

**CHAINS OF REINSURANCE:
NON-COOPERATIVE
EQUILIBRIA AND PARETO OPTIMALITY***

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Abstract

The purpose of this paper is to extend Gerber's seminal paper on chains of reinsurance in three directions: (i) to throw some light on the rationale of finite chains under the Stackelberg equilibria introduced by Gerber, (ii) to propose new non-cooperative equilibrium concepts, of the Cournot type, which allow the first reinsured to transfer more than 50 percent of its risk if the risk aversion is the same for all companies, and (iii) to compare the Pareto optimal and the different equilibrium transfers of risk between reinsures.

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1. INTRODUCTION

In a recent and stimulating article, Gerber (1984) considers a hierarchical system of reinsurance which consists of a chain of proportional reinsurances in which a member is in direct contact with only two other members : the company he reinsures and the company which reinsures him. Assuming a normal distribution for the original risk and that each company maximizes its (exponential) utility function, it is possible to compute the values of the quotas and the loadings for each link of the chain whenever a behavior rule is specified for the companies. Gerber first proposed that each reinsured let his reinsurer know how much he is ready to buy as a function of the loading and the reinsurer, knowing this function, determines the loading. In fact, and by analogy with the theory of games, the resulting equilibrium is "à la Stackelberg", in the sense that the reinsurer acts as a leader, knowing the reaction function revealed by the reinsured, the follower. Because of this information advantage for the reinsurer, we note this equilibrium Stackelberg 2, or S2.

Unfortunately, it can be shown that under these assumptions the first reinsured *always* retains more than 50 percent of his risk. Therefore, Gerber introduces another equilibrium concept of the Stackelberg type where the reinsured is now assumed to be the leader. In this case, depending on the risk aversion of the companies, the amount of risk retained by the first company can vary between 0 and 100 percent. We note this equilibrium concept Stackelberg 1, or S1.

Our purpose is to extend Gerber's paper in three directions. First, we focus our attention on the determinants of the length of a chain (Section 3). Taken alone, transaction costs can of course explain why infinite chains do not occur in real world situations. To avoid as long as possible this kind of tautological explanation, we try to determine endogenously the finiteness of the chains. More precisely, we establish how the expected utility of each company is related to the length of the chain and to its position in the chain. In order to do that,

we have to assume that all companies have the same positive risk aversion.

But, when risk aversions are identical, it appears that, *even for Stackelberg 1*, the first reinsured retains more than 50 percent of the original risk. Because we feel that identical risk aversions in no way is an unrealistic assumption for a large range of real world situations, Gerber's introduction of S1 does not perfectly solve the problem. Therefore, we propose new non-cooperative equilibrium concepts which are of the Cournot type and where the quota of the first link can vary between 50 and 100 percent if risk aversions are equal (Section 4).

Finally, we determine the Pareto optimal values of the quotas and the loadings and check how they differ from the equilibrium values (Section 5).

Let us first recall the assumptions of the model together with Gerber's results about both Stackelberg equilibria.

2. ASSUMPTIONS AND STACKELBERG EQUILIBRIA

2.1. Assumptions

We consider $n+1$ companies which form a chain of reinsurance with n links.

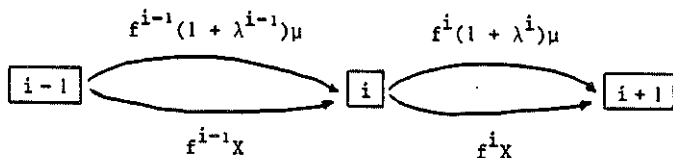
$$C(t) = \ln E[e^{tX}] = \mu t + \frac{1}{2} \sigma^2 t^2$$

is the cumulant generating function of the normally distributed original risk X (in terms of potential claims), where $\mu = E[X]$ and $\sigma^2 = \text{var}[X]$. For each company, we suppose an exponential utility function for money

$$u_i(x) = -\frac{1}{a_i} (1 - e^{-a_i x}) \quad (i = 1, 2, \dots, n+1),$$

where a_i , the absolute risk aversion, is a positive constant. Company i ($i = 2, 3, \dots, n$) sells coverage to company $i-1$ for a fraction f^{i-1} of

X and buys coverage from company $i+1$ for a fraction f^i . The first company does not sell any reinsurance coverage, and the terminal reinsurer does not buy any. If λ^i is the loading rate for the transaction i , the company i receives the premium $f^{i-1}(1 + \lambda^{i-1})\mu$ and pays the premium $f^i(1 + \lambda^i)\mu$.



