Embedded patterns and extended array admittance matrix

Jean Cavillot, Denis Tihon, Ha Bui Van, Christophe Craeye

***Abstract*—Sparse arrays of antennas can sometimes be modelled without accounting for the mutual coupling. In that case, the array radiation pattern can be expressed as the product of two factors: the isolated antenna radiation pattern and the array factor. Using the array impedance matrix, this formalism can be extended to account for the mutual coupling when the array is made of single-mode antennas. In this paper, we propose an extension of that formalism to the case of multi-mode antennas. We show that, contrary to the single-mode case, the array impedance matrix and isolated element patterns are not sufficient to rigorously account for mutual coupling. Building on these concepts, we propose a new quantity, the extended array admittance matrix. Combined with a modal decomposition of the currents on the antennas, it can be used to rigorously account for mutual coupling between multi-mode antennas in an array. The proposed formalism is illustrated by numerical examples.**

***Index Terms*—Mutual coupling, antenna arrays, array impedance matrix, array admittance matrix, extended admittance matrix.**

I. Introduction

Antenna arrays allow a flexible shaping and scanning of the radiation pattern and are hence precious to communication, radar and sensing technologies. For arrays of identical antennas, the array radiation pattern is generally written as the product between the element pattern and the array factor [1]. This model only holds when all antennas have very similar "element" patterns and hence provides a relatively crude model of the beam. Indeed, when a given element is excited, it induces currents on the neighboring elements, which also contribute to the "embedded" pattern of the excited element. Since the different elements do not stand in the same environment (cf. antennas near the edges of a regular array or elements in a non-regular array), the embedded patterns are in essence different. From a reception perspective, on may equivalently talk about the "array manifold", which corresponds to the vector of voltages at each (terminated) antenna port under excitation of a plane wave incident from a given direction. Arrays of wideband elements can exhibit differences of up to 5 dB among their embedded patterns [2]. Dramatic effects can also arise in regular arrays supporting eigenmodes [3].

Yet, the theory based on array factors is quite convenient and the question rapidly arises as to whether it would be possible to correct for the effects of mutual coupling. For instance, for a receiving array, it would be interesting to transform the vector of received voltages, affected by mutual coupling, into a vector that would correspond to those received through embedded patterns equal to the isolated pattern, i.e. voltages not affected by mutual coupling. Ideally, that transformation would be linear, i.e. it could be algebraically represented by a matrix $\mathbf{T}$. Such a transformation would be extremely convenient, since it would allow the use of direction-finding or imaging algorithms relying on the array factor theory. Does such a matrix exist ?

It has been shown by [4] that, under the assumption of single-mode antennas, such a matrix can be readily derived from the array impedance matrix. That approximation is equivalent to saying that the transformation matrix $\mathbf{T}$, derived from the array impedance matrix $\mathbf{Z}$ can transform the embedded element pattern into the isolated element pattern and vice-versa [5], [6]. Unfortunately, matrix $\mathbf{T}$ has been considered in many references without checking the single-mode assumption. An appropriate warning regarding the deficiency of coupling matrices has been launched in [7].

Unfortunately, antennas not being simple circuits, the single-mode assumption is seldom valid, especially for wideband arrays, in which antennas need to occupy a larger volume and hence can support more modes. Hence, a standard answer consists of saying that it will be necessary to resort to full-wave electromagnetic simulation and to more complex processing algorithms able to handle a wide variety of embedded patterns.

Another path may consist of trying to see whether the theory of transformation matrix $\mathbf{T}$ can be extended: is there another (maybe larger) matrix able to transform isolated patterns (or similar quantites, known a priori) into the embedded element patterns ? If such a matrix can be identified, it may for instance guide a minimal set of measurements allowing the inclusion of the effects of coupling in the array response. Such a compact, yet accurate, model is looked for in this paper, starting from elements described by some of the authors in [8], accompanied by examples related to arrays of printed antennas. A parallel will be established with the Macro Basis Functions approach [9] used in numerical methods to reduce the number of unknowns per antenna.

