

Bibliothèque des sciences et technologies

UCL

Place des Sciences 3

BE - 1348 Louvain - la - Neuve (Belgium)



Université Catholique
de Louvain
Faculté de Sciences Appliquées



Centre for Systems Engineering
and Applied Mechanics

Contribution to modeling and control design for wheeled mobile robots

Prof. A. De Herde (président)
Prof. G.Campion (promoteur)
Prof. B. D'Andréa-Novel (encadrement)
Prof. G.Bastin (encadrement)
Prof. P. Fisette
Prof. R. Sepulchre

Université Catholique de Louvain.
Université Catholique de Louvain.
Ecole des mines de Paris.
Université Catholique de Louvain.
Université Catholique de Louvain.
Université de Liège.

Louvain-la-Neuve
Septembre 2002

Il ne faut pas croire ce que j'écris.

Remerciements

Je remercie tout particulièrement le Professeur Guy Campion, promoteur de cette thèse, pour sa disponibilité et son soutien au cours de la réalisation de ce travail. Son enthousiasme et sa bonne humeur ont été, pour moi, une source de motivation permanente. J'ai aussi beaucoup appris sur le plan pédagogique avec lui (certes parfois professeur Tournesol !) toujours prêt qu'il est à se remettre en question face aux difficultés des étudiants. Je remercie aussi les Professeurs Brigitte d'Andréa-Novet et Georges Bastin pour avoir accepté l'encadrement de cette thèse. J'ai particulièrement apprécié leur intervention lors de ma confirmation de thèse pour orienter la suite de mes recherches, ainsi que lors de ma défense privée, pour améliorer le texte final. Les autres membres du jury, Rodolphe Sepulchre et Paul Fisette, ont aussi contribué à l'éclaircissement de plusieurs points dans mon travail. Merci à Rodolphe pour ses explications concernant la partie théorique et pour ses suggestions de mise en perspective du travail. Merci aussi à Paul pour ses conseils et commentaires à propos de diverses justifications pratiques. Je remercie le Professeur André De Herde, doyen FSA, pour sa présidence du jury de thèse.

Merci encore au CESAME et à tous ses membres pour l'extraordinaire ambiance de travail qu'ils créent au profit des jeunes chercheurs. Je pense entre autre à Paul Vandooren, ma référence pour toutes les questions d'algèbre; l'équipe informatique (Guido, Etienne, Yannick et Philippe) pour ses conseils adaptés et ses réponses rapides. Je n'oublie pas les secrétaires (Michou, Isabelle et Lydia) pour leur aide ponctuelle et aussi Victor, pour ses photocopies et reliures toujours impeccables. Merci aussi à mes collègues de bureau, Sophie et Cédric, avec qui j'ai pu partager les problèmes de mes robots, quelques sourires et, avec Cédric, quelques discussions houleuses concernant le C de UCL. Merci encore à Frédéric, Benoît, Cristina, Amaury, Bernard, Gunnar, ... pour leur aide occasionnelle, pour leurs conseils pédagogiques et pour leur soutien durant la très pénible période de rédaction. Un merci particulier à toutes les personnes avec qui j'ai partagé un nombre incalculable de tablettes de chocolat.

Merci enfin à mes proches pour leur soutien durant ces quelques années de recherche. Papa et maman, merci pour avoir toujours été là pour m'encourager, écouter mes rase-bol et vous réjouir de l'avancement de mon travail. Que serais-je sans Sébastien, qui partage ma vie depuis le début de cette thèse ? Merci d'avoir été avec moi dans toutes mes humeurs, dans toutes mes questions et dans tous mes choix.

Summary

The purpose of this thesis is to relax some of the limitations encountered in the literature when dealing with modeling and control of wheeled mobile robots. Two directions are explored, leading to results of totally different nature.

We first relax the assumption that the kinematic constraints are satisfied. The derivation of the dynamical model of the robot then requires to take into account constitutive equations describing the contact forces. Previously, the interaction forces were modeled according to a simple linear model and a singular perturbation formulation of the robot dynamics was derived. Our first contribution is to extend the singular perturbation formulation of the robot dynamical model to a non linear force model based on the general force model of Bakker. Our second contribution is to design feedback control laws taking into account the fast dynamics associated with the contact effects. It requires the feedback of the position, the rolling velocity but no measurement of the sliding velocity is needed. The feedback control law is designed in order to achieve the point tracking control on the corresponding slow manifold. The attractivity of this slow manifold then allows us to ensure the closed-loop performance. The resulting control law appears as a modification of the corresponding ideal model-based control law. Two implementation of this approach are proposed.

First, we use the correction control term to exactly compensate the non ideal perturbation terms. This have been implemented previously on non ideal robotic manipulators. Mathematically, it amounts to linearize the non ideal system on the associated slow manifold. The attractivity of the slow manifold then allows us to prove that this control law ensures semi-global asymptotic convergence of the closed-loop tracking error. However, this control law needs the singular perturbation parameter to be known accurately, corresponding to a perfect knowledge of the stiffness of the wheels. Furthermore, the implementation of the control law supposes it to be developed in power series of the singular parameter and in practice, this development is truncated.

An another way to chose the correction control term is the Liapounov redesign approach. This approach supposes us to define the Liapounov function associated with the nominal law and to build the correction control term and a modified Liapounov function for the non ideal robot on the associated slow manifold. The perturbation terms are then dominated instead of being compensated. In both cases, the theoretical results are illustrated on simulations for the unicycle.

In the last part of the thesis, we address the modeling and posture control for ideal robots equipped only with steering wheels. Two difficulties arise with this type of robots. On one hand, the coordination between the wheels orientations is required in order to ensure the existence of the instantaneous center of rotation. On the other hand, there exist singular configurations where the instantaneous center of rotation coincides with the center of one wheel or where the robot stops. In the neighborhood of such configurations, it may occur that the orientation velocities become too high to be acceptable, even with low robot velocity. Our first contribution is to derive a formulation allowing us to take into account not only the singularities but also the physical limitation of the wheels orientation velocities. We show that this limitation is equivalent to the fact that the robot posture acceleration belongs to a convex set in a 3-dimensional space. We address the posture tracking control problem for such robots. This problem can be solved easily, generically, by a output feedback linearization approach allowing us to assign the tracking error dynamics by an adequate choice of the posture acceleration. Unfortunately, this control law can not be applied when the robot encounters a singularity. Moreover, in the neighborhood of a singularity this generic control law can lead to unacceptable orientation velocities. In the literature, sufficient conditions ensuring that the closed-loop motion does not cross singularities are proposed. These conditions are given under the form of restrictions on the reference trajectory and on the initial tracking error. We extend this result to provide sufficient conditions ensuring that the closed-loop trajectories are posture admissible, i.e. correspond to orientation velocities satisfying the physical constraints. We also propose a modification of the generic control law applicable even when the robot encounters singular configurations. The idea is to assign the posture acceleration as close as possible to the desired acceleration given by the generic control law, but ensuring that the orientation velocities satisfy the physical bounds, i.e. ensuring that the closed-loop trajectory is configuration admissible. The results are illustrated on a four steering wheels mobile robot.

Contents

Introduction	5
I Preliminaries : general description of wheeled mobile robots	9
0.1 Wheeled robot description	11
0.1.1 Robot description	11
0.1.2 Wheels description	12
0.1.3 Wheeled robot description	14
0.2 Constraints' expression	15
0.3 Kinematic model of the ideal robot	15
II Control law design for non ideal mobile robots	19
1 Output linearizing control law for ideal robots	21
1.1 Output feedback linearization of the ideal kinematic model	21
1.2 Dynamical model	23
1.3 The point tracking problem	25
1.4 Reference example : the unicycle	26
1.4.1 Robot description	26
1.4.2 Kinematic constraints	28
1.4.3 Dynamical model	29
1.4.4 Point tracking by input-output feedback linearization	30
2 The non ideal robot singular perturbation analysis	31
2.1 Non ideal kinematic model	31
2.2 Modeling of the contact forces	32
2.2.1 Force model 1 : linear approximation of the forces characteristics	36
2.2.2 Forces model 2 : non linear model	36
2.2.3 Comparison of the two models	38
2.2.4 Normalized formulation	38
2.2.5 Property of the contact force models	40
2.3 Non ideal dynamical model	40

2.4	Properties of the non ideal model	43
2.5	Reference example (continued)	43
2.5.1	Kinematic model	43
2.5.2	Force model 1	44
2.5.3	Force model 2	45
2.5.4	Values of the parameters to make the models comparable	47
2.5.5	Non ideal dynamical model	47
2.6	Singular perturbation analysis	47
2.6.1	Standard singular perturbation model	48
2.6.2	Time-scale expression of the standard model	48
2.6.3	Tikhonov's theorem	50
2.6.4	Singular perturbation analysis for the point tracking problem	51
2.6.5	Reference example (continued)	53
2.6.6	Conclusion	58
2.7	Slow manifold analysis of non ideal robot model	58
2.7.1	Standard slow manifold procedure	58
2.7.2	Existence of the slow manifold	60
2.7.3	Attractivity of the slow manifold	60
3	Control design	63
3.1	Output linearizing feedback control	63
3.1.1	Output linearizability on the slow manifold	63
3.1.2	Control design	64
3.1.3	Closed-loop analysis	65
3.1.4	Practical implementation	66
3.1.5	Reference example (continued)	70
3.1.6	Conclusion	77
3.2	Output tracking by Liapounov redesign	77
3.2.1	Standard Liapounov redesign	78
3.2.2	Smoothed Liapounov redesign	81
3.2.3	Closed-loop analysis	82
3.2.4	Robustness against uncertainties on ε	84
3.2.5	Practical implementation	85
3.2.6	Reference example (continued)	87
3.2.7	Conclusion	97
3.2.8	Remark on the developed simulation program	97
3.3	The drawing robot example	97
3.3.1	Robot characteristics	98
3.3.2	The control parameters	98
3.3.3	Reference trajectory	98
3.3.4	Simulation results for the control laws based on the linearized force model	99

3.3.5	Simulations results for the control laws based on the general force model	101
3.3.6	Conclusion	104
3.4	Discussion of the approach	104
3.4.1	Discussion of the results	104
3.4.2	Possible extensions	105

III Control law design for mobile robots equipped with steering wheels 107

1	Robots equipped with steering wheels : description and properties	109
1.1	Robot description	109
1.2	Kinematic constraints	110
1.3	Kinematic model	113
1.4	Non singular and admissible trajectories	115
2	Posture control design	117
2.1	Posture control	117
2.2	Feedback linearization for posture tracking	118
2.3	Avoiding singularities for the generic control law	119
2.3.1	Sufficient conditions for closed-loop non singular trajectories	119
2.3.2	Sufficient conditions for closed-loop posture admissible trajectories	120
2.4	Control law allowing the robot to go through singularities	121
2.4.1	A constraint minimization problem	121
2.4.2	Practical implementation	122
2.4.3	Tracking error analysis	124
2.5	Control law taking into account some geometrical constraints	125
2.6	Possible extensions of the approach	126
3	Simulation results	127
3.1	Reference example	127
3.2	The velocity constraints control law	127
3.2.1	Generic control law	129
3.2.2	Constrained control law	133
3.3	The geometrical constrained control law	137
3.3.1	ICR outside the X area	138
3.3.2	ICR inside the diamond ace area	142
3.4	Remark on the developed simulation program	144

Conclusions and open issues	145
Appendices	149
A Technical proofs of part 1	149
A.1 Proof of lemma 2.1	149
A.2 Proof of proposition 1.1	149
A.3 Proof of proposition 2.1	151
A.4 Proof of proposition 3.1	152
A.5 Proof of proposition 3.2	154
A.6 Proof of lemma 3.1	156
A.7 Proof of proposition 3.3	156
A.8 Proof of proposition 3.4	157
A.9 Proof of proposition 3.5	157
B Technical proofs of part 2	159
B.1 Expression of $M_s(t)$	159
B.2 Expression of $M_{nas}(t)$	160
Bibliography	165
Index	169

Introduction

A wheeled mobile robot is a vehicle which is capable of an autonomous motion (without external human drive) because it is equipped, for its motion, with motors that are driven by an embarked computer. Over the past twenty years, wheeled mobile robots have become much more important in applications because they can replace human beings in a lot of dangerous situations : security, planetary exploration, nuclear waste transportation, ... The mobile robots under study in this work are made up of a rigid frame and they are moving on an horizontal ground. Such a robot is said to be ideal if, for each wheel, the wheel/ground contact area is reduced to a point and the sliding velocity of this point is zero. Wheeled mobile robots may be equipped with different types of wheels and different number of wheels of each types. There exists therefore numerous wheeled mobile robots. However it has been proven in [1] that they may be grouped in only five classes, each of them being described by a particular form for the kinematic as well as dynamical model. This result simplifies the control law design problem.

The design of control law for wheeled mobile robots is not a new question : a lot of literature proposes solutions to this problem.

In the open-loop approach the control law is designed a priori in order to solve some control problems : to optimize the reference trajectory to avoid some obstacles [2], to optimize the time parametrization of the trajectory to minimize the cover time [3], ... These control laws can be based on the property of differential flatness of the wheeled robot model : this means that the state and input variables are linked by algebraic relations to the evolution of a flat output and its time derivatives (see [4]). The control design problem reduces then to the selection of the trajectory in the output domain.

The closed-loop control laws take into account some measurements during the motion to improve the robustness against perturbation and model uncertainties. The design of the feedback control law are essentially depending on the robot task. The robot may have to follow a given reference trajectory with or without time parametrization, or to be stabilized at a given position. For the stabilization as for the tracking task, the control objective may be to stabilize or to track the full posture of the robot (position and orientation) or only the position. In the first case, we talk about posture stabilization or tracking and in the second case, we talk about point stabilization or tracking. For the posture stabilization task, Brockett showed in [5] that it was not possible to achieve the problem with a smooth and time independent control law. The posture stabilizing control laws have thus to be either non smooth or time-dependent.

Under the assumption that the robot is ideal, the control design problem has been dis-

cussed intensively in the literature. A first approach is the linearization by diffeomorphism and feedback. Necessary and sufficient conditions for full linearizability of non linear systems have been proven and, for the systems that are not fully linearizable, it is possible to find the largest linearizable subset of variables. These results are detailed for example in [6]. Wheeled mobile robots are not fully linearizable by static state feedback but for each class of robot, partially linearizing outputs have been proposed in [7]. The linearization approach may be combined with the dynamic extension algorithm (see [6]) in order to ensure full linearization by dynamic feedback. Linearization by diffeomorphism and static feedback gives thus an answer to the tracking and point stabilization problem but due to Brockett's obstruction, may not be used to solve the posture stabilization problem. For the dynamical feedback, due to the fact we have to make a change of variables, a singularity intrinsic to the approach appears when the robot stops at a cusp point : then the diffeomorphism is singular and the control law fails to work. The dynamic feedback linearization may therefore not be used to solve the posture stabilization problem.

A second way to design control laws is the Liapounov approach. Several Liapounov functions have been proposed to solve the tracking problem for mobile robots. For the tracking of time dependent trajectories, the Liapounov function is based on the tracking error vector (see for instance [8]). For the tracking of reference trajectories that are not parametrized with respect to time, the formalism of curvilinear coordinates and Frenet frame are applied to design a control law and the associated Liapounov function. For instance in [9], such an approach is implemented taking into account the fact that the wheeled mobile robots model may be written in chained form.

The posture stabilization problem is more challenging due to Brockett's obstruction : the stabilizing control laws have either to be discontinuous or time-dependent. A first answer is obtained with discontinuous control laws, for example in [10]. New coordinates are introduced so that the closed-loop system is discontinuous and a Liapounov argumentation allows the authors to conclude to the exponential stability. An other non smooth solution is given in [11] with a piecewise continuous control law. Hybrid feedback laws switching between various time-periodic control laws also may be constructed to solve this problem (see [12]). The other approach is to design smooth time-varying control laws. The work in [8], [9] and [13] led to the development of smooth time-varying control laws for more general systems : for instance for systems in chained form. In the above references, asymptotic convergence towards the desired posture is ensured. Recent papers ([14],[15] and [16], for example) propose other Liapounov based control laws with better convergence rate.

For the class of robots that are equipped with at least two steering wheels, some additional difficulties appear in the control design. These robots present intrinsic geometrical singularities : when the instantaneous center of rotation coincides exactly with the center of one wheel the orientation of this wheel may be chosen arbitrarily. Moreover, when the instantaneous center of rotation is close to the center of one wheel, then it may happen that the kinematic constraints impose the orientation of this wheel to become very large, even for low velocities of the robot. For the stabilization task, Thuilot [17] built a control law that avoids singularities. For the tracking control laws tending to make the

distance between the robot and its reference trajectory decrease, without explicitly tracking a precise point of the reference trajectory, the singularity problem is solved choosing a particular Liapounov function to build the control law ([18]). For the tracking of a time-dependent reference trajectory and for the dynamic feedback linearization, a new singularity arises in the control law when the robot stops at a cusp point. Thuilot [17] proposes sufficient conditions on the reference trajectory and on initial position and velocity that ensure the singularities to be avoided. But this answer is not fully satisfying to treat the singularity problem, because, namely, it does not ensure that the orientation velocities are compatible with specified upper bounds.

Up to now, we discussed the control design for ideal mobile robots. But in practice, it happens that the robots slide and few papers propose to take such sliding effects into account. This supposes the contact forces to be modeled what was not necessary in the ideal case. In [19] a linear model is adopted for the interaction forces and a singular perturbation formulation is derived, representing as fast dynamics these wheel/ground contact effects. The robustness of ideal model-based control laws as regards the effect of neglected dynamics is analyzed. Under several bounds on the characteristics of the wheel/ground contact, when the nominal output linearizing control law is applied to an actual robot, the output tracking error remains bounded but does not converge towards zero (see [19]). In [20], the linear force model is implemented to design a velocity feedback control law for the non ideal unicycle.

The aim of this work is to extend existing control laws so that some of the above restrictive assumptions may be relaxed.

For the non ideal robots, we remind the result obtained in [19] and also introduce a more general model for the interaction forces inspired of the Bakker model ([21]). Our first contribution is to extend the singular perturbation analysis for this general force model. Second, we derive feedback control laws, incorporating the structure of fast dynamics, and ensuring the asymptotic convergence towards zero of the output tracking error. Our method is an extension of the slow manifold procedure introduced by Khorasani and Kokotovic in [22] and [23] to take into account the joint flexibility of robotic manipulators. The basic idea is the same : we define an invariant and attractive slow manifold on which the tracking error decreases exponentially. The difference consists in the nature of the fast dynamics, associated, in our case, to the contact forces, instead of joint flexibility. In both cases the resulting control law takes into account the nature of the fast dynamics, needs accurate knowledge of the position and rolling velocity, but does not require any explicit measurement of the corresponding state vector. We characterize such a slow manifold and prove its attractivity. The control design problem amounts then to ensure the tracking error to decrease on the associated slow manifold. We derive state feedback control laws ensuring asymptotic convergence of the output tracking error, incorporating the nature of the fast dynamics but not requiring any measurement of the sliding velocities. A first approach is to aim at the output linearization of the non ideal robot model. We prove the existence of a such a feedback control law and thanks to the attractivity of the slow

manifold, we prove the control law to ensure semi-global asymptotic convergence of the closed-loop tracking error. This control law is expressed as a power series expansion of the singular perturbation parameter, whose first term is the control law corresponding to the ideal model. The implementation of this control has the major drawback that it is based on an exact characterization of the constitutive equations of the contact forces, which are never known with a good accuracy. Moreover, this control law is built up as a series expansion form and in practical implementations, the development has to be truncated. That is the reason why we develop another method allowing us to ensure robust tracking of a reference trajectory, by use of a Liapounov redesign approach. It is robust in the sense that it does no longer require accurate knowledge of the contact force characteristics. The control law is obtained under the form of a modification of the ideal model based output linearizing control law. The additive correction term is chosen in such a way that the Liapounov function corresponding to the ideal closed-loop is still decreasing along the trajectories belonging to the corresponding slow manifold. We show the closed loop tracking error converges towards zero.

The second contribution of this work deals with the singularity problems arising for robots equipped with at least two steering wheels, and with the design of control law ensuring posture tracking for such robots. We introduce the notion of posture admissible trajectory, for a non singular trajectory that ensure the orientation velocities to be compatible with specified upper bounds. For the non singular and posture admissible reference trajectories, we prove there exist initial conditions ensuring the actual trajectory to be posture admissible. For the singular or non posture admissible reference trajectories, we design a control law that still ensure the tracking to be partially achieved but it is not possible to prove the tracking error to tend towards zero. With this control law, the orientation velocities are ensured to remain bounded even at singularity. The resulting trajectories are therefore called configuration admissible. We write this problem under a constrained control form : find the best angular velocities among bounded angular velocities.

We give a description of a general ideal mobile robot in the first part of this work. In the second part, we address the modeling of non ideal robot and the control design problem for such robots. The third part is devoted to the analysis of the singularity problem for robots equipped with several steering wheels and to the control design problem for such robots.

Part I

Preliminaries : general description of wheeled mobile robots

1902

For information of the Board of Directors
of the National Academy of Sciences

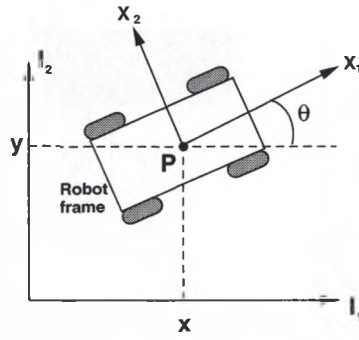


Figure 1: Robot posture definition

The purpose of this preliminary chapter is to remind briefly the framework and assumptions used to describe and analyze wheeled mobile robots. We first assume that the mobile robots are made up of a rigid frame equipped with nondeformable wheels and that they are moving on an horizontal plane. We detail the characteristics of the different types of wheels and highlight the time-dependent variables and the geometric constant parameters necessary to describe the wheeled robot position. We derive the kinematic constraints meaning that the pure rolling and non slipping conditions are satisfied. Robots satisfying these conditions are called *ideal robots*. In part II, this description will be extended to robots not satisfying these conditions. This part is essentially based on reference [1] and [17].

0.1 Wheeled robot description

0.1.1 Robot description

The position of a wheeled robot is completely specified by 3 variables (see figure 1) :

- x, y : cartesian coordinates of a reference point P of the frame in an inertial basis;
- θ : orientation of the robot with respect to the inertial basis.

We define the 3-vector ξ describing the robot posture :

$$\xi \triangleq \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \quad (0.1)$$

and the orthogonal rotation matrix :

$$R(\theta) \triangleq \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (0.2)$$

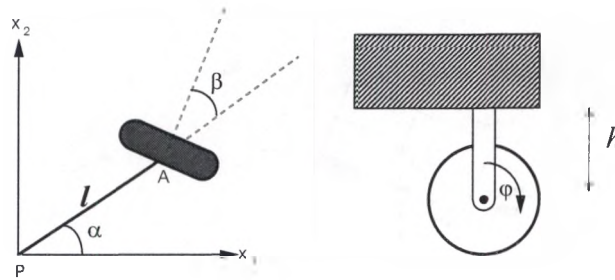


Figure 2: Fixed and steering wheels' description

0.1.2 Wheels description

Our robot can be equipped with 4 types of wheels described in the present subsection :

1) Fixed wheels

When a wheel has a fixed orientation with respect to the frame, it is said to be a fixed wheel. Its configuration is characterized by one variable φ , the wheel rotation angle; and by 5 constant parameters (see figure 2) :

- β , the fixed orientation angle of the wheel;
- r , the radius of the wheel;
- l, α, h , the 3 coordinates of the wheel attach point on the frame.

2) Steering wheels

If a wheel can be oriented in any chosen direction, it is called a steering wheel. The position of this type of wheel is determined by the same parameters as for the fixed wheels but 2 of them are variables, φ and of course β , while the others remain constant (see figure 2).

3) Off-centered wheels

Some wheels are off-centered : this means that the wheel articulation point on the frame is not exactly vertically above the wheel center. In this case, a new parameter is needed to describe the wheel position : the distance d between the wheel center and the vertical axis passing through the articulation point on the frame (see figure 3).

4) Omnidirectional wheels

Omnidirectional wheels are fixed wheels equipped with a special device allowing the wheel to move freely along a fixed direction. The classical example of such wheel is the roller wheel : small free turning rollers are fixed on the contact surface of the wheel allowing it to move orthogonally to the axle of the rollers (see figure 4).

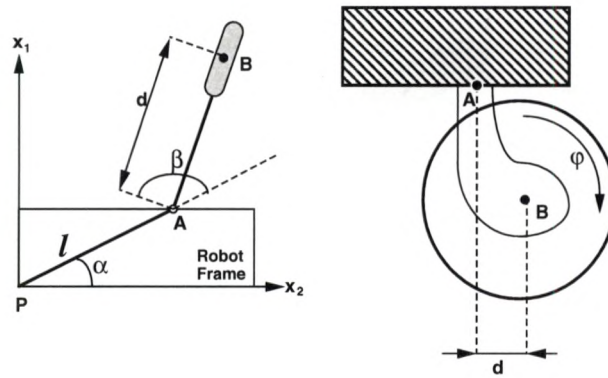


Figure 3: Off-centered wheel description



Figure 4: Photo of a roller wheel of the UCL robot

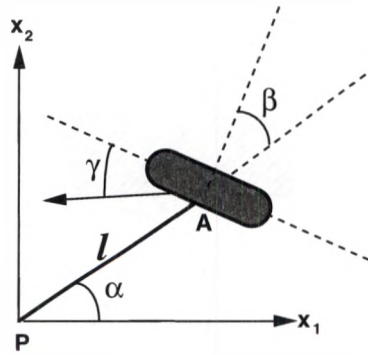


Figure 5: Omnidirectional wheel description

Such wheels have been built to construct mobile robots able to follow any trajectory. Such a prototype has been developed at UCL (PRM). It is equipped with 3 roller wheels placed at the top of an equilateral triangle and can effectively follow any trajectory $x(t), y(t), \theta(t)$ with 3 control actions provided by 3 motors actuating the wheels.

To describe the position of an omnidirectional wheel we introduce a new parameter : the angle γ between the wheel plane and the perpendicular to the direction of the axle of the roller in contact with the ground (see figure 5). The other parameters are the same as for the steering wheel. Obviously an omnidirectional wheel with $\gamma = \frac{\pi}{2}$ has no interest : it has the same properties as a fixed wheel.

4) Summary of the parameters describing each type of wheel

Type of wheel	Variable(s)	Constants
Fixed	φ	β, r, l, α, h
Steering	φ, β	r, l, α, h
Off-centered	φ, β	r, l, α, h, d
Omnidirectional	φ	$\beta, r, l, \alpha, h, \gamma$

0.1.3 Wheeled robot description

Consider a robot equipped with k wheels, among which there are

- k_{fi} fixed wheels;
- k_{or} steering wheels;
- k_{of} off-centered wheels;
- k_{om} omnidirectional wheels;

with $k_{fi} + k_{or} + k_{of} + k_{om} = k$.

Each wheel is numbered from 1 to k in the following order : steering wheels, off-centered wheels, fixed wheels and omnidirectional wheels.

The generalized coordinates vector describing the wheels of our robot has the following form :

$$\Xi = (\beta_1, \dots, \beta_{k_{or}}, \beta_{k_{or}+1}, \dots, \beta_{k_{or}+k_{of}}, \varphi_1, \dots, \varphi_k)^T \quad (0.3)$$

The configuration of a wheeled robot is thus completely described by the *configuration coordinates vector*

$$p = (\xi, \Xi)^T \quad (0.4)$$

of dimension $3 + k_{or} + k_{of} + k$.

0.2 Constraints' expression

For each type of wheel, the ideal case, when the wheels do not slide, impose that the linear velocity at the contact point wheel/ground is zero. These have been developed in [1, 17] and summarized in the following table :

Type of wheel	Expression of the constraint(s)
Fixed	$[-\sin(\alpha + \beta) \quad \cos(\alpha + \beta) \quad l \cos \beta] R(\theta) \dot{\xi} + r \dot{\varphi} = 0 \quad (1)$
	$[\cos(\alpha + \beta) \quad \sin(\alpha + \beta) \quad l \sin \beta] R(\theta) \dot{\xi} = 0 \quad (2)$
Steering	$[-\sin(\alpha + \beta) \quad \cos(\alpha + \beta) \quad l \cos \beta] R(\theta) \dot{\xi} + r \dot{\varphi} = 0 \quad (3)$
	$[\cos(\alpha + \beta) \quad \sin(\alpha + \beta) \quad l \sin \beta] R(\theta) \dot{\xi} = 0 \quad (4)$
Off-centered	$[-\sin(\alpha + \beta) \quad \cos(\alpha + \beta) \quad l \cos \beta] R(\theta) \dot{\xi} + r \dot{\varphi} = 0 \quad (5)$
	$[\cos(\alpha + \beta) \quad \sin(\alpha + \beta) \quad d + l \sin \beta] R(\theta) \dot{\xi} + d \dot{\beta} = 0 \quad (6)$
Omnidirectional	$[-\sin(\alpha + \beta + \gamma) \quad \cos(\alpha + \beta + \gamma) \quad l \cos(\beta + \gamma)] R(\theta) \dot{\xi} + r \cos(\gamma) \dot{\varphi} = 0 \quad (7)$

Remark that the first constraint exactly expresses the fact that the velocity component along the wheel plane is zero. The second one precisely means that the velocity component perpendicular to the wheel plane is zero (except for the omnidirectional wheel).

For the non ideal robots presented in the second part, the sliding velocities are not equal to zero but take small values. The non ideal robot model will then be presented as singular perturbation formulation of the ideal one.

0.3 Kinematic model of the ideal robot

For our ideal k -wheels robot, the constraints can be written in the form

$$D(p) \dot{p} = 0 \quad (0.5)$$

with $D(p)$ a matrix of dimension $(k + k_{of} + k_{or} + k_{fi}) \times (k + k_{of} + k_{or} + 3)$.

In the constraints of type (2) and (4) only appears the posture velocity vector $\dot{\xi}$ so that they restrict the direction of $\dot{\xi}$ and thus the mobility of the robot. For all $\dot{\xi}$ such that the constraints of type (2) and (4) are satisfied, there exists $\beta, \dot{\phi}$ such that constraints of type (1),(3),(5),(6),(7) are satisfied. The last constraints therefore do not restrict the mobility of the robot.

Robots equipped only with off-centered wheels and omnidirectional wheels ($k_{fi} = 0 = k_{or}$) are therefore called *full mobility robots*. On the contrary, if $(k_{fi}, k_{or}) \neq (0, 0)$, the vector $\dot{\xi}$ belongs to a restricted dimension subspace and the robot is called *restricted mobility robot*.

We now study constraints of type (2) and (4) and with this intention we adopt the following notations :

$$D_{1fi} \dot{\xi} = 0$$

denotes constraints of type (2) (with D_{1fi} a $k_{fi} \times 3$ matrix);

$$D_{1or} \dot{\xi} = 0$$

denotes constraints of type (4) (with D_{1or} a $k_{or} \times 3$ matrix);

$$D_1 \dot{\xi} = 0 \tag{0.6}$$

denotes these two types of constraints (with D_1 a $(k_{fi} + k_{or}) \times 3$ matrix).

If $\text{rank}(D_1) = 3$, the only solution to equation (0.6) is $\dot{\xi} = 0$. It means that no motion of the robot is possible. This case is not interesting and is not considered furthermore. We make thus the assumption that $\text{rank}(D_1) \leq 2$ and this condition has a geometrical interpretation. At each instant the motion of the robot can be viewed as an instantaneous rotation around the instantaneous center of rotation (ICR) whose position with respect to the frame may be time-varying. Hence at each instant the velocity vector of any point of the frame is orthogonal to the straight line joining this point and the ICR. In particular this is true for the centers of the fixed and steering wheels. This implies that at each time instant, the horizontal rotation axes of all the fixed and steering wheels are concurrent at the ICR. This fact is illustrated in Fig 6.

Obviously the rank of the matrix D_1 depends on the design of the mobile robot. We define the *degree of mobility* δ_m of a mobile robot as

$$\delta_m = 3 - \text{rank}(D_1).$$

Let us now examine the case $\text{rank}(D_{1fi}) = 2$ which implies that the robot has at least 2 fixed wheels and, if there are more than 2, that their axes are concurrent to the ICR whose position with respect to the frame is fixed. In such a case, it is clear that the

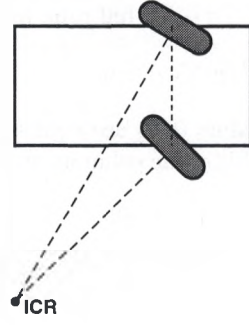


Figure 6: Instantaneous center of rotation

only possible motion is a rotation around its fixed ICR. Obviously this limitation is not acceptable in practice and we thus assume that $\text{rank}(D_{1fi}) \leq 1$. Our robot admits thus 1 fixed wheel or several fixed wheels with the same rotation axle.

With the same arguments, we obtain that the number of steering wheels that can be oriented independently is not more than 2. This means that $\text{rank}(D_{1or}) \leq 2$. We call this number the *degree of steerability* δ_s :

$$\delta_s = \text{rank}(D_{1or})$$

It has been shown in [1] that only five non singular structures of robot were of practical interest and they are characterized by the following admissible pairs of values for δ_m and δ_s :

δ_m	3	2	2	1	1
δ_s	0	0	1	1	2

We finally define the *degree of manoeuvrability* δ_M as

$$\delta_M = \delta_m + \delta_s$$

that represents the number of degrees of freedom of the system.

Keeping in mind all the rank properties of our constraint matrix, we write the constraints (0.5) in a more compact form :

$$\Lambda^T(p) \dot{p} = 0 \quad (0.7)$$

with $\Lambda^T(p)$ a $(m \times n)$ full rank matrix (with $n = k + k_{or} + k_{of} + 3$ and $m = n - \delta_M$) such that $\text{Ker}(\Lambda^T) = \text{Ker}(D^T)$.

Let us now define an associated $n \times (n - m)$ full rank matrix $T(p)$ such that

$$\Lambda^T(p) T(p) = 0$$

where $T(p)$ is not unique. The constraints (0.7) are equivalent to the fact that at each instant $\dot{p}$ belongs to the space generated by the columns of $T(p)$ i.e.

$$\dot{p} \in \text{Im} [T(p)]$$

or equivalently, to the existence of a $(n - m)$ vector ζ which meaning depends on our choice of $T(p)$ such that

$$\dot{p} = T(p) \zeta \tag{0.8}$$

This equation constitutes the *state space kinematic model* of the system.

Part II

Control law design for non ideal mobile robots





Chapter 1

Output linearizing control law for ideal robots

In this chapter we address the point tracking problem for ideal robots, i.e. robots satisfying the kinematic constraints using an input-output feedback linearization approach. We first consider the output feedback linearizing control for the kinematic model of the robot. We introduce particular generalized coordinates that are convenient to design the tracking control law. We then derive the dynamical model of the robot using the Lagrange formalism. Using output feedback linearization, we design a control law that ensures exponential decrease of the closed-loop tracking error. We illustrate all these results on the unicycle that is the reference robot along all this part.

1.1 Output feedback linearization of the ideal kinematic model

The generalized coordinates vector p can be partitioned into two parts :

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

where

- p_1 is the n_1 -vector made by ξ and a set of orientation angles that can be oriented independently ($n_1 = 3 + \delta_S$);
- p_2 is the n_2 -vector made by the remaining configuration variables.

With this partition, the constraints (0.7) can be divided into two subsets :

$$\Lambda_{11}^T(p_1) \dot{p}_1 = 0 \quad \text{and} \quad \Lambda_2^T(p) \dot{p} = \begin{pmatrix} \Lambda_{21}^T(p) & \Lambda_{22}^T(p) \end{pmatrix} \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = 0 \quad (1.1)$$

where the first part correspond to constraints of type (2),(4) for independently oriented wheels.

The $T(p)$ matrix is also partitioned as follows

$$T(p) = \begin{pmatrix} T_1(p_1) \\ T_2(p) \end{pmatrix}$$

where T_1 depends only on p_1 because we assume the matrix $\Lambda(p)$ in (0.7) is built with in its first lines the constraints of type (2) and (4) and it verifies :

$$\Lambda_{11}^T(p_1) T_1(p_1) = 0 \quad (1.2)$$

$$\Lambda_{21}^T(p) T_1(p_1) + \Lambda_{22}^T(p) T_2(p) = 0. \quad (1.3)$$

Remark that if $k_{fi} = k_{or} = 0$ then Λ_{11} does not exist and we can take $T_1 = I$ for example. With these notations the state kinematic model (0.8) becomes :

$$\begin{cases} \dot{p}_1 = T_1(p_1) \zeta \\ \dot{p}_2 = T_2(p) \zeta, \end{cases} \quad (1.4)$$

where ζ is a vector of dimension δ_M .

Proposition 1.1 (Output linearizing feedback in the ideal case)

The general kinematic model (1.4) can be generically transformed by diffeomorphism and input transformation into a controllable linear system of the following form :

$$\begin{cases} \dot{q}_1 = \eta \\ \dot{q}_2 = \Sigma^T(q) \eta \end{cases} \quad (1.5)$$

with the properties below :

- q_1 has dimension δ_M and q_2 dimension $n - \delta_M$;
- two components of q_1 are the cartesian coordinates of a material point of the robot;
- $\Sigma^T(q)$ is bounded uniformly with respect to q_1, q_2 .

The main developments of this proof are given in appendix A.2.

The generalized coordinates vector that is used from now on is the q vector diffeomorphic to p , and that may be partitioned into 2 parts:

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad (1.6)$$

where

- q_1 is a δ_M -vector constituted by the linearizable output;

- q_2 is a vector of dimension $(n - \delta_M)$ constituted by configuration variables (rotation and/or orientation angles of the wheels).

With this definition of q the ideal constraints are written as follows

$$A^T(q) \dot{q} = 0 \quad \text{with} \quad A(q) = \begin{pmatrix} -\Sigma(q) \\ I_m \end{pmatrix} \quad (1.7)$$

where • $m = n - \delta_M$;

- $\Sigma(q)$ is a $((n - m) \times m)$ full rank matrix.

The corresponding kinematic model is then written

$$\dot{q} = \begin{pmatrix} I_{n-m} \\ \Sigma^T(q) \end{pmatrix} \eta \triangleq S(q) \eta \quad (1.8)$$

where η is a $n - m (= \delta_M)$ vector of generalized pseudo-velocities.

It follows from these definitions that $A^T(q) S(q) = 0$.

1.2 Dynamical model

In this section, we derive the dynamical model of the ideal robot. It is described in detail in [19]. The only difference is that we use the q coordinates introduced in the preceding section.

The Lagrange equations of the system have the following standard form :

$$M(q) \ddot{q} = C(q, \dot{q}) \dot{q} + Q + B(q)u \quad (1.9)$$

where

- $M(q)$ is the inertia matrix of the system;
- $C(q, \dot{q})\dot{q}$ is the term of centripetal and Coriolis torques and of gravity and resistant torques;
- u is the $(n - m)$ -vector of the input forces or torques applied by the actuators. We assume that the system is fully actuated, i.e. $(n - m)$ independent control actions are available;
- $B(q)$ is an $n \times (n - m)$ matrix which is supposed to be full rank;
- Q is the n -vector of the generalized forces, corresponding to the constraint forces, and ensuring that the constraints are satisfied.

The fulfillment of the ideal constraints is ensured by the constraint forces (the forces applied by the ground to the wheels) that take 'magically' the adequate value. The evaluation of these forces requires the derivation of the dynamical model of the system.

It is possible to give more information concerning the structure of the constraint generalized forces : the power of these forces is equal to zero for any motion satisfying the constraints, i.e.

$$A^T(q) \dot{q} = 0 \Rightarrow Q^T \dot{q} = 0.$$

This implies that there exists an m -vector, called the Lagrange multipliers vector λ such that

$$Q = A(q) \lambda. \quad (1.10)$$

The system is then completely described by the kinematic model (1.8) and the Lagrange equation (1.9) where Q is expressed according to (1.10). It is possible to eliminate the Lagrange multipliers, by left-multiplying equation (1.9) by the matrix $S^T(q)$:

$$S^T(q) M(q) \ddot{q} = S^T(q) [C(q, \dot{q}) \dot{q} + B(q)u]. \quad (1.11)$$

Then using equation (1.8), we write :

$$\ddot{q} = S(q) \ddot{\eta} + \dot{S}(q) \dot{\eta}. \quad (1.12)$$

Injecting this expression in (1.11), we obtain

$$S^T(q) M(q) S(q) \ddot{\eta} = S^T(q) [-M(q) \dot{S}(q) \dot{\eta} + C(q, S(q) \dot{\eta}) S(q) \dot{\eta} + B(q)u] \quad (1.13)$$

where

$$\dot{S}(q) = \begin{pmatrix} 0 \\ \dot{\Sigma}^T(q) \end{pmatrix}. \quad (1.14)$$

Since $M(q)$ is a symmetric positive definite matrix, we can deduce expression of $\ddot{\eta}$ in the following form :

$$\ddot{\eta} = J_1^{-1} S^T [-M \dot{S} \dot{\eta} + C \dot{q} + Bu] \quad (1.15)$$

where

$$J_1(q) = S^T(q) M(q) S(q). \quad (1.16)$$

Equations (1.8) and (1.15) constitute the full description of the system. It is referred to as the *dynamic state space model*. Notice that the state vector is the following $(2n-m)$ vector

$$\begin{pmatrix} q_1 \\ q_2 \\ \eta \end{pmatrix}.$$

We assume furthermore that the system is actuated in such a way that the square matrix $S^T(q)B(q)$ is non singular. The dynamical state space model can be rewritten as follows :

$$\dot{q}_1 = \eta \quad (1.17)$$

$$\dot{\eta} = J_1^{-1} S^T [-M \dot{S} \eta + C \dot{q}] + J_1^{-1} S^T B u \quad (1.18)$$

$$\dot{q}_2 = \Sigma^T \eta \quad (1.19)$$

and the system is therefore in normal form (see [24]). Different techniques may therefore be implemented to design control laws for such systems. We choose to linearize the dynamic model but we could have follow a Liapounov approach such as in [24] where control Liapounov functions are used.

We choose to implement the following linearizing feedback law u_0

$$u_0 = (S^T B)^{-1} (J_1 v + S^T M \dot{S} \eta - S^T C \dot{q}) \quad (1.20)$$

that ensures the following dynamics for η :

$$\dot{\eta} = v$$

v being an arbitrary reference input.

1.3 The point tracking problem

In this section, we consider the problem of tracking a reference trajectory for a given material point of the robot. In a number of instances, a full control of the posture ξ of the robot is not required while it is sufficient (or even desirable) to control only the position $(x, y)^T$ of a material point P of the robot.

The *point tracking problem* is to find a state feedback controller that achieves the tracking, with stability, of a given reference trajectory $(x_r(t), y_r(t))$ which is assumed to be twice differentiable. In this work, we consider that the control law feedback suppose to measure the posture position and velocity.

We develop hereafter one method to achieve our purpose for the ideal robot : output feedback linearizing control. Other methods could be implemented for example taking into account the fact that the dynamical model is in normal form (see [24]).

We consider the linearizing outputs of proposition 1.1 and we define a reference trajectory $q_{1r}(t)$ such that the first two components of $q_{1r}(t)$ are $x_r(t)$ and $y_r(t)$ and $q_{2r}(t) = \dot{q}_{1r}(t)$. If the robot has a manoeuvrability degree δ_M of dimension 3, the third component of the linearizable output may be used to achieve a secondary objective. Possible secondary objectives include making the orientation of the robot constant or bounded, to make the wheels orientation velocity bounded, ...

The following result has been obtain in reference [25] :

If the reference trajectory is such that $q_{1r}(t)$, $\dot{q}_{1r}(t)$, $\ddot{q}_{1r}(t)$ are bounded for every t , the point tracking problem is solvable for restricted mobility robot, by using the following auxiliary control :

$$v = \ddot{q}_{1r}(t) - K_1 \dot{\bar{q}}_1 - K_2 \bar{q}_1 \quad (1.21)$$

where $\bar{q}_1 \triangleq q_1 - q_{1r}$, K_1 and K_2 are $(\delta_M \times \delta_M)$ matrices such that

$$\Gamma = \begin{pmatrix} 0 & I \\ -K_1 & -K_2 \end{pmatrix} \quad (1.22)$$

is Hurwitz stable. More precisely we have :

R1 : $\bar{q}_1, \dot{\bar{q}}_1$ exponentially converge to zero;

R2 : $\dot{q}_2(t)$ is bounded uniformly with respect to t .

Result R2 is explained by the fact that $q_2(t)$ are angles appearing only as arguments of sine and cosine functions. This result implies that no finite escape time occurs for $q_2(t)$.