Finally, let us remark that because of the normal distribution of X and of the exponential utility functions, we have

$$\text{Max } E[u_i(\alpha X + \beta)] \Leftrightarrow \text{Min } C(-a_i \alpha) \quad (1)$$

and

$$E[u_i(\alpha X + \beta)] \geq E[u_i(\gamma X + \delta)] \Leftrightarrow C(-a_i \alpha) - a_i \beta \leq C(-a_i \gamma) - a_i \delta, \quad (2)$$

where α , β , γ and δ are positive constants.

2.2. Stackelberg equilibria

In the following, f_j^i denotes the reaction function of company j in the transaction i .

Stackelberg 2

Company 1 (the first reinsured) maximizes its expected utility function

$$E[u_1(-(1 - f^1)X - f^1(1 + \lambda^1)\mu)]$$

with respect to f^1 which determines its reaction function $f_1^1 = f_1^1(\lambda^1)$ it communicates to company 2 (the first reinsurer). Knowing this, company 2 maximizes its expected utility with respect to λ^1 and f^2 which determines its reaction function $f_2^2 = f_2^2(\lambda^2)$ it communicates to company 3, etc ...

In other words, company i ($i = 2, 3, \dots, n$), knowing $f_{i-1}^{i-1}(\lambda^{i-1})$, maximizes its expected utility

$$E\{u_i(- (f_{i-1}^{i-1} - f^i)X + f_{i-1}^{i-1}(1 + \lambda^{i-1})\mu - f^i(1 + \lambda^i)\mu)\}$$

with respect to λ^{i-1} and f^i . Finally, company n communicates its reaction function $f_n^n(\lambda^n)$ to the last reinsurer, company $n+1$, which maximizes its expected utility

$$E\{u_{n+1}(- f_n^n X + f_n^n(1 + \lambda^n)\mu)\}$$

with respect to λ^n , i.e. it chooses the value of λ^n . By successive substitutions, Gerber, using (1), then gives to f^n , λ^{n-1} , f^{n-1} , ..., λ^1 , f^1 a value in a recurrent form. If we assume identical risk aversions ($a_i = a$ for $i = 1, 2, \dots, n+1$), it is possible to find an explicit solution for $i = 1, 2, \dots, n$:

$$f^i = 2 - \frac{2^{i+1} - 2}{2^{n+1} - 1} - \sum_{k=i}^n \frac{2^i - 1}{2^k - 1} \quad (3)$$

$$\lambda^i = \frac{\sigma^2 a}{\mu} \left[\frac{2^i}{2^{n+1} - 1} - 1 + \sum_{k=i}^n \frac{2^{i-1}}{2^k - 1} \right] \quad (4)$$

Stackelberg 1

Company $n+1$ (the last reinsurer) maximizes its expected utility with respect to f^n which determines its reaction function $f_{n+1}^n = f_{n+1}^n(\lambda^n)$ it communicates to company n (the last reinsured). Knowing this, company n maximizes its expected utility with respect to λ^n and f^{n-1} which determines its reaction function $f_n^{n-1} = f_n^{n-1}(\lambda^{n-1})$ it communicates to company $n-2$, etc... Finally, company 2 communicates its reaction function $f_2^1(\lambda^2)$ to the first reinsured which determines the value of λ^1 . By successive substitutions, Gerber, using (1), then gives to f^1 , λ^2 , f^2 , ..., λ^n , f^n an explicit value which takes the following form if risk aversions are identical:

$$f^i = \frac{2^{n+1-i} - 1}{2^{n+1} - 1}, \quad (5)$$

$$\lambda^i = \frac{\sigma^2 a}{\mu} \cdot \frac{2^{n-i}}{2^{n+1} - 1}, \quad (6)$$

for $i = 1, 2, \dots, n$.

3. THE DETERMINANTS OF THE LENGTH OF A CHAIN UNDER STACKELBERG EQUILIBRIA

As long as chains are considered, there are two ways for extending them, either by addition of a *last* link (the last reinsurer becoming the last reinsured), or by addition of an *intermediate* link (between two companies in direct connection in the original chain). In order to explain the finiteness of the chains under Stackelberg equilibria, we first determine how the expected utility of the companies inside and outside the original chain is affected by its extension. If it is positively affected for all companies, it militates in favor of infinite chains, *ceteris paribus*. If interests compete, the finiteness of the chains is explained, in some sense.

