \left(\alpha_X \frac{f_X(s)}{s}\mu_Y + \alpha_Y \frac{f_Y(s)}{s}\mu_X\right)\left(1 + o\left(1\right)\right)} \\
&= \frac{c}{1+c} s \frac{f_Y\left(s\right) + \frac{f_Y(s)}{s}\left(\left(\alpha - 1\right)\mu_Y + \frac{\mu_X}{c}\right)\left(1 + o\left(1\right)\right)}{f_Y\left(s\right) + \frac{f_Y(s)}{s}\left(\alpha\mu_Y \frac{c}{1+c} + \mu_X \alpha \frac{1}{1+c}\right)\left(1 + o\left(1\right)\right)} \\
&= \frac{c}{1+c} s \left(1 + \frac{1}{s}\frac{\left(\mu_Y\left(\frac{\alpha}{1+c} - 1\right) + \mu_X\left(\frac{1}{c} - \frac{\alpha}{1+c}\right)\right)\left(1 + o\left(1\right)\right)}{1 + \frac{1}{s}\left(\alpha\mu_Y \frac{c}{1+c} + \mu_X \alpha \frac{1}{1+c}\right)\left(1 + o\left(1\right)\right)}\right) \\
&= \frac{c}{1+c} s \left(1 + \frac{1}{s}\left(\frac{\alpha}{1+c}\left(\mu_Y - \mu_X\right) + \frac{\mu_X}{c} - \mu_Y\right)\left(1 + o\left(1\right)\right)\right).
\end{aligned}$$

This ends the proof. $\square$

If $X \stackrel{\mathrm{d}}{=} Y$, since $c = 1$ and $\mu_X = \mu_Y$, the term $\frac{1}{s}\left(\frac{\alpha}{1+c}\left(\mu_Y - \mu_X\right) + \frac{\mu_X}{c} - \mu_Y\right)$ in (4.3) is zero and the expression provided in Proposition 4.9 corresponds to $m_X(s) = \frac{s}{2}$ as expected.

Example 4.10. *Consider two independent random variables $X \sim P(IV)(\theta_X, \alpha, \vartheta, \lambda_X)$ and $Y \sim P(IV)(\theta_Y, \alpha, \vartheta, \lambda_Y)$ with $\alpha > 3$. Without loss of generality we can assume $\vartheta = 0$. Here, the slowly varying function associated to the density $f_X(\cdot)$ is given by:*

$$L_X(x) = (\alpha - 1)\theta_X^{-\frac{\alpha-1}{\lambda_X}} \frac{x^{\alpha}(x)^{-1+\frac{\alpha-1}{\lambda_X}}}{\left(1 + \left(\frac{x}{\theta_X}\right)^{\frac{\alpha-1}{\lambda_X}}\right)^{\lambda_X+1}}.$$

Note that $\frac{f_X(s)}{f_Y(s)} = \frac{L_X(s)}{L_Y(s)}$. Therefore, we can write $f_X(s) = cf_Y(s)\left(1 + g_{IV}\left(x\right)\right)$, where

$$g_{IV}(x) = x^{-\frac{(\alpha-1)(\lambda_X - \lambda_Y)}{\lambda_X\lambda_Y}} \theta_X^{\frac{1-\alpha}{\lambda_X}} \left(1 + \left(\frac{x}{\theta_X}\right)^{\frac{\alpha-1}{\lambda_X}}\right)^{-(1+\lambda_X)} \left(\frac{\theta_X}{\theta_Y}\right)^{1-\alpha} \theta_Y^{\frac{\alpha-1}{\lambda_Y}} \left(1 + \left(\frac{x}{\theta_Y}\right)^{\frac{\alpha-1}{\lambda_Y}}\right)^{1+\lambda_Y} - 1,$$

verifying $g_{IV}(x) = o(1)$ and $c = \left(\frac{\theta_X}{\theta_Y}\right)^{\alpha-1}$. In particular, if $\lambda = \lambda_X = \lambda_Y$ then

$$g_{IV}(x) = \left(\frac{\theta_Y^{\frac{\alpha-1}{\lambda}} + x^{\frac{\alpha-1}{\lambda}}}{\theta_X^{\frac{\alpha-1}{\lambda}} + x^{\frac{\alpha-1}{\lambda}}}\right)^{(1+\lambda)} - 1.$$

Let us consider two independent random variables $X \sim P(IV)(\theta_X, \alpha, \vartheta, \lambda)$ and $Y \sim P(IV)(\theta_Y, \alpha, \vartheta, \lambda)$ such that $\alpha = 7$, $\theta_X = 1$, $\theta_Y = 2$, $\lambda = 2$ and $\vartheta = 0$. Figure 11 shows the same asymptotic behavior of $m_X(\cdot)$ and

$$\widehat{m}_X(s) = \frac{c}{1+c} s \left(1 + \frac{1}{s}\left(\frac{\alpha}{1+c}\left(\mu_Y - \mu_X\right) + \frac{\mu_X}{c} - \mu_Y\right)\right), \quad c = \frac{1}{64}.$$

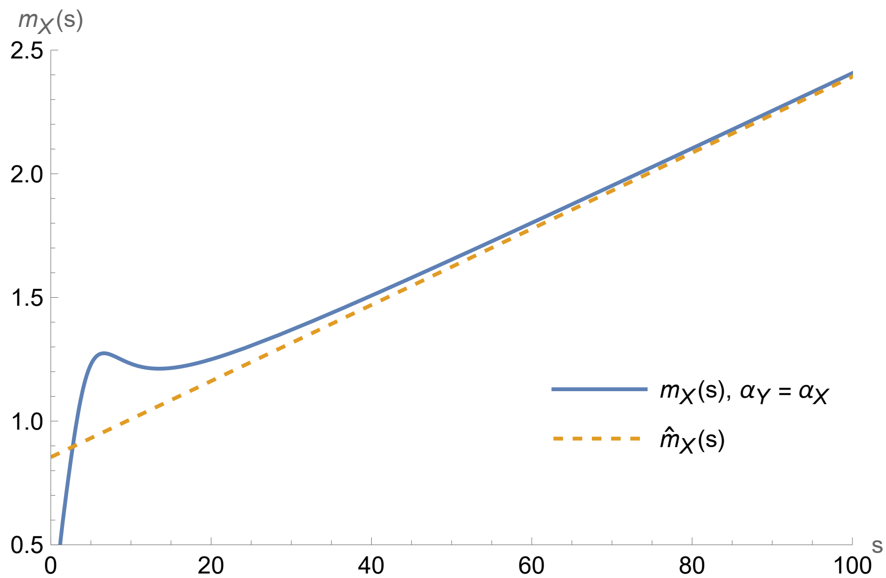


Figure 11: Conditional expectation $m_X(\cdot)$ when $X \sim P(IV)(\theta_X, \alpha, \vartheta, \lambda)$ and $Y \sim P(IV)(\theta_Y, \alpha, \vartheta, \lambda)$ with $\alpha = 7$, $\theta_X = 1$, $\theta_Y = 2$, $\lambda = 2$ and $\vartheta = 0$.

From Proposition 4.9, when the densities have equal tail indices and they are asymptotically proportional, $m_X(\cdot)$ can be approximated by a function where the term which dominates is $\frac{c}{1+c}s$. However, from Proposition 4.7, if $1 > \alpha_X - \alpha_Y > 0$ the asymptotic behavior of $m_X(\cdot)$ can be approximated by a function where the dominant term is $s^{\alpha_Y+1-\alpha_X}\frac{L_X(s)}{L_Y(s)}$, with $1 > \alpha_Y + 1 - \alpha_X > 0$. Hence, if $0 < \alpha_X - \alpha_Y < 1$, asymptotically, $m_X(\cdot)$ increases faster than it does if the indices are equal. And, the nearer the indices, the faster that $m_X(\cdot)$ asymptotically increases.

