Strategic asset allocation with switching dependence.

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Abstract

This paper revisits the problem of the strategic asset allocation between stocks and bonds. The novelty of our approach is to model the influence of economic cycles on the marginal distributions of asset returns and their dependence structure by a single hidden Markov chain. After a brief review of selected statistical distributions (Student't and Weibull) and copulas (elliptic and Archimedian), we describe how the switching regime model is calibrated using two indices: the CAC 40 for stocks and the SGI Bond 10 years, for bonds. We then propose a dynamic investment policy based on the estimated probabilities of sojourn in each state of the Markov chain. Even though the Markov chain ruling the assets dynamics is hidden, a Bayesian procedure can be used to infer the probabilities of being in a certain state of the economy. The asset allocation can then be adapted to provide the highest yield given the most likely state. Having calibrated and estimated the parameters of the model, the performance of static and dynamic strategies are compared by conducting Monte Carlo simulations. Our results show that dynamic strategies, which exploit the additional information relating the probable regime state, perform better than static policies with a limited risk and an acceptable number of reallocations.

Keywords : copula, switching regime.
JEL Classification: C6

1 Introduction.

Strategic asset allocation is a matter of concern for all institutional investors, such as insurance companies and pension funds, with long term commitments to their clients. Ideally, the investment policy should optimize the percentage of stocks and bonds that the company holds to achieve a reasonable return at acceptable risk levels whatever the trends followed by the markets.

One of the key issues to determining an efficient asset allocation policy is that of modelling correctly the benchmarks driving the categories of investment and their dependence structure. Copulas, which enable us to separate the dependence model from the marginal distributions, can be used to achieve this goal. We refer the interested reader to Nelsen (1999) for an introduction on this topic. Recent publications have focused on the modeling of non linear dependence with copulas driven by switching regime. E.g. Guidolin & Timmermann (2005, 2006) have modeled the joint distribution of US stock and bond returns in the presence of regime switching dynamics. In a similar framework, Hennessy and Lapan (2002) have studied the asset allocation problem when the dependence is modeled by an Archimedian copula. The theory of switching regimes was developed by Hamilton (1989) to model the discrete shifts observed in the growth rate of

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non stationary series. This approach is particularly well suited to modelling economic cycles and related changes of interdependence level, such as those observed during the last major crisis of 2008. Rodriguez (2007) analysed the financial contagion occurring during crisis using a two-state switching copula. Pelletier (2006) has studied the change of correlation regimes in a Gaussian framework. More recently, Okimoto (2008) has brought new evidence of the existence of distinct switching regimes for the US and UK equity markets.

The main contribution of this work is to consider the problem of the strategic asset allocation between stocks and bonds when marginal returns and their structure of dependence are influenced by fluctuations of the economy. In our approach modifications of the economic conjuncture are triggered by the switches of regimes of a hidden Markov chain. To each value of the Markov chain, corresponds a certain state of the economy which determines the performance and dependence of assets. We begin with a presentation of candidate statistical distributions (switching Student't and Weibull) for modelling assets returns. This is followed by a brief review of the switching copulas selected to model the dependence. Two categories are investigated: the Archimedian and the elliptic copulas. As marginal distributions of returns and their dependence both depend on the same unobservable Markov chain, both the marginal returns and the related copulas must be calibrated simultaneously. We therefore present a version of the Hamilton filter suited to this task. To illustrate this approach, we have fitted a selection of models to real daily stocks and bonds returns. As benchmark for stocks, we took the French index, CAC 40, of the forty biggest French companies. Whilst the benchmark for tracking bonds was taken to be the SGI Bond 10 years, which is an index computed by the French bank "Société Générale". This index duplicates a portfolio of Euro bonds continuously rebalanced to keep a constant maturity of 10 years.

Finally, we propose a dynamic investment policy that uses the information produced by the calibration procedure, i.e. the likely state of the economy, to build a dynamic investment policy. The idea underlying this approach is relatively straightforward. If the Hamilton filter detects that the economy is in a certain state with a high probability we adapt the asset allocation to optimize the expected yield in this likely state. The one year performance of this policy is compared with the static approach by using Monte-Carlo simulations. The tests show that such policies outperform the returns achieved by static strategies, with a limited risk and acceptable number of portfolio reallocations. This shows that the proposed model is more meaningful in that it captures essential features of the underlying processes and that it enables the use of more effective and pertinent strategies.

2 The states of the world.

We assume that the behavior of assets is ruled by a hidden Markov process representing a finite number of states of the economy. Each value of the chain, also called state in the remainder of this paper, corresponds to a particular economic conjuncture which influences the performance and the risk of assets. As it is impossible for an economic agent to determine the current state of the economy with certainty, the Markov chain is not directly observable. The main advantage of this framework is the ability to model the influence of economic cycles on returns of financial assets that switch from time to time, that is to say between a stable low-volatility state and a more unstable high volatility regime. In the next sections, each state of the Markov process corresponds to a set of parameters for assets returns densities and dependence structure. The hidden Markov process driving the economy is noted α_t and takes its value in the set $\mathcal{N} = \{1, 2, ...N\}$. The generator of α_t is a $N \times N$ matrix Q, whose elements, noted $q_{i,j}$, satisfy the following conditions:

$$q_{i,j} \geq 0 \quad \forall\, i \neq j \qquad \sum_{j=1}^{N} q_{i,j} = 0 \quad \forall i \in \mathcal{N} \ . \tag{2.1}$$

The probabilities of transition between times t and $s \geq t$ are obtained as the (matrix) exponential of Q :

$$P(t, s) = \exp\left(Q(s-t)\right) . \tag{2.2}$$

The elements of the matrix $P(t, s)$ are noted $p_{i,j}(t, s)\ i, j \in \mathcal{N}$. And $p_{i,j}(t, s)$ is the probability of going from state i at time t to state j at time s :

$$p_{i,j}(t, s) \quad = \quad P(\alpha_s = j \,|\, \alpha_t = i) \quad i, j \in \mathcal{N} . \tag{2.3}$$

The probability of being in state i at time t, noted $p_i(t)$ depends upon the initial probabilities $p_{k=1..N}(0)$ at time $t = 0$, as follows:

$$p_i(t) \quad = P(\alpha_t = i) = \sum_{k=1}^{N} p_k(0) p_{k,i}(0, t) \quad \forall i \in \mathcal{N} . \tag{2.4}$$

However, if the Markov process has been running for a sufficiently long enough period of time, we can show that this state probability distribution is independent of the initial state:

$$p_i = \lim_{t \to +\infty} p_i(t) \qquad \forall i \in \mathcal{N} . \tag{2.5}$$

This, so called "stationary property", is a standard characteristic of time-homogeneous Markov chains. The probabilities $p_{i=1...N}$ can be computed from the limit of equation (2.2) (note that the vector of stationary probabilities is also an eigenvector of the matrix of transition probabilities). In the calibration procedure, we chose to work with three states $N = 3$. It is theoretically possible to work with more states but it greatly increases the number of parameters to estimate and is a source of numerical instability as there are relatively few transitions between regimes. We note, but do not develop any further however, the use by Calvet and Fisher (2004) of tight parameterizing techniques for modelling relations between regimes in multi-fractal processes.

3 Marginal distribution of assets.

The aim of this research is to study the asset allocation problem between stocks and bonds when assets returns are influenced by switches of economic conjunctures. This requires the modelling of the dependence structures between the underlying classes of assets for all possible regime states. The theory of copula provides a way of formulating a multivariate distribution in terms of separately defined marginal distributions and a copula function that captures the structure and nature of their mutual dependence. For this reason, copula functions are widely used to model the dependency between multiple assets such as stocks and bonds. This section deals with the first stage in the process: the selection of appropriate marginal distributions. The copulas chosen to model the dependence are detailed in the subsequent section.