To clarify the presentation and because the results under S2 are not as general as those under S1, they are confined in footnotes.

The assumption of identical risk aversions is maintained throughout this section.

3.1. Addition of a last link

It is possible to show that under S1, it is always in an outsider's interest to reinsure the last company of an existing chain. Indeed, it is proved if

$$E[u_{n+2}(0)] < E[u_{n+2}(-f^{n+1}X + f^{n+1}(1 + \lambda^{n+1})\frac{1}{2})],$$

i.e., using (2), if

$$0 > \sigma^2 a(f^{n+1})^2 - 2 f^{n+1} \lambda^{n+1} \mu. \quad (7)$$

Substituting f^{n+1} and λ^{n+1} in (7) by their value from (5) and (6), one can check that the inequality holds¹.

Moreover, it can be shown that the expected utility of *each* company belonging to the original chain increases when a last link is added. Indeed,

$$E[u_i(\cdot)]_{(n+1)} < E[u_i(\cdot)]_{(n+2)} \quad (i = 1, 2, \dots, n+1),$$

where subscripts $(n+1)$ and $(n+2)$ indicate the number of companies in the chain, is equivalent to

$$E[u_i(-(f^{i-1} - f^i)X + f^{i-1}(1 + \lambda^{i-1})\mu - f^i(1 + \lambda^i)\mu)]_{(n+1)} < E[u_i(\cdot)]_{(n+2)}$$

which reduces, using (2), to

$$[\sigma^2 a(f^{i-1} - f^i)^2 - 2f^{i-1}\lambda^{i-1}\mu + 2f^i\lambda^i\mu]_{(n+1)} > [\cdot]_{(n+2)}. \quad (8)$$

Substituting in (8) f and λ by their values from (5) and (6), the result follows^{2,3}.

Taken together, these results explain why chains exist. Moreover, they militate in favor of infinite chains : addition of a last link is to everybody's advantage. However, it is worthwhile to determine how the expected utility of a company is related to its place in the chain. Indeed, if this expected utility decreases with the place, the many existence of positive transaction costs can explain the finiteness of the chain. And one can establish that this is precisely the case :

$$E[u_{i-1}(\cdot)] > E[u_i(\cdot)] \quad (i = 3, 4, \dots, n+1).$$

Using (2), this inequality is verified if

$$\begin{aligned} \sigma^2 a(f^{i-2} - f^{i-1})^2 - 2f^{i-2}\lambda^{i-2}\mu + 2f^{i-1}\lambda^{i-1}\mu < \\ \sigma^2 a(f^{i-1} - f^i)^2 - 2f^{i-1}\lambda^{i-1}\mu + 2f^i\lambda^i\mu. \end{aligned} \quad (9)$$

Substituting in (9) f and λ by their values from (5) and (6), the result follows^{4,5}.

3.2. Addition of an intermediate link

Because we have shown that it is always to the advantage of an outsider to become the last reinsurer of an existing chain and that the expected utility of a company decreases with its place in the chain, it follows that it is always to the advantage of an outsider to join the chain between company i and company $i+1$ ($1 \leq i < i+1 \leq n+1$).

But would all companies agree with this kind of extension of the chain? The answer is yes for all companies h ($1 \leq h \leq i$) because this is formally equivalent to an addition of a last link. The answer is a priori ambiguous for companies k ($i+1 \leq k \leq n+1$) because this addition of an intermediate link extends the chain *but* displaces them. It can be shown that the final effect is negative :

$$E[u_i(\cdot)]_{(n+1)} > E[u_{i+1}(\cdot)]_{(n+2)} \quad (i = 2, 3, \dots, n+1)$$

i.e., using (2), that

$$\begin{aligned} [\sigma^2 a(f^{i-1} - f^i)^2 - 2f^{i-1}\lambda^{i-1}\mu + 2f^i\lambda^i\mu]_{(n+1)} < \\ [\sigma^2 a(f^i - f^{i+1})^2 - 2f^i\lambda^i\mu + 2f^{i+1}\lambda^{i+1}\mu]_{(n+2)}. \end{aligned} \quad (10)$$

Substituting in (10) f and λ by their values from (5) and (6), the result follows^{6,7}.

As long as the addition of an intermediate link is considered, the finiteness of the chain is thus explained by the competing interests of

companies placed "before" and "after" the new intermediary.