Observe the above statement is less restrictive than the result presented in [25]. The additive assumption that $\dot{q}_{1r}$ is $\mathcal{L}_1$ is superfluous in the proof. For example, a sine reference trajectory can be followed by an ideal robot applying the control law in (1.20) and (1.21).

We conclude that the point tracking problem is solvable by partial linearization by static feedback for any restricted mobility robot in the ideal case. The global feedback control $u_0(q, \eta, t)$ ensuring the point tracking is obtained by combining (1.21) and (1.20).

1.4 Reference example : the unicycle

1.4.1 Robot description

We consider a wheeled robot equipped with two fixed wheels on the same axle (wheels 1 and 2) and one off-centered wheel (wheel 3). The system is represented in figure 1.1. The distance between the two fixed wheels is denoted $2l$ and the wheels radius is written r . We suppose that the mass of wheel 3 is negligible and denote by m_T the global mass of the robot. This robot is referred to as the unicycle. We assume that the unicycle moves on a smooth and horizontal ground so that we can introduce an orthonormal basis $\{O, I_1, I_2\}$ of the motion plane. This robot is thus characterized by $(\delta_m, \delta_s) = (2, 0)$ and $\delta_M = 2$.

The position of this robot is described by the following vector :

$$p = \begin{pmatrix} x \\ y \\ \theta \\ \varphi_1 \\ \varphi_2 \end{pmatrix} \quad (1.23)$$

- (x, y) are the coordinates of a reference point on the frame in the basis $\{O, I_1, I_2\}$. This point is chosen in the middle of the axle of the driving wheels;
- θ is the angle between the axle of wheel 1 and 2 and horizontal direction;
- φ_1, φ_2 are rotation angles respectively of wheels 1 and 2.

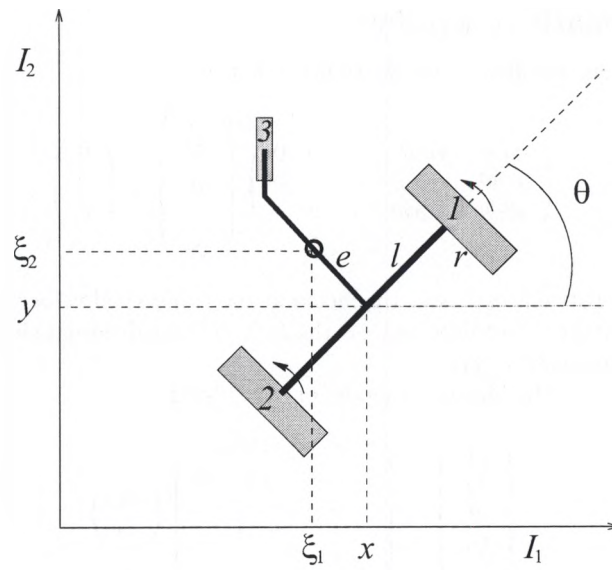


Figure 1.1: Unicycle posture definition

However, it has been proven in [1] that output linearization of this system is possible if we consider as linearizing output the coordinates of a reference point of the frame

$$\xi_1 = x - e \cos \theta \quad (1.24)$$

$$\xi_2 = y + e \sin \theta \quad (1.25)$$

with $e \neq 0$ (see figure 1.1).

The position of the unicycle is thus described by the following vector :

$$q = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \theta \\ \varphi_1 \\ \varphi_2 \end{pmatrix}. \quad (1.26)$$

The generalized coordinates vector is partitioned into

$$q_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \quad q_2 = \begin{pmatrix} \theta \\ \varphi_1 \\ \varphi_2 \end{pmatrix}$$

corresponding to $n = 5$ and $m = 3$.

1.4.2 Kinematic constraints

The kinematic constraints are here written as follows :

$$\begin{pmatrix} \cos\theta & \sin\theta & e & 0 & 0 \\ \sin\theta & -\cos\theta & -l & r & 0 \\ \sin\theta & -\cos\theta & l & 0 & r \end{pmatrix} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\theta} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (1.27)$$

where the first constraint expresses the fact that the transversal velocity of wheel 1 and 2 is zero and the other constraints express the fact that longitudinal slipping velocities of wheel 1 and 2 are equal to zero.

It allows us to derive the kinematic model of our system :

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\theta} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} -\sin\theta & -\frac{e}{l}\cos\theta \\ \cos\theta & -\frac{e}{l}\sin\theta \\ 0 & \frac{1}{l} \\ \frac{1}{r} & \frac{1}{r} \\ \frac{1}{r} & -\frac{1}{r} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \quad (1.28)$$

where $\zeta_1 = l\dot{\theta}$ and ζ_2 is the longitudinal velocity of the reference point. As $\delta_M = 2$, Proposition 1 tells us that we can linearize the components $\dot{\xi}_1$ and $\dot{\xi}_2$. We impose thus the following transformation

$$\begin{pmatrix} -\sin\theta & -\frac{e}{l}\cos\theta \\ \cos\theta & -\frac{e}{l}\sin\theta \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \quad (1.29)$$

This transformation is clearly invertible (as far as $e \neq 0$) and the inverse transformation is

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ -\frac{l}{e}\cos\theta & -\frac{l}{e}\sin\theta \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \quad (1.30)$$

The vector η corresponds physically to the velocity of the reference point $(\dot{\xi}_1, \dot{\xi}_2)$. The ideal kinematic model expressed in term of the linearizing coordinates becomes

$$\dot{q} = \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\theta} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} I_2 \\ \Sigma^T(\theta) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = S(\theta)\eta \quad (1.31)$$

with the following expression for matrix $\Sigma(\theta)$:

$$\Sigma(\theta) = \begin{pmatrix} -\sin\theta & -\frac{l}{e}\cos\theta \\ \cos\theta & -\frac{l}{e}\sin\theta \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{r} & \frac{1}{r} \\ \frac{1}{l} & \frac{1}{r} & -\frac{1}{r} \end{pmatrix}. \quad (1.32)$$

1.4.3 Dynamical model

For simplification of the equations, we make the assumption that the reference point is the center of mass of the unicycle. The Lagrange equations take then the following form

$$M \ddot{q} = Q + Bu \quad (1.33)$$

where

- the mass matrix is given by

$$M = \begin{pmatrix} m_T & 0 & 0 & 0 & 0 \\ 0 & m_T & 0 & 0 & 0 \\ 0 & 0 & I_\theta & 0 & 0 \\ 0 & 0 & 0 & I_\varphi & 0 \\ 0 & 0 & 0 & 0 & I_\varphi \end{pmatrix}$$

with I_θ is the moment of inertia around the vertical axle at (x, y) and I_φ is the rotational moment of inertia of each wheel.

- the control inputs are the torques providing the rotation of the two wheels :

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

because the control acts on angles φ_1 and φ_2 .

Premultiplying the above expression by matrix $S(\theta)$ defined in (1.31) allows us to simplify it as explained in chapter 1 in equation (1.10) and (1.11) :

$$S^T(\theta) M \ddot{q} = S^T(\theta) B u. \quad (1.34)$$

We now develop expression for $\ddot{q}$

$$\ddot{q} = \left(\frac{\partial S}{\partial \theta} \right) \dot{\theta} \eta + S(\theta) \dot{\eta}, \quad (1.35)$$

and inject it in equation (1.34) to obtain the ideal dynamical model of the unicycle :

$$\dot{q} = S \eta \quad (1.36)$$

$$\dot{\eta} = (S^T M S)^{-1} \left[S^T B u - S^T M \frac{\partial S}{\partial \theta} \dot{\theta} \eta \right]. \quad (1.37)$$

1.4.4 Point tracking by input-output feedback linearization

In the beginning of the chapter, we made a transformation ensuring that

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \quad (1.38)$$

and our linearizing control thus verifies

$$\dot{\eta} = (S^T M S)^{-1} \left[S^T B u_0 - S^T M \left(\frac{\partial S}{\partial \theta} \right)^T \dot{\theta} \eta \right] = v. \quad (1.39)$$

It allows us to derive the expression of this nominal linearizing control law :

$$u_0 = (S^T B)^{-1} S^T M \left[S v + \left(\frac{\partial S}{\partial \theta} \right)^T \dot{\theta} \eta \right] \quad (1.40)$$

where v is the auxiliary control defined in (1.21) and ensuring the tracking of the reference trajectory.

Chapter 2

The non ideal robot singular perturbation analysis

In practical situations, the assumptions relative to ideal wheeled mobile robots are not satisfied. This is due to the fact that the wheels are deformable so that the contact area with the ground is not restricted to a single point. The ideal kinematic constraints are no longer satisfied and the forces no longer take "magically" the adequate values. It is necessary to introduce constitutive equations relating the contact forces to the sliding effects. Two different models for the ground/wheels contact forces are described and used to derive the dynamical model of the robots. The force models are rewritten under a form allowing us to consider a family of stiffness characteristics including as limit case the ideal robot. This enables us to derive an equivalent singular perturbation formulation of the dynamical model. Applying the singular perturbation analysis, we obtain a robustness result of the nominal linearizing control law when applied in the non ideal case. We also introduce the slow manifold theory and prove the slow manifold associated with the robot model to be attractive. These are the key arguments to build new control laws in chapter 3.

2.1 Non ideal kinematic model

We consider a robot in the more realistic case where the velocity constraints are not satisfied. For now on, we use coordinates introduced in chapter 1, Proposition 2.1, related to the output feedback linearization of the ideal model.

Here, the generalized velocities vector $\dot{q}$ does not belong to the subspace $Im[S(q)]$. As the columns of $S(q)$ associated with the columns of $A(q)$ cover the full space $\mathbb{R}^n$, $\dot{q}$ can be expressed as follows :

$$\dot{q} = S(q) \eta + A(q) \dot{\mu} \quad (2.1)$$

where $\bullet S(q) = \begin{pmatrix} I_{n-m} \\ \Sigma^T(q) \end{pmatrix}$ and $A(q) = \begin{pmatrix} -\Sigma(q) \\ I_m \end{pmatrix}$;

- $\check{\mu}$ is a m -vector that characterizes the violation of the kinematic constraints according to

$$A^T(q)\dot{q} = A^T(q)A(q)\check{\mu}. \quad (2.2)$$

The sliding velocities of each wheel are directly related to the components of this $(n - m)$ -vector.

2.2 Modeling of the contact forces

We consider the motion of one wheel on an horizontal ground, and we introduce different notations that are needed to model the forces. The figure 2.1 gives a geometrical representation of the conventions introduced below.

The velocity of the wheel center $\vec{v}$ may be decomposed as follows,

$$\vec{v} = \vec{v}_S + \vec{v}_R \quad (2.3)$$

where $\vec{v}_S$ is the sliding velocity and $\vec{v}_R$ is the rolling velocity. If the rotation angle of the wheel is denoted φ and the radius of the wheel is denoted r , we have that $v_R = r\dot{\varphi}$. The transversal and longitudinal components of the sliding velocity $\vec{v}_S$ are respectively denoted $\vec{v}_T$ and $\vec{v}_L$.

The interaction force at the contact point is denoted $\vec{f}$ and its transversal and longitudinal components are respectively denoted $\vec{f}_T$ and $\vec{f}_L$. The sliding velocities' components $\vec{v}_T$ and $\vec{v}_L$ are oriented in the same direction but in opposite sense to the associated force components $\vec{f}_T$ and $\vec{f}_L$.

The complete description of the wheel/ground contact also includes a self-aligning torque that encourages the wheel plane to be in the direction of the velocity $\vec{v}$, as well as a camber effect. In order to simplify the analysis, we neglect these effects and assume the wheel plane is always orthogonal to the ground plane. We furthermore assume that the vertical loads are constant on each wheel. These effects should be taken into account without affecting the nature of the results below, but at the price of a more complex development. For robots with center of mass that is not too high with respect to the axes of wheels, the last simplification is not restrictive.

The interaction forces models presented in the literature consist in relations between the sliding velocities and the forces : the transversal component and the longitudinal component of the contact force applied by the ground to a given wheel are related respectively to the transversal slip ratio and the longitudinal slip ratio of this wheel. These ratios are respectively defined as $\frac{\|\vec{v}_T\|}{\|\vec{v}\|}$ and $\frac{\|\vec{v}_L\|}{\|\vec{v}\|}$. Typical shapes of these relations are given in figure 2.2 representing $\|\vec{f}_T\|$ and $\|\vec{f}_L\|$ as functions of the associated slip ratios.

It must be pointed out that these models are only valid for non zero velocity of the center of the wheel. When v becomes zero, the force should be defined according to a static force model and take "magically" the adequate value. In practice, the transition from the dynamical force model to the static force model is smooth. In simulations, it is not easy

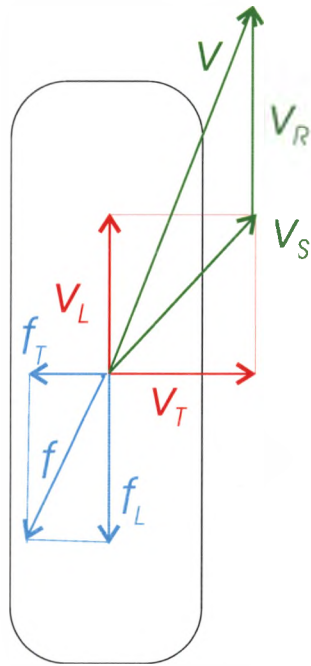


Figure 2.1: The different velocity and force components for a given wheel

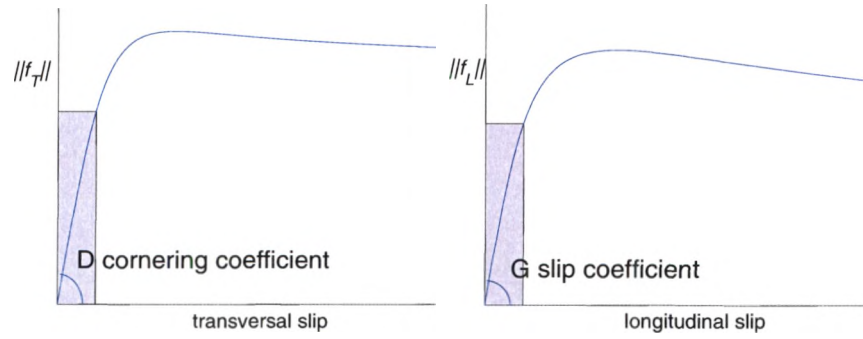


Figure 2.2: Evolution of the transversal and longitudinal forces with respect to the transversal and longitudinal slip ratios

to know when to switch from one model to the other without leading to discontinuities in the forces values. In this part, we only consider the point tracking problem so that the robot stops eventually one instant before to go back. As far as the dynamical force model is not valid eventually for a very short time, we decide not to treat the connection from dynamical force model to static force model. It has been treated for example in [26] for the cam/follower type of contact.

Finally remark that the transversal and longitudinal components of the slip velocity of each wheel are related to the components of the m -vector $A^T(q)\dot{q}$ which are identically equal to zero for the ideal case (see equation (2.2)). For an easy writing of the robot model, we adopt a slight different definition of the slip ratios : $\alpha_T = \frac{\|v_T\|}{\|v_R\|}$ and $\alpha_L = \frac{\|v_L\|}{\|v_R\|}$ where v_R is the rolling part of the velocity not depending on $\check{\mu}$. This approached model is valid as far as the slip ratios are small. Our later study of the different control laws only concerns small values of the slip ratios. The advantage of this convention is that the slip ratios depend linearly on $\check{\mu}$.

Moreover, to avoid the transition in the force model when $v_R = 0$, we choose to define the slip ratios by

$$\alpha_T = \frac{\|v_T\|}{\|v_R\| + \sigma} \quad \alpha_L = \frac{\|v_L\|}{\|v_R\| + \sigma} \quad (2.4)$$

where σ is a small positive number ($\sigma = 0.01$ in the later simulations).

For each wheel (index i), let us define :

- $f_i = \begin{pmatrix} f_{T_i} \\ f_{L_i} \end{pmatrix}$, the components (in the transversal and longitudinal direction with respect to the wheel) of this contact force;

- v_{Ti}, v_{Li} , the components of the sliding velocity, i.e. of the velocity of the point of the wheel in contact with the ground.

These components are related to $\dot{q}$ (and $\ddot{\mu}$) by the following relation

$$\begin{pmatrix} v_{Ti} \\ v_{Li} \end{pmatrix} = L_i A^T(q) \dot{q} = L_i A^T(q) A(q) \ddot{\mu} \quad (2.5)$$

where L_i is a $(2 \times m)$ full rank matrix.

- α_{Ti}, α_{Li} , the transversal and longitudinal slip ratios

$$\alpha_{Ti} = \frac{\|v_{Ti}\|}{\|v_{Ri}\| + \sigma}, \quad \alpha_{Li} = \frac{\|v_{Li}\|}{\|v_{Ri}\| + \sigma} \quad (2.6)$$

where v_{Ri} is the rolling velocity depending only on q and η .

According to the definition of the slip ratios, the power of interaction forces is zero for any motion that satisfies the kinematic constraints :

$$A^T(q) \dot{q} = 0 \Rightarrow Q^T \dot{q} = 0. \quad (2.7)$$

This amounts to consider that all the points of the contact area have the same sliding velocity so that we may neglect the forces dissipation in the contact area when the kinematic constraints are satisfied. As far as the contact area is small, this assumption is not restrictive.

It implies that there exists an m -vector F such that

$$Q = -A(q)F \quad (2.8)$$

where the vector F may be viewed as a Lagrange multiplier vector.

The power of the contact force for wheel i is given by

$$W_i = -\dot{q}^T A(q) L_i^T f_i \leq 0 \quad (2.9)$$

and the global power by

$$W = \sum_i W_i = -\dot{q}^T A(q) \sum_i L_i^T f_i \leq 0. \quad (2.10)$$

This allows us to express the generalized forces as $-A(q)F$ with

$$F = \sum_i L_i^T f_i. \quad (2.11)$$

2.2.1 Force model 1 : linear approximation of the forces characteristics

From the result of Bakker in [21], it is known that for "small" values of the slip ratios (i.e. for slipping ratios less than 10% of the maximal force slip ratios), the transversal and longitudinal components of the force can be approached by a linear relationship i.e.

$$\|f_T\| = D\alpha_T \quad \text{and} \quad \|f_L\| = G\alpha_L \quad (2.12)$$

where D and G are called the cornering and slip coefficients.

The first model we analyze is based on this proportionality assumption and is therefore valid only for small values of the slip ratios.

We may express the global power of this contact forces as a negative definite function of the sliding velocities, i.e.

$$W = - [\dot{q}^T A(q)] \bar{K}(q, \dot{q}) [A^T(q) \dot{q}] \quad (2.13)$$

where $\bar{K}(q, \dot{q})$ is a $(m \times m)$ positive definite matrix depending on the cornering and slip coefficients D and G . The corresponding expression for the generalized forces vector then becomes

$$F = \bar{K}(q, \dot{q}) A^T(q) \dot{q}. \quad (2.14)$$

2.2.2 Forces model 2 : non linear model

If we want to describe a global model of the interaction forces, we may use the Bakker model presented in [21] : an analytic expression of the model is computed based on experiments for given contact characteristics.

For each wheel (we remove the index for simplicity) the components f_T, f_L are related to the sliding velocity by constitutive equations. The two components are related to the corresponding slip ratio according to

$$f_{T,L}(\alpha_{T,L}) = D_{T,L} \sin(C_{T,L} \text{Arctg} \phi_{T,L}) \quad (2.15)$$

where

$$\phi_{T,L} = B_{T,L}(1 - E_{T,L})\alpha_{T,L} + E_{T,L} \text{Arctg}(B_{T,L}\alpha_{T,L}). \quad (2.16)$$

In these expressions the index (T, L) denotes either T or L , $B_{T,L}$ are stiffness factors, $C_{T,L}$ are shape factors, $D_{T,L}$ are peak factors and $E_{T,L}$ are curvature factors of the characteristics of the wheel.

The generalized forces F is then obtained by injecting the expressions (2.15) with (2.16) in (2.11). In our later simulations, we adopt this force expression in the robot dynamics. The control design will be either based on the linearized force model (model 1), or on the non linear force model (model 2).

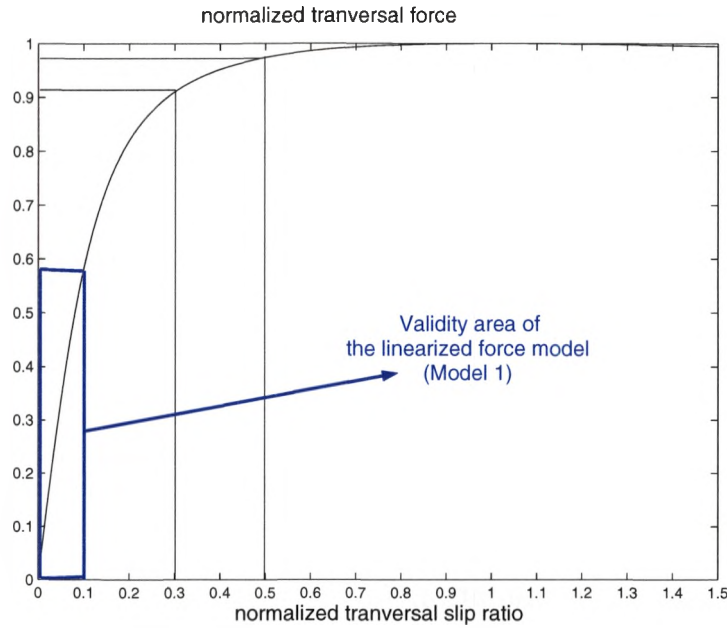


Figure 2.3: Normalized transversal force curve analysis

Such a decoupled model is not fully realistic because in practice, the transversal force also depends on the longitudinal slip. However, it is likely possible to extend the later results for a coupled force model.

We now analyze the transversal force curve in detail. Let us adopt the notation f_{Tmax} for the maximal value of the transversal force and α_{Tmax} for the associated slip ratio. We analyze the relation between the slip ratio and the force using the normalized plot of figure 2.3 : the X-axis has been scaled so that $\alpha_{Tmax} = 1$ and the Y-axis so that $f_{Tmax} = 1$. We observe that when the slip ratio is 10% of α_{Tmax} , the associated force is already more than 50% of the f_{Tmax} value. This justifies the fact that the linearized force model (model 1) is only valid on this area. **For a slip ratio value of 30% of α_{Tmax} , the force already reaches more than 90% of f_{Tmax} .** At 50% of α_{Tmax} , the force value is more than 95% of f_{Tmax} . This needs to be kept in mind to analyze the slip ratio plots in the presented simulations. The longitudinal force curve should be studied the same way and lead to the same type of conclusion : due to the shape of the curve, the slip ratio remains small unless the force is close to the maximal value.

2.2.3 Comparison of the two models

For small values of $\alpha_{T,L}$, the constitutive equations of the non linear force model in (2.15) and (2.16) can be linearly approached by

$$f_{T,L} = \left. \frac{\partial f_{T,L}}{\partial \alpha_{T,L}} \right|_{\alpha_{T,L}=0} \alpha_{T,L} = B_{T,L} C_{T,L} D_{T,L} \alpha_{T,L}. \quad (2.17)$$

This gives us a link between the linearized force model parameters D, G and the non linear force model parameters $B_{T,L}, C_{T,L}, D_{T,L}$:

$$D = B_T C_T D_T; \quad (2.18)$$

$$G = B_L C_L D_L. \quad (2.19)$$

These relations ensure that the linear force model is exactly the linearized non linear force model. The two models therefore coincide for small values of the slip ratios, that is for slip ratios less than 10% of the maximal forces value associated slip ratios (see figure 2.3).

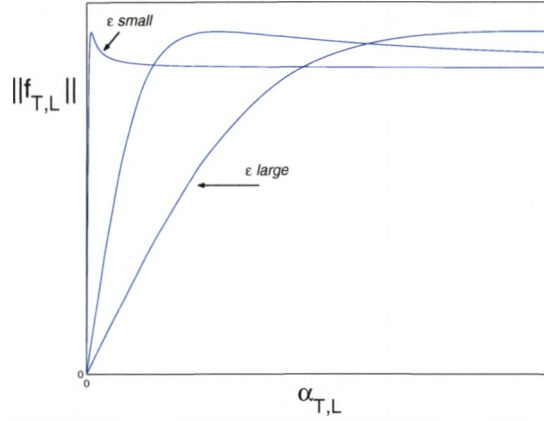
2.2.4 Normalized formulation

We would like to rewrite the equations (2.15) in order to analyze the effect of various values of the stiffness factors, including as a limit situation the case of ideal robots where $\ddot{\mu} = 0$, but where the generalized forces, given by the Lagrange multipliers are not equal to zero. This can be achieved by defining a scaling coefficient ε ($\varepsilon \geq 0$), for the stiffness factor as well as for the sliding velocity, in such a way that the particular value $\varepsilon = 0$ allows us to recover the ideal case. We now detail the development for the two above force models.

Force model 1

It is convenient to consider a family of contact characteristics depending on the value of a parameter ε . We use the same parameter to characterize the slip ratios of all the wheels of the robot. We therefore introduce a normalization parameter $\varepsilon \geq 0$ such that $D = \frac{D_0}{\varepsilon}$ and $G = \frac{G_0}{\varepsilon}$. For simplicity in the later theoretical developments, we introduce the same parameter ε to characterize the families of transversal and longitudinal slip ratios for all the wheels. This corresponds to the assumption that the ratio $\frac{D}{G}$ is the same for all the wheels of all the robots of the family. Different parameters $\varepsilon_{T,i}$ and $\varepsilon_{L,i}$ should be introduced for each wheel i ($i = 1, \dots, k$) to characterized independently the transversal and longitudinal effects. But this would lead to some complications in the equations writing without affecting the final results.

The chosen D_0 and G_0 are the reference stiffness coefficients of the family. Small values of ε correspond to stiff contact coefficients (D and G large) while an increasing ε corresponds to smaller values of the stiffness coefficients.

Figure 2.4: Forces shapes for various values of ε

The expression of matrix $\bar{K}$ appearing in the interaction forces is derived using this normalization convention of the slip and cornering coefficient :

$$\bar{K}(q, \dot{q}) = \frac{K(q, \dot{q})}{\varepsilon}, \quad (2.20)$$

so that the forces are expressed by

$$F = \frac{K(q, \dot{q})}{\varepsilon} A^T(q) \dot{q}. \quad (2.21)$$

For a given robot, with a given value of ε , we can define μ such that

$$\check{\mu} = \varepsilon \mu. \quad (2.22)$$

According to equation (2.1), the final expression of the generalized forces associated with the linear model is

$$F = K(q, \dot{q}) A^T(q) A(q) \mu. \quad (2.23)$$

Force model 2

For each value of the stiffness coefficient B , we define a normalized value B_0 , and ε such that $B = \frac{B_0}{\varepsilon}$. We still introduce the same normalization ε for the longitudinal and transversal sliding parameters for each wheel. To each value of ε corresponds a curve $f(\alpha)$, with a slope at the origin proportional to $\frac{1}{\varepsilon}$. Typical shapes of forces are showed in figure 2.4, corresponding to various values of ε .

For a given robot, with a given value of ε , we can define μ such that

$$\check{\mu} = \varepsilon \mu. \quad (2.24)$$

Obviously the value $\varepsilon = 0$ corresponds to $\check{\mu} = 0$. With these definitions, the quantities $B\alpha$ involved in (2.15-2.16) can be expressed as functions of q, η, μ independently of ε :

$$\begin{pmatrix} B_T \alpha_T \\ B_L \alpha_L \end{pmatrix} = \frac{1}{\|v_R(q, \eta)\|} \begin{pmatrix} B_{0T} & 0 \\ 0 & B_{0L} \end{pmatrix} L A^T(q) A(q) \mu. \quad (2.25)$$

We therefore can consider $f_{T,L}$ and also F as functions of (q, η, μ) independently of the value of the parameter ε . This parameter can be viewed as a measure of the dilatation along the horizontal axis, of the curves of figure 2.4. For given q and η , all the robots of the family are characterized by the same dependence in μ . To a given value of the forces corresponds one precise vector μ , but of course, different sliding effects depending on the actual value of ε , according to (2.24).

This formulation generalizes what we proposed in the previous subsection where the constitutive equations take the simple form of a linear relationship between α_T, α_L and f_T, f_L . It can be extended easily to other contact forces models, using the same conjuring trick, provided that it is possible to identify a stiffness coefficient which is always multiplied by the sliding ratio (i.e. terms of the form $B\alpha$ in (2.15-2.16)).

2.2.5 Property of the contact force models

We give here a technical result (proven in appendix A.1) about the general forces model (Model 2) that is useful in the last section of this chapter.

Lemma 2.1 *Consider a robot characterized by $\varepsilon > 0$. If μ is such that for each wheel the (2×2) matrix $\frac{\partial f_i}{\partial \alpha_i}$ (evaluated at $\alpha_i = \frac{1}{\|v_{Ri}\|} L_i A^T A \varepsilon \mu$) is positive definite, then the matrix $A^T A \frac{\partial F}{\partial \mu}$ is also positive definite.*

The above condition on $\frac{\partial f_i}{\partial \alpha_i}$ means that for each wheel and each force component, the slip ratios remain on the increasing part of the force characteristics. This property is recalled in the section 2.4 summarizing all the properties of the robot non ideal dynamical model.

Of course, for the approached model (Model 1), the condition on $\frac{\partial f_i}{\partial \alpha_i}$ is always satisfied because

$$A^T A \frac{\partial F}{\partial \mu} = A^T A K A^T A \quad (2.26)$$

which is a positive definite matrix.

If a more general force model was considered, where the coupling effects are taken into account, the above conditions would likely still be satisfied but the matrices K (of force model 1) or $\frac{\partial F}{\partial \mu}$ (of force model 2) would not be diagonal matrices.

2.3 Non ideal dynamical model

The dynamical model is still obtained using Lagrange equations

$$M(q) \ddot{q} = C(q, \dot{q}) \dot{q} - A(q) F + B(q) u \quad (2.27)$$

with an explicit model for the wheel/ground contact forces.

We express $\ddot{q}$ under the form below

$$\ddot{q} = [S(q) \ A(q)] \begin{pmatrix} \dot{\eta} \\ \varepsilon \dot{\mu} \end{pmatrix} + s(q, \eta, \varepsilon \mu) + a(q, \eta, \varepsilon \mu)$$

with the following definitions :

$$s(q, \eta, \varepsilon \mu) = \dot{S} \eta \text{ with } \dot{S} = \frac{\partial S}{\partial q}(S \eta + A \varepsilon \mu),$$

$$a(q, \eta, \varepsilon \mu) = \dot{A} \varepsilon \mu \text{ with } \dot{A} = \frac{\partial A}{\partial q}(S \eta + A \varepsilon \mu).$$

After left-multiplication of (2.27) by the matrix $\begin{pmatrix} S^T(q) \\ A^T(q) \end{pmatrix}$ we obtain the following equation :

$$\begin{aligned} J(q) \begin{pmatrix} \dot{\eta} \\ \varepsilon \dot{\mu} \end{pmatrix} &= \begin{pmatrix} S^T(q) \\ A^T(q) \end{pmatrix} [C(q, S \eta + A \varepsilon \mu)(S \eta + A \varepsilon \mu) + B(q)u] \\ &\quad - \begin{pmatrix} S^T(q) \\ A^T(q) \end{pmatrix} M(q) [s(q, \eta, \varepsilon \mu) + a(q, \eta, \varepsilon \mu)] \\ &\quad + \begin{pmatrix} 0 \\ -A^T(q)A(q)F \end{pmatrix} \end{aligned} \quad (2.28)$$

where $J(q)$ is a matrix defined as:

$$J(q) = \begin{pmatrix} S^T(q)M(q)S(q) & S^T(q)M(q)A(q) \\ A^T(q)M(q)S(q) & A^T(q)M(q)A(q) \end{pmatrix}. \quad (2.29)$$

Equations (2.1) and (2.28) constitute a singular perturbation formulation of the non ideal dynamical model of the robot. This means that when $\varepsilon = 0$, the equations are equivalent to the ideal dynamical model. The main results of singular perturbation theory are recalled in Sections 6 and 7 and applied to the above model. A formulation of these equations where the dependance in ε appears explicitly is needed. We therefore pursue the development of the model hereafter.

The $\dot{q}$ -quadratic (and therefore ε -quadratic) function $C(q, \dot{q})\dot{q}$ is developed in power of ε as follows

$$C(q, \dot{q})\dot{q} = C_0(q, \eta) + \varepsilon C_1(q, \eta, \mu) + \varepsilon^2 C_2(q, \eta, \mu) \quad (2.30)$$

where $C_1(q, \eta, \mu)$ and $C_2(q, \eta, \mu)$ are respectively linear and quadratic in μ .

We also develop the term $[s(q, \eta, \varepsilon \mu) + a(q, \eta, \varepsilon \mu)]$ in the following way

$$\begin{aligned} [s(q, \eta, \varepsilon \mu) + a(q, \eta, \varepsilon \mu)] &= \dot{S} \eta + \dot{A} \varepsilon \mu \\ &= s_0(q, \eta) + \varepsilon s_1(q, \eta, \mu) + \varepsilon a_1(q, \eta, \mu) + \varepsilon^2 a_2(q, \eta, \mu) \end{aligned} \quad (2.31)$$

where $s_1(q, \eta, \mu)$, $a_1(q, \eta, \mu)$ are linear functions in μ and $a_2(q, \eta, \mu)$ is quadratic in μ .

Using these developments, equation (2.28) is written as follows :

$$J(q) \begin{pmatrix} \dot{\eta} \\ \varepsilon \dot{\mu} \end{pmatrix} = \Psi_0(q, \eta) + \varepsilon \Psi_1(q, \eta, \mu) + \varepsilon^2 \Psi_2(q, \eta, \mu) + \begin{pmatrix} S^T \\ A^T \end{pmatrix} B(q)u + \begin{pmatrix} 0 \\ -A^T(q)A(q)F \end{pmatrix} \quad (2.32)$$

where • $\Psi_0(q, \eta) = \begin{pmatrix} S^T \\ A^T \end{pmatrix} [C_0(q, \eta) - Ms_0(q, \eta)]$;
 • $\Psi_1(q, \eta, \mu) = \begin{pmatrix} S^T \\ A^T \end{pmatrix} [C_1(q, \eta) - Ms_1(q, \eta, \mu) - Ma_1(q, \eta, \mu)]$
 and $\Psi_1(q, \eta, \mu)$ is a linear function in μ ;
 • $\Psi_2(q, \eta, \mu) = \begin{pmatrix} S^T \\ A^T \end{pmatrix} [C_2(q, \eta) - Ma_2(q, \eta, \mu)]$
 and $\Psi_2(q, \eta, \mu)$ is a quadratic function in μ .

It must be noticed that for $\varepsilon = 0$ equations (2.1) and (2.32) reduce to the model of the ideal robot in (1.8) and (1.15).

In order to obtain an explicit expression for $\dot{\eta}$ and $\varepsilon \dot{\mu}$, we multiply equation (2.32) by $J^{-1}(q) \triangleq R(q) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$ and we obtain

$$\begin{cases} \dot{\eta} &= f_0(q, \eta) + \varepsilon f_1(q, \eta, \mu) + \varepsilon^2 f_2(q, \eta, \mu) + B_1(q)u - R_{12}A^TAF \\ \varepsilon \dot{\mu} &= g_0(q, \eta) + \varepsilon g_1(q, \eta, \mu) + \varepsilon^2 g_2(q, \eta, \mu) + B_2(q)u - R_{22}A^TAF \end{cases} \quad (2.33)$$

$$\begin{aligned} \text{where } R\Psi_0 &= \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}; \quad R\Psi_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}; \quad R\Psi_2 = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}; \\ R \begin{pmatrix} S^T \\ A^T \end{pmatrix} B &= \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \text{ with } B_1 \text{ a full rank matrix.} \end{aligned} \quad (2.34)$$

Equations (2.1) and (2.33) constitute another way to write the singular perturbation formulation of the non ideal dynamical model of the robot that is used later.

We also use a shorter expression for the dynamical model that is given below :

$$\begin{cases} \dot{q} &= S(q)\eta + A(q)\varepsilon\mu \\ \dot{\eta} &= f(q, \eta, \mu, \varepsilon, u) \\ \varepsilon \dot{\mu} &= g(q, \eta, \mu, \varepsilon, u). \end{cases} \quad (2.35)$$

2.4 Properties of the non ideal model

We summarize here the properties of the non ideal model,

P1 : The robot is actuated in such a way that $S^T B$ is non singular and thus B_1 is a full rank matrix. In practice, this assumption corresponds to the fact that the robot is equipped with enough motors on the wheels and that these motors are adequately placed;

P2 : The inertia matrix $J(q)$ is positive definite,

$$J(q) \succ 0 \Rightarrow R(q) \succ 0 \Rightarrow R_{22}(q) \succ 0;$$

P3 : The functions f_1, g_1 are linear in μ and f_2, g_2 are quadratic in μ ;

P4 : All the quantities involved in equations (2.33) as well as their partial derivatives are bounded uniformly with respect to q . For instance, for $f_1(q, \eta, \mu)$, there exists functions $\sigma_i(\eta, \mu)$, $i = 1, \dots, 4$, such that

$$\|f_1(q, \eta, \mu)\| \leq \sigma_1(\eta, \mu), \quad \left\| \frac{\partial f_1}{\partial q} \right\| \leq \sigma_2(\eta, \mu),$$

$$\left\| \frac{\partial f_1}{\partial \eta} \right\| \leq \sigma_3(\eta, \mu), \quad \left\| \frac{\partial f_1}{\partial \mu} \right\| \leq \sigma_4(\eta, \mu),$$

and similar upper bounds for $J(q), R(q), R_{22}(q), f_0(q, \eta), f_2(q, \eta, \mu), g_0(q, \eta), g_1(q, \eta, \mu), g_2(q, \eta, \mu), B_1(q), B_2(q), K(q, \eta)$. This last property can be understood by the fact that invoked quantities depend on some components of q through sine and cosine functions.

P5 : If for each wheel and each force component, the slip ratios in the increasing part of the force characteristics (see figure 2.5), then the matrix $A^T A \frac{\partial F}{\partial \mu}$ is positive definite.

This property is crucial for the later developments but is only satisfied when the slip ratios are small enough. When all the slip ratios are such that they remain in the increasing part of the associated force component, we say that *the slip ratios are "small enough"* (see figure 2.5). For the linear forces model (model 1), this problem does not occur. However, the linearized force model has a limited physical validity area.

2.5 Reference example (continued)

2.5.1 Kinematic model

The three components of the vector $A^T(q)\dot{q}$ are proportional to the transversal sliding velocity that is the same for wheels 1 and 2, to the longitudinal sliding velocity of wheel 1 and to the longitudinal sliding velocity of wheel 2.

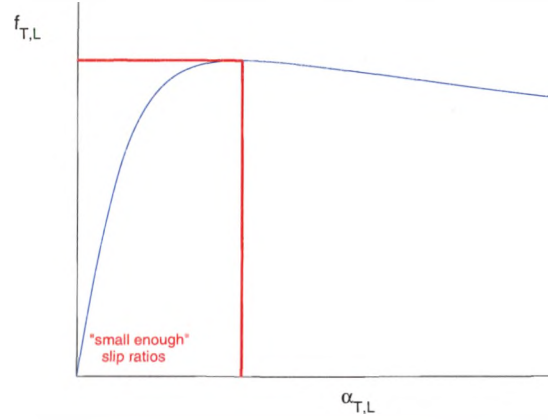


Figure 2.5: Illustration of the notion of "small enough" slip ratios

The non ideal kinematic model can be written under the following form :

$$\dot{q} = S(\theta) \eta + A(\theta) \varepsilon \mu \quad (2.36)$$

where • $S(\theta) = \begin{pmatrix} I_2 \\ \Sigma^T(\theta) \end{pmatrix}$ and $A(\theta) = \begin{pmatrix} -\Sigma(\theta) \\ I_3 \end{pmatrix}$;

• $\Sigma(\theta)$ is defined in equation (1.32);

$$\bullet \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

We now develop the two possible force models in detail.

2.5.2 Force model 1

According to (2.14), the generalized forces are modeled under the following form

$$F = \frac{K(q, \eta)}{\varepsilon} A^T(q) \dot{q}. \quad (2.37)$$

To express the stiffness matrix K , we derive the equations for the wheels velocities (2.5) and here the matrix L_i take the following form :

$$T = \begin{pmatrix} \frac{1}{\varepsilon} & 0 & 0 \\ \frac{l}{er} & -\frac{1}{r} & 0 \\ -\frac{l}{er} & 0 & -\frac{1}{r} \end{pmatrix} \quad L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} T^{-1} \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} T^{-1}. \quad (2.38)$$

In the linear approximation, both longitudinal and transversal effects are decoupled and the two components of the contact force are given by :

$$\begin{pmatrix} f_{Ti} \\ f_{Li} \end{pmatrix} = \frac{1}{\|v_{Ri}\| + \sigma} \begin{pmatrix} D & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} v_{Ti} \\ v_{Li} \end{pmatrix} \quad (2.39)$$

and are proportional to the violation of the constraints.

The power of the interaction forces f_{Ti} and f_{Li} associated with wheel i has the form :

$$W_i = \dot{q}^T A L_i^T \frac{1}{\|v_{Ri}\| + \sigma} \begin{pmatrix} D & 0 \\ 0 & G \end{pmatrix} L_i A^T \dot{q}. \quad (2.40)$$

We finally deduce expression for the generalized force associated to wheel i :

$$f_i = \frac{K_i(\theta, \eta)}{\varepsilon} A^T(\theta) \dot{q} \quad (2.41)$$

where $K_i(\theta, \eta)$ is defined by

$$K_i(\theta, \eta) = \frac{1}{\|v_{Ri}\| + \sigma} \begin{pmatrix} D_0 & 0 \\ 0 & G_0 \end{pmatrix} L_i \quad (2.42)$$

and where D_0 and G_0 are the reference parameter characterizing the family of contact forces we consider. The coefficients D and G are positive to make K_i a positive definite matrix. Making ε tend to zero is equivalent to make D and G tend to infinity.

We make the assumption that the cornering and slip coefficients are the same for both wheels. Then, neglecting the effect of the off-centered wheel, the generalized force is written :

$$F = L_1^T f_1 + L_2^T f_2 = \frac{K(\theta, \eta)}{\varepsilon} A^T(\theta) \dot{q} \quad (2.43)$$

where $K(\theta, \eta)$ is defined by

$$K(\theta, \eta) = T^{-T} \begin{pmatrix} \frac{D_0}{\|v_{R1}\| + \sigma} + \frac{D_0}{\|v_{R2}\| + \sigma} & 0 \\ 0 & \frac{G_0}{\|v_{R1}\| + \sigma} \\ 0 & 0 & \frac{G_0}{\|v_{R2}\| + \sigma} \end{pmatrix} T^{-1} \quad (2.44)$$

where $\sigma = 0.01$ allows us to avoid numerical problem for small values of $\|v_{Ri}\|$.

2.5.3 Force model 2

We model the forces using exactly the expression given in [21] for a dry asphalt road except that we do not consider in our model the self-aligning torque. We still make the assumption that the cornering and slip coefficients are the same for both wheels.

For the transversal component, the parameters are fixed as follows :

$$f_T = D_T \sin(C_T \arctan(B_T \phi_T)); \quad (2.45)$$

with

$$\phi_T = (1 - E_T) \alpha_T + \frac{E_T}{B_T} \arctan(B_T \alpha_T); \quad (2.46)$$

$$C_T = 1.30; \quad (2.47)$$

$$D_T = a_1 f_z^2 + a_2 * f_z; \quad (2.48)$$

$$B_{0T} = \frac{a_3 \sin(a_4 \arctan(a_5 f_z))}{C_T D_T}; \quad (2.49)$$

$$B_T = \frac{B_{0T}}{\varepsilon}; \quad (2.50)$$

$$E_T = a_6 f_z^2 + a_7 f_z + a_8; \quad (2.51)$$

$$a_1 = -22.1; \quad a_2 = 1011; \quad a_3 = 1078; \quad a_4 = 1.82; \quad (2.52)$$

$$a_5 = 0.208; \quad a_6 = 0.000; \quad a_7 = -0.354; \quad a_8 = 0.707; \quad (2.53)$$

where f_z represents the vertical load expressed in kN so that for our robot $f_z = 2.5kN$. In this expressions, the slip ratio α_T is expressed in degree.

