COOPERATIVE PRODUCT GAMES

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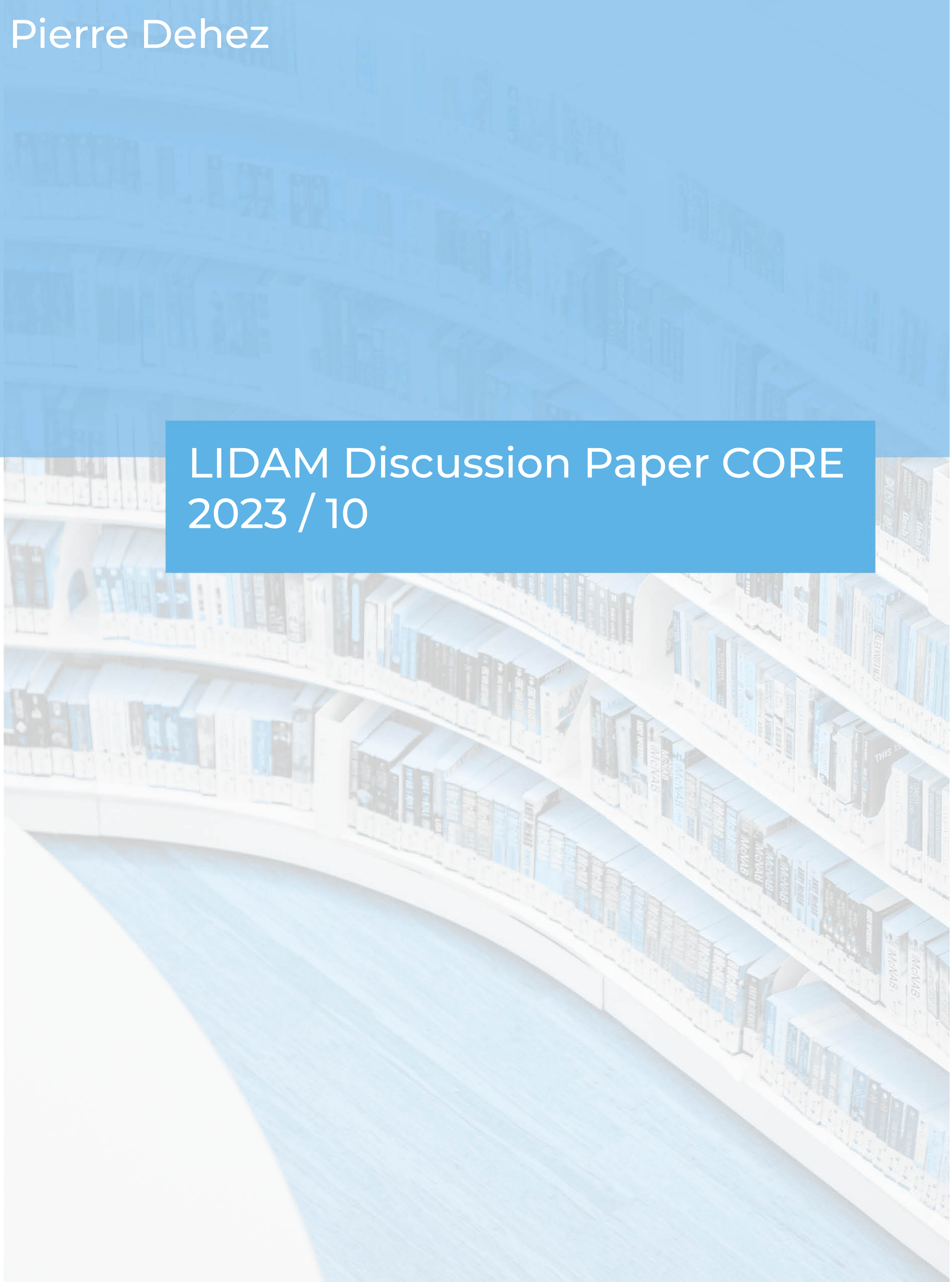

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Cooperative product games

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Abstract

A cooperative product game is a transferable utility game defined by associating the product of given players' weights to coalitions. This idea has been introduced by Rosales (2014) in an unpublished memo. Here we analyze a modified version and cover the Shapley value to which we provide an axiomatic foundation.

