Is the partial age an increasing function of the size of the associated subdomain?

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Let Ω represent the World Ocean, which is delimited by boundary Γ. The latter is made up of the ocean-atmosphere interface, Γ^s, and impermeable boundaries, Γ^i, i.e. the ocean bottom and "lateral" boundaries. The aforementioned surfaces obviously satisfy the following properties: $\Gamma = \Gamma^s \cup \Gamma^i$ and $\varnothing = \Gamma^s \cap \Gamma^i$.

Let x represent the position vector. At steady state, the partial age associated with subdomain Ω' ($\Omega' \subset \Omega$), which is denoted $a'(\mathbf{x})$, is the solution of the following partial differential problem:

$$\begin{cases} \nabla \bullet (\mathbf{K} \bullet \nabla a' - \mathbf{v}a') = \begin{cases} -1, & \text{if } \mathbf{x} \in \Omega' \\ 0, & \text{if } \mathbf{x} \in \Omega \setminus \Omega' \end{cases} \\ \left[a'(\mathbf{x})\right]_{\mathbf{x}\in\Gamma^s} = 0, \quad \left[(\mathbf{K} \bullet \nabla a' - \mathbf{v}a') \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \end{cases} \tag{1}$$

where **v** and **K** represent the velocity and the diffusivity tensor, respectively; **n** is the outward unit normal vector to the domain boundary. The partial age is non negative (e.g. Mouchet et al. 2016).

To solve problem (1), it is helpful to introduce Green's function $G(\mathbf{x}\,|\,\mathbf{x}_0)$, which is the solution of

$$\begin{cases} \nabla \bullet (\mathbf{K} \bullet \nabla G - \mathbf{v}G) = -\delta(\mathbf{x} - \mathbf{x}_0) \\ \left[G(\mathbf{x}\,|\,\mathbf{x}_0)\right]_{\mathbf{x}\in\Gamma^s} = 0, \quad \left[(\mathbf{K} \bullet \nabla G - \mathbf{v}G) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma^i} = 0 \end{cases} \tag{2}$$

where δ is the (three-dimensional) Dirac impulse function. This implies that the physical dimension of the Green's function of the problem under consideration is $(\text{length})^{-3} \times \text{time}$.

The partial age defined above is readily seen to be

$$a'(\mathbf{x}) = \int_{\Omega'} G(\mathbf{x}\,|\,\mathbf{x}_0)\, d\mathbf{x}_0 \tag{3}$$

This is because expression (3) satisfies the boundary conditions obeyed by the partial age and the following relation

$$\nabla \bullet \left[\mathbf{K} \bullet \nabla \left(\int_{\Omega'} G\, d\mathbf{x}_0 \right) - \mathbf{v} \left(\int_{\Omega'} G\, d\mathbf{x}_0 \right) \right] = -\int_{\Omega'} \delta(\mathbf{x} - \mathbf{x}_0)\, d\mathbf{x}_0 = \begin{cases} -1, & \text{if } \mathbf{x} \in \Omega' \\ 0, & \text{if } \mathbf{x} \in \Omega \setminus \Omega' \end{cases} \tag{4}$$

In view of the mathematical developments of Mouchet et al. (2016), I take it for granted that Green's function $G(\mathbf{x}\,|\,\mathbf{x}_0)$ is non negative. Therefore, increasing the size of the subdomain Ω' will cause the partial age to increase at any point of the domain of interest Ω.

QED.

Mouchet A., F. Cornaton, E. Deleersnijder and E.J.M. Delhez, 2016, Partial ages: diagnosing transport processes by means of multiple clocks, *Ocean Dynamics*, 66, 367-386