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Ruin problems in non-standard risk models.

THÈSE

présentée en vue de l'obtention du titre de docteur en sciences, orientation sciences actuarielles.

par

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"Il n'y a qu'une chose qui puisse rendre un rêve impossible, c'est la peur d'échouer",
Paolo Coelho.

I dedicate this work to Michel Malray and Guy Trufin.

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Contents

Introduction

The computation of ruin probabilities constitutes a central topic in risk theory. It goes back to the pioneering works by the Scandinavian actuaries at the beginning of the twentieth century, in particular Filip Lundberg (1903) and Harald Cramér (1930). Standard references include Asmussen (2000) and Rolski, Schmidli, Schmidt and Teugels (1999), while Dickson (2005) offers a gentle introduction to the topic. Even though the study of the ruin problem resulted in a huge amount of published articles (including in the closely related fields of queueing theory and dam and storage processes), some questions have never been studied in the literature, until recently. This thesis aims at contributing to the risk theory by looking into various problems in line with the real management of companies.

General overview

This work is divided into two parts. In the first part, which constitutes the main part of this work, the purpose is to relax some classical assumptions to account for situations of practical relevance. Indeed, the vast majority of existing results require independent increments for the stochastic process describing the aggregate claim amounts filed against the insurance company. But there is a variety of situations where this independence assumption appears rather unrealistic. This is the case in presence of business cycles, delay in claim reporting or settlement, or as the result of the implementation of experience rating mechanisms by the insurance company. In such settings, the financial result of a given calendar year depends on the result of one or several preceding years. So, the first part of this thesis consists to deal with non-classical risk models allowing to reflect such basic insurance realities.

In the second part, the aim is to study a risk measure derived from actuarial ruin theory. One historical purpose of studying ruin probabilities was to obtain the amount of initial capital needed to guarantee to the insurance company a given probability of solvency. The Value-at-Risk (VaR) and the expected shortfall are natural concepts in this setting, and are widely used by the practitioners. However, these two notions are defined in terms of a given time horizon. Consequently, both the VaR and the expected shortfall do not reflect the possible adverse financial situations in between or beyond the specified time interval. Thus, the smallest capital allowing to ensure that the infinite-time ruin probability is less than some acceptance level appears as a good alternative to quantify the insured risk. Indeed, this time, it will suffice to cope with the risk at all times according to the given level. Let us mention that it is this type of robustness that makes the ruin probability still nowadays an interesting quantity in this context. The study of the properties of this smallest capital constitutes the subject of the second part of this thesis.

Part 1

This thesis is of course not the first attempt to study risk processes with dependent increments.

Let us mention Müller and Pflug (2001), where the way the marginal distributions of the annual net profits and their dependence structure affect Lundberg's coefficient is examined. The ordering of Lundberg coefficients yields to an asymptotic ordering of ruin probabilities for some fixed amount of initial capital. Those results are in accordance with Gerber (1981, 1982) where the ruin problem is studied for annual gains forming a linear time serie. See also Yuen and Guo (2001) and Cossette, Landriault and Marceau (2003) for related results. Also, risk processes with dependent inter-claim times, allowing for possible dependence between the arrival process and the claim sizes are discussed in Asmussen (2000). Those processes contain the so-called Markov-modulated models as a special case (see also Asmussen et al. (1994)). In these models, the underlying intensity model follows a finite state Markov chain, enabling for instance the modelling of underlying changes in the economy in general or the market in particular. Other papers devoted to ruin models with dependence between the waiting time for a claim and its actual size include Albrecher and Boxma (2003) and Albrecher and Teugels (2006).

On the contrary to most of the references quoted just above, let us insist on the fact that this work does not aim to study ruin models with general dependence structures. Indeed, the first part of this thesis consists of three chapters, each focusing on a ruin model dealing with a phenomenon of practical relevance.

Chapter 1

The first chapter of this first part consists in studying ruin problems in the presence of business cycles. Underwriting profit rates and insurance prices in the property-liability insurance industry have been cyclical in the postwar period. A "hard market", where prices and profits are high and coverage is restricted tends to be followed by a "soft market", where prices and profits are low and coverage is expanded. The underwriting cycle in the property-liability insurance business is a recurring pattern of ups and downs in insurance prices and profits. A number of studies presenting empirical assessments of this cycle have appeared in the literature over the past 30 years. Assuming that the annual results of the insurance company is subject to underwriting cycles, ruin probabilities will be computed, and compared with the corresponding values obtained under the independence assumption for the annual net gains. This study will allow us in particular to determine if the presence of business cycles increases or not the risk for the insurer.

Chapter 2

Allowing the premium amount to depend on past claims experience through an appropriate credibility mechanism is in line with insurance practice in most lines of business. That is why the second chapter takes into account experience rating. The reason comes from the fact that the insured risk is not completely known for the insurer's point of view. In this chapter, the uncertainty is modelled by considering that the insurance company will inherit randomly from a portfolio belonging to a set of known portfolios, according to a certain distribution also known by the insurer. Let us mention that this approach will be motivated in this chapter. Hence, to calculate premiums, practitioners often use a mix between a quantity based on past claim experience and a market premium level. Here, the Bühlmann's credibility model has been chosen to model these credibility adjusted premiums. In this setting, for large initial capital, the main objective is to study the impact on the ultimate ruin probabilities of the number of past claim experience years taken into account to value the premiums. In order to decrease its risk, the insurer will have to react differently according to the nature of its annual claim amounts distribution.

Chapter 3

As a third step, ruin problems in the presence of delay in claim settlement are investigated. In non-life insurance, all the claims referring to a calendar year are not settled the same year. In particular, we can mention the time required to consolidate an injury, or the duration of some legal procedures. Consequently, several payments spaced in time can be observed before entirely settled a claim. The ruin model considered in this third step allows to explicitly use the Chain-Ladder development factors, widely used in practice to account for IBNR (incurred but not reported) claims. In particular, we assume that the payment of each annual claim is spread over several years according to the development factors (where the spread is applied in a deterministic fashion). This leads to a moving average model for the annual claims that describes in a certain way the dependence of annual claims caused through delayed claims. This model provides a transparent comparison of ruin probabilities with and without IBNR dynamics, allowing us to conclude if the IBNR practice is an advantage or not for the insurance company.

Part 2

In general, over the last years the relative position and relation between (insurance) risk measures that fulfill a certain list of axioms on the one hand, and classical tools in ruin theory to assess the riskiness of an insurance activity in the collective framework on the other hand, has often been a matter of debate. Dhaene, Goovaerts and Kaas (2003) give ample motivation for an exponential risk measure inherited from the Cramér-Lundberg upper bound for the ruin probability in a discrete-time ruin model. Cheridito, Delbaen and Kupper (2006) mention (in an application of their study on coherent risk measures for unbounded stochastic processes) a Value-at-Risk (VaR) risk measure based on the infinite-time ruin probability itself.

So, in this second part, we would like to take up this issue and investigate in more detail some properties of the VaR-type risk measure based on the ruin probability that is mentioned in Cheridito, Delbaen and Kupper (2006). For ease of exposition, we will restrict the considerations to the classical compound Poisson process. In particular, the question of knowing whether this risk measure, which can be interpreted as an economic capital, recognizes a diversification effect in some cases or not is studied. Also, examples where the insurer finds interest to merge two of its portfolios to decrease the amount of money allocated will be showed.

Remark

Before beginning, let us mention that each chapter of this thesis constitutes an article in itself. The first chapter will be published in The North American Actuarial Journal, while the others are submitted. Notice that chapters 1, 3 and 4 are joint works with Professors Hansjörg Albrecher and Michel Denuit, while the Chapter 2 is a joint work with Professor Stéphane Loisel.

Part I

Chapter 1

Impact of underwriting cycles on the solvency of an insurance company

This chapter is published in the North American Actuarial Journal (see Trufin, Albrecher and Denuit (2009)).

Abstract

This paper studies the solvency of an insurance firm in presence of underwriting cycles. A small or medium size insurance company with a price-taker position in the market is considered. Its premium income is assumed to obey an autoregressive process with cycles. Specifically, the premium income for a specific calendar year is influenced by the market experience for the past couple of years. Under this classical AR(2) dynamics governing the premium income, an explicit expression for the ultimate ruin probability is derived, using a martingale approach, in the light-tailed claims case. Furthermore, the logarithmic asymptotic behavior of the ultimate ruin probability as well as the typical path to ruin are investigated. Then, a comparison is made with the classical case where the same company operates on a market without such cycles. Asymptotically, the presence of market cycles is shown to increase the risk for the company. Numerical illustrations are performed on Canadian motor insurance market data and support the theoretical analysis.

1.1 Introduction and Motivation

The computation of ruin probabilities is at the core of actuarial mathematics. It goes back to the pioneering works by the Scandinavian actuaries, including Harald Cramér and Filip Lundberg in the early 1900s. Standard references include Asmussen (2000), Dickson (2005), and Rolski, Schmidli, Schmidt and Teugels (1999). The vast majority of existing results require independent increments for the stochastic process describing the aggregate claim amounts filed against the insurance company. There are a variety of situations where this independence assumption appears rather unrealistic. This is the case in presence of business cycles, for instance. The present work aims to relax the independence assumption to account for underwriting cycles, a situation of practical relevance.

This is not the first attempt to study risk processes with dependent increments. Let us briefly mention the following works, which will be useful for our study. Gerber (1982) examined the ruin probability in a model where the annual gains of an insurance company are dependent random variables. The model he used is the linear model (well known in time-series analysis) which includes the autoregressive model and the moving average model as special cases. It is also shown in Gerber (1982) that a certain credibility model can be interpreted as a first-order model of the mixed type. Promislow (1991) generalized some results of Gerber (1982) by removing the boundedness restriction for the underlying distribution, by allowing for a weaker convergence condition for the coefficients in the infinite order case, and by relaxing the stationarity requirement of the model. In Müller and Pflug (2001), the way the marginal distributions of the annual net profits and their dependence structure affect the Lundberg coefficient (also called adjustment coefficient) is examined. The ordering of Lundberg coefficients yields an asymptotic ordering of ruin probabilities for some fixed amount of initial capital. Those results are in accordance with Gerber (1981, 1982).

The present paper studies ruin problems in the presence of business cycles. Underwriting profit rates and insurance prices in the property-liability insurance industry have been cyclical in the postwar period. A "hard market", where prices and profits are high and coverage is restricted tends to be followed by a "soft market", where prices and profits are low and coverage is expanded. The underwriting cycle in the property-liability insurance business is a recurring pattern of ups and downs in insurance prices and profits. Chapter 12 in the book by Daykin, Pentikäinen and Pesonen (1994) provides a review of the topic. A number of studies presenting empirical assessments of this cycle have appeared in the literature over the past 30 years. Assuming that the annual results of the insurance company are subject to underwriting cycles, ruin probabilities will be computed, and compared with the corresponding values obtained under the independence assumption for the annual net gains.

As pointed out by Jones and Ren (2008), there is little in the risk theory literature providing tools to quantify the additional risk associated with underwriting cycles. These authors recently analyzed the impact of underwriting cycles on an insurer's surplus. Specifically, they modelled the cycles by assuming that the risk loading and the claim rate follow a deterministic cyclic function. They then study different strategies which an insurer may choose in such a cyclic environment.

In the present paper, we aim to derive analytical results for the ultimate ruin probability in presence of underwriting cycles, modelled by assuming that the premium income is stochastic and obeys an AR(2) process with cycles. This is the classical dynamics induced by underwriting cycles after the influential work by Cummins and Outreville (1987). The AR(2) specification has been justified by sound theoretical arguments, as briefly discussed in Section 2. It is furthermore supported by empirical data. Cummins and Outreville (1987) successfully fitted data from the Canadian and US markets over calendar years 1957-1979. In this paper, we adopt the AR(2) process of Cummins and Outreville (1987) to model the underwriting cycles and we update the estimated parameters on the basis of recent data.

The paper is organized as follows. Section 2 discusses underwriting cycles and treats a practical case. Specifically, we consider the Canadian motor insurance market over calendar years 1971-2005. As in Cummins and Outreville (1987), an AR(2) model appears appropriate to describe the dynamics of the inverse loss ratio. Section 3 describes the dynamics of the insurer's surplus. Then, Section 4 studies the ultimate ruin probability for light-tailed claims in the presence of underwriting cycles. Provided that the insurance market is viable, an explicit expression for this probability is derived. A logarithmic asymptotics expression for the ruin probability is also studied in this section. A comparison is performed in Section 5 with the classical case where the premium income is not subject to cycles. Section 6 is devoted to a numerical study based on the Canadian market and the behavior for the heavy-tailed claims case is also briefly discussed. The final Section 7 concludes. The extension to constant interest rate is discussed there, as well as the possible influence of past surplus.

1.2 Underwriting cycles

1.2.1 Nature of the phenomenon

The underwriting cycle refers to a repeating series of conditions that affect property-casualty insurance markets. The typical description of the cycle includes four stages, described in terms of industry accounting profitability. The first stage is marked by several years of low profitability. This is followed by a sudden change to rapidly increasing profitability. The second stage is often called the insurance crisis. In the third stage, profitability remains high but is no longer increasing. Profitability gradually declines during the fourth stage as the industry returns to a period of low profitability. These cycles are closely related to underwriting strategies (hence their name). In the rising phase of the underwriting cycle, competition for business tends to encourage underwriters to ease up on underwriting standards. This tends to produce adverse loss experience and the cycle ultimately turns down. The decline in underwriting profits is stopped by a combination of tightened underwriting standards and rate increases and the cycle begins its upward movement once again.

The inverse loss ratio (premiums divided by losses) shows the price that an insurance firm gets for each unit of losses paid, and is taken as an approximation of the unit price. It gives an indication of what customers get back from insurers for the premiums they have paid and may be regarded as a proxy for the output of the insurance industry. Even if the inverse loss ratio seems to have been widely used in the literature, there are some variations in the definition of losses and premiums, and whether the discounting is taken into account or not.

1.2.2 Possible explanations

Explanations for the insurance cycle can be divided into several broad camps. In one set of explanations, simple models of insurer behavior are developed in which feedback mechanisms are used to induce autoregressive return series. For example, Venezian (1985) shows that setting prices around loss forecasts generated from previous loss experience is capable of generating autocorrelated underwriting returns. Similarly, Berger (1988) develops a model in which insurance capacity depends on the current level of equity that, through retained earnings, is determined largely by pricing decisions made in the previous period. This process may again be sufficient to produce an autoregressive process for underwriting returns. Cummins and Outreville (1987) show that simple lags in data collection or price regulation may be sufficient to produce cyclical performance even in a rational expectations setting.

In this paper, we follow Cummins and Outreville (1987) who hypothesized that the cycle is apparent in the sense that it has nothing to do with the underlying economic and statistical characteristics of insurance markets but rather is attributable to institutional and regulatory rigidities. Among the intervening factors are data collection lags, regulatory lags, policy renewal lags, and statutory accounting rules.

1.2.3 Autoregressive modelling

One of the major assumptions in the model of Cummins and Outreville (1987) is that cycles are determined not only by the supply side (as is often assumed) but can be explained by an equilibrium model with competitive markets and rational expectations. Under the assumption that information is available with lags of about two years and that the endogeneous variable is the only relevant variable, Cummins and Outreville (1987) derive their model as an AR(2) process:

$$\Pi_t = a_0 + a_1\Pi_{t-1} + a_2\Pi_{t-2} + \epsilon_t, \tag{1.2.1}$$

where Π_t is the unit price (i.e. the ratio of premiums to losses, or inverse loss ratio) in period t, and ϵ_t is an error term (the ϵ_t's are independent and identically distributed), with $\mathbb{E}[\epsilon_t] = 0$ and $\mathbb{V}[\epsilon_t] = \sigma_\epsilon^2$. We also assume that there exists a positive constant d such that $\Pr[|\epsilon_t| < d] = 1$. In practice, this is of course not a severe restriction in our model. In this AR(2) model, cycles occur if $a_1 > 0$, $a_2 < 0$ and $a_1^2 + 4a_2 < 0$, i.e. if complex roots exist. The period of cycles is then equal to $2\pi/\arccos(a_1/2\sqrt{-a_2})$. ARMA analysis can be misleading if the underlying time series is not stationary. Several articles have examined the stationarity of some insurance series. Harrington and Yu (2003) established that loss ratios do not contain a unit root (they explained why the previous literature erroneously reached the opposite conclusion, based on unappropriate use of statistical tests). Loss ratios should therefore be considered to be stationary.

Requiring that the mean $\mathbb{E}[\Pi_t]$ and variance $\mathbb{V}[\Pi_t]$ are constant over time, and the covariance for any two time periods (t and $t+k$, say) depends only on the interval between two time periods (here k), not on the starting time (here t), is equivalent to require that the roots of the polynomial $p(z) = 1 - a_1 z - a_2 z^2$ are located outside of the unit circle in the complex plane. This implies $a_2 > -1$ because $a_1^2 + 4a_2 < 0$. The constraints on the parameters a_1 and a_2 are thus

$$a_1 > 0,\ \ -1 < a_2 < 0 \text{ and } a_1^2 + 4a_2 < 0. \tag{1.2.2}$$

To ensure the viability of the market, we also assume that the mean inverse loss ratio exceeds 1, that is,

$$\mu_\Pi := \mathbb{E}[\Pi_t] = \frac{a_0}{1 - a_1 - a_2} > 1, \tag{1.2.3}$$

where $\mathbb{E}[\Pi_t]$ means the unconditional expectation for the $\{\Pi_t\}$ process (from now, for a random variable X, we will denote by μ_X and σ_X^2 its mean and variance, respectively).

1.2.4 Empirical illustration on Canadian motor insurance market data

ARMA modelling

We use data from the Canadian motor insurance market for calendar years 1971-2005. The inverse loss ratio is displayed on Figure 1.2.1. The data do not show any marked time trend. Figure 1.2.1 displays the autocorrelation function (ACF) and the partial autocorrelation function (PACF) of the observed inverse loss ratio. The ACF at lag k gives the correlation coefficient between the observations made at times t and $t+k$. It measures the linear predictability of the series at time $t+k$ using only the value at time t. We recognize on Figure 1.2.1 the classic signature for a stationary series, with a set of correlations that decay at a geometric rate as the lag length increases. In addition to this graphical check, we have also performed formal tests for (non)stationarity. Both the Kwiatkowski-Philips-Schmidt-Shin test and the Augmented Dickey-Fuller test support stationarity.

There are a few basic steps to fitting ARMA models to time series data. The main point is to identify the preliminary values of the autoregressive order p and the moving average order q. Preliminary values of p and q are chosen on the basis of the shape of the ACF and PACF. Considering Figure 1.2.1, values of p and q up to 2 might be of interest. To decide among possible competitors which model we retain for the inverse loss ratio, we use the Akaike's Information Criterion (AIC). The values of this criterion are displayed in the next table:

ARMA(p,q)	AIC		
	q=0	q=1	q=2
p=0	-14.34766	-40.92149	-43.25771
p=1	-35.75842	-43.42793	-42.41657
p=2	-44.49098	-43.32897	-42.26215

Selecting the optimal ARMA model on the basis of the AIC criterion from the ARMA(p,q) candidates with p and q at most equal to 2 gives the AR(2) model suggested by Cummins and Outreville (1987). The parameter estimates are $\widehat{a}_0 = 0.6197$, $\widehat{a}_1 = 0.9945$, $\widehat{a}_2 = -0.4699$ and $\widehat{\sigma}_\epsilon^2 = 0.01602$. Notice that the required constraints (1.2.2) on parameters a_1 and a_2 are fulfilled by these estimations. Moreover, $\widehat{\mu}_\Pi = \widehat{a}_0/(1-\widehat{a}_1-\widehat{a}_2) = 1.3035 > 1$, which means that the loading on the Canadian motor insurance market is approximately 30%.

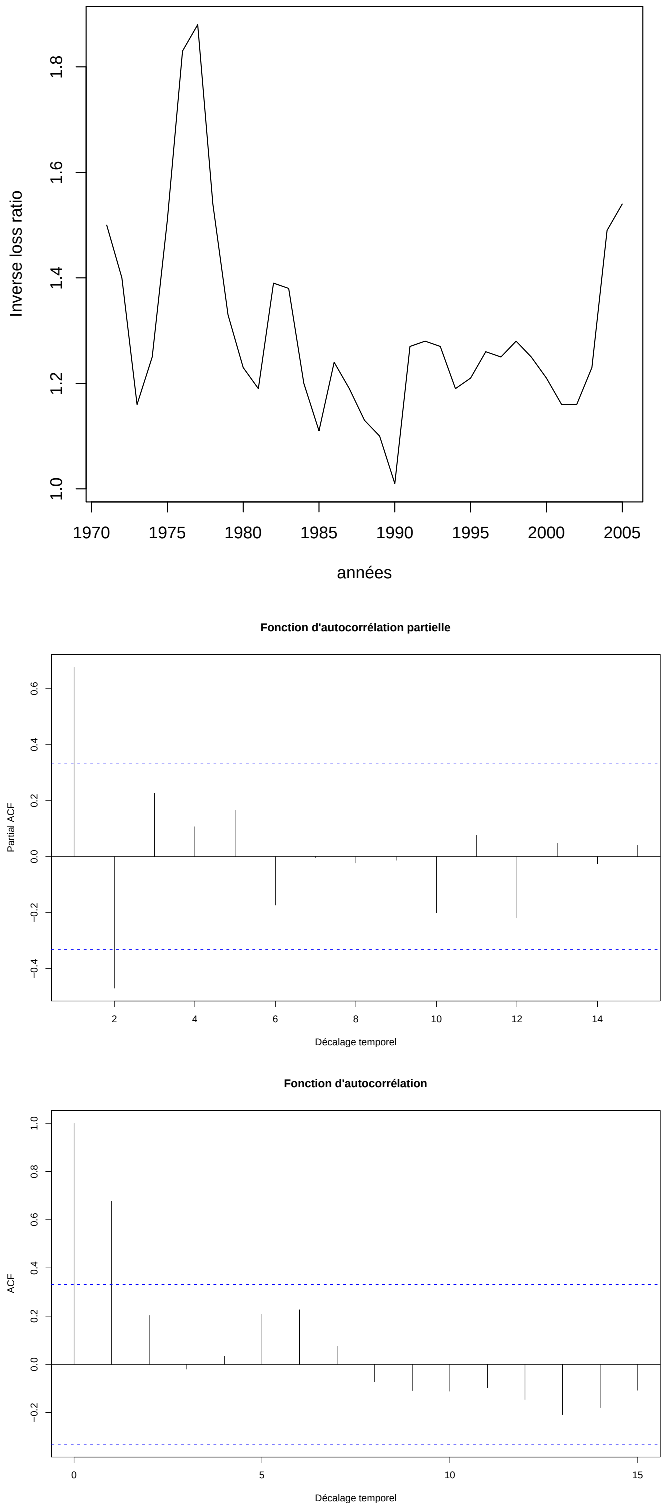

Figure 1.2.1: Inverse loss ratio for the Canadian motor insurance market, calendar years 1971-2005, together with ACF and PACF.

Validation

The inspection of residuals is an important check for the suitability of the model. These residuals can first be inspected visually for the absence of pattern, as well as their autocorrelation function and partial autocorrelation function. As it can be seen from Figure 1.2.2, no significant correlation remains in the series of residuals. Furthermore, the assumption of uncorrelatedness can be formally tested with the Ljung-Box Portmanteau test that considers the magnitude of the autocorrelations in the residuals as a group. It is useful to detect situations where each sample autocorrelation is small in magnitude but collectively these values are large. The null hypothesis of the Ljung-Box test is for model adequacy. Figure 1.2.2 indicates that the large Ljung-Box p-values support the AR(2) modelling.

1.3 Dynamics of the insurer's surplus

Let us consider a small or medium size insurance company which cannot influence the underwriting cycles. More precisely, the technical result of the company does not have any significant impact on the $\{\Pi_t\}$ process. We assume that this company aims to maintain its existing portfolio subject to market cycles. Hence, it has to account for underwriting cycles to avoid loosing policies in favor of competitors. This means that the premium income of this company is driven by the $\{\Pi_t\}$ process. Now, consider that this insurance company starts its business with an initial capital u. At time $k+0$, it collects an amount of premium C_{k+1} subject to insurance cycles. The company covers the claims originated during the period $(k, k+1]$, with total amount Y_{k+1}. The Y_k's are assumed to be independent and identically distributed and we denote as Y a generic random variable distributed as the Y_k's. Hence, the insurer's surplus at time n is given by

$$U_n = u + \sum_{k=1}^{n} C_k - \sum_{k=1}^{n} Y_k, \tag{1.3.1}$$

with $C_k = \mu_Y \Pi_k$, where the Π_k's obey the AR(2) dynamics (1.2.1). This means that the premium income of the insurance company is also AR(2), since

$$C_k = a_1 C_{k-1} + a_2 C_{k-2} + \tilde{\epsilon}_k, \tag{1.3.2}$$

where $\tilde{\epsilon}_k = \mu_Y(a_0 + \epsilon_k)$. The $\tilde{\epsilon}_k$ are independent and identically distributed. Henceforth, we denote as $\tilde{\epsilon}$ a generic random variable distributed as the $\tilde{\epsilon}_k$'s. We further assume that $\{Y_k,\, k = 1, 2, \dots\}$ and $\{\tilde{\epsilon}_k,\, k = 1, 2, \dots\}$ are mutually independent. This means that the premium income of the insurance company is not influenced by a sudden increase in its own claim amounts. This is in line with the assumption of a small to medium sized insurer with

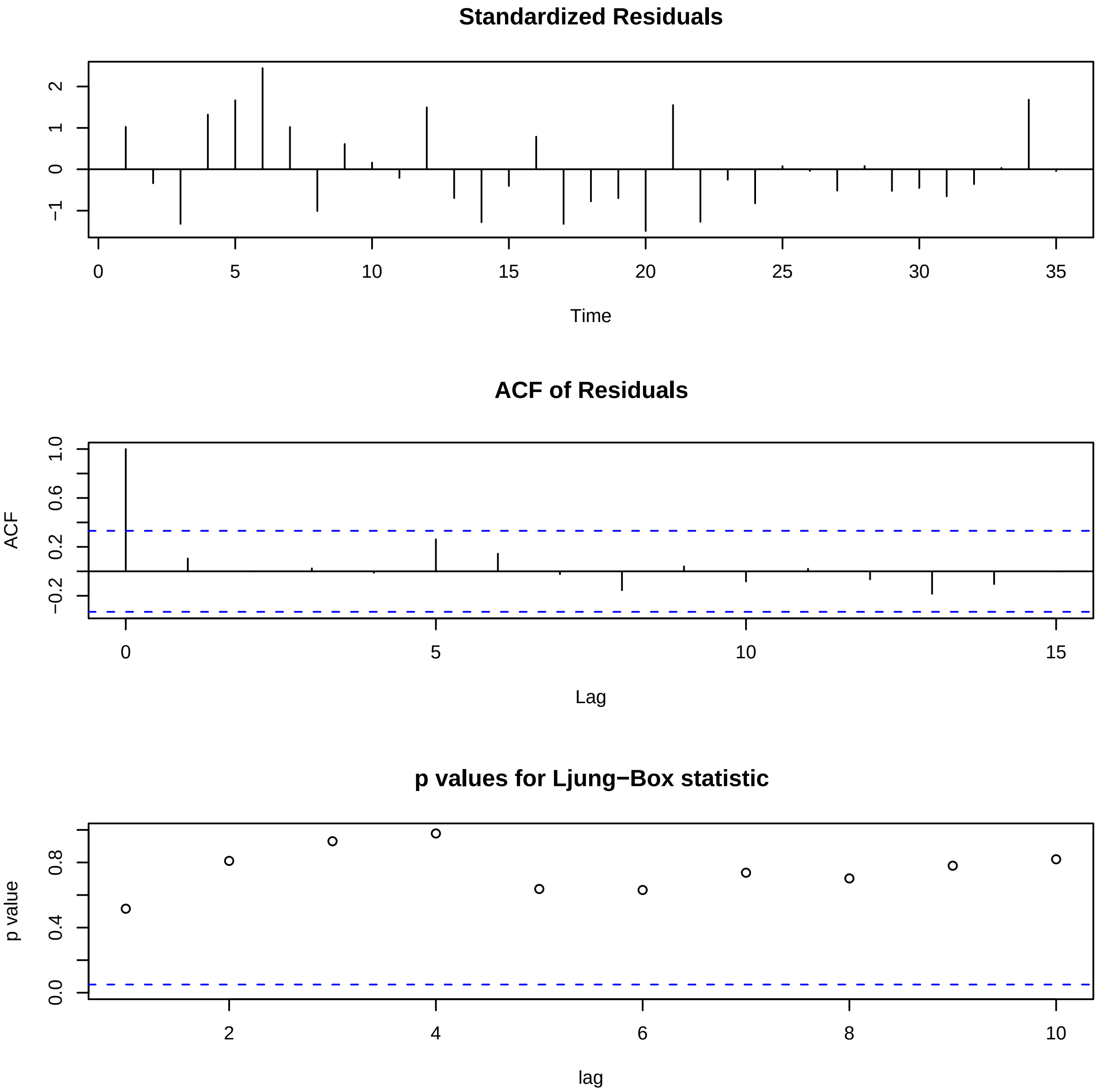

Figure 1.2.2: Diagnostics for the AR(2) fit to the observed inverse loss ratio on the Canadian motor insurance market.

a price-taker position on the market.

The condition $\Pr[|\epsilon_k| < d] = 1$ ensures that there exists a constant c such that $\Pr[|C_k| < c] = 1$ (see Gerber (1981)), since $\{C_k\}$ is a stationary process and $\Pr[|\tilde{\epsilon}_k| < \mu_Y(a_0 + d)] = 1$. Even if our model allows negative C_k's, this happens with such a small probability for realistic values of the parameters that this is not problematic for applications.

1.4 Ruin probability for light-tailed claims

1.4.1 Expression for the ultimate ruin probability

We follow the same approach as Gerber (1982) to deal with ruin probabilities when annual gains obey an autoregressive process. Notice that here, the premiums C_k follow an AR(2) process whereas the annual claim amounts Y_i are independent and identically distributed. Hence, the assumptions of Gerber (1982) are not fulfilled in our case. Using a similar approach, we can nevertheless derive an expression for the ruin probability when the premiums and the claim amounts obey their own dynamics. This is the topic of the present section.

Let us denote the ultimate ruin probability as

$$\psi(u, c_0, c_{-1}) = \Pr[U_n < 0 \text{ for some } n | U_0 = u, C_0 = c_0, C_{-1} = c_{-1}].$$

Then, we have the following result.

Theorem 1.4.1. *Assume that there exists a positive constant ρ such that*

$$\mathbb{E}[e^{\left(\frac{-\rho\tilde{\epsilon}}{1-a_1-a_2}\right)}]\,\mathbb{E}[e^{\rho Y}] = 1. \tag{1.4.1}$$

Then we have

$$\psi(u, c_0, c_{-1}) = \frac{e^{-\rho\hat{u}}}{\mathbb{E}[e^{-\rho\hat{U}_T} | T < \infty]}, \tag{1.4.2}$$

where $\hat{U}_n = U_n + \frac{a_1+a_2}{1-a_1-a_2}C_n + \frac{a_2}{1-a_1-a_2}C_{n-1}$ with $\hat{u} = \hat{U}_0$, and $T = \inf\{n | U_n < 0\}$.

Proof. Let us denote as Φ_n the sigma-algebra generated by $\{\tilde{\epsilon}_i, i = 1, 2, \ldots, n\}$ and $\{Y_i, i = 1, 2, \ldots, n\}$. We then have

$$\begin{aligned}
& \mathbb{E}[e^{-\rho\hat{U}_n} | \Phi_{n-1}] \\
= \; & \mathbb{E}[e^{-\rho(U_{n-1}+C_n-Y_n+\left(\frac{a_1+a_2}{1-a_1-a_2}\right)C_n+\left(\frac{a_2}{1-a_1-a_2}\right)C_{n-1})} | \Phi_{n-1}] \\
= \; & \mathbb{E}[e^{-\rho(U_{n-1}+a_1C_{n-1}+a_2C_{n-2}+\tilde{\epsilon}_n-Y_n+\left(\frac{a_1+a_2}{1-a_1-a_2}\right)\left(a_1C_{n-1}+a_2C_{n-2}+\tilde{\epsilon}_n\right)+\left(\frac{a_2}{1-a_1-a_2}\right)C_{n-1})} | \Phi_{n-1}] \\
= \; & \mathbb{E}[e^{\rho Y}]\,\mathbb{E}[e^{-\rho(U_{n-1}+C_{n-1}\left(a_1+\frac{a_1(a_1+a_2)}{1-a_1-a_2}+\frac{a_2}{1-a_1-a_2}\right)+C_{n-2}\left(a_2+\frac{a_2(a_1+a_2)}{1-a_1-a_2}\right)+\tilde{\epsilon}_n\left(1+\frac{a_1+a_2}{1-a_1-a_2}\right))} | \Phi_{n-1}] \\
= \; & \mathbb{E}[e^{\rho Y}]\,\mathbb{E}[e^{\left(\frac{-\rho\,\tilde{\epsilon}}{1-a_1-a_2}\right)}]\,e^{-\rho\hat{U}_{n-1}} = e^{-\rho\hat{U}_{n-1}}.
\end{aligned}$$

Hence, the process $\{e^{-\rho \hat{U}_n}\}$ is a martingale.