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1. Introduction

Consider a set of players and assume that, by joining a coalition, a player multiplies the coalition's payoff by some coefficient, its "weight". Associating each coalition with the product of the coefficients of its members then defines a class of transferable utility games called "product cooperative games", following a terminology introduced by Rosales (2014).

Product games are convex and have therefore a nonempty core. They belong to the class of positive games and, in the absence of null player, they are strictly convex. The Shapley value is easily obtained from the Harsanyi dividends. It allocates to each player an amount that is proportional to his weight.

A product game can be viewed as a production game whose technology exhibits increasing returns to scale. The Shapley value then determines a fair remuneration of the participants. Health is a potential field of application of product games where players are assimilated to factors which, when combined, favor the development of certain pathologies. The Shapley value then offers a measure of the actual share of each of these factors. And uncertainty can be accommodated by assuming that weights are random variables whose distributions are known.

The paper is organized as follows. The definition and properties of product games are covered in Section 2. The core and the Shapley value are the object of Sections 3 and 4. Section 5 is devoted to an axiomatization of the Shapley value on the class of product games based on the proportionality property. Section 6 is devoted to illustrations and concluding remarks are given in the last section.

2. Product games: definition and properties

A transferable utility game is a pair (N,v) where $N=\{1,...,n\}$ is the set of players and v is a function defined on the subsets of N, where $v(S)$ is the "worth" of coalition S that measures the minimum coalition S can hope to obtain if it decided to form. By convention, $v(\varnothing)=0$.

Notation: Finite sets are denoted by upper-case letters and lower-case letters are used to denote their cardinals: $t = |T|$, $s = |S|$, … For a given vector x, $x(S)$ denotes the sum of its coordinates over the set S. By convention, a sum over an empty set is equal to 0 and the product over an empty set is equal to 1. Set inclusion (non-strict) is denoted by $\subset$ and we define successively $\Omega(S)=\{T\subset S \mid T\neq\varnothing\}$ and $\Omega_i(S)=\{T\subset S \mid i\in T\}$. Braces are sometimes omitted for coalitions: $v(i)$ replaces $v(\{i\})$, replaces $v(i,j)$ replaces $v(\{i,j\})$,...

The central question covered by the theory of transferable utility games concerns the allocation of the worth of the grand coalition among the players. For a given game (N,v), an allocation $x\in\mathbb{R}^n$ is efficient if $x(N)=v(N)$. Requiring in addition that $x_i \geq v(i)$ for all i is a minimum requirement. It defines the *imputation set* $I(N,v)$, the set of individually rational and efficient allocations.

We consider a situation where a weight is attached to each player, a_i for player i. Weights are assumed to be real numbers not smaller than 1: $a_i \geq 1$ for all $i\in N$. A product game is defined by the following characteristic function: [1]

$$v(S)=\prod_{i\in S} a_i - 1 \text{ for all } S\subset N. \tag{1}$$

[1] Rosales actually uses the product of the a_i instead. This would be coherent if he had assumed that $v(\varnothing) = 1$.

In particular, considering an individual player i, $v(i) = a_i - 1 \geq 0$. By definition, a player contributes only if his weight exceeds 1. We identify a product game by a pair (N, a) where $a \in A_n$, $A_n = [1, +\infty)^n$.

Example 1 The 3-player game associated to the weight vector a = (1.2, 1.4, 1.5) is defined by $v =$ (0.20, 0.40, 0.50 | 0.68, 0.80, 1.10 | 1.52).