5 Extensions to zero-augmented random variables

Zero-augmented distributions combine a continuous distribution on the positive real line and a point probability mass at zero. In the context of risk-sharing schemes, zero-augmented distributions are often encountered since the probability mass at zero corresponds to participants who do not not file any claim against the scheme. When considering signals, there is usually a point mass probability at zero due to technical reasons, as the event to measure may be present but its signal is below the limit of detection. For instance, zero-augmented distributions are often used for modeling signals related to environmental factors, as in De Luca et al. (2020), or to biomarkers as in Gleiss et al. (2015). In parametric insurance, a point mass may result from a threshold that has to be exceeded by the index under consideration.

In this section we show that adding a probability mass at 0 to a distribution with regularly varying density, tail indices still play a role in the limit behavior of the conditional mean given the sum. The next result extends Proposition 4.1 and Proposition 4.2 to the zero-augmented case.

Proposition 5.1. *Let X and Y be non-negative random variables such that $X = I_X C_X$ and $Y = I_Y C_Y$ where I_X, C_X, I_Y, and C_Y are independent, I_X and I_Y are Bernoulli distributed* *with mean p_X and p_Y, $0 < p_X, p_Y \leq 1$, and C_X and C_Y possess asymptotically smooth density functions $f_{C_X}(\cdot)$ and $f_{C_Y}(\cdot)$ with tail indices α_{C_X} and α_{C_Y} such that $\min\{\alpha_{C_X}, \alpha_{C_Y}\} > 2$. Then, as s tends to infinity,*

$$\begin{aligned} f_{X+Y}(s) &= p_X f_{C_X}(s)\left(1 + p_Y \alpha_{C_X} \frac{\mu_{C_Y}}{s}(1+o(1))\right) \\ &\quad + p_Y f_{C_Y}(s)\left(1 + p_X \alpha_{C_Y} \frac{\mu_{C_X}}{s}(1+o(1))\right). \end{aligned}$$

In addition, if $\min\{\alpha_{C_X}, \alpha_{C_Y}\} > 3$, $\mu_{\widetilde{C}_X}, \mu_{\widetilde{C}_Y}$ are finite and the conditional mean given the sum satisfies that, as s tends to ∞,

$$m_X(s) = \frac{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(1 + \alpha_Y \frac{\mu_{\widetilde{C}_X}}{s}(1+o(1))\right) + \frac{f_{C_X}(s)}{\mu_{C_X}}\left(\frac{s}{p_Y \mu_{C_Y}} + (\alpha_X - 1)(1+o(1))\right)}{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(\frac{1}{p_X \mu_{C_X}} + \frac{\alpha_Y}{s}(1+o(1))\right) + \frac{f_{C_X}(s)}{\mu_{C_X}}\left(\frac{1}{p_Y \mu_{C_Y}} + \frac{\alpha_X}{s}(1+o(1))\right)} \qquad (5.1)$$

Proof. As the sum of two random variables obeying zero-augmented distributions, $X+Y$ also obeys a zero-augmented distribution with probability mass $(1-p_X)(1-p_Y)$ at 0. For $s > 0$, the density function of $X+Y$ can be written as

$$f_{X+Y}(s) = p_X(1-p_Y) f_{C_X}(s) + (1-p_X) p_Y f_{C_Y}(s) + p_X p_Y f_{C_X + C_Y}(s).$$

Now, Proposition 4.1 applies to $f_{C_X+C_Y}(\cdot)$ so that

$$\begin{aligned} f_{X+Y}(s) &= p_X(1-p_Y) f_{C_X}(s) + (1-p_X) p_Y f_{C_Y}(s) \\ &\quad + p_X p_Y \left(f_{C_X}(s)\left(1 + \alpha_{C_X} \frac{\mu_{C_Y}(s)}{s}(1+o(1))\right) \right. \\ &\quad \left. + f_{C_Y}(s)\left(1 + \alpha_{C_Y} \frac{\mu_{C_X}(s)}{s}(1+o(1))\right)\right), \end{aligned}$$

which ends the first part of the proof.

In order to obtain the expression of the conditional mean, it is known from Proposition 2.2 iii) in Denuit (2019) that

$$m_X(s) = \frac{p_X \mu_{C_X} f_{\widetilde{C}_X + Y}(s)}{p_X \mu_{C_X} f_{\widetilde{C}_X + Y}(s) + p_Y \mu_{C_Y} f_{\widetilde{C}_Y + X}(s)} s = \frac{s}{1 + \frac{p_Y \mu_{C_Y}}{p_X \mu_{C_X}} r(s)}, \qquad (5.2)$$

with $r(s) = \frac{f_{\widetilde{C}_Y + X}(s)}{f_{\widetilde{C}_X + Y}(s)}$. Analogously as above, $\widetilde{C}_Y + X$ is a mixture of $\widetilde{C}_Y$ (with probability $1-p_X$) and $\widetilde{C}_Y + C_X$ (with probability p_X). A similar representation holds for $\widetilde{C}_X + Y$. Therefore, we can write:

$$r(s) = \frac{(1-p_X) f_{\widetilde{C}_Y}(s) + p_X f_{\widetilde{C}_Y + C_X}(s)}{(1-p_Y) f_{\widetilde{C}_X}(s) + p_Y f_{\widetilde{C}_X + C_Y}(s)}.$$

Since $\widetilde{C}_Y, \widetilde{C}_X, C_Y$ and C_X are positive random variables with $\alpha_{\widetilde{C}_Z} = \alpha_{C_Z} - 1$ for $Z = X, Y$, using the asymptotic expansion provided in Proposition 4.1, we can obtain

$$
\begin{aligned}
g(s) &= \frac{(1-p_X) f_{\widetilde{C}_Y}(s) + p_X \left(f_{\widetilde{C}_Y}(s) + f_{C_X}(s) + \left(\alpha_{\widetilde{C}_Y} \frac{f_{\widetilde{C}_Y}(s)}{s} \mu_{C_X} + \alpha_{C_X} \frac{f_{C_X}(s)}{s} \mu_{\widetilde{C}_Y} \right) (1 + o(1)) \right)}{(1-p_Y) f_{\widetilde{C}_X}(s) + p_Y \left(f_{\widetilde{C}_X}(s) + f_{C_Y}(s) + \left(\alpha_{\widetilde{C}_X} \frac{f_{\widetilde{C}_X}(s)}{s} \mu_{C_Y} + \alpha_{C_Y} \frac{f_{C_Y}(s)}{s} \mu_{\widetilde{C}_X} \right) (1 + o(1)) \right)} \\
&= \frac{s \frac{f_{C_Y}(s)}{\mu_{C_Y}} + p_X f_{C_X}(s) + p_X \left((\alpha_{C_Y} - 1) \frac{f_{C_Y}(s)}{\mu_{C_Y}} \mu_{C_X} + \alpha_{C_X} \frac{f_{C_X}(s)}{s} \mu_{\widetilde{C}_Y} \right) (1 + o(1))}{s \frac{f_{C_X}(s)}{\mu_{C_X}} + p_Y f_{C_Y}(s) + p_Y \left((\alpha_{C_X} - 1) \frac{f_{C_X}(s)}{\mu_{C_X}} \mu_{C_Y} + \alpha_{C_Y} \frac{f_{C_Y}(s)}{s} \mu_{\widetilde{C}_X} \right) (1 + o(1))}.
\end{aligned}
$$