We first describe the main features of marginal distributions chosen to model assets log-returns in the numerical application section: namely the Student t and the Weibull distributions. The copulas chosen to model the dependence are detailed below. Note that, in this framework, both parameters of marginal distributions and copulas depend on the state in which the hidden Markov chain, α_t, lies. This approach requires the simultaneous calibration of the marginal laws and their governing copula. We will come back to this point later. In the remainder, it is assumed that stocks and bonds track two benchmarks whose the values are respectively noted S^1 and S^2, and observed at equidistant times $t_i \quad i = 1, ..., T$. The size of time intervals is noted Δt. On the interval of time $[t_i, t_{i+1}]$, the log-returns are noted $Y^k_{t_{i+1}}$, for $k = 1, 2$.

3.1 Student t distribution.

The first marginal distribution considered was the Student t distribution whose the parameters $\mu_k(.)$, $\sigma_k(.)$, $n_k(.)$ depend on the state of world at time t_i, α_{t_i}:

$$Y^k_{t_{i+1}} = \log \frac{S^k_{t_{i+1}}}{S^k_{t_i}} \quad = \quad \mu_k(\alpha_{t_i}) + \sigma_k(\alpha_{t_i})\, T^k_{n_k(\alpha_{t_i})} \qquad k = 1, 2$$

and where $T^k_{n_k(\alpha_{t_i})}$ is a Student t random variable having $n_k(\alpha_{t_i})$ degrees of freedom. The Student t distribution includes the normal distribution (whether $n_k(.)$ tends to $+\infty$) and the Cauchy distribution (whether $n_k(.) = 1$) as special cases. The choice of this distribution is mainly motivated by its ability to capture the excess kurtosis of stocks returns. The marginal density function of the k^{th} asset return on the period $[t_i, t_{i+1}]$, $Y^k_{t_{i+1}}$, is noted $f_k(t_{i+1}, .)$ and is given by the following expression :

$$f_k(t_{i+1}\,,\, y) \quad = \quad \frac{\Gamma((n_k(\alpha_{t_i}) + 1)/2)}{\sqrt{\pi n_k(\alpha_{t_i})\sigma_k^2(\alpha_{t_i})}\Gamma(n_k(\alpha_{t_i})/2)} \left(1 + \frac{\left(\frac{y-\mu_k(\alpha_{t_i})}{\sigma_k(\alpha_{t_i})}\right)^2}{n_k(\alpha_{t_i})} \right)^{-\frac{n_k(\alpha_{t_i})+1}{2}} \tag{3.1}$$

where $\Gamma(.)$ is the gamma function. The standard moments of returns in state α_t are:

$$\begin{aligned}
\mathbb{E}\left(Y^k_{t_{i+1}}\right) &= \mu_k(\alpha_{t_i}) \\
Std\left(Y^k_{t_{i+1}}\right) &= \sqrt{\frac{n_k(\alpha_{t_i})}{n_k(\alpha_{t_i}) - 2}}\sigma(\alpha_{t_i}) \\
Sk\left(Y^k_{t_{i+1}}\right) &= 0 \\
Ku\left(Y^k_{t_{i+1}}\right) &= 3 + \frac{6}{n_k(\alpha_{t_i}) - 4}
\end{aligned}$$

The Student t distribution, like the Gaussian, is a symmetric function and is therefore unable to capture the slight asymmetry sometimes observed in the distribution of returns. As a consequence, we also consider the Weibull distribution as a canditate for marginal returns, .

3.2 Weibull distribution.

The Weibull distribution is also called the Stretched-exponential distribution because it can decay more slowly than an exponential distribution (Its complementary cumulative distribution is a stretched exponential function) . As for the student t distribution, we assume that its parameters depend on the Markov process α_t. In this case, the log-returns are assumed to be:

$$Y^k_{t_{i+1}} = \log \frac{S^k_{t_{i+1}}}{S^k_{t_i}} \quad = \quad \mu_k(\alpha_{t_i}) + W^k_{\lambda_k(\alpha_{t_i}), n_k(\alpha_{t_i})} \qquad k = 1, 2$$

where $W^k_{\lambda_k(\alpha_{t_i}), n_k(\alpha_{t_i})}$ is a 2 parameter Weibull distribution whose the parameters $\lambda_k(.)$, $n_k(.)$ depend upon the state of world, α_{t_i}, at time t_i. The marginal density function of the asset k, $f_k(t_{i+1}, .)$, is given by the following expression:

$$f(t_{i+1}\,,\, y) \quad = \quad \frac{n_k(\alpha_{t_i})}{\lambda_k(\alpha_{t_i})} \left(\frac{y - \mu_k(\alpha_{t_i})}{\lambda_k(\alpha_{t_i})} \right)^{n_k(\alpha_{t_i})-1} e^{-((y-\mu_k(\alpha_{t_i}))/\lambda_k(\alpha_{t_i}))^{n_k(\alpha_{t_i})}}$$

All the moments of the Weibull distribution are finite despite its fat tailness. The first four moments of $Y^k_{t_{i+1}}$ in state α_t are:

$$\begin{aligned}
\mathbb{E}\left(Y^k_{t_{i+1}}\right) &= \mu_k(\alpha_{t_i}) + \lambda_k(\alpha_{t_i})\Gamma\left(1+\frac{1}{n_k(\alpha_{t_i})}\right)\\
Std\left(Y^k_{t_{i+1}}\right) &= \sqrt{\lambda_k^2(\alpha_{t_i})\Gamma\left(1+\frac{2}{n_k(\alpha_{t_i})}\right)-\left(\mathbb{E}\left(Y^k_{t_{i+1}}\right)\right)^2}
\end{aligned}$$

$$\begin{aligned}
Sk\left(Y^k_{t_{i+1}}\right) &= \frac{\lambda_k^3(\alpha_{t_i})\Gamma(1+\frac{3}{n_k(\alpha_{t_i})}) - 3\mathbb{E}\left(Y^k_{t_{i+1}}\right) Std\left(Y^k_{t_{i+1}}\right)^2 - \mathbb{E}\left(Y^k_{t_{i+1}}\right)^3}{Std\left(Y^k_{t_{i+1}}\right)^3}\\
Ku\left(Y^k_{t_{i+1}}\right) &= \frac{\lambda_k^3(\alpha_{t_i})\Gamma(1+\frac{4}{n_k(\alpha_{t_i})}) - 4\,Sk\left(Y^k_{t_{i+1}}\right) Std\left(Y^k_{t_{i+1}}\right)^3 \mathbb{E}\left(Y^k_{t_{i+1}}\right)}{Std\left(Y^k_{t_{i+1}}\right)^4}\\
&\quad - \frac{6\,\mathbb{E}\left(Y^k_{t_{i+1}}\right)^2 Std\left(Y^k_{t_{i+1}}\right)^2 + \mathbb{E}\left(Y^k_{t_{i+1}}\right)^4}{Std\left(Y^k_{t_{i+1}}\right)^4}
\end{aligned}$$

As mentioned in Frisch, U. and D. Sornette (1997) and in Malevergne & Sornette (2006), from a theoretical point of view, the use of this class of distributions is motivated in part by the fact that the large deviations of multiplicative processes tend to towards stretched exponential distributions. The next section presents the different structures of dependence between marginal distributions that will be used later.