Product games are *monotonic*: $S \subset T$ implies $v(S) \leq v(T)$. A game (N, v) is *superadditive* if $S \cap T = \varnothing$ implies $v(S \cup T) \geq v(S) + v(T)$.

Proposition 1 Product games are superadditive.

Proof For any given two subsets S and T, define $\Delta(S,T) = v(S \cup T) + v(S \cap T) - v(S) - v(T)$. Knowing that $a_i \geq 1$ for all $i \in N$, $S \cap T = \varnothing$ implies:

$$\Delta(S,T) = \prod_{i \in S} a_i \prod_{i \in T} a_i - \prod_{i \in S} a_i - \prod_{i \in T} a_i + 1 = \left(\prod_{i \in S} a_i - 1 \right) \left(\prod_{i \in T} a_i - 1 \right) \geq 0.$$

Superadditivity then follows. □

A game is *essential* if $\sum v(i) < v(N)$. Product games are essential if $a_i > 1$ for at least one player. A game is *convex* if $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$ for all $S, T \subset N$.[2] Convexity implies superadditivity. For product games, convexity and superadditivity are actually *equivalent* properties.

Proposition 2 Product games are convex.

Proof Assume superadditivity holds. For any two subsets S and T, we have:

$$\begin{aligned} \Delta(S,T) &= \prod_{i \in S \cup T} a_i + \prod_{i \in S \cap T} a_i - \prod_{i \in S} a_i - \prod_{i \in T} a_i \\ &= \prod_{i \in S \setminus T} a_i \prod_{i \in S \cap T} a_i \prod_{i \in T \setminus S} a_i + \prod_{i \in S \cap T} a_i - \prod_{i \in S \setminus T} a_i \prod_{i \in S \cap T} a_i - \prod_{i \in T \setminus S} a_i \prod_{i \in S \cap T} a_i \\ &= \left(1 + \prod_{i \in S \setminus T} a_i \prod_{i \in T \setminus S} a_i - \prod_{i \in S \setminus T} a_i - \prod_{i \in T \setminus S} a_i \right) \prod_{i \in S \cap T} a_i \\ &= \left(v(S \setminus T \cup T \setminus S) - v(S \setminus T) - v(T \setminus S) \right) \prod_{i \in S \cap T} a_i. \end{aligned}$$

The non-negativity of $\Delta(S,T)$ then follows from superadditivity applied to the disjoint sets $T \setminus S$, and $S \setminus T$. □

Convexity can be defined in terms of marginal contributions. The *marginal contribution* of player i to coalition S is defined by $MC_i(S) = v(S) - v(S \setminus i)$. A game is convex if marginal contributions are nondecreasing with respect to set inclusion: $MC_i(S) \leq MC_i(T)$ whenever $i \in S \subset T$. For a product game, if $i \in S$, we have:

$$MC_i(S) = (a_i - 1) \prod_{j \in S \setminus i} a_j$$

confirming that players contribute once their weight exceeds one.

[2] The notion of convex (or supermodular) games was introduced by Shapley (1971).

Knowing that $a_i \geq 1$ for all *i*, marginal contributions are nondecreasing:

$$v(T) - v(T \setminus i) = (a_i - 1) \prod_{j \in T \setminus i} a_j \geq (a_i - 1) \prod_{j \in S \setminus i} a_j = v(S) - v(S \setminus i) \tag{2}$$

Two players *i* and *j* are *substitutable* if they marginal contributions coincide: $i, j \in S$ implies $MC_i(S) = MC_j(S)$ for all $S \subset N$. A player is *null* if his marginal contributions are equal to zero: $MC_i(S) = 0$ for all $S \subset N$. In a product game, *i* and *j* are *substitutable* players if (and only if) $a_i = a_j$ and *i* is a null player if $a_i = 1$.

A game is *strictly convex* if marginal contributions are strictly increasing with coalition size: $i \in S \subsetneq T \Rightarrow MC_i(S) < MC_i(T)$. According to (2), product games are strictly convex if (and only if) no player is null: $a_i > 1$ for all $i \in N$.

