Kato's inequality when Δu is a measure

L'inégalité de Kato lorsque Δu est une mesure

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Abstract

We extend the classical Kato's inequality in order to allow functions $u \in L^1_{\mathrm{loc}}$ such that Δu is a Radon measure. This inequality has been recently applied by Brezis, Marcus, and Ponce [5] to study the existence of solutions of the nonlinear equation $-\Delta u + g(u) = \mu$, where μ is a measure and $g : \mathbb{R} \to \mathbb{R}$ is an increasing continuous function. *To cite this article: H. Brezis, A.C. Ponce, C. R. Acad. Sci. Paris, Ser. I XXX (2004).*

Résumé

Nous étendons l'inégalité de Kato classique à des fonctions $u \in L^1_{\mathrm{loc}}$ telles que Δu est une mesure de Radon. Cette inégalité a été récemment utilisée par Brezis, Marcus et Ponce [5] pour étudier l'existence des solutions de l'équation elliptique non linéaire $-\Delta u + g(u) = \mu$, où μ est une mesure et $g : \mathbb{R} \to \mathbb{R}$ est une fonction croissante et continue. *Pour citer cet article : H. Brezis, A.C. Ponce, C. R. Acad. Sci. Paris, Ser. I XXX (2004).*

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Soient $N \geq 1$ et $\Omega \subset \mathbb{R}^N$ un ouvert borné quelconque. Selon l'inégalité de Kato classique (voir [8]), étant donné $u \in L^1_{\mathrm{loc}}(\Omega)$ tel que $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$, alors Δu^+ est une mesure de Radon et, de plus,

$$\Delta u^+ \geq \chi_{[u \geq 0]} \Delta u \quad \text{dans } \mathcal{D}'(\Omega). \tag{1}$$

Nous étendons (1) à des fonctions $u \in L^1_{\mathrm{loc}}(\Omega)$ telles que $\Delta u \in \mathcal{M}(\Omega)$, où $\mathcal{M}(\Omega)$ désigne l'espace des mesures de Radon définies sur Ω.

Rappelons que toute mesure $\mu \in \mathcal{M}(\Omega)$ peut être décomposée de façon unique comme une somme de deux mesures de Radon sur Ω (voir e.g. [7]) : $\mu = \mu_{\mathrm{d}} + \mu_{\mathrm{c}}$, avec

$$\begin{aligned} \mu_{\mathrm{d}}(A) = 0 &\quad \text{pour tout borélien } A \subset \Omega \text{ tel que } \operatorname{cap}(A) = 0, \\ |\mu_{\mathrm{c}}|(\Omega \backslash F) = 0 &\quad \text{pour un ensemble } F \subset \Omega \text{ fixé tel que } \operatorname{cap}(F) = 0, \end{aligned}$$

où cap dénote la capacité newtonienne ($W^{1,2}$). Les mesures $\mu_{\rm d}$ et $\mu_{\rm c}$ sont mutuellement singulières ; en particulier, $(\mu_{\rm d})^+ = (\mu^+)_{\rm d}$ et $(\mu_{\rm c})^+ = (\mu^+)_{\rm c}$.

Notre théorème principal est le suivant :

Théorème 0.1 *Soit $u \in L^1_{\rm loc}(\Omega)$ tel que $\Delta u \in \mathcal{M}(\Omega)$. Alors, $\Delta u^+ \in \mathcal{M}(\Omega)$ et, de plus,*

$$(\Delta u^+)_{\rm d} \geq \chi_{[u\geq 0]}(\Delta u)_{\rm d} \quad \textit{sur } \Omega, \tag{2}$$

$$(-\Delta u^+)_{\rm c} = (-\Delta u)^+_{\rm c} \quad \textit{sur } \Omega. \tag{3}$$

Le membre de droite dans (2) est bien défini, car la fonction u est quasicontinue (voir [1], voir aussi [4, Lemme 1]).

1. Introduction and main result

Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded open subset. The classical Kato's inequality (see [8]) states that given any function $u \in L^1_{\rm loc}(\Omega)$ such that $\Delta u \in L^1_{\rm loc}(\Omega)$, then Δu^+ is a Radon measure and the following holds:

$$\Delta u^+ \geq \chi_{[u\geq 0]}\Delta u \quad \text{in } \mathcal{D}'(\Omega). \tag{4}$$

Our main result in this paper (see Theorem 1.1 below) extends (4) to the case $\Delta u \in \mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ denotes the space of Radon measures on Ω. In other words, $\mu \in \mathcal{M}(\Omega)$ if and only if, for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ such that $\left|\int_\Omega \varphi \, d\mu\right| \leq C_\omega \|\varphi\|_\infty$, $\forall \varphi \in C_0^\infty(\omega)$.