4 The copulas.

A copula is a function linking the multivariate distribution, in our case of returns, to their marginal distributions. In this model, the copula parameters are marginal densities directly influenced by the state of the economy, α_t . We first introduce the five copulas tested in the forthcoming numerical application. Three of them belong to the family of Archimedian copulas. We also present the Gaussian and the Student t copulas, which are directly related to multivariate distributions of the same name. Before, we define some additional notation. In particular, we denote the marginal cumulative distributions, Student t or Weibull, of benchmarks log-returns by $F_{k=1,2}(t_{i+1}, y)$. While the copula is noted $C(.\,|\,\alpha_{t_i})$ and depends on the state of α_{t_i}. This copula is related to the multivariate distribution of log-returns, noted $H(.\,|\,\alpha_{t_i})$ as follows:

$$P\left(Y^1_{t_{i+1}} \leq y^1,\, Y^2_{t_{i+1}} \leq y^2\right) = H(y^1,\, y^2\,|\,\alpha_{t_i}) = C\left(F_1(t_{i+1}, y^1)\, F_2(t_{i+1}, y^2)\,|\,\alpha_{t_i}\right) \tag{4.1}$$

For a detailled introduction to copulas, we refer the interest reader to Nelsen (1999).

4.1 Gaussian copula.

The Gaussian copula is the dependence function related to the multidimensional Gaussian distribution. $\rho(\alpha_{t_i})$ is the parameter of correlation depending on the state of the Markov process α_t at time t_i. The symmetric positive matrix $\Sigma(\alpha_{t_i})$ is defined as follows:

$$\Sigma(\alpha_{t_i}) = \begin{pmatrix} 1 & \rho(\alpha_{t_i}) \\ \rho(\alpha_{t_i}) & 1 \end{pmatrix}$$

Let us note $\Phi^{-1}(.)$, the inverse of the cumulative probability function of an univariate $N(0,1)$ random variable. We also write $\Phi_\Sigma(.,.)$, the cumulative probability function of a 2D $N(0,\Sigma(\alpha_t))$.

The Gaussian copula, linking benchmarks returns observed on the interval of time $[t_i,\, t_{i+1}]$, is then defined by the following expression:

$$C(u_1, u_2 \,|\, \alpha_{t_i}) \quad = \quad \Phi_{\Sigma(\alpha_{ti})}\left(\Phi^{-1}(u_1)\,,\, \Phi^{-1}(u_2)\right)$$

The Kendall's τ for the period of time $[t_i,\, t_{i+1}]$ is given by:

$$\tau(\alpha_{t_i}) \quad = \quad \frac{2}{\pi} \arcsin \rho(\alpha_{t_i})$$

As shown in Roncalli (2004), the density of the Gaussian copula is given by the following expression:

$$c((u_1, u_2 \,|\, \alpha_{t_i})) \quad = \quad \frac{1}{\sqrt{1-\rho(\alpha_{t_i})^2}} \exp\left(-\frac{1}{2(1-\rho(\alpha_{t_i})^2)}\left(x_1^2 + x_2^2 - 2\rho(\alpha_{t_i})x_1x_2\right) + \frac{1}{2}\left(x_1^2 + x_2^2\right)\right)$$

where $x_1 = F_1^{-1}(t_i, u_1)$ $x_2 = F_2^{-1}(t_i, u_2)$. The density will be particularly useful during the phase of calibration. Note that whether $\rho(\alpha_{t_i}) < 1$, the upper and lower tail dependence are null:

$$\lambda_U(\alpha_{t_i}) = \lambda_L(\alpha_{t_i}) \quad = \quad 0$$

In this sense, the Gaussian copula does not allow to correlate extremal values.

4.2 Student t copula.

The student t copula is the dependence function related to the Student t multidimensional distribution. As in the previous paragraph, $\rho(\alpha_{t_i})$ is the correlation parameter depending on the state of the Markov process α_t at time t_i. $\Sigma(\alpha_{t_i})$ is defined as:

$$\Sigma(\alpha_{t_i}) = \begin{pmatrix} 1 & \rho(\alpha_{t_i}) \\ \rho(\alpha_{t_i}) & 1 \end{pmatrix}$$

Let us note $F^{-1}_{\nu(\alpha_{t_i})}(.)$, the inverse of the cumulative probability function of an univariate Student t , having $\nu(\alpha_{t_i})$ degrees of freedom at time t_i. Furthermore, let $F_{\Sigma(\alpha_{t_i}),\nu(\alpha_{t_i})}(.,.)$ be the cumulative probability function of a 2D Student t, of parameters $\Sigma(\alpha_{t_i})$, $\nu(\alpha_{t_i})$. The Student copula, linking stocks and bonds returns observed on the interval of time $[t_i,\, t_{i+1}]$, is defined as follows:

$$C(u_1, u_2 \,|\, \alpha_{t_i}) \quad = \quad F_{\Sigma(\alpha_{ti}),\nu(\alpha_{ti})}\left(F^{-1}_{\nu(\alpha_{ti})}(u_1)\,,\, F^{-1}_{\nu(\alpha_{ti})}(u_2)\right)$$

The density of the Student t copula is given by:

$$\begin{aligned} c((u_1, u_2 \,|\, \alpha_{t_i})) \quad = \quad & \frac{1}{\sqrt{1-\rho(\alpha_{t_i})^2}} \frac{\Gamma\left(\frac{\nu(\alpha_{ti})+2}{2}\right)\Gamma\left(\frac{\nu(\alpha_{ti})}{2}\right)}{\Gamma\left(\frac{\nu(\alpha_{ti})+1}{2}\right)^2} \\ & \frac{\left(1+\frac{1}{\nu(\alpha_{ti})}\left(x_1^2+x_2^2-2\rho(\alpha_{t_i})x_1x_2\right)\right)^{-\frac{\nu(\alpha_{ti})+2}{2}}}{\prod_{k=1}^2\left(1+\frac{x_k^2}{\nu(\alpha_{ti})}\right)^{\frac{\nu(\alpha_{ti})+1}{2}}} \end{aligned}$$

To our knowledge, there is no closed form expression of the Kendall's τ (even if practitioners tend to observe that the numerical values of the Kendall's τ are close to those of a Gaussian copula with the same correlation matrix Σ). As the student t copula is symmetrical, its upper and lower tail dependence are identical:

$$\lambda_U(\alpha_{t_i}) = \lambda_L(\alpha_{t_i}) = 2 - 2\, t_{\nu(\alpha_{t_i})+1}\left(\left(\frac{(\nu(\alpha_{t_i})+1))\,(1-\rho(\alpha_{t_i}))}{(1+\rho(\alpha_{t_i}))}\right)^{\frac{1}{2}}\right) \geq 0$$

Note that if the parameter $\nu(\alpha_{t_i})$ tend to infinity, the Student t copula tends to a Gaussian copula.