4. EQUILIBRIA "A LA COURNOT"

The aim of this section is to propose new equilibrium concepts which allow the quota of the first link to be higher than 50 percent of the original risk when risk aversions are identical⁸. These equilibrium concepts are of the Cournot type, i.e. each company acts as a follower, taking the relevant loadings as constants. The first company determines its reaction function $f_1^1 = f_1^1(\lambda^1)$, company i ($1 < i < n+1$) determines its reaction functions $f_i^{i-1} = f_i^{i-1}(f_1^1, \lambda^{i-1})$ and $f_i^i = f_i^i(f_1^1, \lambda^i)$, and the last company determines its reaction function $f_{n+1}^n = f_{n+1}^n(\lambda^n)$.

Two kinds of Cournot equilibrium can be considered, depending on how the f_i^i , λ^i are computed, i.e. how the information circulates.

4.1. Trivial Cournot equilibrium

In the first case, we simply take the $2n$ ($1+2(n-1)+1$) reaction functions as a system with $2n$ unknowns (f_i^i, λ^i ; $i = 1, 2, \dots, n$). The implicit behavioral assumption behind this equilibrium is rather poor and we therefore denote it Trivial Cournot equilibrium, or Cournot T. It is easy to find the f_i^i and λ^i in a recurrent form :

$$f_i^i = \frac{\sum_{k=1}^{n+1} \frac{1}{a_k}}{\sum_{k=1}^{n+1} \frac{1}{a_k}} \quad (i = 1, 2, \dots, n) \quad (11)$$

$$\lambda^i = \frac{\sigma^2}{\mu} \cdot \frac{1}{\sum_{k=1}^{n+1} \frac{1}{a_k}} \quad (i = 1, 2, \dots, n) \quad (12)$$

It can be checked that f^1 can vary between 0 and 1 depending on the risk aversions, but that if they are identical, f^1 increases from 1/2 to 1, as n increases : if a chain exists under Cournot T ($n > 1$), the first reinsured never retains more than 50 percent of the original risk.

For reasons that will be apparent in the next section, it is worthwhile to rewrite (11). First, note that it implies the following equality :

$$-a_i(f^{i-1} - f^i) + a_{i+1}(f^i - f^{i+1}) = 0 \quad (i = 1, 2, \dots, n), \quad (13)$$

which defines a system of *difference equations* with boundary conditions $f^0 = 1$ and $f^{n+1} = 0$. Using the *method of particular solutions*, we find that

$$f^i = A + B \left(\frac{a_i}{a_{i+1}} \right)^i \quad (i = 1, 2, \dots, n) \quad (14)$$

if all risk aversions are not identical and $n > 1^9$ and where

$$A = 1 - \frac{a_{n+1}}{a_{n+1} \cdot Y - a_1 \cdot Z},$$

$$B = \frac{a_1 a_{n+1}}{a_{n+1} \cdot Y - a_1 \cdot Z},$$

$$Y = \frac{a_1^2 a_3^2 + a_1 a_2 a_3^2 - a_2^4}{a_2 a_3^2},$$

$$Z = (a_n + a_{n+1}) \cdot \left(\frac{a_n}{a_{n+1}} \right)^n - a_n \left(\frac{a_{n-1}}{a_n} \right)^{n-1},$$

and that

$$f^i = 1 - \frac{i}{n+1} \quad (i = 1, 2, \dots, n) \quad (15)$$

if all risk aversions are identical.

4.2. Non-trivial Cournot equilibria

A more attractive way to define a Cournot equilibrium is to consider that the equilibrium values "à la Cournot" of the f^i and λ^i are determined link by link, as under the Stackelberg equilibria. Then, it is not surprising that two kinds of equilibria will similarly emerge, depending on which company (the first or the last one) is supposed to first determine its reaction function.

