STOCHASTIC MODELLIZATION OF HYBRID PUBLIC PENSION PLANS (PAYG) UNDER DEMOGRAPHIC RISKS WITH APPLICATION TO THE BELGIAN CASE

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Stochastic Modellization of Hybrid Public Pension Plans (PAYG) under Demographic Risks with Application to the Belgian Case

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Abstract

Aging is an important challenge for pension schemes, especially for social security plans mainly financed by PAYG (Pay as you go) and based on a DB formula (Defined Benefit). In particular, demographic risks induce important increases of the contributions and threaten the financial sustainability of such schemes. On the other hand, switching to Defined Contribution plans can be a solution in terms of funding but introduce significant risks in terms of social adequacy. The purpose of this paper is to study hybrid solutions between DB and DC in a stochastic environment. In particular, we simulate for various risk sharing strategies, the evolution of contributions and benefits by introducing risk factors influencing the demographic dependence ratio (fertility, longevity, baby boom). We study the mean evolution of these processes as well as their value at risk.

Keywords. PAYG Pensions; Dependency Ratio; Musgrave; Convex Combination; Value at Risk

1. Introduction

In the study of Public Pension design, the most important factors we have to model are the demographic risks. To do this we need to model fertility and the mortality of the country coupled with other major factors like, baby boom, migration and many others so as to mimic the dynamics of the demography under study.

In this area of research there has been a lot of studies trying to model the demography of many countries. Mostly in the the literature we have the works of Meijdam and Verbon (1995), Fanti and Gori (2013), Cipriani (2014), Yang (2015), Buyse and Van De Kerckhove (2017), Rosado Cebrián et al. (2020) mainly using the approach of overlapping generation framework.

Also extensions for Stochastic overlapping generation models can be seen in the works of Maußner (2009) and Ohtaki (2011). However, there is a lack of time continuity component in these models, hence the need for multi period models for modeling the demography and consequently tackling the PAYG system. Recently the work of Peris et al. (2018) has implemented a multi period model for Defined contribution. Also time continuity extensions can be seen in Alonso-García and Devolder (2009).

Another modification is the introduction of the point system for modeling PAYG as seen in Gurtovaya and Nisticò (2018) and Schokkaert et al. (2017).

In terms of the fairness, inter-generational redistribution, the social justice of the public Pension scheme, risk sharing of longevity risk and the sustainability of the PAYG are seen in Queisser and Whitehouse (2006), Tabellini (1991), Morsomme and Devolder (2018), Erik and Van Parijs (2003) and Belloni and Maccheroni (2013b).

Also, there has been the notion of automatic adjustment mechanism looking at how to balance the PAYG equilibrium automatically. Here the works of Knell (2010) , Gannon et al. (2018) and Vidal-Meliá et al. (2009) shows some new development in that area.

In terms of the actuarial fairness and the capturing of the longevity risk, we look at the paper by Vidal-Meliá et al. (2018).

A multi period model has a lot of advantages over the overlapping generation models (OLG). Simply because it treats the different generations in the demography separately in terms of age and evolution in time. Hence it can to some extent capture their individual characteristics and traits that are specific for different generations. This methodology is more realistic and can mimic the real demography better than the overlapping generation models, since they consider the generations together.

Looking at the literature we therefore attempt to propose a stochastic model, that can model the demography of Belgium and that is consistent with macro economic existing projections. Essentially we intend to model the dependency ratio[1] correctly. When this is achieved, we can use it to model the pension scheme and make projections and analysis of all the Pension systems.

Let us briefly describe the methodology. Here we look at a Pension model where there are many generations of active contributors as well as retires. Hence we specify at time 0, the initial number of people in all the different generations in the demography of Belgium. We divide the demography into active generation and retires. In our model, we consider active working age to be 25 years

[1] Dependency ratio is ratio of the number of retires and the number of active contributors in a pension scheme.

and legal retirement age to be 65 years. Also the oldest person in our model is 110 years old. Hence we have 40 generations of active contributors as well as 46 generations of retires. Finally, at the maximum age of 110, people leave the system entirely when they die. Hence we end up with a dynamic and stochastic model, that follows some set of assumptions to mimic the real life demography of Belgium.

Our contribution in this paper is the introduction of a multi period stochastic pension design that inculcates, stochastic fertility, stochastic mortality and baby boom effect as well as its impact on the risk analysis of various pension schemes. . We use mean reverting Vasicek model for the stochastic fertility and Ornstein Uhlenbeck process for the stochastic mortality model. To capture the baby boom effect, we use the Belgian figures. The data used is from the Belgian Bureau du plan from 2019 and consist of the number of active contributors and number of retires separately.

The rest of this paper is structured as follows: In section 2, we discuss the public pension scheme (PAYG) and derive it equilibrium equation. In section 2.1, we discuss the risk sharing mechanisms that exist in PAYG system in terms of financing. Here we discuss the DB plan, DC plan, Musgrave and some convex plans. In section 3, we discuss the proposed multi period stochastic model which incorporates low fertility, low mortality and the baby effect in the demography Belgian. Here we show the results of the active population, retired generation and consequently the propose dependency ratio. Then we compare the results of the dependency ratio to the Belgian projections on the same period of time. Section 3.1, we show some simulations for the proposed model and show some results for DB, DC, Musgrave and the convex combination. In section 4, we look at risk analysis of convex plans using some VaR constraints. Finally we conclude in section 5.

2. General Pay As You Go (PAYG) Framework

The Public Pension Scheme (PAYG), also known as Social Security is the Pension Scheme used by many states. We note that the contributing age, retirement age, the rate of contribution to the system and benefit ratio differ from country to country. These differences and many other factors combined, bring about the different types of Pension plans that exist. The discussion on types of Pension plans is in section (2.1). PAYG runs on the idea of a safety net to individuals in the demography so that they can have a better future after retirement. The mechanism is based on the principle of the state collecting contributions at a stipulated age from individuals to the national account and simultaneously paying people who have retired with the money collected. In a pure PAYG system, the income accrued from contributions is transferred to expense directly without any connections to the financial market. The income of the system is simply, the number of people who qualify to contribute to the system multiplied by the contribution rate and the average wage as:

$$\text{Income} = \text{Number of contributors } \times \text{Contribution Rate} \times \text{Mean Wage} \tag{2.0.1}$$

Also, the expenses of the system are simply the number of retires multiplied by the mean pension:

$$\text{Expenses} = \text{Number of Retires} \times \text{Mean Pension} \tag{2.0.2}$$

The equilibrium of the system is achieved if incomes equals the expenses: Hence we equate equations (2.0.1) and (2.0.2) to get the following:

$$\text{Number of Contributors } \times \text{Contribution Rate} \times \text{Mean Wage} = \text{Number of Retires} \times \text{Mean Pension}$$

$$\text{Contribution Rate} = \frac{\text{No. of Retires}}{\text{No. Contributors}} \times \frac{\text{Mean Pension}}{\text{Mean Wage}}$$

$$\underbrace{\pi(t)}_{\text{Contribution Rate}} = \underbrace{D(t)}_{\text{Dependency Ratio}} \times \underbrace{\delta(t)}_{\text{Benefit Ratio}} \tag{2.0.3}$$

Equation (2.0.3) is the equilibrium of PAYG system. Essentially it means that, the amount contributed to the system, should be enough to pay all the pension liabilities at any time (t) for this equilibrium to hold. Hence ideally the dependency ratio should remain steady. However, for our case study of Belgium and as in many other countries it is not the case. Also, the dependency ratio is a very important factor in any Pension designs. In section (3), we explore some stochastic models for modeling the dynamics of the dependency ratio. To have a fair idea about how the situation is for our Belgian case study, we show the projection for the dependency ratio in the next 53 years (2019-2070 inclusively)[1].

From the graph, we see that the dependency ratio is 36% now using the data from the bureau du plan [2]. However, in the next 30 years the dependency ratio will keep increasing drastically to a

[1](source : Bureau du Plan)

[2]The data collected is the total population of Belgian. We calculate the dependency ratio by using the retired population divided by the active contributing population. Here we use contributing age to be 25 years and retired population to be 65 years. Also the oldest in the population is 110 years old and the duration of data is from 2019-2071. We note that we have used the notion of **demographic dependency ratio** since we did not take into consideration other economic factors like inflation, unemployment etc in the economy.

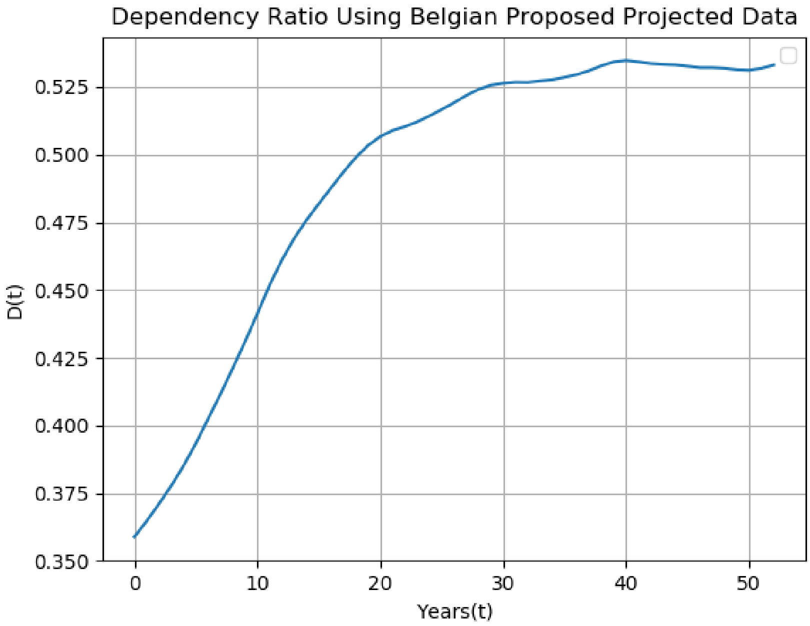


Figure 2.1: Dependency Ratio in Action: (Source of data: Bureau du Plan, 2019-2071)

predicted maximum of 52%. Clearly the state does not have much control over the dynamics of the demography. Hence to achieve the equilibrium of PAYG in equation(2.0.3), with this massive demographic shock we need to adjust the other factors in the PAYG system, that is the contribution rate ($\pi(t)$) and benefit ratio ($\delta(t)$). Essentially the major questions we ask about this imbalance is how to share the risk between all the parties involved? How to be fair to all parties involved? How to sustain the system socially and economically? This brings us to section (2.1) where we study some risk sharing mechanisms that exist to make sure that this equilibrium is balanced in a PAYG setting.