4.3 Archimedian copulas.

The class of Archimedian copulas is particularly popular because the their structures are easy to handle and they can take into account extreme dependence events. We assume that the copula defining the multivariate distribution of returns belongs to the family of Archimedian copulas, indexed by the process α_t. These are defined as follows:

$$C(u_1, u_2 \mid \alpha_t) = \begin{cases} \varphi^{-1}\left(\varphi(u_1, \alpha_t) + \varphi(u_2, \alpha_t)\,,\, \alpha_t\right) & if\ \sum_{k=1}^{M2} \varphi(u_k, \alpha_t) \leq \varphi(0, \alpha_t) \\ 0 & else \end{cases} \tag{4.2}$$

where $\varphi(u, \alpha_t)$ is a continuous strictly decreasing and convex function such $\varphi(1, \alpha_t) = 0$ and $\varphi(0, \alpha_t) = \infty$ in all states of the world. $\varphi(.,.)$ is called the generator of the copula. Furthermore $\varphi^{-1}(u, \alpha_t)$ must be completely monotonic on $[0, +\infty[$:

$$(-1)^d \frac{\partial^k}{\partial u^k} \varphi^{-1}(u) \geq 0 \qquad for\, k = 1, 2, ...$$

The function φ is called the generator of the copula. Genest and Mackay (1986) have proved that the Kendall's tau of those distributions in each state of the world is easily computed by the formula:

$$\tau = 1 + 4 \int_0^1 \frac{\varphi(u, \alpha_t)}{\varphi'(u, \alpha_t)} du\,.$$

They have also shown that the density of this category of copula has the following expression:

$$c(u_1, u_M \mid \alpha_t) = -\frac{\varphi''(C(u_1, u_2 \mid \alpha_t), \alpha_t)\varphi'(u_1, \alpha_t)\varphi'(u_2, \alpha_t)}{\left[\varphi'(C(u_1, u_2 \mid \alpha_t), \alpha_t)\right]^3} \tag{4.3}$$

In particular, we focus on three copulas presented in table 4.1 :

	$\varphi(.)$	$C(u,v)$
Gumbel	$(-\ln u)^{\theta(\alpha_t)}$	$\exp\left(-\left((-\ln u)^{\theta(\alpha_t)} + (-\ln v)^{\theta(\alpha_t)}\right)^{1/\theta(\alpha_t)}\right)$
Franck	$-\ln \frac{e^{-\theta(\alpha_t)u}-1}{e^{-\theta(\alpha_t)}-1}$	$-\frac{1}{\theta(\alpha_t)} \ln\left(1 + \frac{(e^{-\theta(\alpha_t)u}-1)(e^{-\theta(\alpha_t)v}-1)}{e^{-\theta(\alpha_t)}-1}\right)$
Clayton	$\frac{1}{\theta(\alpha_t)}\left(u^{-\theta(\alpha_t)} - 1\right)$	$\left(u^{-\theta(\alpha_t)} + v^{-\theta(\alpha_t)} - 1\right)^{-1/\theta(\alpha_t)}$

Table 4.1: Archimedian copulas

where $\theta(\alpha_t)$ is a positive process depending upon the state of the world and taking its value in the set $\{\theta_1, \theta_2, \ldots, \theta_N\}$. The Kendall tau for these copulas, the upper and lower tail dependencies and the range of parameters[1] are provided in table 4.2.

	Parameter range	τ	λ_u	λ_l
Gumbel	$\theta(\alpha_t) \geq 1$	$1 - 1/\theta(\alpha_t)$	$2 - 2^{1/\theta(\alpha_t)}$	0
Franck	$\theta(\alpha_t) \in \mathbb{R}$	$1 - 4\theta(\alpha_t)^{-1}(1 - D_1(\theta(\alpha_t)))$	0	0
Clayton	$\theta(\alpha_t) \geq -1$	$\frac{\theta(\alpha_t)}{\theta(\alpha_t)+2}$	0	$2^{-1/\theta(\alpha_t)}\ \theta(\alpha_t) > 0$ $0 \quad \theta(\alpha_t) \leq 0$

Table 4.2: Kendall tau, upper & lower tail dependencies

[1] $D_1(\theta)$ is the Debye function $D_1(\theta) = \theta^{-1} \int_0^\theta t/(exp(t) - 1)dt$

5 Calibration of the switching regime model.

The common calibration method for regime switching models is the Hamilton filter (1989), which estimates parameters by maximum likelihood estimation (MLE), from past returns. The Hamilton Filter is a Bayesian procedure, based on an iterative algorithm that is similar in spirit to the Kalman filter (1960), used to fit system driven by a hidden Markov chain. Let O_1, O_2, ...,O_T be T observed log-returns of benchmarks:

$$O_i \quad = \quad \left(y^1_{t_i}\,;\,y^2_{t_i}\right) \qquad i=1,...,T\,.$$

We denote Θ the set of parameters of our model:

$$\Theta \quad = \quad \{\mu^{k=1,2}_{i=1...N},\, \sigma^{k=1,2}_{i=1...N}\,,\, n^{k=1,2}_{i=1...N},\, q_{i=1...N,j=1...N}\}$$

The log-likelihood function is defined as follows:

$$\begin{aligned}\log\mathcal{L} \;&= \log \quad f(O_1|\Theta) + \log f(O_2|\Theta, O_1) + \log f(O_3|\Theta, O_1, O_2)\\ &\qquad + \ldots + \log f(O_T|\Theta, O_1, \ldots, O_{T-1})\end{aligned}$$

where $f(O_k|\Theta, O_1, \ldots O_{k-1})$ is the multivariate density function of log-returns on the k^{th} period, conditionally to a set of given model parameters Θ and to previous observations $O_1, \ldots O_{k-1}$. The best fitting parameters for the switching regime model are the ones maximizing the log-likelihood function. The conditional density, involved in the calculation of $f(O_k|\Theta, O_1, \ldots O_{k-1})$, may be recursively calculated as follows:

$$\begin{aligned}&f(O_k|\Theta, O_1, \ldots O_{k-1})\\ &\quad = \sum_{i=1}^{N}\sum_{j=1}^{N} p_i(t_{k-1}|\Theta, O_1, \ldots O_{k-1})\, p_{i,j}(t_{k-1}, t_k|\Theta)\, f(O_k|\Theta, \alpha_{t_{k-1}} = i\,,\, \alpha_{t_k} = j)\end{aligned}$$

where

- $f(O_k|\Theta, \alpha_{t_{k-1}} = i\,,\, \alpha_{t_k} = j)$ is the multivariate density of $\left(Y^1_{t_k}, Y^2_{t_k}\right)$ in state j,
- $p_{i,j}(t_{k-1}, t_k|\Theta)$ is the probability of transition, as defined by equation (2.3), from state i at time t_{k-1} to state j at time t_k for the set of parameters Θ,
- $p_i(t_{k-1}|\Theta, O_1, \ldots O_{k-1})$ is the probability of being in state i at time t_{k-1}, conditionally to previous observations.

The multivariate density of $\left(Y^1_{t_k}, Y^2_{t_k}\right)$ in state j is equal to the following expression:

$$f(O_k|\Theta, \alpha_{t_{k-1}} = i\,,\, \alpha_{t_k} = j) \quad = \quad c(F_1(t_k, y^1_{t_k}), F_2(t_k, y^1_{t_k}))\, f_1(t_k, y^1_{t_k})\, f_2(t_k, y^2_{t_k})$$

where $c(.)$, $F_{1,2}(.)$, $f_{1,2}(.)$ are respectively the density of copula, the marginal cumulative and probability density functions of returns. The probability $p_i(t_{k-1}|\Theta, O_1, \ldots O_{k-1})$ are inferred recursively from $f(O_{k-1}|\Theta, O_1, \ldots O_{k-2})$ as follows:

$$\begin{aligned}&p_i(t_{k-1}|\Theta, O_1, \ldots O_{k-1}) =\\ &\qquad \frac{\sum_{j=1}^{N} p_j(t_{k-2}|\Theta, O_1, \ldots O_{k-2})\, p_{j,i}(t_{k-2}, t_{k-1}|\Theta)\, f(O_{k-1}|\Theta, \alpha_{t_{k-1}}, \alpha_{t_{k-2}})}{f(O_{k-1}|\Theta, O_1, \ldots O_{k-2})}\end{aligned} \tag{5.1}$$

In order to initiate the recursion, we need to determine $f(O_1|\Theta)$. Hamilton assumes that the Markov chain has been running for a sufficiently long enough period of time, so as to apply the stationary property of Markov chains, mentioned in section 3. In particular, we get that:

$$f(O_1|\Theta) = \sum_{i=1}^{N} p_i(\Theta)\, f(O_1|\Theta, \alpha_{t_0} = i),$$

where $p_i(\Theta)$ are the stationary probabilities of the Markov chain $\alpha(t)$, as defined by equation. (2.5), for the set of parameters Θ.

