On the Index and Asymptotic Stability of Dynamics

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Abstract

Given a compact connected component of zeros of a vector field, we give a necessary condition for its asymptotic stability in terms of its index and Euler characteristic.

1 Introduction

Usually many properties of dynamics are given assuming they are generic or that their singularities lie in a finite codimensional subset, the study of the Morse-Smale flows, Anosov flows, and the bifurcation theory developed by Arnold are standard examples. However, in certain fields such as game theory one has to deal with dynamics that are highly non generic and typically vanish on whole connected components, see e.g. the remarks at the end of Section 2 in [5].

A general issue in the field concerns the asymptotic stability of a component of zeros of a vector field, as has been studied in [5], where asymptotic stability has been related to several game theoretic properties of Nash equilibria.

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In this paper, we give a necessary condition for asymptotic stability of a component in terms of its index (see [4] and [1] for a definition in the context of game theory). In particular an isolated point or a convex component of zeros is asymptotically stable only if its index is 1, in a suitable normalization. This generalizes the obvious criterion for stability of a regular zero. This criterion is particularly useful because from [1] and [2] it follows that for a large class of dynamics the index is actually independent of the dynamics and can be expressed as the degree of the inverse of the Nash correspondence.

In a forthcoming paper we will deal with the economic applications of this result: we will show how one can define the concept of "learnable set". A learnable set is roughly a component of Nash equilibria that has some chance of being learned in at least one learning dynamics; although this concept seems to be quite broad it will be shown that learnable sets always contain stable set in the sense of Mertens, so they satisfy the strictest requirements for rationality. It will also be shown how several related results on the relation between asymptotic stability and Mertens stability such as those in [5] and [6] can be deduced from this one.

2 Notation

Let v be a vector field on a smooth orientable n-dimensional manifold M and let K be a connected compact component of zeros of v. To avoid pathologies, we shall assume that v is quasi-algebraic, or subanalytic, so that K will be homeomorphic to a polyhedron and will have neighborhood retracts. This is the case that arises in applications to game theory.

In the sequel, all homology and cohomology groups will be taken with coefficients in $\mathbb{Q}$. Let U be a relatively compact neighborhood of K such that $v^{-1}(0) \cap U = K$, let $\xi_K \in H^n(U; U \setminus K)$ be the orientation class around K and let $\tau \in H^n(TM; TM \setminus \{0\})$ be the Thom class. The index i_K is defined as the integer i_K such that:

$$v^*(\tau) = (-1)^n i_K \xi_K. \tag{1}$$

Where v^* is the induced map in cohomology. If K is a non-degenerate zero of v, and so it is reduced to a point p, we have:

$$i_p = sign(det(-J_v)), \tag{2}$$

where J_v is the Jacobian matrix of v in some coordinate chart. Moreover, if $\tilde{v}$ is a perturbation of v equal to v outside U, we have

$$i_K(v) = \sum_{K_l \subset U} i_{K_l}(\tilde{v}), \tag{3}$$

where K_l runs over the connected components of $\tilde{v}^{-1}(0)$ contained in U.

If v is a vector field, φ_t will be its associated flow. Note that if p is an isolated regular zero of v,

$$i_p(v) = index_p(\varphi_t), \tag{4}$$

where $index_p(\varphi_t)$ is the index of a fixed point of a diffeomorphism as it appears in the Lefschetz formula; see Chapter 8 of [3].

3 Main Result

Let v, φ_t, and K be as before. Recall the definition of asymptotic stability as given in [5].

Definition 1 *A closed invariant set K is asymptotically stable if for every neighborhood V of K there exists a neighborhood $U \subset V$ such that $\forall t \geq 0, \forall x \in U$, $\varphi_t(x) \in V$ and $d(\varphi_t(x); K) \to 0$ as $t \to \infty$. Where d is the metric inducing the standard topology on the manifold.*

Of course, if K is a set of zeros it is closed and invariant. We are now ready to prove our main theorem.

Theorem 1 *Let K be a component of zeros that is asymptotically stable for φ_t, if $\chi(K)$ is the Euler characteristic of K, we have:*

$$\chi(K) = i_K(v).$$

Proof. Before giving the proof, let us note that if p is a regular zero of v, the theorem is obvious by linearization and says that $i_p(v) = 1$; the theorem is also easy to prove if K admits an invariant neighborhood N of which it is a deformation retract and such that $(N; \partial N)$ is a smooth manifold with boundary, see [1]. It is not clear (at least to the author) that a neighborhood that is both invariant and a smooth manifold with boundary always exists; to avoid this difficulty we proceed as follows.