- Remark about Benefit Ratio Vs Replacement rate

Benefit Ratio Vrs Replacement Rate. Benefit Ratio is a macroeconomic indicator given by the ratio between the mean pension and the mean wage.

$$\delta(t) = \frac{\text{Mean Pension}}{\text{Mean Wage}}$$

Whereas, replacement rate is a micro-economic indicator given by the ratio between the first pension of an individual say X and the last wage of the individual X.

$$\gamma_X(t) = \frac{\text{First Pension of an individual X}}{\text{Last Wage of an Individual X}}$$

For convenient in this paper we use benefit ratio ($\delta(t)$) for all the calculations.

2.1 From DB, DC to Hybrid Pension Plans

Here we consider, the PAYG equilibrium in equation (2.0.3). This is one equation with two unknowns, the contribution rate ($\pi(t)$) and the benefit ratio ($\delta(t)$). The number of retires and number of active contributors are known, hence the dependency ratio is known at any time (t). Hence this equation has infinitely many solutions to balance the PAYG equilibrium as shown below. The major questions we ask here are, which solutions are fair? financially sustainable? and socially sound? We explore some of these solutions and discuss some of the consequences to the economy and the different parties involved in the Pension scheme when adopted. Clearly to balance the PAYG equilibrium, we can increase or decrease the contributions and or the benefit ratio. Alternatively, we could maintain the contributions and reduce the benefit or maintain the benefit ratio and increase the contribution rate. More fairly, we could adjust both the contribution rate and the benefit ratio, this strategy is what we call hybrid plans. Furthermore we could increase the legal retirement age, thereby increasing the duration for contributing to the PAYG system. But this will not be considered in this paper.

$$\Uparrow \pi(t) \Downarrow \;=\; \underbrace{D(t)}_{\text{Known}} \times \Uparrow \delta(t) \Downarrow \tag{2.1.1}$$

2.1.1 Defined Benefit (DB) vrs Defined Contribution (DC).

- DB

A pure Defined Benefit (DB) is a pension architecture, where the risk sharing is done in such a way that retires are not affected by the demographic shocks. Here the benefit ratio is kept constant and all the shocks of the system are absorbed by the active contributors to the system. Clearly retires are protected by a DB plan whilst workers suffer the risk. Using the PAYG equilibrium the DB system is represented as follows:

$$\underbrace{\pi(t)}_{\text{Adjusted Contributions}} = \underbrace{D(t)}_{\text{Known}} \times \underbrace{\delta(0)}_{\text{Fixed Benefit Ratio}} \tag{2.1.2}$$

- DC

In a pure Defined contribution (DC) system, the contributions are fixed and the benefit ratio are adjusted to balance the PAYG equilibrium as follows:

$$\underbrace{\pi(0)}_{\text{Fixed Contributions}} = \underbrace{D(t)}_{\text{Known}} \times \underbrace{\delta(t)}_{\text{Adjusted Benefit Ratio}} \tag{2.1.3}$$

Here we see that the risk is borne by retires whilst contributors are not affected by the demographic shock. Clearly, there is no actuarial fairness between generations in a DC plan as well.

The imbalance of DB and DC plans has lead to the birth of hybrid plans that seek to achieve some level of fairness in sharing the demographic shock. The two major hybrid plans we discuss in this paper are Musgrave and the convex combination.

2.1.2 Hybrid Plans: Musgrave and Convex Combination.

- Musgrave Rule

Generally for any time t, the Musgrave rule is based on a constant ratio between the benefit ratio and 1 minus the contribution rate Musgrave (1986).

$$M = \frac{\delta(t)}{(1-\pi(t))} \tag{2.1.4}$$

Here we note that at any given time, the Musgrave constant M will need to adapt simultaneously the contribution rate and the benefit ratio. At $t = 0$, the Musgrave constant, is given as:

$$M = \frac{\delta(0)}{(1-\pi(0))} \tag{2.1.5}$$

We note that at $t = 0$, $\delta(0)$ is known and the initial dependency ratio (D(0)) is known. Hence we can calculate the contribution rate as follows:

$$\pi(0) = D(0)\delta(0) \tag{2.1.6}$$

For example, if we take $\delta(0) = 65\%$, $D(0) = 36\%$ and $\pi(0) = 0.65 * 0.36 = 0.234 = 23.4\%$. Hence at t=0, Musgrave is given as:

$$M = \frac{0.65}{1-0.234} = 0.849$$

In the next section, we will show that the Musgrave rule is just one of the infinitely many possible solutions that exist in the PAYG equilibrium. More interesting, we will also show that the Musgrave framework is closer to DB than DC, hence retires are more protected with this plan than pure DC.

- Convex Combination

The convex combination principle generalizes the Musgrave rule and has been studied for instance in Devolder and de Valeriola (2020). Here we fix a value of the parameter omega and we allow the following expression K_ω to remain constant over time.

$$K_\omega = \omega\delta(t) + (1-\omega)\pi(t), \text{ where } 0 \leq \omega \leq 1 \tag{2.1.7}$$

Where

$$K_\omega = \omega\delta(0) + (1-\omega)\pi(0), \text{ where } 0 \leq \omega \leq 1 \tag{2.1.8}$$

K_ω is a constant and ranges between the contribution rate and the benefit ratio. We use the PAYG equilibrium in equation (2.1.1) and the convex framework in equation (2.1.7) to solve for the dynamic relations for contribution rate and benefit ratio as follows:

First by inserting equation (2.0.3) into equation (2.1.7) to get:

$$\begin{aligned} K_\omega &= \omega\delta(t) + (1-\omega)D(t) \times \delta(t) \\ \delta(t) &= \frac{K_\omega}{\Big(D(t)(1-\omega)+\omega\Big)} \end{aligned} \tag{2.1.9}$$

From equation (2.0.3) we get the following relation for benefit ratio as follows:

$$\delta(t) = \frac{\pi(t)}{D(t)} \tag{2.1.10}$$

Also substituting equation(2.1.10) into equation (2.1.7) to get the following:

$$\begin{aligned} \omega\frac{\pi(t)}{D(t)} + (1-\omega)\pi(t) &= K_\omega \\ \pi(t)\left(\frac{\omega}{D(t)} + (1-\omega)\right) &= K_\omega \\ \pi(t) &= \frac{K_\omega}{\left(\frac{\omega}{D(t)} + (1-\omega)\right)} \\ \pi(t) &= \frac{K_\omega D(t)}{\big(D(t)(1-\omega)+\omega\big)} \end{aligned} \tag{2.1.11}$$

Equations (2.1.9) and (2.1.11) represent the dynamic relations for the contribution rate and the benefit ratio respectively. Here, for $\omega = 0$, $\pi(t) = K_w$, which is purely DC system, because contributions are constant. Whilst, when $\omega = 1$, $\delta(t) = K_w$ is purely DB, since the benefit ratio is a constant. Hence the two extremes, $\omega = 0$ and $\omega = 1$ represent DC and DB plan respectively.

For an example, using the intermediary $\omega = 0.5$, the convex relation is as follows:

$$\begin{aligned} K_\omega &= \omega\delta(t) + (1-\omega)\pi(t) \\ K_\omega &= \frac{1}{2}\Big(\delta(t) + \pi(t)\Big) \end{aligned} \tag{2.1.12}$$

Clearly, the pension system is neither DB or DC but in-between.

- Link between Musgrave and Convex plan

The Musgrave rule is in fact a particular case of convex combination

$$\begin{aligned} M &= \frac{\delta(t)}{(1-\pi(t))} \\ \delta(t) &= M \times (1-\pi(t)) \end{aligned} \tag{2.1.13}$$

The convex combination of the benefit ratio and the contribution rate is given as:

$$\left(\frac{1}{1+M}\right)\delta(t) + \left(1 - \frac{1}{(1+M)}\right)\pi(t) = \left(\frac{\delta(t)}{(1+M)}\right) + \pi(t) - \left(\frac{\pi(t)}{1+M}\right) \tag{2.1.14}$$

We substitute $(\pi(t))$ into equation (2.1.14) to get:

$$\frac{1}{1+M}\Big(\delta(t) - \delta(t)D(t)\Big) + \pi(t)) = \frac{1}{1+M}\Big(\delta(t) - \pi(t) + \pi \times (1+M)\Big) = \frac{1}{1+M}\Big(\delta(t) + \pi(t)M\Big) \tag{2.1.15}$$

We use equation (2.1.13) to get:

$$\frac{1}{1+M}\delta(t) + \frac{M}{1+M}\pi(t) = \frac{1}{1+M}(\delta(t) + M\pi(t)) = \frac{1}{1+M}(M - M\pi(t) + M\pi(t)) = \frac{M}{1+M} \tag{2.1.16}$$

Relation (2.1.16) shows that the Musgrave rule corresponds to a convex plan with omega = 1/(1+M)

Hence we see that in the framework of Musgrave, the ω-convex combination of the contribution and the benefit ratio, remains a constant.