In our approach, as marginal distributions of returns and their dependence both depend on the same unobservable Markov chain, the calibration of marginal returns cannot be dissociated from fitting the copula. This is an important factor that limits the number of assets that could be handled in this approach. Indeed, if we fit a Gaussian copula and student 't returns for two assets, assuming that the Markov chain has three states, we have to simultaneously estimate 27 parameters[2] by Log-likelihood maximization. This is a very large number of optimization variables for any maximization procedure.

6 Numerical application.

In this section, we discuss the calibration of ten models, obtained by combining the different copulas and marginal distributions presented earlier, to daily returns of stocks and bonds indexes. For stocks, we chose the CAC 40, which is the French index of the forty biggest French companies. The SGI Bond 10 years (SGIXBE10), which is an index computed by the bank "Société Générale", was selected as the benchmark for bonds. This index duplicates a portfolio of Euro bonds continuously rebalanced to keep a constant maturity of 10 years. The marginal distributions and copulas were fitted to daily returns collected from the 2/1/2003 to the 30/11/2009. The statistics of daily returns are presented in table 6.1. The number of hidden states of the Markov chain α_t was set to three ($N = 3$).

Daily returns	CAC 40	SGIXBE10
Mean	0,0117%	0,0097%
Standard Deviation	1,4741%	0,1596%
Skewness	0,0900	-0,3237
Kurtosis	7,9470	7,3183

Table 6.1: Statistics of daily returns

The log-likelihoods obtained after calibration of switching regime models are presented in table 6.2. The results reveal that modelling the marginal returns of assets by a 3-state Student t leads to a higher log-likelihood, for all the copulas. Despite its ability to capture asymmetry and leptokurticity, the Weibull distribution is less efficient than the Student t for modelling daily returns.

	Student t	Weibull
Gumbel	14 362	14 330
Clayton	14 363	14 331
Franck	**14 391**	14 359
Gauss	**14 392**	14 359
Student	**14 391**	14 358

Table 6.2: Log likelihood

According to these figures the Gausian copula is the most appropriate structure to model the dependence between the CAC 40 and the SGIXBE10. In effect, the parameters of the Gumbel and the Clayton copulas are respectively equal to 1 and 0 which corresponds to the degenerate case of statistical independence between assets. This clearly shows that Gumbel and Clayton copulas are not applicable to the joint dynamics of indexes. We also observe that the degrees of freedom

[2] $3 \times 3 \times 2$ parameters for Students't, 3 correlations for the Gaussian Copula, and 6 transition probabilities

of the Student copula tend to infinity in each state that is to say it converges towards a Gaussian copula. For these reasons, we subsequently focused on Franck and Gauss copulas with Student t margins.

6.1 Gaussian copula and Student t marginal distributions.

This section presents the results of the calibration procedure when the index returns are assumed Student t distributed and dependent trough a Gaussian copula. The table 6.3 indicates that in two states on three, the daily average return of the CAC 40 is negative. while the return of the SGIXBE10 is on average positive whatever the state. The states 1, 2 and 3 correspond respectively to bear, bull markets and to a deep crisis. The standard deviation reaches a peak of 3.25% in state 3. The kurtosis of both categories of assets is close to 3 (which is the kurtosis of a Gaussian) in states 1 and 2. In the state of crisis, the kurtosis of the stocks index reaches 4. This indicates that the distribution of stocks return has fat tails in this case. The bonds index has the opposite behavior.

		$\mu(\alpha_t)$	$\sigma(\alpha_t)$	$\nu(\alpha_t)$	Kurtosis $\kappa(\alpha_t)$
State $\alpha_t = 1$	CAC 40	-0.04%	1.47%	911.99	3.0066
	SGIXBE10	0.01%	0.15%	11.45	3.8052
State $\alpha_t = 2$	CAC 40	0.09%	0.70%	13.80	3.6125
	SGIXBE10	0.01%	0.11%	26.81	3.2630
State $\alpha_t = 3$	CAC 40	-0.31%	3.25%	9.09	4.1795
	SGIXBE10	0.05%	0.21%	2.93	-2.6317

Table 6.3: Parameters of marginal distributions

The table 6.4 of daily probabilities of transition $p_{i,j}(t, t + 1\,day)$, shows that there is no way to go from state 2, a bull market, to state 3, the situation of crisis, without transiting by a bear market, state 2. This result is rather intuitively satisfying: as seen during the internet bubble crash and the credit crunch crisis, markets go first through a decreasing phase before falling in a deep crisis. Note that the probability of remaining in the same state, the next day, is above 94%, whatsoever the state. This observation is particularly useful to build a dynamic strategy.

	State $\alpha_t = 1$	State $\alpha_t = 2$	State $\alpha_t = 3$
State $\alpha_t = 1$	0.9838	0.0063	0.0099
State $\alpha_t = 2$	0.0057	0.9943	0
State $\alpha_t = 3$	0.0577	0	0.9423

Table 6.4: Daily probabilities of transition

The table 6.5 reveals that the dependence, measured by the Kendall τ , is negative and nearly identical in state 1 and 3. In a bull market, the dependence nearly half as important, but still negative. This negative dependence confirms that a small diversification effect exists between bonds and stocks.

	Gaussian Copula $\rho(\alpha_t)$	$\tau(\alpha_t)$
State $\alpha_t = 1$	-0.2192	-0.1407
State $\alpha_t = 2$	-0.1349	-0.0861
State $\alpha_t = 3$	-0.2296	-0.1475

Table 6.5: Parameters of the Gaussian copula

The figures 6.1 , 6.2 , 6.3 present the probability of being in states 1, 2 and 3 at time t_{k-1}, conditionally to previous observations, noted $p_i(t_{k-1}|\Theta, O_1, \ldots O_{k-1})$ in section 5. These graphs are of particular interest as they plainly show the periods during which the market was depressed or bullish. The financial crises linked to the Irak war in 2003 and the credit crunch that began in 2008 are clearly identifiable in the probabilities of sojourn in state 3.

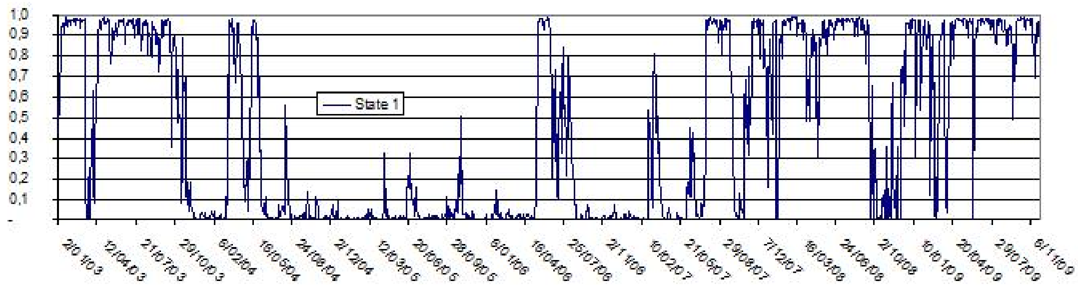

Figure 6.1: Probabilities of being in state 1.