Let $U_1 \supset N \supset U_2 \supset U_3$ be relatively compact open neighborhoods of K with the following properties:

1) U_i deformation retracts onto K for $i = 1, 2, 3,$; (this is possible since v is semi-algebraic);

2) $(N; \partial N)$ is a smooth manifold with boundary;

3) $\varphi_t(N) \subset U_1$ $\forall t \geq 0$;

4) $\exists T > 0$ such that $\varphi_t(U_1) \subset U_2$ $\forall t \geq T$;

5) $\varphi_t(U_3) \subset \varphi_T(N)$ $\forall t \geq 0$ with T as in 4).

In order to satisfy condition 3,4, and 5, we use the asymptotic stability of K. Note that the maps φ_t can be thought, for $t > T$, as maps from N into N.

Lemma 1 *The homology groups $H_i(N)$ split as direct sums $H_i(K) \oplus L_i$, we call L_i the kernel of the map b_* defined in the proof, the map $(\varphi_t)_*$ induced by $\varphi_t : N \to N$, $t \geq T$ as before, is the identity on the $H_i(K)$ factor image $(\varphi_t)_* \subset H_i(K)$. It follows that if $L(\varphi_t)$ is the Lefschetz number $\sum_i (-1)^i trace(\varphi_t)_{*|H_i(N)}$, we have:*

$$L(\varphi_t) = \chi(K), t \geq T.$$

Proof of the Lemma. Consider the chain of inclusions:

$$U_2 \overset{a}{\hookrightarrow} N \overset{b}{\hookrightarrow} U_1 \tag{5}$$

and the induced maps in homology:

$$H_i(U_2) \xrightarrow{a} H_i(N) \xrightarrow{b} H_i(U_1).$$

By property 1) $H_i(U_1) \approx H_i(U_2) \approx H_i(K)$ and so $b_* \circ a_*$ is the identity. So if we identify $H_i(K)$ with the image of a_* and we call L_i the kernel of b_* the splitting follows. The map φ_t factorizes as:

$$\varphi_t : N \to \varphi_t(N) \hookrightarrow U_2 \overset{a}{\hookrightarrow} N,$$

so $image((\varphi_t)_*) \subset image(a_*) = H_i(K)$. Moreover φ_t is the identity on K so $(\varphi_t)_*$ is the identity on the factor $H_i(K)$ of $H_i(N)$.

The last part of the lemma follows by linear algebra. $\triangle$

It is not hard to see that $(\varphi_t)_*$ is zero on L_i. Deform now v to $\tilde{v}$ by a small homotopy such that:

a) the homotopy is the identity outside U_3;

b) if $\tilde{\varphi}_t$ is the associated flow to v, $\tilde{\varphi}_t$ is Kupka-Smale, so it has finitely many zeros and periodic orbits, and they are all regular, [3], Section 17.2.

Since the homotopy from v to $\tilde{v}$ is the identity outside U_3 for any t, the induced homotopy from φ_t to $\tilde{\varphi}_t$ is such that either it connects $\varphi_t(x)$ to $\tilde{\varphi}_t(x)$ with a path contained in the orbit $\cup_{t\geq 0}\varphi_t(U_3) \subset U_2 \subset N$ or $\varphi_t(x) = \tilde{\varphi}_t(x)$, it follows that for $t > T$ φ_t and $\tilde{\varphi}_t$ are homotopic as maps from N into itself. The homotopy of φ and $\tilde{\varphi}_t$ implies that:

$$L(\varphi_t) = L(\tilde{\varphi}_t). \tag{6}$$

Now choose $t > T$ such that there is no orbit of $\tilde{v}$ of period t or a submultiple of it; this means that the only fixed points of $\tilde{\varphi}_t$ are the zeros of $\tilde{v}$. By the Lefschetz formula applied to the manifold with boundary N (see Chapter 8 of [3]) we have:

$$L(\tilde{\varphi}_t) = \sum_{p_l \in Fix\tilde{\varphi}_t, p_l \subset N} index_{p_l}(\tilde{\varphi}_t),$$

but $index_{p_l}(\tilde{\varphi}_t) = i_{p_l}(\tilde{v})$ by (4) and $\sum_{p_l \in N} i_{p_l}(\tilde{v}) = i_K(v)$ by (3), and now by the lemma:

$$\chi(K) = L(\varphi_t) = L(\tilde{\varphi}_t) = i_K(v).$$

Q.E.D.

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