Graphically, we have the following situation:

$DC = \omega = 0$ $\omega = 0.5$ = DC +DB $\omega = \frac{1}{1+M}$ DB=$\omega = 1$

2.1.3 Remark. We note that in the Musgrave framework we can write the equivalence of $\omega = \frac{1}{1+M}$. Using the case of Belgium, $\delta(0) = 65\%, D(0) = 36\%$ and $\pi(0) = 23.4\%$, then $M = 0.849$. Hence $\omega = \frac{1}{1+0.849} =$ 0.540. Here we can see that with the initial conditions adapted, in the Musgrave framework, ω is closer to DB than DC. Generally in Musgrave since $M < 1$ always, ω is always greater than 0.5, so the Musgrave framework is always closer to DB than DC. We note Musgrave is just one of the infinitely many solutions that exist in the convex framework.

We note the following:

- $\omega = 1$: DB , $\delta(t)$ fixed and $\pi(t)$ adapted
- $\omega = 0$: DC , $\pi(t)$ fixed and $\delta(t)$ adapted
- $\omega = 0.5$: DC + DB hybrid scheme
- $\omega < 0.5$: DC + DB hybrid schemes closer to DC than DB
- $\omega > 0.5$: DC + DB hybrid schemes closer to DB than DC

2.1.4 Remark. Here we note that in this work, we have used a constant omega throughout. However, in the event of transitioning from an omega value in an extreme to another over a life span, one can consider the notion of omega as a function of time or even stochastic. For instance in the case of progressive transition from DB to Musgrave.

3. The Multi Period Demographic Model

The purpose of this section is to present a stochastic model of evolution of the Belgian population . We assume an entry age of 25 and a retirement age of 65. Hence we have 40 generations of active contributors $N_{ax}(t)$ as well as 46 generations of retires $N_{rx}(t)$. Here the maximum age for people to die in our model is $x = 110$. The model can be graphically represented as follows :

$N_{a25}(0), t = 0$	$N_{a25}(1), t = 1$	$N_{a25}(2), t = 2$	$\cdots$	$N_{a25}(n), t = n$
$N_{a26}(0)$	$N_{a26}(1)$	$N_{a26}(2)$	$\cdots$	$N_{a26}(n)$
$\vdots$	$\vdots$	$\vdots$		
$N_{a64}(0)$	$N_{a64}(1)$	$N_{a64}(2)$	$\cdots$	$N_{a64}(n)$
$N_{r65}(0)$	$N_{r65}(1)$	$N_{r65}(2)$	$\cdots$	$N_{r65}(3)$
$N_{r66}(0)$	$N_{r66}(1)$	$N_{r66}(2)$	$\cdots$	$N_{r66}(n)$
$\vdots$	$\vdots$	$\vdots$		
$N_{r110}(0)$	$N_{110}(1)$	$N_{r110}(2)$	$\cdots$	$N_{r110}(n)$

- Baby Boom effect

Post world war II, the number of babies born globally increased drastically and Belgium is no exception. This phenomenon created what is known in literature as the baby boom effect. One can not discuss the issue of Pensions modeling without discussing baby boom effect. This is because people who were born post war II, if alive will retire at the same time . Hence it is vital to inculcate baby boom effect to any pension model to capture this phenomenon of many retires at the same time. The practical technique to make sure we capture the real effect of baby boom in a model is by using real time data that has inherent baby boom effect. Here, since Belgian is the case study, we use the Belgian 2019 population data released by the Bureau du Plan on the official website https://www.plan.be/databases/databases.php?lang=en. [1].

[1] For the multi period model the different generation data is considered separately. We use ages(x), $x = 25 - 64$

- Dependence Ratio

In the multi-period model, we have the different contributors $N_{ax}(t)$ and retires $N_{rx}(t)$ for each generation. The corresponding dependency ratio is the ratio of all $N_r(t)$ and $N_a(t)$ in totality. The resulting time dependent dependency ratio is as follows:

$$D(t) = \frac{N_r(t)}{N_a(t)} = \frac{\sum_{x=65}^{110} N_{rx}(t)}{\sum_{x=25}^{64} N_{ax}(t)} \tag{3.0.1}$$

3.0.1 Model with Baby Boom, Stochastic Fertility and Stochastic Mortality. The aim of this section is to inculcate, baby boom, low fertility effect and mortality effect more precisely in the Public Pension Model. The baby boom effect has been taken into account as detailed before. . Here we will concentrate on the stochastic fertility and the stochastic mortality model. In Europe, the fertility rate is dropping whilst the mortality rate is also decreasing coupled with the inherent baby boom effect from the population. We consider the Vasicek model for fertility rate $\rho(t)$ and Ornstein Uhlenbeck process without mean reverting for the mortality process $\mu_x(t)$.

We start with the baby boom process as follows:

- The Model

We model the number of new entrances in the population as follows:

$$N_{a25}(t) = N_{a25}(t-1) \times e^{\int_{t-1}^{t} \rho(s)ds} \approx N_{a25}(t-1) \times e^{\rho(t)}, \text{(Entrance at } x = 25) \tag{3.0.2}$$

Where:

$$d\rho(s) = a_f(b_f - \rho(s))ds + \sigma_f dW_f \tag{3.0.3}$$

The discretized solution for this fertility rate is given by :

$$\rho(t) = e^{(-a_f)} \times \rho(t-1) + b_f \times (1 - e^{-a_f}) + \sigma_f \times \sqrt{\frac{(1 - e^{(-2\times a_f)})}{2\times a_f}} \times \mathcal{N}_f(0,1) \tag{3.0.4}$$

Here σ_f is the volatility, b_f asymptotic rate of increase and a_f the spread of the fertility rate $\rho(t)$. With the above modification, all the entrance for the matrix of $Na(t)$ will be adapted to have a stochastic fertility rate. We note also, the fertility rate will only affect the entrance elements of the equation of active people. We note the motivation for using the Vasicek model with drift is to model the decrease of the fertility rate in the Belgian data set used.

Once the entrance is modified with a stochastic fertility rate, we are now interested in the evolution of the active contributors. Mathematical, we are interested in a growth rate that will inculcate the idea of a mortality effect. We use the link between Survival probability (**P**) and mortality rate $\mu_x(t)$ to captured this effect accurately. Since we adopt the Belgian Population data, which is aging with low mortality rate we need to use a stochastic process that can model low mortality. Here we adapt the Ornstein Uhlenbeck process to capture these effects.

In what follows we show the calculation of survival rate $I(x-1, t-1)$ in our model using the Ornstein Uhlenbeck process to model the mortality rate $\mu_x(t)$:

Here we note that the entrance matrix at age 25 is modified first with the above model(Vasicek with drift) to capture the low fertility effect, so the modification for mortality starts at age $x = 26$. Generally this

as the active generation and $x = 65 - 110$ as the retired generation. The first column of the multi period model at $t = 0$ is the initial data as shown in 3. Here we show that people who are 64 years from the active generation are automatically graduated to the retired generation the following year modification is done as follows:

$$N_{ax}(t)_{x>25} = N_{a(x-1)}(t-1)I(x-1,t-1) \tag{3.0.5}$$

where:

$$I(x-1,t-1) = e^{\left(-\int_{t-1}^{t}\mu_{x-1}(s)ds\right)} \approx e^{-\mu_{x-1}(t-1)} \tag{3.0.6}$$

For the mortality rate we use the following model :

$$d\mu_x(s) = a_x\mu_x(s)ds + \sigma_x dW_x(s) \tag{3.0.7}$$

The solution being given by :

$$\mu_x(t) = \mu_{x-1}(0)e^{a_x t} + \sigma_x \int_0^t e^{a_x(t-s)}dW(s) \tag{3.0.8}$$

The discretized solution is :

$$\mu_x(t) = \mu_{x-1}(t-1)e^{a_x} + \sigma_x \times \sqrt{\frac{(1-e^{(-2\times a_x)})}{2\times a_x}} \times \mathcal{N}_x(0,1)) \tag{3.0.9}$$

Hence from equation (3.0.6), $I(x-1,t-1)$ is the survival process in the multi period Pension model. This rate is different for each specific generation and age.

In what follows we show an example of how to calculate $I(x-1,t-1)$. We do this by calculating $I(25,0)$ as follows:

$$\mu_{x-1}(t-1) = \mu_{x-1}(0) = \mu_{25}(0) = -\ln\left(\frac{l_{26}}{l_{25}}\right) = -\ln\left(\frac{986.643}{987.273}\right) = 0.0006383246 \tag{3.0.10}$$

consequently the approximation of the stochastic survival process is given by:

$$I(25,0) \approx e^{\left(-\int_{t-1}^{t}\mu_{x-1}(s)ds\right)} = e^{(-\mu_{25}(0))} = e^{-(0.0006383246)} = 0.999361879 \tag{3.0.11}$$

In totality, for $x = 26-64$ inclusively, the evolution for active contributors for $t = 1$ with stochastic survival process is as follows:

$$\begin{aligned}
N_{a26}(1) &= N_{a25}(0) \times I(25,0) \\
N_{a27}(1) &= N_{a26}(0) \times I(26,0) \\
N_{a28}(1) &= N_{a27}(0) \times I(27,0) \\
N_{a29}(1) &= N_{a28}(0) \times I(28,0) \\
&\vdots \\
N_{a64}(1) &= N_{a63}(0) \times I(63,0)
\end{aligned} \tag{3.0.12}$$