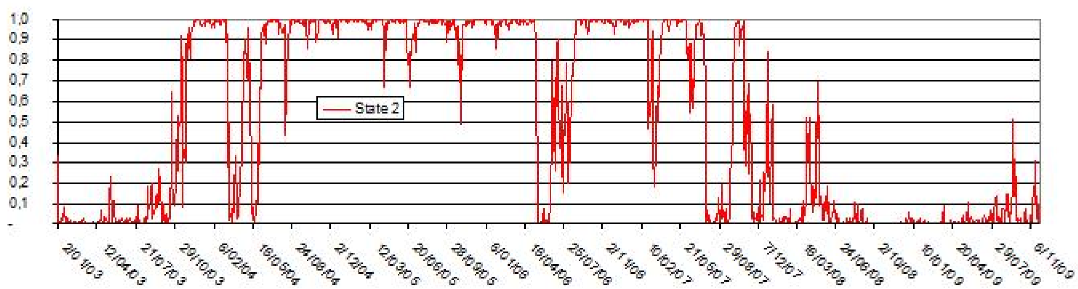

Figure 6.2: Probabilities of being in state 2.

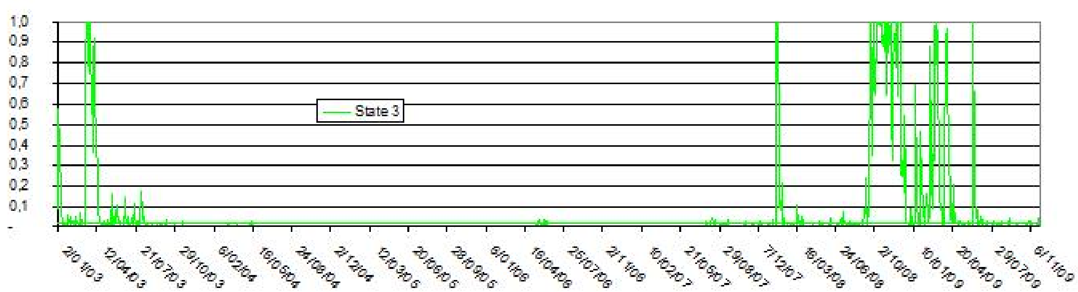

Figure 6.3: Probabilities of being in state 3.

6.2 Frank copula and Student t marginal distributions.

This section presents the results of the calibration procedure when the dependence between Student t marginal returns is modeled by a Frank copula. The parameters of marginal returns, in table 6.6, are very close to those obtained when the dependence is modeled by a Gaussian copula. The transition probabilities in table 6.7 are also nearly identical to those obtained with a Gaussian dependence. This resemblance is intuitively satisfying and shows the robustness of the calibration procedure.

		$\mu(\alpha_t)$	$\sigma(\alpha_t)$	$\nu(\alpha_t)$	Kurtosis $\kappa(\alpha_t)$
State $\alpha_t = 1$	CAC 40	-0.04%	1.474%	911.98	3.0066
	SGIXBE10	0.01%	0.14%	11.41	3.8052
State $\alpha_t = 2$	CAC 40	0.09%	0.70%	13.84	3.6125
	SGIXBE10	0.01%	0.11%	30.46	3.2630
State $\alpha_t = 3$	CAC 40	-0.27%	2.88%	6.01	4.1795
	SGIXBE10	0.04%	0.21%	3.42	-2.6317

Table 6.6: Parameters of marginal distributions

	State $\alpha_t = 1$	State $\alpha_t = 2$	State $\alpha_t = 3$
State $\alpha_t = 1$	0.9844	0.0065	0.0091
State $\alpha_t = 2$	0.0059	0.9941	0
State $\alpha_t = 3$	0.0466	0	0.9534

Table 6.7: Daily probabilities of transition

The table 6.8 shows that the dependence, measured by the Kendall τ , is negative in each state. Contrary to what we have observed, the dependence is slightly more negative in period of crisis than in a bear market. We also notice that the diversification effect is less important in state 2.

	Franck Copula $\theta(\alpha_t)$	$\tau(\alpha_t)$
State $\alpha_t = 1$	-1.2607	-0.1379
State $\alpha_t = 2$	-0.7888	-0.0871
State $\alpha_t = 3$	-1.4936	-0.1624

Table 6.8: Parameters of the Franck copula

The plots of past states probabilities are not shown as they are very close to those presented in figures 6.1, 6.2, 6.3. For ease of comparison we have plotted the copulas in figure 6.4 when the Markov chain is in state 3. Both copulas have a saddle shape but differ in that the curvature of the Frank copula in the central area is higher than that of the Gauss copula.

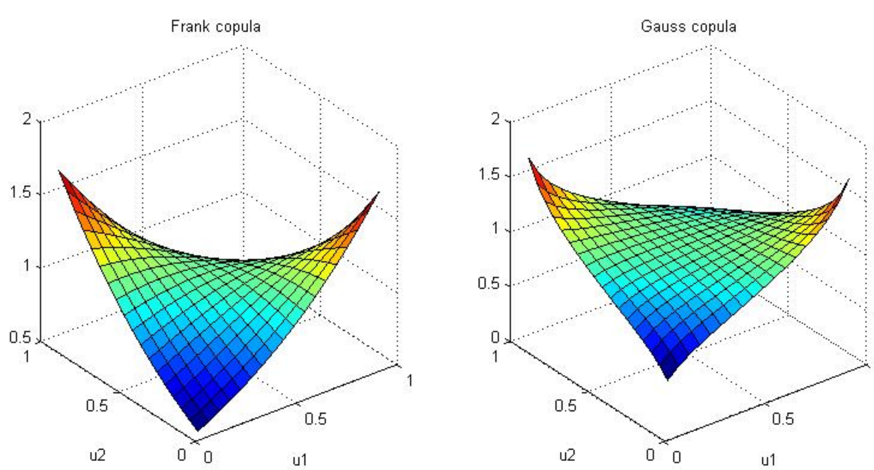

Figure 6.4: Frank and Gauss copulas in state 3.

7 Static Asset Allocation.

In this section, we study the performance and the risk over one year of static investment strategies. This analysis is performed by the means of Monte Carlo simulations. The procedure to simulate returns is rather intuitive. For each simulated trajectory of returns, we generated the daily evolution of a 3-state Markov chain, whose the daily transition probabilities are shown in tables 6.4 and 6.7. Three cases were considered: each one related to the value of α_t taken in (1, 2 or 3). Once that a path for α_t is computed, we simulate the marginal returns of assets (Student t distributed) with the Frank copula and the Gaussian copula. The algorithm used to simulate Archimedian and Gaussian copulas is explained in appendix A.