Let m be a positive integer. By the Optional Stopping Theorem (considering the stopping time $T \wedge m = \min\{T, m\}$), we get

$$e^{-\rho \hat{U}_0} = \mathbb{E}[e^{-\rho \hat{U}_{T\wedge m}}] = \mathbb{E}[e^{-\rho \hat{U}_T} I_{T\leq m}] + \mathbb{E}[e^{-\rho \hat{U}_m} I_{T>m}].$$

Hence, since there exists a constant c such that $|C_k| < c$ for all k, and in view of (1.2.2), we have that the random variables $\exp(-\rho \hat{U}_m) I_{T>m}$ converge to zero pointwise for $m \to \infty$ and are bounded by the constant $\exp(\frac{\rho a_1}{1-a_1-a_2} c)$ if $a_1 + a_2 > 0$, or by $\exp(-\frac{\rho(a_1+2a_2)}{1-a_1-a_2} c)$ otherwise. Thus,

$$\lim_{m\to\infty} \mathbb{E}[e^{-\rho \hat{U}_m} I_{T>m}] = 0.$$

Hence, letting $m \to \infty$, we get

$$e^{-\rho \hat{U}_0} = \mathbb{E}[e^{-\rho \hat{U}_T} | T < \infty] \ \Pr[T < \infty] = \mathbb{E}[e^{-\rho \hat{U}_T} | T < \infty] \, \psi(u, c_0, c_{-1}),$$

where we recognize the announced expression for the ultimate ruin probability. □

Example 4.1

If the annual claims are Exponentially distributed (with parameter α, say), due to the lack-of-memory property the deficit at ruin $|U_T|$ is again exponential (independent of the other quantities of the risk process), so that (1.4.2) simplifies to

$$\psi(u, c_0, c_{-1}) = \frac{(\alpha - \rho) e^{-\rho \hat{u}}}{\alpha \, \mathbb{E}\left[e^{-\rho \frac{a_1+a_2}{1-a_1-a_2} C_T - \rho \frac{a_2}{1-a_1-a_2} C_{T-1}} | T < \infty\right]}.$$

The premiums C_T and C_{T-1} will still depend on u through the dependence of T on u, but this dependence will (in particular in the light of stationarity) often be quite weak, in which case it may be reasonable to argue that one "almost" has $\psi(u) \sim const.e^{-\rho u}$.

Remark 4.1

Every reasonable trajectory of the AR-model for the premium series C_k will give $C_T > 0$ and $C_{T-1} > 0$ with a probability very close to 1. Hence, if one assumes $C_{T-1} > 0$ and $C_T > 0$ almost surely, then the additional condition $a_1 + a_2 < 0$ (recall that $a_1 + a_2 < 1$ (see (1.2.2))) would suffice to deduce from (1.4.2) the upper bound

$$\psi(u, c_0, c_{-1}) \leq \ e^{-\rho \hat{u}}, \tag{1.4.3}$$

since $a_2 < 0$ and $\Pr[U_T < 0 | T < \infty] = 1$. For Exponentially distributed claims the inequality (1.4.3) can be improved to $\psi(u, c_0, c_{-1}) \leq \frac{\alpha-\rho}{\alpha} e^{-\rho \hat{u}}$.

1.4.2 Existence and approximation of ρ

The next result gives conditions to ensure the existence of the adjustment coefficient ρ.

Proposition 1.4.1. *Assume that the moment-generating functions $\mathbb{E}[e^{rY}]$ and $\mathbb{E}[e^{-r\tilde{\epsilon}}]$ exist for all $r > 0$ in a neighborhood of the origin and are steep. If the insurance market is viable, that is, if the inequality* (1.2.3) *is satisfied, then there exists a unique positive constant ρ such that*

$$\mathbb{E}[e^{\left(\frac{-\rho\tilde{\epsilon}}{1-a_1-a_2}\right)}]\,\mathbb{E}[e^{\rho Y}] = 1. \tag{1.4.4}$$

Proof. Define the function h as

$$h(r) = \mathbb{E}[e^{\left(\frac{-r\tilde{\epsilon}}{1-a_1-a_2}\right)}]\,\mathbb{E}[e^{rY}] = \mathbb{E}\left[e^{rZ}\right], \tag{1.4.5}$$

where $Z = Y - \tilde{\epsilon}/(1-a_1-a_2)$. We clearly have $h(0) = 1$. Moreover $h'(r) = \mathbb{E}[e^{rZ}Z]$, so that $h'(0) = \mu_Y\big(1 - a_0/(1-a_1-a_2)\big) < 0$ by assumption. The function h is convex since

$$h''(r) = \mathbb{E}[e^{rZ}Z^2] > 0 \text{ for all } r.$$

To ensure the existence of ρ satisfying (1.4.4), we just need to show (since $\mathbb{E}[e^{rY}]$ and $\mathbb{E}[e^{-r\tilde{\epsilon}}]$ are steep) that there exists a positive constant k_0 for which $h'(r) > 0$ for all $r > k_0$. From

$$h'(r) = \mathbb{E}[e^{rZ}Z\,I_{Z<0}] + \mathbb{E}[e^{rZ}Z\,I_{Z\geq 0}],$$

and $\mathbb{E}[e^{rZ}Z\,I_{Z<0}] > \mathbb{E}[Z\,I_{Z<0}] < 0$, we see that $h'(r) > \mathbb{E}[Z\,I_{Z<0}] + \mathbb{E}[e^{rZ}Z\,I_{Z\geq 0}]$ which becomes ultimately positive. □

Note that the assumption on Y in Proposition 1.4.1 implies light-tailed claims.

Now, let $\eta > 0$ denote the mean safety, defined by $\eta = \mu_\Pi - 1$. The next result gives an approximation of ρ for sufficiently small η's.

Proposition 1.4.2. *Assume that $\rho \to 0$ as $\eta \to 0$. Then, we have*

$$\rho(\eta) = \frac{2\mu_Y}{\sigma_Y^2 + \frac{\mu_Y^2\sigma_{\tilde{\epsilon}}^2}{(1-a_1-a_2)^2}}\eta + O(\eta^2). \tag{1.4.6}$$

Proof. By Taylor expansion of the equality (1.4.4) with respect to ρ up to second order, we get

$$\left(1-\rho\mu_Y(1+\eta)+\frac{\rho^2\mu_Y^2}{2}\left((1+\eta)^2+\frac{\mu_{\epsilon,2}}{(1-a_1-a_2)^2}\right)+O(\rho^3)\right)\left(1+\rho\mu_Y+\frac{\rho^2}{2}\mu_{Y,2}+O(\rho^3)\right) = 1,$$

where $\mu_{\epsilon,2} = \mathbb{E}[\epsilon^2]$ and $\mu_{Y,2} = \mathbb{E}[Y^2]$. Then, after a little algebra, (1.4.6) follows. □

In Section 1.5, we prove that $\rho \to 0$ as $\eta \to 0$ so that the assumption of Proposition 1.4.2 is indeed fulfilled.

1.4.3 Asymptotic behavior for the ultimate ruin probability and typical path to the ruin

The denominator in (1.4.2) will usually depend on u and is difficult to study in general (although its dependence on u can be expected to be weak in many situations). However, the following result makes clear that the exponential decay rate of the numerator of (1.4.2) provides the significant asymptotic behavior in u of $\psi(u, c_0, c_{-1})$.

Theorem 1.4.2. *Assume that the assumptions of Theorem 1.4.1 are fulfilled. Then, given that for all n, the sum of the first n annual losses $Z_n = \sum_{k=1}^{n}(Y_k - C_k)$ possesses a finite moment-generating function $\mathbb{E}[e^{rZ_n}]$ for $0 < r < r_0$, where r_0 is some constant larger than ρ, we have*

$$\lim_{u\to\infty} \frac{1}{u} \ln \psi(u, c_0, c_{-1}) = -\rho. \tag{1.4.7}$$

Proof. As a consequence of the Gärtner-Ellis Theorem from large deviations (which is due to Glynn and Whitt (1994), see also Nyrhinen (1998) and Müller and Pflug (2001)), one only has to show the two following properties:

(A1) $\kappa(r) := \lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}[e^{rZ_n}]$ exists for $0 < r < r_0$,

(A2) $\kappa(\rho) = 0$ and $\kappa'(\rho) > 0$.

Iterating (1.3.2), and with the convention $\sum_1^0 = 0$, we get

$$C_k = \tilde{\epsilon}_k + \sum_{i=1}^{k-1} b_i \tilde{\epsilon}_{k-i} + b_k(c_0 - a_1 c_{-1}) + b_{k+1}c_{-1} = \tilde{\epsilon}_k + \sum_{i=1}^{k-1} b_i \tilde{\epsilon}_{k-i} + b_k c_0 + (b_{k+1} - a_1 b_k)c_{-1},$$

where $b_1 = a_1$, $b_2 = a_2 + a_1^2$ and $b_k = a_1 b_{k-1} + a_2 b_{k-2}$ for $k \geq 3$. Hence, for $0 < r < r_0$,

$$\begin{aligned}
\frac{1}{n} \ln \mathbb{E}[e^{rZ_n}] &= \frac{1}{n} \ln \mathbb{E}[e^{r\sum_{k=1}^{n} Y_k}] + \frac{1}{n} \ln \mathbb{E}[e^{-r\sum_{k=1}^{n} C_k}] \\
&= \ln \mathbb{E}[e^{rY}] + \frac{1}{n} \ln \mathbb{E}[e^{-r\sum_{k=1}^{n}(\tilde{\epsilon}_k + \sum_{i=1}^{k-1} b_i\tilde{\epsilon}_{k-i} + b_k c_0 + (b_{k+1}-a_1 b_k)c_{-1})}] \\
&= \ln \mathbb{E}[e^{rY}] + \frac{1}{n} \sum_{k=1}^{n} \ln \mathbb{E}[e^{-r\tilde{\epsilon}_k(b_0+\cdots+b_{n-k})}] - \frac{r}{n} \sum_{k=1}^{n} \big(b_k c_0 + (b_{k+1} - a_1 b_k)c_{-1}\big),
\end{aligned}$$

where $b_0 = 1$. Letting $n \to \infty$, and since $1/(1 - a_1 - a_2) = 1 + \sum_{k=1}^{\infty} b_k$, we then get

$$\kappa(r) = \lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}[e^{rZ_n}] = \ln \mathbb{E}[e^{\left(\frac{-r\tilde{\epsilon}}{1-a_1-a_2}\right)}] + \ln \mathbb{E}[e^{rY}] \quad \text{for } 0 < r < r_0,$$

which shows that (A1) holds true. Since $\kappa(r) = \ln h(r)$, it now follows immediately from Proposition 1.4.1 that (A2) is also fulfilled. □

Now, given that ruin occurs, we can use large deviations results of Glynn and Whitt (1994) to see that for large u, ruin occurs roughly at time $\lfloor u/\kappa'(\rho)\rfloor$. Furthermore, the cumulant generating function $\kappa_{\lfloor u/\kappa'(\rho)\rfloor}(r)$ of $Z_{\lfloor u/\kappa'(\rho)\rfloor}$ is asymptotically changed from $\kappa_{\lfloor u/\kappa'(\rho)\rfloor}(r)$ to

$$\begin{aligned}\kappa_{\lfloor u/\kappa'(\rho)\rfloor}(r+\rho)-\kappa_{\lfloor u/\kappa'(\rho)\rfloor}(\rho) &= \sum_{k=1}^{\lfloor u/\kappa'(\rho)\rfloor} \ln \frac{\mathbb{E}[e^{-(r+\rho)\tilde{\epsilon}_k(b_0+\cdots+b_{\lfloor u/\kappa'(\rho)\rfloor-k})}]}{\mathbb{E}[e^{-\rho\tilde{\epsilon}_k(b_0+\cdots+b_{\lfloor u/\kappa'(\rho)\rfloor-k})}]} \\ &\quad - r\sum_{k=1}^{\lfloor u/\kappa'(\rho)\rfloor}(b_k c_0+(b_{k+1}-a_1 b_k)c_{-1})+\lfloor u/\kappa'(\rho)\rfloor \ln\frac{\mathbb{E}[e^{(r+\rho)Y}]}{\mathbb{E}[e^{\rho Y}]}.\end{aligned}$$

This means that a surplus path leading to ruin is again of the same form, but with a claim size distribution Y that is exponentially tilted by factor ρ and also each of the variables $-\tilde{\epsilon}_k(b_0+\cdots+b_{\lfloor u/\kappa'(\rho)\rfloor-k})$ tilted by the same factor ρ. Note that exponential tilting of a random variable corresponds to its Esscher transform. This also suggests that a sample path leading to ruin asymptotically has a drift

$$-\kappa'(\rho)=\frac{\mathbb{E}[\frac{\tilde{\epsilon}}{1-a_1-a_2}e^{\left(\frac{-\rho\tilde{\epsilon}}{1-a_1-a_2}\right)}]}{\mathbb{E}[e^{\left(\frac{-\rho\tilde{\epsilon}}{1-a_1-a_2}\right)}]}-\frac{\mathbb{E}[Ye^{\rho Y}]}{\mathbb{E}[e^{\rho Y}]}<0.$$

1.4.4 Finite-time ruin probabilities

Let $\psi(u,M)$ denote the probability of ruin up to time horizon M starting with an initial amount of capital u and define for $a>0$ the quantity r_a as the unique solution of $\kappa'(r_a)=1/a$. Then, invoking Nyrhinen (1998) allows us to get a large deviation estimate for the finite-time ruin probability for our risk process in the presence of market cycles, namely

$$\lim_{u\to\infty}\frac{1}{u}\ln\psi(u,a\,u)=-\rho_a,$$

with

$$\rho_a=\begin{cases} r_a-a\,\kappa(r_a), & a<\frac{1}{\kappa'(\rho)} \\ \rho, & a\geq\frac{1}{\kappa'(\rho)}\end{cases},$$

which is consistent with the result of the previous section.

1.5 Comparison with the classical case for light-tailed claims

Let us now investigate the impact of underwriting cycles on ultimate ruin probabilities. To this end, we compare the ultimate ruin probabilities in the presence of such cycles to the ruin probabilities $\widetilde{\psi}(u) = \Pr[\widetilde{U}_n < 0 \text{ for some } n | \widetilde{U}_0 = u]$, where

$$\widetilde{U}_n = u + n\,\mu_Y\,\mu_\Pi - \sum_{k=1}^{n} Y_k,$$

is the standard surplus process with averaged premium income. Let us define

$$\widetilde{h}(r) = e^{-r\,\mu_Y\,\mu_\Pi}\,\mathbb{E}[e^{rY}]. \tag{1.5.1}$$

Then, we can show the following result.

Proposition 1.5.1. *If $\widetilde{\rho}$ denotes the adjustment coefficient of the standard model, i.e. $\widetilde{h}(\widetilde{\rho}) = 1$, then the inequality $\widetilde{\rho} > \rho$ holds true.*

Proof. The announced result is true if we can show that $\widetilde{h}(r) < h(r)$ for $r > 0$. This is indeed the case since

$$\begin{aligned} h(r) &= \mathbb{E}[e^{\left(\frac{-r\tilde{\epsilon}}{1-a_1-a_2}\right)}]\,\mathbb{E}[e^{rY}] \\ &> e^{\left(\frac{-r\mu_{\tilde{\epsilon}}}{1-a_1-a_2}\right)}\,\mathbb{E}[e^{rY}] \text{ by Jensen inequality} \\ &= e^{\left(\frac{-r\mu_Y a_0}{1-a_1-a_2}\right)}\,\mathbb{E}[e^{rY}] \\ &= e^{-r\mu_Y\mu_\Pi}\,\mathbb{E}[e^{rY}] \\ &= \widetilde{h}(r). \end{aligned}$$

Since $\widetilde{h}(0) = h(0) = 1$; $\widetilde{h}'(0) < 0$ and $h'(0) < 0$; $\widetilde{h}''(r) > 0$ and $h''(r) > 0$, we necessarily have $\widetilde{\rho} > \rho$. □

This result ensures that there exists a level u_0 such that for all $u \geq u_0$, the inequality $\widetilde{\psi}(u) < \psi(u, c_0, c_{-1})$ holds whatever c_0 and c_{-1}. This shows that market cycles in general increase the risk for insurers.

Remark 5.1

Since $\widetilde{\psi}(u) \to 1$ as $\eta \to 0$, it follows from the Lundberg inequality $\widetilde{\psi}(u) \leq e^{-\widetilde{\rho} u}$ that $\widetilde{\rho} \to 0$. Hence, since $\widetilde{\rho} > \rho$, we have $\rho \to 0$ as $\eta \to 0$, which confirms as expected that the assumption of Proposition 1.4.2 is satisfied.

Remark 5.2

By equation (1.4.6) with $\sigma_\epsilon = 0$, we know that $\tilde{\rho}(\eta) = \frac{2\mu_Y\eta}{\sigma_Y^2} + O(\eta^2)$. So, for $\eta \to 0$, we have

$$\rho(\eta) - \tilde{\rho}(\eta) \approx \frac{2\mu_Y\eta}{\sigma_Y^2}\left(\frac{1}{1 + \frac{\mu_Y^2\sigma_\epsilon^2}{\sigma_Y^2(1-a_1-a_2)^2}} - 1\right) < 0, \tag{1.5.2}$$

which illustrates Proposition 1.5.1. Now, let us define η_c by the equality $\tilde{\rho}(\eta_c) = \rho(\eta)$. We can say, at least for large initial capital u, that $\Delta\eta = \eta - \eta_c$ represents the mean excess loading dictated by the market due to the presence of the cycles, or in other words, the mean cost for policyholders caused by the underwriting cycles. At first order, i.e. for small η, we find

$$\Delta\eta \approx \left(1 - \frac{1}{1 + \frac{\mu_Y^2 \sigma_\epsilon^2}{\sigma_Y^2 (1-a_1-a_2)^2}}\right)\eta. \tag{1.5.3}$$

From (1.5.2), we have

$$\frac{\rho(\eta) - \tilde{\rho}(\eta)}{\tilde{\rho}(\eta)} \approx \left(\frac{1}{1 + \frac{\mu_Y^2 \sigma_\epsilon^2}{\sigma_Y^2 (1-a_1-a_2)^2}} - 1\right) \quad \text{as} \quad \eta \to 0. \tag{1.5.4}$$

The following comments apply for small η:

(i) For a given company and fixed parameters a_0, a_1 and a_2 such that η is close to 0, relation (1.5.4) shows that the larger σ_ϵ^2 is (i.e. the more pronounced the market cycles are), the larger is the relative difference $\left|\frac{\rho(\eta)-\tilde{\rho}(\eta)}{\tilde{\rho}(\eta)}\right|$.

(ii) On the other hand, for a given market (and more precisely for a given line of business), the larger the coefficient of variation σ_Y/μ_Y of the company is, the larger is the relative difference $\left|\frac{\rho(\eta)-\tilde{\rho}(\eta)}{\tilde{\rho}(\eta)}\right|$. This suggests that a company with a diversified portfolio, which means here a large number of policy-holders in the line of business of interest, will be more affected, at least asymptotically, by the underwriting cycles. Intuitively, this is not surprising, as a lower value of σ_Y/μ_Y (and hence lower individual risk of the company) will give the risk coming from the cycles (the systematic risk) a more important role within the global uncertainty (the individual risk plus the systematic risk) of the company. Of course, there are other ways to diversify insurance risks for large companies. Writing policies in several lines of business subject to different cycles is expected to lower the systematic risk compared to a company selling just one type of contracts

1.6 Numerical illustrations

Let us assume that the annual claim amount Y for some insurance company operating on the Canadian motor insurance market is Gamma distributed with mean $q_1 q_2$ and variance $q_1 q_2^2$. To fix the values of q_1 and q_2, let us denote as Y^{market} the annual claim amount for the entire Canadian motor insurance market. If n is the number of insured vehicles in the Canadian motor insurance market, each of them generating an amount K_i of claims on a yearly basis, then $Y^{\text{market}} = \sum_{i=1}^n K_i$. The K_i's are assumed to be independent and identically distributed and we denote as K a generic random variable distributed as the K_i's. Let us fix $n\,\mu_K = 1000$, say. If the market share of the insurance company is equal to $x\%$, then

$$\begin{cases} \mu_Y = x\%\,n\,\mu_K = x\%1000 \\ \frac{\sigma_Y}{\mu_Y} = \frac{\sqrt{x\%\,n}\,\sigma_K}{x\%\,n\,\mu_K} = \frac{1}{\sqrt{x\%}}\frac{\sigma_{Y\text{market}}}{\mu_{Y\text{market}}} \end{cases}.$$

Hence, the values of q_1 and q_2 for a particular insurer are deduced from the market coefficient of variation $\frac{\sigma_{Y\text{market}}}{\mu_{Y\text{market}}}$, which is estimated to 0.17 for the Canadian motor insurance market.

Let us consider an insurance company with a market share of 5%. The parameters of the Gamma distribution governing its annual claim amounts are $q_1 = 1.7099$ and $q_2 = 29.2410$. These values are obtained as explained above with $x = 5$. The next table gives the value of the ultimate ruin probability computed from Theorem 1.4.1, where for the determination of the denominator of (1.4.2) we used simulation (note that using formula (1.4.2) and just simulating the deficit at ruin will lead to a better and more efficient estimator than fully simulating the ruin probability itself).

	(c_{-1}, c_0)	u=0	u=50	u=100	u=150	u=200	u=250
With market cycles $\psi(u, c_0, c_{-1})$	$(62.5, 64.0)$	0.7746	0.3802	0.1772	0.0841	0.0410	0.0194
	$(64.0, 57.5)$	0.7513	0.3827	0.1777	0.0777	0.0385	0.0192
	$(57.5, 60.5)$	0.7661	0.3772	0.1794	0.0831	0.0440	0.0209
	$(60.5, 58.0)$	0.7672	0.3676	0.1836	0.0678	0.0375	0.0185
	$(58.0, 58.0)$	0.7574	0.3811	0.1778	0.0898	0.0418	0.0207
	$(58.0, 61.5)$	0.7760	0.3848	0.1839	0.0668	0.0351	0.0166
	$(61.5, 74.5)$	0.7523	0.3769	0.1772	0.0879	0.0426	0.0204
	$(74.5, 77.0)$	0.7566	0.3786	0.1804	0.0653	0.0305	0.0162
Without market cycle $\widetilde{\psi}(u)$	/	0.5059	0.2528	0.1256	0.0587	0.0297	0.0141

Ultimate ruin probabilities obviously decrease as the amount of initial capital increases. We also see that the ultimate ruin probability $\psi(u, c_0, c_{-1})$ depends on the cycle when the insurer starts its business. Nevertheless, also for small values of u, $\psi(u, c_0, c_{-1})$ always exceed the corresponding $\widetilde{\psi}(u)$ whatever the initial position with respect to the underwriting cycle (that is, whatever c_0 and c_{-1}). This is even stronger than what is indicated by Proposition 1.5.1.

Let us now examine finite-time ruin probabilities. Specifically, let $u^\star$ be the amount of initial capital such that $\Pr[\inf_{n\leq 10} \widetilde{U}_n < 0|\widetilde{U}_0 = u^\star] = 0.01$. Thus, $u^\star$ is the initial capital needed to avoid ruin with a probability of 99% over the first 10 years, without underwriting cycle. Then, we compute $\Pr[\inf_{n\leq t} \widetilde{U}_n < 0|\widetilde{U}_0 = u^\star]$ and $\Pr[\inf_{n\leq t} U_n < 0|U_0 = u^\star]$ for $t = 1, 2, \ldots, 20$. The computation of $\Pr[\inf_{n\leq t} U_n < 0|U_0 = u^\star]$ is performed as follows:

(Step 1) The values of c_{-1} and c_0 are fixed at random;
(Step 2) Then, a sequence of premium income is generated for $t = 1, 2, \ldots, 20$ from (1.3.2);
(Step 3) Given these premiums, the finite-time ruin probability is computed for the horizons $t = 1, 2, \ldots, 20$ from the Lefèvre-Picard formula (see Picard, Lefèvre and Coulibaly, 2003);
(Step 4) The preceding steps are repeated a large number of times (20 000 times in the numerical illustrations) and the results are averaged to get $\Pr[\inf_{n\leq t} U_n < 0|U_0 = u^\star]$.

Let us consider the four situations summarized in the next table:

Law for Y_t	Market share	Parameters
Gamma(q_1, q_2)	5%	$q_1 = 1.7099$, $q_2 = 29.2410$
	10%	$q_1 = 3.4199$, $q_2 = 29.2410$
LogNormal(μ, σ)	5%	μ=3.6818, σ=0.6786
	10%	μ=4.4769, σ=0.5065

The Gamma and LogNormal distributions are discretized by rounding up, which is a conservative strategy for the computation of ruin probabilities.

Figures 1.6.1-1.6.4 show the finite-time ruin probabilities for horizons 1 to 20, annual claim amounts obeying the Gamma and the LogNormal distributions, and for a market share of 5 and 10%, respectively.

In all the cases, we see that the underwriting cycles increase the finite-time ruin probabilities for all the considered horizons, compared to the classical case where the premium income is not subject to ups and downs. For a given distribution governing the annual claim amount, increasing the market share magnifies the impact of underwriting cycles on the finite-time ruin probabilities. This is in accordance with Remark 5.2 (ii) in Section 5, since when the market share increases, we expect a smaller coefficient of variation σ_Y/μ_Y. To better illustrate this phenomenon, the extreme cases are represented in Figures 1.6.5 and 1.6.6, where we have considered the entire market (even if this contradicts our assumption that the company has a price-taker position). Notice that it is not surprising to have a smaller capital $u^\star$ for the entire market than for companies with market share of 5 and 10%, since on the one hand, the diversification effect is maximal when we consider the entire market, and on the other hand, the mean loading of the Canadian motor insurance market (approximately equal to 30%) is relatively large compared to $\sigma_{Y^{\text{market}}}/\mu_{Y^{\text{market}}} \approx 17\%$.

For a given market share, the impact of cycles on the finite-time ruin probabilities is stronger for the Gamma case than for the LogNormal case. This suggests that market cycles primarily affect insurers covering small claims, and that the thicker the claims distribution,

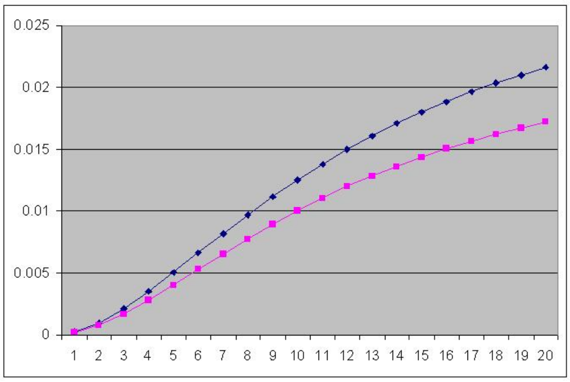

Figure 1.6.1: Finite-time ruin probabilities in the classical case (squares) and in presence of underwriting cycles (diamond) for an insurance company with 5% market share, Gamma distributed annual claim amounts, initial capital $u^\star = 236.7632$.

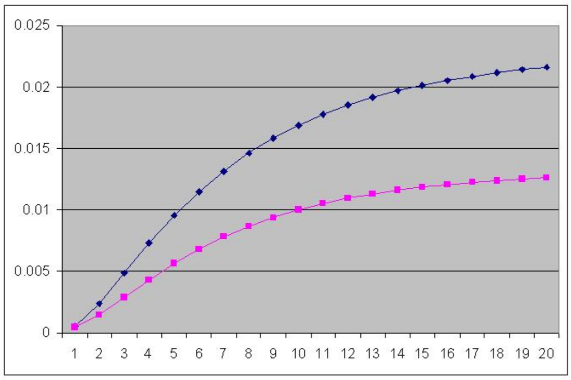

Figure 1.6.2: Finite-time ruin probabilities in the classical case (squares) and in presence of underwriting cycles (diamond) for an insurance company with 10% market share, Gamma distributed annual claim amounts, initial capital $u^\star = 252.8196$.

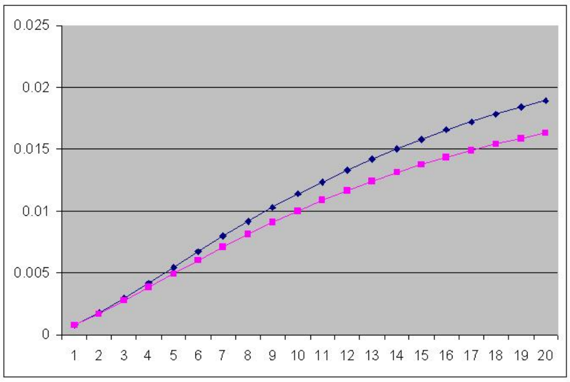

Figure 1.6.3: Finite-time ruin probabilities in the classical case (squares) and in presence of underwriting cycles (diamond) for an insurance company with 5% market share, LogNormally distributed annual claim amounts, initial capital $u^\star = 279.2331$.

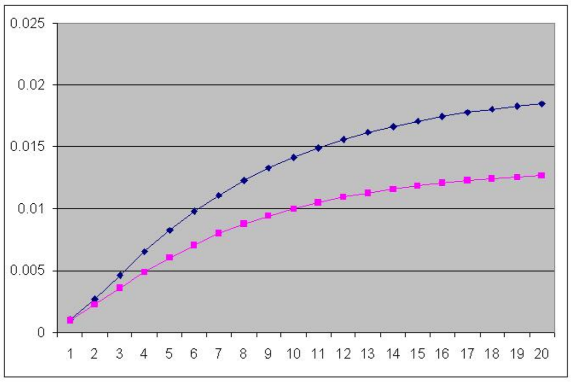

Figure 1.6.4: Finite-time ruin probabilities in the classical case (squares) and in presence of underwriting cycles (diamond) for an insurance company with 10% market share, LogNormally distributed annual claim amounts, initial capital $u^\star = 295.2331$.

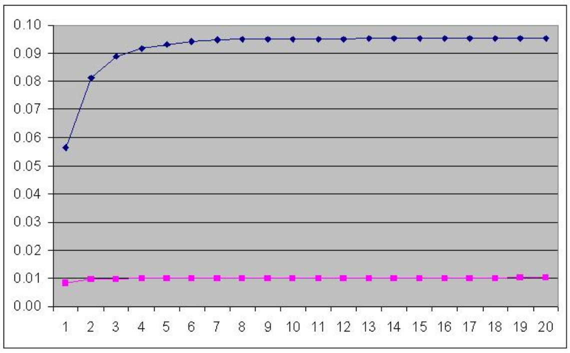

Figure 1.6.5: Finite-time ruin probabilities in the classical case (squares) and in presence of underwriting cycles (diamond) for the entire market, Gamma distributed annual claim amounts, initial capital $u^\star = 152.8662$.

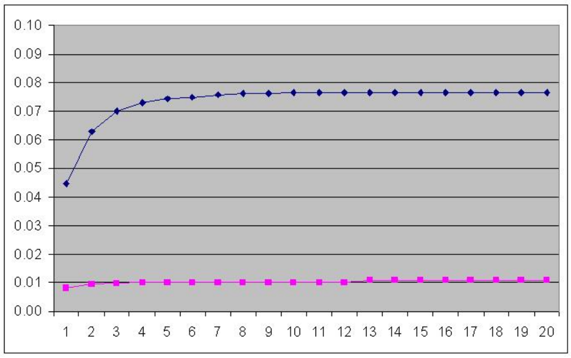

Figure 1.6.6: Finite-time ruin probabilities in the classical case (squares) and in presence of underwriting cycles (diamond) for the entire market, LogNormally distributed annual claim amounts, initial capital $u^\star = 179.7837$.

the lower the impact of market cycles. Once again, this is confirmed by Remark 5.2 (ii), since for the same value of μ_Y, a larger σ_Y increases the coefficient of variation σ_Y/μ_Y. Furthermore, this is in line with the asymptotic results given in Asmussen, Schmidli and Schmidt (1999) who give conditions for short-range dependence, under which the subexponential claims asymptotic for ruin probabilities from the independent case stays insensitive. Concretely, the model of this paper fulfills the conditions of their Theorem 4.3. To see that, note that instead of having yearly claim amounts Y_k and premiums C_k, one can (for the ultimate ruin probability) equivalently interpret our risk process as a continuous risk process with Y_k as individual claims and C_k as a time span with two claims (with constant premium intensity 1). Then, checking for survival after each claim exactly corresponds to checking for survival at the end of the year in our original model. As a consequence, the ruin probability in the heavy-tailed case asymptotically behaves as in the independent case.