Also at the same time $t = 1$ for $x = 65-110$ inclusively, the evolution of the retires with a stochastic mortality rate is:

$$\begin{aligned}
N_{r65}(1) &= N_{a64}(0) \times I(64,0) \\
N_{r66}(1) &= N_{r65}(0) \times I(65,0) \\
N_{r67}(1) &= N_{r66}(0) \times I(66,0) \\
N_{r68}(1) &= N_{r67}(0) \times I(67,0) \\
&\vdots \\
N_{r110}(1) &= N_{r109}(0) \times I(109,0)
\end{aligned} \tag{3.0.13}$$

With the above construction, the resulting matrix for the active contributors $N_a(t)$ is given by the following:

$$N_a(t) = \begin{pmatrix} Age, Time & t=0 & t=1 & t=2 & \cdots \\ x=25 & N_{a25}(0) & N_{a25}(1) = N_{a25}(0)e^{\rho(1)} & N_{a25}(2) = N_{a25}(1)e^{\rho(2)} \cdots & \\ x=26 & N_{a26}(0) & N_{a26}(1) = N_{a25}(0) \times I(25,0) & N_{a26}(2) = N_{a25}(1) \times I(25,1) & \cdots \\ x=27 & N_{a26}(0) & N_{a27}(1) = N_{a26}(0) \times I(26,0) & N_{a27}(2) = N_{a26}(1) \times I(26,1) & \cdots \\ x=28 & N_{a27}(0) & N_{a28}(1) = N_{a27}(0) \times I(27,0) & N_{a28}(2) = N_{a27}(1) \times I(27,1) & \cdots \\ x=29 & N_{a28}(0) & N_{a29}(1) = N_{a28}(0) \times I(28,0) & N_{a29}(2) = N_{a28}(1) \times I(28,1) & \cdots \\ x=30 & N_{a29}(0) & N_{a30}(1) = N_{a29}(0) \times I(29,0) & N_{a30}(2) = N_{a29}(1) \times I(29,1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ x=64 & N_{a64}(0) & N_{a64}(1) = N_{a63}(0) \times I(63,0) & N_{a64}(2) = N_{a63}(1) \times I(63,1) & \cdots \end{pmatrix} \tag{3.0.14}$$

We note following from equation (3.0.14) above:

- The first row shows the different generations (Time (t)).
- The second row shows the active contributors with fertility effect, entrance of the model.
- The first column shows the age of the different generations.
- Here the second column shows the real Belgian 2019 population projection data and has the baby boom effect.
- The diagonals of equation (3.0.14) shows the evolution of the different cohorts.

Also, using the same stochastic mortality model, the matrix for retires $N_r(t)$ become:

$$N_r(t) = \begin{pmatrix} Age, Time & t=0 & t=1 & t=2 & \cdots \\ x=65 & N_{r65}(0) & N_{r65}(1) = N_{a64}(0) \times I(64,0) & N_{r65}(2) = N_{a64}(1) \times I(64,1) & \cdots \\ x=66 & N_{r66}(0) & N_{r66}(1) = N_{r65}(0) \times I(65,0) & N_{r66}(2) = N_{r65}(1) \times I(65,1) & \cdots \\ x=67 & N_{r67}(0) & N_{r67}(1) = N_{r66}(0) \times I(66,0) & N_{r67}(2) = N_{r66}(1) \times I(66,1) & \cdots \\ x=68 & N_{r68}(0) & N_{r68}(1) = N_{r67}(0) \times I(67,0) & N_{r68}(2) = N_{r67}(1) \times I(67,1) & \cdots \\ x=69 & N_{r69}(0) & N_{r69}(1) = N_{r68}(0) \times I(68,0) & N_{r69}(2) = N_{r68}(1) \times I(68,1) & \cdots \\ x=70 & N_{r70}(0) & N_{r70}(1) = N_{r69}(0) \times I(69,0) & N_{r70}(2) = N_{r69}(1) \times I(69,1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ x=110 & N_{r110}(0) & N_{r110}(1) = N_{r109}(0) \times I(109,0) & N_{r110}(2) = N_{r109}(1) \times I(109,1) & \cdots \end{pmatrix} \tag{3.0.15}$$

3.0.2 Demography Parameters. We will use the following parameters:

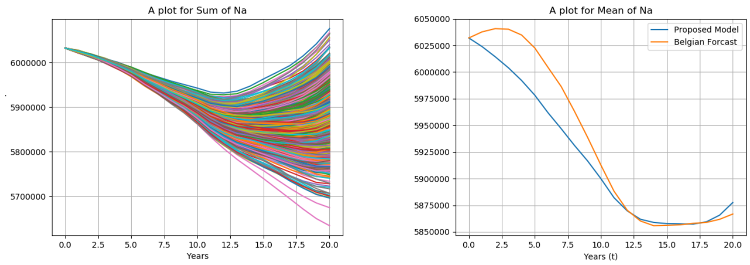
- $D(0) = 36.0\%$ (The Initial Dependency Ratio of the Belgian 2019 population ,Projection data, active generation x = 25-64, retired, x = 65-110)
- $\rho(0) = 0.8\%$ (Initial Increase of Active people)
- $a_f = 0.005$ (mean return force of fertility)
- $b_f = -0.012$ (Asymptotic Rate Increase for Stochastic Fertility Model)
- $\sigma_f = 0.001$ (Volatility for the Stochastic Fertility Model)
- $T = 20$ years (Forecast for 20 years)
- $N = 1000$ (We do 1000 Monte Carlo simulations)
- $a_x = 0.08$ (drift of mortality)
- $\sigma_x = 0.005$ (Volatility for Stochastic Mortality Model)

Here we assume independence between the fertility process and mortality process.

In the next sections , we show the evolutions of how the active contributors evolve with the proposed stochastic model discussed. We expect the number of active contributors to reduce with time. We also show the evolution of retired generation $N_r(t)$ and the sum of the retired and active generation $N_a(t) + N_r(t)$. This sum is expected to somewhat remain stable as predicted by the Belgian projection. The evolution of the proposed model for the demography of Belgium is compared with the existing Belgian projection.

- Evolution of Active Generation: $N_a(t)$

Simulation of evolution of the number of active people and their mean compared to the real Belgian projections.

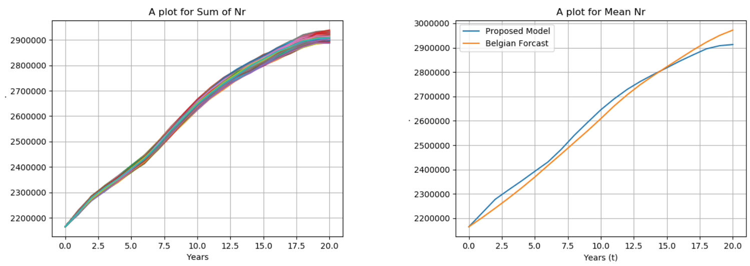


(a) Sum of $N_a(t)$ (b) Mean of Sum $N_a(t)$

Figure 3.1: Na(t), Sum of Active contributors and the Mean

- Evolution of Retired Generation :$N_r(t)$

Simulation of evolution of the number of retires and their mean compared to the real Belgian projections.



(a) Sum of $N_r(t)$ (b) Mean of sum $N_r(t)$

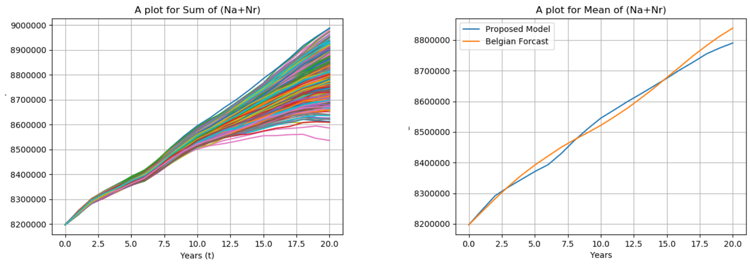
Figure 3.2: $N_r(t)$, Sum of Retires and the Mean

- Evolution of $N_a(t) + N_r(t)$

Simulation of evolution of the sum number of active people and retires and their mean compared to the real Belgian projections.

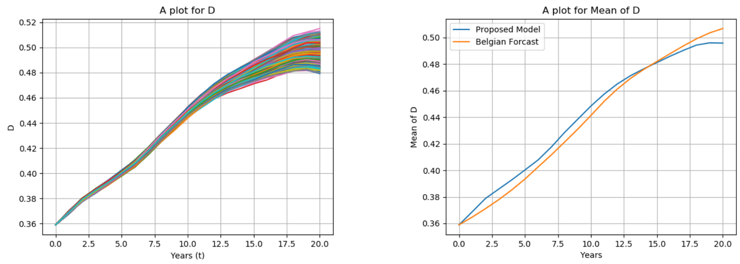
- Dependence Ratio for Proposed Model Versus Belgian Forecast

After simulating N_a and N_r, we simulate the dependence ratio for the propose model presented in section (3). We also compare this model to the Belgian projection for the dependency ratio as follows:



(a) Sum of $N_a(t) + N_r(t)$ (b) Mean Sum of $N_a(t) + N_r(t)$

Figure 3.3: $N_a(t) + N(r)$, Sum of Active Contributors and Retires and their Mean



(a) Dependence Ratio (D) (b) Mean of Dependency Ratio

Figure 3.4: Dependency Ratio ($D(t)$)

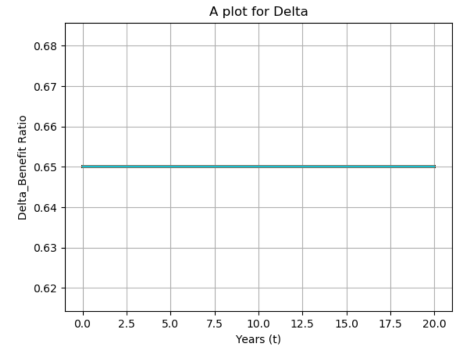
3.1 Pension Simulations And Interpretation

In this section we show the simulation of various pension schemes under the proposed model with stochastic fertility, stochastic mortality and baby boom effect as discussed in section (3). Here we present the model results for all the pension designs discussed in section (2.1) using a time horizon of 20 years.