Given a set of players *N*, the set *G*(*N*) of all characteristic functions on *N* is equivalent to the vector space $\mathbb{R}^{2^n - 1}$. In proving the uniqueness of his value, Shapley (1953) shows that the collection of *unanimity games* (N, u_T) defined by

$$\begin{aligned} u_T(S) &= 1 \quad \text{if } T \subset S, \\ &= 0 \quad \text{if } T \not\subset S, \end{aligned}$$

forms a basis of *G*(*N*): for any given function $v \in G(N)$, there exists a unique $(2^n - 1)$-dimensional vector $\alpha = (\alpha_T)_{T \in \Omega(N)}$ such that:

$$v(S) = \sum_{T \in \Omega(N)} \alpha_T \, u_T(S) = \sum_{T \in \Omega(S)} \alpha_T. \tag{3}$$

The α_T can be defined recursively, starting with $\alpha_\varnothing = 0$, as follows:

$$\alpha_T = v(T) - \sum_{\substack{S \in \Omega(T) \\ S \neq T}} \alpha_S \;\Rightarrow\; \alpha_T = \sum_{S \in \Omega(T)} (-1)^{t-s} v(S). \tag{4}$$

The dividends of a product game are given by:

$$\begin{aligned} &\alpha_{\{i\}} = a_i - 1, \\ &\alpha_{\{i,j\}} = a_i\, a_j - a_i - a_j + 1 = (a_i - 1)(a_j - 1), \\ &\alpha_{\{i,j,k\}} = a_i\, a_j\, a_k - a_i\, a_j - a_i\, a_k - a_j\, a_k + a_i + a_j + a_k - 1 = (a_i - 1)(a_j - 1)(a_k - 1), \ldots \end{aligned}$$

and, by extension,

$$\alpha_T = \prod_{i \in T} (a_i - 1). \tag{5}$$

Dividends are nonnegative. Hence the following proposition.

Proposition 3 Product games are positive.

Positive games form an interesting class of games on which solution concepts tend to converge: the core coincides with the set of weighted Shapley values as well as the set of distributions of Harsanyi dividends (for details, see Dehez, 2017).

Example 1 (continued) The Harsanyi dividends associated to the weight vector *a* = (1.2, 1.4, 1.5) are given by $(0.20, 0.40, 0.50 \,|\, 0.08, 0.10, 0.20 \mid 0.04)$.

3. The core of a product game

Imputations are individually rational and efficient allocations. Superadditivity ensures that the imputation set of a product game is nonempty. Extending rationality from individuals to coalitions leads to the concept of *core*: it is the set of efficient allocations that gives to each coalition S at least its worth $v(S)$.[3] The core of a product game is then given by:

$$C(N,a)=\left\{x\in\mathbb{R}^n \;\middle|\; \sum_{i\in N}x_i=\prod_{i\in N}a_i-1,\ \sum_{i\in S}x_i\geq\prod_{i\in S}a_i-1 \text{ for all } S\subset N\right\}.$$

If nonempty, the core is a convex polyhedron. Convex games have a nonempty (and typically large) core and its vertices are the *marginal contributions vectors* associated to players' ordering. The vector $\mu^{\pi}(N,v)$ associated to the ordering $\pi=(i_1,i_2,...,i_n)$ is defined by:

$$\mu^{\pi}_{i_1}(N,v)=v(i_1),$$
$$\mu^{\pi}_{i_k}(N,v)=v(i_1,...,i_k)-v(i_1,...,i_{k-1}),\ k=2,...,n-1$$
$$\mu^{\pi}_{i_n}(N,v)=v(N)-v(i_1,...,i_{n-1}).$$

There is n! marginal contribution vectors, not necessarily distinct, except in the case of a strictly convex game. By convexity, the core coincides with the *Weber set* defined as the convex hull of the marginal contribution vectors (Weber, 1988). For 3-player product games, they are given by the following table:

π	1	2	3
(1,2,3)	a_1-1	$a_1(a_2-1)$	$a_1a_2(a_3-1)$
(1,3,2)	a_1-1	$a_1a_3(a_2-1)$	$a_1(a_3-1)$
(2,1,3)	$a_2(a_1-1)$	a_2-1	$a_1a_2(a_3-1)$
(2,3,1)	$a_2a_3(a_1-1)$	a_2-1	$a_2(a_3-1)$
(3,1,2)	$a_3(a_1-1)$	$a_1a_3(a_2-1)$	a_3-1
(3,2,1)	$a_2a_3(a_1-1)$	$a_3(a_2-1)$	a_3-1

The marginal contribution vectors are the natural allocations when the order in which players act matters. Clearly, being last to act is the best situation for a player while being first is the worse.

Example 1 (continued) The game associated to the weight vector a = (1.2, 1.4, 1.5) is strictly convex and the marginal contribution vectors are given by (0.20, 0.48, 0.84), (0.20, 0.72, 0.60), (0.28, 0.40, 0.84), (0.42, 0.40, 0.70), (0.30, 0.72, 0.50) and (0.42, 0.60, 0.50).

4. The Shapley value of a product game

Shapley (1953) proves that there is a unique efficient and additive allocation rule that allocates zero to null players and identical amounts to substitutable players. There are several formulations of the Shapley value. We retain the formulation based on Harsanyi dividends.

Following Harsanyi (1959), α_T is the *dividend* accruing to coalition T once all sub-coalitions have received their dividends. By (3), the sum of all dividends is equal to $v(N)$. An allocation can then

[3] The term core was introduced by Gillies (1953) and was later introduced as an independent solution concept by Shapley (1955). See Zhao (2018) for a clarification of the respective contributions of Gillies and Shapley as to the core.

be obtained by distributing the dividends of each coalition among its members. Harsanyi shows that the Shapley value gives to each player the sum of the *per capita* dividend of the coalitions of which he is a member:

$$SV_i(N,v) = \sum_{T\in\Omega_i(N)} \frac{1}{t}\alpha_T(N,v) \quad (i = 1,\ldots,n). \tag{6}$$

Introducing (5) into (6), we obtain the Shapley value of a product games:

$$SV_i(N,a) = \sum_{T\in\Omega_i(N)} \frac{1}{t}\prod_{j\in T}(a_j - 1) = (a_i - 1)\sum_{T\in\Omega_i(N)} \frac{1}{t}\prod_{j\in T\setminus i}(a_j - 1). \tag{7}$$

In the 3-player case, the amount allocated to player 1 by the Shapley value is given by:

$$\begin{aligned} SV_1(N,a) &= (a_1-1) + \frac{1}{2}(a_1-1)(a_2-1) + \frac{1}{2}(a_1-1)(a_3-1) + \frac{1}{3}(a_1-1)(a_2-1)(a_3-1) \\ &= \frac{a_1-1}{6}(2 + a_2 + a_3 + 2a_2a_3). \end{aligned} \tag{8}$$

By convexity, the Shapley value defines a core allocation. It can also be written as the average of the n! marginal contribution vectors, whether distinct or not. As a result, the Shapley value of a *strictly* convex product game coincides with the average of its core's vertices: it is the natural allocation when the orders in which players act are all considered.

Looking at (7), we observe that $SV_i(N,a)$ is *proportional* to $a_i - 1$:

$$SV_i(N,a) = f(a_{-i})(a_i - 1)$$

where $f:\mathbb{R}^{n-1}\to\mathbb{R}$ is a *player independent* and *symmetric* function given by?[4]

$$f(z) = 1 + \sum_{T\in\Omega(\{1,\ldots,n-1\})} \frac{1}{t+1}\prod_{j\in T}(z_j - 1). \tag{9}$$

Example 1 (continued) The Shapley value associated to the weight vector a = (1.2, 1.4, 1.5) is given by (0.303, 0.553, 0.663). By strict convexity, it coincides with the average of the marginal contribution vectors.