1.7 Discussion

1.7.1 Extension with a constant interest rate

The results derived in Sections 4.1-4.2 extend to the case of a constant compound interest rate $i \geq 0$, which has to be interpreted as the risk-free rate proposed by the market.

Let us consider the following dynamics for the insurer's surplus, inspired from Yang (1998):

$$U_n^{(i)} = u(1+i)^n + \mu_Y \sum_{j=1}^{n} \Pi_j (1+i)^{n-j+1} - \sum_{j=1}^{n} Y_j (1+i)^{n-j}. \tag{1.7.1}$$

The premiums are assumed to be paid at the beginning of each year while the claims are assumed to be paid at the end of the year. Denote by

$$\psi^{(i)}(u, c_0, c_{-1}) = \Pr[U_n^{(i)} < 0 \text{ for some } n | U_0^{(i)} = u, C_0 = c_0, C_{-1} = c_{-1}]$$

the ultimate ruin probability relative to the new surplus dynamics (1.7.1). One can show that if there exists a positive constant $\rho^{(i)}$ such that

$$\mathbb{E}[e^{\left(\frac{-\rho^{(i)}\tilde{\epsilon}}{1-a_1 v - a_2 v^2}\right)} \mathbb{E}[e^{\rho^{(i)} v Y}] = 1, \tag{1.7.2}$$

then, we have

$$\psi^{(i)}(u, c_0, c_{-1}) \leq \frac{e^{-\rho^{(i)} \hat{u}^{(i)}}}{\mathbb{E}[e^{-\rho^{(i)} v^T \hat{U}_T^{(i)}} | T < \infty]}, \tag{1.7.3}$$

where $v = 1/(1+i)$, $\hat{U}_n^{(i)} = U_n^{(i)} + \frac{a_1 + a_2 v}{1-a_1 v - a_2 v^2} C_n + \frac{a_2}{1-a_1 v - a_2 v^2} C_{n-1}$, with $\hat{u}^{(i)} = \hat{U}_0^{(i)}$ and $T = \inf\{n : U_n^{(i)} < 0\}$. Indeed, by similar arguments as in the proof of Theorem 1.4.1, we have

$$\mathbb{E}[e^{-\rho^{(i)}v^n\hat{U}_n^{(i)}}|\Phi_{n-1}] = \mathbb{E}[e^{\rho^{(i)}v^nY}]\,\mathbb{E}[e^{\left(\frac{-\rho^{(i)}v^{n-1}\tilde{\epsilon}}{1-a_1v-a_2v^2}\right)}]\,e^{-\rho^{(i)}v^{n-1}\hat{U}_{n-1}^{(i)}}, \tag{1.7.4}$$

where Φ_n still denotes the sigma-algebra generated by $\{\tilde{\epsilon}_k, k = 1,2,\ldots,n\}$ and $\{Y_k, k = 1,2,\ldots,n\}$. Now, as $i \geq 0$, it suffices to note that

$$\mathbb{E}[e^{-\rho^{(i)}v^{n-1}\left(\frac{\tilde{\epsilon}}{1-a_1v-a_2v^2}-vY\right)}] \leq \mathbb{E}[e^{-\rho^{(i)}\left(\frac{\tilde{\epsilon}}{1-a_1v-a_2v^2}-vY\right)}]^{v^{n-1}} \tag{1.7.5}$$

by Jensen's inequality. Consequently, $\{e^{-\rho^{(i)}v^n\hat{U}_n^{(i)}}, n \geq 0\}$ is a supermartingale, since

$$\mathbb{E}[e^{-\rho^{(i)}\left(\frac{\tilde{\epsilon}}{1-a_1v-a_2v^2}-vY\right)}] = 1 \tag{1.7.6}$$

by assumption. In view of the final arguments of the proof of Theorem 1.4.1, the inequality (3.6.3) follows.

Of course, the inequality (3.6.3) is meaningful only if the adjustment coefficient $\rho^{(i)}$ exists. It is easily proved that this is the case if the viability condition (1.2.3) in Proposition 1.4.1 is replaced by

$$\frac{a_0}{1-a_1v-a_2v^2} > v. \tag{1.7.7}$$

Notice that, with $i > 0$, equation (1.7.7) does not reflect the market viability condition anymore, which is now given by

$$\frac{a_0}{1-a_1-a_2} > v, \tag{1.7.8}$$

since we assume a delay of one year between premiums and claims settlement. Nevertheless, for $a_1 > -2a_2$, which is the case for the Canadian motor insurance market, equation (1.7.7) always implies the viability condition (1.7.8).

Finally, an approximation can also be obtained for $\rho^{(i)}$. Let us denote by $\rho^{(i,a_0,a_1,a_2,\epsilon)}$ the adjustment coefficient corresponding to a market characterized by the parameters a_0, a_1 and a_2, with an error term distributed as ϵ and a risk-free rate $i \geq 0$. Then, from (3.6.2) and (1.4.4), it follows that

$$\rho^{(i,a_0,a_1,a_2,\epsilon)}v = \rho^{(0,\frac{a_0}{v},a_1v,a_2v^2,\frac{\epsilon}{v})}. \tag{1.7.9}$$

Now, let us set $\tilde{a}_0 = a_0/v$, $\tilde{a}_1 = a_1v$, $\tilde{a}_2 = a_2v^2$ and $\tilde{\eta} = \tilde{a}_0/(1-\tilde{a}_1-\tilde{a}_2) - 1$, which is not the mean market loading when $i > 0$. From (1.4.6) and (1.7.9), we find that

$$\rho^{(i,a_0,a_1,a_2,\epsilon)}(\tilde{\eta}) = \frac{1}{v}\,\frac{2\mu_Y}{\sigma_Y^2 + \frac{\mu_Y^2\frac{\sigma_\epsilon^2}{v^2}}{(1-\tilde{a}_1-\tilde{a}_2)^2}}\,\tilde{\eta} + O(\tilde{\eta}^2). \tag{1.7.10}$$

1.7.2 The small company assumption

Over the last decades, a substantial body of insurance literature has developed attempting to explain underwriting cycles in property-liability insurance. There is no generally accepted view of what the causes might be, but Meier and Outreville (2006) summarized the work into three main schools of thought:

(1) disequilibrium between supply and demand;

(2) external shocks; and

(3) general business influences.

These views are not necessarily mutually exclusive since the rational expectations hypothesis implies that premiums are influenced by many factors other than the present value of expected future losses (therefore, it is also not possible to test the theories against each other).

Some authors have been skeptical about the assumption that insurers decide to cut prices or raise rates. Rather, prices may not only depend on expected future loss payments but also on present and past values of capital and surplus. This implies that shocks to capital affect the price and quantity of insurance supplied in the short-run and thus supply and demand of capital temporarily move out of equilibrium. For more details, we refer the reader to Niehaus and Terry (1993) or Cummins and Danzon (1997).

Here, we consider the point of view of a small, price-taker insurance company, which has essentially no influence on the market. Therefore, underwriting cycles are driven by large insurers and may be caused by variations in their capital and surplus. The case of a large insurance company, influencing market cycles, could be the subject of future research.

1.7.3 Conclusion

This paper has been devoted to the study of ruin probabilities when the premium income of the insurance company is subject to underwriting cycles. It is well-documented that underwriting cycles can have a significant impact on the stability of property-casualty insurance companies. This is why dynamic financial analysis accounts for such cycles assuming that the underwriting environment shifts among soft and hard markets according to a Markov chain. In this paper, we have kept the standard AR(2) modelling for the inverse loss ratio supported by many empirical studies since Cummins and Outreville (1987), and we have studied the ruin probability of an insurance company with a premium income subject to market cycles. Several analytical results have been derived, and the presence of underwriting cycles has been shown to in general increase the risk for the insurer. Numerical results support these theoretical findings.

Prospects and limitations

Sensitivity of the ruin probability to the market cycle model

The AR(2) process is the classical dynamics induced by underwriting cycles after the influential work by Cummins and Outreville (1987). Although it is also supported by empirical data coming from the Canadian motor insurance market, the table comparing different ARMA specifications appearing in Section 1.2.4 could also suggest for example an ARMA(1,1) dynamics for the $\{\Pi_k\}$ process. Let us assume that we erroneously choose the AR(2) model, while true dynamics is actually ARMA(1,1). What would be the impact on the ultimate ruin probability?

In the literature, as far as we know, there is no result analyzing the sensitivity of the ruin probability to such dependences on premiums. Gerber (1982) studied ruin probabilities for models where annual gains obey ARMA processes, but not premiums only. Thus, let us try to say something by ourself on this subject, on the basis of theoretical arguments.

If we consider that the ARMA(1,1) model is the right dynamics for the $\{\Pi_k\}$ process, we would actually have

$$\Pi_k = b_0 + b_1\Pi_{k-1} + b_2\Theta_{k-1} + \Theta_k,$$

where the Θ_k's are i.i.d., with $\mathbb{E}[\Theta_k] = 0$ and $\mathbb{V}[\Theta_k] = \sigma^2_\Theta$. Consequently, (1.3.2) would become

$$C_k = \tilde{b}_0 + b_1 C_{k-1} + b_2\tilde{\Theta}_{k-1} + \tilde{\Theta}_k,$$

with $\tilde{b}_0 = \mu_Y b_0$ and $\tilde{\Theta}_k = \mu_Y\Theta_k$. Once again, the stationarity of the $\{\Pi_k\}$ process (i.e. $|b_1| < 1$), the viability of the market (i.e. $b_0/(1-b_1) > 1$) and the existence of a positive constant d such that $\Pr[|\Theta_k| < d] = 1$ are assumed to be guaranteed. Now, in this setting, the ultimate ruin probability would be denoted as

$$\psi(u, c_0, \theta_0) = \Pr[U_n < 0 \text{ for some } n | U_0 = u, C_0 = c_0, \Theta_0 = \theta_0].$$

Hence, assuming that there exists a positive constant $\bar{\rho}$ such that

$$\mathbb{E}[e^{-\frac{\bar{\rho}}{1-b_1}(\tilde{b}_0+(1+b_2)\tilde{\Theta})}]\,\mathbb{E}[e^{\bar{\rho}Y}] = 1,$$

where $\tilde{\Theta}$ is a generic random variable distributed as the $\tilde{\Theta}_k$'s, we would then have

$$\psi(u, c_0, \theta_0) = \frac{e^{-\bar{\rho}\bar{u}}}{\mathbb{E}[e^{-\bar{\rho}\bar{U}_T}|T < \infty]},$$

where $\bar{U}_n = U_n + \frac{b_1}{1-b_1}C_n + \frac{b_2}{1-b_1}\tilde{\Theta}_n$ with $\bar{u} = \bar{U}_0$, and $T = \inf\{n|U_n < 0\}$. Indeed, it suffices to see that $\{e^{-\bar{\rho}\bar{U}_n}\}$ is a martingale, which is easy to show.

The comparison of this new ruin probability $\psi(u, c_0, \theta_0)$ with $\psi(u, c_0, c_{-1})$ derived in (1.4.2) would allow to quantify the error made by adopting the AR(2) dynamics. However, this comparison is still complicated, in particular because of the denominators. Therefore, let us compare the Lundberg coefficients ρ and $\bar{\rho}$.

To this end, let us introduce the random variables $X^{(1)}$ and $X^{(2)}$, distributed as $\frac{\mu_Y}{1-a_1-a_2}(a_0+\epsilon_k)$ and $\frac{\mu_Y}{1-b_1}(b_0 + (1 + b_2)\Theta_k)$ respectively. In the following, for simplicity, the ϵ_k's and the Θ_k's will be considered as normally distributed. The two first moments of $X^{(1)}$ and $X^{(2)}$ are given by

$$\mathbb{E}[X^{(1)}] = \mu_Y \frac{a_0}{1 - a_1 - a_2}$$
$$\mathbb{V}[X^{(1)}] = \mu_Y^2 \frac{1}{(1 - a_1 - a_2)^2}\sigma_\epsilon^2,$$

and

$$\mathbb{E}[X^{(2)}] = \mu_Y \frac{b_0}{1 - b_1}$$
$$\mathbb{V}[X^{(2)}] = \mu_Y^2 \frac{(1 + b_2)^2}{(1 - b_1)^2}\sigma_\Theta^2.$$

Let us assume that

$$\frac{a_0}{1 - a_1 - a_2} = \frac{b_0}{1 - b_1},$$

i.e. that the two adjusted models (AR(2) and ARMA(1,1)) will give us the same unconditional expectations for the $\{\Pi_k\}$ process. In this case, $\mathbb{E}[X^{(1)}] = \mathbb{E}[X^{(2)}]$. Since ρ and $\bar{\rho}$ are respectively solutions of

$$\mathbb{E}[e^{-rX^{(1)}}]\,\mathbb{E}[e^{rY}] = 1$$

and

$$\mathbb{E}[e^{-rX^{(2)}}]\,\mathbb{E}[e^{rY}] = 1,$$

it is easy to see that $\rho < \bar{\rho}$ if and only if $\mathbb{V}[X^{(1)}] > \mathbb{V}[X^{(2)}]$, i.e. iff $\frac{1}{(1-a_1-a_2)^2}\,\sigma_\epsilon^2 > \frac{(1+b_2)^2}{(1-b_1)^2}\,\sigma_\Theta^2$. Moreover, this is confirmed for small η ($\eta = \mathbb{E}[\Pi] - 1$). Indeed, in the same way as in Proposition 1.4.2, we can show that

$$\bar{\rho}(\eta) = \frac{2\mu_Y}{\sigma_Y^2 + \frac{\mu_Y^2 \sigma_\Theta^2 (1+b_2)^2}{(1-b_1)^2}}\eta + O(\eta^2).$$

Now, let us define δ by

$$\delta = \left| \frac{1}{(1-a_1-a_2)^2}\,\sigma_\epsilon^2 - \frac{(1+b_2)^2}{(1-b_1)^2}\,\sigma_\Theta^2 \right|.$$

Then, asymptotically, or in other words for large initial capital, the larger δ is, the greater will be the impact on the ultimate ruin probability of the mistake made by adopting the AR(2) model instead of the ARMA(1,1) model.

Let us look at our example concerning the Canadian motor insurance market. For the AR(2) model, the parameter estimates are $\widehat{a}_0 = 0.6197$, $\widehat{a}_1 = 0.9945$, $\widehat{a}_2 = -0.4699$ and $\widehat{\sigma}_\epsilon^2 = 0.01602$, while for the ARMA(1,1) model, the parameter estimates are $\widehat{b}_0 = 0.6651$, $\widehat{b}_1 = 0.4846$, $\widehat{b}_2 = 0.6315$ and $\widehat{\sigma}_\Theta^2 = 0.01498$. Hence, we have

$$\frac{\widehat{a}_0}{1-\widehat{a}_1-\widehat{a}_2} = 1.3035 \quad \approx \quad \frac{\widehat{b}_0}{1-\widehat{b}_1} = 1.2904,$$

and

$$\frac{1}{(1-\widehat{a}_1-\widehat{a}_2)^2}\,\widehat{\sigma}_\epsilon^2 = 0.0709 \quad < \quad \frac{(1+\widehat{b}_2)^2}{(1-\widehat{b}_1)^2}\,\widehat{\sigma}_\Theta^2 = 0.1501.$$

Clearly, in this case, it appears that asymptotically, the AR(2) model, selected on the basis of the AIC criterion, underestimates the risk compared to the ARMA(1,1) model.

In conclusion, if the ARMA(1,1) model was the right dynamics for the $\{\Pi_k\}$ process, our AR(2) approach would not be conservative for our Canadian example. Intuitively, an explanation could be that, according to the model considered, the nature of the $\{\Pi_k\}$ process appears rather different. Indeed, on the one hand, the ARMA(1,1) model, contrary to the AR(2) model, does not reflect a cyclic dynamics. On the other hand, a cyclic dynamics would allow to catch a kind of "return to the average effect", which would be intuitively safer for the insurer. This intuitive argument could allow to think that every fitted model which does not explain a cyclic movement would be riskier compared to our AR(2) model.

Interest rate problem

Some economic models suggest that interest rate could at least partially explain market cycles, or in other words, could be a cause of cycles.

Briefly, the motivation is that as premiums are the outcome of discounted future losses, any changes in the interest rates would induce changes in premiums (see for example Lamm-Tennant and Weiss (1997) or Fung et al. (1998)). Hence, interest rates cycles, observed on the financial markets, could generate underwriting cycles. So, in the light of this explanation, our model would also implicitly take into account interest rates cycle.

However, let us notice that techniques implemented in this chapter would not allow to study ruin models where underwriting cycles and interest rates cycles are both explicitly considered. Indeed, any attempt to generalize for example the results derived in Yang and Zhang (2006), who studied, by also using martingale techniques, a discrete-time risk model with random interest rate and where the annual claim amounts and the premiums are i.i.d, was unsuccessful. With our approach, only constant interest rates can be treated simultaneously with underwriting cycles, as illustrated in Section 1.7 of this chapter. Consequently, other methods should then be envisaged to deal with the generalized ruin model described just above. This could be the subject of future research.

The small company assumption

In this chapter, we take the point of view of a small, price-taker insurance company, which has essentially no influence on the market and hence on the business cycles. Underwriting cycles are then assumed to be driven by large insurers. The case of a large insurance company, influencing market cycles, is clearly also interesting. Nevertheless, this case must be treated separately. Indeed, because the insurer constitutes this time a major player of the market, it seems obvious that the premium loading will not depend anymore only on premium loadings of the other market players, an assumption which seemed realistic for a small company, but also now of the level of its own surplus. So, for a big insurer, it would be advisable rather to envisage a ruin model where premium loadings would depend on the one hand on the AR(2) dynamics of the loading of the other players of the market, and on the other hand on the level of its own reserve. Naturally, the study of this more general ruin model constitutes in itself the subject of a new research.

The main difficulty would be the dependence between premiums and the surplus level. In this context, no martingale can be identified, and no explicit expression of the ultimate ruin probability can be derived. Let us mention the paper of Asmussen and Nielsen (1995). They provided only an upper bound on the ultimate ruin probability, via local adjustment coefficients, for a compound Poisson continuous-time risk model, where the premium rate is assumed to depend upon the current reserve. While this result can be generalized with a discrete-time ruin model, it is not so obvious that a similar result holds for an AR(2) dynamics for the premiums. Consequently, even an upper bound appears difficult to obtain for this generalized model.

However, it is reasonable to think that the ruin probability calculated in this way would be smaller than the one obtained on base of the implemented method during this chapter, because any unfavourable event for the insurer would this time be compensated in the next years with an increase of the loading. Consequently, we could thus say that the approach envisaged in this thesis is conservative for a large company, because this one would overestimate the incurred risk.

Chapter 2

Ultimate ruin probability in discrete time with Bühlmann credibility premium adjustments

This chapter is submitted.

Abstract

In this paper, we consider a discrete-time ruin model where experience rating is taken into account. The main objective is to determine the behavior of the ultimate ruin probabilities for large initial capital in the case of light-tailed claim amounts. The logarithmic asymptotic behavior of the ultimate ruin probability is derived. Typical paths leading to ruin are studied. An upper bound is derived on the ultimate ruin probability in some particular case. The influence of the number of data points taken into account is analyzed, and numerical illustrations support the theoretical findings. Finally, we investigate the heavy-tailed case. The impact of the number of data points used for the premium calculation appears to be rather different from the one in the light-tailed case.

2.1 Introduction

Computation of ruin probabilities is an important topic in actuarial science. Standard references include Asmussen (2000), Dickson (2005), Grandell (1990), Kaas et al. (2008) and Rolski et al. (1999). Even if the study of the ruin problem resulted in a huge amount of published articles (including in the closely related fields of queueing theory and dam and storage processes), some assumptions have rarely been questioned in the literature, until recently.

The majority of existing results require independent increments for the stochastic process describing the aggregate claim amounts filed against the insurance company. However, a lot of practical issues violate this independence assumption, as the result of the implementation of experience rating mechanisms by the insurance company. In this case, the financial result of a given calendar year then depends on the result(s) of one or several previous years.

In this paper, experience rating is taken into account. Allowing the premium amount to depend on past claims experience through an appropriate credibility mechanism is in line with insurance practice in most lines of business.

In the literature, only some authors have studied ruin problem with credibility dynamics. In Asmussen (1999), the ruin model allows adapted premiums rules. More precisely, a continuous-time risk process is considered with compound Poisson claims, where the premium rate is exclusively calculated on the basis of past claims statistics, with no reference to the market. In practice, one would use a mixture between a quantity based on past claim experience and a market premium level. The easiest way to compute credibility adjusted premiums is to use Bühlmann's credibility model. This is what Tsai and Parker (2004) have done in a discrete-time risk model. They have studied, by means of Monte-Carlo simulations, the impact of the number of past claim experience years taken into account for the calculation of the premiums on the ultimate ruin probabilities for light-tailed claims.

In this paper, we deal with a framework that is similar to the one of Tsai and Parker (2004). One of the main observation made by these authors is that for an insurance company with a large initial capital and light-tailed claims, the use of a bigger (finite) number of past claim experience years to calculate the premiums decreases the ultimate ruin probability. Among other results, until a certain number of past claim experience years, this observation is proved in this paper. We also show that the story is different for heavy-tailed claim amounts.

2.2 The model

Let us consider an insurance market made up of n portfolios, where portfolio j $(j = 1, \ldots, n)$ is characterized by independent and identically distributed annual aggregated claim amounts $Y_k^{(j)}$, with common distribution function F_j. For a randomly chosen insurance company operating on this market, let p_j be the probability that the portfolio held by this company is the portfolio j.

To motivate this setting, let us consider for example a simple market made up of 2 categories of policyholders, with n_g "good" policyholders and n_b "bad" policyholders, say. Then a company with $s \leq \min(n_g, n_b)$ policyholders would have $s+1$ possible portfolios, i.e. $n = s+1$. Furthermore, the probability p_j (associated to portfolio j $(j = 1, \ldots, s+1)$, characterized by $j-1$ good policyholders) would be given by

$$p_j = \frac{s!}{(s-j+1)!(j-1)!} \prod_{i=0}^{j-1} \frac{n_g - i}{n_g + n_b - i} \prod_{i=0}^{s-j+1} \frac{n_b - i}{n_g + n_b - j - i}, \tag{2.2.1}$$

with $\prod_0^0 = 1$.

We assume that the company uses credibility premiums in order to determine, on the basis of its own claim amounts, the pure premium corresponding to its unknown portfolio. In this setting, the model proposed in Bühlmann (1967) and (1969) is largely used in practice. In this paper, we focus on this well-known credibility model.

Let us denote by Y_k the annual claim amounts of the company. The a priori mean of the Y_k's is of course given by

$$\mu = \mathbb{E}[Y_k] = \sum_{j=1}^{n} \mathbb{E}[Y^{(j)}]\, p_j, \tag{2.2.2}$$

where $Y^{(j)}$ is a random variable with distribution function F_j, while the a priori variance breaks up in two terms:

$$\sigma^2 = \mathbb{V}[Y_k] = a + \nu, \tag{2.2.3}$$

with

$$a = \sum_{j=1}^{n} (\mathbb{E}[Y^{(j)}] - \mu)^2\, p_j \tag{2.2.4}$$

and

$$\nu = \sum_{j=1}^{n} \mathbb{V}[Y^{(j)}]\, p_j. \tag{2.2.5}$$

Parameter a measures the part of the variance coming from the heterogeneity of the market, whereas parameter ν measures the part of hazard in the a priori variance.

As mentioned in Tsai and Parker (2004), some casualty insurers only consider a finite number of most recent periods of claim experiences to renew the premium. Let us denote by $m \in \mathbb{N}^+ \cup \{\infty\}$ the so-called "horizon of credibility", which corresponds to the number of periods taken into account for the calculation of the premiums.

So, depending on the choice of m, with the convention that $\sum_{i=1}^{0} = 0$, the premium of the company in time $k-1$ is given by

$$C_{k,m} = \left((1 - z_{k,m})\mu + z_{k,m} \frac{\sum_{i=\max(k-m,1)}^{k-1} Y_i}{\min(m, k-1)} \right) (1+\eta), \tag{2.2.6}$$

where

$$z_{k,m} = \frac{\min(m, k-1)\, a}{\nu + \min(m, k-1)\, a} \tag{2.2.7}$$

is Bühlmann's credibility factor and $\eta > 0$ is the premium security loading. Notice that our premium rule depends on the market (via the parameters μ, a and ν) contrary to Asmussen (1999). For simplicity, we have assumed that no historical data is available. This simplifying assumption does not influence the asymptotic results derived in the sequel.

Now, if the portfolio held by the insurer is portfolio j, then the dynamics of the insurer's surplus obeys to the equation

$$U_{k,m}^{(j)} = u + \sum_{i=1}^{k} C_{i,m} - \sum_{i=1}^{k} Y_i^{(j)}, \tag{2.2.8}$$

where the $C_{i,m}$'s are described by equation (2.2.6) with $Y_i = Y_i^{(j)}$, and where u is the initial capital of the company. The corresponding ultimate ruin probability may be written as

$$\psi_m^{(j)}(u) = \Pr\left[U_{k,m}^{(j)} < 0 \text{ for some } k \,|U_{0,m}^{(j)} = u \right]. \tag{2.2.9}$$

Immediately, it appears the existence of a subset of "bad" portfolios, denoting b_m, such that for all $j \in b_m$, $\psi_m^{(j)}(u) = 1$ for all u. Indeed, let us define

$$\eta_m^{(j)} = \frac{(1+\eta)((1-z_m)\mu + z_m \mathbb{E}[Y^{(j)}])}{\mathbb{E}[Y^{(j)}]} - 1, \tag{2.2.10}$$

where

$$z_m = \frac{m\, a}{\nu + m\, a}. \tag{2.2.11}$$

Obviously, for $m < \infty$, $\eta_m^{(j)}$ corresponds to the average security loading of the premium for $k > m$ while for $m = \infty$, $\eta_m^{(j)} = \eta$, which corresponds this time to the security loading for $k \to \infty$. The subset b_m is then defined as

$$b_m = \{j = 1, \ldots, n : \eta_m^{(j)} \leq 0\}. \tag{2.2.12}$$

Also, in the following, the complementary subset of ("not bad") portfolios is denoted by $\bar{b}_m$, and we shall say that $j \in g$ if and only if $\mathbb{E}[Y^{(j)}] \leq \mu$. Clearly, we have $g \subset \bar{b}_m$.

The purpose of this paper is to investigate the behavior of the ultimate ruin probabilities $\psi_m^{(j)}(u)$ for large u in the case of light-tailed annual claim amounts. First, the logarithmic asymptotic behavior of $\psi_m^{(j)}(u)$ is derived. Then, typical paths leading to ruin are studied. Also, an upper bound on $\psi_m^{(j)}(u)$ is derived in some particular cases. Afterwards, we are able to determine, for each portfolio j, the influence of "strategy" m on $\psi_m^{(j)}(u)$ for large u. Next, whatever the portfolio j, optimal strategies, in a sense to be defined in the sequel, are deduced. As an illustration, numerical simulations are also performed. Finally, we also study the influence of m in first order on $\psi_m^{(j)}(u)$ in the heavy-tailed case. As we shall see, the conclusion is totally different than in the light-tailed case.

2.3 Limit results for light-tailed claims

In this section, for simplicity, let us assume that for all $j = 1, \ldots, n$, there exists $r^{(j)} > 0$ such that $\mathbb{E}[e^{rY^{(j)}}]$ exists for all $0 < r < r^{(j)}$ and such that $\lim_{r \to r^{(j)}} \mathbb{E}[e^{rY^{(j)}}] = \infty$. Clearly, this assumption implies necessarily light-tailed claims.

2.3.1 The case $m < \infty$

Asymptotic behavior of $\psi_m^{(j)}(u)$

Theorem 2.3.1. *Suppose that there exists a unique positive solution to the equation*

$$e^{-r(1-z_m)\mu(1+\eta)}\mathbb{E}\left[e^{rY^{(j)}(1-(1+\eta)z_m)}\right] = 1, \tag{2.3.1}$$

which we denote $\rho_m^{(j)}$. Furthermore, assume that for all k, the sum of the first k annual losses $Z_k = \sum_{i=1}^{k}(Y_i^{(j)} - C_{i,m})$ possesses a finite moment-generating function $\mathbb{E}[e^{rZ_k}]$ for $0 < r < r_0$, with $\rho_m^{(j)} < r_0$. Hence, we have

$$\lim_{u \to \infty} \frac{1}{u} \ln \psi_m^{(j)}(u) = -\rho_m^{(j)}. \tag{2.3.2}$$

Proof. As a consequence of the Gärtner-Ellis Theorem from large deviations (which is due to Glynn and Whitt (1994), see also Nyrhinen (1998) and Müller and Pflug (2001)), one only has to show the two following properties:

(A1) $\kappa_m(r) := \lim_{k\to\infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}]$ exists for $0 < r < r_0$,

(A2) $\rho_m^{(j)}$ is the unique positive value such that $\kappa_m(\rho_m^{(j)}) = 0$.

We have

$$\begin{aligned}\ln \mathbb{E}[e^{rZ_k}] &= -r(1+\eta)\mu\sum_{i=1}^{k}(1-z_{i,m}) + \ln \mathbb{E}\left[e^{r\sum_{i=1}^{k}(Y_i^{(j)}-(1+\eta)\bar{z}_{i,m}\sum_{l=\max(1,i-m)}^{i-1}Y_l^{(j)})}\right] \\ &= -r(1+\eta)\mu\sum_{i=1}^{k}(1-z_{i,m}) + \sum_{i=1}^{k}\ln \mathbb{E}\left[e^{rY^{(j)}(1-(\bar{z}_{i+1,m}+\bar{z}_{i+2,m}+\cdots+\bar{z}_{\min(i+m,k),m})(1+\eta))}\right]\end{aligned}$$

where $\bar{z}_{k,m} = \frac{z_{k,m}}{\min(m,k-1)}$. Consequently

$$\lim_{k\to\infty}\frac{1}{k}\ln \mathbb{E}[e^{rZ_k}] = -r(1-z_m)\mu(1+\eta) + \ln \mathbb{E}\left[e^{rY^{(j)}(1-(1+\eta)z_m)}\right].$$

Hence, (A1) holds true. Concerning (A2), notice that $\kappa_m(r) = \ln h_m(r)$, where

$$h_m(r) = e^{-r(1-z_m)\mu(1+\eta)}\mathbb{E}\left[e^{rY^{(j)}(1-(1+\eta)z_m)}\right].$$

□

Notice that since $\mathbb{E}[e^{rY^{(j)}}]$ exists for all $0 < r < r^{(j)}$, result (2.3.2) requires in fact only the existence and the uniqueness of $\rho_m^{(j)}$, with the condition $\rho_m^{(j)} < r^{(j)}$.

The Lundberg coefficient $\rho_m^{(j)}$

Let us define

$$\tilde{\eta}_m^{(j)} = \frac{(1-z_m)\mu(1+\eta)}{\mathbb{E}[Y^{(j)}](1-(1+\eta)z_m)} - 1. \tag{2.3.3}$$

The equation (2.3.1) is equivalent to

$$e^{-r(1-(1+\eta)z_m)\,\mathbb{E}[Y^{(j)}](1+\tilde{\eta}_m^{(j)})}\mathbb{E}\left[e^{r(1-(1+\eta)z_m)Y^{(j)}}\right] = 1, \tag{2.3.4}$$

and consequently

$$\rho_m^{(j)} = \frac{\tilde{\rho}_m^{(j)}}{1-(1+\eta)z_m}, \tag{2.3.5}$$

where $\tilde{\rho}_m^{(j)}$ is the classical Lundberg coefficient defined as the unique positive solution of the equation

$$e^{-r\,\mathbb{E}[Y^{(j)}](1+\tilde{\eta}_m^{(j)})}\mathbb{E}[e^{rY^{(j)}}] = 1. \tag{2.3.6}$$

Now, from classical results, it is then obvious that the inequality $(\infty >)\tilde{\eta}_m^{(j)} > 0$ is a necessary condition for the existence of $\rho_m^{(j)}$. Let us prove that the inequality $(\infty >)\tilde{\eta}_m^{(j)} > 0$ is equivalent to $m < \frac{\nu}{a\eta}$ and $j \in \bar{b}_m$.