The longitudinal component is modeled according to the following relations and choices of parameters :

$$f_L = D_L \sin(C_L \arctan(B_L \phi_L)); \quad (2.54)$$

with

$$\phi_L = (1 - E_L) \alpha_L + \frac{E_L}{B_L} \arctan(B_L \alpha_L); \quad (2.55)$$

$$C_L = 1.65; \quad (2.56)$$

$$D_L = b_1 f_z^2 + b_2 f_z; \quad (2.57)$$

$$B_{0L} = \frac{b_3 f_z^2 + b_4 f_z}{C_L D_L \exp(b_5 f_z)}; \quad (2.58)$$

$$B_L = \frac{B_{0L}}{\varepsilon}; \quad (2.59)$$

$$E_L = b_6 f_z^2 + b_7 f_z + b_8; \quad (2.60)$$

$$b_1 = -21.3; \quad b_2 = 1144; \quad b_3 = 49.6; \quad b_4 = 226; \quad (2.61)$$

$$b_5 = 0.069; \quad b_6 = -0.006; \quad b_7 = 0.056; \quad b_8 = 0.486; \quad (2.62)$$

where the slip ratio α_L is expressed in percent.

To obtain the generalized forces expression for the unicycle we only have to inject these relations in (2.11).

2.5.4 Values of the parameters to make the models comparable

For $f_z = 2.5kN$, the linearized model (model 1) parameter corresponding to the same slope at the origin as the non linear model (model 2) are $D_0 = 47300N$ and $G_0 = 73600N$. The equivalence may be verified injecting the above values in equation (2.18)-(2.19) for $f_z = 2.5kN$.

2.5.5 Non ideal dynamical model

The Lagrange equations for the non ideal unicycle are

$$M \ddot{q} = -AF + Bu \quad (2.63)$$

where the generalized forces term is known from the previous subsection.

According to equation (2.28), the non ideal dynamical model is derived under the form

$$\begin{pmatrix} \dot{\eta} \\ \varepsilon \dot{\mu} \end{pmatrix} = J^{-1} \begin{pmatrix} S^T \\ A^T \end{pmatrix} (Bu - M(s + a) - AF) \quad (2.64)$$

where $J(\theta)$ is the matrix defined by:

$$J = \begin{pmatrix} S^T M S & S^T M A \\ A^T M S & A^T M A \end{pmatrix} \quad (2.65)$$

with the following definitions :

$$s(q, \eta, \varepsilon \mu) = \dot{S} \eta \text{ with } \dot{S} = \frac{\partial S}{\partial \theta} \dot{\theta},$$

$$a(q, \eta, \varepsilon \mu) = \dot{A} \varepsilon \mu \text{ with } \dot{A} = \frac{\partial A}{\partial \theta} \dot{\theta}.$$

In order to obtain explicit expression for $\dot{\eta}$ and $\varepsilon \dot{\mu}$, we introduce the definition

$$J^{-1}(q) \triangleq R(q) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \text{ and we obtain}$$

$$\dot{q} = S \eta + A \varepsilon \mu \quad (2.66)$$

$$\dot{\eta} = (R_{11} S^T + R_{12} A^T) (Bu - M(s + a)) - A^T AF \quad (2.67)$$

$$\varepsilon \dot{\mu} = (R_{21} S^T + R_{22} A^T) (Bu - M(s + a)) - A^T AF. \quad (2.68)$$

2.6 Singular perturbation analysis

In this section we remind essential definitions and theorems about singular perturbation approach. This development is based on [27] (Ch9). This approach applied on the non ideal robot commanded with the nominal linearizing control law leads to a robustness result for the nominal law : the tracking error is ensured to be bounded (proven in [19]). This result is illustrated with some simulations for the unicycle.

2.6.1 Standard singular perturbation model

The singular perturbation model of a dynamical system is a state-space model in which the derivatives of some of the states are multiplied by a small positive parameter ε ; that is

$$\dot{x} = f(x, z, \varepsilon, t), \quad x \in \mathbb{R}^n \quad (2.69)$$

$$\varepsilon \dot{z} = g(x, z, \varepsilon, t), \quad z \in \mathbb{R}^m. \quad (2.70)$$

We assume that the functions f and g are continuously differentiable in their arguments for $(x, z, \varepsilon, t) \in D_1 \times D_2 \times [0, \varepsilon_0] \times [0, t_1]$, where $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^m$ are open connected sets. When we set $\varepsilon = 0$ in (2.69)-(2.70), the dimension of the state equation reduces from $n + m$ to n because the differential equation (2.70) degenerates into the equation

$$0 = g(x, z, 0, t). \quad (2.71)$$

We say that the model (2.69)-(2.70) is *in standard form* if and only if (2.71) has $k \geq 1$ isolated roots

$$z_i = h_i(x, t), \quad i = 1, \dots, k \quad (2.72)$$

for each $(x, t) \in D_1 \times [0, t_1]$. This assumption ensures that a well-defined n -dimensional reduced model corresponds to each root of (2.71). To obtain the i th reduced model, we substitute (2.72) into (2.69), at $\varepsilon = 0$, to obtain

$$\dot{x} = f(x, h(x, t), 0, t) \quad (2.73)$$

where we have dropped the subscript i from h . It is clear from the context which root of (2.72) we are using. This model is sometimes called *quasi-steady-state model* because z , whose velocity $\dot{z} = \frac{g}{\varepsilon}$ can be large when ε is small and $g \neq 0$, may rapidly converge to a root of (2.71) which is the equilibrium of (2.70). The model (2.73) is also known as the *slow model*.

2.6.2 Time-scale expression of the standard model

Let us consider the problem of solving the state equation

$$\dot{x} = f(x, z, \varepsilon, t), \quad x(t_0) = x_0 \quad (2.74)$$

$$\varepsilon \dot{z} = g(x, z, \varepsilon, t), \quad z(t_0) = z_0 \quad (2.75)$$

where $t_0 \in [0, t_1]$. Let $x(t)$ and $z(t)$ denote the solution of the full problem (2.74)-(2.75). The corresponding reduced model (2.73) is here written

$$\dot{x} = f(x, h(x, t), 0, t), \quad x(t_0) = x_0. \quad (2.76)$$

We denote the solution of (2.76) by $\bar{x}(t)$. Since the variable z is excluded from the reduced model and substituted by its "quasi-steady-state" $h(x, t)$, the only information we can obtain about z by solving (2.76) is to compute

$$\bar{z}(t) \stackrel{\text{def}}{=} h(\bar{x}(t), t) \quad (2.77)$$

which describes the quasi-steady-state behavior of z when $x = \bar{x}$. By contrast to the original variable z starting at t_0 from a prescribed z_0 , $\bar{z}$ is not free to start from a prescribed value, and there may be a large discrepancy between its initial value

$$\bar{z}(t_0) = h(x_0, t_0)$$

and the prescribed initial state z_0 . Thus, $\bar{z}(t)$ can not be a uniform approximation of $z(t)$. The best we expect is that the estimate

$$z(t) - \bar{z}(t) = O(\varepsilon)$$

holds on an interval excluding t_0 , that is, for $t \in [t_b, t_1]$ where $t_b > t_0$. On the other hand, it is reasonable to expect the estimate

$$x(t) - \bar{x}(t) = O(\varepsilon)$$

to hold uniformly for all $t \in [t_0, t_1]$ since $x(t_0) = \bar{x}(t_0) = x_0$.

It is more convenient in the analysis to perform the change of variables

$$y = z - h(x, t) \quad (2.78)$$

that shifts the quasi-steady-state of z to the origin. In the new variables (x, y) , the full problem is

$$\dot{\bar{x}} = f(x, y + h(x, t), \varepsilon, t) \quad \text{with} \quad x(t_0) = x_0 \quad (2.79)$$

$$\begin{aligned} \varepsilon \dot{\bar{y}} &= g(x, y + h(x, t), \varepsilon, t) - \varepsilon \frac{\partial h}{\partial x} f(x, y + h(x, t), \varepsilon, t) - \varepsilon \frac{\partial h}{\partial t} \\ &\quad \text{with} \quad y(t_0) = z_0 - h(x_0, t_0) = y_0. \end{aligned} \quad (2.80)$$

Let us note that $\varepsilon \dot{\bar{y}}$ remains finite even when ε tends to zero and $\dot{\bar{y}}$ tends to infinity. We set

$$\varepsilon \frac{dy}{dt} = \frac{dy}{d\tau}; \quad \text{hence} \quad \frac{d\tau}{dt} = \frac{1}{\varepsilon}$$

and use $\tau = 0$ as the initial value at $t = t_0$. The new time variable $\tau = \frac{t-t_0}{\varepsilon}$ is "stretched"; that is, if ε tends to zero, τ tends to infinity.

The variables t and x are slowly varying since, in the τ time scale, they are given by

$$\begin{aligned} t &= t_0 + \varepsilon\tau, \\ x &= x(t_0 + \varepsilon\tau). \end{aligned}$$

Equation (2.80) in the new τ time scale and for $\varepsilon = 0$ is written as

$$\frac{dy}{d\tau} = g(x, y + h(x, t), 0, t) \quad (2.81)$$

where (x, t) are treated as fixed parameters. We refer to (2.81) as the *boundary layer system*. The solution of this system is denoted $\bar{y}(\tau)$. The crucial stability property we need for this system is exponential stability of the origin, uniformly in the frozen parameters. This is discussed in the next subsection.

2.6.3 Tikhonov's theorem

Theorem 1 (Tikhonov) Consider the singular perturbation problem (2.69)-(2.70) and let $z = h(x, t)$ be an isolated root of (2.71). Assume that the following conditions are satisfied

T1: There exists $r, \rho, \varepsilon_0, t_1$ such that

- f, g and their partial derivatives with respect to x, y and ε are continuous in $B_r \times B_\rho \times [0, \varepsilon_0] \times [t_0, t_1]$ with $B_r = \{x \in \mathbb{R}^{2n-m} \mid \|x\| \leq r\}$ and $B_\rho = \{y \in \mathbb{R}^m \mid \|y\| \leq \rho\}$;
- the function $h(x, t)$ and $\frac{\partial g}{\partial y}(x, y, 0, t)$ have continuous first partial derivatives;
- the reduced system (2.76) has a unique solution $\bar{x}$ defined on $[t_0, t_1]$ which belongs to B_r .

T2: $\hat{y} = 0$ is an exponentially stable equilibrium of the boundary layer system (2.81), uniformly in the parameters x_0 and t_0 . Furthermore $y_0 - \bar{y}(t_0)$ belongs to its domain of attraction. This condition implies that $\hat{y}$ exists for $\tau \geq t_0$, and that

$$\lim_{\tau \rightarrow \infty} \hat{y}(\tau) = 0.$$

Then there exist positive constants ν_1, ν_2 and ε^* such that if $\|x_0\| < \nu_1$, $\|y_0 - h(x_0, t_0)\| < \nu_2$ and $\varepsilon < \varepsilon^*$, the following approximations are valid for $t \in [t_0, t_1]$:

$$x(t) = \bar{x}(t) + O(\varepsilon) \quad (2.82)$$

$$y(t) = \bar{y}(t) + \hat{y}(\tau) + O(\varepsilon) \quad (2.83)$$

Equation (2.83) implies that there exists $t_b > t_0$ such that the approximation

$$y(t) = \bar{y}(t) + O(\varepsilon)$$

is valid for $t \in [t_b, t_1]$.

Furthermore, if in addition the following condition holds,

T3: The origin of the reduced system (2.76) is exponentially stable;

then the approximations (2.82)-(2.83) are satisfied uniformly with respect to t_1 .

The proof of this theorem is detailed in chapter 9 of [27] p681-686.

2.6.4 Singular perturbation analysis for the point tracking problem

In this section, we apply the singular perturbation theory to analyze the closed-loop performance of the robot whose dynamics is given by (2.1) and (2.33) and for linearizing control law u_0 defined in (1.20)-(1.21). This is studied in detail in [19] for the approximated force model (Model 1) and we give here the key ideas of the development and extend these results to the generalized force model (Model 2).

Consider for the moment a general smooth control law $u(q, \eta, t)$ or $u(t)$. When $\varepsilon = 0$ in (2.33), the second equation becomes

$$g_0(q, \eta) + B_2(q)u(q, \eta, t) - R_{22}A^T AF(q, \eta, \mu) = 0 \quad (2.84)$$

and this allows us to compute uniquely the associated expression of μ . We denote this expression according to

$$\mu = H_0(q, \eta, t). \quad (2.85)$$

The assumption T1 of Tikhonov's theorem is thus satisfied.

We define the vector $y = \mu - H_0(q, \eta, t)$ and the dynamics in term of these new variables is written :

$$\begin{aligned} \varepsilon \dot{y} = & g_0(q, \eta) + \varepsilon g_1(q, \eta, y + H_0) + \varepsilon^2 g_2(q, \eta, y + H_0) + B_2(q)u(q, \eta, t) \\ & - R_{22}A^T(q)A(q)F(q, \eta, y + H_0) - \varepsilon \frac{\partial H_0}{\partial q} \dot{q} + \varepsilon \frac{\partial H_0}{\partial \eta} \dot{\eta} + \varepsilon \frac{\partial H_0}{\partial t}. \end{aligned} \quad (2.86)$$

For any given t_0 , we introduce the new time variable $\tau = \frac{t-t_0}{\varepsilon}$ so that the boundary layer system is written

$$\frac{dy}{d\tau} = g_0(q_0, \eta_0) + B_2(q_0)u(q_0, \eta_0, t_0) - R_{22}A^T AF(q_0, \eta_0, y + H_0) \quad (2.87)$$

where $q_0 = q(t_0)$ and $\eta_0 = \eta(t_0)$. The solution of this system is denoted $\hat{y}(\tau)$. From the definition of $H_0(q, \eta, t_0)$ in (2.84) and (2.85), it is clear that $\hat{y} = 0$ is an equilibrium of the system. The linear approximation of this system depends essentially of the term $F(q_0, \eta_0, y + H_0)$ that may be developed as follows

$$F(q, \eta, y + H_0) = F(q, \eta, H_0) + \left. \frac{\partial F}{\partial \mu} \right|_{y=0} y, \quad (2.88)$$

and using the equations (2.84) and (2.85), the boundary layer system (2.87) becomes

$$\frac{dy}{d\tau} = -R_{22}A^T A \left. \frac{\partial F}{\partial \mu} \right|_{(q_0, \eta_0, H_0)} y. \quad (2.89)$$

As far as the slip ratios are "small enough", property P5 implies that assumption T2 is satisfied.

For our choice of control law u_0 in (1.20) with the auxiliary control v in (1.21), assumption T3 is satisfied but only for the subsystem composed by equations for $\tilde{q}_1 = q_1 - q_{1r}$ and $\tilde{\eta} = \eta - \dot{q}_{1r}$:

$$\begin{pmatrix} \dot{\tilde{q}}_1 \\ \dot{\tilde{\eta}} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -K_1 & -K_2 \end{pmatrix} \begin{pmatrix} \tilde{q}_1 \\ \tilde{\eta} \end{pmatrix}. \quad (2.90)$$

Applying Tikhonov's theorem gives us a stability result only for q_1 and η . It can be verified that the dynamic in q_2 does not explode because of property P4 of the non ideal model.

The following theorem results in application of Tikhonov's theorem :

Robustness result of the nominal linearizing control law u_0

When applying the nominal feedback law u_0 defined by (1.20) and (1.21) to the non ideal system (2.28), for a given trajectory $(q_{1r}(t), \dot{q}_{1r}(t))$, there exist four positive constants ν_1 , ν_2 , ν_3 and ε^* such that tracking errors $\tilde{q}_1$ and $\tilde{\eta}$ have the following properties:

$$\begin{aligned} \text{For } & \|\tilde{q}_1\|_{t=t_0} < \nu_1 \\ & \|\tilde{\eta}\|_{t=t_0} < \nu_2 \\ & \|\mu - H_0(\tilde{q}_1, q_2, \eta, t)\|_{t=t_0} < \nu_3 \\ & \varepsilon < \varepsilon^* \end{aligned}$$

for $\hat{\mu}(\tau)$ the solution of the boundary layer system (2.89), we have that

- the standard approximations :

$$\begin{pmatrix} \tilde{q}_1(t) \\ \tilde{\eta}(t) \end{pmatrix} = e^{\Gamma t} \begin{pmatrix} \tilde{q}_1(t_0) \\ \tilde{\eta}(t_0) \end{pmatrix} + O(\varepsilon)$$

$$\mu(t) - H_0(\tilde{q}, \tilde{\eta}, t) = \hat{\mu}(\tau) + O(\varepsilon)$$

are valid on infinite time intervals;

- q_2 remains bounded;

where Γ is the following Hurwitz matrix :

$$\Gamma = \begin{pmatrix} 0 & I \\ -K_1 & -K_2 \end{pmatrix}.$$

It means that $\tilde{q}_1$, $\tilde{\eta}$ and the difference $\mu - H_0$ remain bounded if the kinematic constraints are not too much violated ($\varepsilon < \varepsilon^*$).

2.6.5 Reference example (continued)

This is now verified on simulations for the unicycle. We choose the physical parameters of the robot as follows :

$$e = 1m; \quad r = 0.35m; \quad l = 1m; \quad m_T = 1000kg; \quad I_\theta = 500kg.m^2; \quad I_\phi = 1.6kg.m^2. \quad (2.91)$$

We apply the nominal linearizing control law (1.20)-(1.21) to the non ideal unicycle and we choose the control gain matrices as $K_1 = \text{diag}(\omega^2, \omega^2)$, $K_2 = \text{diag}(2\zeta\omega, 2\zeta\omega)$ so that the tracking error dynamics are given for the two components of $\tilde{q}_1$ by an equation of a second order system

$$\ddot{x} = -\omega^2 x - 2\zeta\omega \dot{x}.$$

For all the simulations, the values of the gains are $\omega = 0.9(1/s)$ and $\zeta = 0.7$ so that the associated time constant is $\frac{1}{\omega} = 1.6s$.

The reference trajectory (in yellow) is a sine trajectory of amplitude 10 meters :

$$\begin{aligned} \xi_{1r}(t) &= t; \\ \xi_{2r}(t) &= 10\sin(\omega_{ref}t), \end{aligned}$$

where different values of the reference pulsation ω_{ref} are tested on simulations. The centripetal acceleration in the turns is proportional to ω_{ref}^2 and also the transversal component of the interaction forces. The larger the pulsation ω_{ref} is, the larger the transversal slip ratio is.

The initial position of the robot is at the origin and the initial orientation and velocity are *compatible*, i.e. the *initial position and velocity* correspond to the reference trajectory :

$$\begin{aligned} \xi_1(0) &= 0; \\ \xi_2(0) &= 0; \\ \dot{\xi}_1(0) &= 1; \\ \dot{\xi}_2(0) &= 10\omega_{ref}; \end{aligned} \quad (2.92)$$

$$\mu = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T. \quad (2.93)$$

Remark the choice of ε affects the slip ratios but does not influence the values of the contact forces necessary to follow the reference trajectory. This results from the introduction of the parameter ε as a compression coefficient in the forces model. For that reason, we always present the curves of the slip ratios normalized with respect to ε that is $\frac{1}{\varepsilon} \frac{\|v_{T,i}\|}{\|v_H\| + \sigma}$. For the considered vertical load (2500N), the value corresponding to the maximum of the force characteristics are equal to 2400 N for the transversal force, and 2700N for the longitudinal force. The associated normalized slip ratios are respectively equal to 0.16 and 0.115. These bounds are presented in all the slip ratios plots to show how far from the maximal value the sliding effects are.

The simulation results are presented for different values of the reference pulsation ω_{ref} to test the efficiency of the nominal linearizing control law and for the general force model

in dynamics. We also test the influence of different types of wheels changing the value of parameter ε . Remark here that the model choice does not influence the design of the control law : we saw in chapter 1 §1.2 that the force term could be eliminated from the dynamics equations.

Influence of the forces to apply

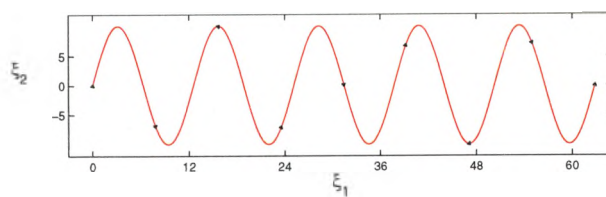
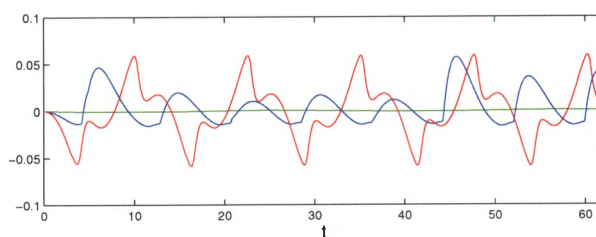
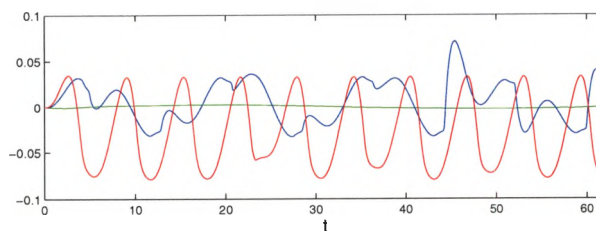
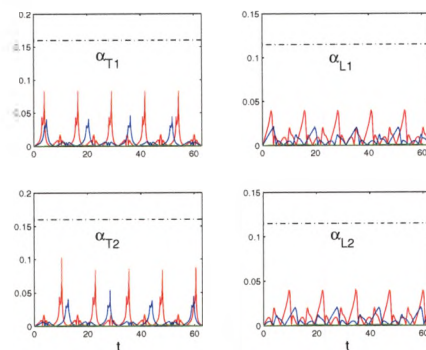
We present the simulations for three values of ω_{ref} and a fixed value of ε ($\varepsilon = 1$). In green, we present the curves associated with the simulations with $\omega_{ref} = 0.1(1/s)$; in blue, the curves associated with $\omega_{ref} = 0.4(1/s)$; and in red, the curves associated with $\omega_{ref} = 0.5(1/s)$.

u_0	$\varepsilon = 1$	$\omega_{ref} = 0.1(1/s)$
		$\omega_{ref} = 0.4(1/s)$
		$\omega_{ref} = 0.5(1/s)$

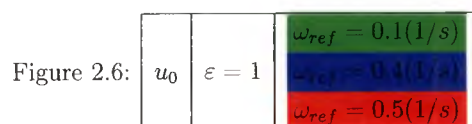
The corresponding values for the slip and corner coefficients are :

$$\begin{aligned} D &= 47300N, \\ G &= 73600N. \end{aligned}$$

We observe in figure 2.6 that the nominal linearizing control law achieves the tracking problem in a valuable way for small ω_{ref} . For higher values, corresponding to larger contact forces (see subsection 2.2.2 and figure 2.3), the closed-loop performance is degraded.

a) Trajectory (meters) for $\varepsilon = 1$ and $\omega_{ref} = 0.5(1/s)$ b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)



Influence of the wheel stiffness

We present the simulations for three values of ε and a fixed value of ω_{ref} ($\omega_{ref} = 0.1(1/s)$). In green, we present the curves associated with the simulations with $\varepsilon = 1$. The curves relative to the simulations with $\varepsilon = 10$ are presented in blue. For $\varepsilon = 100$, the curves are plotted in red.

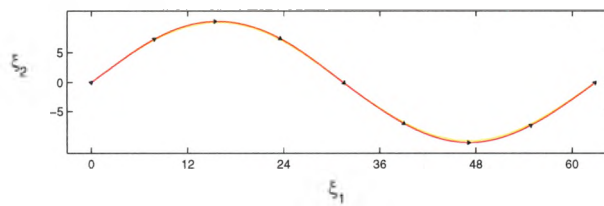
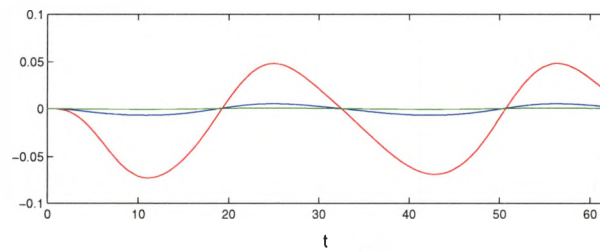
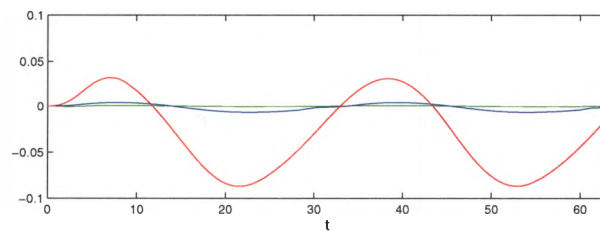
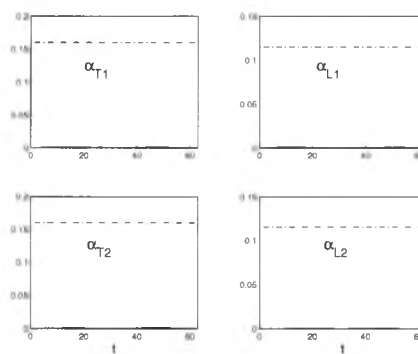
u_0		$\omega_{ref} = 0.1(1/s)$
	$\varepsilon = 1$ $\varepsilon = 10$ $\varepsilon = 100$	

Initial conditions and normalized stiffness coefficients remain the same so that comparison is possible. The corresponding values for the slip and corner coefficients are :

$$D = \frac{47300}{\varepsilon} N,$$

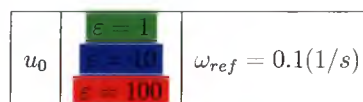
$$G = \frac{73600}{\varepsilon} N.$$

All the associated curves are plotted in figure 2.7 and even if the normalized slip ratios are tiny, we observe that for a larger value of ε , the performance of the ideal closed loop is resulting in increasing tracking error.

a) Trajectory (meters) for $\varepsilon = 100$ and $\omega_{ref} = 0.1(1/s)$ b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)

Figure 2.7:



2.6.6 Conclusion

Applying singular perturbation analysis to the robot allows us to show that if the dynamics of the robot are “good” enough and if we are not too far from the ideal case, the nominal linearizing control law ensures the tracking error to be bounded but not to converge towards zero. This fact is illustrated in the simulation results.

The nominal linearizing control law is designed on the basis of the ideal model, which is nothing but the full model (2.33) developed in power of ε , truncated at order zero. The assumptions on the model allow us to show that for small ε , the perturbation term (corresponding to term ε and ε^2) does not cause the divergence of the closed-loop tracking error.

In the following section, as well in the next chapter, use is made of power series developments, but not truncated at order zero.

2.7 Slow manifold analysis of non ideal robot model

In this section, we give a geometric interpretation of the two-time-scale behavior of the solution of a singularly perturbed problem. Further details can be found in [28] and chapter 9 of [27]. We extend to other fast dynamics the slow manifold procedure analyzed in [27]. We prove the robot model to have an associated manifold on which the dynamics is slow and this manifold has the important property to be the attractive.

2.7.1 Standard slow manifold procedure

We consider the singularly perturbed system (2.69)-(2.70):

$$\dot{x} = f(x, z, \varepsilon, t), \quad (2.94)$$

$$\varepsilon \dot{z} = g(x, z, \varepsilon, t). \quad (2.95)$$

Let $z = h(x, t)$ be an isolated root of $0 = g(x, z, 0, t)$ and suppose that assumptions of Tikhonov's theorem are satisfied for this root. Equation $z = h(x, t)$ describes an n -dimensional manifold in the $(n + m + 1)$ -dimensional space of (x, z, t) . It is an integral manifold of the system

$$\dot{x} = f(x, z, 0, t) \quad (2.96)$$

$$0 = g(x, z, 0, t) \quad (2.97)$$

because a trajectory of (2.96), (2.97) that starts in the manifold remain in the manifold for all future time. The motion on this manifold is described by the reduced model

$$\dot{x} = f(x, h(x, t), 0, t) \quad (2.98)$$

A consequence of Tikhonov's theorem is that trajectories of (2.96), (2.97) which start in an $O(\varepsilon)$ neighborhood of $z = h(x, t)$ remain within an $O(\varepsilon)$ neighborhood of $z = h(x, t)$.

This motivates the question : Is there an analog of the integral manifold $z = h(x, t)$ for $\varepsilon > 0$? We seek an integral manifold for (2.94)-(2.95) in the form

$$z = H_\varepsilon(x, t) \quad (2.99)$$

where $H_\varepsilon(x, t)$ is a sufficiently smooth function. For equation (2.99) to define an integral manifold, it must be true that

$$z(0) - H_\varepsilon(x(0), 0) = 0 \Rightarrow z(t) - H_\varepsilon(x(t), t) \equiv 0, \quad \forall t \in J \subset [0, \infty[\quad (2.100)$$

where J is any time interval over which the solution exists. Differentiating both sides of (2.99) with respect to t , multiplying by ε , and using (2.94)-(2.95) we obtain the *manifold condition*

$$\varepsilon \frac{\partial H_\varepsilon}{\partial x} f(x, H_\varepsilon(x, t), \varepsilon, t) + \varepsilon \frac{\partial H_\varepsilon}{\partial t} = g(x, H_\varepsilon(x, t), \varepsilon, t) \quad (2.101)$$

that $H_\varepsilon(x, t)$ must satisfy for all x in the region of interest and all $\varepsilon \in [0, \varepsilon_0]$. At $\varepsilon = 0$, the partial differential equation (2.101) degenerates into

$$0 = g(x, H_0(x, t), 0, t) \quad (2.102)$$

which shows that $H_0(x, t) = h(x, t)$. Since $0 = g(x, z, \varepsilon, t)$ may have more than one isolated root $z = h(x, t)$, we seek an integral manifold for (2.94)-(2.95) in the neighborhood of each root. It can be shown (see reference [6] in appendix B) that there exists $\varepsilon_1^* > 0$ and a function $H_\varepsilon(x, t)$ satisfying the manifold condition (2.101) for all $\varepsilon \in [0, \varepsilon_1^*]$ and

$$H_\varepsilon(x, t) - h(x, t) = O(\varepsilon) \quad (2.103)$$

for all $\|x\| < r$. The integral manifold $z = H_\varepsilon(x, t)$ is called a *slow manifold* for (2.94)-(2.95). For each slow manifold, there corresponds a slow model

$$\dot{x} = f(x, H_\varepsilon(x, t), \varepsilon, t) \quad (2.104)$$

which describes exactly the motion on that manifold.

In most cases, we can not solve the manifold condition (2.101) exactly, but we can approach $H_\varepsilon(x, t)$ arbitrarily closely as a power series of ε . The approximation procedure starts by substituting in (2.101) a power series of ε for $H_\varepsilon(x, t)$,

$$H_\varepsilon(x, t) = H_0(x, t) + \varepsilon H_1(x, t) + \varepsilon^2 H_2(x, t) + \dots \quad (2.105)$$

and calculating $H_0(x, t)$, $H_1(x, t)$, and so on, by equating terms of like powers of ε . This requires the functions f and g to be continuously differentiable in their arguments a sufficient number of times.

2.7.2 Existence of the slow manifold

Here, we study existence of the slow manifold and develop the manifold condition for our system.

Henceforth we consider smooth time-varying state feedback control of the form $u_\varepsilon^*(q, \eta, t)$, parametrized by ε , and not depending explicitly on μ . Then, according to subsection 2.7.1, the conditions ensuring the applicability of Tikhonov's theorem also guarantee the existence (for sufficiently small ε) of an invariant, integral manifold described by :

$$\mu = H_\varepsilon^*(q, \eta, t) \quad (2.106)$$

where H_ε^* is a smooth function parametrized by ε and depending on the choice of the control law $u_\varepsilon^*(q, \eta, t)$.

The invariance of the manifold implies that

$$\begin{aligned} \varepsilon \left[\frac{\partial H_\varepsilon^*}{\partial q} (S\eta + A\varepsilon H_\varepsilon^*) + \frac{\partial H_\varepsilon^*}{\partial \eta} (f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + B_1 u_\varepsilon^* - R_{12} A^T A F(q, \eta, H_\varepsilon^*)) + \frac{\partial H_\varepsilon^*}{\partial t} \right] \\ = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + B_2 u_\varepsilon^* - R_{22} A^T A F(q, \eta, H_\varepsilon^*) \end{aligned} \quad (2.107)$$

where the functions f_1 , g_1 , f_2 and g_2 are evaluated at $(q, \eta, H_\varepsilon^*(q, \eta, t))$.

In the ideal case ($\varepsilon = 0$) this condition reduces to

$$g_0 + B_2 u_0^* - R_{22} A^T A F(q, \eta, H_0^*) = 0 \quad (2.108)$$

and, for "small enough" slip ratios, from properties P2, P4 and P5, this ensures the existence and uniqueness of H_0^* for any u_0^* .

2.7.3 Attractivity of the slow manifold

We now show that, independently of the choice of the feedback control law $u_\varepsilon^*(q, \eta, t)$, the corresponding slow manifold is attractive, for sufficiently small ε and adequate value of the initial conditions, provided η remains bounded and provided the slip ratios are "small enough".

Proposition 2.1 *Consider a control law $u_\varepsilon^*(q, \eta, t)$ and the corresponding $H_\varepsilon^*(q, \eta, t)$ defining the associated slow manifold.*

For any $M > 0$, there exists $\varepsilon_2^*(M)$, and a function $R(\varepsilon)$, nonnegative, non increasing on $]0, \varepsilon_2^*]$ such that

- *for any ε verifying $0 < \varepsilon < \varepsilon_2^*(M)$;*
- *for any initial condition such that*

$$\|\mu(t_0) - H_\varepsilon^*(q(t_0), \eta(t_0), t_0)\| < R(\varepsilon);$$

there exists $c > 0$, such that

$$\|\mu(t) - H_\varepsilon^*(q(t), \eta(t), t)\| \leq c \|\mu(t_0) - H_\varepsilon^*(q(t_0), \eta(t_0), t_0)\| \quad (2.109)$$

and

$$\lim_{t \rightarrow \infty} \|\mu(t) - H_{\#}^*(q(t), \eta(t), t)\| = 0 \quad (2.110)$$

provided $\forall t \geq t_0$, $\|\eta(t)\| \leq M$ and provided the slip ratios remain "small enough".

The proof of this theorem is given in appendix A.3 and is strongly inspired by the proof given in [22]. Remark that the relation (2.109) only expresses the fact that no finite escape time occurs. The key argument allowing us to ensure the attractivity is the dissipativeness of the interaction forces (expressed by property P5).

As we proved the slow manifold to be attractive, we now intend to build a control law that achieves the tracking of a reference trajectory on the associated slow manifold. That is the subject of the next chapter.

Chapter 3

Control design

In this chapter we exploit the attractivity of the slow manifold associated with the non ideal robot model to design new control laws. The slow manifold approach is used in [22] and [23] to design a feedback control law taking into account the joint flexibility in robotic manipulators. This procedure is here extended to another type of fast dynamics, related to the contact effects for wheeled mobile robots. The new control laws are based on the ideal-model based linearizing control law and we build a correction term taking into account the fact that the robot is not ideal. These new control laws ensure the tracking error to be exponentially stable on the associated slow manifold and the attractivity property allows us to proof closed-loop stability. We propose two different extensions of the nominal linearizing control law. We first augment the nominal law with the objective to obtain output linearization of the system on the slow manifold. This approach was introduced in [22] and [23] for robotic manipulators. Second, we augment the nominal law applying a Liapounov redesign procedure. We present simulation results for the reference example and for the two new control laws.

3.1 Output linearizing feedback control

We derive here a state feedback control law, ensuring the asymptotic convergence of the output tracking error, incorporating the nature of the fast dynamics. We assume that the available measurements are the position of the robot, the rolling velocity, but not the sliding velocity. This control law is expressed as a series expansion in powers of ε , the singular perturbation parameter, whose first term is the control law corresponding to the ideal model. The result presented in this section has been published in [29] and [30] but these articles deal only with the linear force model (model 1).

3.1.1 Output linearizability on the slow manifold

In order to analyze the output linearization we define new variables :

$$y_1 = q_1 \tag{3.1}$$

$$y_2 = \dot{q}_1 = \eta - \Sigma(q)\varepsilon\mu, \quad (3.2)$$

such that the dynamics are described by the equations below

$$\dot{y}_1 = y_2 \quad (3.3)$$

$$\dot{y}_2 = \begin{pmatrix} I & -\Sigma \end{pmatrix} \begin{pmatrix} \dot{\eta} \\ \varepsilon\dot{\mu} \end{pmatrix} - \dot{\Sigma}\varepsilon\mu \quad (3.4)$$

where $\dot{\eta}$ and $\dot{\mu}$ are expressed according to equations (2.67) and (2.68). Our purpose is to construct a feedback control u_ε such that on the associated slow manifold (denoted H_ε), the system is output linearized as follows :

$$\dot{y}_1 = y_2 \quad (3.5)$$

$$\dot{y}_2 = v \quad (3.6)$$

where v is chosen the classical way to ensure the tracking. The existence of the linearizing control law u_ε is ensured by the proposition below.

Proposition 3.1 (Existence of the linearizing control law) *If in the dynamical equations (3.3)-(3.4) of the non ideal robot, the involved functions are C^1 ; Then there exists a real $\varepsilon_3 > 0$ such that for all $\varepsilon \in [0, \varepsilon_3]$ there exists a control law u_ε that linearizes the system on the corresponding slow manifold H_ε .*

The main arguments in the proof of this proposition are the existence of the linearizing control law in the ideal case and the continuity of all the functions with respect to parameter ε . The proof is obtained by extension of the results presented in [23] and may be found in appendix A.4.

3.1.2 Control design

We now come back to the output tracking problem : we look for a state feedback control law such that the output y_1 tracks a reference output signal $y_r(t)$. For the closed-loop dynamics, it amounts to ensure that the tracking error vector

$$\tilde{y} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} y_1 - y_r \\ y_2 - \dot{y}_r \end{pmatrix}$$

asymptotically converges towards zero.

We assume that measurements of q and $\dot{q}$ are available. This allows us to deduce the value of η , according to equation (2.1) :

$$\eta = (S^T S)^{-1} S^T \dot{q}. \quad (3.7)$$

In fact this is a projection of $\dot{q}$ on the subspace generated by the columns of $S(q)$. This projection filters out the slipping velocities components. These slipping velocities are

small and it can be expected that they can be deduced from $q, \dot{q}$ with a bad accuracy. It is the reason why we only use q and η in the feedback loop.

The control law is designed in order to assign linear stable dynamics for the tracking error on the associated slow manifold. Taking advantage of the linearizability property on the slow manifold, this goal is achieved by the following choice for v :

$$v = \ddot{y}_r(t) - K_1(y_1 - y_r) - K_2(\eta - \varepsilon \Sigma H_\varepsilon(q, \eta, t) - \dot{y}_r) \quad (3.8)$$

where $(\eta - \varepsilon \Sigma H_\varepsilon)$ is equal to $\dot{y}_1$ on the slow manifold. The auxiliary control can be rewritten as

$$v = v_0 + \varepsilon K_2 \Sigma H_\varepsilon(q, \eta, t) \text{ with } v_0(y_1, \eta, t) = \ddot{y}_r(t) - K_1(y_1 - y_r) - K_2(\eta - \dot{y}_r) \quad (3.9)$$

where v_0 that is the auxiliary control law of the ideal-model based control law. Our aim is now to build an additive correction term to the nominal linearizing control law so that the control law that linearizes the system on the slow manifold has the form :

$$u_\varepsilon = u_0 + \varepsilon \bar{u}. \quad (3.10)$$

3.1.3 Closed-loop analysis

With our control law, the closed-loop tracking error reduces to

$$\dot{\tilde{y}} = \Gamma \tilde{y} + \Pi(\varepsilon, q, \eta, \mu, t) \quad (3.11)$$

where

$$\Gamma = \begin{pmatrix} 0 & I \\ -K_1 & -K_2 \end{pmatrix} \quad (3.12)$$

$$\Pi(\varepsilon, q, \eta, \mu, t) = \begin{pmatrix} 0 \\ \Pi_b(\varepsilon, q, \eta, \mu, t) \end{pmatrix} \quad (3.13)$$

with

$$\begin{aligned} \Pi_b(\varepsilon, q, \eta, \mu, t) = & \left(f(q, \eta, \mu, \varepsilon, u_\varepsilon) - f(q, \eta, H_\varepsilon, \varepsilon, u_\varepsilon) \right) \\ & - \Sigma \left(g(q, \eta, \mu, \varepsilon, u_\varepsilon) - g(q, \eta, H_\varepsilon, \varepsilon, u_\varepsilon) \right) \\ & - \frac{d\Sigma}{dq} (S\eta + A\varepsilon\mu)^T \varepsilon\mu \\ & + \frac{d\Sigma}{dq} (S\eta + A\varepsilon H_\varepsilon)^T \varepsilon H_\varepsilon \\ & - K_2 \varepsilon \Sigma (\mu - H_\varepsilon). \end{aligned} \quad (3.14)$$

From property P4, the function $\Pi(\varepsilon, q, \eta, \mu, t)$ depends on the factor $(\mu - H_\varepsilon)$ according to the relation below

$$\begin{aligned} \mathbf{P6} : \|\Pi(\varepsilon, q, \eta, \mu, t)\| \leq & k_1(q, \eta, t) \|\mu - H_\varepsilon(q, \eta, t)\| \\ & + k_2(q, \eta, t) \|\mu - H_\varepsilon(q, \eta, t)\|^2 \end{aligned} \quad (3.15)$$

where k_1 and k_2 are bounded uniformly with respect to q for a given (η, t) . Consequently it is verified that $\Pi(\varepsilon, q, \eta, H_\varepsilon, t) = 0$.

The closed-loop analysis is led according to a Liapounov based approach in appendix A.5 and the resulting semi-global stability result is the following :

Proposition 3.2 (Closed-loop analysis) *For bounded reference output*

$$\forall t > t_0, \left\| \begin{pmatrix} y_r(t) \\ \dot{y}_r(t) \end{pmatrix} \right\| \leq c_r,$$

for given initial conditions $\bar{y}(t_0)$ and $(\mu(t_0) - H_\varepsilon(q_0, \eta_0, t_0))$ as far as the slip ratios remain "small enough"; there exists $\varepsilon_4 > 0$ such that $\forall \varepsilon < \varepsilon_4$,

- $\bar{y}$, η and μ are bounded;
- $\lim_{t \rightarrow \infty} (\mu - H_\varepsilon) = 0$ and $\lim_{t \rightarrow \infty} \bar{y} = 0$.

It must be noticed that the dynamics of the q_2 part of the generalized coordinates vector is not linearized :

$$\dot{q}_2 = \Sigma^T \eta + \varepsilon \mu. \quad (3.16)$$

Since q_2 is constituted only by angles involved in sine and cosine functions, there is no necessity to ensure boundedness of q_2 . But what is crucial is that for the closed-loop system η and μ are bounded, ensuring therefore the boundedness of $\dot{q}_2$.

3.1.4 Practical implementation

Two conditions must be verified for u_ε and the corresponding H_ε : the slow manifold condition (2.107) and the following output linearization condition

$$v_0 + \varepsilon K_2 \Sigma H_\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + B_1 u_\varepsilon - R_{12} A^T A F - \Sigma \varepsilon \dot{H}_\varepsilon - \dot{\Sigma} \varepsilon H_\varepsilon \quad (3.17)$$

where F, f_1, f_2, g_1, g_2 are evaluated at $(q, \eta, H_\varepsilon(q, \eta, t))$.

For $\varepsilon = 0$ (ideal case), these conditions reduce to

$$g_0 + B_2 u_0 - R_{22} A^T A F(q, \eta, H_0) = 0 \quad (3.18)$$

and

$$v_0 = f_0 + B_1 u_0 - R_{12} A^T A F(q, \eta, H_0). \quad (3.19)$$

This ensures the existence of a unique solution $u_0(q, \eta, t)$ and $H_0(q, \eta, t)$.

We now suggest a procedure to construct the linearizing control law as well as the corresponding H_ε assuming these functions to be analytic. We develop these functions under the form of power series expansion with respect to ε :

$$\begin{cases} u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \\ H_\varepsilon = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \dots \end{cases} \quad (3.20)$$

where $u_i = \frac{1}{i!} \left(\frac{\partial^i u_\varepsilon}{\partial \varepsilon^i} \right)_{\varepsilon=0}$ and $H_i = \frac{1}{i!} \left(\frac{\partial^i H_\varepsilon}{\partial \varepsilon^i} \right)_{\varepsilon=0}$.

