

Two-edge connected subgraphs with bounded rings: Polyhedral results and Branch-and-Cut

Abstract. We consider the network design problem which consists in determining at minimum cost a 2-edge connected network such that the shortest cycle (a “ring”) to which each edge belongs, does not exceed a given length K . We identify a class of valid inequalities, called cycle inequalities, valid for the problem and show that this inequalities together with the so-called cut inequalities yield an integer programming formulation of the problem in the space of the natural design variables. We then study the polytope associated with that problem and describe further classes of valid inequalities. We give necessary and sufficient conditions for these inequalities to be facet defining. We study the separation problem associated with these inequalities. In particular, we show that the cycle inequalities can be separated in polynomial time when $K \leq 4$. We develop a Branch-and-Cut algorithm based on these results and present extensive computational results.

1. Introduction

The design of cost effective survivable networks is of major concern to the telecommunication industry. Survivable networks must satisfy some connectivity requirements that would permit the restoration of the network service in the event of a catastrophic failure such as the loss of a transmission link or the failure of a switching node. The service could be restored by rerouting the traffic through other existing network edges and nodes.

In the case of fiber optic networks, a 2-connected topology has shown to provide an adequate level of survivability. Since this topology yields the design of sparse “cycle-like” networks (Hamiltonian cycles) [12], in case of edge (or node) failure, the traffic might be rerouted through all the (remaining) edges (nodes) of the network. This is however undesirable in practice.

To limit the length of the rerouting, one can impose a limit on the length of the shortest cycle containing each edge. And hence the survivable network problem in this case would consist in establishing a minimum 2-connected network where each edge belongs to a cycle not exceeding a given length. In this paper we consider this problem.

A graph $G = (V, E)$ is said to be *k-edge connected* if between every pair of nodes of G , there are at least k edge-disjoint paths. Given a graph $G = (V, E)$, with edge costs $c_e \geq 0$, for all $e \in E$, and an integer $K \geq 3$, the *2-edge connected subgraph with bounded rings problem (2ECSBR)* is to find a minimum cost 2-edge connected subgraph $H = (V, F)$ such that each edge of F belongs to a cycle of length less than or equal to K . Such a cycle is called a *feasible cycle*. This problem is a generalization of the classical minimum 2-edge connected subgraph problem. In fact, the latter is nothing but 2ECSBR when $K \geq |V|$.

Fortz et al. [12] consider the node version of 2ECSBR, that is the problem of finding a minimum cost 2-node connected spanning subgraph where every edge belongs to a cycle of length less than or equal to K . They derive valid and facet defining inequalities for the associated polytope, and devise separation procedures. They also present a cutting plane algorithm along with experimental results. In [10], Fortz and Labbé give a formulation for the problem based on a set covering approach. They provide further classes of facets and discuss the associated separation problems. They also report computational results with a cutting plane algorithm. Actually, the results in [10] are related to a project funded by Belgacom, the Belgian telecommunication operator, dealing with the development of new models and optimization techniques for the long-term planning of the backbone network. Related work can also be found in [11]. For an extensive survey of this problem, see Fortz [9].

In this paper we study the 2ECSBR. We identify a class of valid inequalities, called cycle inequalities, and we show that these inequalities together with the cut and the integrality constraints yield a 0 – 1 integer programming

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formulation for the problem in the space of the natural design variables. We show that some facet defining inequalities for the 2-node connected polytope with bounded rings are also facet defining inequalities for the 2ECSBR polytope. We describe new classes of valid inequalities for the 2ECSBR polytope and give necessary and sufficient conditions for most of these inequalities to be facet defining. We show that the separation problem for the cycle inequalities when $K \leq 4$ can be reduced to solving a max-flow problem and then can be solved in polynomial time, and we give a heuristic for solving this problem for $K \geq 5$. We also devise separation procedures for the other classes of inequalities. These are used in a Branch-and-Cut algorithm for which we report computational results on random and real-world problems.

In addition to the fact that we study here the edge connectivity version of the problem, which is also of practical interest, we bring several contributions that were not part of the previous work of Fortz et al on the node connectivity version. The major new contribution is the introduction of cycle inequalities that are sufficient, with cut constraints, to formulate the problem as a 0 – 1 integer program. Cycle partition inequalities are also new. The characterization of facet defining cyclomatic inequalities extend the work in [11] to the full class of inequalities, while only a special case was considered in [11]. Another major difference is the fact that we consider the polytope associated to the problem and not its dominant as in [10, 11].

The minimum 2-edge connected subgraph problem has been along the last fifteen years the subject of extensive research. In [26] Monma et al. studied that problem in the metric case (generally considered in practice), that is when the underlying graph is complete and the cost vector c satisfies the triangle inequality (i.e. $c_{ij} \leq c_{ik} + c_{kj}$ for any triple of distinct nodes i, j and k). They showed in this case that the weight of an optimal traveling salesman tour is no greater than $\frac{4}{3}Q$ where Q is the weight of an optimal 2-edge connected spanning subgraph of G . They also showed that an optimal 2-edge (2-node) connected spanning subgraph of G can be chosen so that every vertex has degree 2 or 3, and that removing one or two edges (nodes) does not leave the resulting connected components all 2-edge (2-node) connected. This shows that an optimal 2-edge connected spanning subgraph has a structure that is very close to that of a Hamiltonian cycle. And this is the motivation to impose a bound on the length of the cycles covering the optimal graph. Further structural properties and worst case analysis are given in Bienstock et al. [5], Cunningham [7] and Goemans and Bertsimas [4].

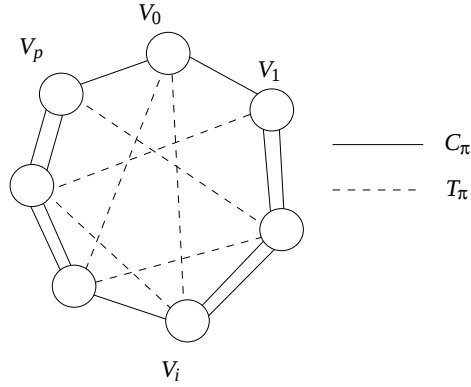
In [24] Mahjoub gave a complete description of the 2-edge connected polytope when the underlying graph is series-parallel. In [3] Barahona and Mahjoub characterized this polytope in the class of Halin graphs. Further polyhedral results related to k -edge (k -node) connected graphs can be found in [5, 6, 8].

The 2-edge connected subgraph problem is a special case of the more general survivable network design problem. Given a graph $G = (V, E)$ with weights $(w_e, e \in E)$ on its edges and integers $(r_i, i \in V)$ associated to its nodes, called connectivity type, the survivable network design problem is to determine a minimum weight subgraph of G such that between every pair of nodes i and j , there are $r(i, j) = \min(r_i, r_j)$ edge-disjoint paths. Grötschel et al. [16–18] studied this problem. In [17] they derived valid and facet defining inequalities for the associated polytope. In [16, 18], they devised cutting plane algorithms and presented some experimental results.

Approximation algorithms have also been devised for the survivable network design problem. These algorithms are based on the function f that associates to $S \subseteq V$, the minimum number of edges crossing $\delta(S)$. Williamson et al. [28] gave a polynomial time $2 f_{\max}$ -approximation algorithm where the function f is proper and $f_{\max} = \max\{f(S), S \subseteq V\}$. (A function f is proper if $f(V) = 0$, $f(S) = f(V \setminus S)$ for each $S \subseteq V$, and $f(A \cup B) \leq \max\{f(A), f(B)\}$ whenever A and B are disjoint.) In [13], Goemans et al. improved this by presenting an approximation algorithm with a performance guarantee of $2\mathcal{H}(f_{\max})$ where $\mathcal{H}(f_{\max}) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{f_{\max}}$ is the harmonic function. And when the function f is weakly supermodular, Jain [22] proposed a factor 2 approximation algorithm. (A function f is supermodular if $f(V) = 0$ and for every $A, B \subseteq V$, at least one of the following holds: $f(A) + f(B) \leq f(A \setminus B) + f(B \setminus A)$ or $f(A) + f(B) \leq f(A \cup B) + f(B \cap A)$.)

The paper is organized as follows. In the following section we give a 0 – 1 integer programming formulation for the 2ECSBR and study the dimension of the associated polytope. In Section 3 we describe various classes of valid inequalities for this polytope and give necessary and sufficient conditions for these inequalities to be facet defining. We also discuss separation algorithms for these inequalities. In section 4 we devise a Branch-and-Cut algorithm based on these results for the 2ECSBR. In Section 5 we present some computational results and in Section 6 we give some concluding remarks.

The rest of this section is devoted to more definitions and notations. The graphs we consider are undirected and loopless. We denote a graph by $G = (V, E)$ where V is the node set and E the edge set. Throughout this paper, $n := |V|$ and $m := |E|$ will denote the number of nodes and edges of G .

Fig. 1. C_π and T_π

Given a graph $G = (V, E)$ and $W \subset V$, the edge set

$$\delta(W) := \{ij \in E \mid i \in W, j \in V \setminus W\}$$

is called the *cut* induced by W . If $V_0, \dots, V_p$ are pairwise disjoint node subsets of V , then $\delta(V_0, \dots, V_p)$ denotes the set of edges having one node in V_i and the other in V_j for $i \neq j$. The set of edges having one endnode in V_i and the other in V_j , $i \neq j$ is denoted by $[V_i, V_j]$. If $\pi = (V_0, \dots, V_p)$ is a partition of V , then we let $C_\pi = \cup_{i=0}^{p-1} [V_i, V_{i+1}] \cup [V_0, V_p]$ and $T_\pi = \delta(V_0, \dots, V_p) \setminus C_\pi$ (see Figure 1).

The *degree* of a node v is the cardinality of $\delta(v)$. Given a node subset W of V , we denote by $E(W)$ the set of edges having both endnodes in W . If $F \subseteq E$ is an edge subset, we denote by $V(F)$ the set of nodes incident to edges of F . We denote by $G(W) = (W, E(W))$ the subgraph induced by edges having both endnodes in W . G/W is the graph obtained from G by contracting the nodes in W to a new node w (retaining parallel edges). Contracting an edge set F consists in contracting the nodes of $V(F)$.

We denote by $V - z := V \setminus \{z\}$ and $E - e := E \setminus \{e\}$ the subsets obtained by removing one node or one edge from the set of nodes or edges, and $G - z$ denotes the graph $G(V - z)$, i.e. the graph obtained by removing a node z and its incident edges from G .

If $x \in \mathbb{R}^m$, the support graph of x , denoted by G_x , is the graph with vertex set V and edge set $\{e \mid x_e > 0\}$.

2. Formulation and associated polyhedron

In this section, we formulate 2ECSBR as a 0 – 1 integer program in the space of the natural variables x . We also determine the dimension of the associated polyhedron. To this end, we first introduce some notation and a family of valid inequalities for 2ECSBR.

We denote by $\mathcal{F}(G)$ the set of edge-sets that induce feasible solutions of 2ECSBR. Given an edge subset $F \subset E$, we denote by $x^F \in \mathbb{R}^m$ the *incidence vector* of F , given by:

$$x_e^F = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, each vector $x \in \{0, 1\}^m$ induces a subset

$$F^x := \{e \in E \mid x_e = 1\}.$$

The convex hull of the incidence vectors of all the solutions of 2ECSBR is

$$\mathcal{P}(G, K) := \text{conv}\{x \in \{0, 1\}^m \mid F^x \in \mathcal{F}(G)\}.$$

Finally, for any subset of edges $F \subseteq E$ we define $x(F) := \sum_{e \in F} x_e$.

Theorem 1. Let $G = (V, E)$ be a graph and $K \geq 3$. Let $\pi = (V_0, \dots, V_p)$ be a partition of V such that $p \geq K$ and let $e \in [V_0, V_p]$. Then, the inequality

$$x(T_\pi^e) \geq x_e \quad (2.1)$$

is valid for $\mathcal{P}(G, K)$, with

$$T_\pi^e := T_\pi \cup ([V_0, V_p] \setminus \{e\}). \quad (2.2)$$

Proof. Let $F \subseteq E$ be an edge set defining a solution of 2ECSBR. If x^F does not satisfy (2.1), then $e \in F$ and $F \cap (T_\pi^e) = \emptyset$. By the definition of T_π^e , any cycle in F using e is completely included in C_π , and is therefore of length greater than p . As $p \geq K$, e does not belong to a feasible cycle, which leads to a contradiction since F is feasible and $e \in F$. $\square$

Inequalities (2.1) are called *cycle inequalities*.