We know that the Shapley value satisfies symmetry: it allocates identical amount to players with identical weights. The converse is true. Indeed, we have:

$$SV_i(N,p) - SV_j(N,p) = (a_i - a_j)[1 + \sum_{\substack{T\subset\Omega(N)\\ i,j\notin T}} \frac{1}{t+1}\prod_{h\in T}(a_h - 1)]$$

where the term between brackets is positive. Hence, the following proposition.

Proposition 4 The Shapley value of a product game (N,a) satisfies the following property:

$$SV_i(N,a) = SV_j(N,a) \text{ if and only if } a_i = a_j.$$

We observe that the differences between the amounts allocated by the Shapley value are proportional to the differences in weights.

[4] A function is symmetric if permuting its arguments leaves its value unchanged.

5. Axiomatization of the Shapley value of a product game

Here, we cannot rely on additivity simply because the sum of two product games is not a product game. It turns out that proportionality, together with efficiency and symmetry, characterize a unique allocation rule on the class of product games. It then must be the Shapley value since it satisfies these three properties. Given a player set N and a vector $a \in A_n$, we only retain the rules $\varphi : (N,a) \to \mathbb{R}^n$ that satisfy the following three properties.

A.1 *Efficiency*: $\sum_{i \in N} \varphi_i(N,a) = \prod_{i \in N} a_i - 1.$

A.2 *Symmetry*: $a_i = a_j \Rightarrow \varphi_i(N,a) = \varphi_j(N,a).$

A.3 *Proportionality*: for all $i \in N$, $\varphi_i(N,a) = \lambda_i(a_i - 1)$ where λ_i is independent of a_i.

Proposition 5 The Shapley value of a product game (N,a) is the unique allocation rule that satisfies A.1, A.2 and A.3.

Proof Consider a situation (N,a) and an allocation rule φ satisfying A.1, A2 and A.3. According to A.3, there exist n functions $f_1,..., f_n$ such that $\varphi_i(N,a) = f_i(a_{-i})(a_i - 1)$ for all i. We first show that, under A.2, the proportionality functions are independent of players' identities. We then show that under A.1, the common proportionality function f is uniquely defined and symmetric. As a result, φ must be the Shapley value and f is given by (9).

Consider the players $i \in \{1,...,n-1\}$ and $j = i+1$. For any given vector $a \in A_n$ such that $a_i = a_j$, we have $a_{-i} = a_{-j} = b$ and $f_i(b) = f_j(b)$ by A.2. Independence then follows:

$$f_i(z_1,...,z_{n-1}) = f(z_1,...,z_{n-1}) \text{ for all } (z_1,...,z_{n-1}) \in A_{n-1} \text{ and all } i \in N$$

We now proceed iteratively to show that the function f is uniquely defined, using A.1 and A.3. We start by evaluating $f(z,...,z)$ for some $z > 1$:

$$n\, f(z,...,z)(z-1) = z^n - 1 \;\Rightarrow\; f(z,...,z) = \frac{z^n - 1}{n(z-1)} = \frac{1}{n}\left(z^{n-1} + z^{n-2} + ... + z + 1\right). \quad (10)$$

For $n = 2$, $f(z) = (z+1)/2.$ For $n \geq 3$, we proceed to the evaluation of $f(z_1, z,...,z)$ for some z_1 and $z > 1$:

$$f(z,...,z)(z_1 - 1) + (n-1)\, f(z_1, z,...,z)(z-1) = z_1\, z^{n-1} - 1$$

or

$$f(z_1, z,...,z) = \frac{z_1\, z^{n-1} - 1 - f(z,...,z)(z_1 - 1)}{(n-1)(z-1)}. \quad (11)$$