Proposition 2.3.1.

$$(\infty >)\tilde{\eta}_m^{(j)} > 0 \Leftrightarrow m < \frac{\nu}{a\eta} \textit{ and } j \in \bar{b}_m. \tag{2.3.7}$$

Proof. ($\Rightarrow$): If $\infty > \tilde{\eta}_m^{(j)} > 0$, then, obviously $z_m < \frac{1}{1+\eta}$, or equivalently $m < \frac{\nu}{a\eta}$, and $(1-z_m)\mu(1+\eta) > \mathbb{E}[Y^{(j)}](1-(1+\eta)z_m)$. Thus

$$\begin{aligned}
\eta_m^{(j)} &= \frac{((1-z_m)\mu + z_m\mathbb{E}[Y^{(j)}])(1+\eta)}{\mathbb{E}[Y^{(j)}]} - 1 \\
&> \frac{\mathbb{E}[Y^{(j)}](1-(1+\eta)z_m) + z_m\mathbb{E}[Y^{(j)}](1+\eta)}{\mathbb{E}[Y^{(j)}]} - 1 \\
&= (1-(1+\eta)z_m) + z_m(1+\eta) - 1 = 0.
\end{aligned}$$

($\Leftarrow$): Since $j \in \bar{b}_m$, we have $((1-z_m)\mu + z_m\mathbb{E}[Y^{(j)}])(1+\eta) > \mathbb{E}[Y^{(j)}]$. Thus, $(1-z_m)\mu(1+\eta) > \mathbb{E}[Y^{(j)}](1-(1+\eta)z_m)$. Now, since $m < \frac{\nu}{a\eta}$, $z_m < \frac{1}{1+\eta}$. So, $\tilde{\eta}_m^{(j)} < \infty$ and $\mathbb{E}[Y^{(j)}](1-(1+\eta)z_m) > 0$. Consequently

$$\infty > \tilde{\eta}_m^{(j)} = \frac{(1-z_m)\mu(1+\eta)}{\mathbb{E}[Y^{(j)}](1-(1+\eta)z_m)} - 1 > \frac{\mathbb{E}[Y^{(j)}](1-(1+\eta)z_m)}{\mathbb{E}[Y^{(j)}](1-(1+\eta)z_m)} - 1 = 0.$$

□

This result highlights the obvious fact that the existence of $\rho_m^{(j)}$ implies that j must belong to $\bar{b}_m$, since $\psi_m^{(j)}(u) = 1$ for $j \in b_m$.

Let us explain the reason why the horizon of credibility $\lceil \frac{\nu}{a\eta} \rceil$, denoted by $m^{(c)}$ from now, constitutes a critical value for the existence of the Lundberg coefficient and consequently for the logarithmic asymptotic behavior of $\psi_m^{(j)}(u)$.

From $m = m^{(c)}$, the global impact of each $Y_k^{(j)}$ on the surplus process for $k > m$, i.e. $Y_k^{(j)}(1-(1+\eta)z_m)$, becomes negative, which is, of course, a drastic change in the nature of the insured risk. Also, $b_m = \emptyset$ for all $m \geq m^{(c)}$.

More fundamentally, in first order, the only effects selected of the credibility dynamics compared to the classical case $m = 0$ are a modification of the security loading and a decreasing of the claim amounts. Indeed, equations (2.3.4) and (2.3.5) teach us that $\rho_m^{(j)}$ is identical to that of the classical ruin model with a premium security loading equals to $\tilde{\eta}_m^{(j)}$ and annual claim amounts given by $Y_k^{(j)}(1-(1+\eta)z_m)$. Therefore, in this associated classical ruin model, from the critical value $m^{(c)}$, no ruin can be observed, which explains the reason why a Lundberg coefficient does not exist in this case.

So, since $\mathbb{E}[e^{rY^{(j)}}]$ exists for all $0 < r < r^{(j)}$ with $\lim_{r \to r^{(j)}} \mathbb{E}[e^{rY^{(j)}}] = \infty$, it comes that the two following conditions are of course necessary but also sufficient conditions to guarantee the existence and the uniqueness of $\rho_m^{(j)}$:

(C1) $m < m^{(c)}$,

(C2) $j \in \bar{b}_m$.

Let us recall that result (2.3.2) requires not only the existence and the uniqueness of $\rho_m^{(j)}$ but also the additional condition $\rho_m^{(j)} < r^{(j)}$. Firstly, we have the following property for the Lundberg coefficient $\rho_m^{(j)}$:

Property 2.3.2. *Assume that $\rho_{m_1}^{(j)}$ and $\rho_{m_2}^{(j)}$ exist, with $0 \leq m_1 < m_2$. Then, $\rho_{m_1}^{(j)} < \rho_{m_2}^{(j)}$.*

Proof. Since $z_{m_1} < z_{m_2}$, we have $\tilde{\eta}_{m_1}^{(j)} < \tilde{\eta}_{m_2}^{(j)}$ and consequently, by classical results, $\tilde{\rho}_{m_1}^{(j)} < \tilde{\rho}_{m_2}^{(j)}$. By (2.3.5), we deduce $\rho_{m_1}^{(j)} < \rho_{m_2}^{(j)}$. □

Secondly, let us define

$$\bar{m}^{(j)} = \min\{m < m^{(c)} : \rho_m^{(j)} \text{ exist and } \rho_m^{(j)} \geq r^{(j)}\}. \tag{2.3.8}$$

If $\rho_m^{(j)}$ does not exist for all $m < m^{(c)}$, then we put $\bar{m}^{(j)} = 0$. Also, if for all $m < m^{(c)}$ such that $\rho_m^{(j)}$ exists, we always have $\rho_m^{(j)} < r^{(j)}$, then we put $\bar{m}^{(j)} = m^{(c)}$. Hence, if we replace condition (C1) by the stronger condition $m < \bar{m}^{(j)}$, we have now necessary and sufficient conditions to guarantee the existence and the uniqueness of $\rho_m^{(j)}$ such that $\rho_m^{(j)} < r^{(j)}$, i.e. result (2.3.2).

Large deviations path to the ruin

Let us assume that $m < \bar{m}^{(j)}$ and $j \in \bar{b}_m$, i.e. that result (2.3.2) is valid. The goal here is to understand how ruin occurs for large u. More particularly, we want to determine the typical shape of a path leading to ruin when the initial capital of the company is large. To this end, let us use large deviations results of Glynn and Whitt (1994). First of all, for large u, we know that ruin occurs roughly at time $\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor$. Next, the cumulant generating function $\gamma_{\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor}(r)$ of $Z_{\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor}$ is asymptotically changed from $\gamma_{\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor}(r)$ to

$$\begin{aligned}\gamma_{\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor}(r+\rho_m^{(j)}) - \gamma_{\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor}(\rho_m^{(j)}) &= -r(1+\eta)\mu \sum_{i=1}^{\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor} (1 - z_{i,m}) \\ &+ \sum_{i=1}^{\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor} \ln \frac{\mathbb{E}\left[e^{(r+\rho_m^{(j)})Y^{(j)}\left(1-(\bar{z}_{i+1,m}+\bar{z}_{i+2,m}+\cdots+\bar{z}_{\min(i+m,\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor),m})(1+\eta)\right)}\right]}{\mathbb{E}\left[e^{\rho_m^{(j)}Y^{(j)}\left(1-(\bar{z}_{i+1,m}+\bar{z}_{i+2,m}+\cdots+\bar{z}_{\min(i+m,\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor),m})(1+\eta)\right)}\right]}.\end{aligned} \tag{2.3.9}$$

Consequently, given that ruin occurs, the claim size distribution $Y_k^{(j)}$ is exponentially tilted by the time-dependent factor $\rho_m^{(j)}(k)$, with

$$\rho_m^{(j)}(k) = \begin{cases} \tilde{\rho}_m^{(j)}, & \text{for } k = 1, \ldots, \lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor - m \\ \left(1 - \left(\frac{\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor - k}{m}\right) z_m(1+\eta)\right)\rho_m^{(j)}, & \text{for } k = \lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor - m + 1, \ldots, \lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor \end{cases}.$$

Hence, for large u, the drift of the path to ruin at time k ($k = 1, \ldots, \lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor$), denoted as $\delta_m(k)$, is given by

$$\delta_m(k) = (1 - z_m)\mu(1+\eta) - (1-(1+\eta)z_m)\frac{\mathbb{E}\left[Y^{(j)}e^{\rho_m^{(j)}(k)\,Y^{(j)}}\right]}{\mathbb{E}\left[e^{\rho_m^{(j)}(k)\,Y^{(j)}}\right]} < 0. \tag{2.3.10}$$

However, notice that since $u \to \infty$, the time period $(\lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor - m + 1, \lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor)$ where $\delta_m(k)$ decreases with k is negligible compared with the time period $(1, \lfloor u/\kappa'_m(\rho_m^{(j)})\rfloor - m)$ where $\delta_m(k)$ remains constant.

This analysis highlights that ruin is caused by a succession of small claims, followed by a succession of moderately large claims.

Upper bound on $\psi_m^{(j)}(u)$

In this subsection only, we take into account a possible claim experience of h years say, i.e. $(Y_{-h+1}^{(j)}, Y_{-h+2}^{(j)}, \ldots, Y_0^{(j)})$, because, this time, as we see in the following, it will impact the result. Hence, equation (2.2.6) becomes

$$C_{k,m} = \left((1 - z_{k,m})\mu + z_{k,m}\frac{\sum_{i=\max(k-m,1-h)}^{k-1} Y_i}{\min(m, h+k-1)}\right)(1+\eta), \tag{2.3.11}$$

with

$$z_{k,m} = \frac{\min(m, h+k-1)\,a}{\nu + \min(m, h+k-1)\,a}. \tag{2.3.12}$$

Let $b = \min(m, h)$, and let us denote

$$\psi_m^{(j)}(u, y_0, \ldots, y_{1-b}) = \Pr\left[U_{k,m}^{(j)} < 0 \text{ for some } k \;|U_{0,m}^{(j)} = u, Y_0^{(j)} = y_0, \ldots, Y_{1-b}^{(j)} = y_{1-b}\right]. \tag{2.3.13}$$

Theorem 2.3.3. *Assume that $m < \bar{m}^{(j)}$ and $j \in \bar{b}_m$. Then*

$$\psi_m^{(j)}(u, y_0, \ldots, y_{1-b}) \leq \frac{e^{-\rho_m^{(j)}\hat{u}}}{\mathbb{E}\left[e^{-\rho_m^{(j)}\hat{U}_T}|T < \infty\right]}, \tag{2.3.14}$$

where $\hat{U}_k = U_{k,m}^{(j)} + (1+\eta)[Y_k^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{k+m,m}) + Y_{k-1}^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{k+m-1,m}) + \cdots + Y_{\max(k-m+1,1-h)}^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{m+\max(k-m+1,1-h),m})]$, $\hat{u} = \hat{U}_0$, *and* $T = \inf\{k|U_{k,m}^{(j)} < 0\}$, *with* $\bar{z}_{k,m} = \frac{z_{k,m}}{\min(m,h+k-1)}$.

Proof. First, we note that

$$\begin{aligned}
\hat{U}_k &= U_{k,m}^{(j)} + (1+\eta)[Y_k^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{k+m,m}) + Y_{k-1}^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{k+m-1,m}) \\
&\qquad + \cdots + Y_{\max(k-m+1,1-h)}^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{m+\max(k-m+1,1-h),m})] \\
&= U_{k-1,m}^{(j)} + C_{k,m} - Y_k^{(j)} \\
&\quad +(1+\eta)[Y_k^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{k+m,m}) + Y_{k-1}^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{k+m-1,m}) \\
&\qquad + \cdots + Y_{\max(k-m+1,1-h)}^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{m+\max(k-m+1,1-h),m})] \\
&= U_{k-1,m}^{(j)} + (1-z_{k,m})\mu(1+\eta) + z_{k,m}(1+\eta)\frac{\sum_{i=\max(k-m,1-h)}^{k-1} Y_i^{(j)}}{\min(m,k+h-1)} - Y_k^{(j)} \\
&\quad +(1+\eta)[Y_k^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{k+m,m}) + Y_{k-1}^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{k+m-1,m}) \\
&\qquad + \cdots + Y_{\max(k-m+1,1-h)}^{(j)}(\bar{z}_{k+1,m} + \cdots + \bar{z}_{m+\max(k-m+1,1-h),m})] \\
&= U_{k-1,m}^{(j)} + (1+\eta)[Y_{k-1}^{(j)}(\bar{z}_{k,m} + \cdots + \bar{z}_{k+m-1,m}) + Y_{k-2}^{(j)}(\bar{z}_{k,m} + \cdots + \bar{z}_{k+m-2,m}) \\
&\qquad + \cdots + Y_{\max(k-m,1-h)}^{(j)}(\bar{z}_{k,m} + \cdots + \bar{z}_{m+\max(k-m,1-h),m})] \\
&\quad +(1-z_{k,m})\mu(1+\eta) - Y_k^{(j)}(1-(1+\eta)(\bar{z}_{k+1,m} + \cdots + \bar{z}_{k+m,m})) \\
&= \hat{U}_{k-1} + (1-z_{k,m})\mu(1+\eta) - Y_k^{(j)}(1-(1+\eta)(\bar{z}_{k+1,m} + \cdots + \bar{z}_{k+m,m})).
\end{aligned}$$

Hence, if we denote by Φ_k the sigma-algebra generated by $\{Y_i^{(j)}, i = 1, 2, \ldots, k\}$, we have

$$\mathbb{E}\left[e^{-\rho_m^{(j)}\hat{U}_k}|\Phi_{k-1}\right] = e^{-\rho_m^{(j)}\hat{U}_{k-1}}e^{-\rho_m^{(j)}(1-z_{k,m})\mu(1+\eta)}\mathbb{E}\left[e^{\rho_m^{(j)}Y^{(j)}(1-(1+\eta)(\bar{z}_{k+1,m}+\cdots+\bar{z}_{k+m,m}))}\right].$$

Now, let us define

$$h_{k,m}(r) = e^{-r(1-z_{k,m})\mu(1+\eta)}\mathbb{E}\left[e^{rY^{(j)}(1-(1+\eta)(\bar{z}_{k+1,m}+\cdots+\bar{z}_{k+m,m}))}\right].$$

Obviously, for $k \geq m-h+1$, $h_{k,m}(r) = h_m(r)$. Since $z_{k-1,m} \leq z_{k,m}$ and $\bar{z}_{k-1,m} \geq \bar{z}_{k,m}$, we clearly have $h_{k-1,m}(r) \leq h_{k,m}(r) \leq h_m(r)$. Consequently, by (2.3.1), $h_{k,m}(\rho_m^{(j)}) \leq h_m(\rho_m^{(j)}) = 1$. Thus, we get

$$\mathbb{E}\left[e^{-\rho_m^{(j)}\hat{U}_k}|\Phi_{k-1}\right] = e^{-\rho_m^{(j)}\hat{U}_{k-1}}h_k(\rho_m^{(j)}) \leq e^{-\rho_m^{(j)}\hat{U}_{k-1}}.$$

So, the process $\{e^{-\rho_m^{(j)}\hat{U}_k}\}$ is a super-martingale.

Let w be a positive integer. By the Optional Stopping Theorem (considering the stopping time $T \wedge w = \min\{T, w\}$),we get

$$e^{-\rho_m^{(j)}\hat{U}_0} \geq \mathbb{E}\left[e^{-\rho_m^{(j)}\hat{U}_{T\wedge w}}\right] = \mathbb{E}\left[e^{-\rho_m^{(j)}\hat{U}_T}I_{T\leq w}\right] + \mathbb{E}\left[e^{-\rho_m^{(j)}\hat{U}_w}I_{T>w}\right] \geq \mathbb{E}\left[e^{-\rho_m^{(j)}\hat{U}_T}I_{T\leq w}\right].$$

Hence letting $w \to \infty$, we have

$$e^{-\rho_m^{(j)}\hat{U}_0} \geq \mathbb{E}\left[e^{-\rho_m^{(j)}\hat{U}_T}|T<\infty\right]\Pr[T<\infty] = \mathbb{E}\left[e^{-\rho_m^{(j)}\hat{U}_T}|T<\infty\right]\psi_m^{(j)}(u, y_0, \ldots, y_{1-b}).$$

$\square$

Remark Inequality (2.3.14) becomes an equality when $h \geq m$. Indeed, in this case, for $k \geq 1$, we have $h_{k,m}(\rho_m^{(j)}) = h_m(\rho_m^{(j)}) = 1$. Thus, the process $\{e^{-\rho_m^{(j)} \hat{U}_k}\}$ is a martingale. Furthermore, the random variables $e^{-\rho_m^{(j)} \hat{U}_w} I_{T>w}$ pointwise converge to zero for $w \to \infty$ and are bounded by 1 since $\hat{U}_w > 0$ for $w < T$.

The case $m \geq \bar{m}^{(j)}$ and $j \in \bar{b}_m$

Now, let us consider the case $m \geq \bar{m}^{(j)}$ and $j \in \bar{b}_m$. For the next result, let us denote $\psi_m^{(j)}(u)$ by $\psi_{m,\eta}^{(j)}(u)$, $\rho_m^{(j)}$ by $\rho_{m,\eta}^{(j)}$, $\tilde{\rho}_m^{(j)}$ by $\tilde{\rho}_{m,\eta}^{(j)}$, $\tilde{\eta}_m^{(j)}$ by $\tilde{\eta}_{m,\eta}^{(j)}$ and $\eta_m^{(j)}$ by $\eta_{m,\eta}^{(j)}$ to reveal explicitly the dependence in η. Obviously, notice that $\tilde{\eta}_{m,\eta}^{(j)}$ increases with η. Hence, we know by classical results that $\tilde{\rho}_{m,\eta}^{(j)}$ also increases with η and consequently $\rho_{m,\eta}^{(j)}$ by (2.3.5). Furthermore, let us define

$$\eta^{(\star,1)} = \min\left\{\tilde{\eta} \leq \eta : \ \eta_{m,\tilde{\eta}}^{(j)} > 0 \text{ and } m < \frac{\nu}{a\tilde{\eta}}\right\}. \tag{2.3.15}$$

and

$$\eta^{(\star,2)} = \max\left\{\tilde{\eta} \leq \eta : \ \eta_{m,\tilde{\eta}}^{(j)} > 0 \text{ and } m < \frac{\nu}{a\tilde{\eta}}\right\}. \tag{2.3.16}$$

Proposition 2.3.2. *Assume that $\rho_{m,\eta^{(\star,1)}}^{(j)} < r^{(j)} \leq \rho_{m,\eta^{(\star,2)}}^{(j)}$. Then, we have*

$$\lim_{u\to\infty} \sup \frac{1}{u} \ln \psi_{m,\eta}^{(j)}(u) \leq -r^{(j)}. \tag{2.3.17}$$

Proof. Obviously, there exists $\eta^{(\star,1)} < \bar{\eta} \leq \eta^{(\star,2)}$ such that $\rho_{m,\bar{\eta}}^{(j)} = r^{(j)}$. For all $\eta^{(\star,1)} \leq \tilde{\eta} < \bar{\eta}$, we have

$$\lim_{u\to\infty} \sup \frac{1}{u} \ln \psi_{m,\eta}^{(j)}(u) \leq \lim_{u\to\infty} \frac{1}{u} \ln \psi_{m,\tilde{\eta}}^{(j)}(u) = -\rho_{m,\tilde{\eta}}^{(j)}.$$

Letting $\tilde{\eta} \uparrow \bar{\eta}$, we obtain (2.3.17). □

As a particular, let us assume that $Y^{(j)} \sim Exp(\lambda_j)$. In this case, $\mathbb{E}[e^{rY^{(j)}}]$ exists all for $r < \lambda_j$ and is equal to $\frac{\lambda_j}{\lambda_j - r}$. Clearly, $r^{(j)} = \lambda_j$. Now, since

$$\lim_{u\to\infty} \frac{1}{u} \ln \Pr[U_{1,m}^{(j)} < 0] = \lambda_j \tag{2.3.18}$$

and that $\psi_m^{(j)}(u) \geq \Pr[U_{1,m}^{(j)} < 0]$, result (2.3.17) becomes

$$\lim_{u\to\infty} \inf \frac{1}{u} \ln \psi_m^{(j)}(u) = \lim_{u\to\infty} \sup \frac{1}{u} \ln \psi_m^{(j)}(u) = -\lambda_j. \tag{2.3.19}$$

Conclusion

In conclusion, on the one hand, if $j \in b_m$ then $\psi_m^{(j)}(u) = 1$ for all u. On the other hand, if $j \in \bar{b}_m$, then for $m < \bar{m}^{(j)}$, the logarithmic asymptotic behavior of $\psi_m^{(j)}(u)$ is given by equation (2.3.2) while for $(\infty) > m \geq \bar{m}^{(j)}$, the logarithmic asymptotic behavior of $\psi_m^{(j)}(u)$ is this time characterized by equation (2.3.17).

2.3.2 The case $m = \infty$

Clearly, with $z_m = 1$, i.e. $m = \infty$, the results of the previous section have no meaning. This is the reason why this important case has to be treated separately.

Asymptotic behavior of $\psi_\infty^{(j)}(u)$

Let us state an analogue of Theorem 2.3.1.

Theorem 2.3.4. *Suppose that there exists a unique positive solution to the equation*

$$\int_0^1 dx \ln \mathbb{E}\left[e^{rY^{(j)}(1+(1+\eta)\ln x)}\right] = 0, \tag{2.3.20}$$

which we denote $\rho_\infty^{(j)}$. If we assume that for all k, $\mathbb{E}[e^{rZ_k}]$ exists for $0 < r < r_0$ (with $\rho_\infty^{(j)} < r_0$), where $Z_k = \sum_{i=1}^k (Y_i^{(j)} - C_{i,\infty})$, we have

$$\lim_{u\to\infty} \frac{1}{u} \ln \psi_\infty^{(j)}(u) = -\rho_\infty^{(j)}. \tag{2.3.21}$$

Proof. Again, it suffices to show that $\kappa_\infty(r) := \lim_{k\to\infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}]$ exists for $0 < r < r_0$, and that $\rho_\infty^{(j)}$ is the unique positive value such that $\kappa_\infty(\rho_\infty^{(j)}) = 0$. We have

$$\ln \mathbb{E}[e^{rZ_k}] = -r(1+\eta)\mu \sum_{i=1}^k (1 - z_{i,\infty}) + \ln \mathbb{E}\left[e^{r\sum_{i=1}^k \left(Y_k^{(j)} - (1+\eta) z_{i,\infty} \frac{\sum_{l=1}^{i-1} Y_l^{(j)}}{i-1}\right)}\right].$$

Now we have

$$\sum_{i=1}^k \left(Y_k^{(j)} - (1+\eta) z_{i,\infty} \frac{\sum_{l=1}^{i-1} Y_l^{(j)}}{i-1}\right) = \sum_{i=1}^k Y_i^{(j)} \left(1 - (1+\eta)\sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l}\right).$$

Hence we get

$$\ln \mathbb{E}[e^{rZ_k}] = -r(1+\eta)\mu \sum_{i=1}^k (1 - z_{i,\infty}) + \ln \mathbb{E}\left[e^{r\sum_{i=1}^k Y_i^{(j)}\left(1-(1+\eta)\sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l}\right)}\right].$$

Thus, we have

$$\begin{aligned} \lim_{k\to\infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}] &= -r(1+\eta)\mu \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^k (1 - z_{i,\infty}) \\ &\quad + \lim_{k\to\infty} \frac{1}{k} \ln \mathbb{E}\left[e^{r\sum_{i=1}^k Y_i^{(j)}\left(1-(1+\eta)\sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l}\right)}\right]. \end{aligned}$$

It is obvious that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^k (1 - z_{i,\infty}) = 0.$$

Furthermore, note that

$$\sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l} = \sum_{l=i}^{k-1} \frac{a}{al+\nu}.$$

Since we have

$$\int_l^{l+1} \frac{a}{al+\nu} dt \leq \frac{a}{al+\nu} \leq \int_{l-1}^{l} \frac{a}{al+\nu} dt,$$

we find that

$$\ln\left(\frac{ak+\nu}{ai+\nu}\right) \leq \sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l} \leq \ln\left(\frac{a(k-1)+\nu}{a(i-1)+\nu}\right).$$

Consequently, letting $k \to \infty$, we get

$$-\sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l} = \ln\left(\frac{ai}{ak+\nu}\right).$$

So, we obtain

$$\lim_{k\to\infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}] = \lim_{k\to\infty} \frac{1}{k} \ln \mathbb{E}\left[e^{r\sum_{i=1}^{k} Y_i^{(j)}\left(1+(1+\eta)\ln\left(\frac{ai}{ak+\nu}\right)\right)}\right].$$

Hence we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}] &= \lim_{k\to\infty} \frac{1}{k} \left(\frac{ak+\nu}{a}\right) \sum_{i=1}^{k} \left(\frac{a}{ak+\nu}\right) \ln \mathbb{E}\left[e^{rY^{(j)}\left(1+(1+\eta)\ln\left(\frac{ai}{ak+\nu}\right)\right)}\right] \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \left(\frac{a}{ak+\nu}\right) \ln \mathbb{E}[e^{rY^{(j)}\left(1+(1+\eta)\ln\left(\frac{ai}{ak+\nu}\right)\right)}] \\
&= \int_0^1 dx \ln \mathbb{E}\left[e^{rY^{(j)}(1+(1+\eta)\ln x)}\right].
\end{aligned}$$

□

It appears that result (2.3.21) requires only the existence and the uniqueness of $\rho_\infty^{(j)}$ (since the condition $\rho_\infty^{(j)} < r^{(j)}$ is always fulfilled).

Equation (2.3.20) and so $\rho_\infty^{(j)}$ does not depend on market parameters μ, a and ν. It means that for $u \to \infty$, the logarithmic asymptotic behavior of $\psi_m^{(j)}(u)$ is insensitive to the initial premium and to the speed with which premiums $C_{k,\infty}$ become $\frac{Y_1^{(j)}+\cdots+Y_{k-1}^{(j)}}{k-1}(1+\eta)$.

As a particular case, let us consider $Y^{(j)} \sim \sum_{i=1}^{N} U_i$, where N is Poisson distributed with parameter β and the U_i's are i.i.d. random variables. We assume that N and the U_i's are independent. We have

$$\begin{aligned}
\mathbb{E}[e^{rY^{(j)}}] &= \mathbb{E}\left[e^{r\sum_{i=1}^{N} U_i}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{r\sum_{i=1}^{N} U_i}|N\right]\right] = \sum_{k=0}^{\infty} \mathbb{E}\left[e^{r\sum_{i=1}^{k} U_i}\right] \Pr[N=k] \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[e^{rU}\right]^k \Pr[N=k] = \sum_{k=0}^{\infty} e^{\ln \mathbb{E}\left[e^{rU}\right]^k} \Pr[N=k] = \sum_{k=0}^{\infty} e^{k\ln \mathbb{E}\left[e^{rU}\right]} \Pr[N=k] \\
&= \mathbb{E}\left[e^{N\ln \mathbb{E}[e^{rU}]}\right] = e^{\beta(\mathbb{E}\left[e^{rU}\right]-1)} \qquad (2.3.22)
\end{aligned}$$

since $\mathbb{E}\left[e^{rN}\right] = e^{\beta(e^r-1)}$. So, the equation (2.3.20) becomes

$$\int_0^1 dx \ln \mathbb{E}\left[e^{rY^{(j)}(1+(1+\eta)\ln x)}\right] = \beta \int_0^1 dx\, \mathbb{E}\left[e^{rU(1+(1+\eta)\ln x)}\right] - \beta = 0. \tag{2.3.23}$$

We recognize the equation (2.2) in Asmussen (1999). In fact, this is not surprizing since $\rho_\infty^{(j)}$ is insensitive to market parameters μ, a and ν.

The Lundberg coefficient $\rho_\infty^{(j)}$

Since $m = \infty$, we know that $\eta_m^{(j)} = \eta > 0$ for all j, i.e. $b_m = \emptyset$.

Proposition 2.3.3. *Assume that $\kappa''_\infty(r) > 0$. Then, $\rho_\infty^{(j)}$ exists and is unique.*

Proof. Obviously, we have $\kappa_\infty(0) = 0$. Denoting $Y^{(j)}(1 + (1+\eta)\ln x)$ by Z_x, we have

$$\begin{aligned}
\kappa'_\infty(r) &= \int_0^1 dx \frac{\mathbb{E}\left[Z_x e^{rZ_x}\right]}{\mathbb{E}\left[e^{rZ_x}\right]} = \int_0^1 dx \frac{\mathbb{E}\left[Z_x e^{rZ_x} I_{Z_x>0}\right] + \mathbb{E}\left[Z_x e^{rZ_x} I_{Z_x\le 0}\right]}{\mathbb{E}\left[e^{rZ_x}\right]} \\
&\ge \int_0^1 dx \frac{\mathbb{E}\left[Z_x e^{rZ_x} I_{Z_x>0}\right] + \mathbb{E}\left[Z_x I_{Z_x\le 0}\right]}{\mathbb{E}\left[e^{rZ_x}\right]}.
\end{aligned}$$

Hence we get

$$\kappa'_\infty(0) = \mathbb{E}[Y^{(j)}]\left(1 + (1+\eta)\int_0^1 dx \ln x\right) = -\mathbb{E}[Y^{(j)}]\eta < 0$$

since $\eta > 0$, and $\kappa'_\infty(r)$ becomes positive for large r. Now, since $\kappa''_\infty(r) > 0$ by assumption, the proof is over. □

Large deviations path to the ruin

As previously, we aim at knowing how ruin occurs for the case $m = \infty$. Of course, the ruin occurs roughly at time $\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor$. Now, the cumulant generating function $\gamma_{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor}(r)$ of $Z_{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor}$ is asymptotically changed from $\gamma_{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor}(r)$ to

$$\begin{aligned}
\gamma_{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor}(r + \rho_\infty^{(j)}) - \gamma_{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor}(\rho_\infty^{(j)}) &= -r(1+\eta)\mu \sum_{i=1}^{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor} (1 - z_{i,\infty}) \\
&+ \sum_{i=1}^{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor} \ln \frac{\mathbb{E}\left[e^{(r+\rho_\infty^{(j)})Y_i^{(j)}\left(1-(1+\eta)\sum_{l=i}^{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor-1}\frac{z_{l+1,\infty}}{l}\right)}\right]}{\mathbb{E}\left[e^{\rho_\infty^{(j)}Y_i^{(j)}\left(1-(1+\eta)\sum_{l=i}^{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor-1}\frac{z_{l+1,\infty}}{l}\right)}\right]}.
\end{aligned} \tag{2.3.24}$$

Consequently, given that ruin occurs, the claim size distribution $Y_k^{(j)}$ is this time exponentially tilted by the factor $\rho_\infty^{(j)}(k)$, where

$$\rho_\infty^{(j)}(k) = \left(1 - (1+\eta) \sum_{l=k}^{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor - 1} \frac{z_{l+1,\infty}}{l}\right) \rho_\infty^{(j)}. \tag{2.3.25}$$

Since $u \to \infty$, we get

$$\rho_\infty^{(j)}(k) = \left(1 + (1+\eta) \ln\left(\frac{k}{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor}\right)\right) \rho_\infty^{(j)}. \tag{2.3.26}$$

The factor $\rho_\infty^{(j)}(k)$ is negative as long as $k < \lfloor u/\kappa'_\infty(\rho_\infty^{(j)})\rfloor e^{-\frac{1}{1+\eta}}$.

We learn that this time, ruin is the consequence of atypically small claim amounts in a first time, followed by atypically large claim amounts in a second time. This result is similar to those obtained in Asmussen (1999) for his continuous time compound Poisson model. A typical path to ruin is shown in Figure 2.3.1.