Theorem 2. Let $G = (V, E)$ be a graph and $K \geq 3$. 2ECSBR is equivalent to the following integer programming problem:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & x(\delta(W)) \geq 2 \quad W \subset V, \emptyset \neq W \neq V, \\ & x(T_\pi^e) \geq x_e \quad e \in [V_0, V_K], \pi = (V_0, \dots, V_K) \text{ is a partition of } V, \\ & \text{and } T_\pi^e \text{ is defined by (2.2),} \end{aligned} \quad (2.3)$$

$$\begin{aligned} & 0 \leq x_e \leq 1 \quad e \in E, \\ & x_e \in \{0, 1\} \quad e \in E. \end{aligned} \quad (2.4)$$

$$\quad (2.5)$$

$$\quad (2.6)$$

Proof. By Theorem 1, the incidence vector of any solution of 2ECSBR satisfies the constraints (2.3)-(2.6). Now, consider an edge set $F \subseteq E$ that does not induce a solution of 2ECSBR. We are going to show that there exists a constraint among (2.3)-(2.6) which is not satisfied by x^F . Obviously, if F does not induce a 2-edge connected subgraph, then at least one of the constraints (2.3) is violated by x^F . Thus, we may suppose that x^F induces a 2-edge connected subgraph of G . Hence, there must exist an edge $e := uv \in F$ that does not belong to a cycle of length less than or equal to K . Let $d(u, i)$ denote the distance from u to i in $F \setminus \{e\}$. Then, $d(u, v) \geq K$ and the cycle constraint (2.1) associated with the partition

$$\begin{aligned} V_i &= \{w \in V \mid d(u, w) = i\} \quad i = 0, \dots, K-1, \\ V_K &= \{w \in V \mid d(u, w) \geq K\} \end{aligned}$$

together with edge e is violated by x^F , which completes the proof of the theorem. $\square$

Inequalities (2.3) are called *cut inequalities*, and inequalities (2.5) are called *trivial inequalities*.

The following remark follows from the proof of Theorem 2.

Remark 1. If for an edge $e = st \in E$, there is no cycle of length $\leq K$ containing e in G , then there is a partition $\pi = (V_0, \dots, V_p)$ with $p \geq K$ and $e \in [V_0, V_p]$, such that $T_\pi^e = \emptyset$. Moreover, this partition can be obtained by a breadth-first search from one of the nodes s and t .

Remark 2. By adding to the formulation given by Theorem 2 the constraints

$$x(\delta_{G-v}(W)) \geq 1, \quad W \subset V \setminus \{v\}, v \in V,$$

we obtain a formulation for the 2-node connected subgraph with bounded rings problem.

By Theorem 2, it follows that

$$\mathcal{P}(G, K) = \text{conv}\{x \in \mathbb{R}^m \mid x \text{ verifies (2.3)-(2.6)}\}.$$

We now study the dimension of $\mathcal{P}(G, K)$. To this aim, we need the following concepts. Given a graph $G = (V, E)$, a constant $K \geq 3$, and a subset of edges $F \subseteq E$, the *restriction of F to bounded rings* (Fortz and Labbé [10]) is defined as

$$F_K := \{e \in F \mid e \text{ belongs to at least one feasible cycle in } F\}.$$

By extension, the subgraph $G_K = (V, E_K)$ is called the *restriction of G to bounded rings*. Remark that an edge $e \in E \setminus E_K$ will never belong to a feasible solution of 2ECSBR and is called an *infeasible edge*.

Given two edges $e, f \in E_K$, ($e \neq f$), we say that e *depends* on f , denoted by $e \prec f$, if and only if

$$e \in F \Rightarrow f \in F, \quad \text{for all } F \in \mathcal{F}(G),$$

or equivalently,

$$x_e^F \leq x_f^F, \quad \text{for all } F \in \mathcal{F}(G).$$

Note that the dependence relation $\prec$ is transitive and reflexive, and therefore defines a semiorder on the edges of E . We also define $\Gamma(e)$ as the set of edges that depend on e , i.e.

$$\Gamma(e) = \{f \in E \mid f \prec e\}.$$

If $e \prec f$ and $f \prec e$, then e and f are *equivalent* and $x_e^F = x_f^F$ for all $F \in \mathcal{F}(G)$. A maximal subset T (with respect to inclusion) of equivalent edges is an *equivalence class* for the semiorder $\prec$, and $\prec$ induces a partition of any set of edges into equivalence classes. As $\Gamma(e) = \Gamma(f)$ if e and f are equivalent, we extend the notation to a subset of equivalent edges T by setting $\Gamma(T) := \Gamma(e)$, $e \in T$. Finally, an edge $e \in E$ is *essential* if e belongs to every solution $F \in \mathcal{F}(G)$.

The dimension of $\mathcal{P}(G, K)$ is stated in the following theorem.

Theorem 3. *Let $G = (V, E)$ be a graph. Let E^* be the set of essential edges. Suppose $\prec$ induces a partition of $E_K \setminus E^*$ into l equivalence classes. Then*

$$\dim(\mathcal{P}(G, K)) = l.$$

Proof. Suppose the l equivalence classes of $E_K \setminus E^*$ are $T_i := \{e_1^i, \dots, e_{k_i}^i\}$ of cardinality k_i for $i = 1, \dots, l$. Since the sets T_i , $i = 1, \dots, l$, form a partition of $E_K \setminus E^*$,

$$\sum_{i=1}^l k_i = |E_K| - |E^*|.$$

Let x be the incidence vector of a feasible solution of 2ECSBR. From the definition of E_K , E^* and $\prec$, it is easy to see that x is a solution of the system

$$(S) \begin{cases} x_e = 0 & e \in E \setminus E_K, \\ x_e = 1 & e \in E^*, \\ x(e_j^i) - x(e_{j+1}^i) = 0 & j = 1, \dots, k_i - 1, i = 1, \dots, l. \end{cases}$$

As $E \setminus E_K$, E^* and T_i , $i = 1, \dots, l$ are obviously pairwise disjoint, the equalities in (S) are linearly independent and

$$\dim(\mathcal{P}(G, K)) \leq |E_K| - |E^*| - \sum_{i=1}^l (k_i - 1) = l.$$

Now consider the $l + 1$ subsets $F_0 := E_K$ and $F_i := E_K \setminus \Gamma(T_i)$, $i = 1, \dots, l$. It is easy to see that $F_i \in \mathcal{F}(G)$ for $i = 0, \dots, l$. As T_i are equivalence classes for the semiorder $\prec$, and by the definition of Γ , there exists a permutation σ of $1, \dots, l$ such that, for all $i = 1, \dots, l - 1$, there exists $e_i \in \Gamma(T_{\sigma(i)})$, $e_i \notin \Gamma(T_{\sigma(j)})$, for $j = i + 1, \dots, l$. It follows that the incidence vectors $x^{\Gamma(T_i)}$ are linearly independent for $i = 1, \dots, l$. Noting that $x^{\Gamma(T_i)} = x^{F_0} - x^{F_i}$, it follows that the incidence vectors of $F_0, \dots, F_l$ are affinely independent, which implies that $\dim(\mathcal{P}(G, K)) \geq l$. $\square$

Corollary 1. *If $G = (V, E)$ is a complete graph with $|V| \geq 4$, then $\mathcal{P}(G, K)$ is full dimensional.*

Proof. It is easy to see that E and $E - e$, for all $e \in E$, induce feasible solutions of 2ECSBR. Therefore, $E = E_K$, $E^* = \emptyset$ and no edge depends on another edge. It follows that there are no equivalent edges, and $E_K \setminus E^* = E$ is partitioned in m equivalence classes. The result follows by Theorem 3. $\square$

In the remainder of the paper, we assume that $G = (V, E)$ is a complete graph. This assumption is not restrictive, since the problem in an incomplete graph can be reduced to the problem in a complete graph by giving a sufficiently high cost to non-existent edges.

3. Valid inequalities and separation

In this section, we introduce some valid inequalities for $\mathcal{P}(G, K)$ and, for several classes of inequalities, we give necessary and sufficient conditions for these inequalities to be facet defining. We also discuss separation algorithms.

3.1. Cut inequalities

We now provide a characterization of facet defining cut inequalities (2.3).

Theorem 4. *Let $G = (V, E)$ be a complete graph, $K \geq 3$ a given constant, and $W \subseteq V$ a subset of nodes, $\emptyset \neq W \neq V$. The inequality*

$$x(\delta(W)) \geq 2$$

defines a facet of $\mathcal{P}(G, K)$ if and only if

- either $K \geq 4$, $|W| \neq 2$ and $|V \setminus W| \neq 2$,
- or $K = 3$, $|W| \notin \{2, 3\}$ and $|V \setminus W| \notin \{2, 3\}$.

Proof. If $W = \{u, v\}$, then $E(W)$ is composed of the single edge $e := uv$ and the cut inequality can be written as the sum of $x(\delta(u)) \geq 2$, $x(\delta(v)) \geq 2$, and $-2x_e \geq -2$, thus it does not define a facet. A similar result holds for $V \setminus W$.

Now, suppose that $K = 3$ and $W = \{u, v, w\}$, and consider a solution $F \in \mathcal{F}(G)$ whose incidence vector lies in the face defined by $x(\delta(W)) \geq 2$. As $|F \cap \delta(W)| = 2$, there is one node in W that is not incident to edges of the solution that belong to the cut. Without loss of generality, let u be this node. As u has a degree at least equal to 2, it must be adjacent to v and w . As uv and uw must belong to a cycle of length at most 3, vw must also belong to the solution and $E(W) \subseteq F$. It follows that the incidence vector of any solution lying in the face defined by $x(\delta(W)) \geq 2$ is a solution of the system

$$\begin{aligned} x(E(W)) &= 3, \\ x(\delta(W)) &= 2. \end{aligned}$$

As this system is of rank 2, the face $x(\delta(W)) \geq 2$ is of dimension at most $|E| - 2$ and cannot define a facet. A similar result holds for $V \setminus W$.

Conversely, let the conditions be satisfied for some inequality $a^T x := x(\delta(W)) \geq 2$. Let $b^T x \geq \beta$ be a facet defining inequality such that the face F_a induced by $a^T x \geq 2$ in $\mathcal{P}(G, K)$ is contained in the face F_b induced by $b^T x \geq \beta$. For any pair of edges $e, f \in \delta(W)$, we define the subset of edges $C_{e,f}$ as

$$C_{e,f} := E(W) \cup E(V \setminus W) \cup \{e, f\}.$$

The solution induced by $C_{e,f}$ is obviously two-edge connected. If e and f are adjacent to a common node, then e and f belong to a cycle of length 3 in $C_{e,f}$. All the other edges of $C_{e,f}$ belong to a cycle of length 3 since $E(W)$ and $E(V \setminus W)$ induce complete subgraphs. Therefore, if e and f are adjacent to a common node, $C_{e,f}$ lies in the face defined by the cut inequality.

Consider two edges e and f in $\delta(W)$ with no common endpoint. Suppose $e := u_1 u_2$ and $f := v_1 v_2$ with u_1, v_1 in W and $u_2, v_2 \in V \setminus W$, and consider edge $g := u_1 v_2$. From the reasoning above, $C_{e,g}$ and $C_{f,g}$ lie in the face defined by the cut inequality, and therefore, $b_e = b_f$.

If e and f are two edges in $\delta(W)$ with a common endpoint, any edge $g \in \delta(W)$ distinct from e and f and sharing the same endpoint will lead to subsets $C_{e,g}$ and $C_{f,g}$ lying in the face defined by the cut inequality, again leading to $b_e = b_f$. Therefore, all the variables corresponding to edges in the cut have the same coefficient.

If $K \geq 4$ and $|W| = 3$, consider an edge $e \in E(W)$, and two edges $f, g \in \delta(W)$ such that e, f and g form a triangle. Let us suppose that $W = \{u, v, w\}$, $e := uv$ and $f := uz$, $g := vz$ with $z \in V \setminus W$. Then f and g together with uw and vw form a feasible cycle, and $C_{f,g} - e$ is feasible and lies in the face. As $C_{f,g}$ lies in the face too, $b_e = 0$.

From now on, we may thus assume that $|W| \geq 4$. Consider three distinct nodes $u, v, w \in W$ and a node $z \in V \setminus W$, and the edges $e := uv$, $f := uz$ and $g := wz$. The edges f and g belong to a cycle of length 3 with the edge uw , while any edge in $E(W) - e$ belongs to a feasible cycle, as well as any edge in $E(V \setminus W)$, since $|W| \geq 4$. It follows that $C_{f,g} - e$ is feasible and lies in the face. As $C_{f,g}$ lies in the face too, $b_e = 0$.

A similar proof leads to $b_e = 0$ for $e \in E(V \setminus W)$.

We have proved that $b = \gamma a$. Since F_b cannot define a facet if $b \leq 0$, we have $\gamma > 0$. Thus $x(\delta(W)) \geq 2$ and $b^T x \geq \beta$ define the same facet $F_a = F_b$. $\square$

The separation of cut constraints can be carried out by computing a minimum cut in the graph, with capacities given by the current LP solution. This can be done in polynomial time, e.g. by the Hao-Orlin algorithm [20] that requires one maximum flow computation. But this algorithm does not output the Gomory-Hu tree [15] that provides the minimum cut between all the pairs of nodes. For this we use Gomory-Hu algorithm [15] for separating these inequalities.

To speed up the computation, we have also developed a simple heuristic to separate the cut inequalities. This works as follow: we repeatedly contract edges with high values until we obtain either a graph of weight less than p , where p is the number of nodes, or a graph on two nodes. If the former case holds, then at least one of the cuts induced by the nodes of the graph is violated. If the latter case holds, we just check whether the cut yielded by the graph is violated.

3.2. Cycle inequalities

The next theorem characterizes which cycle inequalities are facet defining.

Theorem 5. *Let $G = (V, E)$ be a complete graph and $K \geq 3$. Let $\pi = (V_0, \dots, V_p)$ be a partition of V such that $p \geq K$. Let $e := uv$ be an edge of $[V_0, V_p]$. Then the cycle inequality $x(T_\pi^e) \geq x_e$ associated with π and e defines a facet of $\mathcal{P}(G, K)$ if and only if*

1. $p = K$,
2. $|V_0| = |V_p| = 1$,
3. $|V_i| + |V_{i+1}| \geq 3$ for $i = 0, \dots, p-1$.