We evaluate $f(z, z_2,...,z)$ similarly. We then proceed to the evaluation of $f(z_1, z_2, z,...,z)$ for z_1, z_2 and $z > 1$:

$$f(z, z_2,...,z)(z_1 - 1) + f(z_1, z,...,z)(z_2 - 1) + (n-2)\, f(z_1, z_2, z,...,z)(z-1) = z_1\, z_2\, z^{n-2}$$

or $f(z_1, z_2, z,...,z) = \dfrac{z_1\, z_2\, z^{n-2} - f(z, z_2,...,z)(z_1 - 1) - f(z_1, z,...,z)(z_2 - 1)}{(n-2)(z-1)}.$

And so on, until the evaluation of the complete function $f(z_1, z_2, z_3,..., z_{n-1})$ on A_{n-1}. □

Notice that the resulting function is symmetric: all along the proof, the relative positions of $z_1, z_2,...$ are arbitrary.

Here is an illustration for different values of n. For n = 2 and $i \neq j,$ we have:

$$x_i = \frac{a_j + 1}{2}(a_i - 1) = \left(1 + \frac{a_j - 1}{2}\right)(a_i - 1).$$

For n = 3, using (10) and (11), we have successively $f(z,z) = (z^2 + z + 1)/3$ and

$$f(z_1, z_2) = \frac{z_1 z_2^2 - 1 - (z_2^2 + z_2 + 1)(z_1 - 1)/3}{2(z_2 - 1)} = \frac{1}{6}\left(2 + z_1 + z_2 + 2z_1 z_2\right)$$

$$= 1 + \frac{1}{2}(z_1 - 1) + \frac{1}{2}(z_2 - 1) + \frac{1}{3}(z_1 - 1)(z_2 - 1).$$

We then retrieve (8): $SV_1(N,a) = f(a_2, a_3)(a_1 - 1).$

6. Illustrations

6.1 Production game

A product game (N, v) can be viewed as a production game whose technology associates to each coalition S the payoff given by (1). Returns to scale are increasing, in line with the convexity of the resulting game. Every player whose weights exceed 1 contributes by increasing the quantity produced and the Shapley value determines a remuneration that is proportional to his contribution. The results obtained so far differ when other elements enter into account. Consider for instance the situation described by Shapley and Shubik (1967) where player 1 owns the technology but has no multiplicative effect. Then setting $a_1 = 1,$ we have the following game:

$$\begin{aligned} v(S) &= \prod_{i \in S} a_i - 1 \quad \text{if } 1 \in S, \\ &= 0 \qquad\qquad\quad \text{if } 1 \notin S. \end{aligned}$$

The Harsanyi dividends are similar. Using (4), they are given by:

$$\begin{aligned} \alpha_T &= \prod_{j \in T \setminus 1} (a_j - 1) \quad \text{if } 1 \in T \text{ and } t \geq 2, \\ &= 0 \qquad\qquad\qquad \text{otherwise,} \end{aligned}$$

and, using (6), the Shapley value is given by:

$$SV_i(N,a) = \sum_{\substack{T \subset N \\ 1,i \in T}} \frac{1}{t} \prod_{j \in T \setminus 1} (a_j - 1) \qquad (i = 1, \ldots, n).$$

Proportionality still holds for the workers:

$$SV_i(N,a) = (a_i - 1) \sum_{\substack{T \subset N \\ 1,i \in T}} \frac{1}{t} \prod_{j \in T \setminus \{1,i\}} (a_j - 1) \qquad (i = 2, \ldots, n).$$

In the 3-worker case, the Shapley value is given by:

$$SV_1(N,a) = \frac{1}{12}(a_2 + a_3 + a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4 + 3a_2 a_3 a_4 - 9)$$

$$SV_i(N,a) = \frac{1}{12}(a_i - 1)(1 + a_j + a_k + 3a_j a_k) \quad j,k \neq 1,i.$$

Example 2 Consider the case of three workers with weights (3, 5, 7). The associated product game is defined by $v = (0,0,0,0 \mid 2,4,6,0,0,0 \mid 14,20,34,0 \mid 104)$ and the Shapley value is given by $SV(N,a) = (32.66, 19.66, 24.66, 27)$.