The statistics considered are the mean return, the standard deviation, Value at Risk and Tail VaR, on a time horizon of 252 trading days. Portfolios tested contain bonds and stocks, with weights running from 10% to 90%, by step of 10%. Tables 7.1 to 7.6 display the statistics related to the return of static portfolios for different values of the Markov chain α_t at time $t = 0$ and for different structures of dependence.
If the hidden Markov chain is initially in state one ($\alpha_{t=0} = 1$), which corresponds to a bear market, the higher one year return and the smaller risk are obtained by a portfolio overweighted in bonds, irrespective of the copula chosen to model the dependence. The main issue faced by the investor is that $\alpha_{t=0}$ is not directly observable. However, a probability of being in this state can be inferred by applying the calibration procedure developed in section 5. If this probability is high (above a certain trigger chosen by the investor), it is a good indicator that the best static portfolio contains mainly bonds. This observation is at the basis of the dynamic strategy proposed in the next subsection.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	-1.4%	-0.9%	-0.3%	0.3%	0.8%	1,4%	1,9%	2,5%	3,1%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7,8%	5,3%	3.3%
VaR 5%	-39.0%	-34.3%	-29.2%	-24.1%	-19.6%	-14.7%	-10.1%	-5.5%	-2.0%
TVaR 5%	-46.6%	-40.8%	-35.0%	-29.3%	-23.6%	-18.1%	-12,7%	-7.6%	-3.6%

Table 7.1: Returns after 252 days, initial state: 1, Gaussian copula.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	-4.5%	-3.6%	-2.7%	-1.7%	-0.8%	0.1%	1.1%	2.0%	2.9%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7.8%	5.3%	3.3%
VaR 5%	-46%	-40.2%	-34.5%	-29.0%	-23.3%	-17.6%	-12.3%	-7.2%	-2.7%
TVaR 5%	-53.8%	-47.2%	-40.7%	-34.2%	-27.7%	-21.4%	-15.2%	-9.5%	-4.5%

Table 7.2: Returns after 252 days, initial state: 1. Frank copula.

When α_t is initially in state two, which corresponds to a bull market, the best average return is obtained by overweighting stocks. However, the risk increases proportionally with the fraction of stocks held in portfolio. In this situation, the choice of the optimal static allocation is driven by the risk aversion of the investor. If the one year acceptable VaR level is 10%, the best assets mix is 70% of bonds and 30% of stocks. We note again that statistics are not significantly different for Gaussian and Frank copulas.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	10.7%	9.8%	8.9%	8.0%	7.2%	6.3%	5.4%	4.5%	3.7%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7.8%	5.3%	3.3%
VaR 5%	-30.0%	-26.3%	-22.5%	-18.6%	-14.7%	-10.6%	-6.9%	-3.9%	-1.5%
TVaR 5%	-37.6%	-33.0%	-28.4%	-23.8%	-19.2%	-14.7%	-10.3%	-6.3%	-2.9%

Table 7.3: Returns after 252 days, initial state: 2. Gaussian copula.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	11.0%	10.1%	9.2%	8.3%	7.4%	6.5%	5.6%	4.6%	3.7%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7.8%	5.3%	3.3%
VaR 5%	-28.1%	-24.4%	-20.9%	-17.4%	-13.9%	-10.3%	-6.9%	-3.5%	-0.9%
TVaR 5%	-37.6%	-32.8%	-28.1%	-23.4%	-18.7%	-14.1%	-9.7%	-5.5%	-2.2%

Table 7.4: Returns after 252 days, initial state: 2. Frank copula

In times of crisis, which corresponds to $\alpha_{t=0} = 3$, the investor should overweight bonds to minimize risk and improve the one year expected return. Again, this conclusion is valid for both Gaussian and Frank copulas.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	-4.3%	-3.4%	-2.4%	-1.4%	-0.4%	0.5%	1.5%	2.5%	3.5%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7.8%	5.3%	3.3%
VaR 5%	-43.5%	-37.7%	-32,4%	-26.9%	-21.7%	-16.1%	-10,8%	-6.1%	-2.1%
TVaR 5%	-51.5%	-45.0%	-38.6%	-32.2%	-25.8%	-19.6%	-13.6%	-8.1%	-3.9%

Table 7.5: Returns after 252 days, initial state: 3. Gaussian copula.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	-6.0%	-4.9%	-3.8%	-2.6%	-1.5%	-0.4%	0.7%	1.9%	3.0%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7.8%	5.3%	3.3%
VaR 5%	-45.9%	-40.0%	-33.9%	-28.6%	-22.8%	-16.8%	-11.7%	-6.8%	-2.7%
TVaR 5%	-52.1%	-45.6%	-39.1%	-32.7%	-26.3%	-20.1%	-14.1%	-8.7%	-4.5%

Table 7.6: Returns after 252 days, initial state: 3. Frank copula.

8 Dynamic strategy of investment.

The prevailing state of the economy considerably influences the return and the risk of static investment policies. In state 1 and 3, the fund manager should adopt a defensive strategy and invest mainly in bonds. In state 2, the fund manager seeking a high return should instead invest massively in stocks. Although the Markov chain α_t is not directly observable, the calibration procedure provides an efficient way to calculate the probability of being in a certain state based upon the observed achieved returns. At the end of each day, based upon achieved past returns $O_1, \dots O_{k-1}$, we can compute by equation (5.1) the probabilities:

$$p_i(t_{k-1}|\Theta, O_1, \dots O_{k-1})\ i = 1,\ 2,\ 3\ ,$$

of being in state i. If one of these probabilities is above a certain trigger κ chosen by the investor, we can assume that the economy is effectively in the detected state at time t_{k-1}. If the probability of remaining in this state is high (as in our application, see matrix 6.4), the policy of investment can be adjusted for the next day. E.g. Based upon the observations in the previous subsection, if we detect that the market is in state 2 (bull market) with a high probability, the part of the fund invested in stocks should be increased. On the contrary, if we estimate that the chain is in state 1 or 3 with a high probability, it is preferable to increase the percentage of bonds held in portfolio. This strategy should intuively lead to a better return than a static approach.

To test the performance of this strategy, we performed a Monte-Carlo simulation similar to the one used for static strategies. 1000 scenarios of 252 days of trading were simulated. For each scenario, the daily state probabilities were computed according to equation (5.1) where O_k and Θ are respectively the simulated daily returns of benchmarks and the parameters estimated by MLE in the calibration phase. The trigger level κ was set to 60%. The rules determining the asset allocation are detailed in table 8.1. We chose a bipolar strategy. I.e. in a bull market, nearly the whole portfolio is invested in stocks while in the two other states, 90% of the portfolio is invested in bonds. If none of the $p_i(...)$ is above the trigger, the asset allocation remains unchanged. Transaction costs were not taken into account.

	Bonds	Stocks
$p_1(t_{k-1}\|\Theta, O_1, \ldots) \geq \kappa$	90%	10%
$p_2(t_{k-1}\|\Theta, O_1, \ldots) \geq \kappa$	10%	90%
$p_3(t_{k-1}\|\Theta, O_1, \ldots) \geq \kappa$	90%	10%
$p_1, p_2, p_3 \leq \kappa$	No change	

Table 8.1: Rules for asset allocation.

Tables 8.2 and 8.3 provide statistics about the return and the risk of the proposed dynamic strategy when dependence is modeled by a Gaussian and a Frank copula. As the initial state, $\alpha_{t=0}$, of the Markov chain influences widely the one-year return, we performed Monte-Carlo simulations for each value of the Markov chain α_t at time $t = 0$. The results reveal that in all cases, the return achieved is high (around 8% if $\alpha_{t=0} = 1, 2$ and 16% if $\alpha_{t=0} = 3$) while the risk is relatively low (the tail VaR is around 5.5% irrespective of the initial state). The positive skewness indicates that the right tail of the 1y-return distribution is longer than the left tail. The sample kurtosis when $\alpha_{t=0} = 1, 3$ is above 3, (the sample kurtosis of the normal), i.e. the distribution of the 1y-return has fatter tails than that of a normal distribution. If $\alpha_{t=0} = 2$, the kurtosis is close to the one of a normal distribution.