The remainder of this paper is organized as follows. Section II presents the short circuit pattern and its approximation when considering single mode antennas. The proposed extended modal approach is described in Section III. Section IV deals with the admittance matrices resulting from the modal expansion. A numerical example is given in Section V. Finally, conclusions are drawn in Section VI.

II. Short circuit patterns

Single-mode antennas correspond to radiators always supporting the same current distribution, within a multiplying

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factor. Once their port is left opened, such antennas theoretically support no current, since their port current must be zero. In other words, the open-circuit pattern (the pattern of one element excited while other elements are open) is identical to the isolated pattern. That approximation is what allowed the derivation of matrix $\mathbf{T}$ from matrix $\mathbf{Z}$ [4]. In this paper, we use a different formulation, based on the admittance matrix $\mathbf{Y}$ since it will allow a more compact representation of the pattern transformations.

Let us start from a well-known expression for the link between embedded elements patterns $\mathbf{F}^e$ and open-circuit patterns $\mathbf{F}^{o.c.}$ [10], [11], [12]:

$$\mathbf{F}^{o.c.} = (\mathbf{Z} + \mathbf{Z}_L)\mathbf{F}^e \tag{1}$$

where $\mathbf{Z}$ is the array impedance matrix and $\mathbf{Z}_L$ is a diagonal matrix comprising the series impedances of each generator (or the load impedances in the receiving case). $\mathbf{F}^e$ and $\mathbf{F}^{o.c.}$ are matrices with as many rows as ports and as many columns as directions of observation.

That expression can be made specific to an array connected to generators with zero series impedances [13], which yields $\mathbf{F}^{s.c} = \mathbf{Y}\,\mathbf{F}^{o.c.}$, where $\mathbf{Y} = \mathbf{Z}^{-1}$ is the array admittance matrix and $\mathbf{F}^{s.c}$ is the "short-circuit" embedded pattern ($\mathbf{Z}_L = 0$).

Since, according to the single-mode approximation, the open-circuit patterns are approximated as the isolated patterns, the same approximation leads to

$$\mathbf{F}^{s.c.} \simeq \mathbf{Y}\;\mathbf{F}^{is.} \tag{2}$$

Hence the question asked in the introduction is whether that equation can be made more precise by resorting to an extended version of matrix $\mathbf{Y}$ and an extended set of patterns $\mathbf{F}^{is}$. Once the short-circuit patterns are known, it is easy to translate them into embedded element patterns using Equation (1).

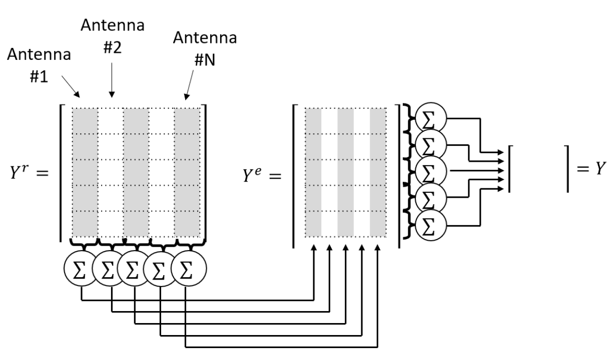

Fig. 1: Partitioning of the $\mathbf{Y}^r$ matrix, and its relation to the $\mathbf{Y}^e$ and $\mathbf{Y}$ matrices.

III. Modal approach

As explained in the introduction, in general, the currents on a given antenna cannot be described by a single mode. That means that several modes will be necessary, i.e. the current distribution on the antenna can be written as a linear combination of a few a priori known current distributions $\bar{J}_j$, whose patterns $\mathbf{F}_j$ are all known and will come in place of $\mathbf{F}^{is.}$ in the new formulation. All patterns are defined here with a common point for the reference of phase. As will be seen below, establishing the link with the array admittance matrix will be greatly simplified by normalizing those current distributions to 1 A at the antenna port[1]. The way in which those modes should be established is outside the scope of this paper, even if some heuristics will be provided below, based on the Macro Basis Functions approach [9]. Things are now in place to establish a system of equations allowing the determination of the coefficients that multiply each mode on each antenna, given a set of voltage generators. In passing, the modes considered on the antennas can be different, even if the antennas are identical.