Cournot 1

By analogy with Stackelberg 1, we denote Cournot 1, or C1, the Cournot equilibrium where the last reinsurer first determines its reaction function $f_{n+1}^n = f_{n+1}^n(\lambda^n)$. Together with the first reaction function of company n : $f_n^n = f_n^n(\lambda^n, f_{n+1}^n)$, this determines the equilibrium values "à la Cournot" of f^n and λ^n as functions of f^{n-1} . Company n then determines its second reaction function $f_n^{n-1} = f_n^{n-1}(\lambda^{n-1})$. Together with the first reaction function of company $n-1$: $f_{n-1}^{n-1} = f_{n-1}^{n-1}(\lambda^{n-1}, f_n^{n-1})$, this determines the equilibrium value of f^{n-1} and λ^{n-1} as functions of f^{n-2} , etc ... Company 2, knowing the equilibrium values of f^2 and λ^2 as function of f^1 , determines its reaction function $f_2^1 = f_2^1(\lambda^1)$. Together with the reaction function of company 1 : $f_1^1 = f_1^1(\lambda^1)$, this gives the equilibrium value of f^1 and λ^1 , and therefore, recursively, the equilibrium values of $f^2, \lambda^2, f^3, \lambda^3, \dots, f^n, \lambda^n$.

Using Gerber's recurrent method for S1, it is possible to derive the equilibrium value of the f^i and λ^i in the identical risk aversions case :

$$f^i = \prod_{k=n-i}^{n-1} \frac{h^k(1)}{h^k(1) + 1} \quad (i=1, 2, \dots, n)$$

$$\lambda^i = \frac{\sigma^2 a}{\mu \cdot h^{n-1}(1)} \cdot \prod_{k=n-1}^{n-i} \frac{h^k(1)}{h^k(1) + 1} \quad (i=1,2,\dots,n)$$

where $h(x) = \frac{(x+1)^2}{2x+1}$ and h^l denotes the function h l times composed, e.g. $h^2(x) = h(h(x))$.

Again, it is possible to check that $f^1 = \frac{h^{n-1}(1)}{h^{n-1}(1) + 1}$ increases from $1/2$ to 1 as n goes from 1 to infinity.

Cournot 2

Here, company 1 first determines its reaction function $f_1^1 = f_1^1(\lambda^1)$ which, together with the first reaction function of company 2 : $f_2^1 = f_2^1(\lambda^1, f^2)$, determines the equilibrium value of f^1 and λ^1 as functions of f^2 , etc... Company n , knowing the equilibrium value of f^{n-1} and λ^{n-1} as functions of f^n , determines its second reaction function : $f_n^n = f_n^n(\lambda^n)$. Together with the reaction function of the last reinsurer : $f_{n+1}^n = f_{n+1}^n(\lambda^n)$, this gives the equilibrium value of f^n and λ^n and then, recursively, of f^{n-1} , λ^{n-1} , ..., f^1 , λ^1 .

Using Gerber's recurrent method for S2, it is again possible to derive the equilibrium value of the f^i and λ^i in the identical risk aversions case. But, because C1 solution is more complex than S1, it is not surprising that C2 solution is still more complex, S2 being already harder to compute than S1. Indeed, it would take more than one page to write down the equilibrium value of f^i and λ^i ! Nevertheless, it is again possible to check that f^1 is an increasing function of n with $f^1 = \frac{1}{2}$ if $n = 1$.

Whatever the Cournot equilibrium (CT, C1 or C2), it then appears that the first reinsured never retains more than 50 percent of the original risk if risk aversions are identical.

5. REINSURANCE AND PARETO OPTIMALITY

In this last section, we first determine the Pareto optimal values of the loadings and the quotas, and then check how they differ from the equilibrium values under S1, S2, CT, C1 and C2.

5.1. Pareto optimality

To find the optimal values of the loadings and the quotas, it is necessary and sufficient to

$$\max_{f^i, \lambda^i} \sum_{k=1}^{n+1} \gamma_k E[u_k] \quad (i=1, 2, \dots, n) \quad (16)$$

where γ_i ($i=1, 2, \dots, n+1$) are arbitrary positive weights. Using (2), (16) is equivalent to

$$\min_{f^i, \lambda^i} \sum_{k=1}^{n+1} \gamma_k \frac{\sigma^2}{2} a_k^2 (f^{k-1} - f^k)^2 - \gamma_k \mu a_k f^{k-1} \lambda^{k-1} + \gamma_k \mu a_k f^k \lambda^k, \quad (i=1, 2, \dots, n)$$

where $f^0 = 1$ and $f^{n+1} = 0$. The first order conditions give for $i = 1, 2, \dots, n$

$$f^i \mu (a_i \gamma_i - a_{i+1} \gamma_{i+1}) = 0 \quad (17)$$

$$\begin{aligned} -\gamma_i \sigma^2 a_i^2 (f^{i-1} - f^i) + \gamma_{i+1} \sigma^2 a_{i+1}^2 (f^i - f^{i+1}) + \\ \mu \lambda^i (a_i \gamma_i - a_{i+1} \gamma_{i+1}) = 0. \end{aligned} \quad (18)$$

(17) and (18) reduce to

$$-a_i (f^{i-1} - f^i) + a_{i+1} (f^i - f^{i+1}) = 0 \quad (i=1, 2, \dots, n) \quad (19)$$

Note that (19) is equivalent to (13), the equation relating the f^i at the Trivial Cournot equilibrium. Since (14) and (15) gave an explicit solution for f^i if (13) holds, we conclude that Pareto optimality implies that, whatever the loadings λ^i , (14) and (15) must be verified..