Hence, from (5.2), and noticing that $\frac{\mu_{\widetilde{C}_Y}}{s} = o(1)$,

$$
\begin{aligned}
m_X(s) &= s \left(1 + \frac{\frac{s f_{C_Y}(s)}{\mu_{C_Y}} \left(\frac{1}{p_X \mu_{C_X}} + \frac{(\alpha_{C_Y} - 1)}{s} (1 + o(1)) \right) + \frac{f_{C_X}(s)}{\mu_{C_X}} \left(1 + \alpha_{C_X} \frac{\mu_{\widetilde{C}_Y}}{s} (1 + o(1)) \right)}{\frac{s f_{C_X}(s)}{\mu_{C_X}} \left(\frac{1}{p_Y \mu_{C_Y}} + \frac{(\alpha_{C_X} - 1)}{s} (1 + o(1)) \right) + \frac{f_{C_Y}(s)}{\mu_{C_Y}} \left(1 + \alpha_{C_Y} \frac{\mu_{\widetilde{C}_X}}{s} (1 + o(1)) \right)} \right)^{-1} \\
&= \frac{\frac{f_{C_Y}(s)}{\mu_{C_Y}} \left(1 + \alpha_{C_Y} \frac{\mu_{\widetilde{C}_X}}{s} (1 + o(1)) \right) + \frac{f_{C_X}(s)}{\mu_{C_X}} \left(\frac{s}{p_Y \mu_{C_Y}} + (\alpha_{C_X} - 1)(1 + o(1)) \right)}{\frac{f_{C_Y}(s)}{\mu_{C_Y}} \left(\frac{1}{p_X \mu_{C_X}} + \frac{\alpha_{C_Y}}{s} (1 + o(1)) \right) + \frac{f_{C_X}(s)}{\mu_{C_X}} \left(\frac{1}{p_Y \mu_{C_Y}} + \frac{\alpha_{C_X}}{s} (1 + o(1)) \right)}.
\end{aligned}
$$

This ends the proof. $\square$

When $p_X = p_Y = 1$, Proposition 5.1 reduces to Propositions 4.1 and 4.2, as expected. Similarly as in Section 3, we can derive the asymptotic level of $m_X(\cdot)$ when we consider zero-augmented random variables. Let us remark that, in order to obtain the asymptotic expression of $m_X(\cdot)$, contrary to the case of continuous random variables, where it is required that $\alpha_X > 3$ and $\alpha_Y > 2$, when considering zero-augmented random variables the hypothesis $\min\{\alpha_{C_X}, \alpha_{C_Y}\} > 3$ is included. This assumption is required because, by Denuit (2019), for zero-augmented distributions, we use the expression $m_X(s) = \frac{p_X \mu_{C_X} f_{\widetilde{C}_X + Y}(s)}{p_X \mu_{C_X} f_{\widetilde{C}_X + Y}(s) + p_Y \mu_{C_Y} f_{\widetilde{C}_Y + X}(s)} s$, instead of the expression $m_X(s) = \mu_X \frac{f_{\widetilde{X}+Y}(s)}{f_{X+Y}(s)}$ for continuous variables. Hence, the expectations of both size-biased transformations of C_X and C_Y are required to be finite.

Corollary 5.2. *Let X and Y be non-negative random variables such that $X = I_X C_X$ and $Y = I_Y C_Y$ where I_X, C_X, I_Y, and C_Y are independent, I_X and I_Y are Bernoulli distributed with mean p_X and p_Y, $0 < p_X, p_Y \leq 1$, and C_X and C_Y possess asymptotically smooth density functions $f_{C_X}(\cdot)$ and $f_{C_Y}(\cdot)$ with tail indices α_{C_X} and α_{C_Y} such that $\min\{\alpha_{C_X}, \alpha_{C_Y}\} > 3$ and respective associated slowly varying functions L_{C_X} and L_{C_Y}.*

- *If $\alpha_{C_X} > \alpha_{C_Y} + 1$, then $\lim_{s\to\infty} m_X(s) = \mathrm{E}[X]$,*
- *If $\alpha_{C_Y} + 1 > \alpha_{C_X} \geq \alpha_{C_Y}$, then $m_X(s) \to \infty$ as $s \to \infty$.*
- *If $\alpha_{C_X} = \alpha_{C_Y} + 1$:*
    - *If $\frac{L_{C_X}(s)}{L_{C_Y}(s)} \to \infty$ as $s \to \infty$, then $m_X(s) \to \infty$ as $s \to \infty$.*

– *If* $\lim_{s\to\infty} \frac{L_{C_X}(s)}{L_{C_Y}(s)} = k$ *with* $k \geq 0$, *then* $\lim_{s\to\infty} m_X(s) = \mathrm{E}[X] + \frac{p_X}{p_Y} k$.

Proof. Let us note that expression 5.1 can be rewritten as

$$m_X(s) = p_X \mu_{C_X} \frac{p_Y f_{C_Y}(s)(1+o(1)) + f_{\widetilde{C}_X}(s)(1+o(1))}{p_Y f_{C_Y}(s)(1+o(1)) + p_X f_{C_X}(s)(1+o(1))}.$$

And, therefore, as $s \to \infty$.

$$m_X(s) \sim p_X \mu_{C_X} \frac{p_Y f_{C_Y}(s) + f_{\widetilde{C}_X}(s)}{p_Y f_{C_Y}(s) + p_X f_{C_X}(s)}. \tag{5.3}$$

As $f_{C_X}(\cdot)$ and $f_{C_Y}(\cdot)$ are asymptotically smooth densities with respective indices $\alpha_{C_X}, \alpha_{C_Y} > 2$, then:

- If $\alpha_{C_X} > \alpha_{C_Y} + 1$, then $f_{C_X}(s) = o(f_{C_Y}(s))$, $f_{\widetilde{C}_X}(s) = o(f_{C_Y}(s))$. Since $\mathrm{E}[X] = p_X \mu_{C_X}$, from (5.3), $\lim_{s\to\infty} m_X(s) = \mathrm{E}[X]$.
- If $\alpha_{C_Y} + 1 > \alpha_{C_X} \geq \alpha_{C_Y}$, then $f_{C_Y}(s) = o\left(f_{\widetilde{C}_X}(s)\right)$ and $f_{C_X}(s) = o\left(f_{\widetilde{C}_X}(s)\right)$. From (5.3), it is direct that $m_X(s) \to \infty$ as $s \to \infty$.
- If $\alpha_{C_X} = \alpha_{C_Y} + 1$, then $f_{C_X}(s) = o(f_{C_Y}(s))$ and we can rewrite (5.3) as, if $s \to \infty$, by

$$m_X(s) \sim p_X \frac{\mu_{C_X} p_Y L_{C_Y}(s) + L_{C_X}(s)}{p_Y L_{C_Y}(s) + o(L_{C_Y}(s))}.$$

  By replacing the limit behavior of $\frac{L_{C_X}(s)}{L_{C_Y}(s)}$, the assertion follows.

This ends the proof. □

As expected, considering $p_X = p_Y = 1$, Corollary 5.2 reduces to the asymptotic levels provided in Section 3. Note that, similarly as the continuous case, we can obtain the asymptotic expansion of the conditional mean given the sum considering zero augmented random variables. However, in order to provide simpler expressions for the zero augmented case, distinctly as in some cases considered in the continuous framework, here we provide expansions considering two significant terms. Therefore, the case $\alpha_Y + 1 > \alpha_X > \alpha_Y$ is divided in two "sub-cases" instead of three as it was considered in Proposition 4.7. Let us note that if $p_X = p_Y = 1$, the asymptotic expressions (except the one mentioned) are the expansions provided in Section 4, as expected.

Corollary 5.3. *Let* X *and* Y *be non-negative random variables such that* $X = I_X C_X$ *and* $Y = I_Y C_Y$ *where* I_X, C_X, I_Y, *and* C_Y *are independent,* I_X *and* I_Y *are Bernoulli distributed with mean* p_X *and* p_Y, $0 < p_X, p_Y \leq 1$, *and* C_X *and* C_Y *possess asymptotically smooth density functions* $f_{C_X}(\cdot)$ *and* $f_{C_Y}(\cdot)$ *with tail indices* α_{C_X} *and* α_{C_Y} *such that* $\min\{\alpha_{C_X}, \alpha_{C_Y}\} > 3$.