To find the amplitude associated with the modes when a given port is excited, one may use an approach inspired from the Method of Moments (MoM) [14]: the electric field radiated by each modal current distribution is computed and boundary conditions (here for conducting antennas) are enforced in an average sense, using the modes themselves as testing functions (Galerkin testing).

$$\sum_j \; x_j \iint \bar{E}\{\bar{J}_j\} \cdot \bar{J}_i \, dS = -\iint \bar{E}^{\text{in}} \cdot \bar{J}_i \, dS \tag{3}$$

where $\bar{J}_j$ is the j^{th} modal current distribution defined on the array, x_j is the associated (unknown) coefficient, $\bar{E}\{\bar{J}_j\}$ the electric field it generates, $\bar{J}_i$ acts as weighting (or *testing*) function i, and $\bar{E}^{\text{in}}$ is the electric field due to the excitation. Using a delta-gap model for the port excitation and given the special normalization of the modes, one obtains $\sum_j \; x_j \iint \bar{E}\{\bar{J}_j\} \cdot \bar{J}_i \, dS = V_{\circ,i}$ with $V_{\circ,i}$ the voltage excitation of the port of the antenna on which mode i is defined. In matrix notation, it yields

$$\mathbf{R}\mathbf{x} = \mathbf{b}. \tag{4}$$

Matrix $\mathbf{R}$ is a *reaction* matrix, whose entry (i, j) reads as $\iint \bar{E}\{\bar{J}_j\} \cdot \bar{J}_i \, dS$. For simplicity, rows (i) and columns (j) of $\mathbf{R}$ related to the same antenna are grouped together. Entry i of $\mathbf{b}$ corresponds to $V_{\circ,i}$. It can be noticed that all the entries of $\mathbf{b}$ associated to a given antenna are equal. The concept of reaction has been exploited already long time ago [15], [16] for the calculation of mutual impedances and mutual admittances. When the modal distributions are decomposed into elementary basis functions [17], they can be viewed as Macro Basis Fucntions [9].

IV. Admittance matrices

From Equation (4), three different admittance matrices will be defined below. The first one simply corresponds to $\mathbf{Y}^r = \mathbf{R}^{-1}$ and allows the determination of modal coefficients $\mathbf{x}$. It can be noticed that $\mathbf{Y}^r$ is partitioned in the same way as $\mathbf{R}$: the columns and rows associated to the same antennas are grouped together. The second admittance matrix, which we will denote as $\mathbf{Y}^e$, is a row-compressed version of $\mathbf{Y}^r$; as will be explained below, it corresponds to the "extended admittance matrix" we are looking for. The third one is the array admittance matrix $\mathbf{Y} = \mathbf{Z}^{-1}$ and corresponds to a column-compressed version of $\mathbf{Y}^e$. The matrix compression steps are schematically depicted in Fig. 1.

[1]In case one mode has a vanishing current at the position of the feed, problems related to the normalization can be avoided by considering a new set of modes made of a linear combination of the previous modes.

Suppose that all the ports of the array are connected to perfect voltage sources. The voltages applied to each source are concatenated in the column vector $\mathbf{v}$. We look for the matrix $\mathbf{Y}^e$ that relates the voltages at the ports to the amplitude $\mathbf{x}$ of the modes: $\mathbf{x} \triangleq \mathbf{Y}^e\mathbf{v}$. To do so, we define a matrix $\mathbf{H}$ such that $\mathbf{b} = \mathbf{H}\mathbf{v}$. The number of rows of $\mathbf{H}$ corresponds to the total number of current modes defined on the array. The number of column corresponds to the number of ports. Each row of $\mathbf{H}$ is filled with zeros, except for one entry equal to unity. In row j, the non-zero entry i corresponds to the index of the port where mode j is defined. Starting from Equation (4), we can write $\mathbf{x} = \mathbf{Y}^r\mathbf{b} = \mathbf{Y}^r\mathbf{H}\mathbf{v} \equiv \mathbf{Y}^e\mathbf{v}$, leading to $\mathbf{Y}^e = \mathbf{Y}^r\mathbf{H}$. Given its definition, it can be noticed that the right-hand side multiplication by $\mathbf{H}$ corresponds to a row-compression in which columns associated to the same antenna are summed up, as illustrated in Fig. 1.