Proof. Suppose that $p > K$. Consider the partition $\pi' = (V'_0, \dots, V'_K)$ defined by $V'_i = V_i$ for $i = 0, \dots, K-1$ and $V'_K = \bigcup_{j=K}^p V_j$. It is easy to see that the cycle inequality associated with π and e is dominated by that associated with π' and e . Hence $x(T_\pi^e) \geq x_e$ does not define a facet.

Now suppose that $p = K$ and assume that $|V_0| > 1$ (the case $|V_p| > 1$ is similar). Let $\pi' = (V'_0, \dots, V'_K)$ be the partition defined by

$$\begin{aligned} V'_0 &= \{u\}, \\ V'_1 &= V_1 \cup (V_0 \setminus \{u\}), \\ V'_i &= V_i \text{ for } i = 2, \dots, K. \end{aligned}$$

Again, the cycle inequality associated with π and e is dominated by that associated with π' and e .

Finally, suppose that there exists $i \in \{0, \dots, p-1\}$ such that $|V_i| + |V_{i+1}| < 3$. Since the subsets defining the partition are nonempty, it follows that $|V_i| = |V_{i+1}| = 1$, and that $[V_i, V_{i+1}]$ contains a single edge f . If $x(T_\pi^e) \geq x_e$ defines a facet, then there exists a feasible solution F such that $x^F(T_\pi^e) = x_e^F = 0$. For otherwise, the face defined by $x(T_\pi^e) \geq x_e$ is included in the face defined by $x_e \leq 1$. But then,

$$\delta\left(\bigcup_{j=0}^i V_j\right) \cap F = [V_i, V_{i+1}] = \{f\},$$

and F is not 2-edge connected, thus not feasible, which leads to a contradiction.

The three conditions are thus necessary for $x(T_\pi^e) \geq x_e$ to define a facet. Suppose they are all satisfied, and let us denote the cycle inequality by $a^T x \geq 0$. Let $b^T x \geq \beta$ be a facet defining inequality such that the face F_a induced by $a^T x \geq 0$ in $\mathcal{P}(G, K)$ is contained in the face F_b induced by $b^T x \geq \beta$.

We first show that $\bar{E} = E \setminus (T_\pi^e \cup \{e\})$ induces a feasible solution of 2ECSBR, and therefore its incidence vector lies in F_a . To do so, we show that every edge of $\bar{E}$ belongs to a cycle of length 3, and then we show that $\bar{E}$ induces a 2-edge connected subgraph.

Consider an edge $g = v_1 v_2 \in \bar{E}$. If g belongs to $E(V_i)$ for some $i \in \{1, \dots, K-1\}$, then for any $w \in V_{i-1}$, the cycle composed of g , $v_1 w$ and $v_2 w$ is included in $\bar{E}$. Otherwise, $g \in [V_i, V_{i+1}]$ for some $i \in \{0, \dots, K-1\}$. By the third condition, the graph induced by $V_i \cup V_{i+1}$, with edge set $E(V_i) \cup E(V_{i+1}) \cup [V_i, V_{i+1}]$, is a complete graph on at least 3 nodes. Therefore g belongs to a cycle of length 3.

As $\bar{E}$ induces a connected graph, each cut contains at least one edge. Since each edge belongs to a cycle, it follows that each cut contains at least two edges, and $\bar{E}$ induces a 2-edge connected subgraph.

Now, consider the edge set $E_f = E \cup \{f, e\}$, with $f \in T_\pi^e$. Clearly, E_f is 2-edge connected as $\bar{E}$ is. To show that E_f induces a feasible solution, it is thus sufficient to show that e and f belong to a ring of length $\leq K$. Suppose $f = v_i v_j \in [V_i, V_j]$ with $i < j$. As $f \in T_\pi^e$, $j \geq i+2$. For $k = 0, \dots, i-1, j+1, \dots, K$, select a node $v_k \in V_k$. As $e = v_0 v_K \in E_f$

and $v_k v_{k+1} \in E_f$ for $k = 0, \dots, i-1, j, \dots, K-1$, the edges e, f and $v_k v_{k+1}$ for $k = 0, \dots, i-1, j, \dots, K-1$ form a cycle of length $2 + i + K - j \leq K$ since $j \geq i + 2$. Therefore, the incidence vector of E_f lies in F_a , thus also in F_b , and $bx^E = bx^{E_f}$. It follows immediately that $b_f = -b_e$ for all $f \in T_\pi^e$.

It remains to show that $b_g = 0$ for all $g \in \bar{E}$. First suppose $g \in E(V_i)$ for some $i \in \{1, \dots, K-1\}$. Let $g = u_1 u_2$, and consider two nodes $w_1 \in V_{i-1}$ and $w_2 \in V_{i+1}$. Observe that $w_1 w_2 \in T_\pi^e$, and let $\tilde{E} = (\bar{E} \setminus \{g\}) \cup \{w_1 w_2, e\}$. As $\tilde{E}$ induces a connected subgraph, to show that it induces a feasible solution, it is sufficient to show that every edge of $\tilde{E}$ belongs to a feasible cycle. Let $h = st \in \bar{E} \setminus \{g\}$. If $h \in E(V_j)$ for some $j \in \{1, \dots, K-1\}$, then clearly, h belongs to a triangle, and hence to a feasible cycle. So suppose that $s \in V_j$ and $t \in V_{j+1}$ for some $j \in \{0, \dots, K-1\}$. If $j \neq i-1$ and $j \neq i$, then by condition 3, we may assume w.l.o.g. that $|V_j| \geq 2$. If $s' \in V_j \setminus \{s\}$, then h belongs to the triangle induced by $\{s, s', t\}$. Now assume that $j = i$ (the case $j = i-1$ is similar). If $|V_{j+1}| \geq 2$, then it is easy to show as before that h belongs to a triangle. If $|V_{j+1}| = 1$, then $w_2 = t$, and h belongs to the triangle induced by $\{s, t, w_1\}$.

Moreover, edges e and $w_1 w_2$ belong to a feasible cycle. Indeed, for $k = 0, \dots, i-2, i+2, \dots, K$, select a node $v_k \in V_k$. Then, $e, w_1 w_2, v_{i-2} w_1, w_2 v_{i+2}$ and $v_k v_{k+1}$ for $k = 0, \dots, i-3, i+2, \dots, K-1$ form a cycle of length $4 + (i-2) + (K-i-2) = K$. In consequence, the incidence vector of $\tilde{E}$ lies in F_a , thus also in F_b , and $bx^{\tilde{E}} = bx^E$. It follows immediately that $b_e + b_{w_1 w_2} - b_g = 0$. As we have shown that $b_{w_1 w_2} = -b_e$, it follows that $b_g = 0$ for all $g \in E(V_i)$, $i = 1, \dots, K-1$.

A similar proof leads to $b_g = 0$ for $g \in [V_i, V_{i+1}]$, $i \in \{0, \dots, K-1\}$, and as a direct consequence, $\beta = bx^{\bar{E}} = 0$.

We have proved that $b = \gamma a$ with $\gamma = -b_e$. Since the complete graph defines a feasible solution not lying in F_b , $bx^E = (m-1)\gamma > 0$, and $\gamma > 0$. Thus $a^T x \geq 0$ and $b^T x \geq \beta$ define the same facet $F_a = F_b$. $\square$

In the following, we discuss the separation problem for the cycle inequalities. In particular, we show that these inequalities can be separated in polynomial time when $K \leq 4$. Note that, based on the proof of Theorem 2, it can be shown that the separation problem for inequalities (2.1) can be easily solved in polynomial time for a 0-1 solution $\bar{x}$.

Suppose instead that the solution contains fractional values. Let $G = (V, E)$ be a graph and s, t two nodes of V . Given a positive integer B , we define an (s, t) - B -path cut to be any edge set C of E that intersects every (s, t) -path of G with at most B edges. Given a weight vector $w \in \mathbb{R}_+^m$, the minimum (s, t) - B -path cut problem (BPCP) is to find an (s, t) - B -path cut of minimum weight. We now show that BPCP is equivalent to the separation problem for inequalities (2.1).

Lemma 1. *Given a solution $\bar{x}$ of $\mathbb{R}_+^m$, the separation problem for inequalities (2.1) reduces to solving BPCP for every edge $e = st \in E$ and $B = K-1$ with respect to the weight vector $\bar{x}$.*

Proof. Suppose that $\bar{x}(C_e) < \bar{x}$ where C_e is a minimum (s, t) - B -path cut with respect to $\bar{x}$. Since there is no cycle of length $\leq K$ in $E \setminus C_e$ containing e , by Remark 1, there exists a partition $\pi = (V_0, \dots, V_p)$ of V with $e \in [V_0, V_p]$ such that $T_\pi^e \subseteq C_e$. As $\bar{x}_f \geq 0$ for all $f \in E$, we then have

$$\bar{x}(T) - \bar{x} \leq \bar{x}(C_e) - \bar{x} < 0,$$

which implies that inequalities (2.1) associated with the partition $(V_0, \dots, V_p)$ and e is violated.

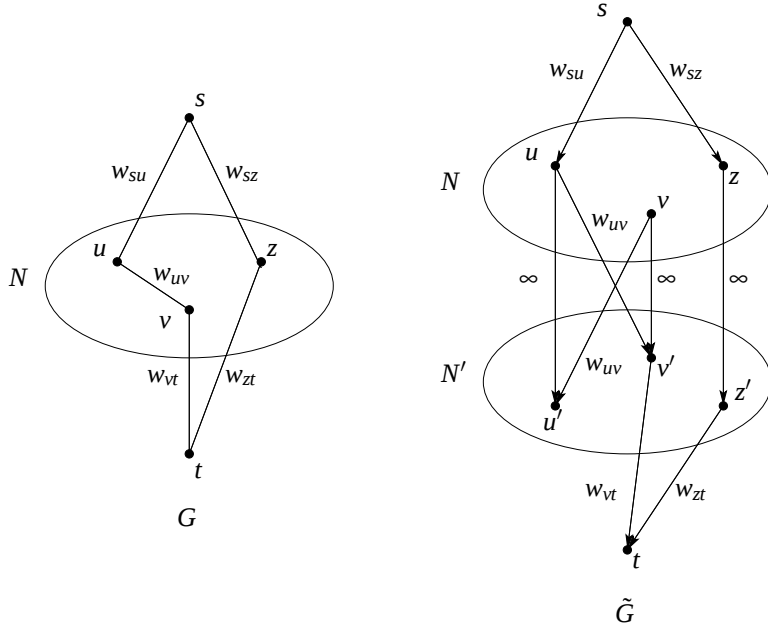
Now suppose that for every edge $e = st \in E$, $\bar{x}(C_e) \geq \bar{x}$ where C_e is a minimum (s, t) -($K-1$)-path cut with respect to $\bar{x}$. We claim that no inequality of type (2.1) is violated by $\bar{x}$. In fact, suppose that for an edge $e = st$, there is a partition $\pi = (V_0, \dots, V_p)$ with $e \in [V_0, V_p]$ and $p \geq K$ such that $\bar{x}(T) - \bar{x} < 0$. Since T_π^e is an (s, t) -($K-1$)-path cut, and hence a solution of BPCP, one should have $\bar{x}(C_e) \leq \bar{x}(T)$, a contradiction. $\square$

If $B = 2$, finding a minimum (s, t) -2-path cut reduces to finding a minimum cut separating s and t in the graph induced by s, t and the nodes adjacent to both s and t .

In what follows, we shall show that the minimum (s, t) - B -path cut problem remains polynomial if $B = 3$. For this case, we used ideas similar to those developed by Itai, Perl, and Shiloach [21] for solving a closely related problem.

Theorem 6. *The BPCP can be solved in polynomial time if $B = 3$.*

Proof. We show that the BPCP for $B = 3$ reduces to a maximum flow in an appropriate graph. To this end, let us note that any node u which is not adjacent whether to s nor to t cannot belong to an (s, t) -path of length at most 3 and so can be deleted. So we may assume that G does not contain such nodes. Let $N = V \setminus \{s, t\}$.

Fig. 2. Construction of $\tilde{G}$

In what follows, we are going to construct a directed graph $\tilde{G} = (\tilde{N}, \tilde{A})$. Let N' be a disjoint copy of N (where we denote the copy of $u \in N$ that is in N' by u'), and set $\tilde{N} = \{s\} \cup \{t\} \cup N \cup N'$. For each edge $su \in E$ with weight w_{su} make arc $(s, u) \in \tilde{A}$ with capacity w_{su} , for each edge $vt \in E$ make arc $(v', t) \in \tilde{A}$ with capacity w_{vt} , and for each edge $uv \in E$ with $u, v \notin \{s, t\}$, make arcs (u, v') and (v, u') , both with capacity w_{uv} . For each $u \in N$ with $u \neq s, t$ make an arc $(u, u') \in \tilde{A}$ with capacity infinity (see Figure 2 for an illustration). Note that there is a 1-1 correspondence between the (s, t) -paths of length ≤ 3 in G and the (s, t) -directed paths of length 3 in $\tilde{G}$.

We claim that there is a correspondence between the minimal feasible (s, t) -3-path cuts in G and the minimal finite capacity cuts in $\tilde{G}$ separating s and t , that preserves objective value. Once this fact is established, it follows that we can solve the BPCP for $B = 3$ by solving a maximum flow problem in $\tilde{G}$.