Injecting these expressions in the linearization condition (3.17) and in the manifold condition (2.107), rearranging the terms in increasing powers of ε we obtain the following condition:

$$\begin{aligned} & (f_0 - \Sigma g_0) - (R_{12} - \Sigma R_{22}) A^T A F(q, \eta, H_0) + (B_1 - \Sigma B_2) u_0 \\ & + \varepsilon \left[\varphi_1(q, \eta, H_0) - (R_{12} - \Sigma R_{22}) A^T A \frac{\partial F}{\partial \mu} \Big|_{H_0} H_1 + (B_1 - \Sigma B_2) u_1 \right] \\ & + \varepsilon^2 \left[\varphi_2(q, \eta, H_0, H_1) - (R_{12} - \Sigma R_{22}) A^T A \frac{\partial^2 F}{\partial \mu^2} \Big|_{H_0} H_2 + (B_1 - \Sigma B_2) u_2 \right] \end{aligned} \quad (3.21)$$

$$+ \varepsilon^3 \dots = v_0 + \varepsilon K_2 \Sigma H_0 + \varepsilon^2 K_2 \Sigma H_1 + \dots \quad (3.22)$$

with adequate definitions of $\varphi_1, \varphi_2, \dots$. Similarly, the manifold condition is rewritten as:

$$\begin{aligned} & g_0 - R_{22} A^T A F(q, \eta, H_0) + B_2 u_0 \\ & + \varepsilon \left[\varrho_1(q, \eta, u_0, H_0, d_0 H_0) - R_{22} A^T A \frac{\partial F}{\partial \mu} \Big|_{H_0} H_1 + B_2 u_1 \right] \\ & + \varepsilon^2 \left[\varrho_2(q, \eta, u_0, u_1, H_0, H_1, d_0 H_0, d_0 H_1) - R_{22} A^T A \frac{\partial^2 F}{\partial \mu^2} \Big|_{H_0} H_2 + B_2 u_2 \right] \end{aligned} \quad (3.23)$$

$$+ \varepsilon^3 \dots = 0$$

where use is made of the operator d_0 that is the derivative with respect to time along the ideal dynamics :

$$d_0 f(q, \eta, t) = \frac{\partial f}{\partial q} S \eta + \frac{\partial f}{\partial \eta} \left(f_0 + B_1 u_0 - R_{12} A^T A F(q, \eta, H_0) \right) + \frac{\partial f}{\partial t}. \quad (3.24)$$

The components u_0, u_1 , are therefore obtained explicitly by solving a sequence of linear systems :

$$\begin{cases} (B_1 - \Sigma B_2) u_0 - (R_{12} - \Sigma R_{22}) A^T A F(q, \eta, H_0) = v_0 - f_0 + \Sigma g_0 \\ B_2 u_0 - R_{22} A^T A F(q, \eta, H_0) = -g_0 \end{cases} \quad (3.25)$$

$$\begin{cases} (B_1 - \Sigma B_2) u_1 - (R_{12} - \Sigma R_{22}) A^T A \frac{\partial F}{\partial \mu} \Big|_{H_0} H_1 = K_2 \Sigma H_0 - \varphi_1 \\ B_2 u_1 - R_{22} A^T A \frac{\partial F}{\partial \mu} \Big|_{H_0} H_1 = -\varrho_1 \end{cases} \quad (3.26)$$

At each step we compute u_i and H_i knowing φ_i, ϱ_i that depend only on q, η, H_{i-1} and $d_0 H_{i-1}$.

Each of these systems admits one unique solution if the associated matrix is invertible. The matrix corresponding to the systems can be decomposed in the form

$$\begin{pmatrix} B_1 - \Sigma B_2 & -(R_{12} - \Sigma R_{22}) A^T A \frac{\partial F}{\partial \mu} \Big|_{H_0} \\ B_2 & -R_{22} A^T A \frac{\partial F}{\partial \mu} \Big|_{H_0} \end{pmatrix} = \begin{pmatrix} I & -\Sigma \\ 0 & I \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} S^T B & 0 \\ A^T B & A^T A \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -\frac{\partial F}{\partial \mu} \Big|_{H_0} \end{pmatrix}$$

where every involved matrix is invertible. It allows us to conclude that each of the systems defining u_0 and H_0 , u_1 and H_1 , ... admits one unique solution.

In a first step we obtain $u_0(q, \eta, t)$ and $H_0(q, \eta, t)$. Then, evaluating φ_1 and ϱ_1 , we obtain $u_1(q, \eta, t)$ and $H_1(q, \eta, t)$ and so on. The augmented linearizing control law is obtained explicitly without solving any partial differential equation.

In practice, we truncate the infinite development (3.20) up to a finite order. We implement the control law developed up to order 1. Although the above analysis of this new control law looks complicated, the control law implementation is rather simple. This is developed below and a summarizing flow chart is presented in figure 3.1.

STEP 0 : Measurement of q , η and computation of v_0

We decide not to measure the sliding velocities (and thus μ) so that we only need the measurements of the position q and of the sound part of the velocity η . With these data we compute the nominal auxiliary control term v_0 according to equation (3.9).

STEP 1 : Computation of u_0 , $F(q, \eta, H_0)$, d_0u_0 and $d_0F(q, \eta, H_0)$

The nominal linearizing control law u_0 and the ideal generalized forces are computed solving the system (3.25). These equations are derived with respect to time along the ideal dynamics equations, to obtain d_0u_0 and $d_0F(q, \eta, H_0)$.

STEP 2 : Computation of H_0 and d_0H_0

For the forces model 1, the link between the generalized forces and H_0 is quite simple :

$$H_0 = (A^T A)^{-1} K^{-1} F(q, \eta, H_0); \quad (3.27)$$

and d_0H_0 is computed deriving this equation with respect to time for the ideal dynamics.

For the force model 2, the generalized forces are related to the slip ratios according to relations (2.11) and (2.15). These equations written for $\mu = H_0$ allows us to compute the associated v_{Ti} and v_{Li} for all i :

$$F(q, \eta, H_0) = \sum_i L_i^T \begin{pmatrix} f_{Ti}(c_{Ti}) \\ f_{Li}(c_{Li}) \end{pmatrix}. \quad (3.28)$$

We saw in (2.5) that these may be linked to H_0 according to equation below

$$\forall i, \begin{pmatrix} v_{Ti} \\ v_{Li} \end{pmatrix} = L_i^T A^T A \varepsilon H_0 \quad (3.29)$$

and it is used to compute H_0 . The expression of d_0H_0 is obtained deriving equations (3.28) and (3.29) along ideal dynamics :

$$d_0F(q, \eta, H_0) = \sum_i L_i^T \begin{pmatrix} d_0f_{Ti}(\alpha_{Ti}) \\ d_0f_{Li}(\alpha_{Li}) \end{pmatrix}, \quad (3.30)$$

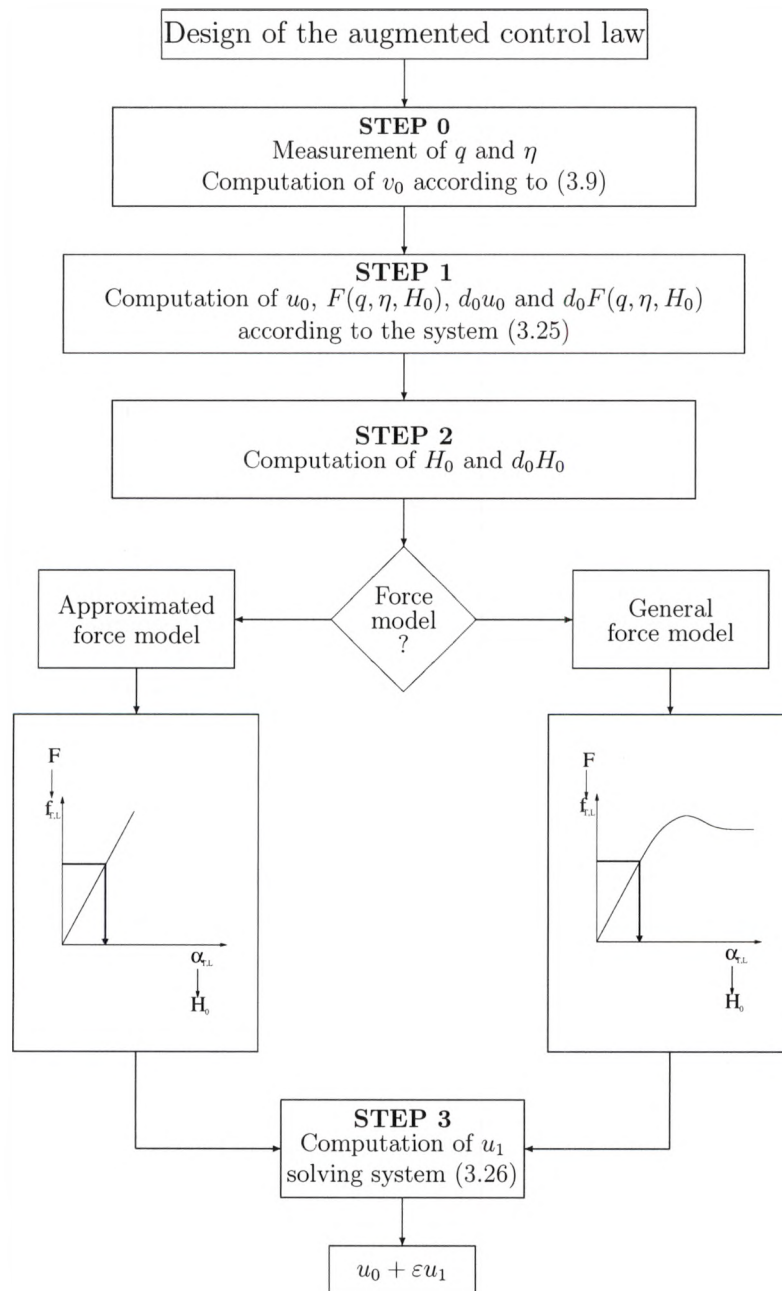


Figure 3.1: Flow chart for augmented linearizing control law design

and this equation allows us to compute $d_0 v_{Ti}$ and $d_0 v_{Li}$ for all i . The expression of $d_0 H_0$ is obtained deriving equation (3.29) for all i :

$$\begin{pmatrix} d_0 v_{Ti} \\ d_0 v_{Li} \end{pmatrix} = L_i^T (d_0 A^T A + A^T d_0 A) \varepsilon H_0 + L_i^T A^T A \varepsilon d_0 H_0. \quad (3.31)$$

STEP 3 : Computation of u_1

The correction control law u_1 is obtained solving the system (3.26) where use is made of the expression of H_0 and $d_0 H_0$.

3.1.5 Reference example (continued)

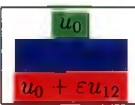
We present simulations for the unicycle and the sine reference trajectory of amplitude 10 meters introduced in chapter 2. For the simulation of the robot behavior, we use the dynamical equations based on the nonlinear force model (Model 2) while for control design, we use both models to compare the corresponding closed-loop performances. We compare the results obtained for the nominal linearizing control law (u_0 in green), for the augmented linearizing control law based on the linear force model (denoted $u_0 + \varepsilon u_{11}$ and associated with color blue in the figures) and for the augmented linearizing control law based on the general force model (denoted $u_0 + \varepsilon u_{12}$ and associated with color red in figures). We first present simulations for different values of ω_{ref} to determine whether it is worth to design the augmented linearizing control law with the interaction forces modeled according to the general model. Second, we analyze the influence of the stiffness of the wheels i.e. of the value of ε . The gains in the auxiliary control law v are chosen at the same value as in the previous simulations so that comparison is possible. The initial conditions are still compatible.

Influence of the force model in the control design

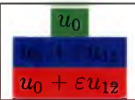
We give here the simulations for a fixed value of ε ($\varepsilon = 10$) and different values of ω_{ref} ($\omega_{ref} = 0.1, 0.4, 0.5(1/s)$).

u_0		
$u_0 + \varepsilon u_{11}$	$\varepsilon = 10$	$\omega_{ref} = 0.1(1/s)$
$u_0 + \varepsilon u_{12}$		

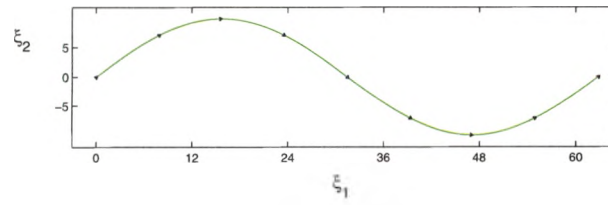
The results are summarized in figure 3.2. A decrease of the tracking error is observed from the use of the augmented control law, and the improvement is of the same order for the two augmented laws. It is not surprising because for small values of ω_{ref} , the contact forces are small and therefore the normalized slip ratios are tiny (see section 2.2.2 and especially figure 2.3). The linear force model is thus a good approximation of the general model.

	$\epsilon = 10$	$\omega_{ref} = 0.4(1/s)$
---	-----------------	---------------------------

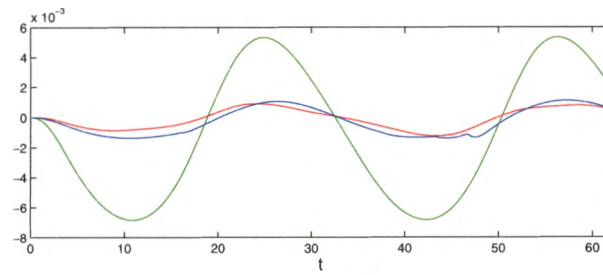
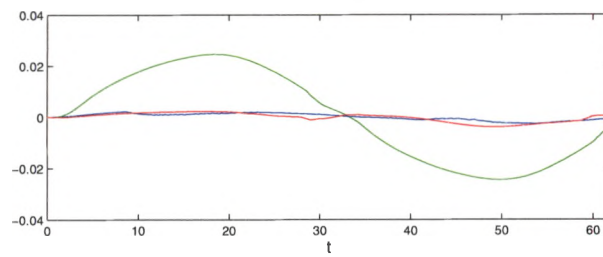
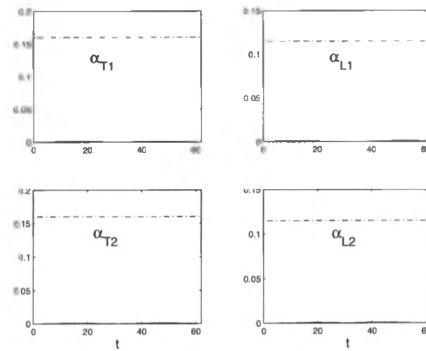
For this larger value of ω_{ref} , the contact forces and the normalized slip ratios necessary to follow the reference trajectory are larger. Then the linear and non linear force models become different. This affects the results of the augmented control law based on the linear force model as it may be observed on figure 3.3.

	$\epsilon = 10$	$\omega_{ref} = 0.5(1/s)$
---	-----------------	---------------------------

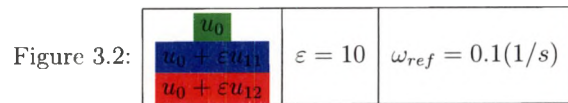
Finally, for reference trajectories that suppose the forces to have values near their maximal values (see figure 3.4), the augmented linearizing control law based on the linear force model becomes totally inefficient while the control law based on the general model still rather improves the closed-loop behaviour.

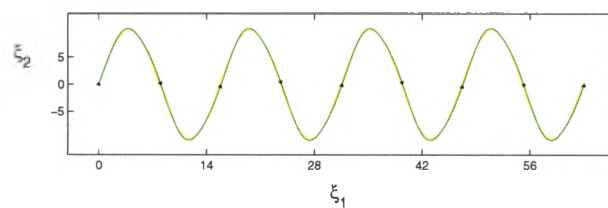


a) Trajectories (meters)

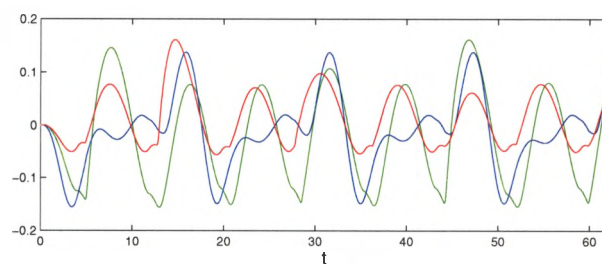
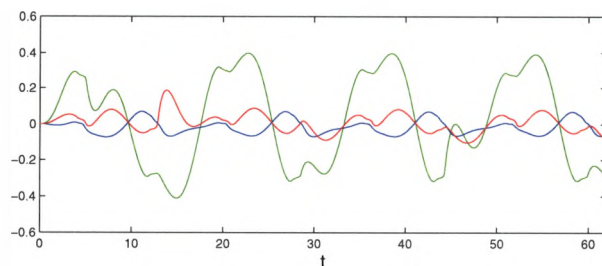
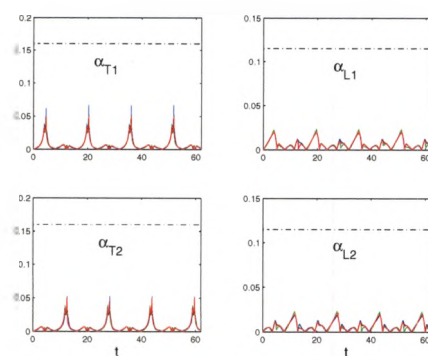
b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)

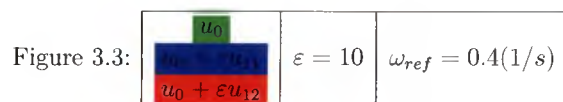


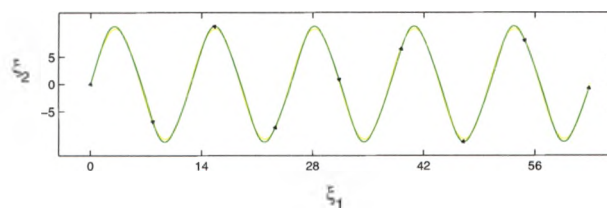


a) Trajectories (meters)

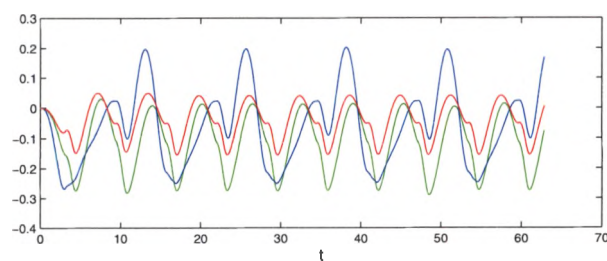
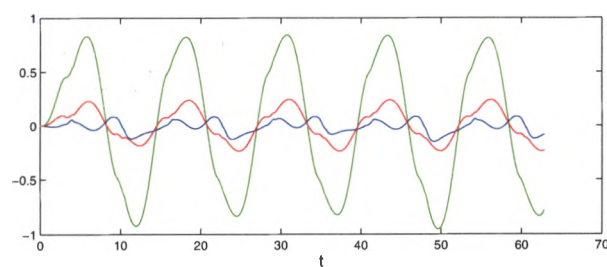
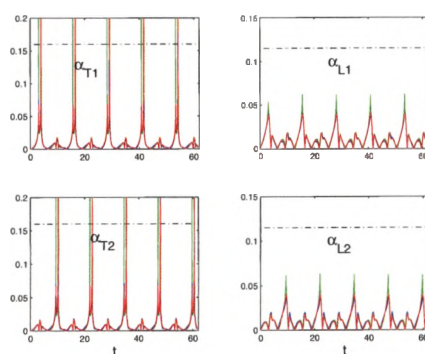
b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)

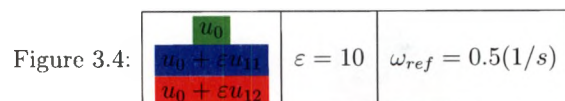




a) Trajectories (meters)

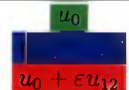
b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)



Influence of the wheels' stiffness

We simulated the non ideal robot for a fixed value of ω_{ref} ($\omega_{ref} = 0.1(1/s)$) and a larger values of the stiffness coefficient ε . Our aim is to observe whether the improvement from the new control law is preserved for low stiffness wheels.

	$\varepsilon = 100$	$\omega_{ref} = 0.1(1/s)$
---	---------------------	---------------------------

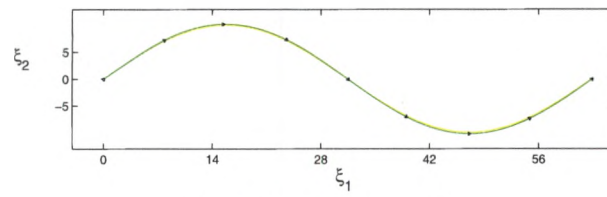
If we compare the results obtained in figure 3.2 associated with $\varepsilon = 10$ with the results of figure 3.5 for $\varepsilon = 100$, we observe that the efficiency of our augmented laws seems to be affected by large values of ε . This is probably due to the effect of the neglected terms in dynamics of order ε^2 that become significant.

Bibliothèque des sciences et technologies

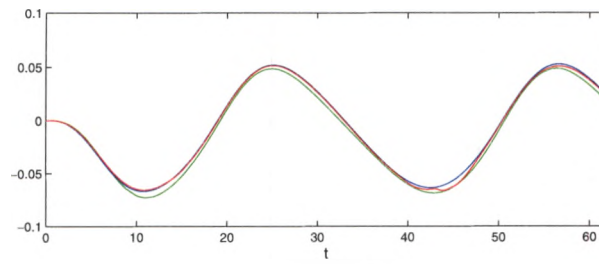
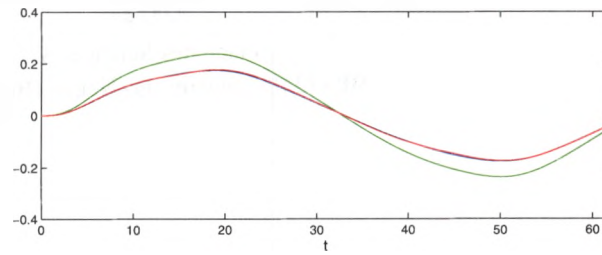
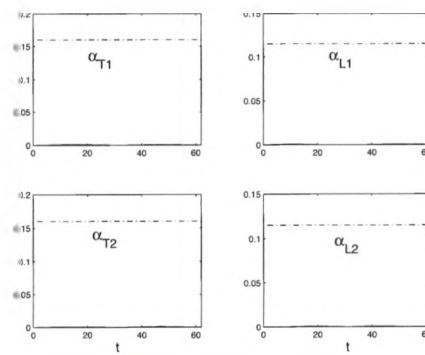
UCL

Place des Sciences 3

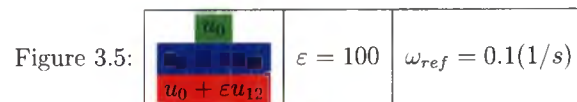
BE - 1348 Louvain-la-Neuve (Belgium)



a) Trajectories (meters)

b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)



3.1.6 Conclusion

The simulations show that for our example, the tracking performance is improved by the use of the augmented linearizing control law. For trajectories leading to large values of the slip ratios, the linear force model is no longer a good approximation of the non linear force model and it seems to be worth to design the correction control term (see section 2.2.2) according to the general force model. The stiffness coefficient ε does affect the efficiency of the linearizing control laws.

We improve the closed-loop behavior even with a truncated development of the control law. It is possible to generate the control term of order ε^2 but it would demand a lot more computations.

A major drawback of this approach is that we have to know the exact value of the stiffness parameter ε to implement the control law. Furthermore, an adaptive version of the control design would suppose us to measure the sliding velocity depending on $\varepsilon\mu$. In the next section, we therefore adopt another approach based on the Liapounov redesign technique.

3.2 Output tracking by Liapounov redesign

In the Liapounov redesign approach, we do not compensate the non linear term, but design the correction law in order to ensure a Liapounov function to be decreasing. The non ideal terms appearing in the model are still considered as perturbations but we try to dominate these terms instead of compensating them exactly. The advantages to be expected from this approach are that it does not require truncature and that it is more robust with respect to force model uncertainties.

In a Liapounov redesign approach the starting point is a nominal control law that exponentially stabilizes the tracking error dynamics of the ideal robot, and a associated Liapounov function. Among the possible Liapounov redesign techniques, constructive nonlinear methods (see [24]) propose to modify the Liapounov function (and of course the control law) in order to ensure the tracking error to be asymptotically decreasing on the slow manifold. In this work, we decide to design the control law in such a way that the original Liapounov function is decreasing on the corresponding slow manifold. We also restrict the analysis to a particular control law : the ideal-model based linearizing control law and the usual associated Liapounov function. But all the developments hold for other nominal control law as far as it exponentially stabilizes the tracking error for the ideal robot dynamics.

3.2.1 Standard Liapounov redesign

For the ideal robot, the control law u_0 in (1.20) with auxiliary control v_0 defined by

$$v_0 = \ddot{y}_r(t) - K_1(y_1 - y_r) - K_2(\eta - \dot{y}_r) \quad (3.32)$$

ensures that the tracking error $\bar{y}$ converges exponentially towards zero.

We therefore define a Liapounov function V_0 associated with the ideal closed-loop with nominal linearizing control law u_0 in (1.20) and (3.32) :

$$V_0(\bar{y}) = \frac{1}{2} \bar{y}^T P \bar{y} \quad (3.33)$$

where P is a real symmetric positive definite matrix. Without loss of generality, we choose P such that

$$\dot{V}_0 = -\|\bar{y}\|^2. \quad (3.34)$$

For a non ideal robot ($\varepsilon > 0$), we define a modified closed-loop control law of the form

$$u_\varepsilon(q, \eta, t) = u_0(q, \eta, t) + \varepsilon \bar{u}(q, \eta, t) \quad (3.35)$$

where the correction term $\varepsilon \bar{u}$ depends on q, η and not on μ . The function $\bar{u}$ is chosen such that on the corresponding slow manifold, characterized by $\mu = H_\varepsilon(q, \eta, t)$, the function V_0 is still a Liapounov function. This will ensure that on the slow manifold the tracking error converges exponentially towards zero. This property together with the attractivity of the slow manifold will enable us to proof the convergence towards zero of the tracking error.

For any choice of the function $\bar{u}(q, \eta, t)$, the function H_ε tends, for ε tending to zero, towards the function $H_0(q, \eta, t)$ defined as the solution of the implicit equation

$$g_0(q, \eta) + B_2(q)u_0(q, \eta, t) = R_{22}(q)A^T(q)A(q)F(q, \eta, H_0(q, \eta, t)). \quad (3.36)$$

This allows us to write H_ε under the following form

$$H_\varepsilon(q, \eta, t) = H_0(q, \eta, t) + \varepsilon H_1(q, \eta, t) + \varepsilon^2 \bar{H}(q, \eta, t, \varepsilon). \quad (3.37)$$

It must be pointed out, however, that the function H_ε , and therefore H_1 and $\bar{H}$, depend on the choice of the function $\bar{u}(q, \eta, t)$.

With the feedback control law (3.35), the η dynamics are described by (2.33) and using the fact that u_0 satisfies (3.19), it becomes

$$\dot{\eta} = v_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon B_1 \bar{u} - R_{12} A^T A (F(q, \eta, H_\varepsilon) - F(q, \eta, H_0)). \quad (3.38)$$

Consequently, the y_1, y_2 dynamics can then be written under the form

$$\dot{y}_1 = y_2 \quad (3.39)$$

$$\dot{y}_2 = \dot{\eta} - \Sigma \varepsilon \dot{\mu} - \dot{\Sigma} \varepsilon \mu \quad (3.40)$$

$$= v_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon B_1 \bar{u} - R_{12} A^T A (F(q, \eta, H_\varepsilon) - F(q, \eta, H_0)) - \Sigma \varepsilon \dot{\mu} - \dot{\Sigma} \varepsilon \mu. \quad (3.41)$$

After a few manipulations, the closed-loop becomes

$$\dot{\bar{y}} = \Gamma \bar{y} + \bar{B}_1 [\varepsilon \bar{u}(q, \eta, t) + \delta(\varepsilon, q, \eta, t, \mu)] \quad (3.42)$$

with

$$\bar{B}_1 = \begin{pmatrix} 0 \\ B_1 \end{pmatrix} \quad (3.43)$$

$$\begin{aligned} \delta(\varepsilon, q, \eta, t, \mu) = B_1^{-1} & [\varepsilon K_2 \Sigma \mu + \varepsilon f_1(q, \eta, \mu) + \varepsilon^2 f_2(q, \eta, \mu) \\ & - R_{12} A^T A (F(q, \eta, \mu) - F(q, \eta, H_0)) \\ & - \varepsilon \Sigma \dot{\mu} - \varepsilon \dot{\Sigma} \mu]. \end{aligned} \quad (3.44)$$

where the perturbation term $\delta(\varepsilon, q, \eta, t, \mu)$ enters the state equation exactly at the point where the control variable enters. The perturbation term is said to satisfy the *matching condition*. This structural property is crucial to design a control law by Liapounov redesign approach.

The Liapounov function behavior for non ideal robot is described as

$$\dot{V}_0 = -\|\bar{y}\|^2 + \bar{y}^T P \bar{B}_1 (\varepsilon \bar{u} + \delta(\varepsilon, q, \eta, t, \mu)). \quad (3.45)$$

For trajectories belonging to the corresponding slow manifold, V_0 is decreasing as far as

$$-\|\bar{y}\|^2 + \bar{y}^T P \bar{B}_1 (\varepsilon \bar{u} + \delta(\varepsilon, q, \eta, t, H_\varepsilon)) \leq 0. \quad (3.46)$$

This is referred to as *the stability condition* and $\bar{u}$ will be designed in such a way that this condition is satisfied on the associated slow manifold.

Remark that the correction term $\varepsilon \bar{u}$ is involved indirectly in the perturbation term δ through its relationship with the function H_ε characterizing the corresponding slow manifold. On the other hand the implementation of the standard Liapounov redesign procedure (see for instance [27] Ch13) requires two properties :

- the perturbation term (here δ) is bounded;
- the effect of the control modification (here $\varepsilon \bar{u}$) on the perturbation is bounded.

Unfortunately, it is not possible in our case to exhibit such bounds because an explicit expression of H_ε is not available. But it is possible to develop the perturbation term δ as a series expansion in powers of ε and to design the correction law considering only the first order term. Obviously the result we expect from this approach is weaker than the standard result.

Lemma 3.1 *For the non ideal robot dynamics defined by equations (3.42), (3.43) and (3.44), $\delta(\varepsilon, q, \eta, t, H_\varepsilon)$ can be expressed as :*

$$\begin{aligned} \delta(\varepsilon, q, \eta, t, H_\varepsilon) = & -\varepsilon B_1^{-1} R_{12} R_{22}^{-1} B_2 \bar{u} \\ & + \varepsilon \Delta(q, \eta, t) + \varepsilon^2 \rho(q, \eta, t, \varepsilon), \end{aligned} \quad (3.47)$$

where

$$\begin{aligned} \Delta(q, \eta, t, H_0) = & B_1^{-1} \left(K_2 \Sigma H_0 + f_1(q, \eta, H_0) - \left(\frac{d\Sigma}{dq} S \eta \right) H_0 \right) \\ & - B_1^{-1} R_{12} R_{22}^{-1} \left(g_1(q, \eta, H_0) - d_0 H_0(q, \eta, t) \right) - B_1^{-1} \Sigma d_0 H_0(q, \eta, t); \end{aligned} \quad (3.48)$$

and where $\rho(q, \eta, t, \varepsilon)$ is polynomial in ε .

The developments of the proof of this lemma are given in appendix A.6.

These relations enable us to rewrite the stability condition under the form below

$$-\|\bar{y}\|^2 + \bar{y}^T P \bar{B}_1 (\bar{B}\bar{u} + \Delta + \varepsilon \rho(q, \eta, t, \varepsilon)) \leq 0 \quad (3.49)$$

where

$$\bar{B} = I - B_1^{-1} R_{12} R_{22}^{-1} B_2 \quad (3.50)$$

is an invertible matrix.

The first idea for this condition to be verified is to choose $\bar{u}$ such that

$$\bar{B}\bar{u} + \Delta = -\bar{B}_1^T P \bar{y}. \quad (3.51)$$

But when the tracking error $\bar{y}$ becomes small, the unknown term $\varepsilon \rho(q, \eta, t, \varepsilon)$ may dominate and make the condition violated.

That is the reason why we consider another expression for $\bar{u}$, ensuring that $\bar{B}\bar{u} + \Delta$ has a "good" direction and is large enough to dominate the $\varepsilon \rho$ term.

This is achieved by the control law introduced in the proposition below :

Proposition 3.3 (Liapounov redesign) *For a non ideal mobile robot described by (2.33), for bounded values of the velocity vector η and for a given positive constant K , there exists $\varepsilon_5 > 0$ such that for all $\varepsilon < \varepsilon_5$, when we apply the Liapounov redesign control law $u = u_0 + \varepsilon \bar{u}$ where u_0 defined by (1.20) and where $\bar{u}$ satisfies*

$$\bar{B}\bar{u} + \Delta = -K \frac{\bar{B}_1^T P \bar{y}}{\|\bar{B}_1^T P \bar{y}\|} \quad (3.52)$$

the stability condition is verified and the function V_0 satisfies

$$\dot{V}_0 \leq -\|\bar{y}\|^2 \quad (3.53)$$

along trajectories belonging to the associated slow manifold.

The proof of this proposition is given in appendix A.7. This result has only a theoretical interest : it does not provide a systematic method to select K ensuring the tracking error convergence towards zero (like for standard Liapounov redesign). It proves that it is worth to implement the method with adequately chosen K . This choice must be oriented by systematic simulation and trial and error procedure.

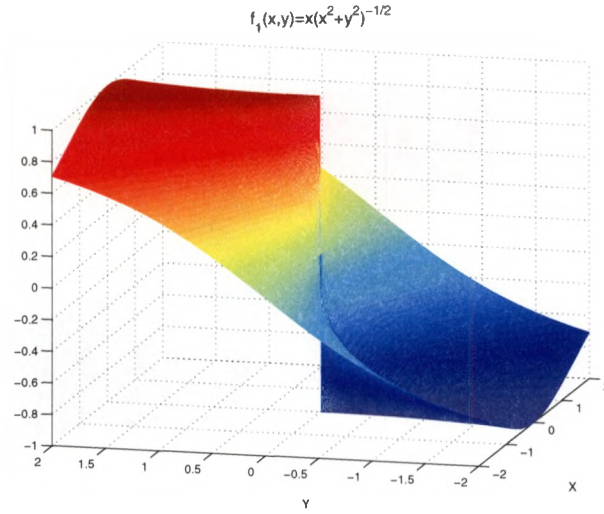


Figure 3.6: Graphic of one component of the function to be smoothed

3.2.2 Smoothed Liapounov redesign

It is important to remark that the expression of $\bar{u}$ in (3.52) depends on the tracking error through a kind of sign function. As we want the tracking error to be as small as possible, the correction control law $\bar{u}$ is discontinuous. This is in contradiction with the assumptions for the slow manifold existence and attractivity. Therefore, the control law has to be smoothed resulting in a slight restriction in the stability result :

Proposition 3.4 (Smoothed Liapounov redesign) *For a non ideal mobile robot described by (2.33), for bounded values of the velocity vector η and for a given positive constant K , there exists $\varepsilon_5 > 0$, such that for all $\varepsilon < \varepsilon_5$, when we apply the smoothed Liapounov augmented control law $u = u_0 + \varepsilon \bar{u}$ where u_0 defined by (1.20) and where $\bar{u}$ satisfies the relation*

$$\bar{B}\bar{u} + \Delta = -K \frac{\bar{B}_1 P \bar{y}}{\sqrt{\|\bar{B}_1 P \bar{y}\|^2 + \gamma^2}} \quad (3.54)$$

with γ that is a small positive constant, the stability condition is satisfied when $\|\bar{y}\|^2 > \varepsilon \frac{K\gamma}{2}$ i.e. outside a ball of radius $\sqrt{\varepsilon \frac{K\gamma}{2}}$. More precisely, the function V_0 satisfies

$$\dot{V}_0 \leq -\|\bar{y}\|^2 + \varepsilon \frac{K\gamma}{2} \quad (3.55)$$

along trajectories belonging to the associated slow manifold.

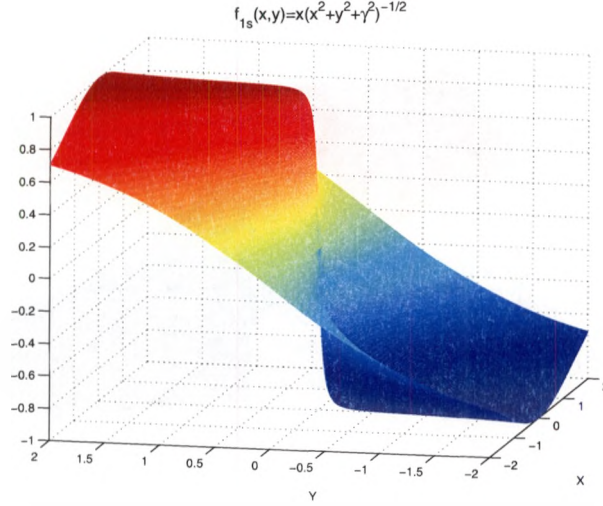


Figure 3.7: Graphic of one component of the smoothed function

The proof of this theorem is given in appendix A.8. Remark that the constants K and associated ε_5 are the same as in the non smoothed version. In return, the stability condition is satisfied only outside a ball.

The smoothing phenomenon is illustrated for dimension 2 vectors comparing the graphic representation of one component of the problematic function $f(x, y) = \frac{(x, y)}{\sqrt{x^2 + y^2}}$. Figure 3.6 presents the graph of the first component of $f(x, y)$: $f_1(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$ where we observe a discontinuity in $(0, 0)$. Figure 3.7 presents the graph of smoothed $f_1(x, y)$ denoted $f_{1s}(x, y)$ defined by $f_{1s}(x, y) = \frac{x}{\sqrt{x^2 + y^2 + \gamma^2}}$ with $\gamma = 0.01$.

Remark that we have to reach an acceptable trade off between the tolerance on the tracking error and the smoothing of the control law. The smoother the control law is, the easier to implement it is but the larger the tracking error tolerance remains.

3.2.3 Closed-loop analysis

From now on, we work with the smoothed Liapounov augmented control law so that the associated slow manifold exists. For this control law, the actual closed-loop tracking error reduces to

$$\dot{\tilde{y}} = \Gamma \tilde{y} + \bar{B}_1 (\varepsilon \bar{u} + \delta(\varepsilon, q, \eta, t, \mu)). \quad (3.56)$$

To analyze the closed-loop dynamics, we make the following transformation in the above equation :

$$\dot{\tilde{y}} = \Gamma \tilde{y} + \bar{B}_1 (\varepsilon \bar{u} + \delta(\varepsilon, q, \eta, t, H_\varepsilon)) + \Pi(\varepsilon, q, \eta, t, \mu) \quad (3.57)$$

where

$$\Pi(\varepsilon, q, \eta, t, \mu) = \bar{B}_1(\delta(\varepsilon, q, \eta, t, \mu) - \delta(\varepsilon, q, \eta, t, H_\varepsilon)). \quad (3.58)$$

Consequently it is verified that $\Pi(\varepsilon, q, \eta, H_\varepsilon, t) = 0$.

Due to property P4, it can be checked that the function $\Pi(\varepsilon, q, \eta, t, \mu)$ depends on the factor $(\mu - H_\varepsilon)$ according to the relation below

$$\begin{aligned} \text{P6 : } \|\Pi(\varepsilon, q, \eta, t, \mu)\| &\leq k_1(q, \eta, t) \|\mu - H_\varepsilon(q, \eta, t)\| \\ &\quad + k_2(q, \eta, t) \|\mu - H_\varepsilon(q, \eta, t)\|^2 \end{aligned} \quad (3.59)$$

where k_1 and k_2 are bounded uniformly with respect to q for a given (η, t) .

For the dynamics (3.57), the function V_0 verifies

$$\dot{V}_0 \leq -\|\tilde{y}\|^2 + \tilde{y}^T P \Pi(\varepsilon, q, \eta, t, \mu) + \varepsilon \frac{K\gamma}{2}, \quad (3.60)$$

because we saw in proposition 3.4 that the smoothed Liapounov augmented control law ensures that the function V_0 satisfies (3.55) on the associated slow manifold. We prove in appendix A.9 that for any initial conditions, the function V_0 is decreasing when $\|\tilde{y}\|^2$ is large enough and when the smoothing parameter γ is small enough. This result is summarized in the following proposition :

Proposition 3.5 (Closed-loop analysis) *For all $K > 0$, Liapounov redesign parameter, there exists $c_r > 0$ such that for bounded reference output*

$$\forall t > t_0, \left\| \begin{pmatrix} y_r(t) \\ \dot{y}_r(t) \end{pmatrix} \right\| \leq c_r,$$

for any initial conditions $\tilde{y}(t_0)$ and $(\mu(t_0) - H_\varepsilon(q_0, \eta_0, t_0))$, there exists $\varepsilon_6, \gamma_{max}, \rho_1, \rho_2 > 0$ and $\alpha > 1$, such that for all $\varepsilon < \varepsilon_6$, for $\gamma < \gamma_{max}$

- $\tilde{y}$, η and μ are bounded;

- $\lim_{t \rightarrow \infty} (\mu - H_\varepsilon) = 0$ and for t tending to infinity, $\|\tilde{y}\|^2$ is upper bounded by a function tending to $\varepsilon \frac{\rho_1 \alpha K \gamma}{2 \rho_2}$

as far as the slip ratios remain "small enough".

Remark that this result is true as far as the velocity of each wheel of the robot does not become too small. If one of the wheel is near to stop, our force model is no longer valid. But the restriction only comes from our force model and not from the control design. The theoretical results do not give a method to obtain the Liapounov parameter K and it is chosen by systematic simulation and trial and error procedure.

3.2.4 Robustness against uncertainties on ε

From the definition of the Liapounov augmented control law, we hope that it is robust with respect to errors on the value of ε . This result is not as evident as thought at first glance because it supposes to know the ε -dependence on the term $\rho(q, \eta, t, \varepsilon)$. This term can not be evaluated so that the result we obtain is only a theoretical result.