Once the coefficients of the modes are known, the array radiation pattern $\mathbf{F}^a$ can be obtained by summing the weighted contribution of each mode. We define $\mathbf{F}^m$ as the matrix whose rows correspond to the radiation pattern of the modes, such that $\mathbf{F}^a = \mathbf{x}^T\mathbf{F}^m$. Expressing $\mathbf{x}$ in terms of voltage excitations yields

$$\mathbf{F}^a = \mathbf{v}^T\mathbf{Y}^{r,T}\mathbf{F}^m \tag{5}$$

with superscript T stands for transposition. The short-circuit pattern corresponds to $\mathbf{F}^a$ when $\mathbf{v}$ contains all zeros, except for a 1 at the position of the element of interest. We hence obtain

$$\mathbf{F}^{\text{s.c.}} = \mathbf{Y}^{e,T}\mathbf{F}^m \tag{6}$$

which is the main result looked for, since it allows modeling the effects of mutual coupling on radiation patterns by linking the (short-circuit) embedded patterns to the patterns associated with each mode.

Finally, we may also compute the total current at port i when a single antenna is excited with a unit voltage. Since the modal currents are normalized to 1 at the level of the ports, the current on a given port simply corresponds to the sum of the amplitudes of the modes associated to that port. Concatenating the port currents into vector $\mathbf{I}$, this can be formulated as $\mathbf{I} = \mathbf{H}^T\mathbf{x}$. Expressing $\mathbf{x}$ in terms of voltage excitation, one obtains

$$\mathbf{I} = \mathbf{H}^T\mathbf{Y}^e\mathbf{v} \equiv \mathbf{Y}\mathbf{v} \tag{7}$$

where the array admittance matrix $\mathbf{Y} = \mathbf{Z}^{-1}$ has been highlighted. Practically, a left-hand side multiplication by $\mathbf{H}$ corresponds to a column compression where rows corresponding to similar antennas are added up, as illustrated in Fig. 1. In passing, that result is consistent with (2) in the single-mode case. Indeed, in that case, $\mathbf{H}$ is the identity matrix, $\mathbf{Y} = \mathbf{Y}^e$ and result (6) is identical to result (2).

V. Example

In this section we provide an example with a small array of printed antennas (depicted in Fig. 2), analyzed with the MoM using 207 RWG elementary basis functions [17] per antenna. The thickness of the grounded slab is 1.667 mm and its relative permittivity is 2.2. The frequency of analysis is 25 GHz. Two cuts in the isolated pattern are given in Fig. 3. Each antenna is connected to a series generator impedance $Z_L = 100\ \Omega$.

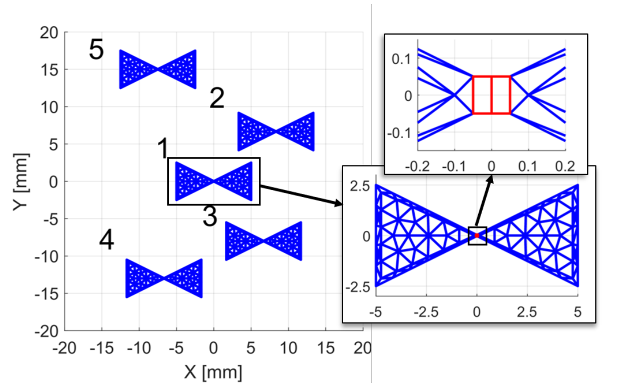

Fig. 2: The antenna array. the geometry of the antennas is detailed in the insets. The (x, y) positions of the antennas are $(0, 0)$, $(8.33, 6.67)$, $(6.67, -8)$, $(-6.67, -13)$ and $(-7.5, 15)$, in mm.