Let C be a minimal (s, t) -3-path cut in G . We say that uv is *gone* if either $uv \in C$ or $uv \notin E$. If $uv \in C$ with $u, v \notin \{s, t\}$, then by minimality of C either we have that the two edges su and vt are gone, or sv and ut are gone, but not both. Let $\tilde{C} \subseteq \tilde{A}$ be the arc set constructed as follows: if su or vt is in C , add arc (s, u) or (v', t) to $\tilde{C}$. If $uv \in C$ with $u, v \notin \{s, t\}$ and su and vt are gone, then add (v, u') to $\tilde{C}$, else add (u, v') to $\tilde{C}$. Then $\tilde{C}$ is a minimal finite capacity cut in $\tilde{G}$, separating s and t with capacity $w(C)$.

Now, suppose that $\tilde{C}$ is a minimal finite capacity cut in $\tilde{G}$, separating s and t . Thus $\tilde{C}$ does not contain any (u, u') arc. By minimality of $\tilde{C}$, an arc (u, v') belongs to $\tilde{C}$ only if both (s, u) and (v', t) are arcs of $\tilde{A} \setminus \tilde{C}$. This implies that if $(u, v') \in \tilde{C}$, then $(v, u') \notin \tilde{C}$. Let C be the edge set that contains each edge su with $(s, u) \in \tilde{C}$, each edge vt with $(v', t) \in \tilde{C}$, and each edge uv such that one of the arcs (u, v') and (v, u') is in $\tilde{C}$. It is not hard to see that C is a minimal (s, t) -3-path cut in G with weight $w(\tilde{C})$, and our theorem is proved. $\square$

Unfortunately, McCormick [25] shows that BPCP is NP-hard for $B \geq 12$. Hence we use a straightforward application of the primal-dual method [27] to get an approximation algorithm for BPCP that we use as a heuristic for separating (2.1). This algorithm, described in Algorithm 3.1, works as follows. It considers the linear programming relaxation of the BPCP and its dual. It first constructs an (s, t) - B -path cut C (i.e. a feasible solution of BPCP), and a dual solution $y = (y_P, P \in \mathcal{P}(B))$ where $\mathcal{P}(B)$ is the set of (s, t) -path of length $\leq B$, that satisfy the primal complementary slackness conditions, that is for all $e \in C$, $\sum_{P: e \in P} y_P = c_e$. After that, the algorithm tries to delete the unnecessary edges of C in order to get a solution with lower weight.

It is easy to see that this algorithm returns an (s, t) - B -path cut and runs in $O(|E|^2)$ time. Also from [27] it can be easily shown that this algorithm is a B -approximation algorithm. If $\bar{x}$ is a fractional solution, we can use this algorithm to separate inequalities (2.1) when $K \geq 5$ as follows. Using this algorithm, for an edge $e = st$, compute an (s, t) -($K-1$)-

Algorithm 3.1 Primal-Dual algorithm for BPCP

Data: a graph $G = (V, E)$, two nodes $s, t \in V$, a weight function w and a bound B .

```

 $l \leftarrow 0; y \leftarrow 0; C \leftarrow \emptyset;$ 
while  $C$  is not an  $(s, t)$ -B-path cut do
   $l \leftarrow l + 1;$ 
  Find a path  $P \in \mathcal{P}(B)$  such that  $P \cap C = \emptyset;$ 
  Increase  $y_P$  until some edge  $e_l \in P$  satisfies  $\sum_{Q: e_l \in Q} y_Q = w_{e_l};$ 
   $C \leftarrow C \cup \{e_l\};$ 
 $C' \leftarrow C;$ 
for  $j \leftarrow l$  down to 1 do
  if  $C' \setminus \{e_j\}$  is still an  $(s, t)$ -B-path cut then
     $C' \leftarrow C' \setminus \{e_j\};$ 
Return  $C';$ 

```

path cut C' in $G - e$. If $\bar{x}(C') < \bar{x}$, then it gives a violated cycle inequality. The partition that induces the inequality can be determined by a breadth-first search of the graph $(V, E \setminus (\{e\} \cup C'))$ from either s or t .

3.3. Subset inequalities

Fortz et al. have introduced in [10] the so-called subset inequalities for the 2-node connected version of the problem. The following result, given without proof, shows that these inequalities can be extended to the 2-edge connected case.

Proposition 1. *Let S be a set of edges such that the graph $G' = (V, E \setminus S)$ does not contain a solution of 2ECSBR. Then the inequality*

$$x(S) \geq 1 \quad (3.1)$$

is valid for $\mathcal{P}(G, K)$.

The next proposition provides sufficient conditions for subset inequalities to define facets.

Proposition 2. *Let $G = (V, E)$ be a complete graph and let S be a set of edges such that the graph $G - S$ does not contain a solution of 2ECSBR. If*

1. *for all $e \in S$, $G - (S \setminus \{e\})$ contains a feasible solution;*
2. *for all $f \in E \setminus S$, there exist $e \in S$ and $F \subseteq E \setminus (S \setminus \{e\})$ such that $f \notin F$ and both F and $F \cup \{f\}$ define feasible solutions of 2ECSBR;*

then the subset inequality $x(S) \geq 1$ defines a facet of $\mathcal{P}(G, K)$.

Proof. Let $b^T x \geq \beta$ be a facet defining inequality such that the face F_a induced by $a^T x := x(S) \geq 1$ in $\mathcal{P}(G, K)$ is contained in the face F_b induced by $b^T x \geq \beta$. By the second condition, for any $f \in E \setminus S$, there exists F , $f \notin F$, such that both the incidence vectors of F and $F \cup \{f\}$ lie in F_b . Therefore, $b^T x^F = b^T x^{F \cup \{f\}}$ and $b_f = 0$.

Now consider an edge $e \in S$. By the first condition, there exists a subset F whose incidence vector lies in F_b and such that $F \cap S = \{e\}$. Since $b_f = 0$ for $f \in E \setminus S$, $b^T x^F = b_e = \beta$.

We have proved that $b = \beta a$. Since F_b cannot define a facet if $b \leq 0$, we have $\beta > 0$. Thus $x(S) \geq 1$ and $b^T x \geq \beta$ define the same facet $F_a = F_b$. $\square$

Remark 3. Given a partition $\pi = (V_0, \dots, V_K)$ with $|V_0| = |V_K| = 1$, if there exists $i \in \{0, \dots, K-1\}$ such that $|V_i| = |V_{i+1}| = 1$, then the constraint $x(T_\pi) \geq 1$ is a subset inequality. Moreover this inequality dominates the cycle inequality associated to the partition $V_0, \dots, V_K$ and the edge between V_0 and V_K .

Proof. Let e and f be the edges between V_0 and V_K , and V_i and V_{i+1} respectively. Suppose there is a solution $F \in \mathcal{F}(G)$ such that $F \cap T_\pi = \emptyset$. As $|F \cap \delta(\bigcup_{j=0}^i V_j)| \geq 2$, it follows that $e, f \in F$. However, the shortest cycle of F containing e (f) is of length $\geq K+1$, a contradiction. Hence, $x(T_\pi) \geq 1$ is a subset inequality. Moreover, as $T_\pi = T_\pi^e$, it is clear that $x(T_\pi) \geq 1$ dominates the cycle inequality $x(T_\pi^e) - x_e \geq 0$ associated with the partition $V_0, \dots, V_K$ and e . $\square$

To separate the subset inequalities we compute for every edge $e := st$ a minimum (s, t) -($K-1$)-path cut C_e . If $\bar{x}(C_e) < 1$, then we compute a partition $\pi = (V_0, \dots, V_K)$ with $V_0 = \{s\}$ by a breadth-first search of the graph $(V, E \setminus (C_e \cup \{e\}))$ from s . Note that $t \in V_K$. If $|V_K| \geq 2$, then we consider the partition $\pi' = (V'_0, \dots, V'_K)$ where $V'_j = V_j$ for $j = 0, \dots, K-2$, $V'_{K-1} = V_{K-1} \cup (V_K \setminus \{t\})$ and $V'_K = \{t\}$. If there is some $i \in \{0, \dots, K-1\}$ where $|V'_i| = |V'_{i+1}| = 1$ then, by Remark 3, it follows that $x(T_{\pi'}) \geq 1$ is a violated subset inequality.

3.4. Metric inequalities

Metric inequalities were introduced by Fortz et al. [12] for the 2-node connected version of our problem with general edge lengths. As their result depends only on the fact that any edge in a solution belongs to a feasible cycle, metric inequalities are also valid for $\mathcal{P}(G, K)$. Hence we have the following.

Proposition 3. Consider an edge $e := ij \in E$ and a set of node potentials $(\alpha_k)_{k \in V}$ satisfying

$$\alpha_i - \alpha_j > K - 1.$$

Then

$$\sum_{f \in E - e} v_f x_f \geq x_e \quad (3.2)$$

is a valid inequality for $\mathcal{P}(G, K)$ where

$$v_f = \min \left(1, \max \left(0, \frac{|\alpha_l - \alpha_k| - 1}{\alpha_i - \alpha_j + 1 - K} \right) \right) \quad (3.3)$$

for all $f := kl \in E - e$.

Inequalities 3.2 are called *metric inequalities*. The next proposition shows that facet-inducing cycle inequalities form a subset of metric inequalities.

Proposition 4. Let $G = (V, E)$ be a graph and $K \geq 3$. Let $\pi = (V_0, \dots, V_p)$ be a partition of V such that $p \geq K$ and let $e := ij \in [V_0, V_p]$. If $p = K$ and $T_\pi^e := T_\pi \cup [V_0, V_p] \setminus \{e\}$, then the cycle inequality

$$x(T_\pi^e) \geq x_e$$

is a metric inequality.

If $p > K$, the cycle inequality is dominated by a metric inequality.

Proof. If $p = K$, it is sufficient to show that there exist node potentials $(\alpha_k)_{k \in V}$ such that v_f defined by (3.3) satisfy $v_f = 1$ for all $f \in T_\pi^e$ and $v_f = 0$ for all $f \in E \setminus (T_\pi^e \cup \{e\})$. Let $\alpha_k = -q$ if and only if $k \in V_q$. Then, $\alpha_i = 0$ and $\alpha_j = -K$, and it is easy to see that (3.3) becomes

$$v_f = \begin{cases} 1 & \text{if } |q - r| > 1, \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in E - e, f \in [V_q, V_r], q, r \in \{0, \dots, p\}$. The result follows immediately.

If $p > K$, the same definition of $(\alpha_k)_{k \in V}$ leads to $\alpha_i = 0, \alpha_j = -p$, and

$$v_f = \begin{cases} 1 & \text{if } |q - r| > p - K + 1, \\ \frac{|q - r| - 1}{p - K + 1} & \text{if } 1 < |q - r| \leq p - K + 1, \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in E - e, f \in [V_q, V_r], q, r \in \{0, \dots, p\}$. Therefore, the coefficient of an edge $f \in E - e$ is the same in the metric and the cycle inequality if $|q - r| > p - K + 1$ or $|q - r| \leq 1$, and it is smaller in the metric inequality if $1 < |q - r| \leq p - K + 1$. It follows that the metric inequality dominates the cycle inequality. $\square$

To separate the metric inequalities, we use the heuristic developed by Fortz et al. [12] for the 2-node connected case. As the metric inequalities are independent from the connectivity type (edge or node connectivity), this algorithm is also valid for our problem.

3.5. Cyclomatic inequalities

Cyclomatic inequalities were introduced by Fortz and Labbé [11] for the 2-node connected version of the problem. These inequalities are not valid for $\mathcal{P}(G, K)$, but can easily be adapted to 2-edge connectivity. For this, we first give the following result stated in [11].

Theorem 7. *Let $G = (V, E)$ be a 2-edge connected network with $n = |V|$ nodes and $m = |E|$ edges. If there exists a covering of the edges of the network by cycles containing at most K nodes each, then there exists such a covering using at most $\mu = m - n + 1$ independent cycles.*

Proposition 5. *Let $G = (V, E)$ be a graph, $K \geq 3$ a given constant, and $V_0, V_1, \dots, V_p$ a partition of V . Then*

$$x(\delta(V_0, \dots, V_p)) \geq M(p, K) := \left\lceil \frac{Kp}{K-1} \right\rceil \quad (3.4)$$

is a valid inequality for $\mathcal{P}(G, K)$.

Proof. Let F be a feasible solution to 2ECSBR and let $\hat{G}$ denote the contracted graph $(V, F)/V_0/\dots/V_p$, with $p+1$ nodes and $\hat{m}$ edges. We show that $\hat{m} \geq M(p, K)$.

It is easy to see that $\hat{G}$ is 2-edge connected and that each edge in $\hat{G}$ belongs to a cycle using at most K edges. Therefore, Theorem 7 holds for $\hat{G}$ and there exists a covering of $\hat{G}$ by at most $\hat{m} - p$ independent feasible cycles. The sum of the number of edges used in each cycle is greater than or equal to $\hat{m}$, since the cycles cover the network. Moreover, since each cycle uses at most K edges, we have

$$\hat{m} \leq (\hat{m} - p)K$$

and, by the integrality of $\hat{m}$,

$$\hat{m} \geq \left\lceil \frac{Kp}{K-1} \right\rceil,$$

which concludes the proof. $\square$

Theorem 8. *Let $G = (V, E)$ be a complete graph, $K \geq 3$ a given constant and $V_0, V_1, \dots, V_p$, $p \geq 2$, a partition of V . The inequality*

$$x(\delta(V_0, \dots, V_p)) \geq M(p, K) := \left\lceil \frac{Kp}{K-1} \right\rceil \quad (3.5)$$

defines a facet of $\mathcal{P}(G, K)$ if and only if the following conditions hold:

- $p \geq K$,
- $|V_i| \neq 2$ for $i = 0, \dots, p$,
- either $|V_i| \neq 3$ for $i = 0, \dots, p$, or $(p+1) \bmod (K-1) \geq 2$.