We design the control terms u_0 according to (1.20) and $\bar{u}$ following smoothed Liapounov redesign approach in (3.54). But we now implement the control law $u_0 + \hat{\varepsilon}\bar{u}$ for the actual robot characterized by the true ε . The stability condition then becomes

$$-\|\tilde{y}\|^2 + \tilde{y}^T P \bar{B}_1 (\hat{\varepsilon} \bar{B} \bar{u} + \varepsilon \Delta + \varepsilon^2 \rho(q, \eta, t, \varepsilon)) \leq 0 \quad (3.61)$$

and the closed-loop tracking error takes the form below

$$\dot{\tilde{y}} = \Gamma \tilde{y} + \bar{B}_1 (\hat{\varepsilon} \bar{u} + \delta(\varepsilon, q, \eta, t, H_\varepsilon)) + \Pi(\varepsilon, q, \eta, t, \mu). \quad (3.62)$$

For the control law (3.54), the stability condition becomes

$$-\|\tilde{y}\|^2 + \tilde{y}^T P \bar{B}_1 \left(-\hat{\varepsilon} K \frac{\bar{B}_1^T P \tilde{y}}{\sqrt{\gamma^2 + \|\bar{B}_1^T P \tilde{y}\|^2}} + (\varepsilon - \hat{\varepsilon}) \Delta + \varepsilon^2 \rho(q, \eta, t, \varepsilon) \right) \leq 0. \quad (3.63)$$

This may be rewritten under the form below

$$-\|\tilde{y}\|^2 + \tilde{y}^T P \bar{B}_1 \left(-\hat{\varepsilon} K \frac{\bar{B}_1^T P \tilde{y}}{\sqrt{\gamma^2 + \|\bar{B}_1^T P \tilde{y}\|^2}} + \hat{\varepsilon}^2 \rho(q, \eta, t, \hat{\varepsilon}) \right) + F(q, \eta, t, \varepsilon, \hat{\varepsilon}) \leq 0, \quad (3.64)$$

with the function $F(q, \eta, t, \varepsilon, \hat{\varepsilon})$ that is defined as

$$F(q, \eta, t, \varepsilon, \hat{\varepsilon}) = (\tilde{y}^T P \bar{B}_1) \left((\varepsilon - \hat{\varepsilon}) \Delta + \varepsilon^2 \rho(q, \eta, t, \varepsilon) - \hat{\varepsilon}^2 \rho(q, \eta, t, \hat{\varepsilon}) \right). \quad (3.65)$$

The stability condition is still satisfied if the condition below is verified

$$-\|\tilde{y}\|^2 - \hat{\varepsilon} K \frac{\|\bar{B}_1^T P \tilde{y}\|^2}{\sqrt{\gamma^2 + \|\bar{B}_1^T P \tilde{y}\|^2}} + \hat{\varepsilon}^2 \|\bar{B}_1^T P \tilde{y}\| \|\rho(q, \eta, t, \hat{\varepsilon})\| + \|F(q, \eta, t, \varepsilon, \hat{\varepsilon})\| \leq 0. \quad (3.66)$$

As far as the functions involved in the dynamics are smooth and uniformly bounded with respect to q (property P4), the function $\rho(q, \eta, t, \varepsilon)$ is smooth and uniformly bounded with respect to q . Therefore, for given $K > 0$ Liapounov design parameter, $M > 0$, when $\|\eta\| < M$, there exists a sufficiently small $\varepsilon_7 > 0$ such that for all $\varepsilon, \hat{\varepsilon} < \varepsilon_7$, the two inequalities below are valid :

$$K > \hat{\varepsilon} \|\rho(q, \eta, \hat{\varepsilon})\| \text{ and } \exists k_1 > 0, \|\varepsilon^2 \rho(q, \eta, t, \varepsilon) - \hat{\varepsilon}^2 \rho(q, \eta, t, \hat{\varepsilon})\| < k_1 |\varepsilon - \hat{\varepsilon}|. \quad (3.67)$$

For $\|\eta\| < M$ and for all $\varepsilon, \hat{\varepsilon} < \varepsilon_7$, we also obtain that the inequality that follows is verified :

$$\exists k_2 > 0, \|F(q, \eta, t, \varepsilon, \hat{\varepsilon})\| < 2k_2 |\varepsilon - \hat{\varepsilon}| \|\tilde{y}\|. \quad (3.68)$$

The above result allows us to ensure the stability condition to be satisfied if the inequality that follows holds :

$$-\|\tilde{y}\|^2 + \hat{\varepsilon}K\|\bar{B}_1^T P\tilde{y}\|\left(-\frac{\|\bar{B}_1^T P\tilde{y}\|}{\sqrt{\gamma^2 + \|\bar{B}_1^T P\tilde{y}\|^2}} + 1\right) + 2k_2|\varepsilon - \hat{\varepsilon}| \|\tilde{y}\| \leq 0. \quad (3.69)$$

Remark that the function $f(x) = \left(-\frac{x}{\sqrt{x^2 + \gamma^2}} + 1\right)x$ appears in this expression and has a maximum in $\bar{x} = \gamma\sqrt{\frac{\sqrt{5}-1}{2}}$ for the value $f(\bar{x}) = \bar{x} \cdot \frac{\sqrt{5}+1}{2} < \frac{\gamma}{2}$. We therefore conclude that the stability condition is satisfied if the inequality below holds

$$-\|\tilde{y}\|^2 + 2k_2|\varepsilon - \hat{\varepsilon}| \|\tilde{y}\| + \hat{\varepsilon}\frac{K\gamma}{2} \leq 0, \quad (3.70)$$

and this is satisfied when $\|\tilde{y}\|$ verifies

$$\|\tilde{y}\| \geq k_2|\varepsilon - \hat{\varepsilon}| + \sqrt{k_2^2|\varepsilon - \hat{\varepsilon}|^2 + \hat{\varepsilon}\frac{K\gamma}{2}}. \quad (3.71)$$

This means that the tracking error decreases outside a ball which radius increases with the smoothing parameter γ and the error on the value of ε . This result is summarized in the following proposition :

Proposition 3.6 (Robustness on errors on ε) *For a non ideal mobile robot described by equations (2.33), for bounded values of the velocity vector η , for a given positive constant K , there exists ε_7 such that for all ε and $\hat{\varepsilon}$ estimation of the wheel/ground contact parameter both smaller than ε_7 , such that when we apply the Liapounov augmented control law $u = u_0 + \hat{\varepsilon}\tilde{u}$ defined by (1.20) and (3.54) the stability condition is verified outside a ball which radius increases with the smoothing parameter γ and the error on the value of ε .*

If we have a "good" estimation of ε , the Liapounov augmented control law still ensures the stability condition to be verified. But we have no way to measure the quality of the estimated parameter. The closed-loop analysis may also be extended for an estimated value of ε (the proof is essentially the same as the proof of proposition 3.5).

Remark there is no reason to over- or underestimate the unknown parameter ε as far as the dependance on ε in $\rho(q, \eta, t, \varepsilon)$ is not known. The restriction introduced by this new result lies in the new constant ε_7 that is smaller than the previous ε_5 of proposition 3.4.

3.2.5 Practical implementation

For the practical implementation of the Liapounov augmented control law, we still need to compute H_0 and d_0H_0 as for the augmented linearizing law. The steps 0 to 2 of the procedure are the same and we only present the flow chart 3.8 to remind them.

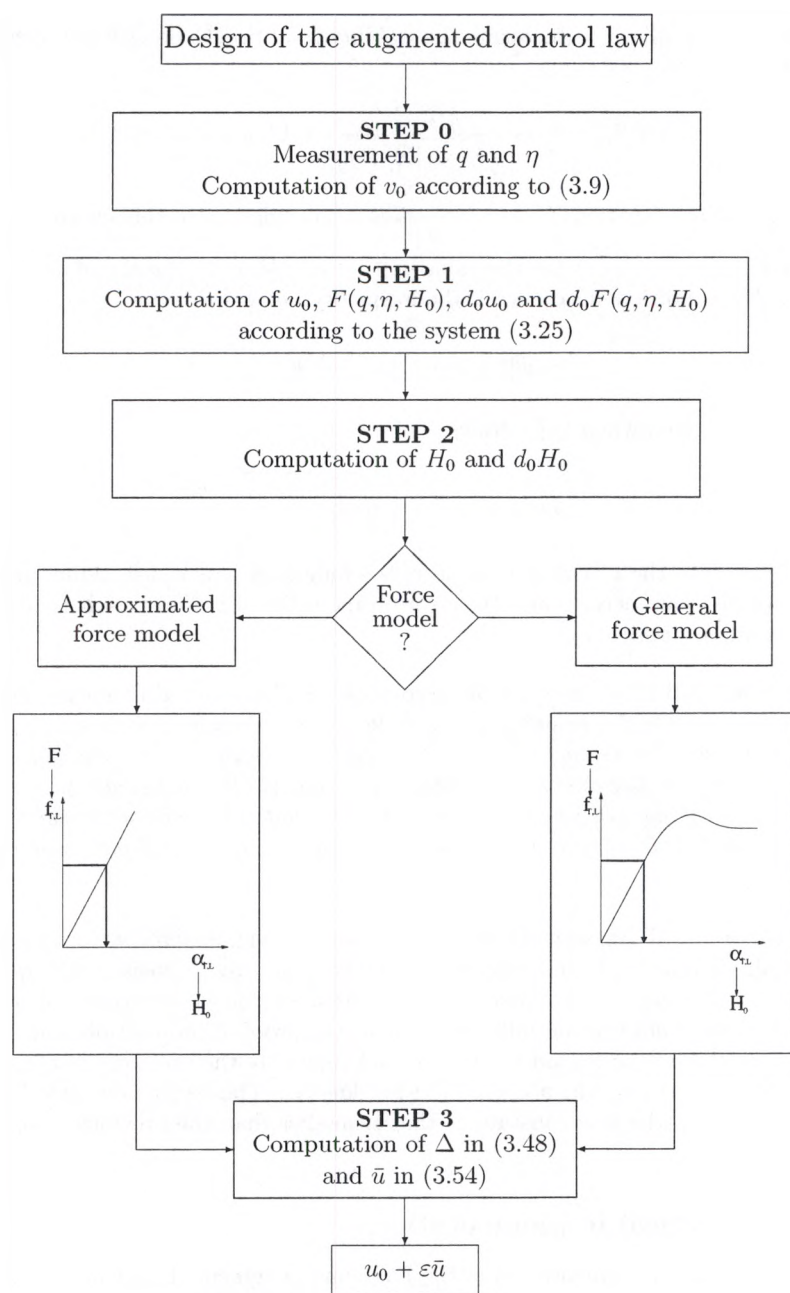


Figure 3.8: Flow chart for Liapounov augmented control law design

STEP 0 : Measurements of q , η and computation of v_0

STEP 1 : Computation of u_0 , $F(q, \eta, H_0)$, $d_0 u_0$ and $d_0 F(q, \eta, H_0)$

STEP 2 : Computation of H_0 and $d_0 H_0$

STEP 3 : Computation of $\bar{u}$

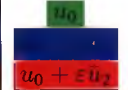
- We evaluate the matrix $\bar{B}$ that is defined in (3.50);
- The expression of the term $\Delta(q, \eta, t, H_0)$ is computed according to equation (3.48);
- Finally, the correction control term is computed following the approach in (3.54).

3.2.6 Reference example (continued)

We still present simulations for the unicycle and the sine reference trajectory of amplitude 10 meters as previously. We compare the results obtained for the nominal linearizing control law (u_0 in green), for the Liapounov augmented control law based on the force model 1 (denoted $u_0 + \varepsilon \bar{u}_1$ and associated with color blue in the figures) and for the Liapounov augmented control law based on the force model 2 (denoted $u_0 + \varepsilon \bar{u}_2$ and associated with color red in figures). We first present simulations for different values of the reference pulsation ω_{ref} to test the influence of the force model used to design the Liapounov control term. Second, we show simulations for different values of the stiffness coefficient ε . The gains in the auxiliary control law v are chosen at the same value as in the previous simulations so that comparison is possible. The Liapounov parameter K is chosen at the same value for all the simulations ($K = 0.0005$) and this choice is discussed in the last subsection. The tolerance on the tracking error is fixed at value 0.001m so that the smoothing parameter γ takes the value $K\gamma = (0.001)^2$, $\gamma = 0.002$. The initial conditions are still "compatible".

Influence of the force model in the control design

We give here the simulations for a fixed value of ε ($\varepsilon = 10$) and different values of ω_{ref} ($\omega_{ref} = 0.1, 0.4, 0.5(1/s)$).

	$\varepsilon = 10$	$\omega_{ref} = 0.1(1/s)$
---	--------------------	---------------------------

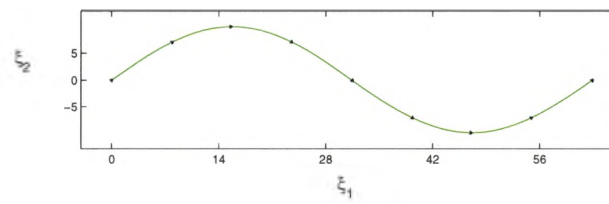
The results are summarized in figure 3.9. An improvement of the tracking error is observed from the use of the augmented control laws. The slip ratios are tiny so that the linear force model is valid (see section 2.2.2). That is the reason why the two augmented control laws present essentially the same closed-loop dynamics.

	$\varepsilon = 10$	$\omega_{ref} = 0.4(1/s)$
---	--------------------	---------------------------

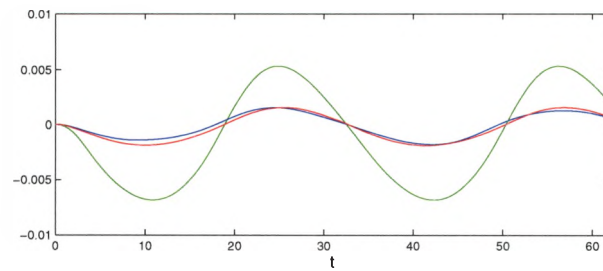
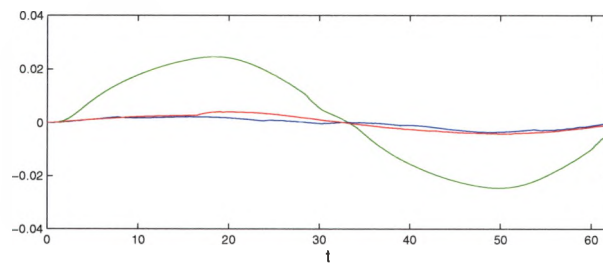
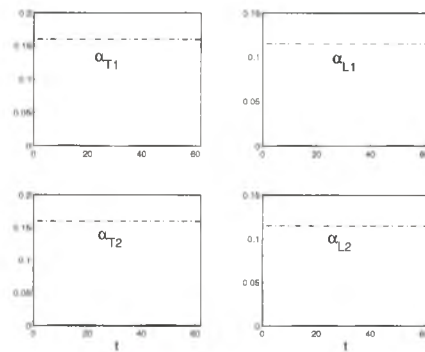
For a larger value of ω_{ref} , the normalized slip ratios necessary to follow the reference trajectory are larger and the linear force model becomes less valid. This affects the results of the Liapounov augmented control law based on the linear force model as it is observed on figure 3.10.

	$\varepsilon = 10$	$\omega_{ref} = 0.5(1/s)$
---	--------------------	---------------------------

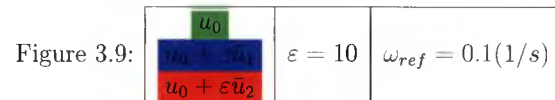
Finally, for reference trajectories that suppose the forces to have values near their maximal values (see figure 3.11), the Liapounov augmented control law based on the linear force model becomes totally unefficient while the control law based on the general model still improves the closed-loop behaviour.

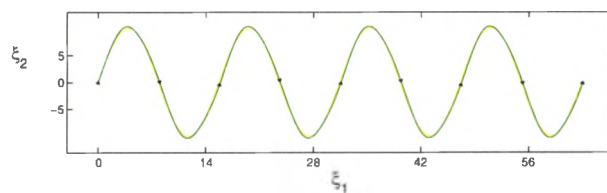


a) Trajectories (meters)

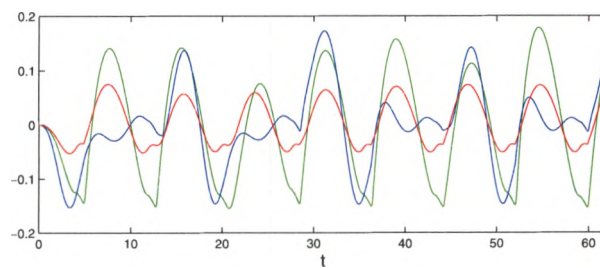
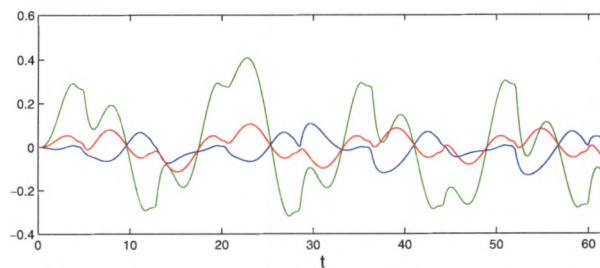
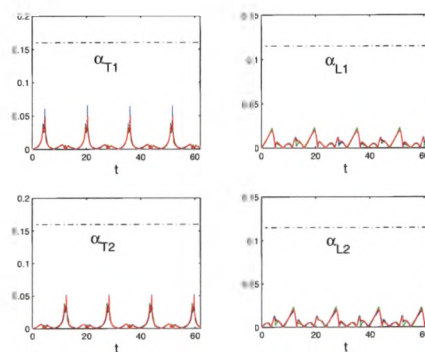
b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)

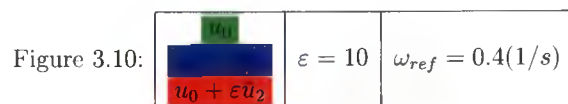


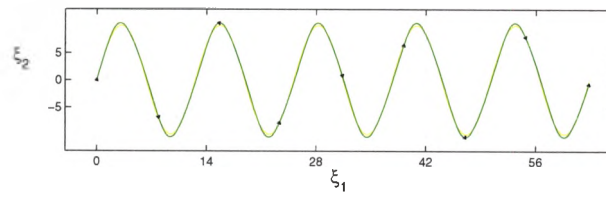


a) Trajectories (meters)

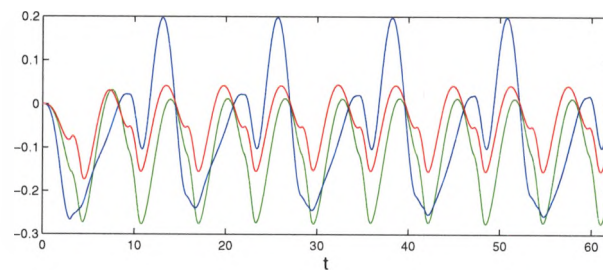
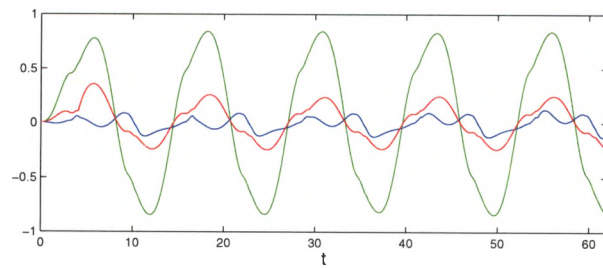
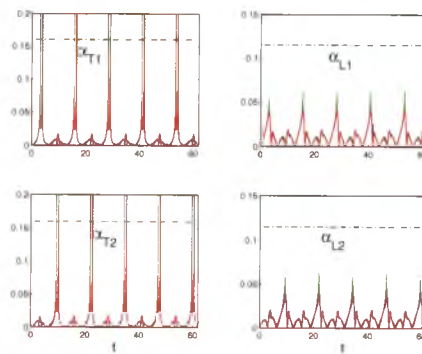
b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)

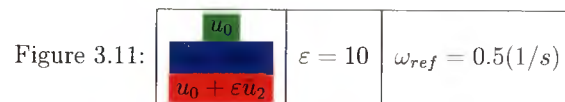




a) Trajectories (meters)

b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)

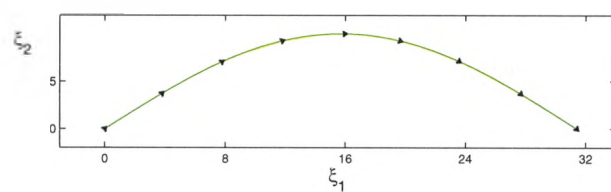


Influence of the wheels' stiffness

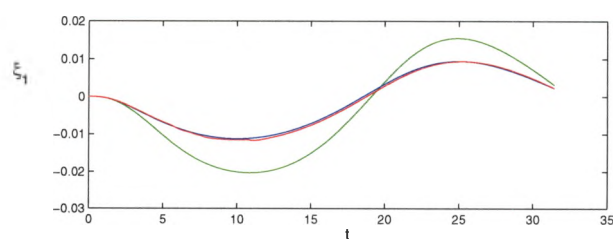
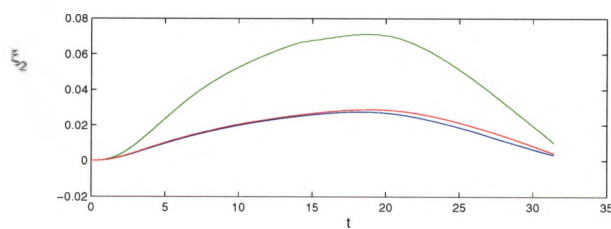
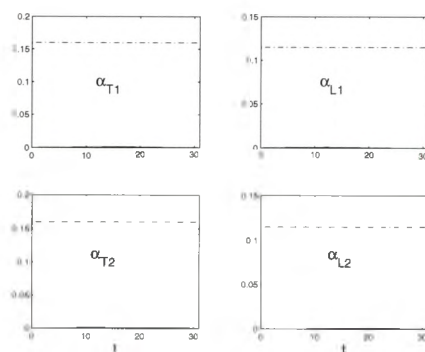
We give a simulation of the non ideal robot for a fixed value of ω_{ref} ($\omega_{ref} = 0.1(1/s)$) and a larger value of the stiffness coefficient ε . Our aim is to observe whether the improvement from the new control law is preserved.

u_0		
$u_0 + \varepsilon \bar{u}_1$	$\varepsilon = 30$	$\omega_{ref} = 0.1(1/s)$
$u_0 + \varepsilon \bar{u}_2$		

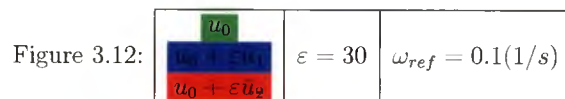
If we compare the results obtained in figure 3.9 associated with $\varepsilon = 10$ with the results of figure 3.12 for $\varepsilon = 30$, we observe that the augmented control laws still improve the performance of the tracking error closed-loop.



a) Trajectories (meters)

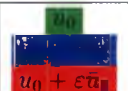
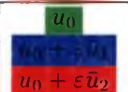
b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)



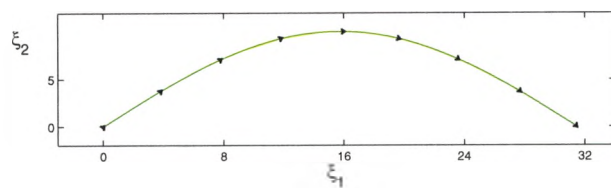
Sensitivity to the Liapounov gain K

In the above simulations, the Liapounov gain was fixed at value $K = 0.0005$. We now study the influence of this choice on the tracking error dynamics. We therefore present simulations for the same experimental conditions as previously but with other values of the parameter K .

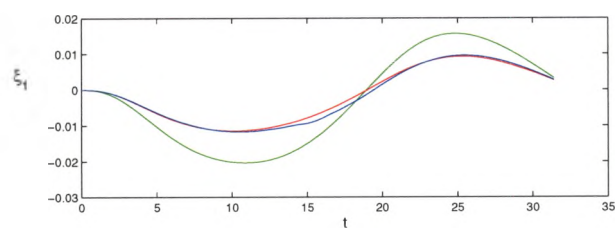
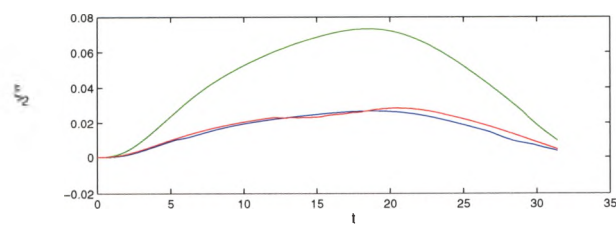
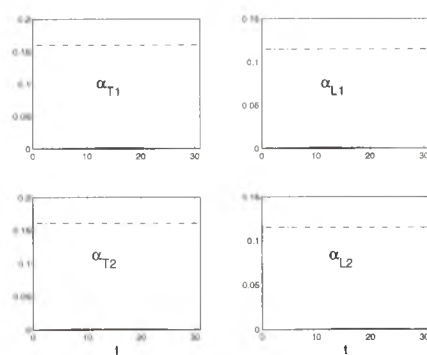
	$\varepsilon = 30$	$\omega_{ref} = 0.1(1/s)$	$K = 0.000005$
	$\varepsilon = 30$	$\omega_{ref} = 0.1(1/s)$	$K = 0.005$

The results are presented in figure 3.13 and 3.14 and may be compared with results obtained for $K = 0.0005$ in figure 3.12. We observe that for $K = 0.000005$, the tracking error dynamics are slightly affected. For the larger K ($K = 0.005$), the tracking error dynamics is degraded, likely due to problems with the existence or with the attractivity of the slow manifold. This is an important result for the practical implementation of the Liapounov augmented law on an actual robot : not all values of the parameter K are acceptable but a wide range of values give satisfying results.

In some sense, it also illustrates the robustness with respect to errors on the stiffness coefficient ε because the correction control term is proportional to εK .

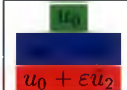


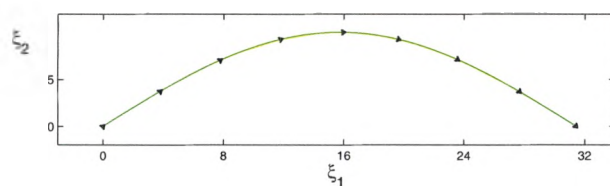
a) Trajectories (meters)

b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

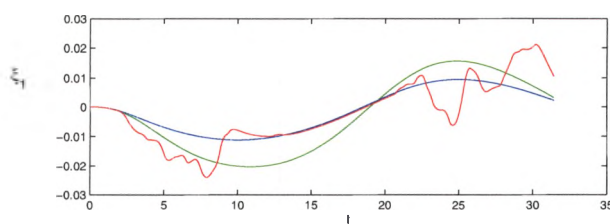
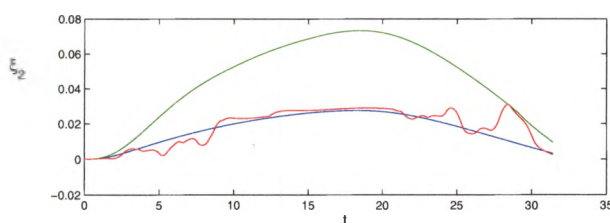
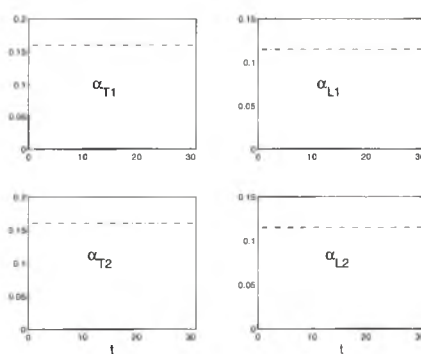
d) Slip ratios (without dimension)

Figure 3.13:

	$\varepsilon = 30$	$\omega_{ref} = 0.1(1/s)$	$K = 0.000005$
---	--------------------	---------------------------	----------------



a) Trajectories (meters)

b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)

Figure 3.14:

	$\varepsilon = 30$	$\omega_{ref} = 0.1(1/s)$	$K = 0.005$
--	--------------------	---------------------------	-------------

3.2.7 Conclusion

Simulations for different values of the parameters show that for our example, the closed-loop performance is improved from the use of the new control laws. These control laws ensure the asymptotic tracking error to be lower than a tolerance level, and there are robust on errors on ε . The simulations for the generalized forces model in control design let us think that in practice, when large values of the normalized slip ratios are needed, the general model is better to be used to design the control law. The choice of the Liapounov parameter K seems not to be crucial : in simulations, it may be chosen in a wide range of values. This result also illustrates the robustness with respect to errors on the value of ε because the correction control law depends on the product $K\varepsilon$.

3.2.8 Remark on the developed simulation program

The simulation program has been developed with matlab and the dynamical model has been integrated using the function *ode15s.m* that is a function for integration of stiff systems. In this part, we only present simulation results for the unicycle and a sine reference trajectory. However, the Matlab functions generating the dynamics and the control laws have been developed for general robots; and other reference trajectories can be tested without needing important modifications. More precisely, the developed functions are valid for all robot type except for robots equipped with several steering wheels characterized by $(\delta_m, \delta_s) = (1, 2)$. For such robots, there does not exist a global kinematic model. The wheels have to be coordinated so that the robot can move. When the pure rolling and non slipping conditions are satisfied for each wheel, the coordination is automatically ensured. When these conditions are not satisfied, the robot may move without being coordinated. Then the kinematic model supposes us to define an associated ideal and therefore coordinated mobile robot. This association is not trivial. Furthermore, such robots present singularities in configuration when one wheel stops or when the robot stops. Then the orientation of the associated wheels is no more defined.

3.3 The drawing robot example

Path tracking precision is essential in mobile robots performing tasks such as welding, painting, gluing, drawing, ..., where some points in the wheeled mobile robot is in contact with navigation surface and shall accurately follow a given path. For such robots, it is realistic to assume we measure the position and rolling velocity accurately. The previous control laws may therefore be implemented when the ideal kinematic constraints are not satisfied.

3.3.1 Robot characteristics

We consider the drawing robot is a small and light robot with the structure of a unicycle, characterized by the parameters below

$$e = 0.03m; \quad r = 0.02m; \quad l = 0.05m; \quad m_T = 1kg \quad I_\theta = 0.003kg.m^2; \quad I_\varphi = 2.10^{-5}kg.m^2.$$

In all the developed simulations, we fix parameter ε to 1. Assuming that Bakker's results in [21] remain valid for small vertical load, the wheel/ground contact parameter choice $\varepsilon = 1$ correspond to a robot equipped with rubber wheels moving on a dry asphalt road.

3.3.2 The control parameters

The gains matrices of the nominal control law in (1.20)-(1.21) are fixed at value $K_1 = \text{diag}(\omega^2, \omega^2)$, $K_2 = (2\zeta\omega, 2\zeta\omega)$ so that the tracking error dynamics are given for the two components of $\bar{q}_1$ by an equation of a second order system

$$\ddot{x} = -\omega^2 x - 2\zeta\omega\dot{x}.$$

For all the simulations, the values of the gains are $\omega = 0.9(1/s)$ and $\zeta = 0.7$ so that the associated time constant is $\frac{1}{\zeta\omega} = 1.6s$.

The Liapounov augmented control law parameter K has been chosen to value 0.4 by trial and error.

3.3.3 Reference trajectory

The reference trajectory (in yellow) is a circle of radius 1 meter with a constant angular velocity :

$$\begin{aligned} \xi_{1r}(t) &= \cos(\omega_{ref}t); \\ \xi_{2r}(t) &= \sin(\omega_{ref}t). \end{aligned}$$

Assuming that the vertical load is equally distributed on the three wheels, i.e. is equal to 3.3N, we can evaluate the maximal transversal and longitudinal forces are each wheel, respectively about $f_{Tmax} = 3.3N$ and $f_{Lmax} = 3N$. The associated transversal and longitudinal slip ratios are respectively $\alpha_{Tmax} = 0.31$ and $\alpha_{Lmax} = 0.15$. On the other hand, for the reference trajectory, the centripetal acceleration is constant ($= R\omega_{ref}^2$), so that the required transversal force for each of the fixed wheel is equal to $\frac{1}{2}mR\omega_{ref}^2 = \frac{1}{2}\omega_{ref}^2$. In the simulations, we shall consider two values of the reference pulsation ω_{ref} ($\omega_{ref} = 2(1/s)$ and $\omega_{ref} = 2.4(1/s)$). For $\omega_{ref} = 2(1/s)$, the transversal forces have to take values about 2N and the associated slip ratio is about 0.04 (about 10 % of α_{Tmax}). For $\omega_{ref} = 2.4(1/s)$, the transversal forces values are about 2.9N and the associated transversal slip is 0.12 (about 40% of α_{Tmax}). To easily understand the above comments on the transversal force

values and the associated slip ratios, the section 2.2.2 should be read and especially the normalized transversal force plot of figure 2.3 should be observed.

The initial conditions are compatible, i.e. the initial position and velocity correspond to the reference trajectory :

$$\begin{aligned}\xi_1(0) &= 0; \\ \xi_2(0) &= 1; \\ \dot{\xi}_1(0) &= 0; \\ \dot{\xi}_2(0) &= \omega_{ref} \\ \mu &= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T.\end{aligned}$$

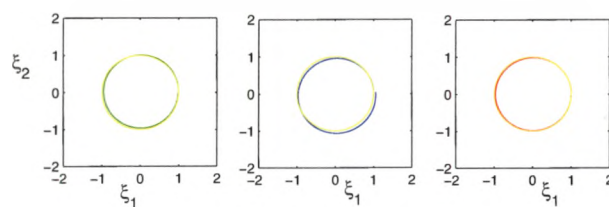
3.3.4 Simulation results for the control laws based on the linearized force model

We propose here to compare the efficiency of the two augmented laws based on the linear force model (Model 1). We therefore present the results associated with the augmented linearizing control law and with the Liapounov augmented control law on the same plots. All the curves associated with the nominal linearizing control law (u_0) are plotted in green; the curves associated with the augmented linearizing control law ($u_0 + \varepsilon u_{11}$) are in blue; and the curves for the Liapounov augmented law ($u_0 + \varepsilon \bar{u}_1$) are red. We give the trajectories plots (with reference trajectory in yellow), the tracking error plots and the slip ratios plots.

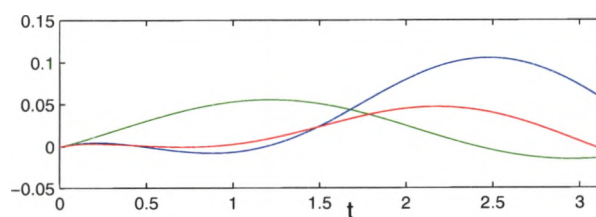
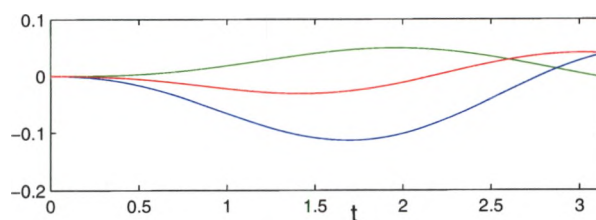
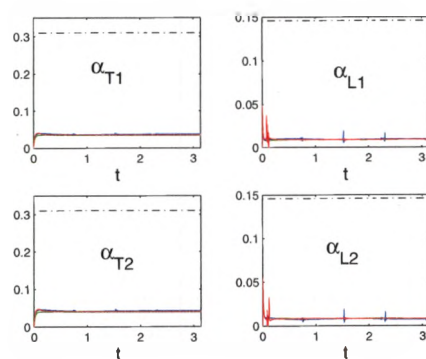
We present the simulations associated with $\omega_{ref} = 2(1/s)$ because it corresponds to transversal forces of about 2N and associated slip ratios is 0.04, that is about 10% of α_{Tmax} , the slip value associated with the force maximum. Remind from section 2.2.2 that the linearized force model is considered to be valid when $\alpha_T < 10\% \alpha_{Tmax}$. The aim of the simulation is therefore to observe whether the augmented laws based on the linearized force model still improve the tracking error dynamics when the slip ratios are about 10% of α_{Tmax} , the theoretical validity bound for the linearized model.

u_0	$\varepsilon = 1$	$\omega_{ref} = 2(1/s)$
$u_0 + \varepsilon u_{11}$		
$u_0 + \varepsilon \bar{u}_1$		

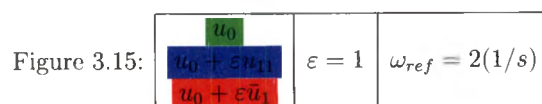
The results associated to $\varepsilon = 1$ and $\omega_{ref} = 2(1/s)$ are presented on figure 3.15. We observe that the transversal slips are about 10% of $\alpha_{Tmax} = 0.31$ and the tracking error behavior is deteriorated for the augmented linearizing control law with respect to the nominal law and no longer visibly improved for the Lipounov augmented law. These observations confirm that the linearized force model becomes non valid for transversal slips larger than 10% of α_{Tmax} . It also let us think that it is better to use the Liapounov augmented control because it improves the tracking error behavior for larger values of the slip ratios.



a) Trajectories (meters)

b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)



3.3.5 Simulations results for the control laws based on the general force model

We now propose some simulations testing the efficiency of the control laws based on the general force model (Model 2). We still present the curves for the nominal linearizing control law (u_0) in green, the curves for the augmented linearizing control law ($u_0 + \varepsilon u_{12}$) in blue and the curves for the Liapounov augmented law ($u_0 + \varepsilon \tilde{u}_2$) in red.

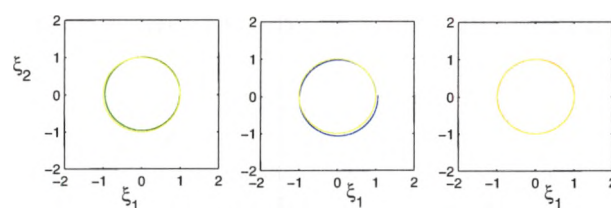
	$\varepsilon = 1$	$\omega_{ref} = 2(1/s)$
---	-------------------	-------------------------

We first present simulation results corresponding to $\omega_{ref} = 2(1/s)$ (figure 3.16). We observe that the performance of the augmented linearizing control are degraded with respect to the nominal control law. This is likely due to the fact that the value $\varepsilon = 1$ is already a too large value for this control modification approach. In fact, for smaller values of ε ($\varepsilon = 0.1$), the linearizing approach gives better results. On the other hand, the Liapounov augmented control law rather improves the tracking error behavior.

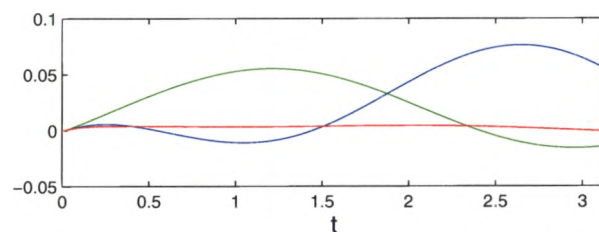
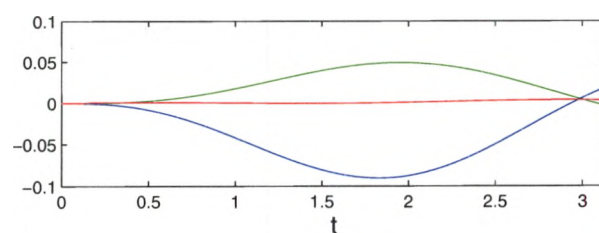
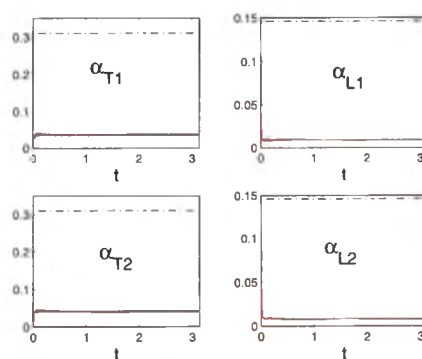
These results can be compared with those of figure 3.15 under the same conditions. It can be seen that the Liapounov augmented control law based on the general force model (model 2) gives better results than the law based on the linearized force model (model 1).

	$\varepsilon = 1$	$\omega_{ref} = 2.4(1/s)$
--	-------------------	---------------------------

We also present simulations for a larger value of ω_{ref} ($\omega_{ref} = 2.4(1/s)$) at figure 3.16. The associated maximal transversal force is $2.9N$ and the associated slip is about 0.12. In that case, the augmented linearizing control law still does not improve the tracking error evolution, likely due to the fact that $\varepsilon = 1$ is already too larger for this control approach. For the Liapounov augmented law, we still observe that the tracking error behavior is better than for the nominal law, but the benefit is lower than for $\omega_{ref} = 2(1/s)$.

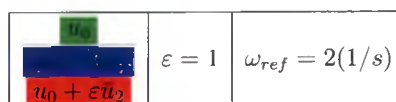


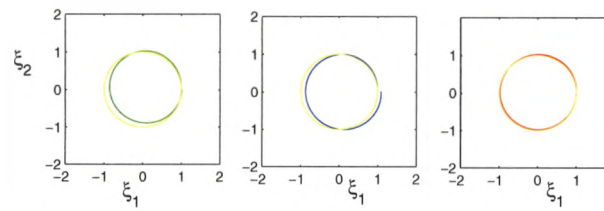
a) Trajectories (meters)

b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

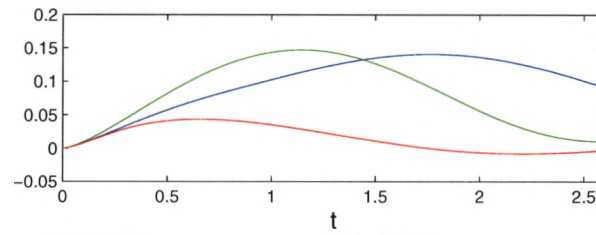
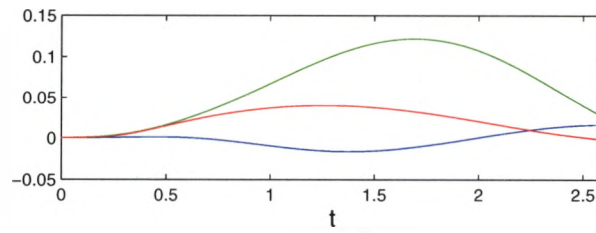
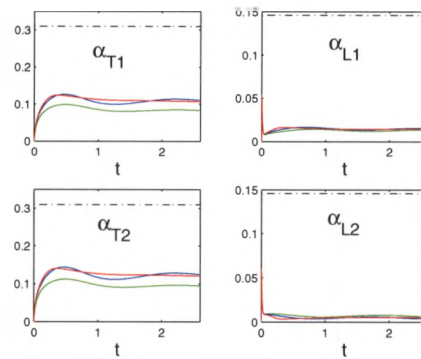
d) Slip ratios (without dimension)

Figure 3.16:





a) Trajectories (meters)

b) Tracking error in ξ_1 (meters)c) Tracking error in ξ_2 (meters)

d) Slip ratios (without dimension)

Figure 3.17:

$\begin{array}{c} u_0 \\ u_0 + \varepsilon \bar{u}_1 \\ u_0 + \varepsilon \bar{u}_2 \end{array}$	$\varepsilon = 1$	$\omega_{ref} = 2.4(1/s)$
--	-------------------	---------------------------

3.3.6 Conclusion

The above simulations confirm the validity of the linearized force model for slip ratios less than 10% of the maximal force value associated slip. For larger values of the slip ratios, it is better to use the augmented control laws based on the general force model. But when the slip ratios become near the maximal force value associated slip, the improvement from the use of the augmented laws diminishes due to the fact the attractivity is lower.

In the presented simulations for $\varepsilon = 1$, the augmented linearizing control law does not improve the tracking error evolution. For smaller ε , improvements from the use of this control law are observed.

3.4 Discussion of the approach

3.4.1 Discussion of the results

In this chapter, we addressed the point tracking control problem for non ideal wheeled mobile robots. The control law is designed in order to achieve point tracking on the corresponding slow manifold. The slow manifold attractivity then ensures the semi-global convergence of the closed-loop tracking error. Our feedback laws supposes us to measure the position and rolling velocity of the robot but takes into account the fast dynamics associated with the wheel/ground effects without requiring measurement of the sliding velocities. The key properties required to implement this approach are the matching condition and the the existence of a smooth control law ensuring exponentially convergence of the tracking error. The resulting control law is then obtained as a modification of this ideal-model based law. Two different implementations are proposed and and analyzed. In the first approach, the control law is designed in order to achieve input-output linearization on the slow manifold. In this case the nominal control law is the output linearizing control law. The second approach is a Liapounov redesign approach. The control law is designed in order to ensure that the Lyapounov function associated with the nominal linearizing control law is still decreasing for the non ideal dynamics. We again augment the nominal input-output linearizing control law but other control laws can be used.

The above results are illustrated on simulations for the unicycle. We compare the efficiency of the control laws when a general force model and when a linearized force model are implemented for the control design. For small values of the slip ratios, the control laws based on linearized model show the same efficiency as the control laws based on the general one. But for larger values of the slip ratios, the control laws based on the linearized model become unefficient. When discussing the interest to use a control law based on the general force model, two things have to be kept in mind :

- The implementation of the control law based on the general force model requires more computation than that one based on the linearizing force model, due to the inversion of the force characteristic;

- From a theoretical point of view, nothing may be expected from the augmented control law when the robot actually skids.

The interest of the law based on the general force model is therefore restricted to situations where the slip ratios are high, but not exceeding the maximal value.

For the nominal control law augmented by Liapounov redesign, we observe that the unknown control parameter K may be chosen in a wide range of values without damaging the closed-loop dynamics. This also illustrates the robustness of the control law with respect to errors on ε because the correction control term depends on the product $K\varepsilon$.

3.4.2 Possible extensions

We now discuss briefly the possible extensions of this control design approach. The two important conditions necessary for such an extension to be possible are that

- the non ideal dynamics is expressed as a singular perturbation formulation of the ideal one and the non ideal perturbation terms satisfy the matching condition;
- for the ideal dynamics, there exists a smooth control law ensuring the control error to be exponentially decreasing.

We propose to design control laws for non ideal robots by extension of an existing ideal-model based control law. We only consider the input-output linearizing control law as nominal law, but other choices can be made as far as the control error is ensured to be exponentially decreasing. The Liapounov redesign extension could be implemented for an other Liapounov based nominal control law obtained for example by backstepping [6] or by a general approach using the notion of control Liapounov function [24].

The posture tracking question for non ideal robots should also be solved applying the presented approaches. The full linearization by static feedback is not possible but the dynamic extension algorithm may be applied to obtain full linearization of an augmented state vector.

The point stabilization problem can be addressed with the same type of control laws as proposed in this thesis. But in that case, the robot may often stop so that our force model becomes often invalid. It is thus necessary to switch from the dynamic force model to the static force model. Such a connection between static and dynamic force model has already been analyzed in [26] for the cam/follower type of contact, but continuity in the model switching is not evident to obtain.

It is much less direct to know whether a posture stabilizing control law could be augmented according to the above developments. Due to Brockett's obstruction [5], the posture stabilizing control laws for ideal robot are either discontinuous either time dependent. If we want to apply the slow manifold approach, we may not apply discontinuous control laws. The recent challenging question for the posture stabilization problem is to

obtain a smooth time-dependent control law that ensures exponential convergence towards the setpoint. In [14], a set of sufficient conditions for generating Lipchitz continuous K -exponentially stabilizers from a general class of driftless systems was developed. This last type of stability is weaker than the usual exponential stability : the state is bounded by a decaying exponential envelope except that the size of the envelope does not scale linearly in the initial condition. This result allowed the authors of [15] to achieve global asymptotical stability with exponential convergence about any desired configuration. Finally in [16] the wheeled mobile robot kinematics is transformed into a dynamic oscillator to promulgate a global exponential regulation property for the transformed kinematic model via a Liapounov type argument. But for example, the control law designed in [16] supposes a particular change of variable and for the dynamics written with respect to these new variables, the non ideal perturbation terms do not satisfy the matching condition. However, we may already say that provided there exists a smooth exponentially stabilizing control law such that the associated non ideal perturbation terms satisfy the matching condition, then all the development we made for the point tracking of a reference trajectory can be transposed for the posture stabilization problem. A possibility would be to find a change of variable and the associated Liapounov function such that in term of the new variables, the non ideal perturbation terms satisfy the matching condition. But this is a non trivial point that oversteps this work.