The first comment this result gives rise to is immediate : the Trivial Cournot equilibrium guarantees a Pareto optimal solution. Moreover, we already noticed that if $n = 1$, i.e. if only two companies belong to the chain,

$$f^1 = \frac{a_1}{a_1 + a_2} \quad (20)$$

under Cournot T. But (20) is precisely, whatever λ^1 , the famous Borch's (1960) theorem in the exponential utility function case. This allows us to consider (14) and (15) as the extension of Borch's theorem in the exponential utility function case to chains with more than two companies.

The last question we then consider is the following : can non-trivial non-cooperative equilibria guarantee Pareto optimal solutions, i.e. Pareto optimal quotas ?¹⁰

5.2. Pareto optimality and non trivial equilibria

A glance at the relevant equilibrium value of f^i reveals that the answer is negative, except under C1 and C2, when only two companies belong to the chain. It is obvious that, in this case, C1, C2 and CT determine the same Pareto optimal equilibrium.

By way of a first summary, we then propose to put in a graph the five non-cooperative equilibria here studied together with the Pareto optimal solutions, when $n = 1$.

It can be checked that

$$f_{CT}^1 = f_{C1}^1 = f_{C2}^1 = \frac{a_1}{a_1 + a_2}$$

$$\lambda_{CT}^1 = \lambda_{C1}^1 = \lambda_{C2}^1 = \frac{\sigma^2 a_1 a_2}{\mu(a_1 + a_2)}$$

$$f_{S1}^1 = \frac{a_1}{a_1 + 2a_2} ; f_{S2}^2 = \frac{a_1}{2a_1 + a_2}$$

$$\lambda_{S1}^1 = \frac{\sigma^2 a_1 a_2}{\mu(a_1 + 2a_2)} ; \lambda_{S2}^1 = \frac{\sigma^2 a_1 (a_1 + a_2)}{\mu(2a_1 + a_2)} .$$

If we define the *reservation quota function* of company i as the quota f_i^* of the original risk which makes it indifferent (in terms of expected utility) between belonging and not belonging to the chain, it is possible to show that

$$f_1^* = 2 - \frac{2\mu \lambda^1}{\sigma^2 a_1}$$

and

$$f_2^* = \frac{2\mu \lambda^1}{\sigma^2 a_2}$$

when $n = 1$. If, for a given λ^1 , $f^1 > f_i^*$ ($i = 1, 2$), the transaction does not occur.

Let us finally recall the relevant reaction functions :

$$f_1^1 = 1 - \frac{\mu \lambda^1}{\sigma^2 a_1}$$

and

$$f_2^1 = \frac{\mu \lambda^1}{\sigma^2 a_2} .$$

Figure 1 restates all this information together with the indifference curves of both companies, when $a_1 > a_2^{11}$. AE is company 1 reaction function with the maximum expected utility at A. OD is company 2