Proof. If $p < K$, then $M(p, K) = p+1$ and the cyclomatic inequality can be written as the sum of the cut inequalities $x(\delta(V_i)) \geq 2$ for $i = 0, \dots, p$, divided by 2.

If $V_i = \{u, v\}$ for some $i \in \{0, \dots, p\}$, then $E(V_i)$ is composed of the single edge $e := uv$. The cyclomatic inequality defined by the partition obtained by replacing V_i by the two subsets $\{u\}$ and $\{v\}$ can be written as

$$x(\delta(V_0, \dots, V_p)) + x_e \geq M(p+1, K).$$

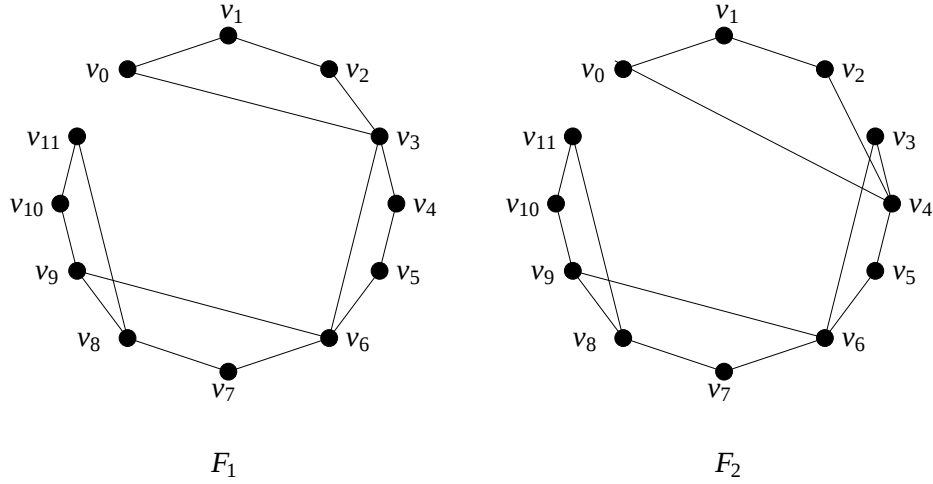
Adding the trivial inequality $-x_e \geq -1$, we obtain that

$$x(\delta(V_0, \dots, V_p)) \geq M(p+1, K) - 1$$

is a valid inequality that dominates (3.5) since $M(p+1, K) \geq M(p, K) + 1$. Thus (3.5) does not define a facet.

Now, suppose that $V_i = \{u, v, w\}$ for some $i \in \{0, \dots, p\}$ and that $(p+1) \bmod (K-1) \leq 1$. The cyclomatic inequality defined by the partition obtained by replacing V_i by the three subsets $\{u\}$, $\{v\}$ and $\{w\}$ can be written as

$$x(\delta(V_0, \dots, V_p)) + x(E(V_i)) \geq M(p+2, K) = \left\lceil \frac{K(p+2)}{K-1} \right\rceil.$$

Fig. 3. F_1 and F_2 for $p = 11$ and $K = 4$

Since $(p+1) \bmod (K-1) \leq 1$, $K-1$ divides either p or $p+1$. By elementary calculus, it follows that

$$\left\lceil \frac{K(p+2)}{K-1} \right\rceil = \left\lceil \frac{Kp}{K-1} \right\rceil + 3,$$

and therefore,

$$x(\delta(V_0, \dots, V_p)) + x(E(V_i)) \geq M(p, K) + 3.$$

Adding the trivial inequality $-x(E(V_i)) \geq -3$, we obtain (3.5) and therefore, (3.5) does not define a facet.

Conversely, let the conditions be satisfied for some inequality $a^T x := x(\delta(V_0, \dots, V_p)) \geq M(p, K)$. Let $b^T x \geq \beta$ be a facet defining inequality such that the face F_a induced by $a^T x \geq M(p, K)$ in $\mathcal{P}(G, K)$ is contained in the face F_b induced by $b^T x \geq \beta$.

We first show that $b_e = \gamma$ for all $e \in \delta(V_0, \dots, V_p)$. To this aim, we consider two distinct cases. Suppose first that $p \geq 2(K-1)$. For all $i \in \{0, \dots, p\}$, select an arbitrary node $v_i \in V_i$, and consider the two following sets of edges:

$$\begin{aligned} F_1 &= \left(\bigcup_{i=0}^p E(V_i) \right) \cup \left(\bigcup_{i=0}^{p-1} \{v_i v_{i+1}\} \right) \\ &\quad \cup \left(\bigcup_{i=1}^{l-1} \{v_{(i-1)(K-1)} v_{i(K-1)}\} \right) \cup \{v_{p-K+1} v_p\}, \\ F_2 &= \left(\bigcup_{i=0}^p E(V_i) \right) \cup \left(\bigcup_{i=0}^{K-3} \{v_i v_{i+1}\} \right) \cup \{v_{K-2} v_K\} \cup \left(\bigcup_{i=K-1}^{p-1} \{v_i v_{i+1}\} \right) \\ &\quad \cup \{v_0 v_K\} \cup \left(\bigcup_{i=2}^{l-1} \{v_{(i-1)(K-1)} v_{i(K-1)}\} \right) \cup \{v_{p-K+1} v_p\}, \end{aligned}$$

where $l = \lceil \frac{p}{K-1} \rceil$. These constructions are illustrated in Figure 3, where we omitted the complete subgraph $E(V_i)$ connected to each node v_i . It is easy to see that F_1 and F_2 define feasible solutions of 2ECSBR. Moreover,

$$x^{F_1}(\delta(V_0, \dots, V_p)) = x^{F_2}(\delta(V_0, \dots, V_p)) = p + l = M(p, K),$$

so the incidence vectors of F_1 and F_2 lie in F_b . The set F_2 is the set F_1 where edges $v_0 v_{K-1}$ and $v_{K-2} v_{K-1}$ have been replaced by $v_0 v_K$ and $v_{K-2} v_K$. Therefore,

$$b_{v_0 v_{K-1}} + b_{v_{K-2} v_{K-1}} = b_{v_0 v_K} + b_{v_{K-2} v_K}.$$

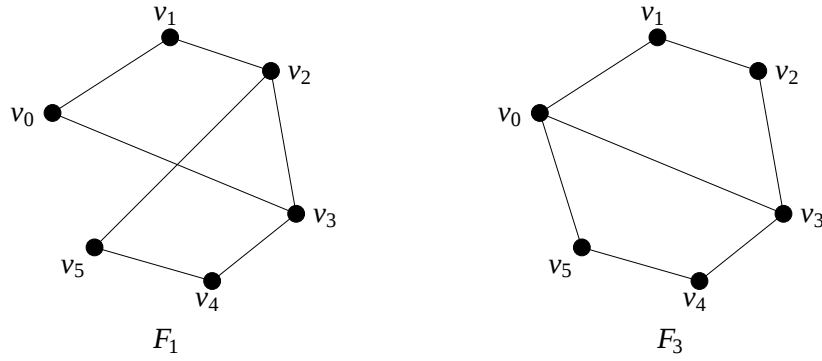


Fig. 4. F_1 and F_3 for $p = 5$ and $K = 4$

Since nodes $v_i, i = 0, \dots, p$ were chosen arbitrarily, and since the numbering of subsets in the partition is also arbitrary, it follows that for any four nodes v_i, v_j, v_k and v_l that belong to different subsets of the partition

$$b_{v_i v_k} + b_{v_j v_k} = b_{v_i v_l} + b_{v_j v_l},$$

by considering an ordering of the partition such that $v_i \in V_0, v_j \in V_{K-2}, v_k \in V_{K-1}$ and $v_l \in V_K$. Similarly,

$$b_{v_i v_k} + b_{v_i v_l} = b_{v_j v_k} + b_{v_j v_l},$$

by considering an ordering of the partition such that $v_i \in V_{K-1}, v_j \in V_K, v_k \in V_0$ and $v_l \in V_{K-2}$. This leads to $b_{v_i v_k} = b_{v_j v_l}$.

Now consider two edges $v_i v_k$ and $v_j v_l$ in $\delta(V_0, \dots, V_p)$ such that v_i and v_j belong to the same subset of the partition (v_i and v_j may be the same node). Since $p \geq 2(K-1) \geq 4$, there exist two nodes v_m and v_n that belong to different subsets of the partition and that do not belong to the same subsets as v_i, v_j, v_k or v_l . By the result above, it follows that $b_{v_i v_k} = b_{v_m v_n} = b_{v_j v_l}$. We can conclude that b_e has the same value γ for all $e \in \delta(V_0, \dots, V_p)$.

If $p < 2(K-1)$, then $l = \lceil \frac{p}{K-1} \rceil = 2$, and F_1 defined as above becomes

$$F_1 = \left(\bigcup_{i=0}^p E(V_i) \right) \cup \left(\bigcup_{i=0}^{p-1} \{v_i v_{i+1}\} \right) \\ \cup \{v_0 v_{K-1}\} \cup \{v_{p-K+1} v_p\}.$$

Replacing edge $v_{p-K+1} v_p$ by $v_0 v_p$, we get the set

$$F_3 = \left(\bigcup_{i=0}^p E(V_i) \right) \cup \left(\bigcup_{i=0}^{p-1} \{v_i v_{i+1}\} \right) \\ \cup \{v_0 v_{K-1}\} \cup \{v_0 v_p\}.$$

F_1 and F_3 are illustrated in Figure 4.

Again, it is not difficult to see that F_1 and F_3 lie in F_b , and therefore, $b_{v_0 v_p} = b_{v_0 v_{p-K+1}}$. Since nodes $v_i, i = 0, \dots, p$ were chosen arbitrarily, and since the numbering of subsets in the partition is also arbitrary, it follows that for any three nodes v_i, v_j and v_k that belong to different subsets of the partition, $b_{v_i v_j} = b_{v_i v_k}$.

If v_i, v_j, v_k and v_l are four nodes belonging to different subsets, we obtain by transitivity that $b_{v_i v_j} = b_{v_i v_k} = b_{v_k v_l}$. Finally, consider two edges $v_i v_k$ and $v_j v_l$ in $\delta(V_0, \dots, V_p)$ such that v_i and v_j belong to the same subset of the partition, with $v_i \neq v_j$. Since $p \geq K \geq 3$, there exists a nodes v_m that does not belong to the same subset as v_i, v_j and v_k . By the results above, it follows that $b_{v_i v_j} = b_{v_k v_m} = b_{v_i v_m} = b_{v_j v_l}$, and b_e has the same value γ for all $e \in \delta(V_0, \dots, V_p)$.

To conclude the proof, note that the conditions of the theorem imply that for any edge $e \in E(V_i)$ for some $i \in \{0, \dots, p\}$, $F_1 \setminus \{e\}$ lies in F_b if $|V_i| \geq 4$, and therefore $b_e = 0$.

Now suppose one of the subsets contains exactly three nodes. As the numbering of the subsets was arbitrary, we may w.l.o.g. assume that it is the last one, so let $V_p := \{v_p, v_{p+1}, v_{p+2}\}$, and let $e := v_p v_{p+2}$. Consider the set of edges

$$F_4 = \left(\bigcup_{i=0}^{p-1} E(V_i) \right) \cup \left(\bigcup_{i=0}^{p+1} \{v_i v_{i+1}\} \right) \cup \left(\bigcup_{i=1}^{l-1} \{v_{(i-1)(K-1)} v_{i(K-1)}\} \right) \cup \{v_{p-K+3} v_{p+2}\},$$

where $l = \lceil \frac{p}{K-1} \rceil$. It is easy to verify that F_4 and $F_4 \cup \{e\}$ both belong to F_b since $(p+1) \bmod (K-1) \geq 2$, so $b_e = 0$.

We have proved that $b = \gamma a$. Since F_b cannot define a facet if $b \leq 0$, we have $\gamma > 0$. Thus $x(\delta(V_0, \dots, V_p)) \geq M(p, K)$ and $b^T x \geq \beta$ define the same facet $F_a = F_b$. $\square$

To separate the cyclomatic inequalities, we developed a heuristic based on Barahona's algorithm [2] (see also [1]) for separating the so-called partition inequalities

$$x(\delta(V_0, \dots, V_p)) \geq p. \quad (3.6)$$

A first separation algorithm for these inequalities has been devised by Cunningham [7] and requires $|E|$ Minimum-Cut computations. Barahona [2] reduced this computing time to $|V|$ Minimum-Cut computations. Both Cunningham and Barahona's algorithms give the most violated inequality.