Assuming that workers are substitutable, we set $a_i = b$ for all $i \neq 1$, and the Shapley value then becomes:

$$SV_1(N,a) = \frac{1}{4}\left(b^3 + b^2 + b - 3\right),$$

$$SV_i(N,a) = \frac{1}{12}(b-1)\left(3b^2 + 2b + 1\right) \quad i = 2,\ldots,n.$$

6.2 Introducing uncertainty

Consider a situation where random variables are associated to players, Y_i for player i, assuming that their distributions are known and defined on the interval $\left[1,+\infty\right[$.

In the case where the n random variables $Y_1,\ldots,Y_n$ are independent, we could use the expectation to define the product game

$$v(S) = E\left[\prod_{i\in S} Y_i\right] - 1 = \prod_{i\in S} E\left[Y_i\right] - 1,$$

on the player set N, associating the coefficient a_i to the expectation $E[Y_i]$. Dropping the independence assumption, we could instead rely on the *geometric* expectation. For an arbitrary random variable Y, it is defined by:

$$E_G[Y] = \mathrm{e}^{E[\log(Y)]}.$$

It is an alternative measure of the central tendencies of random variables defined on $\left]0,+\infty\right[$ and is used in different contexts.[5] The geometric expectation of a discrete random variable Y defined by $\{(r_1,q_1),\ldots,(r_m,q_m)\}$, where $x_h > 0$ for all h and $\sum q_h = 1$, is then simply given by:

$$E_G[Y] = \prod_{h=1}^{m} r_h^{q_h},$$

hence the name "geometric" mean. Among the properties of the geometric expectation, three of them are worth mentioning:

$$E_G[\alpha Y] = \alpha E_G[Y] \text{ for all } \alpha > 0,$$

$$E_G[Y] \leq E[Y],$$

$$E_G[\prod Y_i] = \prod E_G[Y_i].$$

The first one is immediate, the second one is a direct consequence of Jensen inequality while the third one applies whether the random variables Y_i are independent or not. Hence, a product game can be defined by

$$v(S) = E_G\left[\prod_{i\in S} Y_i\right] - 1 = \prod_{i\in S} E_G\left[Y_i\right] - 1,$$

on the player set N, associating the coefficient a_i to the geometric expectation $E_G[Y_i]$.

[5] There is a large literature on the geometric mean and its applications, in particular in finance and insurance. See for instance Jasiulewicz and Kordecki (2016).

7. Concluding remarks

A similar multiplicative structure appears in the probability games introduced by Hou et al. (2018) and later investigated by Dehez (2023). The characteristic function associates to each coalition the probability that it succeeds in reaching a given target. In this framework, the Shapley value appears to be proportional as well.

It is remarkable that this simple property, together with efficiency and symmetry, characterizes the Shapley value on the class of product games: it is the property that differentiates the Shapley value from other efficient and symmetric allocation rules. Additivity does the same on the class of superadditive games. The coefficient a_i *identifies* player *i* and knowledge of the vector *a* is all what is needed to compute an allocation and by, Proposition 4, $SV_i(N,a) = SV_j(N,a)$ *if and only if* $a_i = a_j$. In addition, proportionality can be *equivalently* written as:

$$a,b \in A_n \text{ such that } a_j = b_j \text{ for all } j \neq i \Rightarrow \frac{\varphi_i(N,a)}{\varphi_i(N,b)} = \frac{a_i - 1}{b_i - 1}.$$

Throughout the paper, weights have been assumed to be larger (or equal) to one. If we allow $a_i < 1$ for some players, superadditivity is lost (as well as convexity) but the Shapley value of course remains well defined. It implies a negative payoff for the players whose weights fall below one. This makes sense in certain contexts like for instance health where some factors may have a positive impact.

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