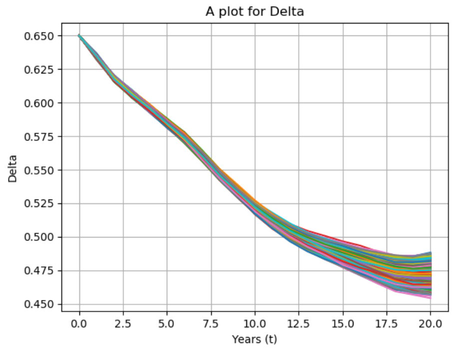
3.1.1 Pension Parameters. We will compare five different pension schemes applied to the Belgian situation :

- Musgrave(M) : $M = (\delta(0))/(1 - \pi[0]) = 0.849$
- $\omega_{(DB)} = 1$ (Pure DB plan)
- $\omega_{(DC)} = 0$ (Pure DC plan)
- $\omega_{(Convex)} = 0.7$ (a convex plan closer to DB than DC)
- $\omega_{(Convex)} = 0.2$ (a convex plan closer to DC than DB)

3.1.2 Simulation Results for the Trajectories of Benefit Ratio. Here we show the trajectories of benefit ratio for all the Pension plans considered:

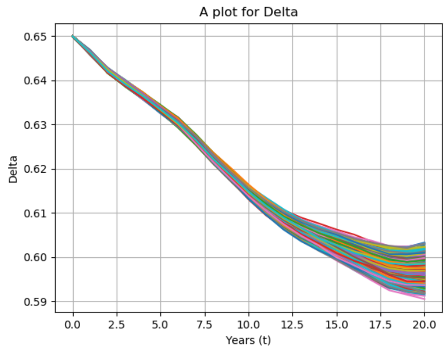


(a) δ in DB

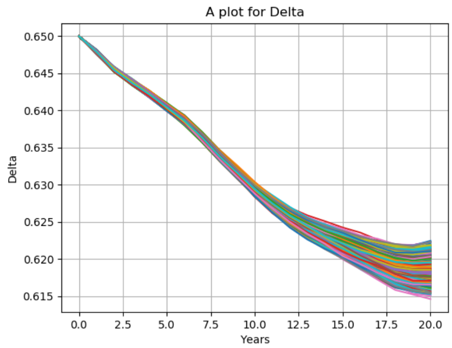


(b) δ in DC

Figure 3.5: Benefit Ratio (δ), DB and DC



(a) δ in Musgrave



(b) δ, omega=0.7

Figure 3.6: Benefit Ratio (δ), Musgarve and convex

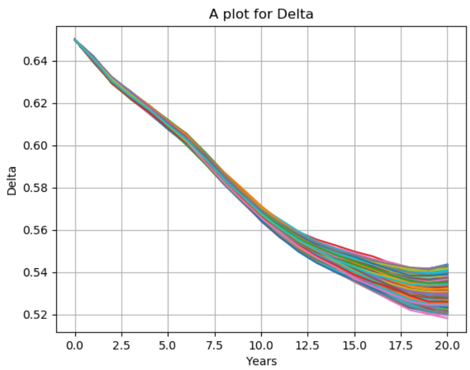
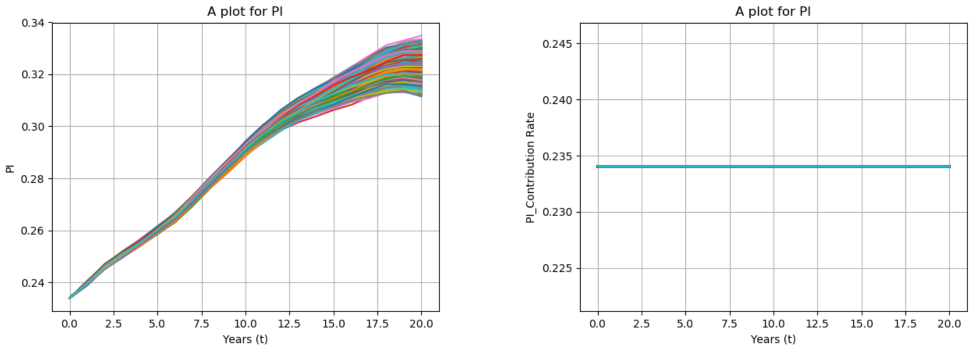


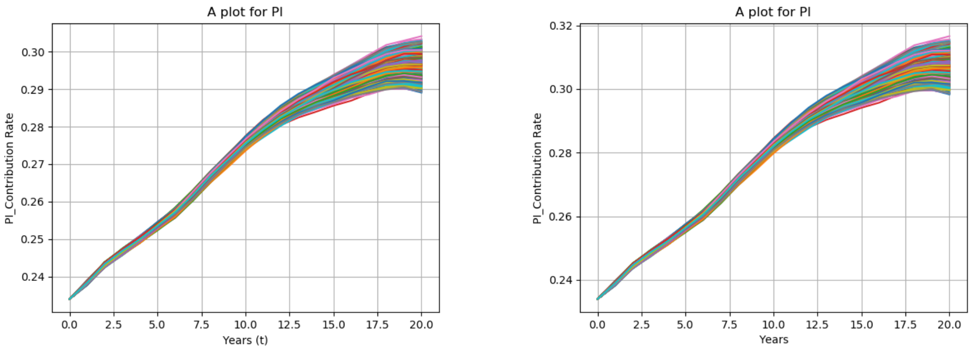
Figure 3.7: δ, omega = 0.2

3.1.3 Simulation Results for the Trajectories of contribution Rate. Here we show the trajectories of contribution rate for all the Pension plans considered:



(a) π in DB (b) π in DC

Figure 3.8: Contribution Rate (π), DB and DC



(a) π in Musgrave (b) π, omega=0.7

Figure 3.9: Contribution Rate (π), omega = 0.7

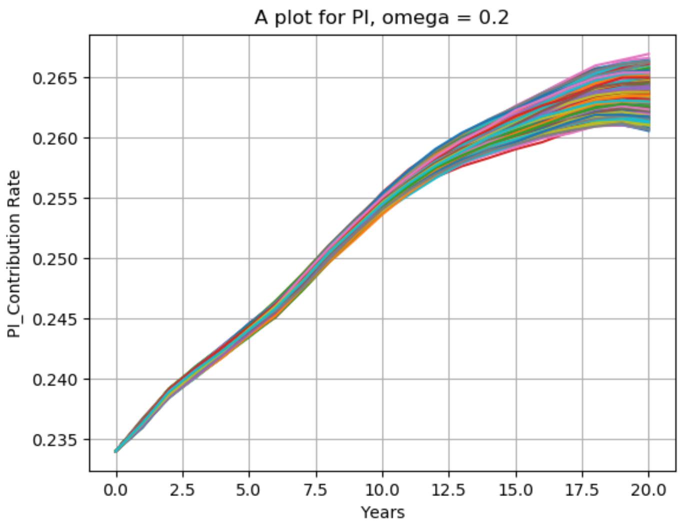


Figure 3.10: Contribution Rate, omega = 0.2

3.1.4 Simulation Results for the Means of Contribution Rate and Benefit Ratio. Here we show the means of the benefit ratio and the contribution rate for all the pension schemes discussed.

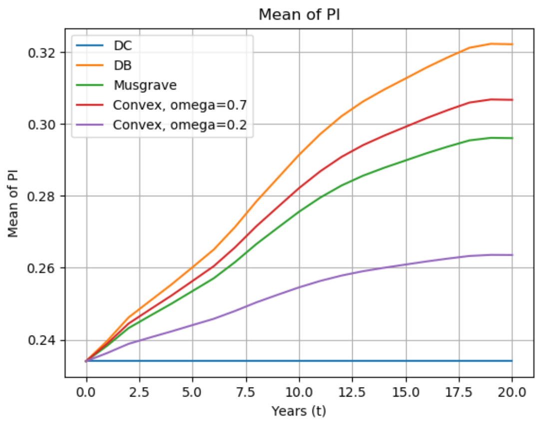


Figure 3.11: Mean Contribution Rate for all Pension Plans Discussed

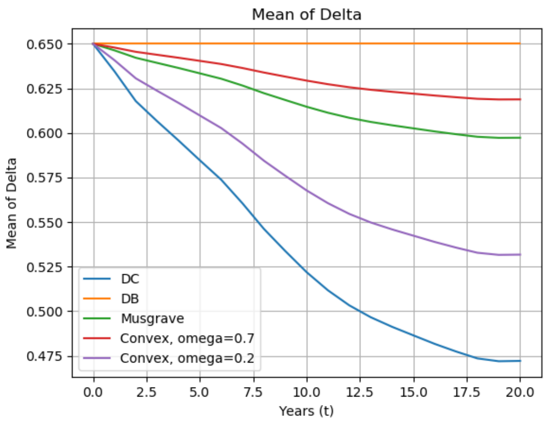


Figure 3.12: Mean Benefit Ratio for all Pension Plans Discussed

3.1.5 Multi Period Pension System in DB . In the DB case, we see from figure (3.11) that the risk sharing is done for just the contribution rate. In the next 20 years, the contribution rate will increase by 37%.

3.1.6 Multi Period Pension System in DC. In a DC plan, we can see that the risk sharing is done on only the benefit ratio whilst the contribution rate remains a constant. Here we see that the benefit ratio decreases by 27% in the next 20 years if a DC plan is adopted.

3.1.7 Multi Period Pension System in Musgrave : M = 0.849, $\omega = 0.544$. In the next 20 years, the retires will have a decrement of 8% in the benefit ratio whilst the contributors will have an increase of 23%. Here the Musgrave framework is biased toward the contributors than the retires. Hence the Musgrave framework is closer to DB than DC.