Consider the following inequalities obtained from the cyclomatic inequalities by deleting the upper integral part from the right hand side.

$$x(\delta(V_0, \dots, V_p)) \geq \frac{Kp}{K-1}. \quad (3.7)$$

Clearly inequalities (3.7) are of type (3.6) (it suffices to set $x' = \frac{K-1}{K}x$). Moreover, if (3.7) is violated, then (3.4) is so. However, it may be that all the inequalities of type (3.7) are satisfied whereas some cyclomatic inequalities are violated. In order to strengthen inequalities (3.7), we consider the inequalities

$$x(\delta(V_0, \dots, V_p)) \geq \frac{Kp}{K-1} + \varepsilon \quad (3.8)$$

and we choose $\varepsilon = \frac{p}{100n}$. Note that $\varepsilon \leq 0.01$ and the right hand side of (3.8) is linear in p . So inequalities (3.8) can be separated using, for instance, Barahona's algorithm. Here inequalities (3.8) can be transformed to inequalities of type (3.6) by setting $x' = (\frac{100nK+K-1}{100n(K-1)})x$. As it is pointed out in the next section, the value we considered for ε gave the best results.

A second and faster heuristic that we have developed for separating the cyclomatic inequalities consists in contracting edges with high values (in particular edges with value 1) until we get either a graph on $p+1$ nodes, with $p \geq K$ and whose weight is less than $\lceil \frac{Kp}{K-1} \rceil$ or a graph on less than $K+1$ nodes. Note that, as given in Theorem 8, cyclomatic inequalities define facets only if $p \geq K$. If the former case holds, then a violated cyclomatic inequality is found. The partition associated to this inequality is given by the resulting graph where each node corresponds to an element of the partition. This heuristic runs in $O(n^3)$ time.

3.6. Cycle partition inequalities

We present here our last class of valid inequalities for $\mathcal{P}(G, K)$, called cycle partition inequalities.

Theorem 9. *Let $G = (V, E)$ be a graph and $\pi = (V_0, V_1, \dots, V_p)$ be a partition of V with $p \geq K$. Then, the inequality*

$$\left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil \right) x(T_\pi) + x(C_\pi) \geq 2p \quad (3.9)$$

is valid for $\mathcal{P}(G, K)$.

Proof. Let F be an edge set inducing a feasible solution of 2ECSBR. If $F \cap T_\pi = \emptyset$, as $p \geq K$, F must contain at least $2p$ edges of C_π , and hence the incidence vector of F , x^F satisfies (3.11). So let us assume that $F \cap T_\pi \neq \emptyset$. Note that the cyclomatic inequality (3.4) can be rewritten as

$$x(T_\pi) + x(C_\pi) \geq p + \left\lceil \frac{p}{K-1} \right\rceil. \quad (3.10)$$

Using this and the fact that $x^F(T_\pi) \geq 1$, it follows that

$$\begin{aligned} & \left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil \right) x^F(T_\pi) + x^F(C_\pi) \\ &= \left(p - \left\lceil \frac{p}{K-1} \right\rceil \right) x^F(T_\pi) + x^F(T_\pi) + x^F(C_\pi) \\ &\geq p - \left\lceil \frac{p}{K-1} \right\rceil + p + \left\lceil \frac{p}{K-1} \right\rceil = 2p, \end{aligned}$$

and the inequality is valid. $\square$

Inequalities (3.9) are called *cycle partition inequalities*. The next lemmas show that the only cycle partition inequalities of interest are those for which $p = K$.

Lemma 2. *Let F be a solution to 2ECSBR. If $|F \cap T_\pi| \geq 2$ then x^F , the incidence vector of F , does not satisfy (3.9) to equality.*

Proof. If $|F \cap T_\pi| \geq 2$, then $x^F(T_\pi) \geq 2$. Combining this and (3.10), we get

$$\begin{aligned} & \left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil \right) x^F(T_\pi) + x^F(C_\pi) \\ &= \left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil \right) x^F(T_\pi) + x^F(T_\pi) + x^F(C_\pi) \\ &\geq 2 \left(p - \left\lceil \frac{p}{K-1} \right\rceil \right) + p + \left\lceil \frac{p}{K-1} \right\rceil \\ &= 3p - \left\lceil \frac{p}{K-1} \right\rceil > 2p, \end{aligned}$$

where the last relation holds since $p \geq K \geq 3$. $\square$

Lemma 3. *If $p > K$, then (3.9) does not define a facet of $\mathcal{P}(G, K)$.*

Proof. Suppose that $p > K$, and consider an edge $f \in [V_1, V_p]$. If (3.9) defines a facet of $\mathcal{P}(G, K)$, as it is different from a trivial inequality, there must exist a feasible solution F containing f whose incidence vector lies in the face defined by (3.9). By Lemma 2 it follows that $(F \cap T_\pi) \setminus \{f\} = \emptyset$. Suppose first that $F \cap [V_i, V_{i+1}] = \emptyset$ for some $i \in \{1, \dots, p-1\}$. For F to be 2-edge connected, one should then have $|F \cap [V_j, V_{j+1}]| \geq 2$ for $j \in \{1, \dots, p-1\} \setminus \{i\}$. This yields to

$$\begin{aligned} & \left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil \right) x^F(T_\pi) + x^F(C_\pi) \\ &\geq \left(p + 1 - \left\lceil \frac{p}{K-1} \right\rceil \right) + 2 + 2(p-2) \\ &= 3p - 1 - \left\lceil \frac{p}{K-1} \right\rceil > 2p, \end{aligned}$$

since $p > K \geq 3$, which leads to a contradiction.

In consequence, $F \cap [V_i, V_{i+1}] \neq \emptyset$, for $i = 1, \dots, p-1$. Furthermore, it follows from the development above that for some $i \in \{1, \dots, p-1\}$, F contains exactly one edge, say g , from $[V_i, V_{i+1}]$. But the shortest cycle in F containing g must go through $V_1, \dots, V_p, V_1$ and hence, it is of length at least $p > K$, a contradiction. $\square$

Following this result, we restrict our attention to partitions $\pi = (V_0, \dots, V_K)$ and the cycle partition inequality can be written as

$$(K-1)x(T_\pi) + x(C_\pi) \geq 2K. \quad (3.11)$$

In what follows, we present necessary and sufficient conditions for these inequalities to define facets of $\mathcal{P}(G, K)$.

Theorem 10. *Let $G = (V, E)$ be a complete graph and $\pi = (V_0, V_1, \dots, V_K)$ be a partition of V . For notational convenience, let $V_{K+1} := V_0$. Then, the cycle partition inequality*

$$(K-1)x(T_\pi) + x(C_\pi) \geq 2K$$

defines a facet for $\mathcal{P}(G, K)$ if and only if

1. $|V_i| + |V_{i+1}| + |V_j| + |V_{j+1}| \geq 5$ for all $i, j \in \{0, \dots, K\}$, $i \neq j$,
2. $|V_i| \neq 2$ for $i = 0, \dots, K$.

Proof. If the first condition does not hold, then there are two edges $f, g \in C_\pi$ such that $\{f, g\}$ induces a 2-edge cutset of the graph $G - T_\pi$. All the cycles using f in $G - T_\pi$ also use g and are included in C_π , thus are of length $K+1$. It follows that $G - T_\pi$ does not contain any feasible solution, and $x(T_\pi) \geq 1$ is a valid subset inequality. As by Lemma 2, any solution F whose incidence vector x^F lies in the face defined by (3.11) contains at most one edge of T_π , it follows that $x^F(T_\pi) = 1$ and the face defined by (3.11) is included in the face defined by $x(T_\pi) \geq 1$, thus (3.11) cannot define a facet of $\mathcal{P}(G, K)$.

If the second condition is not satisfied, there exists some $i \in \{0, \dots, K\}$ such that $V_i = \{u, v\}$. Let us assume w.l.o.g. that $i = 0$ and let $ax := (K-1)x(T_\pi) + x(C_\pi) \geq 2K$ be the inequality (3.11). Consider first the case $p = K \geq 4$. If $ax \geq 2K$ defines a facet there must exist a solution $F \in \mathcal{F}(G)$ containing an edge of $[V_1, V_3]$ such that $ax^F = 2K$. If there exists $i \in \{0, 3, 4, \dots, K\}$ such that $[V_i, V_{i+1}] \cap F = \emptyset$ then by inequalities (2.3) and Lemma 2 it follows that $|[V_j, V_{j+1}] \cap F| \geq 2$ for all $j \in \{0, 3, 4, \dots, K\} \setminus \{i\}$ and $|\delta(V_2) \cap C_\pi \cap F| \geq 2$. Hence $ax^F \geq K-1 + 2K-2$. As $K > 3$, we have $ax^F > 2K$, a contradiction. Thus $|[V_i, V_{i+1}] \cap F| \geq 1$ for all $i \in \{0, 3, 4, \dots, K\}$. In a similar way, we can show that $|[V_i, V_{i+1}] \cap F| = 1$ for all $i \in \{0, 3, 4, \dots, K\}$. Let $f_0 = F \cap [V_0, V_1]$ and $f_K = F \cap [V_K, V_0]$. As the shortest cycle of F containing f_0 and also containing f_K , goes through the sets $V_1, V_3, V_4, \dots, V_K, V_0$ and must be of length $\leq K$, f_0 and f_K must be incident to the same node of V_0 , say u . This implies that $|F \cap \delta(v)| \leq 1$, contradicting the fact that F induces a 2-edge connected spanning subgraph.

Now let us assume that $p = K = 3$. As $ax \geq 2K$ is not a trivial inequality, there exists a solution $F \in \mathcal{F}(G)$ which does not contain uv and such that $ax^F = 2K$. Consider first the case where $F \cap T_\pi = \emptyset$. Thus any cycle of F of length 3 containing edges of C_π intersects exactly two elements of the partition. By the cut constraints this implies that for $W = \{u\}$ ($W = \{v\}$, $W = V_2$), at least one of the following statements holds: $|F \cap [W, V_1]| \geq 2$ and $|F \cap [W, V_3]| \geq 2$. Therefore $|F \cap C_\pi| \geq 6$. If $|F \cap C_\pi| = 6$ it is not hard to see that there is a node set $W' \subset V$ where $\delta(W') \cap F = \emptyset$, which is impossible. Consequently $|F \cap C_\pi| > 6$. But this implies that x^F does not satisfy $ax \geq 2K$ with equality, a contradiction. If $F \cap T_\pi \neq \emptyset$, then by Lemma 2, $|F \cap T_\pi| = 1$. We can show along the same line that in this case, $|F \cap C_\pi| \geq 5$. But this implies that $ax^F > \alpha$, which yields again a contradiction.

Conversely, suppose that both conditions are satisfied for some inequality $a^T x := (K-1)x(T_\pi) + x(C_\pi) \geq 2K$. Let $b^T x \geq \beta$ be a facet defining inequality such that the face F_a induced by $a^T x \geq 2K$ in $\mathcal{P}(G, K)$ is contained in the face F_b induced by $b^T x \geq \beta$.

We first show that $b_e = b_{e'}$ for every $e, e' \in C_\pi$. Consider an edge set $[V_j, V_{j+1}]$ where $|[V_j, V_{j+1}]| \geq 3$. By the two conditions, there is at most one $i \in \{0, \dots, K\}$ such that $|[V_i, V_{i+1}]| = 1$. Therefore, we may suppose w.l.o.g., that $|[V_i, V_{i+1}]| \geq 3$ for $i = 1, \dots, K$ and thus $j \neq 0$. Note that we can have $|V_0| = |V_1| = 1$. Let f_i^1, f_i^2 be two fixed edges of $[V_i, V_{i+1}]$ for $i = 1, \dots, K$, and $f_0^1 \in [V_0, V_1]$. As G is complete, we may suppose that f_i^1 and f_i^2 (f_i^1 and f_{i+1}^1) are adjacent for $i = 1, \dots, K$. Let $\bar{E} = \bigcup_{i=0}^K E(V_i)$. By the second condition, we may assume that $|V_{j+1}| \geq 3$. We may also suppose w.l.o.g., that f_j^1 and f_j^2 are incident to a node, say w , of V_j . Consider the edge sets

$$\begin{aligned} E_1 &= \{f_1^1, f_1^2, f_2^1, f_2^2, \dots, f_K^1, f_K^2\} \cup \bar{E}, \\ E_2 &= (E_1 \setminus \{f_j^1\}) \cup \{f\}, \end{aligned}$$

where $f \in [w, V_{j+1}] \setminus \{f_j^1, f_j^2\}$. As both graphs $G(E_1)$ and $G(E_2)$ are 2-edge connected and every edge of E_1 (E_2) belongs to a cycle of length 3, we have that E_1, E_2 are feasible and belong to the face defined by $ax = 2K$. Thus $bx^{E_1} = bx^{E_2}$ and $b_{f_j^1} = b_f$. Therefore, $b_e = b_{e'}$ for all $e, e' \in [w, V_{j+1}]$.

If $|V_j| = 1$, then we are done. If not, then by symmetry, we also obtain that $b_e = b_{e'}$ for all $e, e' \in [V_j, w']$, $w' \in V_{j+1}$. It follows that $b_e = b_{e'}$ for all $e, e' \in [V_j, V_{j+1}]$.

Now let $i \in \{1, \dots, K\}$ and g be an edge of $[V_i, V_{i+2}]$ adjacent to edges f_{i-1}^1 and f_{i+2}^1 . Note that the edges $g, f_{i+2}^1, \dots, f_{i+K}^1$ form a cycle of length K . Let us examine the sets

$$\begin{aligned} E_3 &= \{f_0^1, f_1^1, \dots, f_K^1, g\} \cup \bar{E}, \\ E_4 &= (E_3 \setminus \{f_{i+1}^1\}) \cup \{f_i^2\}. \end{aligned}$$

Clearly, E_3 and E_4 are feasible and belong to the face defined by $ax = 2K$. Therefore $bx^{E_3} = bx^{E_4}$ and $b_{f_{i+1}^1} = b_{f_i^2}$. This implies that $b_e = b_{e'}$ for all $e \in [V_i, V_{i+1}]$, $e' \in [V_{i+1}, V_{i+2}]$, $i \in \{1, \dots, K\}$.