1. *If* $\alpha_{C_X} > \alpha_{C_Y} + 2$, *then, as* s *tends to infinity,*

$$m_X(s) = p_X \mu_{C_X} + \alpha_{C_Y} \frac{p_X}{s} \mathrm{Var}[C_X](1+o(1)).$$

2. If $\alpha_{C_Y} + 2 > \alpha_{C_X} > \alpha_{C_Y} + 1$, then, as s tends to infinity,

$$m_X(s) = p_X \mu_{C_X} + s \frac{p_X}{p_Y} \frac{f_{C_X}(s)}{f_{C_Y}(s)} (1 + o(1)).$$

3. If $\alpha_{C_Y} + 1 > \alpha_{C_X} > \alpha_{C_Y} + \frac{1}{2}$, then, as s tends to infinity ,

$$m_X(s) = s \, \frac{f_{C_X}(s)}{f_{C_Y}(s)} \left(\frac{p_X}{p_Y} + \frac{p_X \mu_{C_X} f_{C_Y}(s)}{s f_{C_X}(s)} (1 + o(1)) \right).$$

4. If $\alpha_{C_Y} + 1 > \alpha_{C_X} > \alpha_{C_Y} + \frac{1}{2}$, then, as s tends to infinity ,

$$m_X(s) = s \, \frac{p_X}{p_Y} \, \frac{f_{C_X}(s)}{f_{C_Y}(s)} \left(1 - \frac{p_X}{p_Y} \frac{f_{C_X}(s)}{f_{C_Y}(s)} (1 + o(1)) \right).$$

5. If $\alpha_{C_Y} = \alpha_{C_X}$ and $f_{C_X}(s) = c f_{C_Y}(s) (1 + o(1))$ with $c > 0$, then as s tends to infinity

$$m_X(s) = \frac{c}{\frac{p_Y}{p_X} + c} s \left(1 + \frac{1}{s} \left(\frac{\mu_{C_X} p_Y}{c} - p_Y \mu_{C_Y} + \alpha \frac{p_Y}{\frac{p_Y}{p_X} + c} \left(\frac{p_Y}{p_X} \mu_{C_Y} - \mu_{C_X} \right) \right) (1 + o(1)) \right).$$

The proof of the Corollary is provided in Appendix C.

6 Extension to more than two terms in the sum, with application to risk sharing

Until now, we have considered sums of two random variables X and Y. In applications to risk sharing, pools often comprise many participants so that this section studies the non-decreasingness of conditional expectations given sums of n random variables. Precisely, let us consider n independent, non-negative and continuous random variables $X_1, X_2, \dots, X_n$ with regularly varying densities. The following result is useful to study the no-sabotage condition with more than two participants. Proceeding by induction, the result comes as a direct consequence of Theorem 2.1 in Bingham et al. (2006).

Corollary 6.1. *Let $f_{X_1}, f_{X_2}, \dots, f_{X_n}$ be the probability density functions of $X_1, X_2, \dots, X_n$ which are regularly varying with respective indices $\alpha_1, \alpha_2, \dots, \alpha_n$. If*

$$\alpha_j = \min\{\alpha_1, \dots, \alpha_n\} \text{ with } \alpha_j < \alpha_k, \text{ for } k \neq j,$$

so that $f_{X_k}(x) = \mathrm{o}(f_{X_j}(x))$ for $k \neq j$, then the convolution product $f_{X_1+X_2+\dots+X_n}$ of $f_1, f_2, \dots, f_n$ satisfies

$$\lim_{s \to \infty} \frac{f_{X_1+X_2+\dots+X_n}(s)}{f_{X_j}(s)} = 1$$

and is a regularly varying density with index α_j.

Let $S_n = \sum_{j=1}^{n} X_j$ be the total loss of the pool, where the subscript n emphasizes the number of economic agents forming the pool. The contribution paid ex post by participant i is $m_i(s) = \mathrm{E}[X_i \mid S_n = s]$. We are now in a position to determine a sufficient condition for the existence of a sabotage opportunity.

Proposition 6.2. *Consider independent, non-negative and continuous random variables $X_1, \ldots, X_n$ with regularly varying density functions $f_{X_1}, f_{X_2}, \ldots, f_{X_n}$ having respective indices $\alpha_1, \ldots, \alpha_n$. Let us assume that $\alpha_j = \min\{\alpha_1, \ldots \alpha_{i-1}, \alpha_{i+1} \ldots \alpha_n\}$ with $\alpha_j < \alpha_k$ for all $k \neq j$. If $\alpha_i > \alpha_j + 1$ then*

1. $\lim_{s\to\infty} m_i(s) = \mathrm{E}[X_i]$.
2. *There exists a non-empty interval in $(0, \infty)$ where $m_i(\cdot)$ is decreasing.*

Proof. Using the representation of the conditional expectation given the sum in terms of size-biasing provided in Proposition 2.2 in Denuit (2019), we can write

$$m_i(s) = \mathrm{E}\left[X_i\right] \frac{f_{S_n - X_i + \widetilde{X}_i}(s)}{f_{S_n}(s)} = \mathrm{E}\left[X_i\right] \frac{\frac{f_{S_n - X_i + \widetilde{X}_i}(s)}{f_{X_j}(s)}}{\frac{f_{S_n}(s)}{f_{X_j}(s)}}.$$

Hence, by Corollary 6.1,

$$\lim_{s\to\infty} \frac{f_{S_n - X_i + \widetilde{X}_i}(s)}{f_{X_j}(s)} = \lim_{s\to\infty} \frac{f_{S_n}(s)}{f_{X_j}(s)} = 1 \text{ and } \lim_{s\to\infty} m_i(s) = \mathrm{E}[X_i].$$

The latter assertion follows as in the proof of Proposition 3.3, taking into account that $m_i(0) = 0$, $\lim_{s\to\infty} m_i(s) = \mathrm{E}\left[X_i\right]$ and $\mathrm{E}[X_i] = \mathrm{E}[m_i(S_n)]$. This ends the proof. □

A direct consequence of this result is that once any two indices differ in more than one unit, there exists an index i such that $m_i(\cdot)$ decreases in an interval.

Corollary 6.3. *Consider independent, non-negative and continuous random variables $X_1, \ldots, X_n$ with regularly varying density functions having respective indices $\alpha_1, \ldots, \alpha_n$. If there exists i and j such that $\alpha_j = \min\{\alpha_1, \ldots \alpha_{i-1}, \alpha_{i+1} \ldots \alpha_n\}$ with $\alpha_j < \alpha_k$ for $k \neq j$ and $\alpha_j + 1 < \alpha_i$, then there is an interval where $m_i(\cdot)$ is decreasing.*

Corollary 6.3 indicates that in a pool where the risks X_i have regularly varying densities, a necessary condition to avoid sabotage is to remove participants whose index differ by more than one unit from the others. The following example illustrates this situation.