3.1.8 Multi Period Pension in Convex Case : $\omega = 0.7$, $K_\omega = 0.525$. In the convex combination framework, we strive to be between DB and DC depending on the chosen value of ω. For this example we choose $\omega = 0.7$, to depict a hybrid Pension plan, where we are closer to DB than DC but not entirely a DB plan. Intuitively, this Pension plan will protect the retires much better in terms of the risk sharing than the contributors. On a 20 year horizon, retires will lose 5%. However, the rate of contribution will increase by 31%.

3.1.9 Multi Period Pension in Convex Case : $\omega = 0.2$, $K_\omega = 0.3172$. Here we choose $\omega = 0.2$, to depict a hybrid Pension plan, where we are closer to DC than DB but not entirely a pure DC plan. Intuitively, this Pension plan will protect the contributors much better in terms of financing the scheme than the retires, the rate of contribution wont be a constant like in a pure DC but wont increase much over the duration of 20 years considered. However, retires will see a drastic decrease for the PAYG equilibrium to balance. On a 20 year horizon, retires will lose 17% of their benefit ratio but the rate of contribution will increase by 12%.

4. Risk Analysis

Introducing a risk sharing between the contributors and the retires, as proposed in the convex pension plans of section 2.1, generate risk for both parties. The choice of an adequate level of risk sharing is therefore crucial (choice of the parameter omega). In this stochastic context, it seems relevant to look not only at mean evolutions as in the previous section 3 but also at risk measurement of the two main processes: the contribution rates and the benefit ratio. More precisely, we want to analyse and monitor the degree of risk introduced by convex pension plans and compare the exposition to risks for the two stakeholders (retires and contributors). In order to perform this risk management perspective, we will base our measure on value at risks. We carry out three analyses:

• Analysis 1 : univariate analysis of the value at risk of the benefit ratio and of the value at risk of the contribution rate , depending on the level of risk sharing chosen in the scheme (value of the parameter omega);

• Analysis 2 : maximisation of the expectation of the benefit ratio under a value at risk constraint on the contribution rate (how to get good pension benefits without generating too much risk on the contribution side) ; minimisation of the expectation of the contribution rate under a value at risk constraint on the benefit ratio (how to obtain acceptable costs without generating too much risk on the benefit side)

• Analysis 3: bivariate risk analysis by introducing simultaneously value at risk constraints on the contribution process and on the benefit ratio process (how to get a limitation of the risk as well for the contributors as for the retires).

In this section, the following proposition will be handy:

4.0.1 Proposition. *$\delta(t)$ and $\pi(t)$ are increasing function of the parameter ω if $D(t) > D(0)$. The proof is show in appendix (A.0.1).*

4.0.2 Analysis 1: Univariate Condition on Value at Risk.

4.0.3 1a: Univariate Condition on the Benefit Ratio. : We accept the omega automatic adjustment but we restrict the range of possible values of omega to the cases where we are sure with a great probability that the benefit ratio will not fall under a multiple of the initial benefit ratio (for instance 90%). We try to find the interval of values of omega satisfying the VaR constraint:

$$P(\delta(T) \geq (1 - \lambda_1)\delta(0)) \geq 1 - \alpha$$

Or equivalently:

$$P(\delta(T) < (1 - \lambda_1)\delta(0)) < \alpha$$

where:

α safety level (for instance 5%);

λ_1 tolerance level (for instance if 10% : we accept a maximal decrease of 10% of the initial δ) for $0 \leq \lambda_1 \leq 1$.

Taken into account Proposition (4.0.1) the accepted range for omega will be of the form: $[\omega_1(\lambda_1), 1)]$.

Objective: Try to find the form of this function $\omega_1(\lambda_1)$ depending on λ_1 (assuming the safety level α is fixed).

4.0.4 1b: Univariate Condition on the Contribution Rate. : we accept the omega automatic adjustment but we restrict the range of possible values of omega to the cases where we are sure with a great probability that the contribution rate will not increase beyond a multiple of the initial contribution rate (for instance 115%). We try to find the interval of values of omega satisfying the VaR constraint

$$P(\pi(T) < (1+\lambda_2)\pi(0)) \geq 1-\alpha$$

Where:

λ_2 tolerance level (for instance if 10% : we accept a maximal increase of 10% of the initial π) for $(0 \leq \lambda_2 \leq 1)$.

Taking into account proposition (4.0.1) the result of this first analysis will be an interval of the form: $[0, \omega_2(\lambda_2)]$.

Objective: Try to find the form of this function $\omega_1(\lambda_2)$ depending on λ_2 (assuming the safety level α fixed).

4.0.5 Analysis 2: Univariate Condition on Value at Risk with Optimisation Sustainability-Adequacy.

4.0.6 2a: Maximisation of the Benefit Ratio Under a VaR Condition on the Contribution Rate. : We try to find the best omega automatic adjustment such as to have a maximum expected benefit ratio (adequacy aspect) but we restrict to the cases where we are sure with a great probability that the contribution rate will not increase beyond a multiple of the initial contribution rate (for instance 115%) (Sustainability aspect).

The constraint on the value at risk on the contribution rate already shows us that the optimal is in the interval $[0, \omega_2(\lambda_2)]$(See analysis 1b) . Therefore taking into account proposition (4.0.1) the optimal solution will be the upper bound of this interval: $\omega^* = \omega_2(\lambda_2)$.

4.0.7 2b: Minimization of the Contribution Rate Under a VaR Condition on the Benefit Ratio. : We try to find the best omega automatic adjustment such as to have the lowest expected contribution rate (sustainability aspect) but we restrict to the cases where we are sure with a great probability that the benefit ratio will not fall under a multiple of the initial benefit ratio (for instance 90%) (Adequacy aspect).

The constraint on the value at risk on the benefit ratio already shows us that the optimal is in the interval $[\omega_1(\lambda_1), 1]$ see analysis 1a) . Therefore, taking into account proposition. (4.0.1), then the optimal solution will be the lower bound of this interval: $\omega^* = \omega_1(\lambda_1)$.

4.0.8 Analysis 3: Bivariate Condition on Value at Risk.

4.0.9 VaR conditions on the benefit ratio and on the contribution rate. : We accept the omega automatic adjustment but we restrict to the cases where we are sure with a great probability that the contribution rate will not increase beyond a multiple of the initial contribution rate (for instance 115%) and that simultaneously the benefit ratio will not fall under a multiple of the initial benefit ratio (for instance 90%).

Then from analysis 1a and 1b, we can say that we have the two following constraints in order to satisfy the double VaR goal:

- Condition on the contribution rate : $\omega \leq \omega_2(\lambda_2)$

- Condition on the benefit ratio : $\omega \geq \omega_1(\lambda_1)$

Therefore, we have two cases:

- If $\omega_2(\lambda_2) < \omega_1(\lambda_1)$: there is no solution

- If $\omega_1(\lambda_1) < \omega_2(\lambda_2)$ the solution is in the interval $[\omega_1(\lambda_1), \omega_2(\lambda_2)]$

Interesting to represent on the same graph the two functions $\omega_1(\lambda_1)$ and $\omega_2(\lambda_2)$ and look when graphically the solution is non-empty (case 2).

4.0.10 Simulation of VaR with different λ values, $\alpha = 1\%$. Here we consider λ to be $25\%, 23\%, 22\%, 20\%, 18\%, 17\%, 16\%, 15\%, 14\%, 13\%$. Hence we expect for a fixed $\alpha = 1\%$, we expect that the the following ω values, the corresponding $\delta(t)$ will be greater than $1-\lambda$) as shown in the table below:

Table 4.1: VaR in Convex Plan with Different ω and λ for $\delta(t)$, T = 20 years

ω	0, $\lambda = 25\%$	0.1, $\lambda = 23\%$	0.2, $\lambda = 22\%$	0.3, $\lambda = 20\%$	0.4, $\lambda = 18\%$	0.5, $\lambda = 17\%$
$\delta_t \geq (1-\lambda)$	48.75	50.05	50.7	52.0	53.3	53.95

Table 4.2: VaR in Convex Plan with Different ω and λ for $\delta(t)$, T = 20 years

ω	0.6, $\lambda = 16\%$	0.7, $\lambda = 15\%$	0.8, $\lambda = 14\%$	0.9, $\lambda = 13\%$	1, $\lambda = 0\%$
$\delta_t \geq (1-\lambda)$	54.6	55.25	55.9	56.55	65.0