	Initial state		
	State 1	State 2	State 3
Mean	8.1%	15.7%	7.8%
Deviation	9.8%	9.8%	9.8%
VaR 5%	-4.2%	-2.4%	-3.5%
TVaR 5%	-5.8%	-5.3%	-5.6%
Skewness	1.1983	0.6633	1.2147
Kurtosis	4.9317	3.3311	5.1503

Table 8.2: Returns after 252 days, dynamic strategy. Gaussian copula.

	Initial state		
	State 1	State 2	State 3
Mean	8.02%	15.13%	7.43%
Deviation	9.95%	12.3%	9.03%
VaR 5%	-3.56%	-1.96%	-3.59%
TVaR 5%	-5.58%	-4.44%	-6.14%
Skewness	1.3257	0.6275	1.1257
Kurtosis	5.2291	3.0647	4.566

Table 8.3: Returns after 252 days, dynamic strategy. Frank copula.

As mentioned earlier, these results do not take into account transaction fees which can, in reality, seriously decrease the return achieved by a dynamic strategy of investment, particularly if the number of portfolio reallocations is high. Tables 8.4 and 8.5 present some statistics about the number of assets reallocations of our dynamic strategy, for both type of dependence (Gaussian and Frank copula). Regardless of the initial state, the portfolio is rebalanced, on average, seven times during the year of trading. The maximum number of asset reallocations (around 25) is still acceptable in practice in comparison to common strategies such CPPI that usually require at least a weekly rebalancing. It is therefore reasonable to assume, given the limited number of portfolio reallocations, that dynamic strategies similar to the one studied should perform relatively well even when taking into account transaction fees.

	Initial state		
	State 1	State 2	State 3
Mean	8	7	7
Deviation	14	13	17
Min	1	3	0
Max	25	24	25

Table 8.4: Number of assets reallocation after 252 days. Gaussian copula.

	Initial state		
	State 1	State 2	State 3
Mean	8	7	6
Deviation	12	10	14
Min	1	2	0
Max	21	19	19

Table 8.5: Number of assets reallocation after 252 days. Frank copula.

9 Conclusion

This work explores the strategic allocation of a portfolio of stocks and bonds tracking the CAC 40 and the SGI Bond 10 years. The novelty of the approach is to consider that marginal distributions of indices returns and their dependence structure both depend on the same unobservable Markov chain. The model is calibrated by using the Hamilton filter. This procedure, applied to a three-state Markov chain, on daily data of the French market, from 2003 to end 2009, leads to the identification of each state and the past probabilities of sojourn in each state. State one corresponds to a bear market: the return is negative and the volatility is high. State two is equivalent to a bull market: the return is high and risk is low. In the third state, the market is seriously depressed: there is high volatility and stocks return are strongly negative. An analysis of probabilities of sojourn and of transitions in this state clearly highlight the financial crisis related to the Irak war

in 2003 and the recent credit crunch. The loglikelihood is maximized when marginal returns are fitted by a Student t law and when the dependence is modeled either by a Gaussian or either by a Frank copula.
After discussion of the calibration procedure and results, we compare the one year return of a static investment strategy with a dynamic policy. We observe that static investment strategies are strongly influenced by the initial state of the hidden Markov chain and that their one year performance is poor excepted in a bull market. In the, alternative, dynamic policy, the investment strategy is based upon the estimated state probabilities of the Markov chain. If the Markov chain is in a certain state with a probability above a given trigger, the investment policy is adapted to provide the highest yield in this state. The numerical tests show that a good average return can be achieved by investing mainly in stocks if we detect a bull market state and to reverse the position in bonds if we detect any other state. Furthermore, the one year VaR and tail VaR are relatively low and the number of reallocations needed remains reasonable.

We conclude, that modelling the marginal distributions of asset returns and their dependence structure by a single hidden Markov chain leads to a more meaningful representation of the underlying processes and, consequently, more effective investment strategies.

Appendix A Simulation of copulas.

In this section, we briefly review the methods used to simulate Archimedian, Gaussian and Student copulas.

Simulation of an Archimedian copula.

The algorithm to simulate Archimedian copulas is based upon the relation between the Laplace transform of a random variable and the generator function. Let us note $\Lambda(\alpha_t)$ a family of random variable indexed by the Markov chain α_t, defined such that their Laplace transform is equal to the inverse of the generator $\varphi^{-1}(.)$:

$$\varphi^{-1}(u,\alpha_t) \quad = \quad \mathbb{E}\left(e^{-\Lambda(\alpha_t)u}\right)$$

The distribution of $\Lambda(\alpha_t)$, for the three families of copulas introduced in section 4, are presented in table 9.1 where $\theta(\alpha_t)$ is the parameter of the Archimedian copula, indexed by the Markov chain.

Family	Generator $\varphi(.)$	Laplace Transform, $L_\Lambda(u,\alpha_t)$	Distribution of $\Lambda(\alpha_t)$
Clayton	$u^{-\theta(\alpha_t)}-1$	$(1+u)^{-\frac{1}{\theta(\alpha_t)}}$	Gamma law
Gumbel	$(-\ln u)^{\theta(\alpha_t)}$	$\exp\left(-u^{\frac{1}{\theta(\alpha_t)}}\right)$	Positive stable law
Frank	$\ln\frac{e^{\theta(\alpha_t)u}-1}{e^{(\alpha_t)}-1}$	$\frac{1}{\theta(\alpha_t)}\ln\left(1+e^u(e^{\theta(\alpha_t)}-1)\right)$	Law of logarithmic series distribution on the positive integers

Table 9.1: Generator and related Laplace transforms

By definition (4.2), the multivariate cumulative density function of returns, whose the dependence is modeled by a Archimedian copula, is related to marginal distributions as follows:

$$H(y^1,\,y^2\,|\,\alpha_t) \quad = \quad L_\Lambda\left(L_\Lambda^{-1}(F_1(t,y^1),\alpha_t)+L_\Lambda^{-1}(F_2(t,y^2),\alpha_t)\,,\,\alpha_t\right)$$

Based upon this observation, Marshall & Olkin (1988) have proposed the following algorithm to simulate a random hit of $\left(Y_t^1,Y_t^2\right)$:

1. Generate a random variable $\Lambda(\alpha_t)$ whose the Laplace transform is $L_\Lambda(u,\alpha_t)$.

2. Generate two U_1 , U_2 independent uniform (0,1) random numbers.

3. For $k = 1, 2$ calculate

$$u_k = \exp\left(-\left(-\frac{1}{\theta(\alpha_t)}\ln U_k\right)\right) \qquad k = 1, 2$$

4. Invert the marginal distributions of returns to retrieve a random hit of $\left(Y_t^1, Y_t^2\right)$:

$$y^k = F_1^{-1}(t, u_k) \qquad k = 1, 2$$

For details about the law of logarithmic series distribution, the reader can refer to Jeffrey (1988) for details.

Simulation of a Gaussian or Student copula.

Simulating Gaussian or Student copulas is easy because we can take advantage of the underlying multivariate statistical distributions:

1. Generate a random bivariate normal variable, (U_1, U_2), having a null mean and a correlation matrix $\Sigma(\alpha_{t_i})$.

2. To retrieve a random hit of $\left(Y_t^1, Y_t^2\right)$, invert the marginal distributions of returns :

$$y^k = F_1^{-1}(t, U_k) \qquad k = 1, 2$$

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