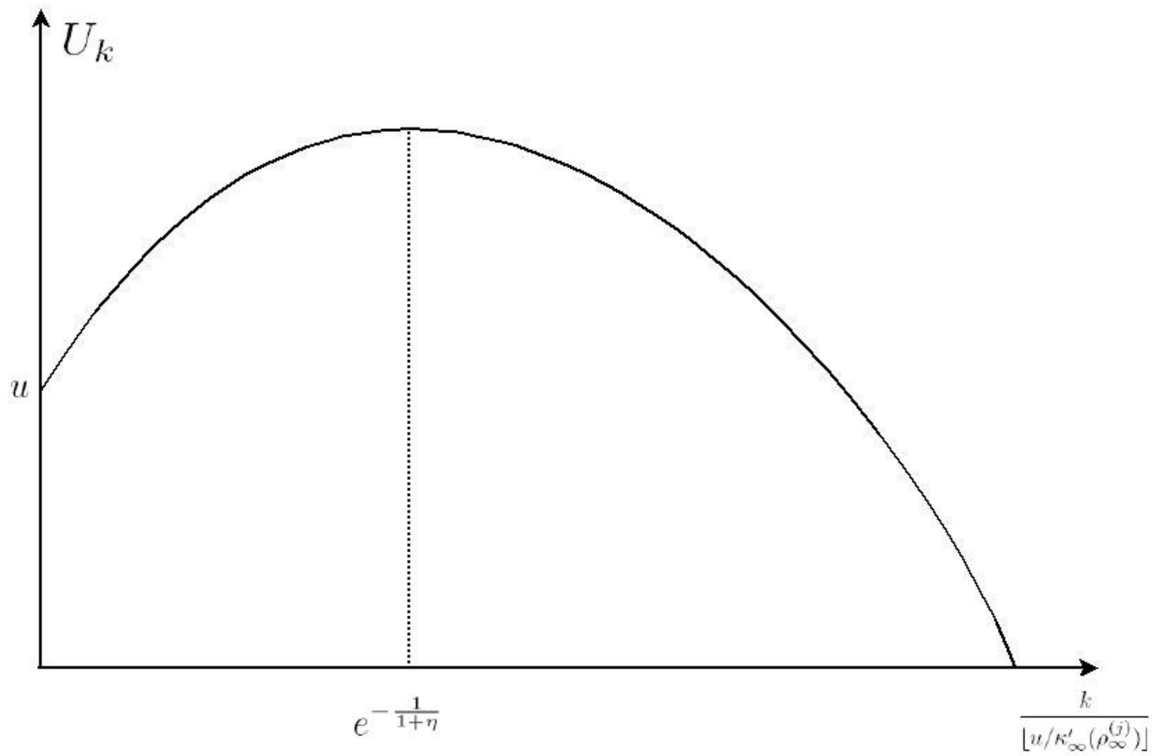

Figure 2.3.1:

2.3.3 Impact of the insurer's strategy m on $\psi_m^{(j)}(u)$ for large u

Comparison of $\psi_m^{(j)}(u)$ and $\psi_{m+1}^{(j)}(u)$ for large u (with $m < \infty$)

Let us assume that the insurer's strategy is to take an horizon of credibility $m < \infty$. The objective is to determine for large u if it is beneficial to take one year more to renew the premium. In other words, the objective is to compare $\psi_m^{(j)}(u)$ and $\psi_{m+1}^{(j)}(u)$ for large u.

For $j \in b_m$, we have the following Proposition:

Proposition 2.3.4.

$$\psi_{m+1}^{(j)}(u) \leq \psi_m^{(j)}(u) = 1 \textit{ for all } u. \tag{2.3.27}$$

Proof. It suffices to show that $b_{m+1} \subseteq b_m$. Let $j \in b_{m+1}$. By definition, $\eta_{m+1}^{(j)} \leq 0$. Let $h(x) = (1-x)\mu + x\mathbb{E}[Y^{(j)}]$. We have $h'(x) = \mathbb{E}[Y^{(j)}] - \mu > 0$ since $\eta_{m+1}^{(j)} \leq 0$ means $\mathbb{E}[Y^{(j)}] \geq (1+\eta)((1-z_{m+1})\mu + z_{m+1}\mathbb{E}[Y^{(j)}])$ and $\eta > 0$. Thus, since $z_{m+1} > z_m$, we have $\eta_m^{(j)} = (1+\eta)\frac{h(z_m)}{\mathbb{E}[Y^{(j)}]} - 1 < (1+\eta)\frac{h(z_{m+1})}{\mathbb{E}[Y^{(j)}]} - 1 = \eta_{m+1}^{(j)} \leq 0$, or equivalently, $j \in b_m$. □

This means that if $j \in b_m$, it is beneficial to pass from an horizon of credibility m to an horizon $m+1$, whatever the initial capital u.

Now, for $j \in \bar{b}_m$, it could seem a priori that the comparison of $\psi_m^{(j)}(u)$ and $\psi_{m+1}^{(j)}(u)$ for large u differs according to the involved portfolio j. Indeed, two opposite a priori behaviours seem to take shape. If $j \in g$, then $h'(x) \leq 0$ and consequently $\eta_m^{(j)} \geq \eta_{m+1}^{(j)}$, which seems to be in favor of $\psi_m^{(j)}(u) \leq \psi_{m+1}^{(j)}(u)$ whatever u and in particular for large u. But, on the other hand, if $j \in \bar{b}_m \setminus g$, then $h'(x) > 0$ and consequently $\eta_m^{(j)} < \eta_{m+1}^{(j)}$, which seems this time to be in favor of the inequality $\psi_{m+1}^{(j)}(u) \leq \psi_m^{(j)}(u)$ whatever u.

For $m < \bar{m}^{(j)}$, we have the next Proposition:

Proposition 2.3.5. *There exists a positive constant u_0 such that for all $u \geq u_0$, we have*

$$\psi_{m+1}^{(j)}(u) \leq \psi_m^{(j)}(u). \tag{2.3.28}$$

Proof. We know that $j \in \bar{b}_m$ and $m < \bar{m}^{(j)}$. Since $b_{m+1} \subseteq b_m$, $j \in \bar{b}_{m+1}$. Hence, we have to consider two cases.

Firstly, let us assume that $m+1 < \bar{m}^{(j)}$. By Property 2.3.2, our result is proved.

Secondly, let us assume that $m+1 \geq \bar{m}^{(j)}$. Hence, we know that

$$\lim_{u\to\infty} \frac{1}{u} \ln \psi_m^{(j)}(u) = -\rho_m^{(j)}$$

and

$$\limsup_{u\to\infty} \frac{1}{u} \ln \psi_{m+1}^{(j)}(u) \leq -r^{(j)},$$

and since $\rho_m^{(j)} < r^{(j)}$, this completes the proof. □

Thus, even if $j \in g$ and hence $\eta_m^{(j)} \geq \eta_{m+1}^{(j)}$, (2.3.28) holds for large u. In fact, this is not surprizing. With light-tailed claims, the event ruin occurs after many atypically large claims. At most m is large, at most the premium reacts favorably to this large claims, i.e. $\eta_{m+1}^{(j)}$ becomes larger than $\eta_m^{(j)}$ in such circumstances. This facts are confirmed by our analysis of the path leading to ruin. Indeed, it teaches us that the ruin is caused by a succession of many atypically large claims more dangerous as m is large since $\tilde{\rho}_m^{(j)}$ increases with m.

For $m \geq \bar{m}^{(j)}$, we can nothing conclude from this kind, since the only result we have is equation (2.3.17).

In conclusion, assuming that the strategy of the company is m, if $j \in b_m$, then it is always beneficial for large u to increase by 1 the horizon of credibility. Furthermore, if $j \in \bar{b}_m$, for $m < \bar{m}^{(j)}$, the same conclusion comes while for $m \geq \bar{m}^{(j)}$, we cannot certify that this is also the case.

Sub-optimal strategies m for large u

Let us investigate sub-optimal strategies m for large u.

Definition 2.3.5. A strategy m_1 is said to be sub-optimal compared to a strategy m_2 if for all portfolio j, we have $\psi_{m_2}^{(j)}(u) \leq \psi_{m_1}^{(j)}(u)$ for $u \to \infty$.

Definition 2.3.6. A strategy m_1 is said to be sub-optimal if there exists a strategy m_2 such that m_1 is sub-optimal compared to m_2.

We have the next Proposition:

Proposition 2.3.6. *$m < \min_{\{j:\bar{m}^{(j)}>0\}} \bar{m}^{(j)}$ like $m = \infty$ are sub-optimal.*

Proof. In view of Propositions 2.3.4 and 2.3.5, it is clear that $m < \min_{\{j:\bar{m}^{(j)}>0\}} \bar{m}^{(j)}$ is sub-optimal. For $m = \infty$, it is also obvious since $\rho_\infty^{(j)} < r^{(j)}$. $\square$

As we have seen, for a given portfolio j, increasing m decreases the number of "bad portfolios" and provides better premium adaptation to scenarios leading to ruin, at least as long as $m < \bar{m}^{(j)}$. An extension of this reasoning to all m would allow to think that all $m < \infty$ is sub-optimal compared to $m = \infty$. However, Proposition 2.3.6 shows that this is not the case. In fact, as $u \to \infty$, all $m < \infty$ becomes negligible compared to the ruin time T. Hence, the premiums will be able to react "quickly" (relatively to T) and "significantly" (at least for m "large" but finite) to a succession of large claim amounts, whatever the time where this claims occur and whatever the claim amounts observed before. But for $m = \infty$, obviously, this is not the case anymore. Indeed, as time passes, the premium reaction will become less and less effective to finish by not be able anymore to thwart several important successive claim amounts. This phenomenon is even truer if the claim amounts observed initially are small. This is well highlighted by the analysis of the path leading to ruin.

A probable extension of Proposition 2.3.6 would be then the following: $m = \infty$ is sub-optimal and all $m < \infty$ is sub-optimal compared to $m + 1$.

Numerical illustration

Let us consider an insurance market made up of 3 portfolios, i.e. $n = 3$. Let us assume that $Y^{(j)}$ ($j = 1, 2, 3$) are Exponentially distributed, with mean $1/\lambda_j$ and variance $1/\lambda_j^2$. We then have $\mu = (1/\lambda_1)p_1 + (1/\lambda_2)p_2 + (1/\lambda_3)p_3$, $a = (1/\lambda_1 - \mu)^2 p_1 + (1/\lambda_2 - \mu)^2 p_2 + (1/\lambda_3 - \mu)^2 p_3$ and $\nu = (1/\lambda_1^2)p_1 + (1/\lambda_2^2)p_2 + (1/\lambda_3^2)p_3$. Let us assume that $1/\lambda_1 = 3/4$, $1/\lambda_2 = 1$, $1/\lambda_3 = 5/4$, $p_1 = p_2 = 1/3$ and $\eta = 0.1$. We have $j = 1, 2 \in g$ and $j = 3 \in b_m$ as long as $m < 30$. The critical horizon of credibility $m^{(c)} = 250$. For our numerical purpose, let us compare strategies $m = 0, 2, 10, 250, 1000$ and ∞. Of course, for all u, $\psi_m^{(3)}(u) = 1$ for $m = 0, 2$ and 10. Carrying out 100 000 simulations on an horizon of 10 000 time periods, we obtain the results below, also shown on Figures 2.4.1 to 2.4.3.

$\psi_m^{(1)}(u)$											
u	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$m = 0$	43.758%	29.996%	20.566%	14.086%	9.844%	6.754%	4.604%	3.218%	2.268%	1.566%	1.120%
$m = 2$	44.438%	30.316%	20.572%	13.850%	9.474%	6.352%	4.232%	2.866%	1.952%	1.326%	0.890%
$m = 10$	45.892%	31.502%	21.348%	14.278%	9.568%	6.308%	4.072%	2.696%	1.772%	1.124%	0.724%
$m = 250$	46.936%	32.404%	22.034%	14.822%	9.920%	6.582%	4.252%	2.810%	1.828%	1.156%	0.758%
$m = 1000$	46.942%	32.408%	22.036%	14.827%	9.922%	6.583%	4.252%	2.810%	1.828%	1.156%	0.758%
$m = \infty$	46.948%	32.412%	22.042%	14.832%	9.924%	6.584%	4.252%	2.810%	1.828%	1.156%	0.758%
u	5.5	6	6.5	7	7.5	8	8.5	9	9.5	10	
$m = 0$	0.764%	0.540%	0.386%	0.260%	0.200%	0.148%	0.086%	0.060%	0.046%	0.036%	
$m = 2$	0.608%	0.408%	0.276%	0.214%	0.150%	0.088%	0.054%	0.042%	0.018%	0.010%	
$m = 10$	0.450%	0.308%	0.220%	0.148%	0.094%	0.046%	0.020%	0.010%	0.006%	0.000%	
$m = 250$	0.478%	0.318%	0.236%	0.160%	0.094%	0.050%	0.024%	0.008%	0.005%	0.000%	
$m = 1000$	0.478%	0.318%	0.236%	0.160%	0.094%	0.050%	0.024%	0.008%	0.004%	0.000%	
$m = \infty$	0.478%	0.318%	0.236%	0.160%	0.094%	0.050%	0.024%	0.008%	0.004%	0.000%	

$\psi_m^{(2)}(u)$											
u	0	1	2	3	4	5	6	7	8	9	10
$m = 0$	82.416%	69.122%	57.854%	48.526%	40.738%	34.052%	28.682%	24.060%	20.122%	16.894%	14.206%
$m = 2$	82.208%	67.124%	55.578%	45.370%	37.024%	30.182%	24.640%	19.976%	16.290%	13.186%	10.594%
$m = 10$	82.932%	68.260%	52.710%	40.310%	30.182%	22.138%	16.162%	11.560%	8.202%	5.890%	4.152%
$m = 250$	91.820%	79.506%	64.450%	49.480%	36.514%	25.882%	18.026%	12.112%	7.926%	5.256%	3.402%
$m = 1000$	92.023%	79.808%	64.905%	49.754%	36.901%	26.228%	18.273%	12.300%	8.109%	5.422%	3.529%
$m = \infty$	92.232%	80.108%	65.160%	50.184%	37.088%	26.424%	18.486%	12.528%	8.278%	5.542%	3.636%
u	11	12	13	14	15	16	17	18	19	20	
$m = 0$	11.792%	9.918%	8.234%	6.904%	5.742%	4.812%	4.070%	3.374%	2.834%	2.372%	
$m = 2$	8.688%	7.016%	5.686%	4.618%	3.746%	3.018%	2.462%	1.970%	1.598%	1.328%	
$m = 10$	2.866%	2.016%	1.452%	1.004%	0.738%	0.538%	0.374%	0.280%	0.188%	0.134%	
$m = 250$	2.190%	1.360%	0.860%	0.554%	0.380%	0.224%	0.140%	0.088%	0.060%	0.038%	
$m = 1000$	2.170%	1.304%	0.823%	0.528%	0.367%	0.203%	0.128%	0.084%	0.058%	0.037%	
$m = \infty$	2.286%	1.395%	0.865%	0.530%	0.365%	0.202%	0.127%	0.084%	0.058%	0.037%	

$\psi_m^{(3)}(u)$											
u	0	2	4	6	8	10	12	14	16	18	20
$m = 0$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%
$m = 2$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%
$m = 10$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%
$m = 250$	99.992%	99.648%	96.894%	88.370%	73.964%	58.720%	42.814%	29.850%	20.238%	13.378%	8.816%
$m = 1000$	99.994%	99.790%	97.490%	89.582%	75.532%	57.010%	41.046%	28.186%	18.682%	12.006%	7.636%
$m = \infty$	99.996%	99.810%	96.894%	88.370%	75.124%	57.010%	41.046%	28.186%	18.682%	12.006%	7.636%
u	22	24	26	28	30	32	34	36	38	40	
$m = 0$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	
$m = 2$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	
$m = 10$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	
$m = 250$	5.714%	3.660%	2.440%	1.566%	1.034%	0.704%	0.486%	0.330%	0.240%	0.142%	
$m = 1000$	4.708%	2.862%	1.784%	1.048%	0.644%	0.412%	0.250%	0.146%	0.098%	0.038%	
$m = \infty$	4.708%	2.862%	1.784%	1.048%	0.644%	0.412%	0.250%	0.146%	0.098%	0.038%	

On the one hand, asymptotically, whatever the portfolio j, we see that the larger m is, the smaller $\psi_m^{(j)}(u)$ is, even for $m \geq \bar{m}^{(j)}(\leq m^{(c)})$. This confirms our theoretical findings and intuition, namely that (also for $m \geq \min_{\{j:\bar{m}^{(j)}>0\}} \bar{m}^{(j)}$) m is sub-optimal compared to $m+1$.

On the other hand, for small initial capital u, a bigger value of m may this time increase in certain cases the ultimate ruin probability. Obviously, this observation is not unexpected.

Remark Clearly, the numerical illustrations can not highlight the fact that $m = \infty$ is sub-optimal.

2.4 Limit results for heavy-tailed claims

We have seen that in the case where the Lundberg coefficient exists, adjusting premium with credibility may increase this one. As typical sample paths for which ruin occurs exhibit a sequence of moderately large claims, there is quite often enough time to compensate early losses with credibility-adjusted future premiums. On the opposite, in the heavy-tailed case, it is often said that ruin is likely to be caused by one large claim (see for example Theorems 1.1 and 1.2 in Asmussen and Klüppelberg (1996) in the classical risk model). It is logical to think that credibility plays a less important role for heavy-tailed claim amount distributions. In this section, we focus on the regular variation case.

Definition 2.4.1. The cumulative distribution function F with support $(0, \infty)$ belongs to the regular variation class if for some $\alpha > 0$

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \quad for\ y > 0. \tag{2.4.1}$$

We note $F \in \mathcal{R}_{-\alpha}$. The convergence is uniform on each subset $y \in [y_0, \infty)$ $(0 < y_0 < \infty)$.

In the light-tailed case, we have studied the influence of m on the Lundberg coefficient

$$\rho_m^{(j)} = -\lim_{u\to\infty} \frac{1}{u} \ln \psi_m^{(j)}(u). \tag{2.4.2}$$

In our heavy-tailed case, the equivalent quantity to be studied is parameter

$$\gamma_m^{(j)} = -\lim_{u\to\infty} \frac{1}{\ln u} \ln \psi_m^{(j)}(u). \tag{2.4.3}$$

As we consider the case $F_j \in \mathcal{R}_{-\alpha^{(j)}}$, and since credibility adjustments involve implicitly that the two first moments of claim amounts exist, we restrict to the case $\alpha^{(j)} > 2$. Now, for $m < m^{(c)}$, we shall prove that, if $j \in \bar{b}_m$, then m does not modify the value of $\gamma_m^{(j)}$, which is in accordance with our intuition.

Theorem 2.4.2. *Suppose that the $Y_k^{(j)}$'s are regularly varying with index $\alpha^{(j)} > 2$. If $m < m^{(c)}$ and $j \in \bar{b}_m$, i.e.*

$$\eta > \frac{\mathbb{E}[Y^{(j)}]}{(1-z_m)\mu + z_m \mathbb{E}[Y^{(j)}]} - 1, \tag{2.4.4}$$

we get

$$\gamma_m^{(j)} = -\lim_{u\to\infty} \frac{1}{\ln u} \ln \psi_m^{(j)}(u) = \alpha^{(j)} - 1. \tag{2.4.5}$$

Proof. In the compound Poisson risk model with sub-exponential claims, Embrechts and Veraverbeke (1982) have shown that the (continuous-time) classical ruin probability $\psi(u)$ with initial surplus $u \geq 0$ (without credibility premium adjustment) behaves as $u \to \infty$ as

$$\psi(u) \sim \frac{\lambda}{c - \lambda\mu_W} \int_u^{+\infty} (1 - F_W(x))dx,$$

where c is the (continuous-time) premium income rate, λ is the intensity of the Poisson process, F_W is the cumulative distribution function of the individual claim amounts and μ_W is the expected individual claim amount. In the $\alpha^{(j)}$-regularly varying case with $\alpha^{(j)} > 2$ (this means that

$$1 - F_W(x) \sim x^{-\alpha^{(j)}} \text{ as } x \to \infty,$$

this corresponds to

$$\psi_{cont}(u) \sim \frac{\lambda}{c - \lambda\mu_W} \frac{1}{\alpha^{(j)} - 1} u^{-\alpha^{(j)}+1}.$$

This result may be adapted to the discrete-time model, with constant 1-period premium income and independent, identically distributed (aggregate) claim amounts on each period, with $\alpha^{(j)}$-regularly varying distribution: if the safety loading is positive, for $\alpha^{(j)} > 1$, there exists $C > 0$ such that the (discrete-time) probability of ruin $\psi_{disc}(u)$ satisfies $\psi_{disc}(u) \sim Cu^{-\alpha^{(j)}+1}$ as $u \to \infty$. To show this, one might for example use arguments developed in Rullière and Loisel (2004) and Lefèvre and Loisel (2008).

To show that there exist $C_1 > 0$ and a function ψ_1 such that

$$\psi_m^{(j)}(u) \geq \psi_1(u) \text{ and } \psi_1(u) \sim C_1 u^{-\alpha^{(j)}+1} \text{ as } u \to \infty,$$

let us consider a modified model, in which future earned premiums due to each claim are anticipated and earned at the claim instant: if claim amount $Y_k^{(j)}$ occurs at time k, the sum of future credibility premiums associated to $Y_k^{(j)}$ corresponds to

$$Y_k^{(j)}(1+\eta)\sum_{i=k+1}^{k+m}\bar{z}_{i,m}.$$

In our modified model, each claim amount $Y_k^{(j)}$ is replaced with

$$\widetilde{Y}_k^{(j)} = Y_k^{(j)}\left[1-(1+\eta)\sum_{i=k+1}^{k+m}\bar{z}_{i,m}\right],$$

and premium income

$$C_{k,m} = \left((1-z_{k,m})\mu + z_{k,m}\frac{\sum_{i=\max(k-m,1)}^{k-1}Y_i^{(j)}}{\min(m,k-1)}\right)(1+\eta)$$

is replaced with

$$\widetilde{C}_{k,m} = (1-z_{k,m})\mu(1+\eta).$$

For each sample path, if ruin occurs in the modified model, then it occurs as well in the initial model as credibility adjusted premiums are the same in both models but are received later in the modified model. Since from $k=m+1$, $\sum_{i=k+1}^{k+m}\bar{z}_{i,m} = z_m$, we have from Embrechts and Veraverbeke (1982) that the ruin probability $\psi_1(u)$ in the modified model satisfies $\psi_1(u) \sim C_1 u^{-\alpha^{(j)}+1}$ for some $C_1 > 0$ as $u \to \infty$ as long as the safety loading is positive in the modified model, which is guaranteed by condition (2.4.4).

To show that there exist $C_2 > 0$ and a function ψ_2 such that

$$\psi_m^{(j)}(u) \leq \psi_2(u) \text{ and } \psi_2(u) \sim C_2 u^{-\alpha^{(j)}+1} \text{ as } u \to \infty,$$

let us rewrite $\psi_m^{(j)}(u)$ as

$$\psi_m^{(j)}(u) = \Pr\left[\left(\exists 1 \leq i \leq m,\, U_{i,m}^{(j)} < 0\right) \text{ or } \left(\exists i > m,\, U_{i,m}^{(j)} < 0\right)\right].$$

This means that

$$\psi_m^{(j)}(u) \leq \Pr\left[\exists 1 \leq i \leq m,\, U_{i,m}^{(j)} < 0\right] + \Pr\left[\exists i > m,\, U_{i,m}^{(j)} < 0\right].$$

The first term is $O(u^{-\alpha^{(j)}})$ and so $o(u^{-\alpha^{(j)}+1})$. The second term

$$\Pr\left[\exists i > m,\, U_{i,m}^{(j)} < 0\right]$$

satisfies

$$
\begin{aligned}
&\Pr\left[\exists i > m,\, U_{i,m}^{(j)} < 0\right] \\
&\le \Pr\left[\exists i > m,\, \sum_{k=1}^{i} Y_k^{(j)} > u/2 + u/2 + z_m(1+\eta)\sum_{l=1}^{i-m} Y_l^{(j)} + \mu(1+\eta)(1-z_m)i\right] \\
&= \Pr\left[\exists i > m,\, \sum_{l=i-m+1}^{i} Y_l^{(j)} + (1 - z_m(1+\eta))\sum_{k=1}^{i-m} Y_k^{(j)} > u/2 + u/2 + (1+\eta)\mu(1-z_m)i\right] \\
&= \Pr\Bigg[\exists i > m,\, \left(\sum_{l=i-m+1}^{i} Y_l^{(j)}\right) + \left((1 - z_m(1+\eta))\sum_{k=1}^{i-m} Y_k^{(j)}\right) \\
&\qquad\qquad > \left(u/2 + \frac{\delta}{2} i\right) + \left(u/2 + \left((1+\eta)\mu(1-z_m) - \frac{\delta}{2}\right) i\right)\Bigg],
\end{aligned}
$$

where

$$
\delta = (1+\eta)\mu(1-z_m) - \mathbb{E}[Y^{(j)}](1 - z_m(1+\eta)) > 0
$$

from Condition (2.4.4).

Hence, since for any sequences of nonnegative r.v.'s $(X_i)_{i\ge1}$ and $(Z_i)_{i\ge1}$, and for any sequences of nonnegative numbers $(a_i)_{i\ge1}$ and $(b_i)_{i\ge1}$, we have

$$
\Pr[\exists i \ge 1, X_i + Z_i > a_i + b_i] \le \Pr[\exists i \ge 1, X_i > a_i] + \Pr[\exists j \ge 1, Z_j > b_j],
$$

we get that

$$
\begin{aligned}
&\Pr\left[\exists i > m,\, U_{i,m}^{(j)} < 0\right] \\
&\le \Pr\left[\exists i > m,\, \sum_{l=i-m+1}^{i} Y_l^{(j)} > u/2 + \frac{\delta}{2} i\right] \\
&\qquad + \Pr\left[\exists i > m,\, (1 - z_m(1+\eta))\sum_{k=1}^{i-m} Y_k^{(j)} > u/2 + \left((1+\eta)\mu(1-z_m) - \frac{\delta}{2}\right) i\right].
\end{aligned}
$$

The second term is equivalent to $C_3 u^{-\alpha^{(j)}+1}$ as $u \to \infty$ for some constant $C_3 > 0$, because it corresponds to a ruin probability with claim size distribution in $RV(\alpha^{(j)})$ and with a positive safety loading (from (2.4.4)). The first term

$$
\Pr\left[\exists i > m,\, \sum_{l=i-m+1}^{i} Y_l^{(j)} > u/2 + \frac{\delta}{2} i\right]
$$

satisfies

$$
\Pr\left[\exists i > m,\, \sum_{l=i-m+1}^{i} Y_l^{(j)} > u/2 + \frac{\delta}{2} i\right] \le \Pr\left[\exists i > m,\, m \max_{l=i-m+1}^{i} Y_l^{(j)} > u/2 + \frac{\delta}{2} i\right],
$$

which leads to

$$\begin{aligned}
\Pr\left[\exists i > m,\ \sum_{l=i-m+1}^{i} Y_l^{(j)} > u/2 + \frac{\delta}{2} i\right] &\leq \Pr\left[\exists i > m,\ mY_i^{(j)} > u/2 + \frac{\delta}{2}(i-m)\right] \\
&= \Pr\left[\exists i > 0,\ mY_i^{(j)} > u/2 + \frac{\delta}{2} i\right].
\end{aligned}$$

This may be rewritten as

$$\Pr\left[\exists i > m,\ \sum_{l=i-m+1}^{i} Y_l^{(j)} > u/2 + \frac{\delta}{2} i\right] \leq \Pr\left[\exists i > 0,\ Y_i^{(j)} > \frac{u}{2m} + \frac{\delta}{2m} i\right].$$

Obviously, we have

$$\Pr\left[\exists i > 0,\ Y_i^{(j)} > \frac{u}{2m} + \frac{\delta}{2m} i\right] \leq \sum_{i=1}^{\infty} \Pr\left[Y_i^{(j)} > \frac{u}{2m} + \frac{\delta}{2m} i\right].$$

Now, by Lemma 5.2 in Foss et al. (2009), we obtain

$$\sum_{i=1}^{\infty} \Pr\left[Y_i^{(j)} > \frac{u}{2m} + \frac{\delta}{2m} i\right] \sim \frac{2m}{\delta} \Pr\left[Y^{(j)} > \frac{u}{2m}\right] \text{ as } u \to \infty,$$

which completes the proof. □

Let us conclude in saying that, on the one hand, if $j \in b_m$ then $\psi_{m+1}^{(j)}(u) \leq \psi_m^{(j)}(u) = 1$ by Proposition 2.3.4 (clearly also valid for the heavy-tailed case), but on the other hand, this time, if $j \in \bar{b}_m$, then an increasing of the horizon of credibility has asymptotically no impact on $\psi_m^{(j)}(u)$ in first order, at least for $m < m^{(c)}$.

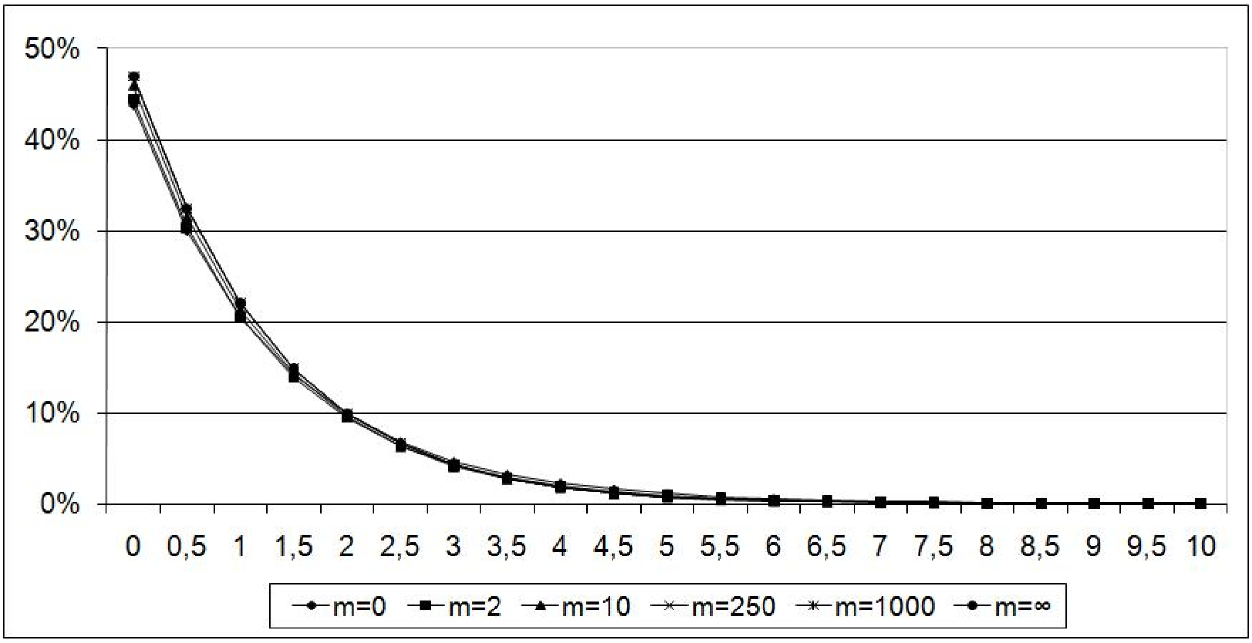

Figure 2.4.1: Numerical results for $\psi_m^{(1)}(u)$

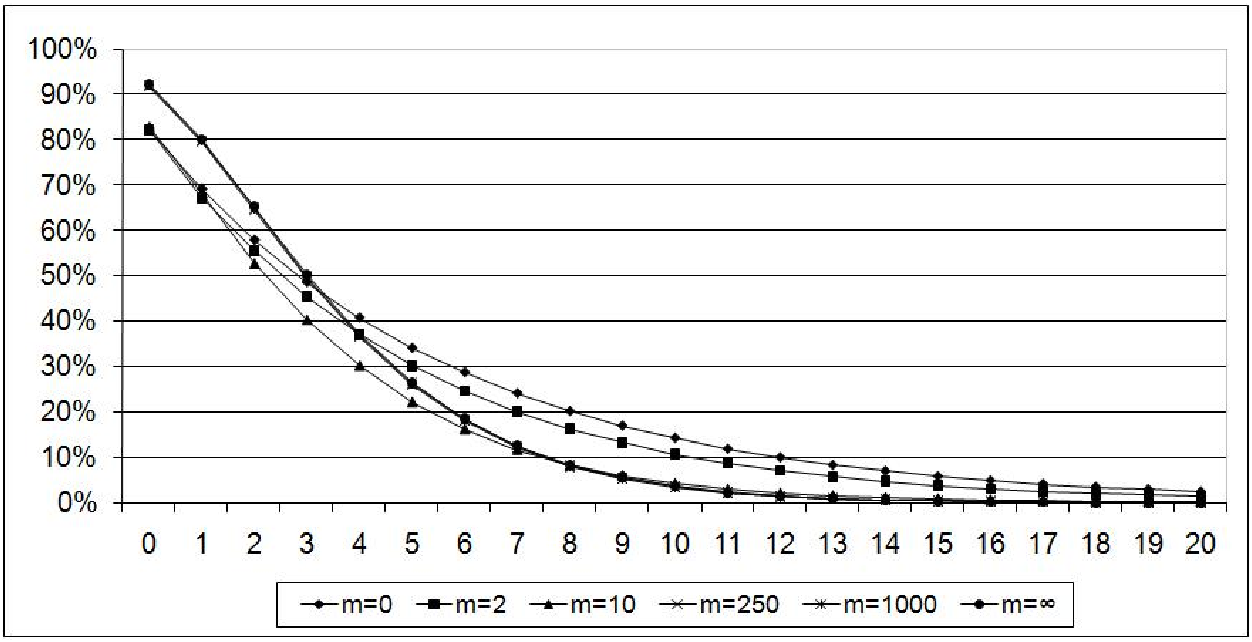

Figure 2.4.2: Numerical results for $\psi_m^{(2)}(u)$

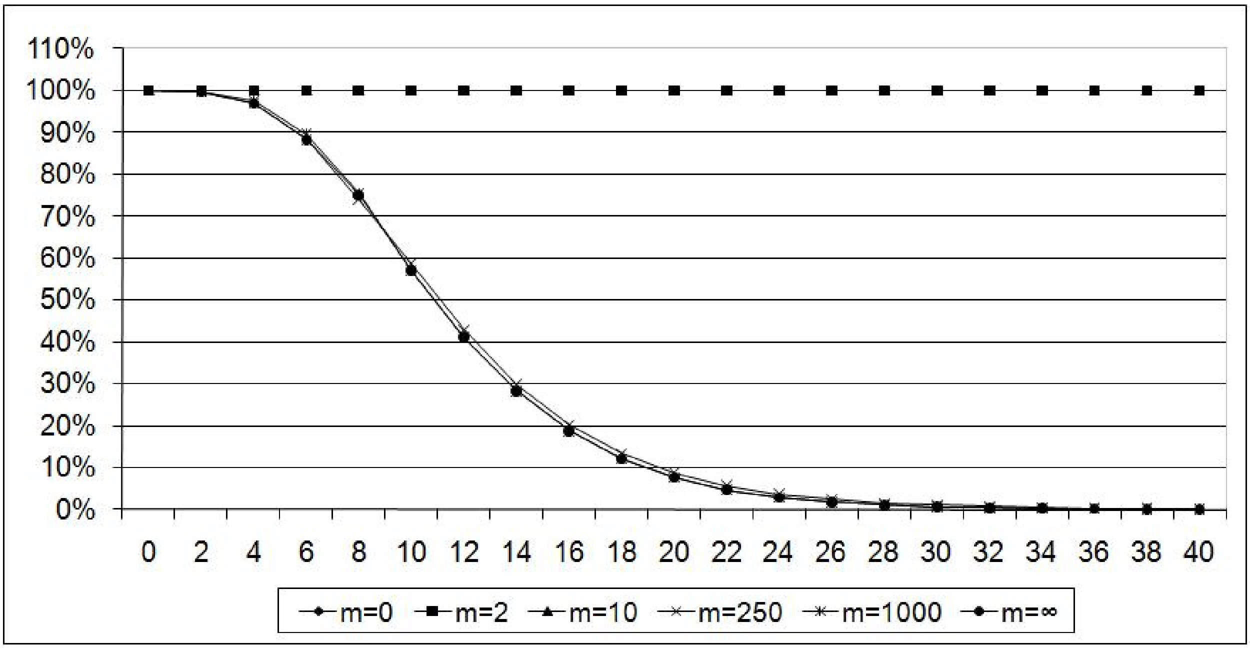

Figure 2.4.3: Numerical results for $\psi_m^{(3)}(u)$

Prospects and limitations

Closed portfolios assumption

An interesting extension would be to consider portfolios whose size fluctuates over time. The number of policyholders would depend for example on the premium level required by the insurer. Indeed, a high premium at a given time could make the policyholders leave the portfolio. However, due to the fact that these transition probabilities would be then correlated with the entire path of claim amounts, since the premiums are described by equation (2.2.6), such a generalization appears non-trivial. Anyway, clearly, in this new setting, techniques developed in this chapter would be inadequate. In the same idea, to illustrate this problem, we refer for example to Yang (2003), who had used for mathematical reasons, in spite of the practical irrelevance, a Markov process independent of the surplus process to represent the credit rating dynamics of an insurance company.