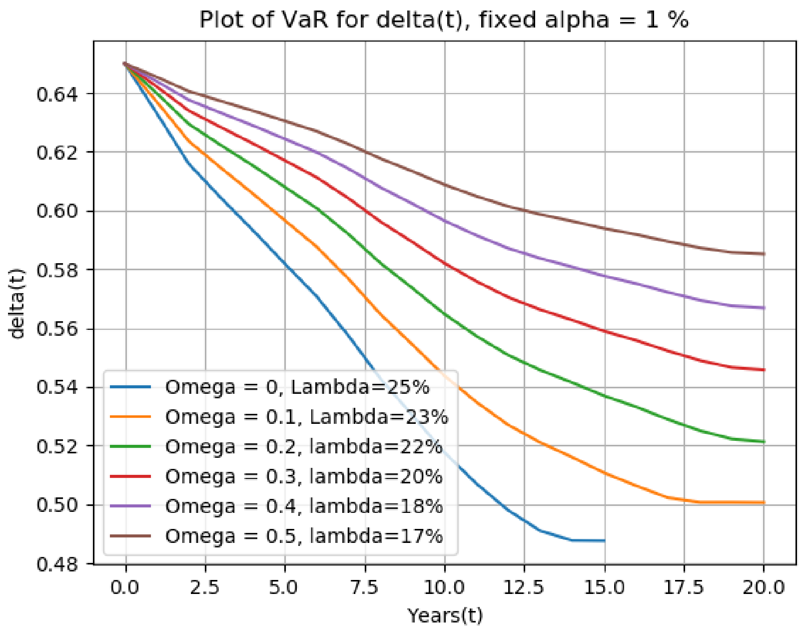


Figure 4.1: Plot of VaR for $\delta(t)$, different λ and $\alpha = 1\%$ fixed

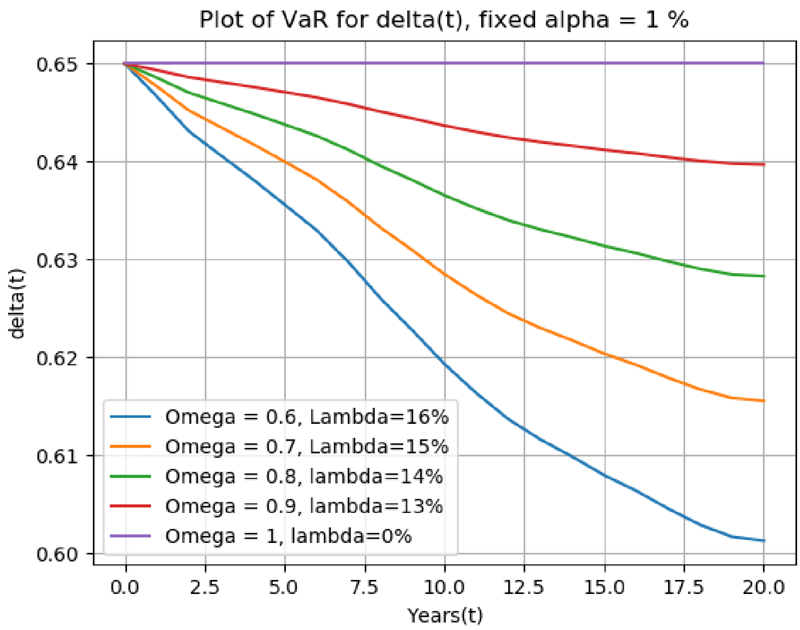


Figure 4.2: Plot of VaR for $\delta(t)$, different λ and $\alpha = 1\%$ fixed

4.0.11 VaR π. Here we consider examples chosen λ to be 35%, 33%, 32%, 30%, 28%, 27%, 25%, 23%, 22%, 21%, 20%. Hence for a fixed $\alpha = 1\%$, we expect the VaR for $\pi(t)$ with different ω values, will be greater than $1+\lambda*(\pi(0))$ as in 1b.

Table 4.3: VaR in Convex Plan with Different ω and λ for $\pi(t)$, T = 20 years

ω	1, $\lambda = 35\%$	0.9, $\lambda = 33\%$	0.8, $\lambda = 32\%$	0.7, $\lambda = 30\%$	0.6, $\lambda = 28\%$	0.5, $\lambda = 27\%$
$\pi(t) < (1+\lambda)$	31.59	31.122	30.888	30.42	29.952	29.718

Table 4.4: VaR in Convex Plan with Different ω and λ for $\pi(t)$, T = 20 years

ω	0.4, $\lambda = 25\%$	0.3, $\lambda = 23\%$	0.2, $\lambda = 22\%$	0.1, $\lambda = 21\%$	0, $\lambda = 0\%$
$\pi(t) < (1+\lambda)$	0.2925	0.28782	0.28548	0.28314	23.4

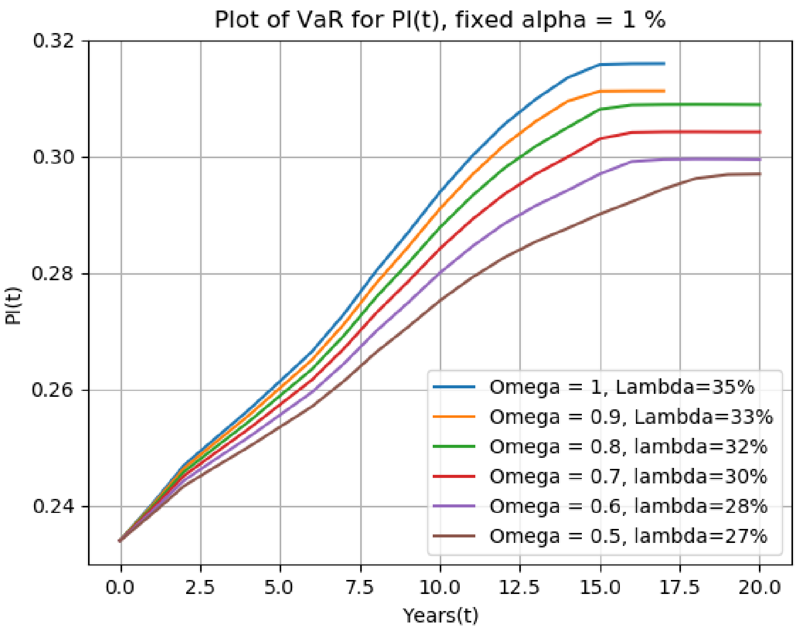


Figure 4.3: Plot of VaR for $\pi(t)$, different λ and $\alpha = 1\%$ fixed

Here we superimpose the two graphs to see the point of intersection λ^*.Here $\omega_1 < \omega_2$ is the point where we have double VaR. This is the solution for analysis (4.0.8).

Here we show an example for the bi-variate condition.

Here we make the following deductions using the relation in section (4.0.2), (4.0.5), (4.0.8): For $\lambda_1 = 15\%$ and a constant $\alpha = 1\%$.

$$P(\delta(T) \geq (1-\lambda_1)\delta(0)) < 1-\alpha$$
$$P(\delta(T) \geq (1-15\%)\delta(0)) < 0.99$$
$$P(\delta(T) \geq (85\%)65\%) < 0.99$$
$$P(\delta(T) \geq 55.25\%) < 0.99$$

Referring from table (4.5) the interval for relation (4.0.3) is given by: $[\omega_1(\lambda_1), 1] = [0.3(15\%), 1] = [55.45\%, 65\%]$

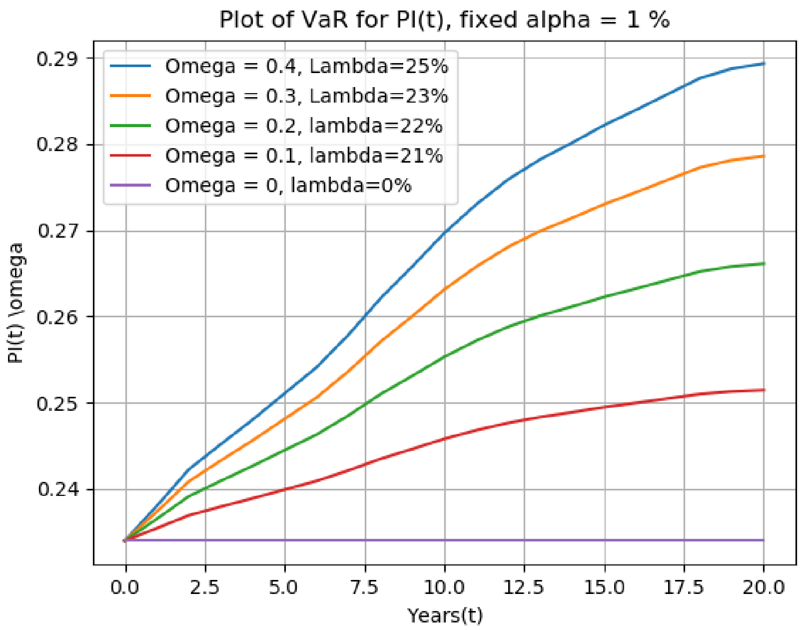


Figure 4.4: Plot of VaR for $\pi(t)$, different λ and $\alpha = 1\%$ fixed

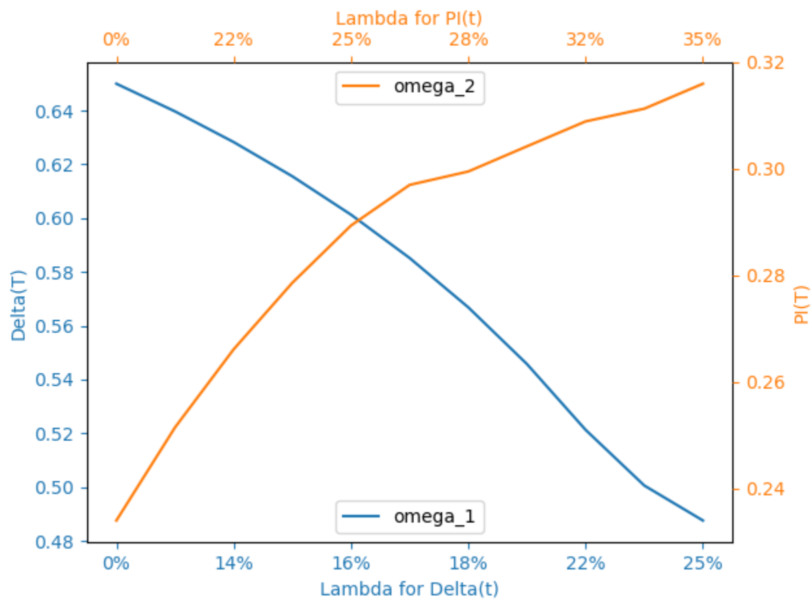


Figure 4.5: Plot of VaR for $\pi(T)$ and $\delta(T)$, different λ and $\alpha = 1\%$ fixed, $T = 20$ years

For the case of $\pi(T)$ using the relation (4.0.4) is given by the following: Here we use $\lambda_2 = 25\%$ and α fixed at 1%.

$$P(\pi(T) < (1 + \lambda_2)\pi(0)) \geq 1 - \alpha$$

$$P(\pi(T) < (1 + 25\%)\pi(0)) \geq 1 - 1\%$$

$$P(\pi(T) < (125\%)23.4\%) \geq 0.99$$

Table 4.5: Optimize Var in Convex Plan with Different ω for $\pi(T)$ and $\delta(T)$, T = 20 years

ω	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\lambda_\delta\%$	25	23	22	20	18	17	16	15	14	13	0
$\lambda_\pi\%$	0	21	22	23	25	27	28	30	32	33	35
$\delta(T)$	48.76	50.47	53.16	55.45	57.40	59.09	60.58	61.89	63.04	64.07	65.0
$\pi(T)$	23.40	25.02	26.37	27.51	28.49	29.34	29.91	30.40	30.86	31.11	31.58

$$P(\pi(T) < 29.25) \geq 0.99$$

Here from table (4.5) the interval for relation (4.0.4) is given by: $[0, \omega_2(\lambda_2)] = [0, 0.4(25\%)] = [23.4\%, 28.49\%]$

Consequently the results for $\delta(t)$ as shown in (4.0.6) is the lower bound $\omega^* = 55.45\%$ and $\pi(t)$ as shown in (4.0.7) is the upper bound $\omega^* = 28.49\%$.