Example 6.4. *Let us consider four independent random variables $X_i \sim P(I)(1, \alpha_i)$ ($i = 1, 2, 3, 4$) with $\alpha_1 = 2.9$, $\alpha_2 = 1.6$, $\alpha_3 = 2.4$, $\alpha_4 = 2$. Since $\alpha_1 > \alpha_2 + 1$, we know from Proposition 6.2 that there must be an interval where $\mathrm{E}[X_1 \mid S_4 = s]$ decreases with s, which is indeed visible on Figure 12. Specifically, considering a pool with the four risks included, Figure 12 shows the contribution of each participant in terms of the total loss, where we can see that $\mathrm{E}[X_1 \mid S_4 = s]$ converges to $\mathrm{E}[X_1]$ and starts decreasing when S_4 exceeds a threshold around 15. Therefore, the no-sabotage condition does not hold.*

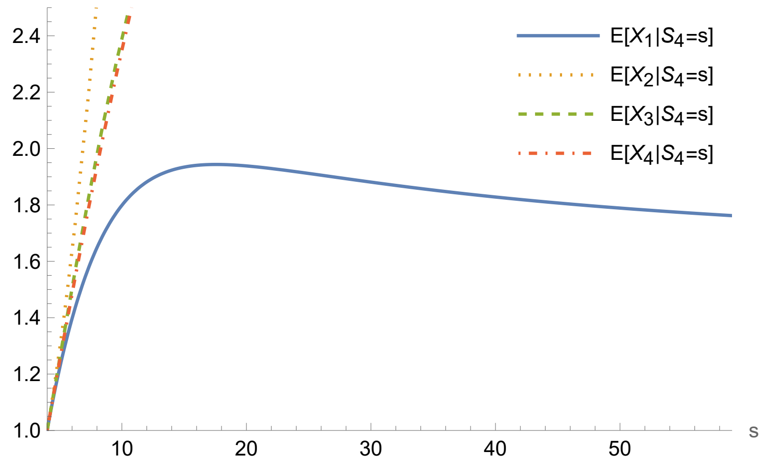
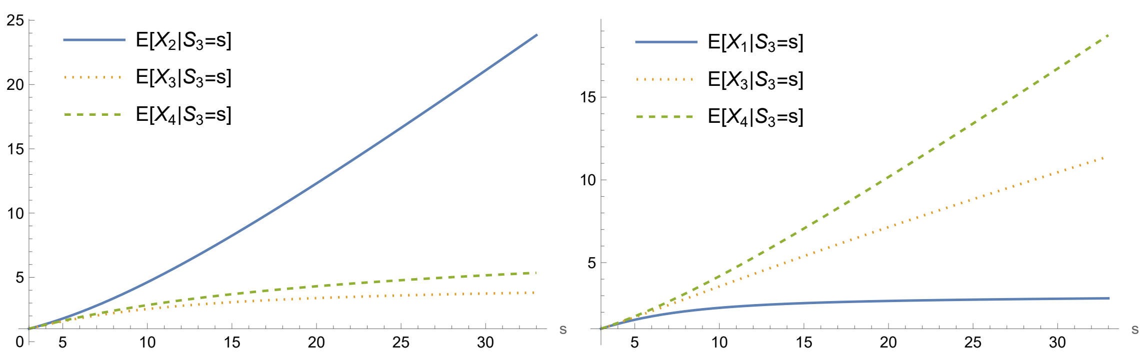


Figure 12: Functions $s \mapsto \mathrm{E}[X_i \mid S_4 = s]$ for $i = 1, 2, 3, 4$ where $S_4 = \sum_{i=1}^{4} X_i$ and $X_i \sim P(I)(1, \alpha_i)$ with $\alpha_1 = 2.9$, $\alpha_2 = 1.6$, $\alpha_3 = 2.4$, $\alpha_4 = 2$.

Removing either X_1 or X_2 from the pool so that no index differs by more than one unit from the others, there is no evidence of a sabotage scenario. The contributions in the new pools, without participant 1 or 2 appear in Figures 13a and 13b, and we can observe that, for this numerical examples, all contributions increase in the sum.



(a) Functions $s \mapsto \mathrm{E}[X_2 \mid S_3 = s], s \mapsto \mathrm{E}[X_3 \mid S_3 = s], s \mapsto \mathrm{E}[X_4 \mid S_3 = s]$.

(b) Functions $s \mapsto \mathrm{E}[X_1 \mid S_3' = s], s \mapsto \mathrm{E}[X_3 \mid S_3' = s], s \mapsto \mathrm{E}[X_4 \mid S_3' = s]$.

Figure 13: Contributions considering $X_i \sim P(I)(1, \alpha_i)$ $(i = 1, 2, 3, 4)$ with $\alpha_1 = 2.9$, $\alpha_2 = 1.6$, $\alpha_3 = 2.4$, $\alpha_4 = 2$ where $S_3 = X_1 + X_3 + X_4$ and $S_3' = X_1 + X_3 + X_4$.

7 Conclusion

The monotonicity of the conditional expectations of variables given their sum is known to be relevant in several contexts. For instance, in signal processing, such property is linked to regression dependence between the variable of interest and the perturbed signal, which is an important aspect for accurately interpreting such signal. It is also of great utility for actuarial applications, as the conditional expectations of variables given their sum corresponds to the conditional mean risk-sharing rule and its monotonicity is referred to as the no-sabotage condition.

In order to ensure the monotonicity of $m_X(s) = \mathrm{E}[X|X+Y=s]$, it is common to assume log-concave densities, meaning that variables are supposed to be light-tailed, an assumption which might be too strong. In this paper, we studied the behavior of $m_X(\cdot)$ when variables are heavy-tailed. In particular, we studied the behavior of the conditional expectation given the sum considering variables with regularly varying densities. We showed that, under the independence assumption, (i) a difference higher than 1 on the respective tail indices of X and Y implies that $m_X(\cdot)$ cannot be monotonic over its domain, converging to the unconditional expected value and (ii) if the difference is smaller than 1, $m_X(\cdot)$ is not bounded as the value of the sum diverges.

Considering a risk-sharing scheme scenario, the result implies that, if the risks associated to the pool members have tail indices differing by more than one unit, there will always exist a sabotage scenario. Hence, it enables practitioners to identify a necessary condition when forming a pool, based on risks having a similar tail behavior. Analogously, if the difference is less than one, even if we cannot state that the contribution of each participant is increasing in the total loss, we know that they diverge. In signal processing, the consequence of the result is that, if the tail index of the variable of interest X exceeds by more than one unit the tail index of the noise, then the regression function $m_X(\cdot)$ tends to the unconditional expected value of X as the signal diverges. This means, that we are not recovering information about X from the perturbed signal. Considering industry loss warranties, the heavy-tailedness of the buyer's loss X with respect to the total loss Y for the other companies on the market thus plays an important role.

Under the assumption that random variables involved have finite second moments, we have also provided the asymptotic expression of $m_X(\cdot)$. This allowed us to study its asymptotic behavior depending on the differences between the tail indices. These results have been extended to zero-augmented distributions. In this case, the difference between the tail indices of the continuous components determines the limit behavior of $m_X(\cdot)$. An extension to several terms has also been proposed, to study the presence of sabotage opportunities within pools gathering more than two participants. Throughout the paper, several numerical examples with parametric families are provided in order to illustrate the results.

The analysis conducted in this paper is restricted to independent random variables X and Y. We acknowledge that this assumption may be restrictive in some applications to insurance. For instance, Gatzert et al. (2019) showed that industry loss warranties play an important role in risk management of natural catastrophes, where X and Y cannot be assumed to be independent. Therefore, extending the results derived in the present paper to correlated random variable X and Y is certainly relevant for future research.

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Appendix

A Examples of asymptotically smooth distributions

Property A.1. *The densities of all distributions listed in Table 2.1 are asymptotically smooth with index α.*

Proof. By Proposition 2.1(ii) in Barbe and Mc Cormick (2005), we know that if the density $f(\cdot)$ is regularly varying and it has an ultimately monotone derivative (monotone on (x_0, ∞) for some $x_0 > 0$), then $f(\cdot)$ is asymptotically smooth. It is thus enough to show that the regularly varying densities of all distributions listed in Table 2.1 have an ultimately monotone derivative. This is done next:

- Pareto Type II (Type I is particular case of Type II with $\vartheta = \theta$). If $X \sim P(II)(\theta, \alpha, \vartheta)$ then the second derivative of the density $f(x) = (x + \theta - \vartheta)^{-\alpha}\,\theta^{\alpha-1}(\alpha - 1)$ is

$$f''(x) = \theta^{\alpha-1}(\alpha-1)\alpha(\alpha+1)\,(x+\theta-\vartheta)^{-(\alpha+2)} \geq 0 \text{ for all } x \geq \vartheta.$$