Part III

Control law design for mobile robots equipped with steering wheels



Chapter 1

Robots equipped with steering wheels : description and properties

The last part of this work deals with robots equipped only with steering wheels and satisfying the pure rolling kinematic constraints (ideal robots). Such robots present a wide range manoeuvrability allowing them to follow any geometric trajectory. In order to exploit this manoeuvrability, we suppose that for each wheel the orientation velocity and the rolling velocity can be manipulated so that the robot is overactuated : there are more control inputs than degrees of freedom. This redundancy however is only apparent : the input cannot be manipulated arbitrarily in order to satisfy the kinematic constraints. The kinematic model of such robots presents singularities when the instantaneous center of rotation goes exactly through the center of one wheel. Then the wheel orientation can be chosen freely and the wheel may have to be reoriented instantaneously when the instantaneous center of rotation moves back. Even when the instantaneous center of rotation passes near the center of one wheel, this wheel may have to be reoriented very fast. In practice, the wheel orientation velocities have to satisfy physical bounds and this imposes additional restrictions to the admissible trajectories, namely in the neighborhood of a singularity. In this chapter, we propose a formulation allowing us to take into account these singularities as well as the physical bounds on the wheels orientation velocities. This results in the definition of admissible posture, admissible configuration and accessible acceleration.

1.1 Robot description

As seen in the first part, from the user point of view, the position of a wheeled mobile robot is specified by the vector ξ :

$$\xi = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}. \quad (1.1)$$

We assume the robot is equipped with k steering wheels that all have the same radius

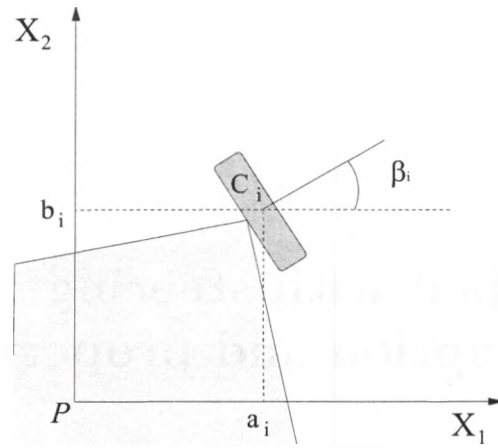


Figure 1.1: Wheel parameters and variables description

denoted r . We define here all the parameters and variables with respect to the moving frame $\mathbf{X}_1, \mathbf{X}_2$ fixed on the trolley at the reference point P . For each wheel centered at point C_i ($i = 1, \dots, k$), we introduce the following notations (see figure 1.1):

- (a_i, b_i) are the cartesian coordinates of the wheel center C_i ;
- β_i is the orientation of the horizontal angle measured with respect to $\mathbf{X}_1$;
- φ_i is the rotation angle.

We define the orientation angles vector and the rotation angles vector as follows :

$$\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_k \end{pmatrix} \quad \text{and} \quad \varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_i \\ \vdots \\ \varphi_k \end{pmatrix}. \quad (1.2)$$

1.2 Kinematic constraints

The velocity of point P may be developed under the form below

$$\vec{v}_P = v_1 \mathbf{X}_1 + v_2 \mathbf{X}_2. \quad (1.3)$$

The velocity of each wheel center is then given by

$$\vec{v}_{C_i} = \vec{v}_P + \dot{\theta} \mathbf{X}_3 \times [a_i \mathbf{X}_1 + b_i \mathbf{X}_2]. \quad (1.4)$$

We do not consider the sliding effects in this part so that the kinematic constraints are satisfied. This means that the component of $v_{Ci}^{\rightarrow}$ orthogonal to the plane of the wheel is zero and that the parallel component is the rotation velocity. Mathematically, this amounts the two following relations to be satisfied

$$v_{Ci}^{\rightarrow} \cdot (\cos \beta_i \mathbf{X}_1 + \sin \beta_i \mathbf{X}_2) = 0 \quad (1.5)$$

$$v_{Ci}^{\rightarrow} \cdot (-\sin \beta_i \mathbf{X}_1 + \cos \beta_i \mathbf{X}_2) = r\dot{\varphi}_i, \quad (1.6)$$

and using (1.3) and (1.4) these are equivalent to

$$\begin{pmatrix} \cos \beta_i & \sin \beta_i \\ -\sin \beta_i & \cos \beta_i \end{pmatrix} \begin{pmatrix} 1 & 0 & -b_i \\ 0 & 1 & a_i \end{pmatrix} \begin{pmatrix} \bar{v}_1 \\ v_2 \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ r\dot{\varphi}_i \end{pmatrix}. \quad (1.7)$$

Introducing the following notations,

$$R_1^T(\beta_i) = (\cos \beta_i \quad \sin \beta_i); \quad (1.8)$$

$$R_2^T(\beta_i) = (-\sin \beta_i \quad \cos \beta_i); \quad (1.9)$$

$$G_i = \begin{pmatrix} 1 & 0 & -b_i \\ 0 & 1 & a_i \end{pmatrix}; \quad (1.10)$$

$$\eta = \begin{pmatrix} v_1 \\ v_2 \\ \dot{\theta} \end{pmatrix}; \quad (1.11)$$

the kinematic constraints (1.7) can be written under a more compact form :

$$\begin{aligned} R_1^T(\beta_i)G_i\eta &= 0 \\ R_2^T(\beta_i)G_i\eta &= r\dot{\varphi}_i \end{aligned} \quad (i = 1, \dots, k). \quad (1.12)$$

With the matrix definitions below

$$H(\beta) = \begin{bmatrix} R_1^T(\beta_1)G_1 \\ \vdots \\ R_1^T(\beta_i)G_i \\ \vdots \\ R_1^T(\beta_k)G_k \end{bmatrix}, \quad H^*(\beta) = \begin{bmatrix} R_2^T(\beta_1)G_1 \\ \vdots \\ R_2^T(\beta_i)G_i \\ \vdots \\ R_2^T(\beta_k)G_k \end{bmatrix}, \quad (1.13)$$

the matrix form of the constraints is the following :

$$H(\beta)\eta = 0, \quad (1.14)$$

$$H^*(\beta)\eta = r\dot{\varphi}. \quad (1.15)$$

From the kinematic constraints (1.14), it must be true that $\eta \in \text{Ker}(H(\beta))$. For a robot equipped with at least three steering wheels, the coordination of the wheels is needed

for the robot to move without sliding. This imposes that $\text{rank}(H(\beta)) = 2$, i.e. that all the wheel axles to point towards one point : the instantaneous center of rotation. We therefore define the *coordination manifold* $\mathcal{M}$ by the condition below

$$\beta \in \mathcal{M} \iff \text{rank} H(\beta) = 2.$$

Thuilot proved in [17] that this two dimensional manifold could not be parametrized with one map.

The kinematic constraints also impose that $H^*(\beta)\eta = r\dot{\phi}$ and for orientation angles in the coordination manifold, the following property is satisfied

$$\text{rank} \begin{bmatrix} H(\beta) \\ H^*(\beta) \end{bmatrix} = 3, \quad (1.16)$$

because the matrix kernel is reduced to the zero vector. The above matrix is therefore right-invertible and any wanted velocity η of the robot may be assigned with adequate reorientation of the wheels β and an adequate rotation velocity $\dot{\phi}$. This last property is a mathematical interpretation of the robot wide manoeuvrability.

If we consider a robot equipped with three or more steering wheels, the wheels have to be coordinated so that the robot can move. If the robot is moving ($\eta \neq 0$) and if the above kinematic constraints are satisfied for each wheel, then the robot coordination is automatically ensured.

We now recall the key properties of the robot model that we use later :

P1 : When the kinematic constraints are satisfied, if $R_2^T(\beta_i)G_i\eta = 0$, then $G_i\eta = 0$. This implies the equivalence below is satisfied :

$$\dot{\phi}_i = 0 \iff G_i\eta = 0.$$

This corresponds to the situation when the instantaneous center of rotation coincides with the center of wheel i : the wheel stops and the robot rotates around this wheel.

P2 : When $k \geq 3$, if $\dot{\phi}_i = \dot{\phi}_j = 0$ with $i \neq j$ then $\dot{\phi}_l = 0$ for $l = 1, \dots, k$. This corresponds to the situation when two of the wheels stop and with the kinematic constraints that are satisfied, this implies that all the wheels stop.

P3 : The coordination manifold $\mathcal{M}$ is a 2-dimensional manifold :

$$\beta \in \mathcal{M} \iff \text{rank} H(\beta) = 2.$$

P4 : The robot manoeuvrability is mathematically interpreted by the relation below :

$$\beta \in \mathcal{M} \implies \text{rank} \begin{bmatrix} H(\beta) \\ H^*(\beta) \end{bmatrix} = 3.$$

1.3 Kinematic model

To exploit the robot mobility, we have to use the orientation angles and rotation angles. It is natural to consider the rotation and orientation velocities as system inputs, i.e.,

$$\dot{\beta} = u, \quad (1.17)$$

$$\dot{\varphi} = v. \quad (1.18)$$

Our system has therefore $2k$ inputs for 3 degrees of freedom. It is overactuated but the extra inputs may not be assigned freely : the kinematic constraints must be verified, and among other things, the orientation angles coordination condition $\beta \in \mathcal{M}$ has to be satisfied.

Our aim is to obtain a state formulation allowing us to generate all the trajectories satisfying the constraints.

The inputs v_i explicitly appear in the constraints $H^*(\beta)\eta = r\dot{\varphi}$ and we derive the constraint $H(\beta)\eta = 0$ to make the inputs u_i appear ($i = 1, \dots, k$) :

$$\begin{cases} u_i R_2^T(\beta_i) G_i \eta + R_1^T(\beta_i) G_i \dot{\eta} = 0, \\ R_2^T(\beta_i) G_i \eta = r v_i; \end{cases} \quad (1.19)$$

where we used the fact that $\dot{R}_1(\beta_i) = \dot{\beta}_i R_2(\beta_i)$.

All the trajectories satisfying the constraints are therefore generated according to the relations

$$\dot{\xi} = R^T(\theta)\eta \quad (1.20)$$

$$\dot{\eta} = w \quad (1.21)$$

$$\dot{\beta}_i = u_i \quad (1.22)$$

$$\dot{\varphi}_i = v_i \quad (1.23)$$

with $\beta(t_0) \in \mathcal{M}$ and with the inputs u_i , v_i and w verifying $\forall i = 1, \dots, k$

$$\begin{cases} u_i R_2^T(\beta_i) G_i \eta + R_1^T(\beta_i) G_i w = 0, \\ R_2^T(\beta_i) G_i \eta = r v_i. \end{cases} \quad (1.24)$$

Remind that matrix $R(\theta)$ is defined by

$$R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1.25)$$

The first above conditions for all the wheels are written under the matrix form below

$$\begin{pmatrix} P_1(\beta_1, \eta) & & 0 \\ & \ddots & \\ 0 & & P_k(\beta_k, \eta) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} + \begin{pmatrix} R_1^T(\beta_1) G_1 \\ \vdots \\ R_k^T(\beta_k) G_k \end{pmatrix} w = 0 \quad (1.26)$$

with $P_i(\beta_i, \eta) = R_2^T(\beta_i)G_i\eta$. Shortly written, it becomes :

$$P(\beta, \eta) u + H(\beta)w = 0 \quad (1.27)$$

where we know from property P3 that $\text{rank}(H) = 2$ and from property P2 that $\text{rank}(P)$ takes value k , $k - 1$ or 0 .

rank(P) = k : This is the generic situation, corresponding to all $G_i\eta \neq 0$ for $i = 1, \dots, k$. The acceleration vector w may be chosen freely and the inputs u_i and v_i satisfying the constraints are given by

$$u_i = -\frac{R_1^T(\beta_i)G_iw}{R_2^T(\beta_i)G_i\eta}, \quad (1.28)$$

$$v_i = \frac{1}{r}R_2^T(\beta_i)G_i\eta, \quad (1.29)$$

for all $i = 1, \dots, k$. In that case, the coordination constraints remain satisfied along the evolution of the state vector choosing adequate inputs u and v .

rank(P) = $k - 1$: There exists one i such that $G_i\eta = 0$. The *wheel* i stops (property P1) and is said to be *singular*. In that case, w is not arbitrary because equation (1.26) for wheel i becomes

$$R_1^T(\beta_i)G_iw = 0, \quad (1.30)$$

meaning that w belongs to a subspace of dimension 2. In that case u_i is chosen arbitrarily and other inputs u_j for $i \neq j$ are defined as in (1.28) so that the constraints are satisfied. The v_i inputs are still defined according to (1.29) for all $i = 1, \dots, k$.

rank(P) = 0 : At least two of the wheels are singular and the robot stops ($\eta = 0$). In that situation, the *robot* is said to be *singular*. The vector w has to satisfy

$$H(\beta)w = 0. \quad (1.31)$$

From property P3, this means that w belongs to a subspace of dimension 1. The inputs u_i ($i = 1, \dots, k$) are chosen so that the coordination constraints remain satisfied in order to allow the robot wheels to move later.

With this description of a coordinated robot, the physical singularity problem is mathematically interpreted by a rank deficiency of matrix P .

In practice, difficulties occur not only at singularity. When the ICR passes near the center of one wheel, this wheel may be reoriented fast. This corresponds mathematically to the fact that there exists i such that $G_i\eta$ is near to zero. In that case, the expression of u_i in (1.28) becomes very large, corresponding to large and non acceptable β_i . It is therefore realistic to consider the inputs u_i to be bounded

$$\forall i = 1, \dots, k \quad |u_i| \leq \dot{\beta}_s. \quad (1.32)$$

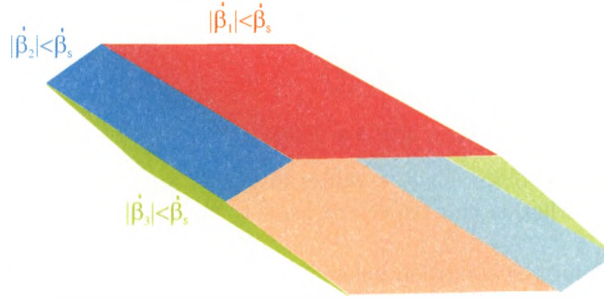


Figure 1.2: Representation of the global accessible set for a 3-steering wheels robot

The above constraints on the inputs u_i imply some restrictions on the accessible w according to equation (1.24).

We introduce the definition of *accessible set for w associated with wheel i* :

$$D_i(\beta, \eta) = \left\{ w \in \mathbb{R}^3 : \exists u_i \text{ with } |u_i| < \dot{\beta}_s \text{ and such that } u_i R_2^T(\beta_i) G_i \eta + R_1^T(\beta_i) G_i w = 0 \right\}. \quad (1.33)$$

A geometrical interpretation of this set is that w lives between 2 parallel planes defining a kind of "millefeuille" in $\mathbb{R}^3$. Remark that all the accessible sets $D_i(\beta, \eta)$ contain a straight line parallel to the kernel of $H(\beta)$.

The *global accessible set for w* is the intersection of all the accessible sets for w associated with each wheel :

$$D(\beta, \eta) = \bigcap_{i=1}^k D_i(\beta, \eta). \quad (1.34)$$

The geometrical interpretation of this set is a prism in $\mathbb{R}^3$ defined as the intersection of all the "millefeuilles" and that contains a straight line parallel to the kernel of $H(\beta)$ (see figure 1.2 for a 3-wheels robot). This set is obviously a convex subset of $\mathbb{R}^3$.

The fact that w is accessible does not prevent the robot or one of its wheels to become singular. This notion allows us to choose w compatible with the kinematic constraints even in case of singularity. This is used in the next chapter to design a control law that allows the robot to go through the singularities.

1.4 Non singular and admissible trajectories

With respect to the configuration variables $(\xi, \eta, \beta, \varphi)$, the dynamics is described by equations (1.20) to (1.23) with condition (1.24). We define an *admissible configuration trajectory* as a trajectory in the configuration space that leads to accessible values of the input w ($w \in D(\beta, \eta)$). Such a trajectory may be singular ($\exists i \in \{1, \dots, k\}, \dot{\varphi}_i = 0$) with bounded wheels orientation velocities ($|\dot{\beta}_i| < \dot{\beta}_s$).

Consider now a trajectory $\xi(t)$ defined in the posture space. It is possible to derive the evolution of $\eta(t)$ and $\dot{\eta} = w$ according to relations

$$\eta = R(\theta)\dot{\xi}, \quad (1.35)$$

$$\dot{\eta} = w = \dot{R}(\theta)\dot{\xi} + R(\theta)\ddot{\xi}. \quad (1.36)$$

The singularities are detected observing the values of $G_i\eta(t)$ for $i = 1, \dots, k$ and it is therefore possible to know whether this trajectory leads to singular configurations.

A trajectory $\xi(t)$ verifying the condition below

$$\forall t \geq t_0, \forall i = 1, \dots, k \quad G_i\eta(t) \neq 0 \quad (1.37)$$

is called *non singular*. For such a trajectory, it is possible to derive the evolution of $\beta(t)$ and the inputs u_i and v_i using (1.28) and (1.29). Indeed, the kinematic model (1.20)-(1.23) is differentially flat with respect to the output ξ .

But it may happen that the reorientation of the wheels is rapid violating the condition (1.32). It has been shown however that the bound is respected provided w remains in D , i.e. w is accessible. Such a non singular trajectory leading to accessible values of the input w is said to be *posture admissible*. If the trajectory defined by $\xi(t)$ is singular, at least one orientation angle is not defined and the accessibility set can not be defined.

These concepts of posture and configuration admissible trajectories together with the differential flatness of the system restricted to the posture trajectories (see [4]) are useful to address the problems of path planning and feedback control in the posture domain. The problem of path planning reduces to the selection of a posture trajectory $\xi(t)$, according to (1.35), (1.36), and satisfying, in addition to path arguments (such as obstacle avoidance or cost minimization), the condition that w is accessible. Then, due to the differential flatness, it is straightforward to link the physical inputs to the chosen posture trajectory. In chapter 2, we discuss in detail the problem of feedback control design for the robot posture, with particular attention for the trajectory tracking problem.

Chapter 2

Posture control design

This chapter deals with the design of feedback control laws for the posture of ideal robots equipped with steering wheels, in the framework introduced in the previous chapter. The posture stabilization and posture tracking problems have been addressed by Thuilot [17] who proposes control laws ensuring that the robot configurations remain non singular along the closed-loop trajectories. But there is no guarantee that the wheel orientation velocities satisfied specified physical bounds. Our concepts of accessible acceleration and admissible trajectories allows to design control laws satisfying these constraints.

In section 1, we show how to assign the posture acceleration w by an adequate choice of the physical inputs. We then restrict the analysis to the posture tracking problem. In section 2, we focus on a particular feedback control design, resulting in a generic auxiliary control w_d which of course may be non accessible. In section 3, we discuss sufficient conditions ensuring that, with this generic control law, the closed-loop trajectories remain non singular, or posture admissible. But these conditions are very restrictive. It is the reason why we propose in section 4 to modify the control law when the generic control w_d becomes non accessible. It is then replaced by the closest accessible w . This allows the closed-loop trajectory to go through singular configurations.

2.1 Posture control

The posture kinematic model in (1.20) and (1.21) may be rewritten in normal form by the change of variable that follows :

$$\xi_1 = \xi, \quad (2.1)$$

$$\xi_2 = R(\theta)^T \eta. \quad (2.2)$$

With respect to the new variables, the kinematic model becomes

$$\dot{\xi}_1 = \xi_2, \quad (2.3)$$

$$\dot{\xi}_2 = \dot{R}^T(\theta)\eta + R^T(\theta)w, \quad (2.4)$$

and the system is written in normal form with w as new input. A lot of control approaches exist for such systems such as linearization, backstepping or non linear control design (see [27], [6], [24] for example) that enable us to assign the input w for the robot to achieve a specified task.

The next problem is to choose the physical inputs u and v in order to assign the desired input w . The input w is not directly involved in the kinematic constraints (1.24). It is therefore necessary to differentiate these relations in order to involve explicitly w : this results in the following relations :

$$P(\beta, \eta)u + H(\beta)w = 0, \quad (2.5)$$

$$H^*(\beta)w = r\dot{\varphi} = r\dot{v}, \quad (2.6)$$

where we used the fact that $H(\beta)\eta = 0$. Using property P3, we conclude that the mapping $(u, \dot{v}) \rightarrow w$ is onto. For any w , there exists u and $\dot{v}$ satisfying the relations (2.5) and (2.6), and for $(u, \dot{v})$ satisfying these relations, the w is unique. Moreover, in the generic case (i.e. not in a singular configuration), the $(u, \dot{v})$ corresponding to a given w are unique

$$u = -P^{-1}(\beta, \eta)H(\beta)w \quad (2.7)$$

$$\dot{v} = \frac{1}{r}H^*(\beta)w. \quad (2.8)$$

When one wheel is singular ($\exists l \in 1, \dots, k$ with $G_l\eta = 0$), $\dot{v}$ is unique, the orientation velocity associated with the non singular wheel ($i \neq l$) are given by

$$u_i = -\frac{R_1^T(\beta_i)G_i w}{R_2^T(\beta_i)G_i \eta} \quad (2.9)$$

and u_l can be chosen freely. However, if w is continuous, it is possible to fix the value of u_l by continuity.

When the robot is singular ($\eta = 0$), $\dot{v}$ is still unique and the orientation velocities can all be chosen freely. In that case we decide to fix the wheels assigning $u = 0$.

2.2 Feedback linearization for posture tracking

The problem we address is the tracking of a reference trajectory $\xi_r(t)$. Exploiting the normal form of the posture model and the fact that the input w can be manipulated from the physical inputs, there are several methods to solve this problem. We focus here on a particular approach : feedback linearization control allowing us to assign stable linear dynamics to the tracking error. It must be clear however that the developments hereafter can easily be extended to other control design resulting in a generic w_d , provided they ensure exponential convergence of the tracking error.

It is straightforward to see that the following choice of the input w_d ensures linear exponentially stable dynamics for the posture tracking error :

$$w_d = R(\theta) \left[z - \dot{R}^T(\theta)\eta \right]. \quad (2.10)$$

with

$$z(\xi, \dot{\xi}, t) = \ddot{\xi}_r - K_1(\xi - \xi_r) - K_2(\dot{\xi} - \dot{\xi}_r) \quad (2.11)$$

where K_1, K_2 are such that the matrix

$$\Gamma = \begin{pmatrix} 0 & I \\ -K_1 & -K_2 \end{pmatrix} \quad (2.12)$$

is a Hurwitz stable matrix.

With respect to the variable $\tilde{y}$ defined as

$$\tilde{y} = \begin{pmatrix} \tilde{\xi} \\ \dot{\tilde{\xi}} \end{pmatrix} = \begin{pmatrix} \xi - \xi_r \\ \dot{\xi} - \dot{\xi}_r \end{pmatrix} \quad (2.13)$$

the closed-loop dynamics has the form below

$$\dot{\tilde{y}} = \Gamma \tilde{y}. \quad (2.14)$$

When out of singularity, any desired value w_d may be assigned. When at singularity, w_d has to belong to $\text{Ker}(H(\beta))$ and may not be assigned freely. Furthermore, if the robot is near to singularity, the desired w_d may become non accessible. This is the subject of the next section to give sufficient conditions for the generic control law to lead to non singular or posture admissible trajectories of the actual robot.

2.3 Avoiding singularities for the generic control law

2.3.1 Sufficient conditions for closed-loop non singular trajectories

Consider a reference trajectory $\xi_r(t)$ which is non singular. In order to characterize the level of non singularity of this trajectory, we adopt the same approach as Thuilot in [17] : we define a *non singularity margin* with respect to the tracking error vector $\tilde{y}$. This is denoted $M_s(t)$ and mathematically defined by the relation

$$\forall t > t_0, \|\tilde{y}(t)\| < M_s(t) \Rightarrow \xi(t) \text{ corresponds to a non singular configuration.} \quad (2.15)$$

The construction of such a function $M_s(t)$ with respect to our variable choice may be found in appendix B.1.

When the robot goes through a singularity, the generic control law leads to instantaneous reorientation of the wheel(s). This is not acceptable in practice. We therefore give the sufficient conditions on the reference trajectory and on the initial conditions for the robot commanded with the generic control law to avoid singularities.

Proposition 2.1 *Given $K(t)$, an exponentially decreasing function ($K(t) = k_1 e^{-k_2(t-t_0)}$ with k_1, k_2 , two positive constants) characterizing the robot closed-loop dynamics for the generic control law, i.e.*

$$\forall t > t_0, \quad \|\tilde{y}(t)\| \leq K(t)\|\tilde{y}(t_0)\|. \quad (2.16)$$

*Given a non singular reference trajectory $\xi_r(t)$ with non singularity margin $M_s(t)$.
Given ε an arbitrarily small positive number.*

If

$$\|\tilde{y}(t_0)\| \leq \min_{t \geq t_0} \left(\frac{M_s(t) - \varepsilon}{K(t)} \right), \quad (2.17)$$

then no singularity problem occurs during the robot motion.

This result has been proven by Thuilot in [17]. The first step is to verify there is no singularity at time t_0 and the second step is to prove there is no singularity for later time. The latter is proven by contradiction. The function $K(t)$ may easily be derived finding the Liapounov function associated with the linearizing control law and following the same development as in [17].

2.3.2 Sufficient conditions for closed-loop posture admissible trajectories

For a posture admissible (and therefore non singular) reference trajectory, if the initial conditions are not too far from the reference initial conditions, we hope that the robot driven with the generic control law will follow a posture admissible trajectory. This analysis supposes us to introduce the notion of *admissibility margin* with respect to the tracking error vector $\tilde{y}$:

$$\forall t > t_0, \quad \|\tilde{y}(t)\| < M_{nas}(t) \Rightarrow \xi(t) \text{ is a non singular} \\ \text{and posture admissible trajectory} \quad (2.18)$$

The construction of such a function $M_{nas}(t)$ associated to the generic control law may be found in appendix B.2.

This definition allows us to formulate the following proposition :

Proposition 2.2 *Given $K(t)$, an exponentially decreasing function ($K(t) = k_1 e^{-k_2(t-t_0)}$ with k_1, k_2 , two positive constants) characterizing the robot closed-loop dynamics for the generic control law, i.e.*

$$\forall t > t_0, \quad \|\tilde{y}(t)\| \leq K(t)\|\tilde{y}(t_0)\|. \quad (2.19)$$

*Given a posture admissible reference trajectory $\xi_r(t)$ that with admissibility margin $M_{nas}(t)$.
Given ε an arbitrarily small positive number.*

If

$$\|\tilde{y}(t_0)\| \leq \min_{t \geq t_0} \left(\frac{M_{nas}(t) - \varepsilon}{K(t)} \right), \quad (2.20)$$

then no singularity problem occurs during the robot motion and the actual trajectory of the robot remains posture admissible.

The proof of this proposition is obtained by the same developments as proof of Proposition 2.1 of Thuilot.

These results show the limitation of application of the generic control law but not really solve the singularity problem. The next section is devoted to an heuristic extension of the control law for trajectories that are singular or non posture admissible. Our objective is to modify the generic law so that it can be assigned in practice even in case of singularity and so that the tracking is as good as possible. This amounts to impose the robot closed-loop trajectory to be configuration admissible or equivalently to impose the assigned acceleration vector to live in the accessibility prism.

2.4 Control law allowing the robot to go through singularities

If the desired w_d is non accessible, it can not be assigned in practice. For w_d leaving $D(\beta, \eta)$, it is necessary to modify the generic control law in order to assign an accessible w , while not affecting too much the evolution of the tracking error. Several approaches are possible. For instance, we may chose $w \in D(\beta, \eta)$ in such a way that the evolution of the Liapounov function associated with the generic design is as close as possible to the normal generic evolution, or at least decreasing along the closed-loop trajectories. We propose here an other approach : when w_d is non accessible, w is chosen as the closest accessible acceleration vector. This choice reduces therefore to a standard minimization problem. The advantage of this solution it that provided w_d evolves continuously, the assigned value w is also continuous. This allows us to define the orientation velocity of the wheel by continuous extension in case of one wheel singularity.

2.4.1 A constraint minimization problem

If w_d is inside the accessibility prism, we assign w to this value because it ensures the tracking error to be exponentially decreasing. When w_d is non accessible, we propose to assign an accessible value of w that is as near as possible of the desired value w_d . We therefore minimize the distance between w and w_d with respect to a given norm, for example the 2-norm :

$$\min_{w \text{ accessible}} \|w - w_d\|_2^2. \quad (2.21)$$

The accessible set of w defines a prism in $\mathbb{R}^3$ that is a convex set. With respect to the 2-norm, the cost function in (2.21) is quadratic in w . The problem is therefore

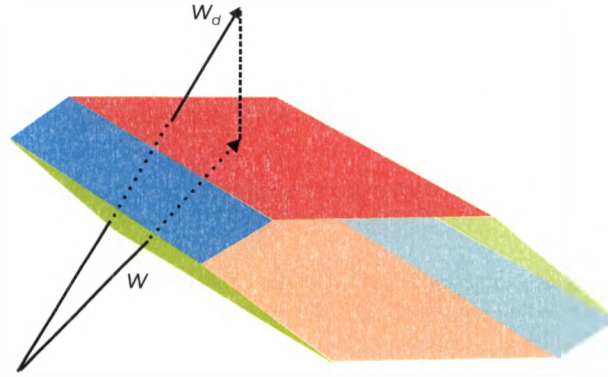


Figure 2.1: Illustration of the constraint minimization solution

easily solved applying classical optimization methods. Several methods of quadratic programming are described in [31], and the *quadprog.m* function of Matlab implements a trust-region method for large-scale problems and an active set method for the small-scale problems. The optimization operation is easily interpreted graphically as a projection on the accessible prism of the desired vector w_d (see figure 2.1).

The cost function should have been defined with respect to an other norm but the 2-norm present several advantages :

- when w_d is inside the accessibility prism, w is assigned to w_d ;
- when w_d goes out of the prism, no discontinuity occurs in the input w and therefore in the physical inputs β and $\dot{\varphi}$;
- as a consequence of the above observation, an input $\dot{\beta}_i$ that becomes undefined at singularity may then be defined by continuous extension.

From now on, we adopt the convention that the robot is said to be *bridled* if w_d is non accessible.

2.4.2 Practical implementation

To implement the control law, we take into account the special structure of our system using the singular value decomposition of our constant rank matrix $H(\beta)$ defined in (1.13) :

$$H = \begin{pmatrix} | & & | \\ l_1 & \dots & l_k \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} - & r_1^T & - \\ - & r_2^T & - \\ - & r_3^T & - \end{pmatrix}. \quad (2.22)$$

The input w and its desired value w_d is expressed with respect to the r_i base :

$$w = \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3, \quad (2.23)$$

$$w_d = \gamma_1 r_1 + \gamma_2 r_2 + \gamma_3 r_3. \quad (2.24)$$

With this conventions, the cost function in (2.21) becomes

$$\|w - w_d\|_2^2 = (\alpha_1 - \gamma_1)^2 + (\alpha_2 - \gamma_2)^2 + (\alpha_3 - \gamma_3)^2 \quad (2.25)$$

that is an elementary quadratic function with respect to variable $\alpha_1, \alpha_2, \alpha_3$.

The vector Pu of equation (1.27) is also developed with respect to the l_i base :

$$Pu = \delta_1 l_1 + \dots + \delta_k l_k. \quad (2.26)$$

The above development and equation (1.27) allows us to deduced that

$$\begin{pmatrix} - & l_1^T & - \\ & \vdots & \\ - & l_k^T & - \end{pmatrix} Pu = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_k \end{pmatrix} = \begin{pmatrix} -\sigma_1 \alpha_1 \\ -\sigma_2 \alpha_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (2.27)$$

Consequently, the kinematic constraints in (1.27) become

$$Pu = -\sigma_1 \alpha_1 l_1 - \sigma_2 \alpha_2 l_2, \quad (2.28)$$

where the vector does not depend on the third component of w , denoted α_3 . The cost function (2.25) is therefore simplified choosing $\alpha_3 = \gamma_3$ because variable α_3 does not affect the accessibility of w .

We now discuss the implementation of the control law for the different possible values of $\text{rank}(P)$:

rank(P) = k : In that case, we assign the input u_i as follows :

$$\forall i = 1, \dots, k \quad u_i = P_i^{-1} (-\sigma_1 \alpha_1 l_{1i} - \sigma_2 \alpha_2 l_{2i}) \quad (2.29)$$

where l_{1i} denotes the i -th component of vector l_1 . The scalars α_1, α_2 are chosen so that the associated value of w is admissible. The minimization problem takes thus the form below

$$\min_{\alpha_1, \alpha_2} (\alpha_1 - \gamma_1)^2 + (\alpha_2 - \gamma_2)^2 \quad (2.30)$$

under constraints

$$P_i^{-1} (-\sigma_1 \alpha_1 l_{1i} - \sigma_2 \alpha_2 l_{2i}) < \beta_s \quad \forall i = 1, \dots, k \quad (2.31)$$

$$-P_i^{-1} (-\sigma_1 \alpha_1 l_{1i} - \sigma_2 \alpha_2 l_{2i}) < \beta_s \quad \forall i = 1, \dots, k \quad (2.32)$$

and it can be solved with classical optimization tools. We used the Matlab *quadprog.m* procedure in our simulations. The dynamics on inputs v_i is defined according to equations (2.6).

rank(P) = $k - 1$: Let us assume that the singular wheel is wheel 1. The first row of the kinematic constraints in (2.28) then takes the form

$$-\sigma_1\alpha_1l_{11} - \sigma_2\alpha_2l_{21} = 0 \quad (2.33)$$

that is a linear constraint in α_1 and α_2 . This equation no longer depends on u_1 that is chosen arbitrarily. We decide to fix the velocity orientation of the singular wheel assigning u_1 constant. The orientation velocities of the non singular wheels are still chosen according to expression (2.29) where the w vector is defined solving the following optimization problem :

$$\min_{\alpha_1, \alpha_2} (\alpha_1 - \gamma_1)^2 + (\alpha_2 - \gamma_2)^2 \quad (2.34)$$

under constraints

$$-\sigma_1\alpha_1l_{11} - \sigma_2\alpha_2l_{21} = 0 \quad (2.35)$$

$$P_i^{-1}(-\sigma_1\alpha_1l_{1i} - \sigma_2\alpha_2l_{2i}) < \dot{\beta}_s \quad \forall i = 2, \dots, k \quad (2.36)$$

$$-P_i^{-1}(-\sigma_1\alpha_1l_{1i} - \sigma_2\alpha_2l_{2i}) < \dot{\beta}_s \quad \forall i = 2, \dots, k. \quad (2.37)$$

This value of w is also used to compute the compatible values of the dynamics in v_i ($i = 1, \dots, k$) according to (2.6).

rank(P) = 0 : In that case the kinematic constraints do not restrict the inputs u_i ($i = 1, \dots, k$). But the inputs have still to be chosen so that the coordination is ensured and we decide to assign $u_i = 0$ for all $i = 1, \dots, k$. The associated values of the v_i inputs are $v_i = 0$ for all $i = 1, \dots, k$ because all the wheels stop.

2.4.3 Tracking error analysis

It can be expected that the proposed method to select the assigned value of w when the robot is bridled produces satisfying results because it minimizes the difference between the desired value of w , resulting from the generic control law, and the actual w . But we are not able to show that for any initial conditions the tracking error globally decreases because there always exist reference trajectories such that the projection we apply makes us loose the good direction.

If the reference trajectory presents finite bridle times that are far enough from each other, we hope the tracking error to be globally decreasing : it decreases rapidly when the robot is not bridled, then maybe increases while the robot is bridled, but leaves the bridle area, then decreases, ... (see figure 2.2). But it also may happen that the bridle periods are too close (see figure 2.3) or that the robot does not leave the bridle area. We can not know the bridle times and error increase rate without knowing the reference trajectory a priori.

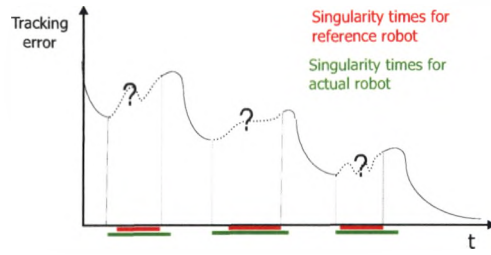


Figure 2.2: Globally decreasing tracking error

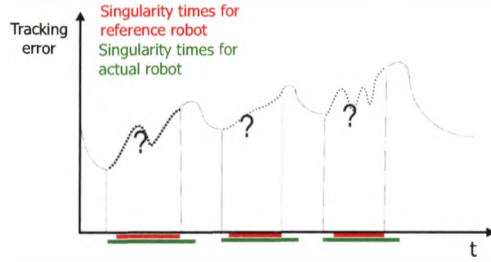


Figure 2.3: Globally increasing tracking error

2.5 Control law taking into account some geometrical constraints

This section is devoted to an other implementation of the previous constrained control law. The robot construction imposes us some practical bounds on the orientation angles of the wheels :

$$\forall i = 1, \dots, k \quad |\beta_i| < \alpha < \frac{\pi}{2}. \quad (2.38)$$

These geometrical constraints are met if we apply the previous control law with additive constraints on the orientation velocities. When one of the orientation angle arises near the maximal value, we impose the orientation velocity associated with this wheel to be negative :

$$\exists i \in \{1, \dots, n\} \text{ such that } \tan(\beta_i - \alpha) < d_{min} \implies \dot{\beta}_i < 0. \quad (2.39)$$

While the angle β_i is not far enough from the maximal bound, the above additive constraint is imposed,

$$\text{while } \tan(\beta_i - \alpha) < d_{leave}, \quad \dot{\beta}_i < 0. \quad (2.40)$$

Similarly, when one orientation angle approaches the minimal bound, we impose the associated orientation velocity to be positive as long as the angle is not far enough of the bound. In term of the accessibility prism, it amounts to project for a time the desired acceleration vector w_d on half of the prism, because one of the "millefeuille" is more thin.

The control law causes thus an hysteresis phenomenon : the orientation velocities depends on the past values of the orientation angles. This phenomenon makes the tracking error analysis much more difficult than in the previous section. For that reason, we do not further detail this question.

2.6 Possible extensions of the approach

The accessibility domain

In this work, we do not consider constraints on the orientation torques because these torques depend essentially on the friction torques of the ground that dominate the effect of wheel inertia. However, it should be taken into account considering the derivative of the acceleration as posture input.

On the other hand, bounds on the rotation torques should easily be taken into account in the accessibility definition. The global accessible set then would become a finite height prism that is a still convex subset.

The robot tasks

Our approach could be implemented to stabilize the robot posture at a given position, but nothing may be hoped from the stabilization error dynamics because the robot tends towards a singular configuration. Remark that in [17], Thuilot constructs a time varying control law that ensures the posture stabilization while avoiding the singularities.

The desired acceleration vector

Other generic control laws could be implemented as far as they ensure the exponential tracking error decrease. A Liapounov design approach could be implemented or backstepping, ...

The assigned accessible acceleration vector

The constrained control law can be implemented for the cost function written in term of another norm. It is also possible to select the admissible value of w according to a Liapounov approach. The Liapounov function associated with the generic control law is imposed to be as decreasing as possible along the admissible trajectories. This cost function is linear in w and the constraint problem is easily solved. But it still does not ensure that the tracking errors always decreases. Furthermore, such an approach may lead to discontinuities in the input w and therefore also in the physical inputs $\dot{\beta}$ and $\dot{\varphi}$.

Chapter 3

Simulation results

In this chapter, we give some simulation results to illustrate the efficiency and limits of the different control laws. The reference robot presented in the first section is a four steering wheels robot. We analyze the tracking error for different reference trajectories, different initial conditions and saturation levels β_s . The simulation results are presented in the next two sections. The second section is devoted to simulations for the constrained control law taking into account the velocity constraints for the reference example. The third section is devoted to simulations taking into account some additional geometrical constraints on the orientation angles of the wheels.

3.1 Reference example

We illustrate our results on a robot equipped with 4 steering wheels with same radius denoted r . The four wheels are attached on the tops of a rectangle with length $2L$ and width $2l$ (see figure 3.1). The reference point P is the center of the rectangle. The moving frame $\mathbf{X}_1, \mathbf{X}_2$ is chosen with axis $\mathbf{X}_1$ parallel to the direction of the width.

The centers of the steering wheels are located by their components with respect to the moving frame : $(a_1, b_1) = (l, L)$, $(a_2, b_2) = (-l, L)$, $(a_3, b_3) = (-l, -L)$ and $(a_4, b_4) = (l, -L)$.

In our simulations, the physical parameters have been fixed to the values below :

$$L = 2.5m, \quad l = 1.5m, \quad r = 0.35m. \quad (3.1)$$

3.2 The velocity constraints control law

In this section, we consider the reference trajectory is a sine function of amplitude 2 meters with constant orientation of the frame :

$$\xi_r(t) = \begin{pmatrix} t \\ \sin(t) \\ \frac{\pi}{4} \end{pmatrix}. \quad (3.2)$$

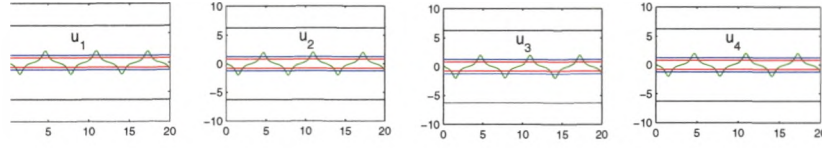


Figure 3.2: Orientation velocities along the reference trajectory (rad/s)

The associated curves are plotted in green in all the graphics.

The evolution of the inputs u_i ($i = 1, 2, 3, 4$) associated with this reference trajectory is presented in figure 3.2. We test the efficiency of the different control laws on this trajectory with different values of the saturation bound β_s . For $\beta_s = 2\pi$ (plotted in black on figure 3.2), the reference trajectory is posture admissible. For $\beta_s = \frac{2\pi}{5}$ rad/s (in blue) and $\beta_s = \frac{\pi}{4}$ (in red), the reference trajectory is posture non admissible.

All the plots corresponding to a particular simulation are presented on one page. We first show the trajectories with the reference trajectory in green and the closed-loop trajectory in blue (graph a). We then show the tracking errors with the position errors expressed in meters and the orientation error expressed in radians (graph b). The rotation velocities (expressed in rad/s) are plotted next to detect the singular configurations (graph c). Finally, the orientation velocities (in rad/s) are given to highlight the non admissibility times of the trajectory (for generic control law) or the bridle times (for the constraint control law) (graph d). These non admissibility or bridle times are easily detected on all the plots looking at the constraint bar color : the blue color is associated with admissibility or non bridle times, the red color corresponds to the non admissible or bridle times.

In the expression of $v(\xi, \dot{\xi}, t)$ defined in (2.11), we choose the matrix gains under the form below :

$$K_1 = \text{diag}(\omega_1^2, \omega_2^2, \omega_3^2) \quad K_2 = \text{diag}(2\omega_1\zeta, 2\omega_2\zeta, 2\omega_3\zeta),$$

so that the tracking errors dynamics are given by a second order equation :

$$\ddot{x} = -\omega^2 x - 2\zeta\omega\dot{x}.$$

For all the simulations, the values of the gains are $\zeta = 0.7$ and $\omega_1 = \omega_2 = 2.86(1/s)$ so that the associated value of the time constant is $\frac{1}{\zeta\omega_{1,2}} = 0.5s$; and $\omega_3 = 0.86(1/s)$ so that the associated time constant is $\frac{1}{\zeta\omega_3} = 1.7s$.