The paths leading to ruin

A priori, it seems surprising to observe asymptotically two distinct behaviors for paths leading to ruin according to whether m is finite or not. The reason comes from the fact that whatever $m < \infty$, even very large, the ratio m/u becomes finally zero for $u \to \infty$. That is why in our analysis the shape of paths for the finite case $m < \infty$ is different from the shape of paths for the infinite case $m = \infty$. Now, if we took a finite u arbitrarily large, we would certainly have, even from $m < \infty$ to $m = \infty$, a "continuous" behavior in the form of the trajectories leading to ruin with respect to m.

Chapter 3

Ruin problems under IBNR dynamics

This chapter is submitted.

Abstract

In this paper, we consider a discrete-time risk process allowing delay in claim settlement, which introduces a certain type of dependence in the process. From martingale theory, an expression for the ultimate ruin probability is obtained, and Lundberg-type inequalities are derived. The impact of delay in claim settlement is then investigated. To this end, a convex order comparison of the aggregate claim amounts is performed with the corresponding non-delayed risk model, and numerical simulations are carried out with Belgian market data.

3.1 Introduction

In classical risk theory, the stochastic process describing the aggregate claim amounts filed against the insurance company is assumed to have independent increments. In a variety of situations this independence assumption appears rather unrealistic. This is for instance the case when considering delay in claim reporting or settlement. The financial result of a given calendar year then depends on the result of one or several preceding years.

In the literature, some risk models allowing for delay in claim settlement have been discussed. Waters and Papatriandafylou (1985) considered a discrete-time risk model with possible delay in claim settlements and derived upper bounds on the ultimate ruin probability. Boogaert and Haezendonck (1989) studied the mathematical properties of the liability process with settlement delay within the framework of an economical environment.

Klüppelberg and Mikosch (1995) proposed to describe delayed claims in terms of Poisson shot noise processes. Ruin estimates for this model have been studied by Brémaud (2000), who used large deviations theory to give Cramér-Lundberg type approximations for the ultimate ruin probability, see also Macci and Torrisi (2004) and Macci et al. (2005). More generally, Ganesh and Torrisi (2006) considered shot-noise Cox processes and obtained ruin probability estimates for heavy-tailed claims.

A risk process combining delay in claim settlement and reserve-dependent premium rate can be found in Ganesh et al. (2007), where asymptotic estimates of the ultimate ruin probability as well as a most likely path leading to ruin are derived. Yuen et al. (2005) consider a compound Poisson risk model in which each claim induces an additional delayed claim (called by-claim).

In this paper, we study ruin problems in the presence of delay in claim settlement with a discrete-time risk model. Our approach is quite different from the one in Waters and Papatriandafylou (1985). In Waters and Papatriandafylou (1985), the claims development referring to a calendar year operates in a stochastic way, while the ruin event occurs when the reserves for IBNR (incurred but not reported) claims that the insurer needs exceed the amount of available cash.

Here, on the other hand, we use a setting that allows to explicitly use the Chain-Ladder development factors, widely used in practice to account for IBNR claims. In particular, we assume that the payment of each annual claim is spread over several years according to the development factors (where the spread is applied in a deterministic fashion). This leads to a moving average model for the annual claims that describes in a certain way the dependence of annual claims caused through delayed claims and that allows a transparent comparison of ruin probabilities with and without IBNR dynamics.

The paper is organized as follows. In Section 2, we describe the risk model used in this paper for the insurer's surplus. In Section 3, we use the martingale method to obtain an explicit expression for the ultimate ruin probability for light-tailed claims. A generalization to the case of a surplus dynamics with a constant interest rate is illustrated in the appendix. Subsequently Lundberg-type inequalities are derived. In Section 4, asymptotic estimates for ultimate ruin probabilities are determined in terms of the Lundberg coefficient, and an insensitivity property of our model is highlighted. This property is also proved for the heavy-tailed case. Finally, in Section 5, the impact of this kind of IBNR dynamics on ruin probability is investigated both for the light-tailed and heavy-tailed claim case. A comparison is made with the associated non-delayed risk model. We establish, for instance, a convex order relation between the IBNR aggregate claims process and the non-delayed aggregate claims process. Numerical illustrations are then performed with Belgian market data.

3.2 The Model

Let n be the number of years needed to settle all the claims related to a given accident year. Let X_i be the amount paid in calendar year i for the claims occurring in the same year. The X_is are assumed to be independent with common distribution function F_X, and we denote as X a generic random variable with distribution function F_X. The annual claim amount of the company is assumed to be given by

$$Y_i = X_i + X_{i-1}(\lambda_1 - 1) + \sum_{j=i-n+1}^{i-2} X_j\,(\lambda_{i-j} - 1) \prod_{l=1}^{i-j-1} \lambda_l, \tag{3.2.1}$$

where the λ_is are called development factors and are larger than 1. In practice, these factors are associated to the Chain-Ladder method for loss reserving. So, for our model, the surplus of the insurance company obeys the dynamics

$$U_k = u + k\,c - \sum_{i=1}^{k} Y_i, \tag{3.2.2}$$

with Y_i described by (3.2.1), where u is the initial amount of capital and c is the annual premium income.

Remark 1: Notice that Y_i can alternatively be written as

$$Y_i = \beta_1 S_i + \beta_2 S_{i-1} + \cdots + \beta_n S_{i-n+1}, \tag{3.2.3}$$

where S_i is the ultimate cost for the claims originated in year i and β_j is the proportion of S_i paid after j years. Some practitioners prefer the representation (3.2.3) rather than (3.2.1).

Assuming that the S_is are independent and identically distributed like S, the equivalence between (3.2.1) and (3.2.3) results from $S_i = X_i \prod_{j=1}^{n-1} \lambda_j$, $\beta_1 = \frac{1}{\prod_{j=1}^{n-1} \lambda_j}$ and

$$\begin{cases} \beta_2 = \frac{\lambda_1 - 1}{\lambda_1} & \text{if } n = 2 \\ \beta_j = \frac{(\lambda_{j-1}-1)\prod_{l=1}^{j-2}\lambda_l}{\prod_{l=1}^{n-1}\lambda_l} \quad j = 2, \ldots, n & \text{if } n > 2. \end{cases} \tag{3.2.4}$$

Remark 2: All the results obtained in the following can be extended without difficulty to the case of link factors depending on accident years, keeping the X_is independent and identically distributed. Such an extension is in line with practice, as speed in settlement may vary over time. The equation (3.2.1) then becomes

$$Y_i = X_i + X_{i-1}(\lambda_1^{(i-1)} - 1) + \sum_{j=i-n+1}^{i-2} X_j\,(\lambda_{i-j}^{(j)} - 1) \prod_{l=1}^{i-j-1} \lambda_l^{(j)}, \tag{3.2.5}$$

where the $\lambda_j^{(i)}$'s, $j = 1, \ldots, n-1$, are now the link ratios for accident year i, subject to the constraint $\prod_{j=1}^{n-1} \lambda_j^{(i)} = d$ for all i.

3.3 Ultimate ruin probability for light-tailed claims

3.3.1 Expression for the ultimate ruin probability

Define

$$\begin{cases} d_1 = \lambda_1 - 1 \\ d_i = (\lambda_i - 1)\prod_{l=1}^{i-1}\lambda_l \quad i = 2, \ldots, n-1, \end{cases} \tag{3.3.1}$$

and $d = 1 + d_1 + \cdots + d_{n-1}$. Clearly, $d = \prod_{j=1}^{n-1} \lambda_j$. Let us denote the ultimate ruin probability as

$$\psi(u, x_0, x_{-1}, \ldots, x_{-n+2}) = \\ \Pr[U_k < 0 \text{ for some } k | U_0 = u, X_0 = x_0, X_{-1} = x_{-1}, \ldots, X_{-n+2} = x_{-n+2}]. \tag{3.3.2}$$

We are now in a position to establish the following result. Its proof is inspired by Gerber (1982).

In the following, for a random variable Z, we will denote by μ_Z and σ_Z^2 its mean and variance, respectively. Furthermore, its distribution function will be denoted by $F_Z = 1 - \overline{F_Z}$.

Theorem 3.3.1. *Assume that there exists a constant $\rho > 0$ such that*

$$e^{-\frac{\rho}{d}c}\,\mathbb{E}[e^{\rho X}] = 1. \tag{3.3.3}$$

Then we have

$$\psi(u, x_0, x_{-1}, \ldots, x_{-n+2}) \leq \frac{e^{-\frac{\rho}{d}\hat{u}}}{\mathbb{E}[e^{-\frac{\rho}{d}\hat{U}_T} | T < \infty]}, \tag{3.3.4}$$

where $\hat{U}_k = U_k - \sum_{l=0}^{n-2} \alpha_l X_{k-l}$ (with $\alpha_l = d_{l+1} + \cdots + d_{n-1}$), $\hat{u} = \hat{U}_0$ and $T = \inf\{k | U_k < 0\}$.

Proof. Let Φ_k be the sigma-algebra generated by $\{X_i, i = 1, 2, \ldots, k\}$. We then have

$$\mathbb{E}[e^{-\frac{\rho}{d}\hat{U}_k}|\Phi_{k-1}] = \mathbb{E}[e^{-\frac{\rho}{d}(U_k-\sum_{l=0}^{n-2}\alpha_l X_{k-l})}|\Phi_{k-1}] = \mathbb{E}[e^{-\frac{\rho}{d}(U_{k-1}+c-Y_k-\sum_{l=0}^{n-2}\alpha_l X_{k-l})}|\Phi_{k-1}].$$

Now

$$Y_k + \sum_{l=0}^{n-2}\alpha_l X_{k-l} = X_k + \sum_{j=1}^{n-1} d_j X_{k-j} + \sum_{l=0}^{n-2}\alpha_l X_{k-l} = dX_k + \sum_{l=0}^{n-2}\alpha_l X_{k-1-l}.$$

Hence,

$$\begin{aligned}
\mathbb{E}[e^{-\frac{\rho}{d}\hat{U}_k}|\Phi_{k-1}] &= \mathbb{E}[e^{-\frac{\rho}{d}(U_{k-1}+c-dX_k-\sum_{l=0}^{n-2}\alpha_l X_{k-1-l})}|\Phi_{k-1}] \\
&= e^{-\frac{\rho}{d}c}\,\mathbb{E}[e^{\rho X}]\,\mathbb{E}[e^{-\frac{\rho}{d}(U_{k-1}-\sum_{l=0}^{n-2}\alpha_l X_{k-1-l})}|\Phi_{k-1}] \\
&= e^{-\frac{\rho}{d}\hat{U}_{k-1}} \text{ by assumption.}
\end{aligned}$$

This shows that the stochastic process $\{e^{-\frac{\rho}{d}\hat{U}_k}, k \geq 1\}$ is a martingale.

Let m be a positive integer. By the Optional Stopping Theorem (considering the stopping time $T \wedge m = \min\{T, m\}$), we get

$$\begin{aligned}
e^{-\frac{\rho}{d}\hat{U}_0} &= \mathbb{E}[e^{-\frac{\rho}{d}\hat{U}_{T\wedge m}}] \\
&= \mathbb{E}[e^{-\frac{\rho}{d}\hat{U}_T} I_{T\leq m}] + \mathbb{E}[e^{-\frac{\rho}{d}\hat{U}_m} I_{T>m}]. \\
&\geq \mathbb{E}[e^{-\frac{\rho}{d}\hat{U}_T}|T \leq m]\ \Pr[T \leq m]
\end{aligned}$$

Letting $m \to \infty$ gives

$$e^{-\frac{\rho}{d}\hat{U}_0} \geq \mathbb{E}[e^{-\frac{\rho}{d}\hat{U}_T}|T < \infty]\,\psi(u, x_0, \ldots, x_{-n+2}),$$

which completes the proof. □

This result can easily be generalized to the case of a constant compound interest rate $i \geq 0$, as is shown in the appendix.

Corollary 3.3.2. *Under the assumptions of Theorem 3.3.1, if there exists a positive constant b such that* $\Pr[X < b] = 1$, *then we have the equality*

$$\psi(u, x_0, x_{-1}, \ldots, x_{-n+2}) = \frac{e^{-\frac{\rho}{d}\hat{u}}}{\mathbb{E}[e^{-\frac{\rho}{d}\hat{U}_T}|T < \infty]}. \tag{3.3.5}$$

Proof. Let $\alpha = \sum_{l=0}^{n-2} \alpha_l$. Since $\Pr[X < b] = 1$, we have

$$e^{-\frac{\varrho}{d}\hat{U}_m}\, I_{T>m} < e^{\frac{\varrho}{d}\, b\, \alpha}.$$

Furthermore, the random variables $e^{-\frac{\varrho}{d}\hat{U}_m}\, I_{T>m}$ converge to zero pointwise for $m \to \infty$. Hence, we get

$$\lim_{m\to\infty} \mathbb{E}[e^{-\frac{\varrho}{d}\hat{U}_m} I_{T>m}] = 0.$$

□

It is well-known that if the security loading $\eta = \frac{c}{d\mu_X} - 1$ is positive and if $\mathbb{E}[e^{rX}]$ exists for all r in the neighborhood of the origin and is steep (which in particular implies light-tailed claims), the Lundberg equation $e^{-r\mu_X(1+\eta)}\mathbb{E}[e^{rX}] = 1$ admits only one positive solution.

3.3.2 Lundberg-type inequalities for the ultimate ruin probability

Lundberg-type inequalities can be derived from Theorem 3.3.1. Note that from $\hat{U}_T < 0$ we have $\mathbb{E}[e^{-\frac{\varrho}{d}\hat{U}_T}|T < \infty] > 1$. Hence, we get the next Proposition.

Proposition 3.3.1. *Under the assumptions of Theorem 3.3.1, we have*

$$\psi(u, x_0, \ldots, x_{-n+2}) \le e^{-\frac{\varrho}{d}\,\hat{u}}. \tag{3.3.6}$$

Now, let us define θ as the largest positive integer such that $\Pr[T \ge \theta] = 1$. Of course, when there is no constant b such that $\Pr[X < b] = 1$, we always have $\theta = 1$. Then, the following Proposition, based on (3.3.4), improves the upper bound (3.3.6).

Proposition 3.3.2. *Under the assumptions of Theorem 3.3.1, we have*

$$\psi(u, x_0, \ldots, x_{-n+2}) \le e^{-\frac{\varrho}{d}\,(\bar{u}+\alpha\mu_X)}. \tag{3.3.7}$$

where

$$\bar{u} = \begin{cases} u - \sum_{l=0}^{n-3} \alpha_l \tilde{x}_{-l} - \alpha_{n-2} x_{-n+2} & \text{if } \theta \le n-2, \\ \hat{u} & \text{otherwise,} \end{cases} \tag{3.3.8}$$

with $\tilde{x}_{-l} = \max(\mu_X, x_{-l})$.

Proof. As $e^{-\frac{\varrho}{d}U_T} > 1$, we have

$$\psi(u, x_0, \ldots, x_{-n+2}) \le \frac{e^{-\frac{\varrho}{d}\hat{u}}}{\mathbb{E}[e^{\frac{\varrho}{d}\sum_{l=0}^{n-2}\alpha_l X_{T-l}}|T < \infty]} \le \frac{e^{-\frac{\varrho}{d}\hat{u}}}{e^{\frac{\varrho}{d}\sum_{l=0}^{n-2}\alpha_l\, \mathbb{E}[X_{T-l}|T<\infty]}}$$

by Jensen's inequality. Since $\mathbb{E}[X_i|T < \infty] \ge \min(x_{\theta-n+2}, \ldots, x_0)$ for $\theta - n + 2 \le i \le 0$, we have $\mathbb{E}[X_{T-l}|T < \infty] \ge \gamma$ for $l = 0, \ldots, n-2$, where

$$\gamma = \begin{cases} \min(x_{\theta-n+2}, \ldots, x_0, \mu_X) & \text{if } \theta \le n-2 \\ \mu_X & \text{otherwise} \end{cases}.$$

Hence

$$\psi(u, x_0, \ldots, x_{-n+2}) \leq e^{-\frac{\rho}{d}(\hat{u}+\alpha\gamma)}, \tag{3.3.9}$$

where we recognize (3.3.7) for $\theta > n-2$. If $\theta \leq n-2$, it suffices to see that $\psi(u, x_0, \ldots, x_{-n+2}) \leq \psi(u, \tilde{x}_0, \ldots, \tilde{x}_{-n+3}, x_{-n+2})$. □

Note that for $\theta \leq n-2$, we have from (3.3.7) and (3.3.9)

$$\psi(u, x_0, \ldots, x_{-n+2}) \leq \min(e^{-\frac{\rho}{d}(\hat{u}+\alpha\gamma)}, e^{-\frac{\rho}{d}(\bar{u}+\alpha\mu_X)}). \tag{3.3.10}$$

But since $\bar{u} \leq \hat{u}$ and $\gamma \leq \mu_X$, we always have $e^{-\frac{\rho}{d}(\hat{u}+\alpha\gamma)} \geq e^{-\frac{\rho}{d}(\bar{u}+\alpha\mu_X)}$.
Clearly, the upper bound $e^{-\frac{\rho}{d}(\bar{u}+\alpha\mu_X)}$ is tighter than $e^{-\frac{\rho}{d}\hat{u}}$.

Based on a recursive formula for finite-time ruin probabilities, one can derive another Lundberg-type bound for $\psi(u, x_0, \ldots, x_{-n+2})$. Denote by $\psi_k(u, x_0, \ldots, x_{-n+2})$ the ruin probability up to time k. Clearly, for $k = 1, 2, \ldots$, we have

$$\begin{aligned}\psi_{k+1}(u, x_0, \ldots, x_{-n+2}) &= \bar{F}_X(u + c - \sum_{i=1}^{n-1} d_i\, x_{1-i}) \\ &\quad + \int_0^{u+c-\sum_{i=1}^{n-1} d_i\, x_{1-i}} \psi_k(u + c - x - \sum_{i=1}^{n-1} d_i\, x_{1-i}, x, x_0, \ldots, x_{-n+3})\, dF_X(x).\end{aligned} \tag{3.3.11}$$

Hence, we have the next upper bound.

Proposition 3.3.3. *Assume that*

$$e^{\frac{\rho}{d}\sum_{i=1}^{n-2} d_{i+1}x_{1-i}}\, \mathbb{E}[e^{\frac{\rho}{d}X(1+d_1)} I_{[X \leq u+c-\sum_{i=1}^{n-1} d_i\, x_{1-i}]}] \leq \mathbb{E}[e^{\rho X} I_{[X \leq u+c-\sum_{i=1}^{n-1} d_i\, x_{1-i}]}]. \tag{3.3.12}$$

Then,

$$\psi(u, x_0, \ldots, x_{-n+2}) \leq \beta e^{-\frac{\rho}{d}(u-\sum_{i=1}^{n-1} d_i\, x_{1-i})}, \tag{3.3.13}$$

where

$$\beta^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{\rho y} dF_X(y)}{e^{\frac{\rho}{d}t}\bar{F}_X(t)}. \tag{3.3.14}$$

Proof. For any $x \geq 0$, we have

$$\begin{aligned}\bar{F}_X(x) &= \left(\frac{\int_x^\infty e^{\rho y} dF_X(y)}{e^{\frac{\rho}{d}x}\bar{F}_X(x)}\right)^{-1} e^{-\frac{\rho}{d}x} \int_x^\infty e^{\rho y} dF_X(y) \\ &\leq \beta e^{-\frac{\rho}{d}x} \int_x^\infty e^{\rho y} dF_X(y) \\ &\leq \beta e^{-\frac{\rho}{d}x} \mathbb{E}[e^{\rho X}].\end{aligned}$$

Then,

$$\begin{aligned}\psi_1(u, x_0, \ldots, x_{-n+2}) &\leq \beta e^{-\frac{\rho}{d}(u+c-\sum_{i=1}^{n-1} d_i\, x_{1-i})}\mathbb{E}[e^{\rho X}] \\ &= \beta e^{-\frac{\rho}{d}(u-\sum_{i=1}^{n-1} d_i\, x_{1-i})}.\end{aligned}$$

Now, let us assume that

$$\psi_k(u, x_0, \ldots, x_{-n+2}) \leq \beta e^{-\frac{\rho}{d}(u-\sum_{i=1}^{n-1} d_i\, x_{1-i})}.$$

Thus

$$\begin{aligned}&\psi_k(u + c - x - \sum_{i=1}^{n-1} d_i\, x_{1-i}, x, x_0, \ldots, x_{-n+3}) \\ &\qquad\qquad \leq \beta e^{-\frac{\rho}{d}(u+c-x-\sum_{i=1}^{n-1} d_i\, x_{1-i}-d_1 x-\sum_{i=1}^{n-2} d_{i+1}\, x_{1-i})} \\ &\qquad\qquad = \beta e^{-\frac{\rho}{d}(u+c-\sum_{i=1}^{n-1} d_i\, x_{1-i})}\, e^{\frac{\rho}{d}(x(1+d_1)+\sum_{i=1}^{n-2} d_{i+1}\, x_{1-i})}.\end{aligned}$$

Furthermore,

$$\bar{F}_X(u + c - \sum_{i=1}^{n-1} d_i\, x_{1-i}) \leq \beta e^{-\frac{\rho}{d}(u+c-\sum_{i=1}^{n-1} d_i\, x_{1-i})} \int_{u+c-\sum_{i=1}^{n-1} d_i\, x_{1-i}}^{\infty} e^{\rho x}\, dF_X(x)$$

Consequently, by (3.3.11), we have

$$\begin{aligned}&\psi_{k+1}(u, x_0, \ldots, x_{-n+2}) \leq \beta e^{-\frac{\rho}{d}(u+c-\sum_{i=1}^{n-1} d_i\, x_{1-i})} \\ &\quad\left(\int_{u+c-\sum_{i=1}^{n-1} d_i\, x_{1-i}}^{\infty} e^{\rho x} dF_X(x) + \int_0^{u+c-\sum_{i=1}^{n-1} d_i\, x_{1-i}} e^{\frac{\rho}{d}(x(1+d_1)+\sum_{i=1}^{n-2} d_{i+1}\, x_{1-i})} dF_X(x)\right)\end{aligned}$$

By (3.3.12) we then get

$$\psi_{k+1}(u, x_0, \ldots, x_{-n+2}) \leq \beta e^{-\frac{\rho}{d}(u-\sum_{i=1}^{n-1} d_i\, x_{1-i})}.$$

Letting $k \to \infty$, we obtain (3.3.13). □

Obviously, for $n = 2$, the assumption (3.3.12) is fulfilled.

This new bound can be more precise than (3.3.7). For example, if $X \sim Exp(\tau)$, then

$$\beta^{-1} = \inf_{t \geq 0} \left(\frac{\tau}{\tau - \rho}\right) e^{\rho(1-\frac{1}{d})t} = \frac{\tau}{\tau - \rho}. \tag{3.3.15}$$

So, for instance, with $n = 2$, (3.3.7) becomes

$$e^{-\frac{\rho}{d}\left(u-\frac{d_1}{\tau}\right)} \tag{3.3.16}$$

while (3.3.13) is

$$\left(\frac{\tau - \rho}{\tau}\right) e^{-\frac{\rho}{d}(u-d_1 x_0)}, \tag{3.3.17}$$

which for some values of parameters τ, η (and hence ρ), d_1 and x_0, is larger than (3.3.16).

3.4 Asymptotic behavior for the ultimate ruin probability

3.4.1 The light-tailed case

The following result makes clear that the exponential decay rate of the numerator in (3.3.4) provides the significant asymptotic behavior in u of $\psi(u, x_0, \ldots, x_{-n+2})$.

Theorem 3.4.1. *Assume that the assumptions of Theorem 3.3.1 are fulfilled. Hence, given that for all m, the sum of the first m annual losses $Z_m = \sum_{i=1}^{m}(Y_i - c)$ possesses a finite moment-generating function $\mathbb{E}[e^{rZ_m}]$ for $0 < r < r_0$, where $r_0 > \frac{\rho}{d}$, we have*

$$\lim_{u\to\infty} \frac{1}{u} \ln \psi(u, x_0, \ldots, x_{-n+2}) = -\frac{\rho}{d}. \tag{3.4.1}$$

Proof. By the Gärtner-Ellis Theorem from large deviations (cf. Glynn and Whitt (1994), Nyrhinen (1998) or Müller and Pflug (2001)), one only has to show that:

(A1) $\kappa(r) := \lim_{m\to\infty} \frac{1}{m}\kappa_m(r) = \lim_{m\to\infty} \frac{1}{m} \ln \mathbb{E}[e^{rZ_m}]$ exists for $0 < r < r_0$,

(A2) $\kappa(\frac{\rho}{d}) = 0$ and $\kappa'(\frac{\rho}{d}) > 0$.

First of all, we prove that

$$\lim_{m\to\infty} \frac{1}{m} \ln \mathbb{E}[e^{rZ_m}] = -r\,c + \ln \mathbb{E}[e^{rdX}].$$

Obviously, we have

$$\lim_{m\to\infty} \frac{1}{m} \ln \mathbb{E}[e^{rZ_m}] = -r\,c + \lim_{m\to\infty} \frac{1}{m} \ln \mathbb{E}[e^{r\sum_{i=1}^{m} Y_i}].$$

Now

$$\lim_{m\to\infty} \sum_{i=1}^{m} Y_i = d \lim_{m\to\infty} \sum_{i=1}^{m} X_i + \sum_{i=1}^{n-1}\sum_{k=i}^{n-1} d_k\, x_{i-k}.$$

Hence,

$$\begin{aligned}
\lim_{m\to\infty} \frac{1}{m} \ln \mathbb{E}[e^{r\sum_{i=1}^{m} Y_i}] &= \lim_{m\to\infty} \frac{1}{m} \ln \mathbb{E}[e^{rd\sum_{i=1}^{m} X_i}\, e^{r\sum_{i=1}^{n-1}\sum_{k=i}^{n-1} d_k\, x_{i-k}}] \\
&= \lim_{m\to\infty} \frac{1}{m} \left(r \sum_{i=1}^{n-1}\sum_{k=i}^{n-1} d_k\, x_{i-k} \right) + \lim_{m\to\infty} \frac{1}{m} \ln \prod_{i=1}^{m} \mathbb{E}[e^{rdX_i}] \\
&= \lim_{m\to\infty} \frac{1}{m} \ln \mathbb{E}[e^{rdX}]^m = \ln \mathbb{E}[e^{rdX}].
\end{aligned}$$

Now, notice that $\lim_{m\to\infty} \frac{1}{m} \ln \mathbb{E}[e^{rZ_m}] = \ln h(r)$, where $h(r) := e^{-r\,c}\mathbb{E}[e^{rdX}]$. Since $h(\frac{\rho}{d}) = 1$ by (3.3.3), we have $\kappa(\frac{\rho}{d}) = \ln h(\frac{\rho}{d}) = 0$ and $\kappa'(\frac{\rho}{d}) > 0$. □

Given that ruin occurs, one can also look at the path leading to ruin. Using large deviations results of Glynn and Whitt (1994), we can see that for large u, ruin occurs roughly at time $\lfloor u/\kappa'(\frac{\rho}{d})\rfloor$. Furthermore, the cumulant generating function of $Z_{\lfloor u/\kappa'(\frac{\rho}{d})\rfloor}$ is asymptotically changed from $\kappa_{\lfloor u/\kappa'(\frac{\rho}{d})\rfloor}(r)$ to

$$\begin{aligned}\kappa_{\lfloor u/\kappa'(\frac{\rho}{d})\rfloor}\left(r+\frac{\rho}{d}\right)-\kappa_{\lfloor u/\kappa'(\frac{\rho}{d})\rfloor}\left(\frac{\rho}{d}\right) &= -r\,\lfloor u/\kappa'\big(\frac{\rho}{d}\big)\rfloor\, c + r\sum_{i=1}^{n-1}\sum_{k=i}^{n-1} d_k\, x_{i-k} \\ &\quad +(\lfloor u/\kappa'\big(\frac{\rho}{d}\big)\rfloor - n+1)\ln\frac{\mathbb{E}[e^{(r+\frac{\rho}{d})dX}]}{\mathbb{E}[e^{\rho X}]} \\ &\quad +\sum_{i=1}^{n-1}\ln\frac{\mathbb{E}[e^{(r+\frac{\rho}{d})(1+\sum_{j=1}^{i-1}d_j)X}]}{\mathbb{E}[e^{\frac{\rho}{d}(1+\sum_{j=1}^{i-1}d_j)X}]}.\end{aligned} \tag{3.4.2}$$

Consequently, given that ruin occurs, on an asymptotic level the claim size distribution X is exponentially tilted by the (time-dependent) factor $\rho(k)$, $k=1,2,\ldots,\lfloor u/\kappa'(\frac{\rho}{d})\rfloor$, with

$$\rho(k)=\begin{cases}\rho, & \text{for } k=1,\ldots,\lfloor u/\kappa'(\frac{\rho}{d})\rfloor-n+1\\ \frac{\rho}{d}(1+\sum_{j=1}^{\lfloor u/\kappa'(\frac{\rho}{d})\rfloor-k}d_j), & \text{for } k=\lfloor u/\kappa'(\frac{\rho}{d})\rfloor-n+2,\ldots,\lfloor u/\kappa'(\frac{\rho}{d})\rfloor\end{cases}. \tag{3.4.3}$$

This suggests that a sample path leading to ruin asymptotically locally (at time k) has a drift $\delta(k)$ given by

$$\delta(k)=c-d\,\frac{\mathbb{E}[X e^{\rho(k)X}]}{\mathbb{E}[e^{\rho(k)X}]}<0. \tag{3.4.4}$$

Notice that the drift can be interpreted as constant during the time period $(1,\lfloor u/\kappa'(\frac{\rho}{d})\rfloor - n+1)$ and increasing at the end, i.e. during the time period $(\lfloor u/\kappa'(\frac{\rho}{d})\rfloor-n+2,\lfloor u/\kappa'(\frac{\rho}{d})\rfloor)$. However, since $u\to\infty$, this last time period is negligible compared to the first one.

