Smart abstraction of dynamical systems via a Cantor-Kantorovich distance between Markov chains

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1 Introduction

In recent years, computing metrics between Markov models has been an active research topic in the computer science community, for example for their applications in reinforcement learning related topics. For example, [3] show that transfer learning algorithms can be improved using different distances between MDPs. In parallel, many recent research efforts focus on abstracting dynamical systems within the control & systems community. More precisely, many works started exploring te possibility of generating those abstractions only from observations of systems (see [2] for example).

In [1], we propose a new notion of distance between labelled Markov chains. It consists of a Kantorovich distance between the distributions of infinitely long random walks. The latter are equipped with the Cantor distance, which is classicly used in symbolic dynamics. In addition, we show in [1] that this metric is natural and meaningful to construct non-trivial abstractions of dynamical systems.

2 The Cantor-Kantorovich distance

We consider Labelled Markov Chains (LMCs), noted $\Sigma = (\mathscr{S}, \mathscr{A}, P, \mu, L)$, where $\mathscr{S}$ is the set of states, $\mathscr{A}$ is a finite alphabet, P is a transition matrix on $\mathscr{S} \times \mathscr{S}$, μ is an initial measure on $\mathscr{S}$ and $L : \mathscr{S} \to \mathscr{A}$ is a labelling function. A LMC generates a probability distribution on sequences of labels, noted $p^n(w)$ for some word $w = a_1 \dots, a_n \in \mathscr{A}^n$.

Given two LMCs Σ_1 and Σ_2 that generate p_1^n and p_2^n, the Kantorovich distance between p_1^n and p_2^n is defined as

$$K(p_1^n, p_2^n) = \min_{\pi^n \in \Pi(p_1^n, p_2^n)} \sum_{w_1, w_2 \in \mathscr{A}^n} C(w_1, w_2) \pi^n(w_1, w_2), \quad (1)$$

where $\Pi(p,q)$ is the set of couplings of distributions p and q and C is the Cantor distance between words w_1 and w_2. The latter is defined as $C(w_1, w_2) = 2^{-l}$ where l is the length of the longest common prefix of w_1 and w_2.

The Cantor-Kantorovich distance between LMCs is defined as $d(\Sigma_1, \Sigma_2) = \lim_{n\to\infty} K(p_1^n, p_2^n)$. In [1], we show that this quantity is well-defined and is indeed a distance. Moreover, we can approximate it efficiently with a finite n because $0 \leq d(\Sigma_1, \Sigma_2) - K(p_1^n, p_2^n) \leq 2^{-n}$.

3 Application to abstraction of dynamical system

We study dynamical systems of the form

$$x_{k+1} = F(x_k), \quad y_k = H(x_k), \quad (2)$$

where $x_k \in \mathscr{X}$ is the state, and $y_k \in \mathscr{A}$ is the output in a finite alphabet. We abstract those systems with LMCs, noted $\Sigma_{\mathscr{W}}$, whose states is a set of words $\mathscr{W} = \{w_1, \dots, w_k\}$ of different lengths. Each state $w_i = a_1 \dots a_{n_i} \in \mathscr{A}^{n_i}$ in the LMC represents a set of states in the state-space defined as $[w_i] = \{x \in \mathscr{X} : H(x_{j-1}) = a_j, j = 1, \dots, n_i\}$. Accordingly, the labels are defined as $L(w) = a_1$ (the current output in the system), and the probabilities μ_w and P_{w_1,w_2} are determined in a Monte-Carlo way by sampling trajectories.

The choice of a meaningful $\mathscr{W}$ is crucial. We propose a data-driven procedure to do this with our Cantor-Kantorovich distance. First, we start with $\mathscr{W} = \{a\}_{a\in\mathscr{A}}$, i.e. all the words of length 1, and we construct $\Sigma_{\mathscr{W}}$. Then, for all $w \in \mathscr{W}$, we define $\mathscr{W}_i'$ for all $w_i \in \mathscr{W}$, defined as the set $\mathscr{W}$ where we removed w_i, and we added the words $w_i a$ for all $a \in \mathscr{A}$. For each $\mathscr{W}_i'$, we construct $\Sigma_{\mathscr{W}_i'}$, and we compute $d(\Sigma_{\mathscr{W}}, \Sigma_{\mathscr{W}_i'})$. We update $\mathscr{W}$ with $\mathscr{W}_i'$ such that the distance computed is the largest, and we repeat the procedure.

In [1], we show that this heuristic leads to interesting numerical results on a toy control problem.

References

[1] A. Banse, L. Romao, A. Abate, and R. M. Jungers. Data-driven abstractions via adaptive refinements and a kantorovich metric [extended version], 2023.

[2] R. Coppola, A. Peruffo, and M. Mazo. Data-driven abstractions for verification of linear systems. *IEEE Control Systems Letters*, 7:2737–2742, 2023.

[3] J. Song, Y. Gao, H. Wang, and B. An. Measuring the distance between finite markov decision processes. In *Proceedings of the 2016 International Conference on Autonomous Agents & Multiagent Systems*, AAMAS '16, page 468476, Richland, SC, 2016. International Foundation for Autonomous Agents and Multiagent Systems.

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