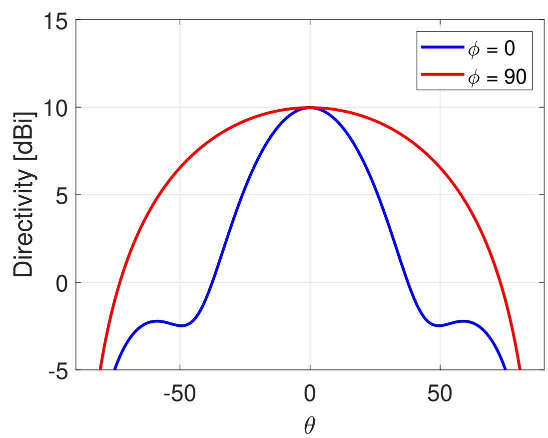

Fig. 3: Isolated element patterns.

The current modes on the different antennas are obtained through a multiple-scattering process [18]. The patterns associated with each mode on each antenna are kept in memory. The extended admittance matrix is computed as explained in Section 4. This allows the computation of the embedded element patterns (EEPs). It can be seen in Fig. 5 that one mode per antenna is not sufficient to obtain accurate embedded patterns. The comparison improves when we consider three modes per antenna. However, the difference is of the order of 1 dB in some directions. 5 modes per antenna are sufficient. The extended admittance matrix $\mathbf{Y}^e$ is provided in Fig. 4(a) considering 3 modes per antenna. From this matrix we can add up the entries corresponding to modes for each antenna over the columns as illustrated in Fig. 1, to obtain an approximate admittance matrix. This approximate matrix is denoted as $\mathbf{Y}'$ and is given along with the exact matrix, which is denoted as $\mathbf{Y}$ in Fig. 4(b). As a consequence of the insufficient number of modes considered, relative errors up to 50% can be observed on some entries. The relative rms error is 5.5×10^{-3}. With 5 modes per antenna, the relative rms error decreases down to 7.07×10^{-5}.

VI. Conclusion

For single-mode antennas, the embedded patterns can be obtained from the isolated patterns and the array admittance matrix. For multi-mode antennas, the former are replaced by the patterns associated with each mode and the latter is

$\mathbf{Y^e} =$

8200+4700i	-130+11i	-43+70i	-68-1000i	-500-130i
-44+91i	-170-1400i	-12-0.21i	8200+5000i	130-25i
-100+100i	36-200i	-440-340i	190+40i	470+85i
-120+190i	130+280i	110-220i	79+13i	-66+17i
120+410i	-77-550i	150+70i	1600+490i	-260-460i
8200+4700i	-130+11i	-43+70i	-68-1000i	-500-130i
-44+91i	-170-1400i	-12-0.21i	8200+5000i	130-25i
-100+100i	36-200i	-440-340i	190+40i	470+85i
-120+190i	130+280i	110-220i	79+13i	-66+17i
120+410i	-77-550i	150+70i	1600+490i	-260-460i
8200+4700i	-130+11i	-43+70i	-68-1000i	-500-130i
-44+91i	-170-1400i	-12-0.21i	8200+5000i	130-25i
-100+100i	36-200i	-440-340i	190+40i	470+85i
-120+190i	130+280i	110-220i	79+13i	-66+17i
120+410i	-77-550i	150+70i	1600+490i	-260-460i

(a)

$\mathbf{Y}' =$

8000+4700i	-220-1300i	-510-440i	130+260i	200-72i
-220-1300i	8200+4900i	550+170i	-110+100i	720+540i
-510-440i	550+170i	8200+4700i	-1300+380i	120+71i
130+260i	-110+100i	-1300+380i	8400+4600i	210+0.94i
200-72i	720+540i	120+71i	210+0.95i	8300+4600i