Define the surplus process

$$\tilde{U}_k=\tilde{u}+k\,c-d\sum_{i=1}^{k}X_i, \tag{3.4.5}$$

which for $\tilde{u}=\hat{u}$ corresponds to the associated non-delayed classical version of (3.2.2). Denote by $\widetilde{\psi}(\tilde{u})$ the corresponding ultimate ruin probability. Clearly, the Lundberg coefficient $\tilde{\rho}$ of this associated non-delayed process is equal to $\frac{\rho}{d}$. Thus, the logarithmic asymptotic behavior of $\widetilde{\psi}(\tilde{u})$ and $\psi(u,x_0,\ldots,x_{-n+2})$ is of the same order. This consitutes an insensitivity property of the model. Due to the short-range dependence induced by the claim settlement, only the distribution of dX is important asymptotically, and not the way to settle the claims (cf. Brémaud (2000), Yuen et al. (2005) and Ganesh and Torrisi (2006) for a similar conclusion for other delayed-claims risk models).

Remark. A large deviation estimate (consistent with (3.4.1)) can be obtained for the finite-time ruin probability for our risk process in the presence of IBNR claims. Indeed, let $\psi(u, M) = \Pr[T < M]$ denote the probability of ruin up to time horizon M and define for $a > 0$ the quantity r_a as the unique solution of $\kappa'(r_a) = \frac{1}{a}$. Then, invoking Nyrhinen (1998),

$$\lim_{u\to\infty} \frac{1}{u} \ln \psi(u, a\,u) = -\rho_a, \tag{3.4.6}$$

with

$$\rho_a = \begin{cases} r_a - a\,\kappa(r_a), & a < \frac{1}{\kappa'(\frac{\rho}{d})} \\ \frac{\rho}{d}, & a \geq \frac{1}{\kappa'(\frac{\rho}{d})} \end{cases}. \tag{3.4.7}$$

3.4.2 The heavy-tailed case

Let us now consider subexponential claim distributions of regularly varying type. By definition, a d.f. F on $\mathbb{R}^+$ belongs to the class R of regularly varying distributions ($F \in R(-\delta)$) if there exists some constant $0 \leq \delta < \infty$ and a positive slowly varying function $L(x)$ such that

$$\overline{F}(x) = x^{-\delta} L(x). \tag{3.4.8}$$

Let $\overline{B_0}(x) = \frac{1}{\mu_Z} \int_x^\infty \overline{F_Z}(z)dz$, where $Z = dX - c$. If $F_X \in R(-\delta - 1)$ for some $\delta > 0$, then

$$\lim_{u\to\infty} \frac{\psi(u, x_0, \ldots, x_{-n+2})}{\overline{B_0}(u)} = 1. \tag{3.4.9}$$

To show (3.4.9), we adapt the proof of Proposition 3.1 in Ganesh and Torrisi (2006) to our setting.

We know by Theorem 2(B) in Veraverbeke (1977) that

$$\lim_{\tilde{u}\to\infty} \frac{\widetilde{\psi}(\tilde{u})}{\overline{B_0}(\tilde{u})} = 1. \tag{3.4.10}$$

Clearly, with $\tilde{u} = \hat{u}$, $\tilde{U}_k \leq U_k$ for all $k = 1, 2, \ldots$. Hence, the inequality

$$\psi(u, x_0, \ldots, x_{-n+2}) \leq \widetilde{\psi}(\hat{u}) \tag{3.4.11}$$

holds. Consequently, we can write

$$\lim_{u\to\infty} \sup \frac{\psi(u, x_0, \ldots, x_{-n+2})}{\overline{B_0}(u)} \leq \lim_{u\to\infty} \frac{\widetilde{\psi}(\hat{u})}{\overline{B_0}(u)} = \lim_{u\to\infty} \frac{\widetilde{\psi}(\hat{u})}{\overline{B_0}(\hat{u})} \lim_{u\to\infty} \frac{\overline{B_0}(\hat{u})}{\overline{B_0}(u)} = \lim_{u\to\infty} \frac{\overline{B_0}(\hat{u})}{\overline{B_0}(u)} \tag{3.4.12}$$

by (3.4.10). Since $F_X \in R(-\delta - 1)$ (and hence $F_Z \in R(-\delta - 1)$), it follows from Karamata's Theorem (see Embrechts et al. (1997)) that $B_0 \in R(-\delta)$, which implies that

$$\lim_{u\to\infty} \frac{\overline{B_0}(\hat{u})}{\overline{B_0}(u)} = 1. \tag{3.4.13}$$

Now, notice that for $k \geq n$

$$\sum_{i=1}^{k} Y_i \geq \sum_{i=1}^{k-n+1} d\,X_i. \tag{3.4.14}$$

This implies that

$$\begin{aligned}
\psi(u, x_0, \ldots, x_{-n+2}) &= \Pr[\sum_{i=1}^{k} Y_i - k\,c > u \text{ for some } k > 0] \\
&\geq \Pr[\sum_{i=1}^{k} Y_i - k\,c > u \text{ for some } k \geq n] \\
&\geq \Pr[\sum_{i=1}^{k-n+1} d\,X_i - k\,c > u \text{ for some } k \geq n] \\
&= \Pr[\sum_{i=1}^{k-n+1} d\,X_i - (k-n-1)\,c > u + (n+1)\,c \text{ for some} k \geq n] \\
&= \Pr[\sum_{i=1}^{k} d\,X_i - k\,c > u + (n+1)\,c \text{ for some } k > 0] \\
&= \widetilde{\psi}(u + (n+1)\,c).
\end{aligned} \tag{3.4.15}$$

Consequently, we get

$$\begin{aligned}
\lim_{u\to\infty} \inf \frac{\psi(u, x_0, \ldots, x_{-n+2})}{\overline{B_0}(u)} &\geq \lim_{u\to\infty} \frac{\widetilde{\psi}(u + (n+1)\,c)}{\overline{B_0}(u)} \\
&= \lim_{u\to\infty} \frac{\widetilde{\psi}(u + (n+1)\,c)}{\overline{B_0}(u + (n+1)\,c)} \lim_{u\to\infty} \frac{\overline{B_0}(u + (n+1)\,c)}{\overline{B_0}(u)} = 1
\end{aligned} \tag{3.4.16}$$

by (3.4.10) and $B_0 \in R(-\delta)$.
So, as in the light-tailed claims case, the ruin probability is asymptotically insensitive to the way to settle the claims.

3.5 Impact of IBNR dynamics on ultimate ruin probability

Inequality (3.4.11) may indicate that the IBNR dynamics decrease the risk for the insurer, as the occurred claims are paid out more evenly over time. But a "fairer" comparison in terms of the invoked initial capital is in order.

Let us return to the equivalent representation (3.2.3) for the annual claim amount and let $Y_i^{(n)}$ denote the annual claim amount for year i and $\beta_j^{(n)}$ the proportion of S_i paid after j years (given that the number of years needed to settle the claims equals n). So, if we define $\psi^{(n)}(u) = \mathbb{E}[\psi(u, S_0, \ldots, S_{-n+2})]$, we want to know whether the inequality

$$\psi^{(n)}(u) \leq \widetilde{\psi}(u) \tag{3.5.1}$$

holds true for all $u \geq 0$.

Before going on, let us briefly recall some notions on ordering (for more details, see e.g. Denuit et al. (2005)). Given two random variables X and Y, X precedes Y in convex order (denoted by $X \leq_{\mathrm{cx}} Y$), if the inequality $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ holds for any convex function g such that the expectations exist. Next, consider a vector $\vec{x} \in \mathbb{R}_+^k$, and its elements ranked in decreasing order $x_{(1:k)} \geq x_{(2:k)} \geq \cdots \geq x_{(k:k)}$. Given $\vec{x}, \vec{y} \in \mathbb{R}_+^k$, the vector $\vec{y}$ is said to majorize $\vec{x}$ (denoted by $\vec{x} \leq_{\mathrm{maj}} \vec{y}$), if

$$\sum_{i=1}^{j} x_{(i:k)} \leq \sum_{i=1}^{j} y_{(i:k)} \text{ for } j = 1, 2, \ldots, k-1 \text{ and } \sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i. \tag{3.5.2}$$

Proposition 3.5.1. *For all $n = 2, 3, \ldots$ and $k = 1, 2, \ldots,$ we have*

$$\sum_{i=1}^{k} Y_i^{(n)} \leq_{cx} \sum_{i=1}^{k} Y_i^{(1)}. \tag{3.5.3}$$

Proof. From (3.2.3), we get

$$\sum_{i=1}^{k} Y_i^{(n)} = \sum_{i=-n+2}^{k} \gamma_{i,k}^{(n)} S_i,$$

with

$$\gamma_{i,k}^{(n)} = \begin{cases} \sum_{j=-i+1}^{\min(n,k-i+1)} \beta_j^{(n)} & \text{if } i \leq 0 \\ 1 & \text{if } 1 \leq i \leq k-n+1 \\ \sum_{j=1}^{k+1-i} \beta_j^{(n)}, & \text{if } i \geq k-n+2 \end{cases}.$$

For any $n \geq 2$, define the $(k+n-1)$-dimensional vectors $\vec{\phi} = (\vec{0}, \gamma_{1,k}^{(1)}, \gamma_{2,k}^{(1)}, \ldots, \gamma_{k,k}^{(1)})$ and $\vec{\chi} = (\gamma_{-n+2,k}^{(n)}, \gamma_{-n+3,k}^{(n)}, \ldots, \gamma_{k,k}^{(n)})$. Obviously,

$$\vec{\chi} \leq_{\mathrm{maj}} \vec{\phi}.$$

Indeed, $\sum_{i=1}^{k+n-1} \chi_i = \sum_{i=-n+2}^{k} \gamma_{i,k}^{(n)} = \sum_{i=1}^{k+n-1} \phi_i = \sum_{i=1}^{k} \gamma_{i,k}^{(1)} = k$. Secondly $\gamma_{i,k}^{(n)} \leq 1$ and $\gamma_{i,k}^{(1)} = 1$. Consequently, by the i.i.d. property of the S_i's and Property 3.4.48 of Denuit et al. (2005), we get

$$\sum_{i=1}^{k} Y_i^{(1)} = \sum_{i=1}^{k+n-1} \phi_i \, S_{i-n+1} \geq_{\text{cx}} \sum_{i=1}^{k+n-1} \chi_i \, S_{i-n+1} = \sum_{i=1}^{k} Y_i^{(n)}.$$

□

This result seems to be in favor of (3.5.1). However, notice that in discrete-time risk models, there does not exist any result showing that the classification in the convex order sense of the aggregate claims processes implies the classification of the ruin probabilities.

Let us examine the inequality (3.5.1) by some numerical illustrations. Consider motor third-party liability insurance (MTPL) and pecuniary losses insurance (PL). On the basis of statistics for the entire Belgian market over the period $1992-2006$ for motor TPL insurance and over the period $1997-2006$ for the pecuniary losses insurance, we get:

MTPL insurance						
β_1	β_2	β_3	β_4	β_5	β_6	β_7
37.79%	26.13%	8.50%	5.94%	4.78%	3.91%	2.96%
β_8	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}
2.43%	2.03%	1.51%	1.44%	1.09%	0.89%	0.60%

PL insurance								
β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
57.46%	37.19%	3.60%	1.34%	0.25%	0.09%	0.04%	0.01%	0.02%

Let us assume that $\mathbb{E}[S] = 10$. Based on 1 000 000 simulations of the claims amounts S_i for $i = -12, \ldots, 500$ for $S \sim Gam(q_1, q_2)$ ($\mathbb{E}[Gam(q_1, q_2)] = q_1 \, q_2$ and $\mathbb{V}[Gam(q_1, q_2)] = q_1 \, q_2^2$) and alternatively for $S \sim LogN(\mu, \sigma)$ ($\mathbb{E}[LogN(\mu, \sigma)] = e^{\mu+\sigma^2/2}$ and $\mathbb{V}[LogN(\mu, \sigma)] = e^{2\mu+\sigma^2} \, (e^{\sigma^2} - 1)$), we have the following numerical results for $\eta = 0.1$:

$S \sim Gam(q_1, q_2)$, with $\mathbb{E}[S] = 10$						
		$u = 0$	$u = 50$	$u = 100$	$u = 150$	$u = 200$
$q_1 = 1$	$\psi^{(\mathrm{MTPL})}(u)$	0.7035	0.2547	0.1048	0.0428	0.0175
	$\psi^{(\mathrm{PL})}(u)$	0.7900	0.3094	0.1288	0.0528	0.0212
	$\widetilde{\psi}(u)$	0.8234	0.3405	0.1407	0.0578	0.0235
		$u = 0$	$u = 25$	$u = 50$	$u = 75$	$u = 100$
$q_1 = 2$	$\psi^{(\mathrm{MTPL})}(u)$	0.6213	0.2150	0.0892	0.0369	0.0154
	$\psi^{(\mathrm{PL})}(u)$	0.7363	0.2923	0.1203	0.0493	0.0199
	$\widetilde{\psi}(u)$	0.7797	0.3269	0.1360	0.0569	0.0236
		$u = 0$	$u = 15$	$u = 30$	$u = 45$	$u = 60$
$q_1 = 3$	$\psi^{(\mathrm{MTPL})}(u)$	0.5608	0.2042	0.0908	0.0411	0.0186
	$\psi^{(\mathrm{PL})}(u)$	0.6931	0.2999	0.1348	0.0607	0.0278
	$\widetilde{\psi}(u)$	0.7460	0.3442	0.1559	0.0701	0.0319
		$u = 0$	$u = 15$	$u = 30$	$u = 45$	$u = 60$
$q_1 = 4$	$\psi^{(\mathrm{MTPL})}(u)$	0.5140	0.1377	0.0474	0.0164	0.0059
	$\psi^{(\mathrm{PL})}(u)$	0.6572	0.2166	0.0744	0.0258	0.0091
	$\widetilde{\psi}(u)$	0.7182	0.2580	0.0892	0.0312	0.0110
		$u = 0$	$u = 10$	$u = 20$	$u = 30$	$u = 40$
$q_1 = 5$	$\psi^{(\mathrm{MTPL})}(u)$	0.4752	0.1504	0.0608	0.0250	0.0104
	$\psi^{(\mathrm{PL})}(u)$	0.6315	0.2475	0.1041	0.0436	0.0183
	$\widetilde{\psi}(u)$	0.6951	0.3009	0.1255	0.0521	0.0216
		$u = 0$	$u = 10$	$u = 20$	$u = 30$	$u = 40$
$q_1 = 6$	$\psi^{(\mathrm{MTPL})}(u)$	0.4399	0.1108	0.0377	0.0132	0.0046
	$\psi^{(\mathrm{PL})}(u)$	0.6055	0.1980	0.0693	0.0233	0.0080
	$\widetilde{\psi}(u)$	0.6736	0.2449	0.0853	0.0295	0.0106
		$u = 0$	$u = 10$	$u = 20$	$u = 30$	$u = 40$
$q_1 = 7$	$\psi^{(\mathrm{MTPL})}(u)$	0.4083	0.0838	0.0236	0.0066	0.0019
	$\psi^{(\mathrm{PL})}(u)$	0.5861	0.1624	0.0469	0.0135	0.0041
	$\widetilde{\psi}(u)$	0.6520	0.1995	0.0583	0.0165	0.0048
		$u = 0$	$u = 5$	$u = 10$	$u = 15$	$u = 20$
$q_1 = 8$	$\psi^{(\mathrm{MTPL})}(u)$	0.3830	0.1358	0.0638	0.0308	0.0150
	$\psi^{(\mathrm{PL})}(u)$	0.5618	0.2612	0.1283	0.0630	0.0313
	$\widetilde{\psi}(u)$	0.6385	0.3347	0.1660	0.0816	0.0404
		$u = 0$	$u = 5$	$u = 10$	$u = 15$	$u = 20$
$q_1 = 9$	$\psi^{(\mathrm{MTPL})}(u)$	0.3606	0.1138	0.0491	0.0201	0.0087
	$\psi^{(\mathrm{PL})}(u)$	0.5431	0.2276	0.1023	0.0465	0.0209
	$\widetilde{\psi}(u)$	0.6221	0.2943	0.1324	0.0613	0.0275
		$u = 0$	$u = 5$	$u = 10$	$u = 15$	$u = 20$
$q_1 = 10$	$\psi^{(\mathrm{MTPL})}(u)$	0.3390	0.0953	0.0369	0.0150	0.0061
	$\psi^{(\mathrm{PL})}(u)$	0.5243	0.2012	0.0848	0.0363	0.0159
	$\widetilde{\psi}(u)$	0.6077	0.2701	0.1118	0.0461	0.0190

$S \sim LogN(\mu, \sigma)$, with $\mathbb{E}[S] = 10$						
		$u = 0$	$u = 1000$	$u = 2000$	$u = 3000$	$u = 4000$
$\mu = 0.2$	$\psi^{(\mathrm{MTPL})}(u)$	0.7444	0.1908	0.0800	0.0400	0.0228
	$\psi^{(\mathrm{PL})}(u)$	0.7777	0.1989	0.0825	0.0411	0.0233
	$\widetilde{\psi}(u)$	0.7950	0.2022	0.0837	0.0417	0.0236
		$u = 0$	$u = 1000$	$u = 2000$	$u = 3000$	$u = 4000$
$\mu = 0.4$	$\psi^{(\mathrm{MTPL})}(u)$	0.7552	0.1672	0.0619	0.0282	0.0151
	$\psi^{(\mathrm{PL})}(u)$	0.8037	0.1756	0.0641	0.0292	0.0157
	$\widetilde{\psi}(u)$	0.8067	0.1775	0.0649	0.0297	0.0159
		$u = 0$	$u = 1000$	$u = 2000$	$u = 3000$	$u = 4000$
$\mu = 0.6$	$\psi^{(\mathrm{MTPL})}(u)$	0.7645	0.1397	0.0445	0.0182	0.0090
	$\psi^{(\mathrm{PL})}(u)$	0.7988	0.1464	0.0462	0.0187	0.0093
	$\widetilde{\psi}(u)$	0.8168	0.1491	0.0469	0.0191	0.0095
		$u = 0$	$u = 500$	$u = 1000$	$u = 1500$	$u = 2000$
$\mu = 0.8$	$\psi^{(\mathrm{MTPL})}(u)$	0.7725	0.2469	0.1084	0.0526	0.0278
	$\psi^{(\mathrm{PL})}(u)$	0.8065	0.2607	0.1138	0.0549	0.0289
	$\widetilde{\psi}(u)$	0.8252	0.2665	0.1162	0.0560	0.0295
		$u = 0$	$u = 500$	$u = 1000$	$u = 1500$	$u = 2000$
$\mu = 1.0$	$\psi^{(\mathrm{MTPL})}(u)$	0.7767	0.2037	0.0746	0.0308	0.0143
	$\psi^{(\mathrm{PL})}(u)$	0.8122	0.2166	0.0787	0.0322	0.0150
	$\widetilde{\psi}(u)$	0.8317	0.2224	0.0807	0.0329	0.0153
		$u = 0$	$u = 250$	$u = 500$	$u = 750$	$u = 1000$
$\mu = 1.2$	$\psi^{(\mathrm{MTPL})}(u)$	0.7782	0.3060	0.1515	0.0781	0.0420
	$\psi^{(\mathrm{PL})}(u)$	0.8155	0.3291	0.1625	0.0836	0.0449
	$\widetilde{\psi}(u)$	0.8353	0.3379	0.1667	0.0858	0.0461
		$u = 0$	$u = 250$	$u = 500$	$u = 750$	$u = 1000$
$\mu = 1.4$	$\psi^{(\mathrm{MTPL})}(u)$	0.7745	0.2388	0.0937	0.0382	0.0166
	$\psi^{(\mathrm{PL})}(u)$	0.8140	0.2587	0.1012	0.0413	0.0178
	$\widetilde{\psi}(u)$	0.8370	0.2671	0.1044	0.0427	0.0184
		$u = 0$	$u = 125$	$u = 250$	$u = 375$	$u = 500$
$\mu = 1.6$	$\psi^{(\mathrm{MTPL})}(u)$	0.7638	0.3072	0.1509	0.0760	0.0387
	$\psi^{(\mathrm{PL})}(u)$	0.8081	0.3400	0.1669	0.0838	0.0428
	$\widetilde{\psi}(u)$	0.8325	0.3544	0.1738	0.0872	0.0447
		$u = 0$	$u = 125$	$u = 250$	$u = 375$	$u = 500$
$\mu = 1.8$	$\psi^{(\mathrm{MTPL})}(u)$	0.7405	0.1863	0.0591	0.0193	0.0064
	$\psi^{(\mathrm{PL})}(u)$	0.7905	0.2126	0.0675	0.0220	0.0072
	$\widetilde{\psi}(u)$	0.8188	0.2246	0.0712	0.0230	0.0076
		$u = 0$	$u = 50$	$u = 100$	$u = 150$	$u = 200$
$\mu = 2.0$	$\psi^{(\mathrm{MTPL})}(u)$	0.6883	0.2129	0.0817	0.0320	0.0124
	$\psi^{(\mathrm{PL})}(u)$	0.7517	0.2565	0.0991	0.0388	0.0151
	$\widetilde{\psi}(u)$	0.7887	0.2791	0.1072	0.0420	0.0166

As expected, the inequality (3.5.1) is satisfied in these examples since we always have $\psi^{(\mathrm{MTPL})}(u) \leq \widetilde{\psi}(u)$ and $\psi^{(\mathrm{PL})}(u) \leq \widetilde{\psi}(u)$. However, in the heavy-tailed case the gap is less pronounced. This is in accordance with the intuition that for heavy tails ruin is the consequence of one big jump, so that the smoothing effect of the annual claim amounts induced by the IBNR dynamics is not as strong as in the light-tailed case.

3.6 Appendix

Let us consider the following dynamics for the insurer's surplus

$$U_k^{(i)} = u + c\sum_{j=1}^{k}(1+i)^j - \sum_{j=1}^{k} Y_j(1+i)^{k-j}. \tag{3.6.1}$$

The premiums are assumed to be paid at the beginning of each year while the claims are assumed to be paid at the end of the year. Denote by

$$\begin{aligned}&\psi^{(i)}(u, x_0, x_{-1}, \ldots, x_{-n+2})\\ &\qquad = \Pr[U_k^{(i)} < 0 \,\text{for some}\; k|U_0^{(i)} = u, X_0 = x_0, X_{-1} = x_{-1}, \ldots, X_{-n+2} = x_{-n+2}]\end{aligned}$$

the ultimate ruin probability relative to the new surplus dynamics (3.6.1). One can prove that if there exists a positive constant $\rho^{(i)}$ such that

$$e^{-\frac{\rho^{(i)}}{d}c}\,\mathbb{E}[e^{\frac{\rho^{(i)}}{d}vX(1+\sum_{j=1}^{n-1}v^j d_j)}] = 1, \tag{3.6.2}$$

then we have

$$\psi^{(i)}(u, x_0, \ldots, x_{-n+2}) \leq \frac{e^{-\frac{\rho^{(i)}}{d}\hat{u}^{(i)}}}{\mathbb{E}[e^{-\frac{\rho^{(i)}}{d}v^T\hat{U}_T^{(i)}}|T < \infty]}, \tag{3.6.3}$$

where $v = \frac{1}{1+i}$, $\hat{U}_k^{(i)} = U_k^{(i)} - \sum_{l=0}^{n-2} X_{k-l}\sum_{j=1}^{n-l-1} v^j d_{l+j}$, with $\hat{u}^{(i)} = \hat{U}_0^{(i)}$ and $T = \inf\{k : U_k^{(i)} < 0\}$. Indeed, we have

$$\begin{aligned}\mathbb{E}[e^{-\frac{\rho^{(i)}}{d}v^k\hat{U}_k^{(i)}}|\Phi_{k-1}] &= \mathbb{E}[e^{-\frac{\rho^{(i)}}{d}v^k(U_{k-1}^{(i)}(1+i)+c(1+i)-Y_k-\sum_{l=0}^{n-2}X_{k-l}\sum_{j=1}^{n-l-1}v^j d_{l+j})}|\Phi_{k-1}]\\ &= \mathbb{E}[e^{-\frac{\rho^{(i)}}{d}v^{k-1}(U_{k-1}^{(i)}+c-v[Y_k+\sum_{l=0}^{n-2}X_{k-l}\sum_{j=1}^{n-l-1}v^j d_{l+j}])}|\Phi_{k-1}],\end{aligned} \tag{3.6.4}$$

where Φ_k is the sigma-algebra generated by $\{X_i, i = 1, 2, \ldots, k\}$. Now

$$v[Y_k + \sum_{l=0}^{n-2} X_{k-l}\sum_{j=1}^{n-l-1} v^j d_{l+j}] = X_k\big(v + \sum_{j=1}^{n-1} v^{j+1}d_j\big) + \sum_{l=0}^{n-2} X_{k-1-l}\sum_{j=1}^{n-l-1} v^j d_{l+j}. \tag{3.6.5}$$

Thus, we get

$$
\begin{aligned}
\mathbb{E}[e^{-\frac{\rho^{(i)}}{d}v^k\hat{U}_k^{(i)}}|\Phi_{k-1}] &= \mathbb{E}[e^{-\frac{\rho^{(i)}}{d}v^{k-1}(c-vX(1+\sum_{j=1}^{n-1}v^j d_j))}] \\
&\quad \mathbb{E}[e^{-\frac{\rho^{(i)}}{d}v^{k-1}(U_{k-1}^{(i)}-\sum_{l=0}^{n-2}X_{k-1-l}\sum_{j=1}^{n-l-1}v^j d_{l+j})}|\Phi_{k-1}] \\
&= \mathbb{E}[e^{-\frac{\rho^{(i)}}{d}v^{k-1}(c-vX(1+\sum_{j=1}^{n-1}v^j d_j))}]\, e^{-\frac{\rho^{(i)}}{d}v^{k-1}\hat{U}_{k-1}^{(i)}}. \qquad (3.6.6)
\end{aligned}
$$

Furthermore, as $i \geq 0$, it suffices to note (cf. Yang (1998)) that

$$
\begin{aligned}
\mathbb{E}[e^{-\frac{\rho^{(i)}}{d}v^{k-1}(c-vX(1+\sum_{j=1}^{n-1}v^j d_j))}] &= \mathbb{E}[e^{(-\frac{\rho^{(i)}}{d}(c-vX(1+\sum_{j=1}^{n-1}v^j d_j)))^{v^{k-1}}}] \\
&\leq \mathbb{E}[e^{-\frac{\rho^{(i)}}{d}(c-vX(1+\sum_{j=1}^{n-1}v^j d_j))}]^{v^{k-1}} \qquad (3.6.7)
\end{aligned}
$$

by Jensen's inequality. Consequently, $\{e^{-\frac{\rho^{(i)}}{d}v^k\hat{U}_k^{(i)}},\ k \geq 0\}$ is a supermartingale, since

$$\mathbb{E}[e^{-\frac{\rho^{(i)}}{d}(c-vX(1+\sum_{j=1}^{n-1}v^j d_j))}] = 1 \qquad (3.6.8)$$

by assumption. In view of the final arguments of the proof of Theorem 3.3.1, the inequality (3.6.3) follows.

Prospects and limitations

Discount interest rate problem

The classical approach in non-life is to work with no interest rates, i.e. in fact at zero rates. However, in a Solvency II perspective, we have to take into account discount rates for the calculation of the premiums. So, premium loadings appear explicitly. If the impact of this discounting could still be neglected when the claims are all settled immediately, it would not be the case anymore when the payments of the claims are spread out during time. Nevertheless, from an economic point of view, the introduction of such discount factors in our model has sense only if, on the one hand, inflation rates and, on the other hand, interests on the surplus are also considered at the same time.

Pervozvansky (1998) studied classical compound Poisson processes where the company continuously invests its surplus at a constant interest force, and where premiums and claim amounts are submitted to a constant inflation force. In this setting, Pervozvansky derived an integro-differential equation for survival probability in a finite time interval.

Also, Delbaen and Haezendonck (1987) gave a general description of the classical risk process when macro-economic factors were taken into account, and they studied their effects on bounds on ruin probabilities.

Obviously, Section 3.6 shows that it would not be complicated to extend our results to the case of constant interest and inflation rates. But notice that the introduction of such rates does not imply in our analysis the existence of a martingale anymore, but rather the existence of a supermartingale.

Clearly, the most interesting case is the one which deals with stochastic rates. Nevertheless, our analysis appears to be limited in this context. Even the martingale approach developed by Yang and Zhang (2006) to include stochastic interest rates in the classical risk model turns out to be unsuccessful in our case. So, a new way to tackle this problem must be considered.

Let us finish by mentioning Frolova et al. (2002). They highlighted the fact that for light-tailed claims, and independently of the safety loading, the investments in an asset with large volatility lead to the bankruptcy with probability one while for a small volatility, the ruin probability decreases as a power function. This means that ruin models with light-tailed claims and risky investments are in some sense equivalent to classical ruin models with heavy-tailed claims! So, this observation underlines the important role played by the financial risk in an insurance company. The recent events clearly illustrated this assertion. Of course, this remark puts in perspective the classical actuarial approach envisaged in this work.

Remark on a model combining the credibility mechanism and the IBNR dynamics

A model combining the credibility mechanism of Chapter 2 and the IBNR dynamics of Chapter 3 could be considered. However, such generalization implies to take the same linkratios for all the possible portfolios. The uncertainty would then be located on the distribution of the claim amounts paid each year for the claims occuring the same year. Obviously, we can expect asymptotic results as those obtained in Chapter 2 due to the insensitivity property showed in Chapter 3.

However, the assumption considering the same linkratios whatever the portfolio appears unrealistic. This remark suggests to introduce stochastic linkratios in our model, where the distribution functions would depend on the given portfolio. Of course, this constitutes a completely new research.

Part II

Chapter 4

Properties of a risk measure derived from ruin theory

This chapter is submitted.

Abstract

This paper studies a risk measure inherited from ruin theory and investigates some of its properties with respect to the axiomatic framework of risk measures. Specifically, we derive various properties of a VaR-type risk measure defined as the smallest initial capital needed to ensure that the ruin probability is less than a given level. A related tail-VaR-type risk measure is also discussed.

4.1 Introduction and motivation

This paper discusses risk measures for insurance portfolios which are derived from actuarial ruin theory. Dhaene, Goovaerts and Kaas (2003) give ample motivation for an exponential risk measure inherited from the Cramér-Lundberg upper bound for the ruin probability in a discrete-time ruin model. Cheridito, Delbaen and Kupper (2006) mention (in an application of their study on coherent risk measures for unbounded stochastic processes) a Value-at-Risk (VaR) risk measure based on the infinite-time ruin probability itself. In general, over the last years the relative position and relation between (insurance) risk measures that fulfill a certain list of axioms on the one hand, and classical tools in ruin theory to assess the riskiness of an insurance activity in the collective framework on the other hand, has often been a matter of debate. In this paper we would like to take up this issue and investigate in more detail some properties of the VaR-type risk measure based on the ruin probability that is mentioned in Cheridito, Delbaen and Kupper (2006). For ease of exposition, we will restrict the considerations to the classical compound Poisson process.

Whereas VaR and expected shortfall are natural and frequently applied concepts in practice, they are usually defined in terms of a given time horizon. In certain applications it may be more natural to look for measures that give a more robust reflection of the risk inherent in a business activity in a random environment. In particular, both the VaR and the expected shortfall for a given time horizon do not reflect the possible adverse financial situations in between or beyond the specified time interval. On the other hand, in an insurance context, starting with the work of Filip Lundberg and Harald Cramér in the early 20th century, ruin theory has always studied somewhat robust measures of insurance risk. In particular, the probability of ruin can be interpreted as the continuous alternative to the VaR.

One historical purpose of studying ruin probabilities was to obtain the amount of initial capital needed to guarantee a certain probability of solvency throughout the lifetime of the process, given that both the premium strategy and the aggregate claim process are stationary over time. In that sense the ruin probability might be interpreted as a measure of dynamic risk in a static environment (meaning that the initial capital and the premium strategy are fixed and later on one is not able to or does not want to interfere during the lifetime of the process). In particular, the correspondingly allocated capital will then suffice to cope with the involved insurance risk at all times according to some acceptance level of ϵ. It is this type of robustness that makes the ruin probability still nowadays an interesting quantity in this context. Most of all, the type of thinking coming from ruin theory is often considered important by practitioners.