It follows that $b_e = \gamma$ for all $e \in C_\pi$.

Since E_1 and E_3 lie in the face defined by $ax = 2K$, $bx^{E_1} = bx^{E_3}$ and $\sum_{i=1}^K b_{f_i^2} = b_g$. It follows that $b_g = (K-1)\gamma$.

As G is complete, for any edge h of T_π , there exist a cycle, say C_h of length $K+1$ going through the sets $V_0, \dots, V_K$ and having h as a chord. Note that $C_h \cap E(V_i) = \emptyset$ for $i = 0, \dots, K$. Let

$$E_h = C_h \cup \{h\} \cup \bar{E}, \text{ for all } h \in T_\pi.$$

Clearly, E_h lies in the face defined by $ax = 2K$ for all $h \in T_\pi$. If h and h' are two edges of T_π , we therefore have $bx^{E_h} = bx^{E_{h'}}$. Since $bx^{C_h} = bx^{C_{h'}}$, it follows that $b_h = b_{h'}$. Thus $b_e = (K-1)\gamma$ for all $e \in T_\pi$.

Next we show that $b_e = 0$ for all $e \in E(V_i)$, $i = 0, \dots, K$. Let $i \in \{0, \dots, K\}$ such that $|V_i| \neq 1$. By 2. it follows that $|V_i| \geq 3$. Let $e := uv \in E(V_i)$. Consider first the case where $|V_i| \geq 4$. W.l.o.g., we may assume that at least one edge among $\{f_{i-1}^1, f_{i-1}^2\}$ ($\{f_i^1, f_i^2\}$) is not incident neither to u nor to v . Let $E'_1 = E_1 \setminus \{e\}$. It is not hard to see that $E'_1 \in \mathcal{F}(G)$. As $ax^{E'_1} = ax^{E_1}$, we have $b_e = 0$. Suppose now that $|V_i| = 3$. Let w be the node in $V_i \setminus \{u, v\}$, $z \in V_{i-1}$ and $z' \in V_{i+1}$. We can assume, w.l.o.g. that f_{i-2}^1, f_{i-2}^2 are incident to z and f_{i+1}^1, f_{i+1}^2 are incident to z' . It is not hard to see that

$$\begin{aligned} E_4 &= \bigcup_{j=0}^K E(V_j) \cup \left(\bigcup_{\substack{j \in \{1, \dots, K\} \\ j \neq i-1, j \neq i}} \{f_j^1, f_j^2\} \right) \cup \{zv, zw, z'u, z'w\}, \\ E'_4 &= E_4 \setminus \{e\}. \end{aligned}$$

define solutions of 2ECSBR that lie in the face defined by $ax \geq 2K$ and thus $b_e = 0$.

Altogether we have shown that

$$b_e = \begin{cases} \gamma & \text{for all } e \in C_\pi, \\ (K-1)\gamma & \text{for all } e \in T_\pi, \\ 0 & \text{for all } e \in E, \end{cases}$$

thus $b = \gamma a$. As, for every edge $f \in E$, $E \setminus \{f\} \in \mathcal{F}(G)$, we have that $\gamma > 0$, which completes the proof. $\square$

To separate the cycle partition inequalities, we developed a heuristic similar to that used for the separation of cyclomatic inequalities. This heuristic works in two phases. First, we contract edges with high values until a graph on $K+1$ nodes is obtained. Each node of this graph corresponds to an element of the partition inducing the inequality. In the second phase, we order the elements of the partition in order to get a partition that provides a minimum left hand side in (3.11). For this, we fix a node say v_0 , which will correspond to V_0 , we determine a node v_1 such that $\bar{x}([v_0, v_1])$ is maximum. Node v_1 correspond to V_1 . After that, a node v_2 , where $\bar{x}([v_1, v_2])$ is maximum is computed and so on until a partition $V_0, \dots, V_K$ is obtained. This is repeated $K+1$ times by changing v_0 . We then consider the partition giving the minimum left hand side. If this is less than $2K$, then a violated cycle partition inequality is found. This heuristic is given in Algorithm 3.2.

4. A Branch-and-Cut algorithm

In this section, we present a Branch-and-Cut algorithm for the 2-edge connected subgraph problem with bounded rings. Our aim is to address the algorithmic applications of the theoretical results presented in the previous sections and describe some strategic choices in order to solve that problem.

Algorithm 3.2 Separation heuristic for cycle partition inequalities

Data: a solution $\bar{x}$, its support graph $G_{\bar{x}} = (V_{\bar{x}}, E_{\bar{x}})$ and a bound K .

```

 $\bar{G} = (\bar{V}, \bar{E}) \leftarrow G_{\bar{x}};$ 
while  $|\bar{V}| > K + 1$  do
   $maxedg \leftarrow 0; maxval \leftarrow 0.0;$ 
  for all  $e \in \bar{E}$  do
    if  $\bar{x}(e) > maxval$  then
       $maxval \leftarrow \bar{x}(e);$ 
       $maxedg = e;$ 
  Let  $maxedg = st;$ 
   $\bar{G} \leftarrow \bar{G} / \{s, t\};$ 
   $\pi^* \leftarrow 0, value^* \leftarrow 0;$ 
  for all  $u \in \bar{V}$  do
     $V' \leftarrow \bar{V} \setminus \{u\};$ 
     $\pi \leftarrow 0; \pi_0 \leftarrow u;$ 
     $l \leftarrow 0; value = 0;$ 
    while  $V' \neq \emptyset$  do
      Let  $v_{max} \in V'$  such that  $\bar{x}([\pi_l, v_{max}]) = \max\{\bar{x}([\pi_l, v]), v \in V'\};$ 
       $value \leftarrow value + \bar{x}([\pi_l, v_{max}]);$ 
       $l \leftarrow l + 1; \pi_l \leftarrow v_{max};$ 
       $V' \leftarrow V' \setminus \{v_{max}\};$ 
       $value \leftarrow value + \bar{x}([\pi_l, \pi_0]);$ 
      if  $value > value^*$  then
         $value^* \leftarrow value; \pi^* \leftarrow \pi;$ 
  for all  $i, j \in \{0, \dots, K\}$  such that  $i + 1 < j$  and  $j - i \neq K$  do
     $value^* \leftarrow value^* + \bar{x}([\pi_i, \pi_j]);$ 
  if  $value < 2K$  then
    {We found a violated cycle partition inequality}
    {The partition is given by the nodes of  $\bar{G}$  in the order given in  $\pi$ }
  else
    {No violated cyclomatic inequality was found}

```

To start the optimization we consider the following linear program given by the cut inequalities associated with the vertices of the graph together with the cyclomatic inequality induced by the trivial partition (where the elements of the partition correspond to the nodes of the graph) and the trivial inequalities, that is

$$\begin{aligned}
 \text{Min } & \sum_{e \in E} c_e x_e \\
 \text{s.t. } & x(\delta(v)) \geq 2, \quad \text{for all } v \in V, \\
 & x(E) \geq \left\lceil \frac{nK}{K-1} \right\rceil, \\
 & 0 \leq x_e \leq 1, \quad \text{for all } e \in E.
 \end{aligned}$$

An important task in the Branch-and-Cut algorithm is to determine whether or not an optimal solution of the relaxation of the 2ECSBR is feasible. An optimal solution $\bar{x}$ of the relaxation is feasible for the 2ECSBR if it is an integer vector that satisfies the cut inequalities and such that every edge of $G_{\bar{x}}$ is contained in a feasible cycle of $G_{\bar{x}}$. Verifying if $\bar{x}$ is feasible for 2ECSBR can be done in polynomial time. We first check if each edge e with $\bar{x}(e) = 1$ belongs to a feasible cycle of $G_{\bar{x}}$ by computing a shortest path between the endnodes of e . And then, by a breadth-first search, we verify if $G_{\bar{x}}$ is connected. If this is the case, as each edge belongs to a cycle, the cut inequalities are also satisfied.

Another important issue in the effectiveness of the Branch-and-Cut algorithm is the computation of a good upper bound. For this, we first try to transform each LP-solution obtained in the Branch-and-Cut to a feasible solution by rounding up to 1 all the variables with fractional value. Then we delete all the edges that do not belong to feasible cycles. And in a final step, we try to reduce the resulting solution F by repeatedly removing edges with high cost c_e such that $F \setminus \{e\}$ still induces a solution of the 2ECSBR.

If an optimal solution $\bar{x}$ of the linear relaxation of the 2ECSBR is not feasible, the Branch-and-Cut algorithm generates further inequalities that are valid for $\mathcal{P}(G, K)$ and violated by $\bar{x}$. The separation of valid inequalities is performed in the following order:

- cut inequalities,
- metric inequalities,
- cycle and subset inequalities,
- cyclomatic inequalities,
- cycle partition inequalities.

We remark that all inequalities are global (i.e. valid in all the Branch-and-Cut tree) and several constraints may be added at each iteration. Moreover, we go to the next class of inequalities only if we do not find any violated inequalities in the current class.

To separate the different inequalities, we use the algorithms described in section 3. All our separation algorithms are applied on the graph $G_{\bar{x}} = (V_{\bar{x}}, E_{\bar{x}})$ where $\bar{x}$ is the current LP-solution.

The exact separation of cut constraints can be done using the Gomory-Hu algorithm [15]. This algorithm produces the so-called Gomory-Hu tree with the property that for all pairs of nodes $s, t \in V_{\bar{x}}$ the minimum (s, t) -cut in the tree is also a minimum (s, t) -cut in $G_{\bar{x}}$. Actually, we use the algorithm developed by Gusfield [19] which requires $|V_{\bar{x}}| - 1$ maximum flow computations. The maximum flow computations are handled by the efficient Goldberg and Tarjan algorithm [14] that runs in $O(mn \log \frac{n^2}{m})$ time. The exact algorithm that permits to separate the cut inequalities is then implemented to run in $O(mn^2 \log \frac{n^2}{m})$ time.

The separations of the cycle and subset inequalities are performed simultaneously. We first compute for an edge $e = st$ a (minimum) (s, t) -($K - 1$)-path cut C_e using either the exact algorithm if $K \leq 4$ or the primal-dual algorithm if $K \geq 5$. If $\bar{x}(C_e) < 1$, we determine a partition $\pi = (V_0, \dots, V_K)$ by a breadth-first search from s in the graph induced by $E_{\bar{x}} \setminus (C_e \cup \{e\})$. If $|V_K| \geq 2$, then we consider the partition $\pi' = (V'_0, \dots, V'_K)$ where $V'_j = V_j$ for $j = 0, \dots, K - 2$, $V'_{K-1} = V_{K-1} \cup (V_K \setminus \{t\})$ and $V'_K = \{t\}$. The idea behind this is to get a cycle inequality that, by Theorem 5, may define a facet. If there is some $i \in \{0, \dots, K - 1\}$ where $|V'_i| = |V'_{i+1}| = 1$ then by Remark 3, $x(T_{\pi'}) \geq 1$ is a violated subset inequality. Note that this inequality dominates the cycle inequality $x(T_{\pi}^e) \geq x_e$. If this is not the case and we have $\bar{x}(T_{\pi}^e) < \bar{x}$, then the cycle inequality corresponding to π' and e is violated. Moreover, this inequality is facet defining. We consider this procedure only for edges e with $\bar{x} \geq 0.5$. If $\bar{x}$ is small, there is little hope to get a violated cycle inequality involving e . This procedure runs in $O(n^3)$ time if $K \leq 4$ and in $O(n^4)$ time if $K \geq 5$.

To separate cyclomatic inequalities, we first use the heuristic based on the contraction of edges. If no violated inequality is found, then we try to generate violated cyclomatic inequalities using the procedure based on Barahona's algorithm [2] for the multicut problem. Both algorithms produce a partition $\pi = (V_0, \dots, V_p)$ with $p \geq K$. For every $i \in \{0, \dots, p\}$ such that $|V_i| = 2$ (resp. $|V_i| = 3$ and either $K = 3$ or $(p + 1) \bmod (K - 1) \leq 1$) then we consider the partition obtained from π by expanding the set V_i . By Theorem 8, The cyclomatic inequality given by this latter partition dominates the one produced by partition π .

When solving instances of the 2ECSBR, we remarked that the separation of cut inequalities using the exact Gusfield algorithm is time consuming. For this we adopted the strategy to use that algorithm only if no constraints of any type could be found using the separation routines presented before.

To store the generated inequalities, we created a pool whose size increases dynamically. All the generated inequalities are put in the pool and are dynamic, i.e. they are removed from the current LP when they are not active. We first separate inequalities from the pool. If all the inequalities in the pool are satisfied by the current LP-solution, we separate the classes of inequalities in the order given above.

5. Computational results

The Branch-and-Cut algorithm described in the previous section has been implemented in C++, using BCP [23] to manage the Branch-and-Cut tree and CPLEX 7.1 as LP-solver. It was tested on a Pentium IV 1,7 GHz with 1 Gb RAM, running under Linux. We fixed the maximum CPU time to 5 hours.