3.2.1 Generic control law

We here show some simulation results that illustrate the limits of the generic control law that is given by equations (1.28), (1.29). The saturation level β_s is given for information on the configuration admissibility of the reference trajectory. In the next section, we present simulations for the constrained control law and the same parameters values and initial conditions. The aim is to compare the results of the two control laws and especially

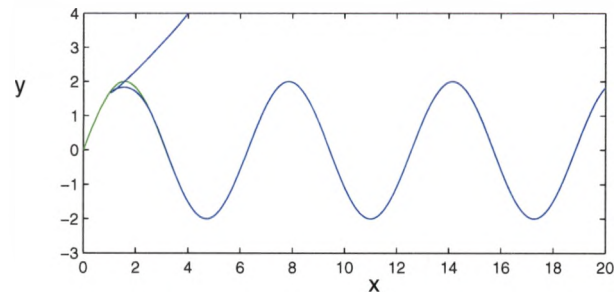
to see how much the tracking error dynamics is degraded when the constrained control law is assigned.

generic control law	$\dot{\beta}_s = 2\pi rad/s$	bad initial conditions
---------------------	------------------------------	------------------------

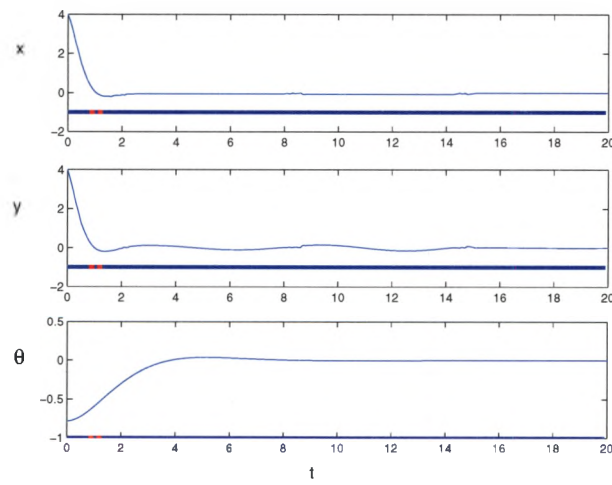
For $\dot{\beta}_s = 2\pi rad/s$, the reference trajectory is non singular and posture admissible. We impose bad initial conditions ($\xi_0 = [4; 4; 0]$ and $\dot{\xi}_0 = [-2; -2; 0]$) meaning that the robot has to make a tight turn and the generic control law leads to a non configuration admissible trajectory (on figure 3.3). We observe the exponential decrease of the tracking error and it may be verified on the v_i plots that the robot is near to stop about at time instant $t = 1s$. The u_i plots show that at this instant the wheels have to be reoriented rapidly because the u_i inputs exceed the admissible bounds.

generic control law	$\dot{\beta}_s = \frac{2\pi}{3} rad/s$	good initial conditions
---------------------	--	-------------------------

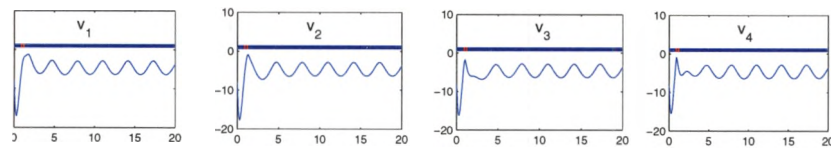
For a lower saturation level, the reference trajectory is non configuration admissible. We give simulations for good initial conditions ($\xi_0 = [0; 0; \frac{\pi}{4}]$ and $\dot{\xi}_0 = [1; 1; 0]$), that is that these are near the reference initial conditions. At figure 3.4, we observe that no singular configuration is encountered but that the orientation velocities of the wheels become too large for the trajectory to remain admissible.



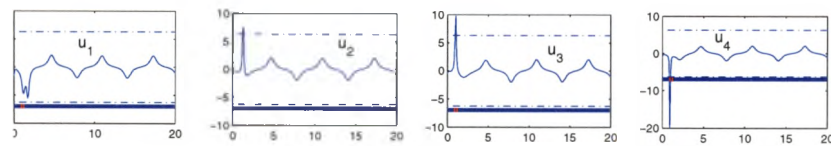
a) Trajectories (meters)



b) Tracking errors (meters and radians)

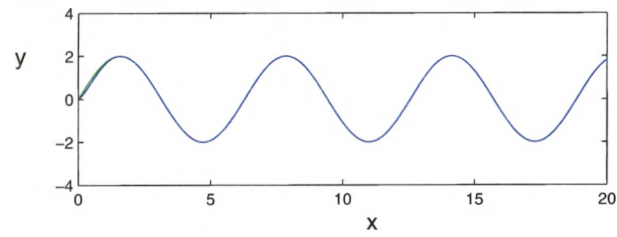


c) Rotation velocities (rad/s)

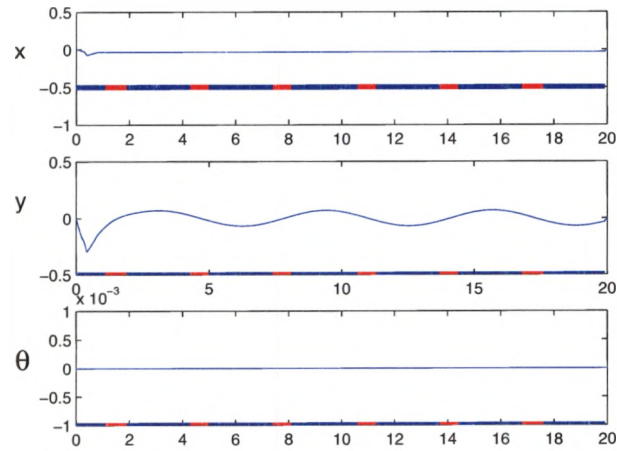


d) Orientation velocities (rad/s)

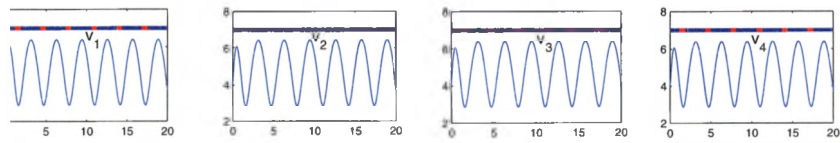
Figure 3.3: generic control law $\beta_s = 2\pi \text{rad/s}$ bad initial conditions



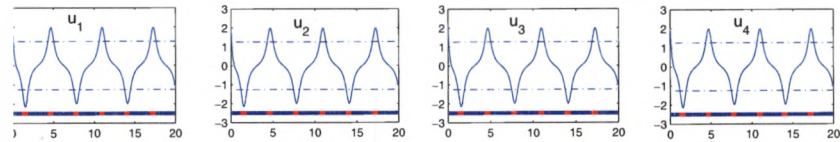
a) Trajectories (meters)



b) Tracking errors (meters and radians)



c) Rotation velocities (rad/s)



d) Orientation velocities (rad/s)

Figure 3.4: generic control law $\dot{\beta}_s = \frac{2\pi}{5} \text{rad/s}$ good initial conditions

3.2.2 Constrained control law

We present here some simulation for the constrained control law designed in section 2.3. We first give the plots associated with the constrained control law for the same parameters values and same initial conditions as previously. It is therefore possible to compare the generic and the constrained control law results.

constrained control law	$\dot{\beta}_s = 2\pi rad/s$	bad initial conditions
-------------------------	------------------------------	------------------------

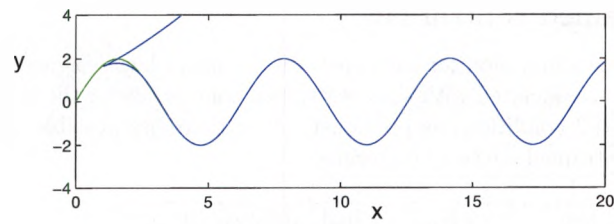
For $\dot{\beta}_s = 2\pi rad/s$ and bad initial conditions, the bridle times are short and the trajectory associated with the constraint control law remains acceptable (see figure 3.5). The tracking errors plots may be compared to the plots obtained with the generic control law in figure 3.3. We observe a slight increase of the tracking error while in bridle area, but it globally decreases. The robot stops and goes back because all the rotation velocities change of sign at the same time instant. In comparison, for the generic control law (in figure 3.3) the robot makes a tight turn but does not stop. For the constraint control law, the orientation velocities remain acceptable, while for the generic control law the closed-loop trajectory is non admissible.

constrained control law	$\dot{\beta}_s = \frac{2\pi}{5} rad/s$	good initial conditions
-------------------------	--	-------------------------

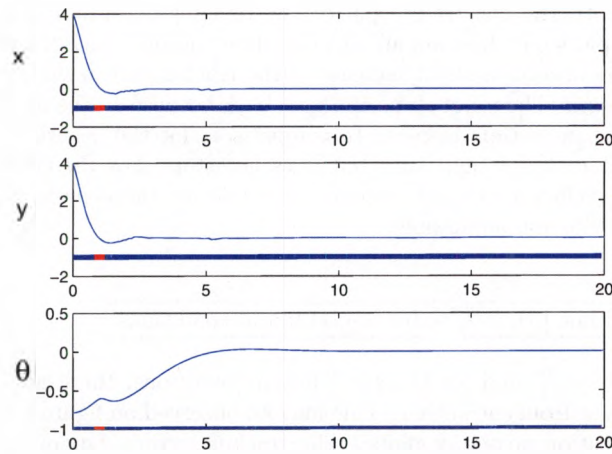
In the case when $\dot{\beta}_s = \frac{2\pi}{5}$ and for the good initial conditions, there are several bridle areas that are quite far from each other. This may be observed on figure 3.6 on the bridle bar or on the orientation velocities plots. The tracking errors dynamics are of course degraded when in bridle area. This may be compared with the results obtained for the generic control law in figure 3.4.

constrained control law	$\dot{\beta}_s = \frac{\pi}{4} rad/s$	good initial conditions
-------------------------	---------------------------------------	-------------------------

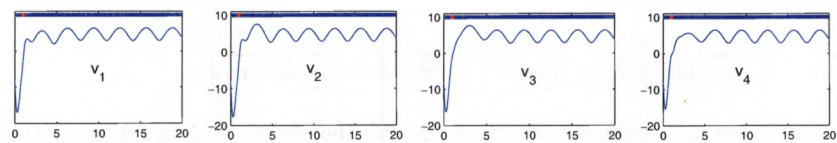
Finally, we present a simulation for $\dot{\beta}_s = \frac{\pi}{4}$ and the same good initial conditions as previously. With this constraint, the admissible trajectory is much farther from the reference trajectory (see figure 3.7) because the robot is too often bridled. The tracking error dynamics is thus degraded and the orientation velocities are very often saturated. Remark the robot two times stops and goes back : the rotation velocities then all change of sign.



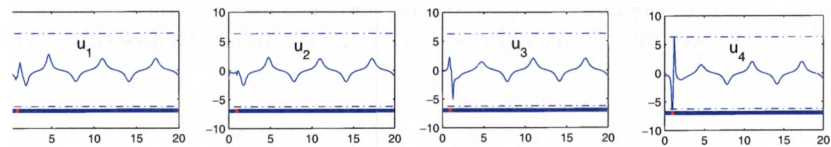
a) Trajectories (meters)



b) Tracking errors (meters and radians)

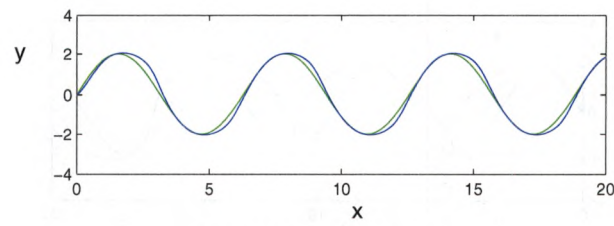


c) Rotation velocities (rad/s)

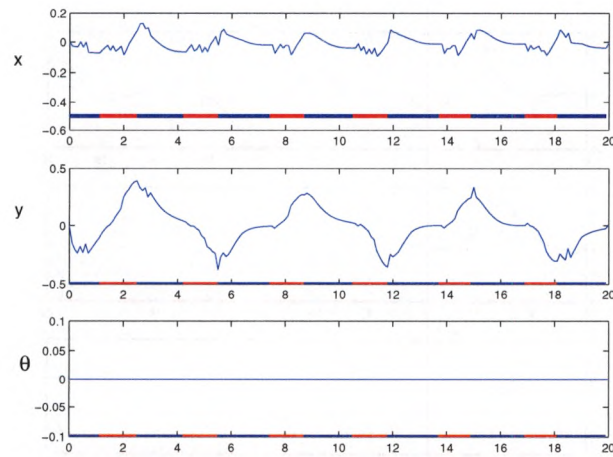


d) Orientation velocities (rad/s)

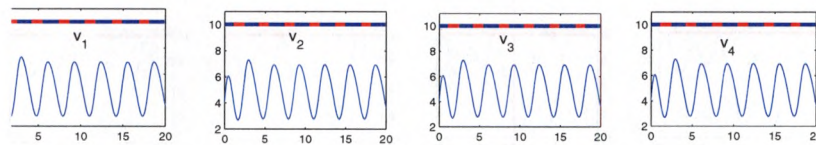
Figure 3.5: constrained control law | $\beta_s = 2\pi rad/s$ | bad initial conditions



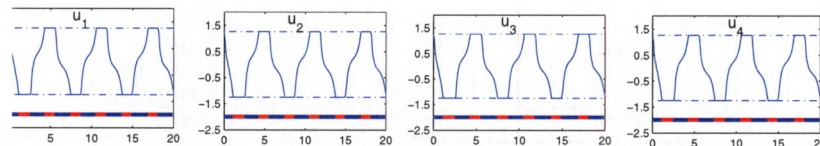
a) Trajectories (meters)



b) Tracking errors (meters and radians)



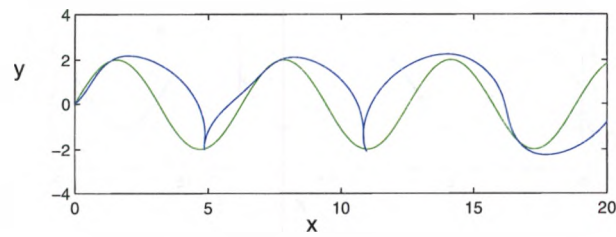
c) Rotation velocities (rad/s)



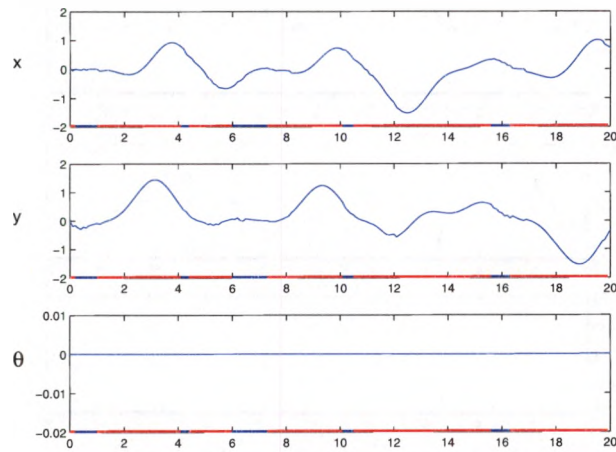
d) Orientation velocities (rad/s)

Figure 3.6:

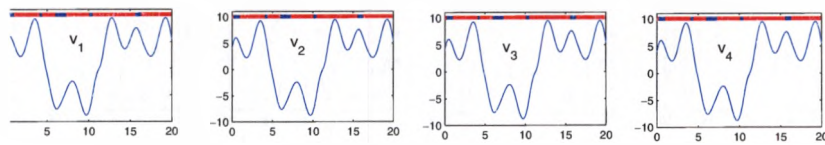
constrained control law	$\dot{\beta}_s = \frac{2\pi}{5} \text{rad/s}$	good initial conditions
-------------------------	---	-------------------------



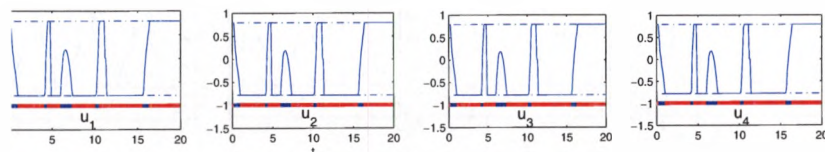
a) Trajectories (meters)



b) Tracking errors (meters and radians)



c) Rotation velocities (rad/s)



d) Orientation velocities (rad/s)

Figure 3.7: constrained control law $\beta_s = \frac{\pi}{4} \text{rad/s}$ good initial conditions



Figure 3.8: Bounded orientation angles impose the ICR to live outside a X-area or inside a diamond ace area

3.3 The geometrical constrained control law

We present simulations for the four steering wheels mobile robot described before. The bounds on the orientation angles impose the ICR to live either outside a X form area or inside a diamond ace form area (see figure 3.8). If the bound on orientation angles α is less than $\frac{\pi}{4}$, the diamond ace area does not exist.

Remark it is possible for the ICR to go from the right area to the left area (or the contrary) while the robot moves. This corresponds to an ICR going to the right infinite and coming back by the left infinite (or the contrary). To go from the left or right area to the diamond ace area (or vice versa), the robot needs to stop and to reorient its wheels before to move again.

In our simulations, the geometrical and control parameters values are the following for all the wheels :

$$\alpha = \frac{4\pi}{10}, \quad \dot{\beta}_s = 2\pi, \quad d_{min} = 0.05m, \quad d_{leave} = 0.25m, \quad (3.3)$$

and the gain matrix K_1, K_2 are unchanged with respect to the previous section.

In this section, the simulation results are still presented on one page with the same graphics as in the previous section. We add plots of the orientation angles tangent with

the maximal and minimal values required in dot line.

3.3.1 ICR outside the X area

We give simulation results associated with a sine reference trajectory with amplitude 2 meters, with different values of the pulsation ω_{ref} and with fixed orientation θ_r :

$$\xi_r(t) = \begin{pmatrix} t \\ 2\sin(\omega_{ref}t) \\ \frac{\pi}{4} \end{pmatrix}. \quad (3.4)$$

The orientation angles and orientation velocities associated with this reference trajectory with $\omega_{ref} = 0.3(1/s)$ are plotted in figure 3.9 and 3.10. The reference trajectory is posture admissible but leads to orientation angles beyond the imposed bounds. For the value $\omega_{ref} = 1(1/s)$, the reference trajectory is admissible (see figure 3.11) but the orientation angles often exceed the imposed bounds (at figure 3.12). These two reference trajectories are now tested on the geometrical constrained control law.

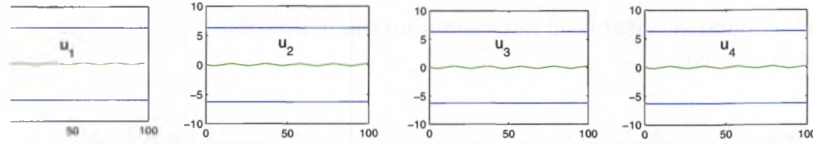


Figure 3.9: Orientation velocities (rad/s) for the reference trajectory with $\omega_{ref} = 0.3(1/s)$

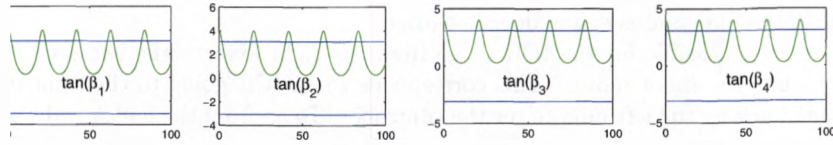


Figure 3.10: Orientation angles tangent for the reference trajectory with $\omega_{ref} = 0.3(1/s)$

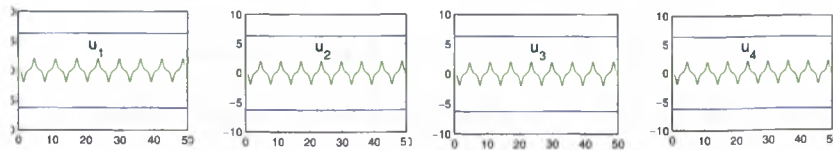
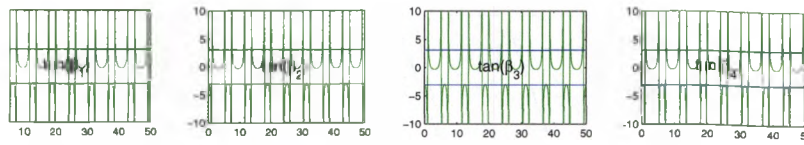


Figure 3.11: Orientation velocities (rad/s) for the reference trajectory with $\omega_{ref} = 1(1/s)$

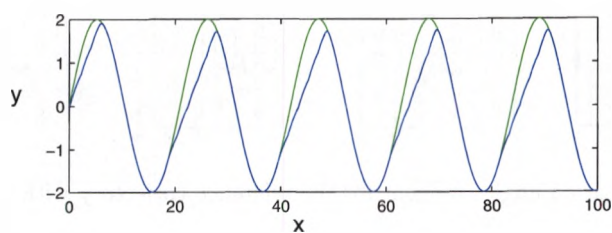
Figure 3.12: Orientation angles tangent for the reference trajectory with $\omega_{ref} = 1(1/s)$

geometrical constrained control law	$\beta_s = 2\pi rad/s$	$\omega_{ref} = 0.3(1/s)$	good initial conditions
-------------------------------------	------------------------	---------------------------	-------------------------

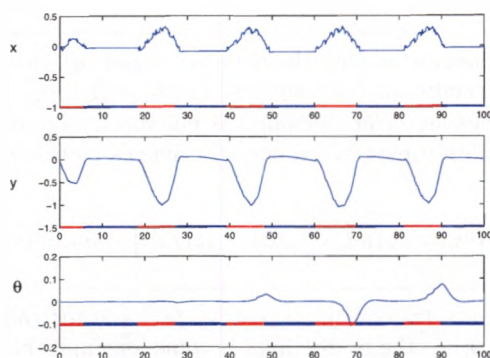
We give a first simulation associated with the reference trajectory characterized by $\omega_{ref} = 0.3(1/s)$ and good initial conditions ($\xi_0 = [0; 0; \frac{\pi}{4}]$ and $\dot{\xi}_0 = [1; 0; 0]$). It may be observed in figure 3.13 that the tracking error dynamics is not much degraded because the bridle times are finite. In bridle times, the orientation angles plots are serrate due to the hysteresis cycles.

geometrical constrained control law	$\beta_s = 2\pi rad/s$	$\omega_{ref} = 1(1/s)$	good initial conditions
-------------------------------------	------------------------	-------------------------	-------------------------

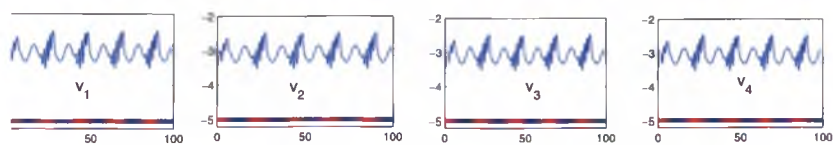
For a more exciting reference trajectory ($\omega_{ref} = 1(1/s)$) and for good initial conditions ($\xi_0 = [0; 0; \frac{\pi}{4}]$ and $\dot{\xi}_0 = [1; 0; 0]$), the bridle times become long and the tracking is severely degraded (see figure 3.14).



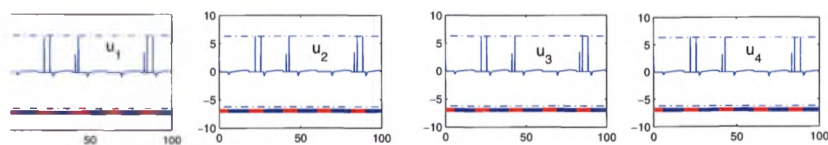
a) Trajectories (meters)



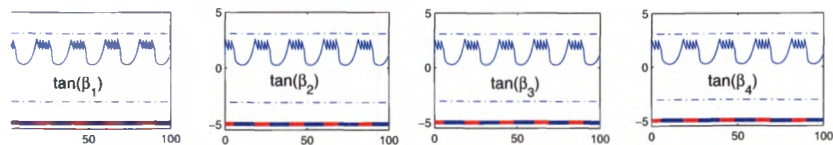
b) Tracking errors (meters and radians)



c) Rotation velocities (rad/s)



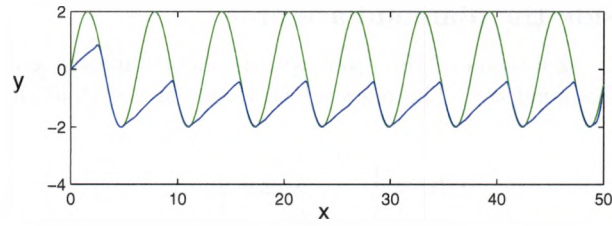
d) Orientation velocities (rad/s)



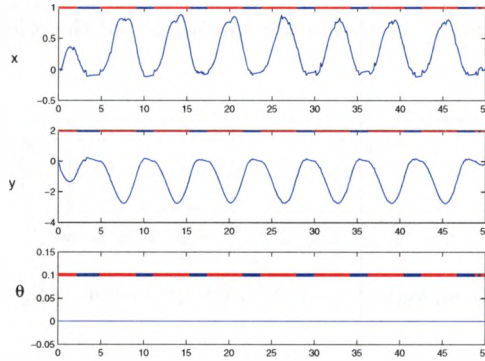
e) Tangent of the orientation angles

Figure 3.13:

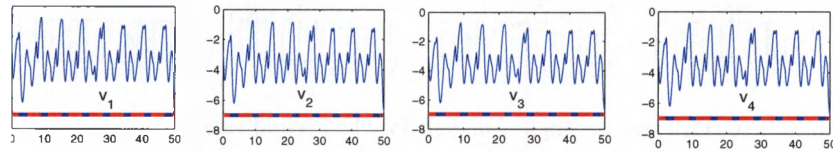
geometrical constrained control law	$\beta_s = 2\pi \text{rad/s}$	$\omega_{ref} = 0.3(1/s)$	good initial conditions
--	-------------------------------	---------------------------	-------------------------



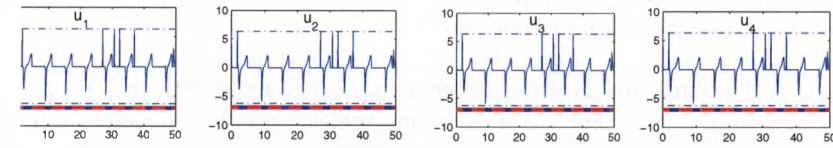
a) Trajectories (meters)



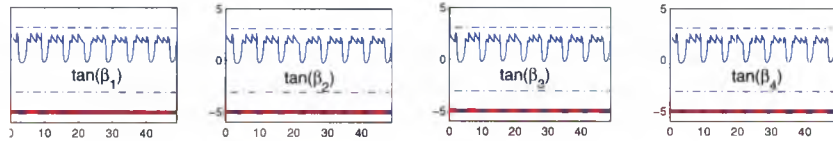
b) Tracking errors (meters and radians)



c) Rotation velocities (rad/s)



d) Orientation velocities (rad/s)



e) Tangent of the orientation angles

Figure 3.14:

geometrical constrained control law	$\beta_s = 2\pi \text{ rad/s}$	$\omega_{ref} = 1(1/s)$	good initial conditions
-------------------------------------	--------------------------------	-------------------------	-------------------------

3.3.2 ICR inside the diamond ace area

We also implement a simulation of the robot with its instantaneous center of rotation situated inside the diamond ace. The reference trajectory is a cycloid characterized by equations below

$$\xi_r(t) = \begin{pmatrix} at - b \sin t \\ a - b \cos t \\ 0 \end{pmatrix} \quad (3.5)$$

with $a = 0.3$, $b = 0.2$. The bound on orientation velocities is $\dot{\beta}_s = 2\pi \text{rad/s}$. The associated evolution of the orientation velocities and angles are given in figure 3.15 and 3.16. The reference trajectory is configuration admissible but the orientation angles often overstep the bounds.

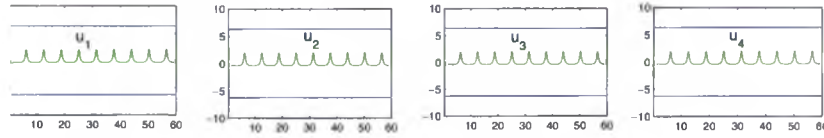


Figure 3.15: Orientation velocities (rad/s) for the cycloid reference trajectory

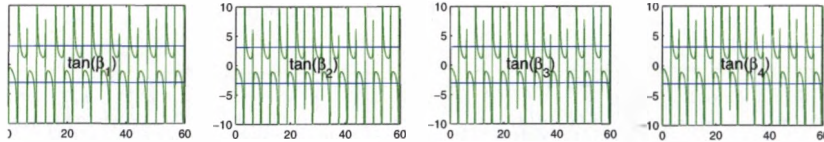
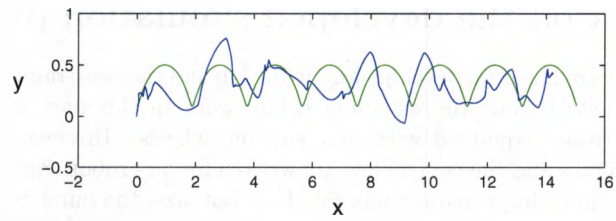


Figure 3.16: Orientation angles tangent for the cycloid reference trajectory

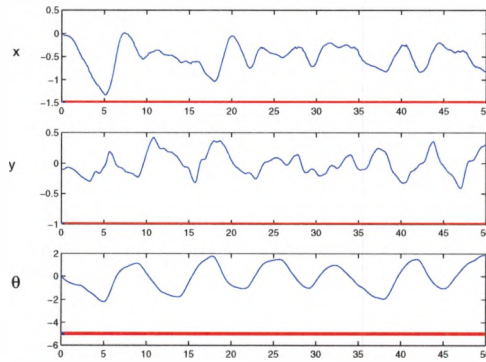
The initial conditions are $\xi_0 = [0; 0; 0]$ and $\dot{\xi}_0 = [0; 0; -1]$. The results are presented on figure 3.17 where we observe that the robot often prefers to loose the good orientation $\theta_r = 0$ to achieve better tracking in position.

This simulation illustrates how to succeed a transversal move with such a robot with bounded orientation angles. For a 5 meters long vehicle, less than 1 meter clearance is sufficient for the robot to handle. But the orientation of the robot oscillates, meaning that the robot "swings" while moving.

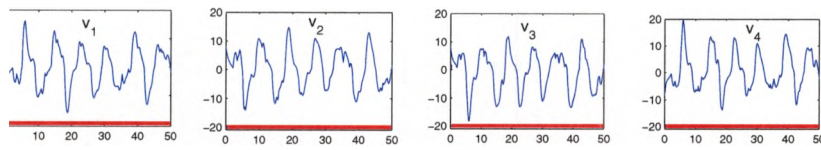
As general observation, it may be told that such a robot equipped with several steering wheels with bounded orientation angles is not much more easy to handle than a car. The trajectory presents cusp points even for slow and smooth reference trajectories.



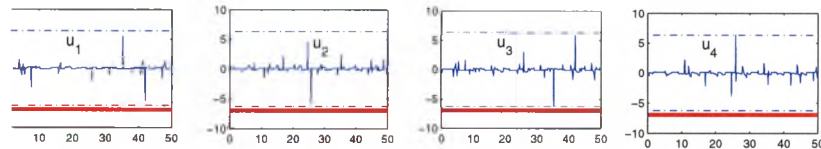
a) Trajectories (meters)



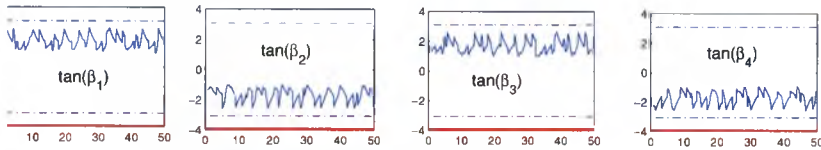
b) Tracking errors (meters and radians)



c) Rotation velocities (rad/s)



d) Orientation velocities (rad/s)



e) Tangent of the orientation angles

Figure 3.17:

geometrical constrained control law	$\beta_s = 2\pi \text{ rad/s}$	cycloid
--	--------------------------------	---------

3.4 Remark on the developed simulation program

The simulation program has been developed with Matlab and the constrained optimization problem has been solved using the *quadprog.m* function. In this part we only present simulations for the robot equipped with four steering wheels. However, the functions generating the dynamics and the control law are written for any robot equipped only with steering wheels. We introduced parameters for the robot size, the number of wheels, the wheels position, the saturation level on the orientation velocities and the initial conditions; so that they can be modified easily. Different reference trajectories may be tested without difficulties.

Conclusions and open issues

The purpose of this thesis is to relax some of the limitations encountered in the literature when dealing with modeling and control of wheeled mobile robots. Two directions are explored, leading to results of totally different nature.

We first relax the assumption that the kinematic constraints are satisfied. The derivation of the dynamical model of the robot then requires to take into account constitutive equations describing the contact forces. Previously, the interaction forces were modeled according to a simple linear model and a singular perturbation formulation of the robot dynamics was derived. Our first contribution is to extend the singular perturbation formulation of the robot dynamical model to a non linear force model based on the general force model of Bakker. Our second contribution is to design feedback control laws taking into account the fast dynamics associated with the contact effects. It requires the feedback of the position, the rolling velocity but no measurement of the sliding velocity is needed. The feedback control law is designed in order to achieve the point tracking control on the corresponding slow manifold. The attractivity of this slow manifold then allows us to ensure the closed-loop performance. The resulting control law appears as a modification of the corresponding ideal model-based control law. Two implementation of this approach are proposed.

First, we use the correction control term to exactly compensate the non ideal perturbation terms. This have been implemented previously on non ideal robotic manipulators. Mathematically, it amounts to linearize the non ideal system on the associated slow manifold. The attractivity of the slow manifold then allows us to prove that this control law ensures semi-global asymptotic convergence of the closed-loop tracking error. However, this control law needs the singular perturbation parameter to be known accurately, corresponding to a perfect knowledge of the stiffness of the wheels. Furthermore, the implementation of the control law supposes it to be developed in power series of the singular parameter and in practice, this development is truncated.

An another way to chose the correction control term is the Liapounov redesign approach. This approach supposes us to define the Liapounov function associated with the nominal law and to build the correction control term and a modified Liapounov function for the non ideal robot on the associated slow manifold. The perturbation terms are then dominated instead of being compensated. This approach is therefore more robust with respect to model uncertainties. In this work, we build the correction control term to ensure the nominal Liapounov function still to be decreasing for the non ideal robot on the slow

manifold. The slow manifold theory supposes the Liapounov correction control term to be smoothed. This results in a slight restriction in the stability result : the closed-loop tracking error is proved to tend towards a ball which radius decreases with the steep of the smoothing. Here again we prove the semi-global stability.

In both cases, the theoretical results are illustrated on simulations for the unicycle. Two force models have been implemented in the control design : a general force model based on Bakker's results and a linearized force model approximation. The control law based on the linearized force model presents good tracking performance for trajectories leading to small values of the slip ratios. When the slip ratios become large, the two force models become different and the control law based on the linearized force model becomes unefficient while the control law based on the general force model still presents good tracking performance. However, the calculation required by the control laws based on the general force model are time-consuming and can cause difficulties for online implementation. Furthermore, it must be kept in mind that when the robot is skidding, i.e. when one of the slip ratios is too high, then the above theoretical results are no longer valid, even for the control laws based on the general force model, because the attractivity of the slow manifold is not ensured. These remarks allow us to discuss the limitations of the augmented control law based on the general force model : more efficient but at the price of increasing computation time, and valid only for restricted values of the slip ratios.

In the last part of the thesis, we address the modeling and posture control for ideal robots equipped only with steering wheels. Two difficulties arise with this type of robots. On one hand, the coordination between the wheels orientations is required in order to ensure the existence of the instantaneous center of rotation. On the other hand, there exist singular configurations where the instantaneous center of rotation coincides with the center of one wheel or where the robot stops. In the neighborhood of such configurations, it may occur that the orientation velocities become too high to be acceptable, even with low robot velocity. Our first contribution is to derive a formulation allowing us to take into account not only the singularities but also the physical limitation of the wheels orientation velocities. We show that this limitation is equivalent to the fact that the robot posture acceleration belongs to a convex set in a 3-dimensional space. We address the posture tracking control problem for such robots. This problem can be solved easily, generically, by a output feedback linearization approach allowing us to assign the tracking error dynamics by an adequate choice of the posture acceleration. Unfortunately, this control law can not be applied when the robot encounters a singularity. Moreover, in the neighborhood of a singularity this generic control law can lead to unacceptable orientation velocities. In the literature, sufficient conditions ensuring that the closed-loop motion does not cross singularities are proposed. These conditions are given under the form of restrictions on the reference trajectory and on the initial tracking error. We extend this result to provide sufficient conditions ensuring that the closed-loop trajectories are posture admissible, i.e. correspond to orientation velocities satisfying the physical constraints. We also propose a modification of the generic control law applicable even when the robot encounters singular configurations. The idea is to assign the posture ac-

celeration as close as possible to the desired acceleration given by the generic control law, but ensuring that the orientation velocities satisfy the physical bounds, i.e. ensuring that the closed-loop trajectory is configuration admissible. This approach can be extended to other type of physical limitations, for example limitations on the maximum orientation angles of the wheels. The results are illustrated on a four steering wheels mobile robot.

For the two studied issues, the control objectives are only satisfied if the reference trajectory does meet some conditions. For the non ideal robot, the reference trajectory velocity can not be too high so that the slip ratios remain acceptable. For the robot equipped with steering wheels, the reference trajectory must be posture admissible for the closed-loop trajectory to be posture admissible. These two approaches therefore open new questions of path planning of the reference trajectories for tracking. For example, the parametrization of the geometrical reference trajectory with respect to time should be determined in order to meet one of the two above conditions (scaling). The complete choice of the reference trajectory could also be determined to satisfy some geometrical constraints (avoid obstacles, ...) and one of the above conditions.

Appendix A

Technical proofs of part 1

A.1 Proof of lemma 2.1

From the definition of the generalized forces vector F in (2.11), we deduce the expression of $\frac{\partial F}{\partial \mu}$ as follows

$$\frac{\partial F}{\partial \mu} = \sum_i L_i^T \frac{\partial f_i}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial \mu} = \varepsilon \sum_i \left[L_i^T \frac{\partial f_i}{\partial \alpha_i} \frac{1}{\|v_{Ri}\|} L_i \right] A^T(q) A(q), \quad (\text{A.1})$$

and therefore, the matrix $A^T A \frac{\partial F}{\partial \mu}$ is symmetrical.

We then easily obtain that the matrix $A^T A \frac{\partial F}{\partial \mu}$ is positive definite by contradiction.

Let us assume that there exists a vector z in $\mathbb{R}^m$ verifying

$$z^T A^T A \frac{\partial F}{\partial \mu} z \leq 0.$$

According to the expression of $\frac{\partial F}{\partial \mu}$ in (A.1), this inequality can be rewritten under the form below

$$\varepsilon \sum_{i=1}^k z^T A^T T L_i^T \frac{\partial f_i}{\partial \alpha_i} \frac{1}{\|r_{Ri}\|} L_i A^T A z \leq 0.$$

For the above inequality to be satisfied, it is necessary that there exists at least one wheel j such that

$$z^T A^T T L_j^T \frac{\partial f_j}{\partial \alpha_j} \frac{1}{\|r_{Rj}\|} L_j A^T A z \leq 0,$$

but this is not possible from the positive definiteness of matrix $\frac{\partial f_j}{\partial \alpha_j}$. We therefore conclude that the matrix $A^T A \frac{\partial F}{\partial \mu}$ is positive definite.

A.2 Proof of proposition 1.1

It has been shown in [7] that there exist a linearizing vector function

$$q_1 = w(p_1)$$

of dimension δ_M which does not depend on ζ such that

$$\dot{q}_1 = \frac{\partial w}{\partial p_1} T_1(p_1) \zeta = W(p_1) \zeta$$

with $W(p_1)$ non singular and such that $W(p_1)$ and $W^{-1}(p_1)$ are bounded for all p_1 . Two of the components of q_1 are the coordinates of the reference point of the frame. The other component depends on the type of the robot and is studied in detail in [25]. We also obtain from [25] that $W(p_1)$ depends only on angles through sine and cosine functions.

We define the following change of coordinates :

$$\begin{aligned} q_1 &= w(p_1) \\ q_2 &= \begin{pmatrix} q_{21} \\ q_{22} \end{pmatrix} \end{aligned}$$

with

- $q_{21} = z(p_1)$ constructed such that the transformation

$$p_1 \rightsquigarrow \begin{pmatrix} w(p_1) \\ z(p_1) \end{pmatrix} = \begin{pmatrix} q_1 \\ q_{21} \end{pmatrix}$$

is a diffeomorphism and where we have the property that $\frac{\partial z}{\partial p_1}$ is bounded;

- $q_{22} = p_2$

so that we have a global diffeomorphism between p and q .

With the following input transformation,

$$\eta = W(p_1) \zeta,$$

the dynamical equation on q_2 becomes

$$\dot{q}_{21} = \frac{\partial z}{\partial p_1} T_1(p_1) \zeta = \frac{\partial z}{\partial p_1} T_1(p_1) W^{-1}(p_1) \eta,$$

$$\dot{q}_{22} = \dot{p}_2 = T_2(p) \zeta = T_2(p) W^{-1}(p_1) \eta,$$

and defining

$$\Sigma^T(q) \triangleq \begin{pmatrix} \frac{\partial z}{\partial p_1} T_1(p_1) W^{-1}(p_1) \\ T_2(p) W^{-1}(p_1) \end{pmatrix}_{p_1=w^{-1}(q_1), p_2=q_{22}}$$

we obtain the formulation (1.5) and $\Sigma(q)$ is clearly bounded.

A.3 Proof of proposition 2.1

In this proof, we use the non ideal dynamical model under the form given in (2.33). Define the following candidate Liapounov function :

$$W(q, \eta, \mu, t) = \frac{1}{2} (\mu(t) - H_\varepsilon^*(q, \eta, t))^T R_{22}^{-1} (\mu(t) - H_\varepsilon^*(q, \eta, t)).$$

From property P2, there exists two positive constants λ_1, λ_2 such that

$$\lambda_1 \|\mu - H_\varepsilon^*\|^2 \leq W \leq \lambda_2 \|\mu - H_\varepsilon^*\|^2. \quad (\text{A.2})$$

Computing $\dot{W}$ along the trajectories of (2.33) we obtain

$$\dot{W} = (\mu - H_\varepsilon^*)^T R_{22}^{-1} (\dot{\mu} - \dot{H}_\varepsilon^*) - \frac{1}{2} (\mu - H_\varepsilon^*)^T R_{22}^{-T} \dot{R}_{22} R_{22}^{-1} (\mu - H_\varepsilon^*).$$

The equations (2.33) allow us to obtain the following intermediate expression

$$\begin{aligned} \varepsilon(\dot{\mu} - \dot{H}_\varepsilon^*) &= \varepsilon(g_1(q, \eta, \mu) - g_1(q, \eta, H_\varepsilon^*)) + \varepsilon^2(g_2(q, \eta, \mu) - g_2(q, \eta, H_\varepsilon^*)) \\ &\quad - R_{22} A^T A (F(q, \eta, \mu) - F(q, \eta, H_\varepsilon^*)). \end{aligned}$$

Using this expression, equation (2.107) and properties P3 and P4, we conclude that there exist two smooth non negative functions, $G_1(q, \eta, \mu - H_\varepsilon^*)$, $G_2(q, \eta, \mu - H_\varepsilon^*)$, respectively of order 1 and 2 with respect to $(\mu - H_\varepsilon^*)$ and such that

$$(\mu - H_\varepsilon^*)^T R_{22}^{-1} (\dot{\mu} - \dot{H}_\varepsilon^*) \leq (\mu - H_\varepsilon^*)^T \left[G_1 + \varepsilon G_2 - \frac{1}{\varepsilon} A^T A (F(q, \eta, \mu) - F(q, \eta, H_\varepsilon^*)) \right].$$

As far as the slip ratios are "small enough", property P5 implies that matrix $A^T A \frac{\partial F}{\partial \mu}$ is positive definite i.e.

$$\exists k_0 > 0, \quad A^T A \frac{\partial F}{\partial \mu} \geq k_0 I.$$

These results and property P4 allows us to conclude that for any M , there exists non negative constants $c_0(M)$, $c_1(M)$ and $c_2(M)$ such that for $\|\eta\| \leq M$,

$$\begin{aligned} \dot{W} &\leq c_1 \|\mu - H_\varepsilon^*\|^2 + \varepsilon c_2 \|\mu - H_\varepsilon^*\|^3 - \frac{c_0}{\varepsilon} \|\mu - H_\varepsilon^*\|^2 \\ &\leq \frac{W}{\varepsilon} \left[\frac{\varepsilon c_1}{\lambda_1} + \varepsilon^2 \frac{c_2}{\lambda_1^{\frac{3}{2}}} W^{\frac{1}{2}} - \frac{c_0}{\lambda_2} \right]. \end{aligned}$$

Define

$$\begin{aligned} F(t) &= \frac{\varepsilon c_1}{\lambda_1} + \varepsilon^2 \frac{c_2}{\lambda_1^{\frac{3}{2}}} W^{\frac{1}{2}} - \frac{c_0}{\lambda_2} \\ \alpha &= \frac{c_0}{4\lambda_2} \\ \varepsilon_2^*(M) &= \min \left(\varepsilon_1^*, \left(-\alpha + \frac{c_0}{\lambda_2} \right) \frac{1}{2} \frac{\lambda_1}{c_1} \right) \\ R(\varepsilon) &= \frac{1}{\varepsilon} \frac{c_1}{c_2} \sqrt{\frac{\lambda_1}{\lambda_2}} \end{aligned}$$

where $R(\varepsilon)$ is a non negative and decreasing function of ε .