Finally the intersection and beyond as shown in the graph (4.5) is the results for the the relation of Bivariate VaR as shown in the relation (4.0.8) where $\omega_1(\lambda_1) < \omega_2(\lambda_2)$.

5. Discussions

Based on T = 20 years projection on the Belgian data set used, we arrive at the following conclusions:

- The proposed multi period model with low stochastic fertility, low stochastic mortality and baby boom effect, predicts the projected demography of Belgium correctly.
- In the next 20 years, we expect the dependency ratio to peak around 51%.
- Clearly DC plans are worse than DB in terms of risk sharing in PAYG. In DC we can not control the major source of funding for PAYG, the contributions. Here we can not ensure social sustainability so retires suffer the most.
- We have seen that Musgrave framework shares the risk better than DB. And hence an improvement on the fairness of sharing the risk but biased towards DB. Contributors are treated unfair in a Musgrave framework. However, this is just one of the infinitely many solutions that exist for financing the PAYG system.
- A hybrid plan with $\omega = 0.2$, is one of the infinitely many solutions that exist in a hybrid plan. Here we are closer to a DC plan than a DB. Hence retires suffer more in terms of reduced benefit ratio to balance reduced contributions in this plan.
- An $\omega = 0.7$ hybrid plan, will be a proposed solution to the strict Belgian DB plan. Here the retired generation will still be protected more than the contributors but the burden on contributors will be slightly loosened as compared to a pure DB plan.
- Finally, Convex plans with flexible ω provide the best risk sharing and fairness plan than DC, DB, and the Musgrave Framework. Here the policy maker can always choose the desired ω based on the policy statement of either putting a cap on the rate of contribution $\pi(t)$ or a floor on the benefit ratio $\delta(t)$ to fit the desired result.
- Here we provide some flexibility in the structure of the Pension plan the state wants to design. All we need is how much risk the country is willing to take on the contributions and benefit ratio. When that is known, a flexible pension plan can be designed to fit the multi period model.
- We can conclude that indeed for an increasing dependence ratio, the benefit ratio and the contribution rates are always increasing functions with respect to an increasing ω.

Appendix A. The benefit ratio and the contribution rate are increasing function of omega

A.0.1 Benefit ratio : for a fixed t we compute the partial derivative of delta with respect to omega and we proof that this derivative is non negative under the aging condition that the dependency ratio at time t is greater than at time 0 . From equation (2.1.9) we have the following:

$$\delta(t,\omega) = \frac{k_\omega}{\big(D(t)(1-\omega)+\omega\big)} \tag{A.0.1}$$

Where K_ω is the constant given by:

$$K_\omega = \omega\delta(0) + (1-\omega)D(0)\times\delta(0)$$

We simplify (A.0.1) to get the following:

$$\delta(t,\omega) = \frac{\omega\delta(0) + (1-\omega)D(0)\times\delta(0)}{\big(D(t)(1-\omega)+\omega\big)} = \frac{\omega\Big(\delta(0)(1-D(0))\Big) + D(0)\delta(0)}{\omega\Big(1-D(t)\Big) + D(t)} \tag{A.0.2}$$

We simplify as follows:

$$A = \delta(0)\Big(1-D(0)\Big), \qquad B = D(0)\delta(0), \qquad C = \Big(1-D(t)\Big), \qquad D = D(t)$$

$$\delta(t,\omega) = \frac{\omega(A)+B}{\omega(C)+D} \tag{A.0.3}$$

Now we differentiate equation (A.0.3) with respect to ω as follows:

$$\frac{\partial\delta(t,\omega)}{\partial\omega} = \frac{A\Big(\omega(C)+D\Big) - C\Big(\omega(A)+B\Big)}{\Big(\omega(C)+D\Big)^2} \tag{A.0.4}$$

We simplify the numerator to get the following:

$$\frac{\partial\delta(t,\omega)}{\partial\omega} = \frac{\omega(C)A + DA - \omega CA - BC}{\Big(\omega(C)+D\Big)^2} = \frac{DA-BC}{\Big(\omega(C)+D\Big)^2} \tag{A.0.5}$$

For equation (A.0.5) to be positive we have to show the numerator is positive since the denominator is obviously positive:

$$\begin{aligned} DA - BC &= D(t)\delta(0)\Big(1 - D(0)\Big) - D(0)\delta(0)\Big(1 - D(t)\Big) \\ &= D(t)\delta(0) - D(t)\delta(0)D(0) - D(0)\delta(0) + D(0)\delta(0)D(t) \\ &= D(t)\delta(0) - D(0)\delta(0) \\ &= \delta(0)\Big(D(t) - D(0)\Big) \end{aligned} \tag{A.0.6}$$

Here if $D(t) > D(0)$ then equation (A.0.6) is positive. Hence we can conclude that $\frac{\partial \delta(t,\omega)}{\partial \omega} > 0$. Therefore we can conclude that $\delta(t, \omega)$ is an increasing function.

A.0.2 Contribution rate : for a fixed t we compute the partial derivative of pi with respect to omega and we proof that this derivative is non negative under the aging condition that the dependency ratio at time t is greater than at time 0. From equation (2.1.11):

$$\pi(t, \omega) = \frac{K_\omega D(t)}{\big(D(t)(1 - \omega) + \omega\big)} \tag{A.0.7}$$

The constant K_ω is given by:

$$K_\omega = \omega \frac{\pi(0)}{D(0)} + (1 - \omega)\pi(0)$$

$$\pi(t, \omega) = \frac{\Big(\omega \frac{\pi(0)}{D(0)} + (1 - \omega)\pi(0)\Big)D(t)}{\big(D(t)(1 - \omega) + \omega\big)} \tag{A.0.8}$$

we simplify to get the following:

$$\pi(t, \omega) = \frac{\omega \frac{\pi(0)}{D(0)} D(t) + \pi(0)D(t) - \omega\pi(0)D(t)}{\big(D(t)(1 - \omega) + \omega\big)} \tag{A.0.9}$$

$$\pi(t, \omega) = \frac{\omega\Big(\frac{\pi(0)}{D(0)} D(t) - \pi(0)D(t)\Big) + \pi(0)D(t)}{\omega\Big(1 - D(t)\Big) + D(t)} = \frac{\omega\Big(\pi(0)\Big(\frac{D(t)}{D(0)} - D(t)\Big)\Big) + \pi(0)D(t)}{\omega\Big(1 - D(t)\Big) + D(t)} \tag{A.0.10}$$

We simplify as follows:

$$A = \pi(0)\Big(\frac{D(t)}{D(0)} - D(t)\Big), \qquad B = \pi(0)D(t), \qquad C = \Big(1 - D(t)\Big), \qquad D = D(t)$$

Now equation (A.0.10) becomes:

$$\delta(t,\omega) = \frac{\omega(A) + B}{\omega(C) + D} \tag{A.0.11}$$

W e differentiate $\pi(t)$ to get the following:

$$\frac{\partial\delta(t,\omega)}{\partial\omega} = \frac{A\Big(\omega(C) + D\Big) - C\Big(\omega(A) + B\Big)}{\Big(\omega(C) + D\Big)^2} \tag{A.0.12}$$

simplify as:

$$\frac{\partial\delta(t,\omega)}{\partial\omega} = \frac{DA - BC}{\Big(\omega(C) + D\Big)^2} \tag{A.0.13}$$

For equation (A.0.13) to be positive we have to show the numerator is positive since the denominator is obviously positive:

$$\begin{aligned}
DA - BC &= D(t)\pi(0)\left(\frac{D(t)}{D(0)} - D(t)\right) - \pi(0)D(t)\Big(1 - D(t)\Big) \\
&= \frac{D(t)^2\pi(0)}{D(0)} - D(t)^2\pi(0) - \pi(0)D(t) + D(t)^2\pi(0) \\
&= \frac{D(t)^2\pi(0)}{D(0)} - \pi(0)D(t) \\
&= \pi(0)D(t)\left(\frac{D(t)}{D(0)} - 1\right)
\end{aligned} \tag{A.0.14}$$

Here if $D(t) > D(0)$ then equation (A.0.14) is positive. Hence we can conclude that $\frac{\partial\pi(t,\omega)}{\partial\omega} > 0$ from equation (A.0.13). Therefore we conclude that $\pi(t,\omega)$ is an increasing function for an increasing $D(t)$.

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