Results are presented here for instances coming from real applications and randomly generated instances. The instances consist in complete graphs with edge costs equal to rounded Euclidean distances. Tests were performed for $K = 3, 4, 5, 6, 7, 10, 13, 16$. Usually in practice, the bound does not exceed 5. The real instances come from the network of the Belgian telecommunications operator Belgacom (52 nodes) and subsets of these nodes. The random problems were generated with 10 to 50 nodes, and five instances of each size were tested. Data on the randomly generated test problems are available at the Web page <http://www.poms.ucl.ac.be/staff/bf/en/2cnbm/data.html>.

In the various tables, the entries are:

Table 1. Results for real instances

$ V $	K	Cu	Cy	Me	Su	Cc	Cp	No	Gap2	Gt	CPU
12	3	2	4	6	2	1	1	1	0.00	0.00	0:00:00
17	3	13	43	41	15	11	6	7	0.51	0.00	0:00:01
30	3	30	91	50	18	49	17	41	0.58	0.00	0:00:10
52	3	101	681	426	84	895	124	3015	1.33	0.00	0:42:36
12	4	4	24	24	16	5	0	5	0.52	0.00	0:00:00
17	4	21	119	88	28	11	0	45	1.76	0.00	0:00:02
30	4	111	6676	3437	514	293	81	8723	3.67	0.00	0:55:07
52	4	141	6028	2325	251	796	117	4653	5.30	3.67	5:00:00
12	5	14	24	24	37	5	0	9	1.77	0.00	0:00:00
17	5	28	97	42	103	16	2	29	2.21	0.00	0:00:01
30	5	159	20487	5157	5597	525	140	32327	4.63	0.86	5:00:00
52	5	134	6151	1302	1144	619	49	4451	7.17	5.27	5:00:00
12	6	7	5	8	15	7	0	7	0.72	0.00	0:00:00
17	6	9	7	11	26	6	2	1	0.00	0.00	0:00:00
30	6	143	17670	3156	11446	331	65	25811	5.17	1.36	5:00:00
52	6	121	6712	791	1840	349	12	3035	7.32	5.69	5:00:00
12	7	22	23	43	142	9	0	29	1.84	0.00	0:00:00
17	7	30	102	78	228	12	0	39	2.81	0.00	0:00:02
30	7	108	3627	419	4333	93	0	2545	3.14	0.00	0:17:42
52	7	146	7745	644	4121	268	0	3113	8.08	6.39	5:00:00
12	10	4	0	20	0	6	0	11	0.83	0.00	0:00:00
17	10	2	0	0	0	2	0	1	0.00	0.00	0:00:00
30	10	81	606	209	1620	22	0	293	1.41	0.00	0:00:57
52	10	126	6058	211	11138	111	1	3701	6.49	4.99	5:00:00
17	13	2	0	0	0	3	1	1	0.00	0.00	0:00:00
30	13	62	227	143	1452	7	0	127	0.92	0.00	0:00:20
52	13	161	6364	228	26030	62	0	3229	6.66	3.60	5:00:00
17	16	2	0	0	0	3	0	1	0.00	0.00	0:00:00
30	16	89	490	964	4848	20	0	535	2.03	0.00	0:01:10
52	16	155	3724	183	20931	67	0	1747	2.44	1.14	5:00:00

- $|V|$: the number of nodes of the problem,
 K : the bound on the cycles,
Cu : the number of generated cut inequalities,
Cy : the number of generated cycle inequalities,
Me : the number of generated metric inequalities,
Su : the number of generated subset inequalities,
Cc : the number of generated cyclomatic inequalities,
Cp : the number of generated cycle partition inequalities,
No : the number of generated nodes in the Branch-and-Cut tree,
o/p : the number of problems solved to optimality over the
number of instances tested (only for random instances),
Gap1 : the gap between the best upper bound (UB) and the lower
bound obtained at the root node of the Branch-and-Cut tree
without adding cycle and cycle partition inequalities.
Gap2 : the gap between UB and the lower bound obtained at the
root node of the Branch-and-Cut tree.
Gt : the gap between UB and the best lower bound found (LB),
CPU : the total time in second.

Table 1 reports results obtained for the real instances, while Table 2 presents the average results for the randomly generated problems.

We remark that for 20 nodes or less, all problems could be solved to optimality. Moreover for $K = 3$, all instances have been solved to optimality within the time limit. Comparing these results to those in [11], it appears that the edge connectivity version of the problem is much easier to solve than the node connectivity version for $K = 3$. However for $4 \leq K \leq 7$ and instances with 30 nodes and more, the problem seems to be harder to solve. In fact only a few instances of this type have been solved in less than 5 hours. For the real instances with 52 nodes and $4 \leq K \leq 7$, we got an average gap of 5.26%. For the random instances with 50 nodes, this average gap is 6.93%. A similar increase can

Table 2. Results for random instances

V	K	Cu	Cy	Me	Su	Cc	Cp	No	o/p	Gap1	Gap2	Gt	CPU
10	3	3.8	9.0	9.8	6.0	4.2	2.8	3.0	5/5	1.32	0.44	0.00	0:00:00
20	3	13.0	27.4	19.8	11.0	9.0	7.0	3.4	5/5	1.39	0.15	0.00	0:00:00
30	3	28.2	98.2	59.8	20.6	33.4	14.8	36.6	5/5	1.80	0.83	0.00	0:00:08
40	3	53.4	311.0	175.8	40.0	192.4	53.6	1175.8	5/5	3.06	1.31	0.00	0:05:55
50	3	80.6	776.0	468.4	101.2	710.0	204.6	8053.8	5/5	3.01	1.88	0.00	1:28:13
10	4	6.6	16.2	18.0	12.2	5.8	2.6	9.4	5/5	1.71	1.26	0.00	0:00:00
20	4	40.8	1066.0	507.8	120.8	56.0	31.2	551.0	5/5	4.78	3.37	0.00	0:01:07
30	4	105.0	16707.6	7316.2	914.6	397.8	172.4	22925.4	1/5	7.19	5.72	1.14	4:49:59
40	4	126.2	10328.8	4098.6	616.2	568.2	168.0	13345.0	0/5	8.55	6.35	2.40	5:00:00
50	4	123.6	6585.6	2364.2	350.6	668.4	109.0	6466.6	0/5	10.43	7.87	5.06	5:00:00
10	5	5.0	6.6	9.0	17.2	3.6	0.4	6.2	5/5	1.16	0.89	0.00	0:00:00
20	5	49.4	2478.6	732.0	1101.8	77.8	28.8	2891.8	5/5	6.37	4.39	0.00	0:05:49
30	5	108.2	17383.6	3512.8	3949.6	368.6	75.6	18396.2	1/5	7.27	6.09	1.57	4:24:57
40	5	131.6	11380.0	2111.8	2156.2	457.2	65.4	9775.4	0/5	9.45	7.94	4.71	5:00:00
50	5	123.2	7240.8	1139.2	1206.8	474.8	55.8	4050.2	0/5	10.38	9.78	7.01	5:00:00
10	6	9.8	2.8	19.6	51.0	6.2	0.6	9.0	5/5	2.74	2.70	0.00	0:00:00
20	6	58.0	2413.6	507.2	2105.0	60.8	17.0	2472.6	5/5	5.33	4.46	0.00	0:06:33
30	6	110.0	17946.4	2536.2	9337.0	232.8	20.6	14776.6	1/5	7.48	6.88	2.37	4:43:21
40	6	127.0	12085.2	1317.0	4560.8	281.2	35.8	7021.0	0/5	9.99	9.15	5.85	5:00:00
50	6	121.8	7495.8	741.2	2395.8	270.6	19.8	2711.8	0/5	10.91	10.53	8.19	5:00:00
10	7	6.2	0.4	14.2	37.6	4.0	0.2	9.0	5/5	1.36	1.25	0.00	0:00:00
20	7	45.8	910.8	166.0	1049.0	34.2	8.4	625.4	5/5	3.73	3.38	0.00	0:01:04
30	7	119.6	16455.2	1953.2	16711.6	203.4	6.0	16300.6	0/5	6.19	5.64	1.69	5:00:00
40	7	137.0	13231.8	1035.8	8448.8	239.2	20.4	6652.6	0/5	7.67	7.61	4.24	5:00:00
50	7	127.6	8159.6	530.8	4238.0	206.8	12.2	2712.6	0/5	9.77	9.49	7.45	5:00:00
10	10	2.8	0.0	0.0	0.0	1.4	0.0	1.0	5/5	0.00	0.00	0.00	0:00:00
20	10	40.2	480.8	117.6	1944.4	25.2	1.4	601.0	5/5	2.30	2.50	0.00	0:00:45
30	10	122.2	9110.8	753.8	28861.8	113.4	0.8	7852.6	4/5	3.79	3.81	0.14	2:57:21
40	10	154.2	12217.8	673.8	21737.4	152.2	0.8	6181.8	1/5	4.16	6.24	3.11	4:41:58
50	10	152.8	10722.8	400.0	16278.2	120.0	1.6	3177.8	0/5	6.45	6.35	4.46	5:00:00
20	13	25.6	86.8	183.6	812.8	14.0	0.2	130.2	5/5	1.72	1.72	0.00	0:00:06
30	13	80.6	1427.6	220.6	7718.4	38.0	0.4	769.4	5/5	2.50	2.24	0.00	0:05:36
40	13	155.2	7688.4	515.6	32764.8	143.6	0.8	5373.4	2/5	4.63	4.61	2.19	4:00:09
50	13	175.0	10648.2	328.0	30746.6	105.4	0.4	3961.8	0/5	4.59	4.55	2.81	5:00:00
20	16	18.8	1.8	67.0	121.4	8.4	0.2	13.0	5/5	0.61	0.60	0.00	0:00:01
30	16	88.2	885.4	306.6	7535.8	56.0	0.4	610.2	5/5	2.57	2.41	0.00	0:03:52
40	16	141.8	3743.8	335.2	24970.8	100.4	0.2	2433.8	4/5	3.51	3.66	1.26	1:32:27
50	16	184.4	8520.6	266.8	43858.0	92.0	0.4	3611.0	0/5	3.23	3.24	1.57	5:00:00

Table 3. Results for real instances without cycle and cycle partition inequalities

V	K	Cu	Cy	Me	Su	Cc	Cp	No	Gap2	Gt	CPU
30	3	32	0	134	22	66	0	93	0.93	0.00	0:00:20
52	3	103	0	1187	104	1220	0	6505	1.99	0.00	1:28:12
30	4	119	0	9687	983	435	0	16183	5.66	0.00	1:45:07
53	4	146	0	5109	572	816	0	8555	7.31	5.48	5:00:00
30	5	152	0	14900	8797	670	0	37329	5.13	0.80	5:00:00
52	5	144	0	4156	2923	649	0	5975	9.02	6.73	5:00:00
30	6	171	0	8361	17601	287	0	33919	5.89	1.50	5:00:00
52	6	150	0	3163	5844	383	0	4385	7.93	6.13	5:00:00

also be observed between real and random instances with 30 nodes. Therefore, it seems that real instances are easier to solve.

In both tables, a significant number of cycle and cycle partition inequalities have been generated for most of the instances with 30 nodes and more when $K \leq 6$. In order to evaluate the impact of these inequalities on the performance of the algorithm, Table 3 reports results obtained for real instances with 30 and 52 nodes and $K = 3, 4, 5, 6$ without the use of the cycle and the cycle partition inequalities. The instances with $K = 3, 4$ that were solved to optimality using the cycle and the cycle partition inequalities are also solved to optimality without using these inequalities, but both the computing time and the size of the branch-and-bound tree more than doubled. For the other instances, it can be seen from Tables 1 and 3 that the use of these inequalities allowed to reduce the overall gap by about 18%. This improvement can also be observed in Table 2 for the random instances by comparing the lower bound obtained at

the root node with **(Gap2)** and without **(Gap1)** these inequalities. It appears that the relative error between the best upper bound and the lower bound at the root increases by about 24% if the cycle and the cycle partition inequalities are not added. However for the instances with $K \geq 7$, the gain is usually not significant. These inequalities seem to play a central role in the resolution of the 2ECSBR with small bounds. For high bounds, the subset inequalities seem to be more effective. In fact, we remark that for these problems, the number of generated subset inequalities is significantly greater than that of the cycle inequalities. Hence our heuristic for separating these inequalities seem to be quite efficient.

Finally, we notice that problems with $K \geq 10$ are easier to solve. Most of the instances on 30 nodes and some on 40 nodes have been solved to optimality. This is due to the fact that for large value of K , the 2ECSBR is closer to the 2-edge connected subgraph problem which can be solved efficiently — for the graph sizes considered in this paper — using the cut constraints only.

6. Concluding remarks

We studied the two-edge connected subgraph problem where every edge must belong to a bounded cycle. We have given an integer programming formulation for this problem. We have identified various classes of valid inequalities and discussed necessary and sufficient conditions for these inequalities to be facet defining. We have provided separation algorithms for these inequalities. In particular, we have shown that the separation problem for the cycle inequalities can be reduced to a maximum flow problem when the cycle bound is ≤ 3 and thus, can be solved in this case in polynomial time. Using these results we have described a Branch-and-Cut algorithm for this problem. Our computational results have shown that the problem could be hard to solve for K between 3 and 7. We could estimate the effect of the cycle and the cycle partition inequalities in the Branch-and-Cut algorithm. We could also measure the performance of our separation techniques.

It would be interesting to extend the results given in this paper to the more general survivable network design problem [18] with bounded rings.

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