Discrete curve fitting on manifolds

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Fitting a regression curve to data points can serve at least two purposes: it may help reduce measurement noise and it can fill gaps in the data. We developed a framework and an algorithm to define and solve a type of discrete regression problem across time-labeled data points on manifolds. Unlike regression in Euclidean spaces, only little work has been done in this area. This abstract is based on [2].

Let $p_1, p_2, \ldots, p_N$ be N data points in the Euclidean space $\mathbb{R}^n$ with time labels $t_1 \leq t_2 \leq \cdots \leq t_N$. Continuous regression consists in searching for a curve $\gamma : [t_1, t_N] \to \mathbb{R}^n$ that simultaneously (i) reasonably fits the data and (ii) is sufficiently "smooth". One common way of formalizing this loose description of a natural need is to express γ as the minimizer of an objective function such as

$$E_c(\gamma) = \frac{1}{2}\sum_{i=1}^{N} \|p_i - \gamma(t_i)\|^2 + \frac{\lambda}{2}\int_{t_1}^{t_N} \|\dot\gamma(t)\|^2 \mathrm{d}t + \frac{\mu}{2}\int_{t_1}^{t_N} \|\ddot\gamma(t)\|^2 \mathrm{d}t, \quad (1)$$

defined over some suitable curve space. The parameters λ and μ tune the balance between fitting and smoothness.

One possible discretization of (1) is to reduce the curve space to the set of sequences of N_d discretization points γ_i (with fixed time labels τ_i) and to consider the following objective function:

$$E(\gamma) = \frac{1}{2}\sum_{i=1}^{N} \|p_i - \gamma_{s_i}\|^2 + \frac{\lambda}{2}\sum_{i=1}^{N_\mathrm{d}-1} \alpha_i \left\| \frac{\gamma_{i+1} - \gamma_i}{\Delta\tau} \right\|^2 + \frac{\mu}{2}\sum_{i=2}^{N_\mathrm{d}-1} \beta_i \left\| \frac{\gamma_{i+1} - 2\gamma_i + \gamma_{i-1}}{\Delta\tau^2} \right\|^2. \quad (2)$$

The indices s_i are chosen such that τ_{s_i} is closest (ideally equal) to t_i. The weights α_i and β_i follow from a suitable numerical integration scheme such as the trapezium rule. We assumed constant time-sampling $(\Delta\tau)$ for ease of notation.

In this work, we consider the more general situation where the data points p_i and the discrete curve γ lie on a manifold.

For $\mathscr{M}$ a nonlinear manifold, classical finite differences no longer make sense. Finite differences are generalized to manifolds in this work, which permits the introduction of a generalization of (2):

$$E(\gamma) = \frac{1}{2}\sum_{i=1}^{N} \mathrm{dist}^2(p_i, \gamma_{s_i}) + \frac{\lambda}{2}\sum_{i=1}^{N_\mathrm{d}-1} \alpha_i \left\| \frac{\mathrm{Log}_{\gamma_i}(\gamma_{i+1})}{\Delta\tau} \right\|_{\gamma_i}^2 + \frac{\mu}{2}\sum_{i=2}^{N_\mathrm{d}-1} \beta_i \left\| \frac{\mathrm{Log}_{\gamma_i}(\gamma_{i+1}) + \mathrm{Log}_{\gamma_i}(\gamma_{i-1})}{\Delta\tau^2} \right\|_{\gamma_i}^2, \quad (3)$$

where dist is the Riemannian distance and Log is the Riemannian logarithmic map [1]. Equation (3) is a slightly simplified version of our general formulation.

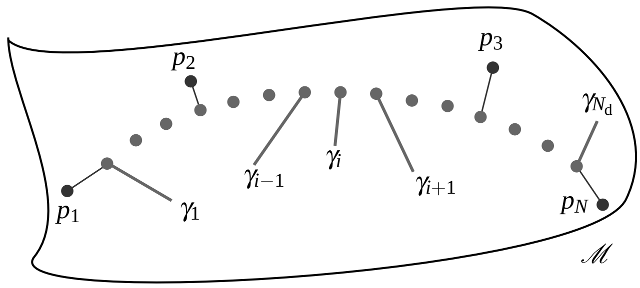

Figure 1: Our algorithms fit a "smooth", discrete curve $(\gamma_1, \ldots, \gamma_{N_\mathrm{d}})$ to N data points p_i on a manifold $\mathscr{M}$.

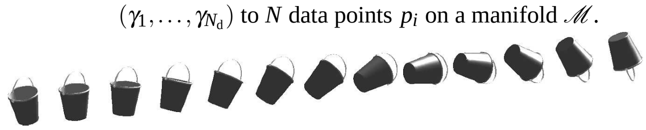

Figure 2: Smooth path of rotations in $\mathbb{R}^3$ taken from [2].

We minimize (3) using a Riemannian Conjugate Gradient method [1] on a few manifolds including the sphere, the special orthogonal group (rotations in $\mathbb{R}^n$, see Figure 2) and the set of positive-definite matrices.

The major challenge now is to develop second-order optimization methods for the objective function (3). In future work, we will explore the use of approximations of the logarithmic map to circumvent this difficulty.

References

[1] P.-A. Absil, R. Mahony, and R. Sepulchre. *Optimization Algorithms on Matrix Manifolds*. Princeton University Press, Princeton, NJ, 2008.

[2] Nicolas Boumal and P.-A. Absil. A discrete regression method on manifolds and its application to data on SO(n). *Technical Report UCL-INMA-2010.059 (submitted), Université catholique de Louvain*, 2010.