$\mathbf{Y} =$

8200+4700i	-180-1300i	-430-420i	110+250i	390-5.3i
-180-1300i	8200+4800i	580+100i	-140+110i	670+510i
-430-420i	580+100i	8400+4500i	-1400+390i	96+89i
110+250i	-140+110i	-1400+390i	8500+4600i	180+11i
390-5.3i	670+510i	96+89i	180+11i	8200+4900i

(b)

Fig. 4: (a) $\mathbf{Y}^e$ matrix of the array shown in Fig. 2 with three modes per antenna. (b) Exact matrix $\mathbf{Y}$ and approximate matrix $\mathbf{Y}'$ obtained by compressing the columns $\mathbf{Y}^e$ of as shown in Fig. 1. The entries of the three matrices are expressed in microSiemens.

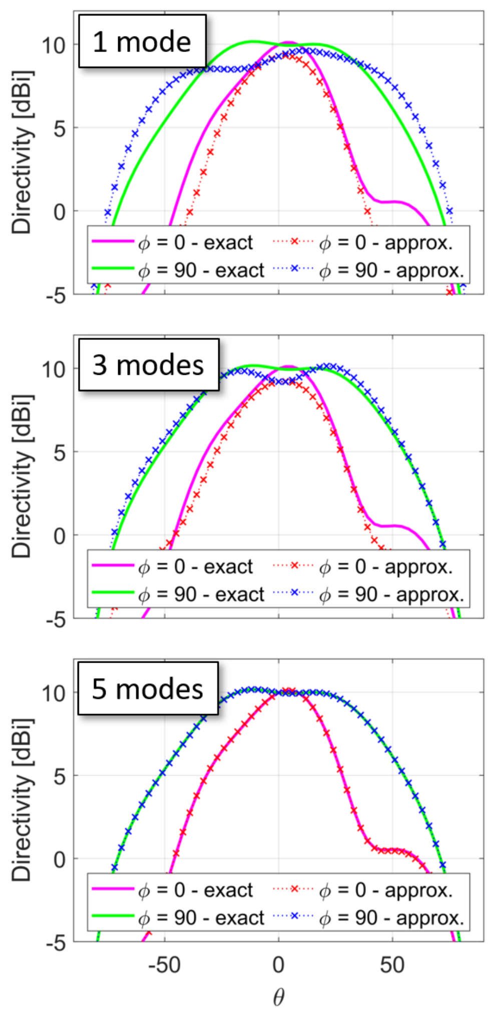

Fig. 5: Embedded pattern of the element at $(x, y) = (0, 0)$ obtained using 1 mode (top), 3 modes (middle) and 5 modes (bottom). The embedded element pattern using the modal decomposition (crosses) is compared with the exact one (continuous lines) in the E-plane ($\phi = 0°$) and H-plane ($\phi = 90°$).

replaced by an extended admittance matrix. The obtained patterns correspond to short-circuit patterns, but they can be translated into patterns with arbitrary terminations using well-established relations. Since, in general, the extended admittance matrix has more rows than columns, it is not truly possible, without any external information regarding the directions of the sources, to determine the voltages that would have been received without mutual coupling from the actually received voltages. In other words, the concept of square mutual coupling matrix often used in the literature as a correction operator is in general not applicable. This has also been observed from a signal processing perspective by Friedlander [7], who proposed to build the array manifold from an antenna manifold and a non-square coupling matrix.

We should stress that it is in principle not necessary to determine the "uncoupled" received voltages. Indeed, the main effect of mutual coupling being a change in the shape of the embedded element patterns, it is useful to remind that powerful algorithms exist for direction finding and imaging [19] which can accommodate different antenna patterns. Finally, the size of the extended mutual admittance matrix informs us about the minimum number of calibration coefficients [20] that are necessary to know the response of the array, assuming that the modal patterns (or corresponding current distributions) are known, either from simulations or measurements. That number corresponds to $M \times N$, where M is the total number of modes defined on all antennas of the array and N is the number of ports. In the single-mode case, the measurement of the array impedance matrix provides the N^2 coefficients required to characterize the array. In the multi-mode case, additional measurements are required to characterize the effects of mutual coupling. Such measurements may also exploit different loading conditions, as done in [21].

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