- Pareto Type IV (Type III is a particular case of Type IV with $\lambda = 1$). If $X \sim P(IV)(\theta, \alpha, \vartheta, \lambda)$ then the second derivative of the density

$$f(x) = (\alpha-1)\theta^{-\frac{\alpha-1}{\lambda}}(x-\vartheta)^{-1+\frac{\alpha-1}{\lambda}}\left(1+\left(\frac{\theta}{x-\vartheta}\right)^{-\frac{\alpha-1}{\lambda}}\right)^{-(\lambda+1)}$$

is given by $f''(x) = g_1(x)\, g_2(x)$ where

$$g_1(x) = \frac{(\alpha-1)\theta^{\frac{1-\alpha}{\lambda}}\left(1+\left(\frac{\theta}{x-\vartheta}\right)^{\frac{1-\alpha}{\lambda}}\right)^{-\lambda}\left(\frac{\theta}{x-\vartheta}\right)^{\alpha/\lambda}(x-\vartheta)^{-3+\frac{\alpha-1}{\lambda}}}{\lambda^2\left(\left(\frac{\theta}{x-\vartheta}\right)^{1/\lambda}+\left(\frac{\theta}{x-\vartheta}\right)^{\alpha/\lambda}\right)^3} > 0$$

and

$$\begin{aligned} g_2(x) &= \alpha(\alpha+1)\lambda^2\left(\frac{\theta}{x-\vartheta}\right)^{2/\lambda} + (2\lambda+1-\alpha)(\lambda+1-\alpha)\left(\frac{\theta}{x-\vartheta}\right)^{\frac{2\alpha}{\lambda}} \\ &\quad +(1+\lambda-\alpha)(\alpha+3\alpha\lambda+\lambda-1)\left(\frac{\theta}{x-\vartheta}\right)^{\frac{1+\alpha}{\lambda}}. \end{aligned}$$

Clearly, g_2 has the same sign as $g_2(x)\left(\frac{\theta}{x-\vartheta}\right)^{-2/\lambda}$. Since

$$\lim_{x\to\infty} g_2(x)\left(\frac{\theta}{x-\vartheta}\right)^{-2/\lambda} = \alpha(\alpha+1)\lambda^2 > 0,$$

we see that g_2 is ultimately positive and, in consequence, $f'(\cdot)$ is ultimately increasing.

- Log-Gamma. If $X \sim LG(\alpha, \lambda)$ then the second derivative of the density $f(x) = \frac{(\alpha-1)^\lambda}{\Gamma(\lambda)}(\log x)^{\lambda-1}x^{-\alpha}$, $x \geq 1$, can be written as $f''(x) = g_1(x)\, g_2(x)$, where

$$g_1(x) = \frac{x^{-(\alpha+2)}(\alpha-1)^\lambda(\log(x))^{\lambda-3}}{\Gamma(\lambda)} > 0$$

and

$$g_2(x) = 2 - 3\lambda + \lambda^2 - (2\alpha+1)(\lambda-1)\log(x) + \alpha(1+\alpha)\log(x)^2.$$

As $\alpha(1+\alpha) > 0$ and $\log(x)$ is an increasing function, $\lim_{x\to\infty} g_2(x) = \infty$ and therefore g_2 is ultimately positive and $f'(\cdot)$ is ultimately increasing.

- Dagun distribution. It can be checked that it has an ultimately increasing density by taking into account that, since $\left(\frac{x}{\theta}\right)^{-(\alpha-1)} = \left(\frac{y-\mu}{\theta}\right)^{\frac{\alpha-1}{\lambda}} \Leftrightarrow y = \theta\left(\frac{x}{\theta}\right)^{-\lambda} + \mu$, we have

$$\overline{F}_{\text{Dagun}}(x) = F_{\text{Pareto(IV)}}\left(\theta\left(\frac{x}{\theta}\right)^{-\lambda} + \vartheta\right).$$

- Davis distribution. If $X \sim Davis(\alpha, b, \vartheta)$ then the density satisfies $f(x) \propto \left(e^{\frac{b}{x-\vartheta}} - 1\right)^{-1}(x-\vartheta)^{-(1+\alpha)}$. Hence, $f''(x) \propto \frac{(x-\vartheta)^{-(5+\alpha)}}{\left(e^{\frac{b}{x-\vartheta}}-1\right)^3}d(x)$, with $\frac{(x-\vartheta)^{-(5+\alpha)}}{\left(e^{\frac{b}{x-\vartheta}}-1\right)^3} > 0$ and where

$$\begin{aligned} d(x) &= b^2e^{\frac{b}{x-\vartheta}}\left(1+e^{\frac{b}{x-\vartheta}}\right) + \left(e^{\frac{b}{x-\vartheta}}-1\right)(1+\alpha)(2+\alpha)(x-\vartheta)^2 \\ &\qquad - 2be^{\frac{b}{x-\vartheta}}\left(e^{\frac{b}{x-\vartheta}}-1\right)(2+\alpha)(x-\vartheta) \\ &= b^2e^{\frac{b}{x-\vartheta}}\left(1+e^{\frac{b}{x-\vartheta}}\right) + \left(e^{\frac{b}{x-\vartheta}}-1\right)(x-\vartheta)(2+\alpha)\cdot h(x), \end{aligned}$$

where $h(x) = \left(e^{\frac{b}{x-\vartheta}}-1\right)(1+\alpha)(x-\vartheta) - 2be^{\frac{b}{x-\vartheta}}$. It can be checked that $b^2e^{\frac{b}{x-\vartheta}}\left(1+e^{\frac{b}{x-\vartheta}}\right) > 0$ and $\left(e^{\frac{b}{x-\vartheta}}-1\right)(x-\vartheta)(2+\alpha) > 0$. In addition, by l'Hôpital rule, $\lim_{x\to\infty}\frac{e^{\frac{b}{x-\vartheta}}-1}{(x-\vartheta)^{-1}} = b$ and $\lim_{x\to\infty} h(x) = b(\alpha-1) > 0$. Therefore, g is ultimately positive and f' ultimately increasing.

This ends the proof. $\square$

B Extension of Lemma 5.2 in Barbe and Mc Cormick (2005)

The proof of Proposition 4.1 requires the following extension of Lemma 5.2 in Barbe and Mc Cormick (2005).

Property B.1. *Let X and Y be continuous positive random variables with respective asymptotically smooth and regularly varying density functions f_X and f_Y with indices α_X and α_Y such that* $\min\{\alpha_X, \alpha_Y\} \geq 2$. *Then,*

$$\begin{aligned} T_X f_Y(t) &= f_Y(t) + \alpha_Y \frac{f_Y(t)}{t} \mu_X(t) \big(1 + o(1)\big) \\ T_Y f_X(t) &= f_X(t) + \alpha_X \frac{f_X(t)}{t} \mu_Y(t) \big(1 + o(1)\big) \end{aligned}$$

as $t \to \infty$. In particular, if $\min\{\alpha_X, \alpha_Y\} > 2$, *as $t \to \infty$,*

$$\begin{aligned} T_X f_Y(t) &= f_Y(t) + \alpha_Y \frac{f_Y(t)}{t} \mu_X \big(1 + o(1)\big) \\ T_Y f_X(t) &= f_X(t) + \alpha_X \frac{f_X(t)}{t} \mu_Y \big(1 + o(1)\big). \end{aligned}$$

Proof. It is enough to establish the representation for $T_Y f_X$. For $\epsilon > 0$, as f_X is asymptotically smooth, there exists $\frac{1}{2} > \delta > 0$ such that for all large t

$$\sup_{0<x\leq\delta t} \left| \frac{f_X(t-x) - f_X(t)}{\frac{x}{t} f_X(t)} - \alpha_X \right| \leq \epsilon. \tag{B.1}$$