The purpose of this paper is to consider certain properties of risk measures that are motivated by ruin theory. The goal is not so much to advocate them for practical use in risk management, but rather to improve the understanding of the connections between these fields and to see whether the quite active research field of ruin theory can offer additional insight for people dealing with practical risk management of insurance companies as well.

Note that Geman (1998) also emphasized the usefulness of the actuarial ruin paradigm and Embrechts and Samorodnitsky (2004) demonstrated that ruin theory can be especially useful for operational risk.

Our paper is organized as follows. In Section 2, we set up the scene by recalling some properties and tools of the compound Poisson risk model. In Section 3, we derive various properties of a VaR-type risk measure, which is defined as the smallest amount of capital needed to ensure a ruin probability below a given probability level. In Section 4 we then discuss the corresponding Tail-VaR-type risk measure. The final Section 5 concludes.

4.2 Risk model and harmonic mean residual life order

4.2.1 Compound Poisson surplus model

This section recalls some basic results about ruin probabilities, which will be useful in our analysis. For more details, we refer the interested reader, e.g., to Asmussen (2000). The surplus process (or risk process) is defined as

$$U_t = u + c\,t - S_t, \quad t \geq 0, \tag{4.2.1}$$

where U_t is the insurer's capital at time t starting from some initial capital $U_0 = u$, c is the (constant) premium income per unit of time and $S_t = \sum_{k=1}^{N_t} X_k$ is the total claim amount up to time t, with N_t the corresponding number of claims, and X_k the size of the kth claim, assumed to be non-negative. The claim number process $\{N_t,\ t \geq 0\}$ is assumed to be Poisson with constant rate λ. The X_ks are independent and distributed as X. They are furthermore assumed to be independent of $\{N_t,\ t \geq 0\}$. We assume that the premium rate is of the form $c = (1+\eta)\lambda\mathbb{E}[X]$ where $\eta > 0$ is called the safety loading. Henceforth, η is assumed to be the same for all the risk processes used in this paper.

The ruin time T is defined as

$$T = \begin{cases} \min\{t \geq 0 \,|\, U_t < 0\}, \\ +\infty \quad \text{if } U_t \geq 0 \text{ for all } t, \end{cases} \tag{4.2.2}$$

and the ruin probability is $\psi(u) = \Pr[T < \infty]$. If needed, we will denote the ruin probability as ψ_X to make explicit the dependence on X. Let $L_t = S_t - ct$ be the aggregate loss at time t, i.e. the amount by which the insurance claims exceed the collected premiums at some time t, the ruin time T can also be expressed as

$$T = \min\{t \geq 0 | u - L_t < 0\}. \tag{4.2.3}$$

Correspondingly, $\psi(u) = \Pr[L > u]$, where $L = \max_{t\geq 0} L_t$ denotes the maximal aggregate loss of the process.

4.2.2 Compound geometric representation of the ruin probability

It is well-known that ψ coincides with the tail function of a compound geometric distribution since L can be decomposed as

$$L = \sum_{j=1}^{M} D_j, \tag{4.2.4}$$

where M follows the geometric distribution with success probability $1-\psi(0) = \frac{\eta}{1+\eta}$ and where $D_1, D_2, \ldots$ are the ladder heights of the loss process which are independent and identically distributed. In the compound Poisson model, the common distribution function of the D_js is given by the integrated tail distribution

$$F_D(y) = \int_0^y \frac{1 - F_X(x)}{\mathbb{E}[X]} dx, \quad y > 0. \tag{4.2.5}$$

The non-ruin probability is then given by

$$1 - \psi(u) = \sum_{m=0}^{\infty} p(1-p)^m F_D^{\star(m)}(u), \tag{4.2.6}$$

where $p = \frac{\eta}{1+\eta}$ and $F_D^{\star(m)}$ is the m-fold convolution of F_D.

Note that the mapping $F_X \to F_D$ is well-known in applied probability, where (apart from the name integrated tail distribution) F_D is often called the stationary forward recurrence time, the equilibrium distribution or the residual lifetime in the literature. The relationship between the moments of X and of D is given by

$$\mathbb{E}[D^k] = \frac{\mathbb{E}[X^{k+1}]}{(k+1)\mathbb{E}[X]}, \quad k = 1, 2, \ldots$$

An important result in our setting is as follows. Recall the definition of the stochastic dominance and of the convex order. Given two random variables X and Y, X precedes Y in the stochastic dominance, denoted as $X \preceq_{\text{st}} Y$, if the inequality $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ holds for any non-decreasing function g such that the expectations exist. Similarly, X precedes Y in the convex order, denoted as $X \preceq_{\text{cx}} Y$, if the inequality $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ holds for any convex function g such that the expectations exist. Then, denoting as F_E the integrated tail distribution associated to F_Y, and as E a random variable with distribution function F_E, we have

$$X \preceq_{\text{cx}} Y \Rightarrow D \preceq_{\text{st}} E.$$

Henceforth, we will also need the increasing convex, or stop-loss, order defined as follows. Given two random variables X and Y, X precedes Y in the increasing convex order, denoted as $X \preceq_{\text{icx}} Y$, if the inequality $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ holds for any non-decreasing convex function g such that the expectations exist.

4.2.3 Harmonic mean residual life order

Michel (1987) defined a stochastic order relation among claim size distributions by reference to the comparison of ruin probabilities. This order in fact compares the ladder-height distribution F_D of the renewal process describing claim occurrences. The harmonic mean residual life order, which proves useful in reliability applications, naturally arises in this context. For other applications of this ordering in actuarial science, we refer the interested reader, e.g., to Lefèvre and Utev (2001).

Given two non-negative random variables X and Y, recall that X is said to be smaller than Y in the harmonic mean residual life order (denoted as $X \preceq_{\text{hmrl}} Y$) when

$$\frac{\int_t^\infty \overline{F}_X(u)\,du}{\mathbb{E}[X]} \leq \frac{\int_t^\infty \overline{F}_Y(u)\,du}{\mathbb{E}[Y]} \quad \text{for all} \quad t \geq 0, \tag{4.2.7}$$

where $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$. The inequality in (4.2.7) can be equivalently written as

$$\frac{\mathbb{E}[(X-t)_+]}{\mathbb{E}[X]} \leq \frac{\mathbb{E}[(Y-t)_+]}{\mathbb{E}[Y]} \quad \text{for all} \quad t \geq 0. \tag{4.2.8}$$

From (4.2.8) it follows that $X \preceq_{\text{hmrl}} Y$ if, and only if,

$$\frac{\mathbb{E}[g(X)]}{\mathbb{E}[X]} \leq \frac{\mathbb{E}[g(Y)]}{\mathbb{E}[Y]} \tag{4.2.9}$$

for all non-decreasing convex functions $g : \mathbb{R}^+ \to \mathbb{R}$, provided the expectations exist.

A useful property for the present study is Theorem 2.B.16 in Shaked and Shanthikumar (2007). It shows that any mixture of two distributions ordered in the $\preceq_{\text{hmrl}}$ order is bounded from above and from below by the components of the mixture. Formally, let X and Y be such that $X \preceq_{\text{hmrl}} Y$ and define the distribution function $F_Z = \beta F_X + (1-\beta)F_Y$ for some $\beta \in (0,1)$. Then,

$$X \preceq_{\text{hmrl}} Z \preceq_{\text{hmrl}} Y. \tag{4.2.10}$$

Remark: Neither of the orders $\preceq_{\text{st}}$ and $\preceq_{\text{hmrl}}$ implies the other. It can nevertheless be shown that

$$X \preceq_{\text{hmrl}} Y \Rightarrow \mathbb{E}[X \big| X > 0] \leq \mathbb{E}[Y \big| Y > 0],$$

so that when X and Y are positive almost surely

$$X \preceq_{\text{hmrl}} Y \Rightarrow \mathbb{E}[X] \leq \mathbb{E}[Y]. \tag{4.2.11}$$

If $\mathbb{E}[X] = \mathbb{E}[Y]$ then (4.2.9) ensures that

$$X \preceq_{\text{hmrl}} Y \Leftrightarrow X \preceq_{\text{cx}} Y. \tag{4.2.12}$$

4.3 VaR-type risk measure

Whereas in Dhaene et al. (2003), essentially the risk measure is the necessary annual premium amount such that for a given initial capital u the resulting ruin probability is bounded by ϵ, in this paper we would like to focus on another approach linking ruin probabilities with the framework of risk measures. We consider the continuous-time risk model (4.2.1) and, instead of fixing u, we assume that the safety loading η is fixed and ask for the amount of initial capital needed in order to bound the ruin probability by ϵ. That is, we define now the ruin-consistent Value-at-Risk risk measure as

$$\rho_\epsilon[X] = \inf\{v \geq 0 | \psi(v) \leq \epsilon\} = \psi^{-1}(\epsilon).$$

In words, $\rho_\epsilon[X]$ is the smallest amount of capital needed such that the ultimate ruin probability ψ for a risk process with individual claim sizes distributed as X is at most equal to some specified probability level ϵ. Working directly on the basis of the ruin probability ψ (instead of on the basis of Lundberg's upper bound used by Dhaene et al. (2003)) has the advantage that the risk measure is exact rather than a bound and secondly, and this is also quite important, the method is then not restricted to insurance risk processes with exponentially bounded claims for which a Lundberg exponent exists. In particular, many realistic descriptions of risk processes are based on heavy-tailed (subexponential) claims for which the Lundberg exponent does not exist.

One possible interpretation of this approach is as follows. Competition among companies (market pressure, underwriting cycles, etc.) determines a safety loading η that can be realized in the insurance market and then $\rho_\epsilon[X]$ reflects the amount of capital needed to ensure a stable business according to the safety level ϵ, assuming "stationary" insurance business (the amount $\rho_\epsilon[X]$ then puts the portfolio into an acceptable position with respect to the safety measure ψ). So in this approach the premium income is considered to be (the market-induced "balancing") part of the insurance risk in the portfolio rather than the control variable. Measuring risk by the infinite-time ruin probability ψ entails that we deal with an unbounded time horizon. Hence an acceptable position in such a framework reflects long-term stability thinking. Naturally, as time evolves, using claim experience and further acquired information, the capital requirements will be readjusted. But compared to the classical one-year time horizon VaR approach, in the above setup the incentive is to always maintain a (with respect to available information) sustainable strategy. It is worth to mention that the vast majority of the results in risk theory consider η as fixed when the effect of switching from one severity distribution to another is studied.

Remark: It may appear surprising that the risk measure ρ_ϵ does not explicitly depend on the claim frequency λ. This is because the ruin probability itself does not depend on λ as we are allowed to switch to any operational time scale. Formally, this is clear from the compound geometric representation (4.2.6). This means that ρ_ϵ is not influenced by the volume of the portfolio reflected in λ, since the fixed value of η ensures that the corresponding value of c compensates for the change in the portfolio volume. This is because the classical risk model values "time diversification". Hence, the number of policies does not matter, but well the distribution of the claim amount they generate, i.e. ρ_ϵ only depends on X (and on η). Depending on the field of application, this may be considered as an advantage or disadvantage of the applicability of ρ_ϵ.

Another way to look at ρ_ϵ is to consider it as a VaR applied to the transformed risk L (the maximal aggregate loss). Specifically, for any probability level ϵ,

$$\rho_\epsilon[X] = F_L^{-1}(1-\epsilon) = \inf\{t \geq 0 | F_L(t) \geq 1-\epsilon\}.$$

Instead of taking the VaR of the claim severity X, we first transform X into L and then compute the VaR of this geometric compound sum. The question now is to determine whether this is a safe strategy, that is, whether L is more dangerous than X, in some sense. To study this problem, let us decompose the switch from X to L in two steps: (i) a change from F_X to the corresponding integrated tail distribution F_D followed by (ii) a geometric compounding.

Step (i): Recall that a random variable is said to be NWUE (or New Worse than Used in Expectation) if $\mathbb{E}[(X-t)_+] \geq \mathbb{E}[X]\overline{F}_X(t)$ for all $t \geq 0$. This basically means that X

dominates the Negative Exponential with the same expectation in the $\preceq_{\rm cx}$ sense. All the standard claim size distributions fulfill this property. According to Theorem 1.A.31 in Shaked and Shanthikumar (2007), we have that X NWUE $\Rightarrow X \preceq_{\rm st} D$. Hence, step (i) in the switch from X to L appears to be conservative.

Step (ii): The second step consists in replacing D with $\sum_{j=1}^{M} D_j$, where the D_js are independent copies of D. Assume that $\eta \in (0,1)$ so that $\mathbb{E}[M] > 1$ and $1 \preceq_{\rm icx} M$. As a $\preceq_{\rm icx}$-ranking of the number of terms induces the same ordering for the resulting compound sums, we get $D \preceq_{\rm icx} L$ so that step (ii) is also conservative. Combining the two steps reveals that $X \preceq_{\rm icx} L$ holds, so that the risk X is first replaced with a more dangerous (in the $\preceq_{\rm icx}$-sense) random variable L before taking a VaR.

Properties of ρ_ϵ

The risk measure ρ_ϵ possesses the following properties.

Property 4.3.1. *(i) The risk measure ρ_ϵ is positively homogeneous, that is, $\rho_\epsilon[aX] = a\rho_\epsilon[X]$ for any constant $a > 0$.*

(ii) The risk measure ρ_ϵ agrees with the stop-loss order, that is $X \preceq_{icx} Y \Rightarrow \rho_\epsilon[X] \leq \rho_\epsilon[Y]$.

(iii) If Y and Z are identically distributed and if Z is more positively dependent on X than Y, in the sense that the inequality $\Pr[X \leq x, Y \leq y] \leq \Pr[X \leq x, Z \leq y]$ holds for all x and y, then the risk measure ρ_ϵ expresses the fact that $X + Z$ is more dangerous than $X + Y$ as $\rho_\epsilon[X + Y] \leq \rho_\epsilon[X + Z]$.

(iv) If Z is a mixture of X, Y and $X+Y$ (that is, $F_Z = p_1F_X + p_2F_Y + p_3F_{X+Y}$ with $p_i \geq 0$ for $i = 1,2,3$ and $\sum_{i=1}^3 p_i = 1$) such that $\alpha_\epsilon Z \preceq_{hmrl} Y$, where $\alpha_\epsilon = \frac{\rho_\epsilon[Y]}{\rho_\epsilon[X]+\rho_\epsilon[Y]}$, then $\rho_\epsilon[Z] \leq \rho_\epsilon[X] + \rho_\epsilon[Y]$.

Proof. Item (i) is obvious since multiplying the annual claim amount by a has the same effect on the annual premium income, so that the initial capital needed to have an ultimate ruin probability of ϵ is also multiplied by a. Formally, replacing X with aX changes L into aL which results in $a\rho_\epsilon[X]$. Considering item (ii), we know that $X \preceq_{\rm icx} Y \Rightarrow \psi_X(u) \leq \psi_Y(u)$ for all u. This classical result of risk theory can be found, e.g., in standard textbooks as Kaas et al. (2008, Section 7.4.2). Next, item (iii) is deduced from (ii) since $X + Y \preceq_{\rm cx} X + Z$. Finally, for item (iv), coming back to the compound geometric representation of the ruin probability, we can write

$$\alpha_\epsilon Z \preceq_{\rm hmrl} Y \Rightarrow \psi_{\alpha_\epsilon Z}(\rho_\epsilon[Y]) \leq \psi_Y(\rho_\epsilon[Y]).$$

Now, multiplying the claim sizes by a fixed coefficient α_ϵ is equivalent to dividing the initial capital by the same factor. Hence,

$$\psi_{\alpha_\epsilon Z}(\rho_\epsilon[Y]) = \psi_Z\left(\frac{\rho_\epsilon[Y]}{\alpha_\epsilon}\right) = \psi_Z(\rho_\epsilon[X] + \rho_\epsilon[Y]) \leq \psi_Y(\rho_\epsilon[Y]) \leq \epsilon$$

so that the inequality $\rho_\epsilon[Z] \leq \rho_\epsilon[X] + \rho_\epsilon[Y]$ indeed holds true, which completes the proof. □

In particular, it is clear, from (iii), that ρ_ϵ is comonotonicity-consistent (in the terminology of Dhaene et al. (2003)). Furthermore, from (iv), it also follows that ρ_ϵ recognizes diversification effect under conditions of exchangeable risks, negative quadrant dependent risks with equivalent sizes, and risks that are members of scale family distributions, whatever the dependence between them. Let us formally establish these results.

Property 4.3.2. *(i) The risk measure ρ_ϵ is subadditive for exchangeable risks, that is, $\rho_\epsilon[X+Y] \leq \rho_\epsilon[X] + \rho_\epsilon[Y]$ if X and Y satisfy $\Pr[X \leq t_1, Y \leq t_2] = \Pr[X \leq t_2, Y \leq t_1]$ for all t_1 and t_2. This is in particular the case if X and Y are independent and identically distributed.*

(ii) If X and Y are negatively quadrant dependent and identically distributed, that is, the inequality $\Pr[X \leq x, Y \leq y] \leq \Pr[X \leq x]\Pr[Y \leq y]$ holds for all x and y then the risk measure ρ_ϵ is subadditive, that is, $\rho_\epsilon[X+Y] \leq \rho_\epsilon[X] + \rho_\epsilon[Y]$.

(iii) If $X = \beta V_1$ and $Y = \gamma V_2$, where V_1 and V_2 are identically distributed as V, then the risk measure ρ_ϵ is subadditive, that is, $\rho_\epsilon[X+Y] \leq \rho_\epsilon[X] + \rho_\epsilon[Y]$.

Proof. Let us consider Property 4.3.1 (iv) with $p_1 = p_2 = 0$. Then, as $\alpha_\epsilon = \frac{1}{2}$, item (i) is obvious since $\frac{X+Y}{2} \preceq_{\text{cx}} Y$. Next, considering (ii), since $\alpha_\epsilon = \frac{1}{2}$, it suffices to note that $\frac{X+Y}{2} \preceq_{\text{cx}} \frac{X^\perp + Y^\perp}{2} \preceq_{\text{cx}} Y$, where $(X^\perp, Y^\perp)$ has independent components with the same marginals as (X, Y). Finally, for item (iii), we have to show that $\beta V_1 + \gamma V_2 \preceq_{\text{cx}} (\beta+\gamma)V$, since $\alpha_\epsilon = \frac{\gamma}{\beta+\gamma}$. Now, as $\Pr[\beta V \leq x, \gamma V \leq y] \geq \Pr[\beta V_1 \leq x, \gamma V_2 \leq y]$ holds for all x and y, we have that, by equation 9.A.19 in Shaked and Shanthikumar (2007), $\beta V_1 + \gamma V_2 \preceq_{\text{cx}} (\beta+\gamma)V$. □

Application in terms of portfolios

Now, let us define the two processes $S_t^{(1)} = \sum_{i=1}^{N_t^{(1)}+N_t} X_i$ and $S_t^{(2)} = \sum_{i=1}^{N_t^{(2)}+N_t} Y_i$, where $\{N_t, t \geq 0\}, \{N_t^{(1)}, t \geq 0\}$ and $\{N_t^{(2)}, t \geq 0\}$ are independent Poisson processes with constant rates λ, λ_1 and λ_2 respectively, and the X_ks (resp. Y_ks) are independent and distributed as the generic random variable X (resp. Y). Furthermore, X_ks and Y_ks are assumed to be independent of $\{N_t, t \geq 0\}, \{N_t^{(1)}, t \geq 0\}$ and $\{N_t^{(2)}, t \geq 0\}$. Then, the properties of ρ_ϵ allow us to deduce some cases where merging two portfolios with cumulative losses described by

$S_t^{(1)}$ and $S_t^{(2)}$ is beneficial.

Assume that the X_ks are independent of the Y_ks. Then, $S_t^{(1)} + S_t^{(2)}$ is a compound Poisson process with intensity $\lambda+\lambda_1+\lambda_2$ and generic claim size Z with distribution function $F_Z = p_1 F_X + p_2 F_Y + p_3 F_{X+Y}$, where $p_1 = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda}$, $p_2 = \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda}$ and $p_3 = \frac{\lambda}{\lambda_1+\lambda_2+\lambda}$. Hence, if $\alpha_\epsilon Z \preceq_{\text{hmrl}} Y$, then, by Property 4.3.1 (iv), a diversification effect at level ϵ is recognized. The next result gives an illustration.

Proposition 4.3.1. *Let $X = \beta V_1$ and $Y = \gamma V_2$, with $\beta \le \gamma$, where V_1 and V_2 are independent and identically distributed. Then $\alpha_\epsilon Z \preceq_{hmrl} Y$.*

Proof. Obviously, we have $X \preceq_{\text{hmrl}} Y \Leftrightarrow \beta \le \gamma$. Hence, it is easily seen from (4.2.8) that $\alpha_\epsilon X \preceq_{\text{hmrl}} Y$ and $\alpha_\epsilon Y \preceq_{\text{hmrl}} Y$. Now, we also know by the proof of Property 4.3.2(iii) that $\alpha_\epsilon(X+Y) \preceq_{\text{hmrl}} Y$.

Hence, we necessarily have $\alpha_\epsilon Z \preceq_{\text{hmrl}} Y$. Indeed,

$$
\begin{aligned}
\frac{\mathbb{E}[(\alpha_\epsilon Z - t)_+]}{\mathbb{E}[\alpha_\epsilon Z]} &= \frac{\mathbb{E}[(\alpha_\epsilon X - t)_+]\, p_1 + \mathbb{E}[(\alpha_\epsilon Y - t)_+]\, p_2 + \mathbb{E}[(\alpha_\epsilon (X+Y) - t)_+]\, p_3}{\mathbb{E}[\alpha_\epsilon X]\, p_1 + \mathbb{E}[\alpha_\epsilon Y]\, p_2 + \mathbb{E}[\alpha_\epsilon (X+Y)]\, p_3} \\
&= \frac{\frac{\mathbb{E}[(\alpha_\epsilon X - t)_+]}{\mathbb{E}[\alpha_\epsilon X]}\, \mathbb{E}[\alpha_\epsilon X]\, p_1 + \frac{\mathbb{E}[(\alpha_\epsilon Y - t)_+]}{\mathbb{E}[\alpha_\epsilon Y]}\, \mathbb{E}[\alpha_\epsilon Y]\, p_2 + \frac{\mathbb{E}[(\alpha_\epsilon (X+Y) - t)_+]}{\mathbb{E}[\alpha_\epsilon (X+Y)]}\, \mathbb{E}[\alpha_\epsilon (X+Y)]\, p_3}{\mathbb{E}[\alpha_\epsilon X]\, p_1 + \mathbb{E}[\alpha_\epsilon Y]\, p_2 + \mathbb{E}[\alpha_\epsilon (X+Y)]\, p_3} \\
&\le \frac{\frac{\mathbb{E}[(Y-t)_+]}{\mathbb{E}[Y]}(\mathbb{E}[\alpha_\epsilon X]\, p_1 + \mathbb{E}[\alpha_\epsilon Y]\, p_2 + \mathbb{E}[\alpha_\epsilon (X+Y)]\, p_3)}{\mathbb{E}[\alpha_\epsilon X]\, p_1 + \mathbb{E}[\alpha_\epsilon Y]\, p_2 + \mathbb{E}[\alpha_\epsilon (X+Y)]\, p_3} \\
&= \frac{\mathbb{E}[(Y-t)_+]}{\mathbb{E}[Y]},
\end{aligned}
$$

which ends the proof. □

Assume that $\lambda = 0$. Hence, $S_t^{(1)}$ and $S_t^{(2)}$ are independent. Then, in this particular case, we have the following result.

Proposition 4.3.2. *Assume that $\lambda = 0$. If $X \preceq_{hmrl} Y$, then $\alpha_\epsilon Z \preceq_{hmrl} Y$.*

Proof. From (4.2.10), we have that $X \preceq_{\text{hmrl}} Y \Rightarrow Z \preceq_{\text{hmrl}} Y$. Also, it is easily seen from (4.2.8) that $\alpha_\epsilon Z \preceq_{\text{hmrl}} Z$ so that $X \preceq_{\text{hmrl}} Y \Rightarrow \alpha_\epsilon Z \preceq_{\text{hmrl}} Y$. □

Let us give some examples where $X \preceq_{\text{hmrl}} Y$ holds true:

(1) If $X \sim Uni(a,b)$ and $Y \sim Uni(a',b')$, then $X \preceq_{\text{hmrl}} Y \Leftrightarrow a+b \le a'+b'$ and $b \le b'$.

(2) If $X \sim Exp(a)$ and $Y \sim Exp(a')$, then $X \preceq_{\text{hmrl}} Y \Leftrightarrow a \ge a'$.

(3) If $X \sim Par(a,b)$ and $Y \sim Par(a',b')$ with $\min(b,b') > 1$, then
$X \preceq_{\text{hmrl}} Y \Leftrightarrow \frac{b-1}{b'-1} \ge \max(\frac{a}{a'}, 1)$

(cf. Heilmann and Schröter (1991)). Notice that the second example is a particular case of the equivalence $X \preceq_{\text{hmrl}} Y \Leftrightarrow \beta \leq \gamma$, where $X = \beta V_1$ and $Y = \gamma V_2$, with V_1 and V_2 are independent and identically distributed.

4.4 Tail-VaR-type risk measure

Finally, we would like to point out that a natural extension of ρ_ϵ is the Tail-VaR (or average VaR) defined as

$$\overline{\rho}_\epsilon[X] = \frac{1}{\epsilon}\int_0^\epsilon \rho_w[X]\, dw,$$

where ρ_w is the VaR-type risk measure at level w discussed in the previous section. By definition we can write

$$\overline{\rho}_\epsilon[X] = \frac{1}{\epsilon}\int_0^\epsilon \psi^{-1}(w)\, dw,$$

where $\psi^{-1}(w)$ is the inverse function of the ruin probability, or further

$$\overline{\rho}_\epsilon[X] = \frac{1}{\epsilon}\int_0^\epsilon F_L^{-1}(1-w)\, dw.$$

Hence, $\overline{\rho}_\epsilon[X]$ appears to be the Tail-VaR of L. Under the conditions discussed above, we now have

$$X \preceq_{\text{icx}} L \Rightarrow \text{TVaR}[X;\epsilon] \leq \text{TVaR}[L;\epsilon] = \overline{\rho}_\epsilon[X].$$

Alternatively one can rewrite

$$\begin{aligned}
\overline{\rho}_\epsilon[X] &= \frac{1}{\epsilon}\int_{\psi^{-1}(0)}^{\psi^{-1}(\epsilon)} w\, d\psi(w) \\
&= \psi^{-1}(\epsilon) + \frac{1}{\epsilon}\int_0^\epsilon \frac{w\, dw}{|\psi'(\psi^{-1}(w))|} \\
&= \psi^{-1}(\epsilon) + \frac{1}{\epsilon}\mathbb{E}\big[(L-\psi^{-1}(\epsilon))_-\big] \\
&= \mathbb{E}[L \big| L > F_L^{-1}(1-\epsilon)].
\end{aligned}$$

The last two equalities follow from general properties of Tail-VaR, as e.g. given in Denuit et al. (2005, Section 2.4) or in Pflug and Römisch (2007).

From the above it becomes clear that the measure $\overline{\rho}_\epsilon[X]$ represents the amount of capital needed to be able to cope "in expectation" also with the insurance loss in those problematic cases that occur with probability less than ϵ. One also observes that explicit knowledge of the ruin probability expression can tremendously simplify the calculations of this risk measure. In Cheridito et al. (2006) it is shown that $\overline{\rho}_\epsilon[X]$ is coherent and hence satisfies several desirable properties. However, on a computational level, $\overline{\rho}_\epsilon$ will typically be harder to deal with than ρ_ϵ and, as often in related contexts, in practice there will be a trade-off between the gain in theoretical attractiveness and the lowering of practicability of the resulting expressions when choosing feasible measures for the insurance risk.

4.5 Conclusion

In this paper we discussed the classical concept of the ruin probability of an insurance portfolio in the framework of risk measures and established some properties of the corresponding naturally motivated VaR-type risk measure ρ_ϵ. The crucial assumption in this approach is that the collective insurance business in the portfolio has a certain degree of stationarity and that the safety loading is fixed (for instance, determined by market conditions). Note that apart from that the results are quite general as they are expressed through the ruin probability directly. This may for instance be particularly welcome in situations with dependence among the insurance risks for which still some results on the ruin probability are available. Finally, we would like to emphasize once more that the purpose of this paper is not to advocate the practical use of ρ_ϵ, but rather to clarify some relations that might be helpful when assessing the riskiness of certain financial positions in the insurance context.

The present approach is somewhat different from Dhaene et al. (2003). Apart from the fact that these authors use the Lundberg bound instead of the ruin probability itself, they fix the initial capital and determine the annual premium in their discrete-time model according to the required level of the ruin probability. Here, we fix the safety loading (and hence the premium relative to the claim rate λ) and try to allocate the appropriate initial capital for the required level of the ruin probability. Motivation for that approach has been given in terms of market competition.

The risk measures discussed in the present paper enjoy nice properties and appear natural in the classical framework of risk measures: they are just a VaR and a Tail-VaR of a random variable, namely the minimum L of a stochastic process. Some of the results established in the present paper might be extended to general renewal models. For instance, the ruin probability is then still obtained from the tail function of a geometric compound distribution, although the ladder height distribution is in general not the integrated tail and not explicitly available anymore.

Prospects and limitations

The Pollaczeck-Khinchine formula

In the literature, equation (4.2.6) is usually called the Pollaczeck-Khinchine formula. This well-known formula constitutes the basis of all the results derived in this final chapter. That is why, the approach implemented here is not generalizable with a ruin model taking into account financial returns produced by the surplus process, since in this case, the Pollaczeck-Khinchine formula is not available anymore. Also, for the same reason, arguments used in this chapter cannot be extended in a finite-time horizon setting as well as in a discrete-time context.

In all these cases, we could work for example on upper bounds of the ruin probability. Besides, for the classical discrete-time ruin model, Dhaene et al. (2003) have investigated properties of our capital risk measure by using Lundberg's upper bounds. Of course, the main drawback of this method is to lose exact risk measure in favor of an approximation.

The portfolio size problem

As already mentioned in this chapter, due to the compound Poisson surplus model, our risk measure does not depend on parameter λ, i.e. the size of the portfolio. Let us notice that a finite time horizon approach as well as a discrete time ruin model would have compensated for this limitation of our work. Thus, in practice, it seems inappropriate to propose this work as an alternative to the Solvency II one-year time horizon approach. Nevertheless, let us insist that our risk measure remains interesting, since it allows, on the basis of the claim amounts X_k and with a perspective of an infinite time horizon, to compare risk processes. But of course, the interpretation in terms of capital appears then really compromised.

Now, if this observation constitutes clearly a limitation of our modelling, let us suggest some lines of thinking.

Firstly, the Solvency II one-year time horizon approach is maybe not so fair anymore as it could appeared at first sight. Indeed, a finite-time horizon calculation can also raise some questions. A large company has much more claims and premium income in one year than a small company. So, a large insurer is comparable to a small one with a longer time horizon. Hence, if the one-year time horizon approach is not wrong, since the resulting solvency capital reflects somehow how diversified the risk is, one can think of whether it would make sense to think about different time horizons for different companies. Clearly, it could be the subject of future research.

Secondly, let us mention that a manner of partially solving this lack of sensitivity of the portfolio size, and then to come back to a capital interpretation, would be to consider the claim amounts X_k as the annual claim amounts of the company. Indeed, with $\lambda = 1$, the continuous reserve process described by equation (4.2.1) would then constitue an approximation of the dynamics of the insurer's surplus. It is a kind of Poisson approximation (see the individual and collective models in Denuit and Charpentier (2004) for example). We would switch from a collective model to an individual model, and we could interpret the compound Poisson as the claim amount for one period of time in the classic surplus process.

Another interesting risk measure

Let us introduce an other interesting risk measure defined as the expected value of the time-aggregated negative part of the risk process until a fixed time-horizon $s > 0$, i.e.

$$\mathbb{E}\left[\int_0^s I_{\{U_t<0\}}\,|U_t|\,dt\right].$$

Notice that in Loisel (2005), for $s = \infty$, a closed-form formula is derived in the Poisson-exponential case. Obviously, this new risk measure, contrary to $\rho_\epsilon[X]$, has the advantage to depend on the size of the portfolio. Furthermore, a priori, it seems to verify the same types of properties, in view of the following rewriting

$$\mathbb{E}\left[\int_0^s I_{\{U_t<0\}}\,|U_t|\,dt\right] = \mathbb{E}\left[\int_0^s (S_t - u - c\,t)_+\,dt\right] = \int_0^s \mathbb{E}\left[(S_t - u - c\,t)_+\right]\,dt.$$

However, new challenges, dedicated to this particular risk measure, will come because of the new role played by the parameter λ. That is why, this will lead to a new collaboration in a near future with Professor Stéphane Loisel.

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