Let us show that for all t , $F(t) < -\alpha$.

Notice first that for $\varepsilon < \varepsilon_2^*(M)$ and $\|\mu - H_\varepsilon\|_{t_0} < R(\varepsilon)$,

$$W(t_0) < \lambda_2 R^2(\varepsilon).$$

This implies that $F(t_0) < -\alpha$.

We now proof by contradiction that $\forall t \geq t_0$, $F(t) < -\alpha$. Suppose there exists $t_1 > t_0$ such that

$$F(t) < -\alpha \text{ for } t_0 \leq t < t_1 \text{ and } F(t_1) = -\alpha.$$

Then, for $t \leq t_1$,

$$\dot{W} \leq -\frac{\alpha}{\varepsilon} W$$

and therefore

$$W(t_1) \leq W(t_0) < \lambda_2 R^2(\varepsilon).$$

This implies that $F(t_1) < -\alpha$ and the contradiction appears.

Consider any ε such that $0 < \varepsilon \leq \varepsilon_2^*(M)$, and an initial condition such that $\|\mu - H_\varepsilon\|_{t_0} < R(\varepsilon)$. Then, as we showed that for all t , $F(t) < -\alpha$, we conclude that

$$\dot{W} \leq -\frac{\alpha}{\varepsilon} W$$

which implies exponential convergence to zero of W . The relation (2.109) easily follows from it and from properties (A.2) :

$$\|\mu(t) - H_\varepsilon^*(q, \eta, t)\|^2 \leq \frac{1}{\lambda_1} W(t) \leq \frac{1}{\lambda_1} W(t_0) \leq \frac{\lambda_2}{\lambda_1} \|\mu(t_0) - H_\varepsilon(q, \eta, t_0)\|^2.$$

A.4 Proof of proposition 3.1

Consider the non ideal dynamical model of the robot under the form (2.33).

We want to prove that there exists a control law u_ε that linearizes this system on the associated slow manifold. It amounts to find u_ε verifying

$$v = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + B_1 u_\varepsilon - R_{12} A^T A F - \Sigma \varepsilon \dot{H}_\varepsilon - \dot{\Sigma} \varepsilon H_\varepsilon.$$

To this aim we introduce the function $V(q, \eta, t, \varepsilon, u_\varepsilon)$ that is defined below

$$\begin{aligned} V(q, \eta, t, \varepsilon, u_\varepsilon) = & f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + B_1 u_\varepsilon - R_{12} A^T A F(q, \eta, H_\varepsilon) \\ & - \Sigma \varepsilon \dot{H}_\varepsilon - \dot{\Sigma} \varepsilon H_\varepsilon. \end{aligned} \quad (\text{A.3})$$

The thesis is equivalent to the following equation to be solved with respect to u_ε

$$V(q, \eta, t, \varepsilon, u_\varepsilon) = v$$

with the resulting function

$$u_\varepsilon = U(q, \eta, t, v, \varepsilon)$$

that is continuous.

This is obtained applying the implicit function theorem. We prove that the Jacobian $\frac{\partial V}{\partial u_\varepsilon}$ exists, is continuous and non singular for all $q \in \mathbb{R}^n, \eta \in \mathbb{R}^{n-m}, t \in \mathbb{R}$. The implicit function theorem allows us to conclude the inverse function $U(q, \eta, t, v, \varepsilon)$ exists and is continuous.

- 1) $\frac{\partial V}{\partial u_\varepsilon}$ exists and is continuous

All the functions involved in the definition of $V(q, \eta, t, \varepsilon, u_\varepsilon)$ are C^1 by construction. This directly implies the thesis.

- 2) $\frac{\partial V}{\partial u_\varepsilon}$ is non singular

The function $V(q, \eta, t, \varepsilon, u_\varepsilon)$ is also continuously differentiable with respect to ε .

The following relation is thus satisfied :

$$\forall q \in \mathbb{R}^n, \eta \in \mathbb{R}^{n-m}, t \in \mathbb{R} \quad \frac{\partial V}{\partial u_\varepsilon}(q, \eta, t, 0, u_0) = \lim_{\varepsilon \rightarrow 0} \frac{\partial V}{\partial u_\varepsilon}(q, \eta, t, \varepsilon, u_\varepsilon).$$

If we prove that $\frac{\partial V}{\partial u_\varepsilon}(q, \eta, t, 0, u_0)$ is non singular, then there exists ε_3 such that for all $\varepsilon \in [0, \varepsilon_3]$, the Jacobian $\frac{\partial V}{\partial u_\varepsilon}(q, \eta, t, \varepsilon, u_\varepsilon)$ is non singular and the proof is ended.

Differentiating the relation (A.3) with respect to u_ε and evaluating it at $\varepsilon = 0$ gives us the wanted expression :

$$\frac{\partial V}{\partial u_\varepsilon}(q, \eta, t, 0, u_0) = B_1 - R_{12} A^T A \left. \frac{\partial F}{\partial \mu} \right|_{H_0} \frac{\partial H_0}{\partial u_0}.$$

We obtain the expression for $\frac{\partial H_0}{\partial u_0}$ differentiating the manifold condition (2.107) with respect to u_ε and evaluating it at $\varepsilon = 0$:

$$B_2 - R_{22} A^T A \left. \frac{\partial F}{\partial \mu} \right|_{H_0} \frac{\partial H_0}{\partial u_0} = 0.$$

According to the above result, we rewrite the previous equation under the form

$$\frac{\partial V}{\partial u_\varepsilon}(q, \eta, t, 0, u_0) = B_1 - R_{12} R_{22}^{-1} B_2.$$

This can also be expressed under a product of three full rank matrices in the following way

$$\frac{\partial V}{\partial u_\varepsilon}(q, \eta, t, 0, u_0) = \begin{pmatrix} R_{22} & -R_{12} \end{pmatrix} R_{22}^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

We conclude that $\frac{\partial V}{\partial u_\epsilon}(q, \eta, t, 0, u_0)$ is non singular and that $\frac{\partial V}{\partial u_\epsilon}(q, \eta, t, \epsilon, u_\epsilon)$ is also non singular for sufficiently small ϵ .

A.5 Proof of proposition 3.2

- a) Let us first prove that for a given $M > 0$, if $\|\eta\| < M$, then the thesis holds and there exists $\epsilon' > 0$ such that

$$\forall \epsilon < \epsilon', \quad \lim_{t \rightarrow \infty} (\mu - H_\epsilon) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \tilde{y} = 0.$$

To this purpose, we define the following candidate Liapounov function

$$W_{CL} = W_L + W_A \quad (\text{A.4})$$

where W_L is a Liapounov function associated with the non ideal model-based linearizing control law and where W_A is the Liapounov function associated with the attractivity of the slow manifold.

The function W_L as the form

$$W_L = \frac{1}{2} \tilde{y}^T P \tilde{y}$$

and its derivative with respect to time verifies

$$\dot{W}_L = -\tilde{y}^T Q \tilde{y} + \frac{1}{2} (\Pi^T P \tilde{y} + \tilde{y}^T P \Pi)$$

where Q is a positive definite matrix and where we used the fact that the closed-loop tracking error dynamics is given by equations (3.11) to (3.14). According to property P4, P6 and knowing that $\|\eta\| < M$, we conclude that there exists positive constants c_1, c_2, c_3 such that

$$\dot{W}_L \leq -c_1 \|\tilde{y}\|^2 + c_2 \|\tilde{y}\| \|\mu - H_\epsilon\| + c_3 \|\tilde{y}\| \|\mu - H_\epsilon\|^2.$$

The function W_A is defined in appendix A.3 and when $\|\eta\| < M$ it is proven to verify

$$\dot{W}_A \leq c_4 \|\mu - H_\epsilon\|^2 + \epsilon c_5 \|\mu - H_\epsilon\|^3 - \frac{c_6}{\epsilon} \|\mu - H_\epsilon\|^2.$$

Finally, the candidate Liapounov function W_{CL} in (A.4) verifies :

$$\begin{aligned} \dot{W}_{CL} \leq & -c_1 \|\tilde{y}\|^2 + c_2 \|\tilde{y}\| \|\mu - H_\epsilon\| + c_3 \|\tilde{y}\| \|\mu - H_\epsilon\|^2 \\ & + c_4 \|\mu - H_\epsilon\|^2 + \epsilon c_5 \|\mu - H_\epsilon\|^3 - \frac{c_6}{\epsilon} \|\mu - H_\epsilon\|^2. \end{aligned} \quad (\text{A.5})$$

For given values of the initial conditions $\tilde{y}(t_0), (\mu(t_0) - H_\epsilon(q_0, \eta_0, t_0))$, we define α_0 and a domain D as follows

$$\alpha_0 = W(\tilde{y}(t_0), \mu - H_\epsilon(q_0, \eta_0, t_0));$$

$$D = \{\bar{y}, (\mu - H_\varepsilon) \text{ such that } W_{CL}(\bar{y}, \mu - H_\varepsilon) \leq \alpha_0\}.$$

It is possible to find $R > 0$ such that $B(R)$, the ball centered at the origin of radius R , contains D . We choose ε' such that

$$\forall \varepsilon < \varepsilon', \quad \forall \bar{y}, (\mu - H_\varepsilon) \in B(R), \quad \dot{W}_{CL} \leq 0.$$

Indeed, in the ball $B(R)$, the left-hand term in expression (A.5) may be upper bounded as written below :

$$\begin{aligned} \dot{W}_{CL} \leq & -c_1 \|\bar{y}\|^2 + c_2 \|\bar{y}\| \|\mu - H_\varepsilon\| + c_3 R \|\bar{y}\| \|\mu - H_\varepsilon\| \\ & + c_4 \|\mu - H_\varepsilon\|^2 + \varepsilon c_5 R \|\mu - H_\varepsilon\|^2 - \frac{c_6}{\varepsilon} \|\mu - H_\varepsilon\| \end{aligned}$$

that is a quadratic form in $\|\bar{y}\|$ and $\|\mu - H_\varepsilon\|$ characterized by the matrix below :

$$\begin{pmatrix} -c_1 & \frac{1}{2}(c_2 + c_3 R) \\ \frac{1}{2}(c_2 + c_3 R) & c_4 + \varepsilon c_5 R - \frac{c_6}{\varepsilon} \end{pmatrix}.$$

It is possible to find ε' such that for all $\varepsilon < \varepsilon'$ the above matrix is negative definite. While $\bar{y}$ and $\mu - H_\varepsilon$ live in $B(R)$ and therefore in D , W_{CL} is decreasing and the trajectories do not leave the domain D . This ends the proof of part a.

b) We now discuss the existence of a bound such that $\|\eta\| \leq M$.

From the definition of y_2 in (3.2), η can be expressed by

$$\eta = y_2 - \Sigma \varepsilon H_\varepsilon - \Sigma \varepsilon (\mu - H_\varepsilon).$$

This implies that

$$\|\eta\| \leq c_r + \|\bar{y}\| + \varepsilon \|\Sigma(q)\| \|\mu - H_\varepsilon\| + \varepsilon \|\Sigma(q)\| \|H_\varepsilon\|.$$

Therefore, from property P4, there exists $\sigma > 0$ such that

$$\|\eta\| \leq c_r + \varepsilon \sigma \|H_\varepsilon\| + \varepsilon \sigma \|\mu - H_\varepsilon\| + \|\bar{y}\|.$$

While in $B(R)$, the above inequality may be upper bounded as follows

$$\|\eta\| \leq c_r + R + \varepsilon \sigma (R + \|H_\varepsilon\|).$$

For given q, η and for any M with $M > c_r + R$, we define a monotonic function

$$F(M) = \sup_{\|\eta\| < M} \|H_\varepsilon(q, \eta, t)\|.$$

Then there exists ε'' such that for all $\varepsilon < \varepsilon''$

$$c_r + R + \varepsilon \sigma (R + F(M)) \leq M.$$

And we therefore obtain that in $B(R)$, for all $\varepsilon < \varepsilon''$, $\|\eta\| < M$. Knowing that η remains bounded in $B(R)$, we conclude from a) that the error vectors remain in $B(R)$ and that η remains bounded.

c) Finally define $\varepsilon_4 = \min(\varepsilon', \varepsilon'')$ and the proposition 3.2 holds.

A.6 Proof of lemma 3.1

Developing the manifold condition (2.107) in powers on ε leads to the following equations for the terms of power ε^0 and ε^1 :

$$g_0(q, \eta) + B_2 u_0(q, \eta, t) - R_{22} A^T A F(q, \eta, H_0) = 0 \quad (\text{A.6})$$

$$g_1(q, \eta, H_0) + B_2 \bar{u}(q, \eta, t) - R_{22} A^T A \bar{F}(q, \eta, t, H_1) = d_0 H_0(q, \eta, t). \quad (\text{A.7})$$

where $\varepsilon \bar{F}(q, \eta, t, H_\varepsilon) = F(q, \eta, H_\varepsilon) - F(q, \eta, H_0)$ and where $H_1(q, \eta, t)$ is defined in (3.37).

The general expression of $\delta(\varepsilon, q, \eta, t, H_\varepsilon)$ in (3.44) is approached up to an error of order ε by the following relation

$$\begin{aligned} \delta(\varepsilon, q, \eta, t, H_\varepsilon) = & \varepsilon B_1^{-1} \left[K_2 \Sigma H_0 + f_1(q, \eta, H_0) - \Sigma d_0 H_0(q, \eta, t) \right. \\ & \left. - \left(\frac{d\Sigma^T}{dq} S \eta \right) H_0 - R_{12} A^T A \bar{F}(q, \eta, t, H_1) \right] \\ & + \varepsilon^2 \rho(q, \eta, t, \varepsilon). \end{aligned}$$

The equation (A.7) allows us to find the expression of $\bar{F}(q, \eta, t, H_1)$. Using this result, we approach the general expression of $\delta(\varepsilon, q, \eta, t, H_\varepsilon)$ under the form given in lemma 1 :

$$\delta(\varepsilon, q, \eta, t, H_\varepsilon) = -\varepsilon B_1^{-1} R_{12} R_{22}^{-1} B_2 \bar{u} + \varepsilon \Delta(q, \eta, t, H_0) + \varepsilon^2 \rho(q, \eta, t, \varepsilon)$$

where $\Delta(q, \eta, t, H_0)$ is defined by

$$\begin{aligned} \Delta(q, \eta, t, H_0) = & B_1^{-1} \left[K_2 \Sigma H_0 + f_1(q, \eta, H_0) - \left(\frac{d\Sigma^T}{dq} S \eta \right) H_0 \right] \\ & - B_1^{-1} R_{12} R_{22}^{-1} (g_1(q, \eta, H_0) - d_0 H_0(q, \eta, t, H_0)) - B_1^{-1} \Sigma d_0 H_0(q, \eta, t) \end{aligned}$$

and where $\rho(q, \eta, t, \varepsilon)$ is polynomial in ε .

A.7 Proof of proposition 3.3

We have to ensure that the stability condition

$$-\|\bar{y}\|^2 + \varepsilon \bar{y}^T P \bar{B}_1 (\bar{B}_2 \bar{u} + \Delta + \varepsilon \rho(q, \eta, t, \varepsilon)) \leq 0$$

is satisfied.

For the control law (3.52), the stability condition is verified when

$$-\|\bar{y}\|^2 + \varepsilon \|\bar{y}^T P \bar{B}_1\| (-K + \varepsilon \|\rho(q, \eta, t, \varepsilon)\|) \leq 0.$$

For any positive constant K and M such that $\|\eta\| < M$, property P4 ensures that there exists $\varepsilon_5(M)$ such that for all $\varepsilon < \varepsilon_5(M)$

$$K > \varepsilon \|\rho(q, \eta, t, \varepsilon)\| \quad (\text{A.8})$$

so that the stability condition is satisfied.

A.8 Proof of proposition 3.4

We have to find the conditions so that the stability condition

$$-\|\tilde{y}\|^2 + \varepsilon \tilde{y}^T P \bar{B}_1 (\bar{B} \bar{u} + \Delta + \varepsilon \rho(q, \eta, t, \varepsilon)) \leq 0 \quad (\text{A.9})$$

is satisfied when the smoothed Liapounov augmented control law is applied.

Knowing the expression for $\bar{u}$ in (3.54), the condition (A.9) becomes :

$$-\|\tilde{y}\|^2 - \varepsilon K \frac{\|\bar{B}_1^T P \tilde{y}\|^2}{\sqrt{\|\bar{B}_1^T P \tilde{y}\|^2 + \gamma^2}} + \varepsilon^2 \tilde{y}^T P \bar{B}_1 \rho(q, \eta, t, \varepsilon) \leq 0.$$

With ε_5 chosen as in the previous proof, the $\varepsilon \rho(q, \eta, t, \varepsilon)$ verifies (A.8). The above condition is therefore satisfied when

$$-\|\tilde{y}\|^2 + \varepsilon K \|\bar{B}_1^T P \tilde{y}\| \left(-\frac{\|\bar{B}_1^T P \tilde{y}\|}{\sqrt{\|\bar{B}_1^T P \tilde{y}\|^2 + \gamma^2}} + 1 \right) \leq 0.$$

The function $f(x) = \left(-\frac{x}{\sqrt{x^2 + \gamma^2}} + 1 \right)x$ appears in the last term of this expression and has a maximum in $\bar{x} = \gamma \sqrt{\frac{\sqrt{5}-1}{2}}$ for the value $f(\bar{x}) = \bar{x} \cdot \frac{\sqrt{5}+1}{2} < \frac{\gamma}{2}$. So that the stability condition is verified when

$$-\|\tilde{y}\|^2 + \varepsilon K \frac{\gamma}{2} \leq 0$$

that is

$$\|\tilde{y}\|^2 \geq \varepsilon \frac{K\gamma}{2}.$$

A.9 Proof of proposition 3.5

The proof is essentially the same as proof of proposition 3.2 in appendix A.5. We first proof that when η is bounded the thesis holds and second show that η remains bounded.

The candidate Liapounov function is of course

$$W_{CL} = V_0 + W_A$$

where W_A is the Liapounov function associated with the attractivity of the slow manifold. From inequality (3.60) and from property P6, we obtain that there exist positive constants c_1, c_2 such that

$$\dot{V}_0 \leq -\|\tilde{y}\|^2 + c_1 \|\tilde{y}\| \|\mu - H_\varepsilon\| + c_2 \|\tilde{y}\| \|\mu - H_\varepsilon\|^2 + \varepsilon \frac{K\gamma}{2}.$$

Following the same development as in proof of proposition 3.2, the function W_{CL} is proven to satisfy

$$\begin{aligned} \dot{W}_{CL} \leq & -\|\tilde{y}\|^2 + c_1 \|\tilde{y}\| \|\mu - H_\varepsilon\| + c_2 \|\tilde{y}\| \|\mu - H_\varepsilon\|^2 + \varepsilon \frac{K\gamma}{2} \\ & + c_3 \|\mu - H_\varepsilon\|^2 + \varepsilon c_4 \|\mu - H_\varepsilon\|^3 - \frac{c_5}{\varepsilon} \|\mu - H_\varepsilon\|^2 \end{aligned}$$

where c_3, c_4 and c_5 are positive constants. We define α_0 , the domain D and the ball $B(R)$ similarly as in appendix A.5, and while in $B(R)$, the inequality below holds :

$$\begin{aligned} \dot{W}_{CL} \leq & -\|\tilde{y}\|^2 + c_1\|\tilde{y}\| \|\mu - H_\varepsilon\| + c_2R\|\tilde{y}\| \|\mu - H_\varepsilon\| + \varepsilon\frac{K\gamma}{2} \\ & + c_3\|\mu - H_\varepsilon\|^2 + \varepsilon c_4R\|\mu - H_\varepsilon\|^2 - \frac{c_5}{\varepsilon}\|\mu - H_\varepsilon\|^2 \end{aligned}$$

that appears as the sum of two terms : a constant $\varepsilon\frac{K\gamma}{2}$ and a quadratic form in $\|\tilde{y}\|$ and $\|\mu - H_\varepsilon\|$ characterized by the matrix below :

$$\begin{pmatrix} -1 & \frac{1}{2}(c_1 + c_2R) \\ \frac{1}{2}(c_1 + c_2R) & c_3 + \varepsilon c_4 - \frac{c_5}{\varepsilon} \end{pmatrix}.$$

It is possible to find $\varepsilon' > 0$ such that for all $\varepsilon < \varepsilon'$ the above matrix is negative definite. There therefore exists $c_6 > 0$ such that

$$\dot{W}_{CL} \leq -c_6(\|\tilde{y}\|^2 + \|\mu - H_\varepsilon\|^2) + \varepsilon\frac{K\gamma}{2}.$$

Knowing that $\exists \sigma_1, \sigma_2 > 0$ such that

$$\sigma_2(\|\tilde{y}\|^2 + \|\mu - H_\varepsilon\|^2) \leq W_{CL} \leq \sigma_1(\|\tilde{y}\|^2 + \|\mu - H_\varepsilon\|^2),$$

the above inequality implies that

$$\dot{W}_{CL} \leq -\frac{c_6}{\sigma_1}W_{CL} + \varepsilon\frac{K\gamma}{2}$$

and W is bounded; and therefore also $\tilde{y}$ and $(\mu - H_\varepsilon)$.

From this result and from equation (3.60), for all $\alpha > 1$, there exists $t_1 > t_0$ such that

$$\forall t > t_1, \quad \dot{V}_0 \leq -\|\tilde{y}\|^2 + \alpha\varepsilon\frac{K\gamma}{2}.$$

As far as V_0 satisfies

$$\rho_1\|\tilde{y}\|^2 \leq V_0 \leq \rho_2\|\tilde{y}\|^2, \quad (\text{A.10})$$

for some positive constants ρ_1, ρ_2 , the above relation also implies that

$$\forall t > t_1, \quad \dot{V}_0 \leq -\frac{V_0}{\rho_1} + \alpha\varepsilon\frac{K\gamma}{2}.$$

This allows us to conclude that the V_0 function evolves verifying

$$\forall t > t_1, \quad V_0(t) \leq V_0(t_1)e^{-\frac{1}{\rho_1}t} + \rho_1\alpha\varepsilon\frac{K\gamma}{2}(1 - e^{-\frac{1}{\rho_1}t}).$$

Finally, using (A.10), we obtain that $\|\tilde{y}\|$ evolves verifying

$$\forall t > t_1, \quad \|\tilde{y}\|^2 \leq \frac{V_0}{\rho_2} \leq \frac{\rho_1}{\rho_2}\|\tilde{y}(t_1)\|^2 e^{-\frac{1}{\rho_1}t} + \frac{\rho_1}{\rho_2}\alpha\varepsilon\frac{K\gamma}{2}(1 - e^{-\frac{1}{\rho_1}t}),$$

and we conclude that $\|\tilde{y}\|^2$ is bounded by a function tending towards $\varepsilon\frac{\rho_1\alpha K\gamma}{2\rho_2}$.

The rest of the proof is essentially the same as in appendix A.5 : we prove there exists $\varepsilon'' > 0$ such that for all $\varepsilon < \varepsilon''$, η remains bounded and finally define $\varepsilon_6 = \min(\varepsilon', \varepsilon'')$ for the proposition to hold.

Appendix B

Technical proofs of part 2

B.1 Expression of $M_s(t)$

From property P1, we know the singularity phenomenon is essentially linked to the value of the velocity vector η . We therefore decide to find a preliminary distance to the singularity with respect to variable η , i.e. to define $M_1(t)$ such that :

$$\|\eta - \eta_r\| < M_1(t) \Rightarrow \forall i = 1, \dots, k, \quad G_i \eta \neq 0.$$

Distance to the nearest η such that $\exists! i, G_i \eta = 0$

Using the expression of G_i in (1.10), it is possible to show that such a singular η has the form

$$\eta = \alpha \begin{pmatrix} b_i \\ -a_i \\ 1 \end{pmatrix}$$

and we have to find α and i such that $\|\eta - \eta_r\|$ is minimal.

With respect to the 2-norm, the cost function becomes

$$\|\eta - \eta_r\|_2^2 = (a_i^2 + b_i^2 + 1)\alpha^2 - 2\alpha(b_i\eta_{1r} - a_i\eta_{2r} + \eta_{3r}) + \eta_{1r}^2 + \eta_{2r}^2 + \eta_{3r}^2 \triangleq f_i(\alpha)$$

that is a quadratic function with respect to α . For each i it is possible to find the minimum value α_i^* of the cost function :

$$\alpha_i^* = \frac{b_i\eta_{1r} - a_i\eta_{2r} + \eta_{3r}}{a_i^2 + b_i^2 + 1}.$$

The final minimum is the minimum of the k possible α_i^* , $i = 1, \dots, k$ and its index is denoted l :

$$l = \arg \min_{i=1, \dots, k} f_i(\alpha_i^*).$$

Distance to $\eta = 0$

In that case, we know exactly the value of the the cost function :

$$\|\eta - \eta_r\| = \|\eta_r\|.$$

The function $M_1(t)$ may thus be defined as follows :

$$M_1(t) = \min(\|\eta_r\|, \sqrt{f_l(\alpha_l^*)}). \quad (\text{B.1})$$

We now have to link this result with a bound on the tracking error vector $\bar{y} = \begin{pmatrix} \tilde{\xi} \\ \tilde{\xi} \end{pmatrix}$.

Using equation (2.10) to express η and η_r , we obtain the inequality below

$$\|\eta - \eta_r\| < \|R(\theta) - R(\theta_r)\| \|\dot{\xi}_r\| + \|R(\theta)\| \|\dot{\xi}\|.$$

It is possible to show that with respect to the 2-norm we have the following properties :

$$\begin{aligned} \|R(\theta) - R(\theta_r)\|_2 &< 3|\theta - \theta_r| < 3\|\tilde{\xi}(t)\|_2, \\ \|R(\theta)\|_2 &= 1, \end{aligned}$$

and therefore

$$\|\eta - \eta_r\|_2 < 3\|\tilde{\xi}(t)\|_2 \|\dot{\xi}_r(t)\|_2 + \|\dot{\xi}(t)\|_2, \quad (\text{B.2})$$

$$< (3\|\dot{\xi}_r(t)\|_2 + 1) \|\tilde{y}\|_2. \quad (\text{B.3})$$

If we define $M_s(t)$ as follows

$$M_s(t) = \frac{M_1(t)}{3\|\dot{\xi}_r(t)\|_2 + 1},$$

when $\|\tilde{y}\|_2 < M_s(t)$ then $\|\eta - \eta_r\| < M_1(t)$ and the trajectory is non singular.

B.2 Expression of $M_{nas}(t)$

We assume the reference trajectory to be admissible i.e. $\forall t > t_0, w_r(t) \in D(\beta_r(t), \eta_r(t))$ so that it is possible to define $d > 0$ such that

$$\forall i = 1, \dots, k \quad \forall t > t_0, \quad |\dot{\beta}_{ri}(t)| < \dot{\beta}_s - d. \quad (\text{B.4})$$

For given η, β, w satisfying $\beta \in \mathcal{M}$ and $\forall i, G_i \eta \neq 0$, we define a class of functions $f_i(\beta, \eta, w)$ as follows :

$$\begin{aligned} f_i : \mathbb{R}^{k+6} &\longrightarrow \mathbb{R} \\ \begin{pmatrix} \beta \\ \eta \\ w \end{pmatrix} &\longrightarrow f_i(\beta, \eta, w) = -\frac{R_1^T(\beta_i)G_i w}{R_2^T(\beta_i)G_i \eta}. \end{aligned}$$

Remark that the fact that $w \in D(\beta, \eta)$ is equivalent to the condition below :

$$\forall i = 1, \dots, k \quad |f_i(\beta, \eta, w)| < \dot{\beta}_s,$$

and that is what we now have to ensure.

We develop the functions $f_i(\beta, \eta, w)$ around (β_r, η_r, w_r) up to order 1 and we obtain

$$\begin{aligned} f_i(\beta, \eta, w) &= f_i(\beta_r, \eta_r, w_r) + \nabla f_{i,\beta}(\beta_r + \delta\bar{\beta}, \eta_r + \delta\bar{\eta}, w_r + \delta\bar{w})\bar{\beta} \\ &\quad + \nabla f_{i,\eta}(\beta_r + \delta\bar{\beta}, \eta_r + \delta\bar{\eta}, w_r + \delta\bar{w})\bar{\eta} \\ &\quad + \nabla f_{i,w}(\beta_r + \delta\bar{\beta}, \eta_r + \delta\bar{\eta}, w_r + \delta\bar{w})\bar{w}, \end{aligned}$$

where $\nabla f_{i,\eta}(\cdot)$ is the gradient vector of f_i with respect to η , and $\delta_i \in (1, 0)$.

For a priori bound for $\|\bar{w}\|_2$, and for non singular trajectories, the gradient vectors are bounded. It is therefore possible to exhibit $c, k_i, l_i, m_i > 0$ verifying

$$\begin{aligned} \|\bar{w}\|_2 < c, \quad \|\bar{y}\|_2 < M_s(t) \implies \forall i = 1, \dots, k \quad \exists k_i, l_i, m_i > 0, \\ \|\nabla f_{i,\beta}(\beta_r + \delta_i\bar{\beta}, \eta_r + \delta_i\bar{\eta}, w_r + \delta_i\bar{w})\|_2 < k_i, \\ \|\nabla f_{i,\eta}(\beta_r + \delta_i\bar{\beta}, \eta_r + \delta_i\bar{\eta}, w_r + \delta_i\bar{w})\|_2 < l_i, \\ \|\nabla f_{i,w}(\beta_r + \delta_i\bar{\beta}, \eta_r + \delta_i\bar{\eta}, w_r + \delta_i\bar{w})\|_2 < m_i. \end{aligned}$$

This allows us to conclude that

$$|f_i(\beta, \eta, w)| < |f_i(\beta_r, \eta_r, w_r)| + k_i\|\bar{\beta}\| + l_i\|\bar{\eta}\| + m_i\|\bar{w}\|$$

and reminding equation (B.4), we also have that

$$|f_i(\beta, \eta, w)| < \bar{\beta}_s - d + k_i\|\bar{\beta}\| + l_i\|\bar{\eta}\| + m_i\|\bar{w}\|.$$

We define $k_{max} = \max_i k_i$, $l_{max} = \max_i l_i$ and $m_{max} = \max_i m_i$. The w vector is accessible for $D(\beta, \eta)$ if the condition below is satisfied

$$k_{max}\|\bar{\beta}\| + l_{max}\|\bar{\eta}\| + m_{max}\|\bar{w}\| < d \quad (\text{B.5})$$

where k_{max}, l_{max} and m_{max} depend on the a priori bound on $\bar{w}$ denoted c . We now link the tracking error vector $\|\bar{y}\|$ with the errors $\|\bar{\eta}\|$, $\|\bar{\beta}\|$ and $\|\bar{w}\|$. This is the subject of the 3 following steps :

Link between $\|\bar{\eta}\|_2$ and $\|\bar{y}\|_2$

In the previous proof, we obtained a link between $\|\bar{\eta}\|_2$ and $\|\bar{y}\|_2$ at equation (B.2). We therefore only remind that

$$\|\bar{\eta}\|_2 < (3\|\dot{\xi}_r\|_2 + 1)\|\bar{y}\|_2.$$

Link between $\|\bar{\beta}\|_2$ and $\|\bar{y}\|_2$

The coordination constraints (1.12) impose the orientation angles β_i ($i = 1, \dots, k$) to verify

$$\forall i = 1, \dots, k \quad \cos \beta_i (\eta_1 - b_i \eta_3) + \sin \beta_i (\eta_2 + a_i \eta_3) = 0.$$

This allows us to express β_i as a continuous and differentiable function of η :

$$\begin{aligned} g_i : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ \eta &\longrightarrow g_i(\eta) = \beta_i. \end{aligned}$$

We develop the functions $g_i(\eta)$ around $\eta = \eta_r$ up to order 1 and we obtain

$$g_i(\eta) = g_i(\eta_r) + \nabla g_{i,\eta}(\eta_r + \delta_i \bar{\eta}) \bar{\eta},$$

where $\delta_i \in (1, 0)$.

It may be proven that as far as $\|\bar{y}\|_2 < M_s(t)$, the gradient vectors remains bounded :

$$\|\bar{y}\|_2 < M_s(t) \implies \forall i = 1, \dots, k \quad \exists c_i > 0, \quad \|\nabla g_{i,\eta}(\eta_r + \delta(\eta - \eta_r))\| < c_i.$$

We therefore may conclude that the tracking error in β verifies

$$\|\bar{\beta}\|_2 \leq \sqrt{3} \|\bar{\beta}\|_\infty = \sqrt{3} \max_{i=1, \dots, k} |g_i(\eta) - g_i(\eta_r)| < \sqrt{3} c_{max} \|\bar{\eta}\|$$

where

$$c_{max} = \max_{i=1, \dots, k} c_i.$$

Link between $\|\bar{w}\|_2$ and $\|\bar{y}\|_2$

We know that the link between w and $\ddot{\xi}$ is given by equation

$$w = R(\theta) (\ddot{\xi} - \dot{R}^T(\theta) \eta).$$

Developing w and w_r according to above equation and adding and subtracting some intermediary term, we obtain that their difference may be expressed by

$$\begin{aligned} w - w_r &= (R(\theta) \ddot{\xi} - R(\theta) \ddot{\xi}_r) + (R(\theta) \ddot{\xi}_r - R(\theta_r) \ddot{\xi}_r) \\ &\quad + (R(\theta) \dot{R}^T(\theta_r) \eta - R(\theta) \dot{R}^T(\theta) \eta) \\ &\quad + (R(\theta_r) \dot{R}^T(\theta_r) \eta - R(\theta) \dot{R}^T(\theta_r) \eta) \\ &\quad + (R(\theta_r) \dot{R}^T(\theta_r) \eta_r - R(\theta_r) \dot{R}^T(\theta_r) \eta). \end{aligned}$$

It is possible to show that with respect to the 2-norm the following bounds are valid when the trajectory is non singular:

$$\begin{aligned} \|R(\theta)\|_2 &= 1 \\ \|R(\theta) - R(\theta_r)\|_2 &< 3|\theta - \theta_r| < 3\|\dot{\xi}\|_2 \\ \|\dot{R}^T(\theta) - \dot{R}^T(\theta_r)\|_2 &< 2|\dot{\theta} - \dot{\theta}_r| + 3|\dot{\theta}_r| \quad |\theta - \theta_r| \\ &< 2\|\dot{\xi}\|_2 + 3|\dot{\theta}_r| \quad \|\dot{\xi}\|_2 \\ \|\eta - \eta_r\|_2 &< M_1(t) \\ \|\eta\|_2 &< \|\eta - \eta_r\|_2 + \|\eta_r\|_2 < M_1(t) + \|\eta_r\|_2 \end{aligned}$$

where the function $M_1(t)$ is the distance to the nearest singular configuration with respect to the η variable. It has been defined in appendix B.1 at equation (B.1).

On the other hand, the generic control law defined for admissible trajectories ensures the tracking error dynamics satisfies

$$\ddot{\tilde{y}} = \Gamma \tilde{y},$$

where the matrix Γ is defined as

$$\Gamma = \begin{pmatrix} 0 & I \\ -K_1 & -K_2 \end{pmatrix}.$$

The tracking error dynamics is therefore exponentially decreasing and verifies the relation below

$$\|\ddot{\tilde{\xi}}\|_2 < \|\dot{\tilde{y}}\|_2 < \lambda_{max}(\Gamma) \|\tilde{y}\|_2$$

where $\lambda_{max}(\Gamma)$ represents the largest eigenvalue of matrix Γ .

The above bounds allows us to deduce that

$$\|w - w_r\|_2 < h(\xi_r, \dot{\xi}_r, \ddot{\xi}_r, \Gamma, t) \|\tilde{y}\|_2$$

with the scalar function h defined by equation below

$$\begin{aligned} h(\xi_r, \dot{\xi}_r, \ddot{\xi}_r, \Gamma, t) = & \lambda_{max}(\Gamma) + 3\|\ddot{\xi}_r\|_2 + (1 + 3|\dot{\theta}_r|)(M_1 + \|\eta_r\|_2) \\ & + 3\|\ddot{R}^T(\theta_r)\|_2(M_1 + \|\eta_r\|_2) + \|\dot{R}^T(\theta_r)\|_2 M_1 \end{aligned}$$

where $\xi_r, \dot{\xi}_r, \ddot{\xi}_r$ and M_1 are time-dependent. Remark this function is strictly positive.

According to the previous results, equation (B.5) is satisfied when $\|\tilde{w}\|_2 < c$ and when

$$\left((k_{max}\sqrt{3}c_{max} + l_{max})(3\|\dot{\xi}_r\|_2 + 1) + m_{max}h(\xi_r, \dot{\xi}_r, \ddot{\xi}_r, \Gamma, t) \right) \|\tilde{y}\|_2 < d.$$

We therefore may define $M_{nas}(t)$ as follows

$$M_{nas}(t) = \min \left(M_s(t), \frac{c}{h(\xi_r, \dot{\xi}_r, \ddot{\xi}_r, \Gamma, t)}, \frac{d}{(k_{max}\sqrt{3}c_{max} + l_{max})(3\|\dot{\xi}_r\|_2 + 1) + m_{max}h(\xi_r, \dot{\xi}_r, \ddot{\xi}_r, \Gamma, t)} \right)$$

to have the distance to the nearest non admissible state in terms of $\tilde{y}$.

Bibliography

- [1] G.Campion, G.Bastin, and B.D'Andréa-Novel, "Structural properties and classification of kinematic and dynamic models of wheeled mobile robots," *IEEE Transactions on robotics and automation*, vol. 12, pp. 47-61, february 1996.
- [2] J.P.Laumond, P.E.Jacobs, M.Taix, and R.M.Murray, "A motion planner for non-holomic mobile robots," *IEEE Transactions on Robotics and Automation*, vol. 10, pp. 577-593, october 1994.
- [3] A.Benayad, *Modélisation et commande des robots mobiles à roues*. PhD thesis, Université Chouaïb Doukkali, september 2000.
- [4] P.Rouchon, M.Fliess, J.Lévine, and Ph.Martin, "Flatness, motion planning and trailer systems," *Proceedings of the 32nd IEEE Conference on Decision and Control*, pp. 2700-2705, 1993. San Antonio.
- [5] R.W.Brockett, "Asymptotic stability and feedback stabilization," *Differential geometric control theory*, pp. 181-191, 1983.
- [6] A.Isidori, *Nonlinear Control Systems*. Communications and Control Engineering, Springer, third ed., 1995.
- [7] B. D'Andréa-Novel, G.Campion, and G.Bastin, "Control of nonholomic wheeled mobile robots by state feedback linearization," *The International Journal of Robotics Research*, vol. 14, pp. 543-559, december 1995.
- [8] C.Samson, "Velocity and torque feedback control of a nonholomic cart," *Proceedings of the international workshop in adaptative and nonlinear control : Issues in robotics*, 1990.
- [9] C.Samson, "Control of chained systems application to path-following and time-varying point-stabilization of mobile robots," *IEEE Transactions on Automatic Control*, vol. 40, pp. 64-77, Januari 1995.
- [10] A.Astolfi, "Exponential stabilization of a wheeled mobile robot via discontinuous control law," *Journal of dynamic systems measurement and control-transactions of the ASME*, vol. 121, pp. 121-126, March 1999.

- [11] O. Canudas de Wit and O.J.Sordalen, "Exponential stabilization of mobile robots with nonholomic constraints," *IEEE Transactions on Automatic Control*, vol. 37, no. 11, pp. 1791-1797, 1992.
- [12] I.Kolmanovsky and N.H.McClamroch, "Hybrid feedback laws for a class of cascade nonlinear control systems," *IEEE Transactions on Automatic Control*, vol. 41, pp. 1271-1282, september 1996.
- [13] J.B.Pomet, "Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift," *Systems & Control Letters*, vol. 18, pp. 147-158, 1992.
- [14] R.T.MCloskey and R.M.Murray, "Exponential stabilization of driftless nonlinear control systems using homogeneous feedback," *IEEE Transactions on Automatic Control*, vol. 42, pp. 614-628, may 1997.
- [15] O.J.Sordalen and O.Egeland, "Exponential stabilization of nonholomic chained systems," *IEEE Transactions on Automatic Control*, vol. 40, pp. 35-49, january 1995.
- [16] W.E.Dixon, Z.P.Jiang, and D.M.Dawson, "Global exponential setpoint control of wheeled mobile robots : a lyapounov approach," *Automatica*, vol. 36, pp. 1741-1746, november 2000.
- X L [17] B.Thuilot, *Contribution à la modélisation et à la commande de robots mobiles à roues*. PhD thesis, Ecole Nationale Supérieure des Mines de Paris, 1995.
- λ L [18] C. A.Micaelli, "Trajectory tracking for two-steering wheels mobile robots," *IFAC Symposium on Robot Control*, pp. 249-256, September 1994.
- [19] B.D'Andréa-Novet, G.Campion, and G.Bastin, "Control of wheeled mobile robot not satisfying ideal velocity constraints : a singular perturbation approach," *International journal of robust and nonlinear control*, vol. 5, pp. 243-267, 1995.
- [20] M.Ellouze, *Modélisation et Commande de Robots Mobiles à Roues en Présence de Glissements*. PhD thesis, Ecole des mines de Paris, 2001.
- [21] E.Bakker, L.Nyborg, and H.B.Pacejka, "Tyre modelling for use in vehicle, dynamic studies," *Society of Automotive Engineers*, pp. 2.190-2.204, 1988.
- [22] K.Khorasani and P.V.Kokotovic, "Feedback linearization of a flexible manipulator near its rigid body manifold," *Systems & Control Letters*, vol. 6, pp. 187-192, 1985.
- [23] M.W.Spong, K.Khorasani, and P.V.Kokotovic, "An integral manifold approach to the feedback control of flexible joint robots," *IEEE Journal of robotics and automation*, vol. 3, pp. 291-300, august 1987.
- [24] R.Sepulchre, M.Janković, and P. Kokotović, *Constructive Nonlinear Control*. Communications and Control Engineering, Springer, 1997.

- X[25] O. Canudas de Wit, B. Siciliano, and G. Bastin, *Theory of Robot Control*. Springer-Verlag, 1996.
- [26] P. Fisette, J.M. Péterkenne, B. Vaneghem, and J.C. Samin, "A multibody loop constraints for modelling cam/follower devices," *Nonlinear dynamics*, no. 22, pp. 335–359, 2000.
- [27] H.K. Khalil, *Nonlinear Systems*. Prentice-Hall, 1996.
- [28] V.A. Sobolev, "Integral manifolds and decomposition of singularly perturbed systems," *Systems & Control Letters*, no. 5, pp. 169–179, 1984.
- [29] I. Motte and G. Campion, "Control of sliding mobile robots : a slow manifold approach," *Proceeding of the Fourteenth Symposium of Mathematical Theory of Networks and Systems*, June 2000. Perpignan.
- [30] I. Motte and G. Campion, "A slow manifold approach for the control of mobile robots," *IEEE Transactions on Robotics and Automation*, vol. 16, pp. 875–880, december 2000.
- [31] R. Fletcher, *Practical Methods of Optimization*. John Wiley & Sons, LTD, second ed., 1987. Chapter 10.

Index

- accessible set associated with wheel i , 115
- admissibility margin, 120
- boundary layer system, 49
- bridled robot, 122
- compatible initial conditions, 53
- configuration admissible trajectory, 115
- configuration coordinates vector, 15
- coordination manifold, 112
- degree of manoeuvrability δ_M , 17
- degree of mobility δ_m , 16
- degree of steerability δ_s , 17
- dynamic state space model, 24
- full mobility robot, 16
- global accessible set, 115
- ideal robot, 11
- manifold condition, 59
- matching condition (perturbation satisfying the), 79
- non singular trajectory, 116
- non singularity margin, 119
- point tracking problem, 25
- posture admissible trajectory, 116
- quasi-steady-state model, 48
- restricted mobility robot, 16
- singular robot, 114
- singular wheel, 114
- slow manifold, 59
- slow model, 48
- small enough slip ratios, 43
- stability condition, 79
- standard form (dynamical model in), 48
- state space kinematic model, 18