As in Barbe and Mc Cormick (2005), we write $T_Y f_X(t) - f_X(t) - \alpha_X \frac{f_X(t)}{t} \mu_Y(t)$ as a sum of four terms,

$$\begin{aligned} &\int_0^{t\delta} \left(f_X(t-x) - f_X(t) \left(1 + \alpha_X \frac{x}{t}\right)\right) f_Y(x)\,\mathrm{d}x + \int_{t\delta}^{t/2} f_X(t-x) f_Y(x)\,\mathrm{d}x \\ &- f_X(t)\,\overline{F}_Y(t\delta) - \alpha_X \frac{f_X(t)}{t} \int_{t\delta}^{t} x f_Y(x)\,\mathrm{d}x. \end{aligned} \tag{B.2}$$

Since $\frac{x}{t} f_X(t)$ is positive, we know from (B.1) that for all $0 < x \leq \delta t$,

$$\left| f_X(t-x) - f_X(t) - \alpha_X \frac{x}{t} f_X(t) \right| \leq \epsilon \frac{x}{t} f_X(t).$$

Therefore, the first term of (B.2) is bounded by

$$\begin{aligned} \int_0^{t\delta} \left(f_X(t-x) - f_X(t) \left(1 + \alpha_X \frac{x}{t}\right)\right) f_Y(x)\,\mathrm{d}x &\leq \int_0^{t\delta} \left| f_X(t-x) - f_X(t) \left(1 + \alpha_X \frac{x}{t}\right)\right| f_Y(x)\,\mathrm{d}x \\ &\leq \int_0^{t\delta} \epsilon \frac{x}{t} f_X(t) f_Y(x)\,\mathrm{d}x \\ &= \epsilon \frac{f_X(t)}{t} \mu_Y(\delta t) \\ &\leq \epsilon \frac{f_X(t)}{t} \mu_Y(t). \end{aligned}$$

Let us consider $f_X^*(t) = \sup_{\frac{t}{2}<x<(1-\delta)t} f_X(x)$. If $t\delta \leq x \leq t/2$, then $t/2 \leq t - x \leq t(1-\delta)$ and

$$\int_{t\delta}^{t/2} f_X(t-x) f_Y(x)\,\mathrm{d}x \leq f_X^*(t) \int_{t\delta}^{t/2} f_Y(x)\,\mathrm{d}x \leq f_X^*(t) \int_{t\delta}^{\infty} f_Y(x)\,\mathrm{d}x = \overline{F}_Y(\delta t) f_X^*(t).$$

Hence (B.2) can be bounded as follows:

$$
\begin{aligned}
&T_Y f_X(t) - f_X(t) - \alpha_X \frac{f_X(t)}{t}\mu_Y(t)\\
&= \int_0^{t\delta} \left(f_X(t-x) - f_X(t)\left(1+\alpha_X \frac{x}{t}\right)\right) f_Y(x)\,\mathrm{d}x + \int_{t\delta}^{t/2} f_X(t-x) f_Y(x)\,\mathrm{d}x\\
&\quad - f_X(t)\,(\overline{F}_Y(t\delta)) - \alpha_X \frac{f_X(t)}{t}\int_{t\delta}^{t} x f_Y(x)\,\mathrm{d}x\\
&\le \frac{f_X(t)}{t}\mu_Y(t)\,\epsilon + \overline{F}_Y(\delta t) f_X^*(t) - f_X(t)\,\overline{F}_Y(t\delta) - \alpha_X \frac{f_X(t)}{t}\int_{t\delta}^{t} x f_Y(x)\,\mathrm{d}x\\
&= \alpha_X \frac{f_X(t)}{t}\left(\mu_Y(t)\frac{\epsilon}{\alpha_X} - \int_{t\delta}^{t} x f_Y(x)\,\mathrm{d}x\right) + \overline{F}_Y(\delta t)\,(f_X^*(t) - f_X(t))\\
&= \alpha_X \frac{f_X(t)}{t}\mu_Y(t)\left(\frac{\epsilon}{\alpha_X} - \frac{1}{\mu_Y(t)}\int_{t\delta}^{t} x f_Y(x)\,\mathrm{d}x + \frac{\overline{F}_Y(\delta t) f_X^*(t)\,t}{\alpha_X f_X(t)\,\mu_Y(t)} - \frac{\overline{F}_Y(\delta t) t}{\alpha_X \mu_Y(t)}\right).
\end{aligned}
$$

Therefore, the announced result follows if we can prove that

$$
\frac{\epsilon}{\alpha_X} - \frac{1}{\mu_Y(t)}\int_{t\delta}^{t} x f_Y(x)\,dx + \frac{\overline{F}_Y(\delta t) f_X^*(t)\,t}{\alpha_X f_X(t)\,\mu_Y(t)} - \frac{\overline{F}_Y(\delta t) t}{\alpha_X \mu_Y(t)} = o(1). \tag{B.3}
$$

Since $f_X(\cdot)$ is regularly varying, $f_X^*(t) = \sup_{\frac{t}{2}<x<(1-\delta)t} f_X(x)$ is $O(f_X(t))$ and $\overline{F}_Y(\delta t))$ is trivially $O(\overline{F}_Y(t))$. In consequence, (B.3) holds if

$$
\overline{F}_Y(t) = o\left(\frac{\mu_Y(t)}{t}\right) \text{ and } \int_{t\delta}^{t} x f_Y(x)\,\mathrm{d}x = o\left(\mu_Y(t)\right).
$$

This follows analogously as done in Lemma 5.2 in Barbe and Mc Cormick (2005) since $\mathrm{E}[Y]$ is finite and as $L(t) = \int_0^t x f_Y(x)\,\mathrm{d}x$ is a slowly varying function. The second assertion follows as $\min\{\alpha_X, \alpha_Y\} > 2$ implies that the random variables X and Y have finite first order moments. □

C Proof of Corollary 5.3

Considering expression (5.1), we can obtain:

1) If $\alpha_{C_X} > \alpha_{C_Y} + 2$, then $f_{C_X}(s) = o(\frac{f_{C_Y}(s)}{s^2})$. Hence, as $s \to \infty$,:

$$
\begin{aligned}
m_X(s) &= p_X \mu_{C_X} \frac{\frac{s f_{C_Y}(s)}{\mu_{C_Y}}\left(1 + \alpha_{C_Y}\frac{1}{s}(\frac{Var[C_X]}{\mu_{C_X}} + \mu_{C_X})\,(1+o(1))\right) + o(1)}{\frac{s f_{C_Y}(s)}{\mu_{C_Y}}\left(1 + \frac{p_X \mu_{C_X}}{s}\alpha_{C_Y}\,(1+o(1))\right) + o(1)}\\
&= p_X \mu_{C_X}\left(1 + \frac{\alpha_Y \frac{1}{s}\frac{Var[C_X]}{\mu_{C_X}}\,(1+o(1))}{1 + \frac{p_X \mu_{C_X}}{s}\alpha_{C_Y}\,(1+o(1))}\right)\\
&= p_X \mu_{C_X}\left(1 + \alpha_{C_Y}\frac{1}{s}\frac{\mathrm{Var}[C_X]}{\mu_{C_X}}\,(1+o(1))\right).
\end{aligned}
$$

2) If $\alpha_{C_Y}+2>\alpha_{C_X}>\alpha_{C_Y}+1$, then $f_{C_X}(s)=o\left(\frac{f_{C_Y}(s)}{s}\right)$ and $f_{C_Y}(s)=o(s^2 f_{C_X}(s))$. Hence, we can write:

$$\begin{aligned}
m_X(s) &= \frac{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(1+\alpha_{C_Y}\frac{\mu_{\widetilde{C}_X}}{s}(1+o(1))\right)+\frac{f_{C_X}(s)}{\mu_{C_X}}\left(\frac{s}{p_Y\mu_{C_Y}}+(\alpha_{C_X}-1)(1+o(1))\right)}{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(\frac{1}{p_X\mu_{C_X}}+\frac{\alpha_{C_Y}}{s}(1+o(1))\right)+o\left(\frac{f_{C_Y}(s)}{s}\right)} \\
&= p_X\mu_{C_X}\frac{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(s+\alpha_{C_Y}\mu_{\widetilde{C}_X}(1+o(1))\right)+s\frac{f_{C_X}(s)}{\mu_{C_X}}\left(\frac{s}{p_Y\mu_{C_Y}}+(\alpha_{C_X}-1)(1+o(1))\right)}{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(s+p_X\mu_{C_X}\alpha_{C_Y}(1+o(1))\right)} \\
&= p_X\mu_{C_X}\left(1+o(1)+\frac{s\frac{f_{C_X}(s)}{\mu_{C_X}}\left(\frac{s}{p_Y\mu_{C_Y}}+(\alpha_{C_X}-1)(1+o(1))\right)}{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(s+p_X\mu_{C_X}\alpha_{C_Y}(1+o(1))\right)}\right) \\
&= p_X\mu_{C_X}+s\frac{p_X}{p_Y}\frac{f_{C_X}(s)}{f_{C_Y}(s)}(1+o(1)).
\end{aligned}$$