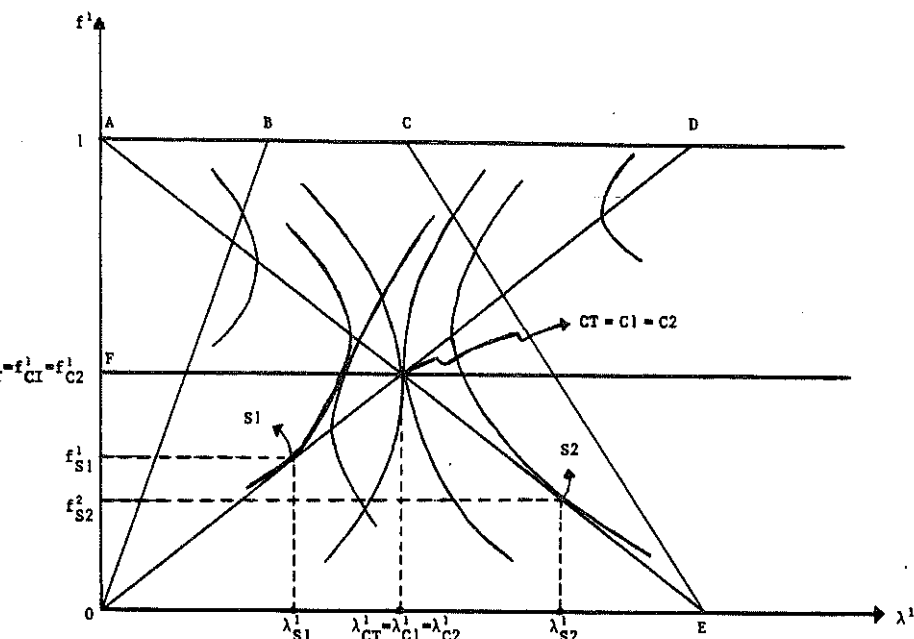


FIGURE 1

reaction function with maximum expected utility at D. At the intersection of these functions are the Cournot equilibria. At the tangency point between company 2 (resp. 1) reaction function and the highest company 1 (resp. 2) indifference curve, is S1 (resp. S2). CE (resp. OB) is the reservation quota function of company 1 (resp 2). For all (f^1, λ^1) to the left of CE, company 1 is then better off than in the no reinsurance case. The same for company 2 for all (f^1, λ^1) to the right of OB. As already mentioned, all equilibria define (f^1, λ^1) acceptable by both companies, i.e. in OBCE. Finally, the horizontal axis at F

gives the Pareto optimal allocations of the risk between the companies, determined by the tangency points of their indifference curves. The Cournot equilibria are one of these allocations, as shown.

6. CONCLUSIONS

Extending Gerber's paper, our purpose was here threefold : (i) to throw some light on the rationale of finite chains of reinsurance under the Stackelberg equilibria introduced by Gerber; (ii) to propose new non-cooperative equilibrium concepts, of the Cournot type, which allow the first reinsured to transfer more than 50 percent of its risk if the risk aversion is the same for all companies, and (iii) to compare the Pareto optimal and the different equilibrium transfers of risk between reinsurers.

FOOTNOTES

¹Substituting f^{n+1} and λ^{n+1} in (7) by their value from (3) and (4), one can check that inequality (7) also holds under S2.

²Note that when $i = 1$, $f^{i-1} = 1$ and $\lambda^{i-1} = 0$. Symmetrically, when $i = n+1$, $f^i = 0$ in the left hand side of inequality (8).

³The same result holds under S2, *except* that the expected utility of the last reinsurer *decreases* if he is reinsured, because he loses, in some sense, part of his leadership with this addition of a last link : he becomes a follower. This confirms that Gerber was not right when, noting that λ^i decreases with the length of the chain, he claimed it is to the advantage of the original chain to add a last link. As inspection of (3) - (6) reveals, it is true that, whatever the Stackelberg equilibrium, the λ^i are decreasing functions of the length of the chain and, moreover, the f^i are increasing functions of this length. But, taken alone, this information cannot be a priori sufficient conditions to say that each company would agree with the extension of the chain.

⁴Note that when $i = n+1$, $f^i = 0$.

⁵While it cannot be proved with full generality, examples suggest that this result also holds under S2.

⁶Note that when $i = n+1$, $f^i = 0$ in the left hand side of (10).

⁷Again, while it cannot be proved with full generality, examples suggest that this result also holds under S2.

⁸One can show that with identical risk aversions, f^1 increases from $1/3$ and approaches $1/2$ (resp. 0,393 ...) under S1 (resp. S2), as n increases.

⁹If $n = 1$, $f^1 = \frac{a_1}{a_1 + a_2}$.

¹⁰Of course, this leaves the question of possible Pareto optimal cooperative equilibria open.

¹¹If a_1 were not greater than a_2 , only the relative positions of f_{S1}^1 and f_{S2}^1 , and of B and C in Figure 1 would change. Note that $AC \leq AD$ as $a_1 \leq 2a_2$.

REFERENCES

- Borch, K. (1960), "The safety loading of reinsurance premiums", *Skandinavisk Aktuarietidskrift*, pp. 163-184.
- Gerber, H.U. (1984), "Chains of reinsurance", *Insurance : Mathematics and Economics*, 3, pp. 43-48.