3-4) If $\alpha_{C_Y}+1>\alpha_{C_X}>\alpha_{C_Y}$, then, we can write:

$$\begin{aligned}
m_X(s) &= \frac{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(1+\alpha_{C_Y}\frac{\mu_{\widetilde{C}_X}}{s}(1+o(1))\right)+\frac{f_{C_X}(s)}{\mu_{C_X}}\left(\frac{s}{p_Y\mu_{C_Y}}+(\alpha_{C_X}-1)(1+o(1))\right)}{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(\frac{1}{p_X\mu_{C_X}}+\frac{\alpha_{C_Y}}{s}(1+o(1))\right)+\frac{f_{C_X}(s)}{\mu_{C_X}}\left(\frac{1}{p_Y\mu_{C_Y}}+\frac{\alpha_{C_X}}{s}(1+o(1))\right)} \\
&= s\,\frac{p_X}{p_Y}\,\frac{f_{C_X}(s)}{f_{C_Y}(s)}\frac{\frac{f_{C_Y}(s)}{f_{C_X}(s)\mu_{C_Y}}\frac{p_Y}{s}(1+o(1))+\frac{1}{\mu_{C_X}}\left(\frac{1}{\mu_{C_Y}}+\frac{p_Y(\alpha_{C_X}-1)}{s}(1+o(1))\right)}{\frac{1}{\mu_{C_Y}\mu_{C_X}}+\frac{p_X\alpha_{C_Y}}{\mu_{C_Y}s}(1+o(1))+\frac{f_{C_X}(s)}{f_{C_Y}(s)\mu_{C_X}}\left(\frac{p_X}{p_Y\mu_{C_Y}}+\frac{p_X\alpha_{C_X}}{s}(1+o(1))\right)} \\
&= s\,\frac{p_X}{p_Y}\,\frac{f_{C_X}(s)}{f_{C_Y}(s)}(1+r(s)).
\end{aligned}$$

where

$$r(s)=\frac{\frac{p_Y}{s}\frac{f_{C_Y}(s)}{f_{C_X}(s)\mu_{C_Y}}(1+o(1))+\frac{1}{s}\left(\frac{p_Y(\alpha_{C_X}-1)}{\mu_{C_X}}-\frac{p_X\alpha_{C_Y}}{\mu_{C_Y}}\right)(1+o(1))-\frac{p_X}{p_Y}\frac{f_{C_X}(s)}{f_{C_Y}(s)}\left(\frac{1}{\mu_{C_Y}\mu_{C_X}}(1+o(1))\right)}{\frac{1}{\mu_{C_Y}\mu_{C_X}}+\frac{p_X\alpha_{C_Y}}{s\mu_{C_Y}}(1+o(1))+\frac{p_X}{p_Y}\frac{f_{C_X}(s)}{f_{C_Y}(s)}\left(\frac{1}{\mu_{C_Y}\mu_{C_X}}+\frac{p_Y\alpha_{C_X}}{\mu_{C_X}s}(1+o(1))\right)}.$$

We can proceed analogously as in Proposition 4.7 and differ among the different frameworks. If $\alpha_{C_X}-\alpha_{C_Y}>\frac{1}{2}$, then, the dominant term in the numerator is $\frac{f_{C_Y}(s)}{sf_{C_X}(s)}$ and we can write $r(s)=\frac{p_Y\mu_{C_X}f_{C_Y}(s)}{sf_{C_X}(s)}(1+o(1))$. However, if $\alpha_{C_X}-\alpha_{C_Y}<\frac{1}{2}$, then the dominant term is $\frac{f_{C_X}(s)}{f_{C_Y}(s)}$ and $r(s)=-\frac{p_X}{p_Y}\frac{f_{C_X}(s)}{f_{C_Y}(s)}(1+o(1))$.

5) If $\alpha = \alpha_{C_Y} = \alpha_{C_X}$ and and $f_{C_X}(s) = cf_{C_Y}(s)\,(1+o\,(1))$ with $c > 0$, then, we can write:

$$
\begin{aligned}
m_X(s) &= \frac{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(1+\alpha\frac{\mu_{\tilde{C}_X}}{s}\,(1+o\,(1))\right)+\frac{cf_{C_Y}(s)}{\mu_{C_X}}\left(\frac{s}{p_Y\mu_{C_Y}}+(\alpha-1)\right)(1+o\,(1))}{\frac{f_{C_Y}(s)}{\mu_{C_Y}}\left(\frac{1}{p_X\mu_{C_X}}+\frac{\alpha}{s}\,(1+o\,(1))\right)+\frac{cf_{C_Y}(s)}{\mu_{C_X}}\left(\frac{1}{p_Y\mu_{C_Y}}+\frac{\alpha}{s}\right)(1+o\,(1))} \\
&= s\,\frac{\frac{1}{s\mu_{C_Y}}\,(1+o\,(1))+\frac{c}{\mu_{C_X}}\left(\frac{1}{p_Y\mu_{C_Y}}+\frac{(\alpha-1)}{s}\right)(1+o\,(1))}{\frac{1}{\mu_{C_Y}}\left(\frac{1}{p_X\mu_{C_X}}+\frac{\alpha}{s}\,(1+o\,(1))\right)+\frac{c}{\mu_{C_X}}\left(\frac{1}{p_Y\mu_{C_Y}}+\frac{\alpha}{s}\right)(1+o\,(1))} \\
&= \frac{c}{\frac{p_Y}{p_X}+c}s\,\frac{\frac{\mu_{C_X}p_Y}{c\,s}\,(1+o\,(1))+\left(1+p_Y\mu_{C_Y}\frac{(\alpha-1)}{s}\right)(1+o\,(1))}{\left(1+\frac{\alpha}{s}\frac{p_Y\left(c\,\mu_{C_Y}+\mu_{C_X}\right)}{\frac{p_Y}{p_X}+c}\right)(1+o\,(1))} \\
&= \frac{c}{\frac{p_Y}{p_X}+c}s\left(1+\frac{1}{s}\frac{\frac{\mu_{C_X}p_Y}{c}\,(1+o\,(1))+(p_Y\mu_{C_Y}(\alpha-1))\,(1+o\,(1))-\alpha\frac{p_Y\left(c\,\mu_{C_Y}+\mu_{C_X}\right)}{\frac{p_Y}{p_X}+c}}{\left(1+\frac{\alpha}{s}\frac{p_Y\left(c\,\mu_{C_Y}+\mu_{C_X}\right)}{\frac{p_Y}{p_X}+c}\right)(1+o\,(1))}\right) \\
&= \frac{c}{\frac{p_Y}{p_X}+c}s\left(1+\frac{1}{s}\left(\frac{\mu_{C_X}p_Y}{c}-p_Y\mu_{C_Y}+\alpha\frac{p_Y}{\frac{p_Y}{p_X}+c}\left(\frac{p_Y}{p_X}\mu_{C_Y}-\mu_{C_X}\right)\right)(1+o\,(1))\right).
\end{aligned}
$$