We first recall that any $\mu \in \mathcal{M}(\Omega)$ can be uniquely decomposed as a sum of two Radon measures on Ω (see e.g. [7]): $\mu = \mu_{\rm d} + \mu_{\rm c}$, where

$$\mu_{\rm d}(A) = 0 \quad \text{for any Borel measurable set } A \subset \Omega \text{ such that } \operatorname{cap}(A) = 0,$$

$$|\mu_{\rm c}|(\Omega \backslash F) = 0 \quad \text{for some Borel measurable set } F \subset \Omega \text{ such that } \operatorname{cap}(F) = 0.$$

Here, cap denotes the Newtonian ($W^{1,2}$) capacity of a set. We observe that $\mu_{\rm d}$ and $\mu_{\rm c}$ are singular with respect to each other. This decomposition is the analog of the classical Radon-Nikodym Theorem, but with respect to cap. Clearly, $(\mu_{\rm d})^+ = (\mu^+)_{\rm d}$ and $(\mu_{\rm c})^+ = (\mu^+)_{\rm c}$.

Using the above notation, we can now state our main result:

Theorem 1.1 *Let $u \in L^1_{\rm loc}(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. Then, $\Delta u^+ \in \mathcal{M}(\Omega)$, and the following holds:*

$$(\Delta u^+)_{\rm d} \geq \chi_{[u\geq 0]}(\Delta u)_{\rm d} \quad \textit{on } \Omega, \tag{5}$$

$$(-\Delta u^+)_{\rm c} = (-\Delta u)^+_{\rm c} \quad \textit{on } \Omega. \tag{6}$$

Note that the right-hand side of (5) is well-defined because u is quasicontinuous. More precisely, if $u \in L^1_{\rm loc}(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$, then there exists $\tilde{u} : \Omega \to \mathbb{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in Ω (see [1] and also [4, Lemma 1]). In (5), we then identify u with its quasicontinuous representative. It is easy to see that $\chi_{[u\geq 0]}$ is locally integrable in Ω with respect to the measure $\left|(\Delta u)_{\rm d}\right|$.

The proof of (5) requires a theorem of Boccardo, Gallouët, and Orsina [2], which says that a Radon measure μ is diffuse (i.e. $\mu_{\rm c} = 0$) if and only if $\mu \in L^1_{\rm loc}(\Omega) + \Delta\big[H^1_{\rm loc}(\Omega)\big]$. Identity (6) relies on (and in fact is equivalent to) the "inverse" maximum principle, recently established by Dupaigne and Ponce [6] (see Theorem 3.1 below).

An equivalent statement of Theorem 1.1 is the following:

Corollary 1.2 *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. Then, $\Delta|u| \in \mathcal{M}(\Omega)$, and the following holds:*

$$(\Delta|u|)_{\mathrm{d}} \geq \operatorname{sgn}(u)\,(\Delta u)_{\mathrm{d}} \quad \text{on } \Omega, \tag{7}$$

$$(\Delta|u|)_{\mathrm{c}} = -|\Delta u|_{\mathrm{c}} \quad \text{on } \Omega. \tag{8}$$

Here, $\operatorname{sgn}(t) = 1$ for $t > 0$, $\operatorname{sgn}(t) = -1$ for $t < 0$, and $\operatorname{sgn}(0) = 0$.

Remark 1 A slight modification of the proof of Theorem 1.1 shows that

$$(\Delta u^+)_{\mathrm{d}} \geq \chi_{[u>0]}(\Delta u)_{\mathrm{d}} \quad \text{on } \Omega. \tag{9}$$

In other words, we can replace the set $[u \geq 0]$ in (5) by $[u > 0]$ and still get the same result.

Here is a simple consequence of (9):

Corollary 1.3 *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. If $u \geq 0$ a.e. in Ω, then*

$$(\Delta u)_{\mathrm{d}} \geq 0 \quad \text{on the set } [u = 0]. \tag{10}$$

2. Proof of (5) in Theorem 1.1

We start with the following:

Lemma 2.1 *Assume $\mu \in \mathcal{M}(\Omega)$ is a diffuse measure with respect to* cap *(i.e. $\mu_{\mathrm{c}} = 0$ on Ω). Let (v_n) be a sequence in $L^\infty(\Omega) \cap H^1(\Omega)$ such that $\|v_n\|_\infty \leq C$ and $v_n \rightharpoonup v$ in H^1. Then,*

$$v_n \to v \quad \text{in } L^1_{\mathrm{loc}}(\Omega; d\mu). \tag{11}$$

Equivalently, there exists a subsequence (v_{n_k}) converging to v $|\mu|$-a.e. in Ω.

Proof. Without loss of generality, we may assume that $|\mu|(\Omega) < \infty$. By Theorem 2.1 of Boccardo, Gallouët, and Orsina [2], we know that $\mu = f - \Delta g$ in $\mathcal{D}'(\Omega)$, for some $f \in L^1(\Omega)$ and $g \in H^1(\Omega)$. Using a standard density argument, we conclude that

$$\int_\Omega w\varphi \, d\mu = \int_\Omega w\varphi f + \int_\Omega \nabla g \cdot \nabla(w\varphi), \quad \forall \varphi \in C_0^\infty(\Omega), \quad \forall w \in L^\infty \cap H^1. \tag{12}$$

By assumption, the sequence $\big(|v_n - v|\big)$ is bounded in $H^1(\Omega)$ and, by Rellich's theorem, $|v_n - v| \to 0$ in $L^2(\Omega)$. Thus,

$$|v_n - v| \rightharpoonup 0 \quad \text{in } H^1. \tag{13}$$

Given $\varepsilon > 0$, let $\omega \subset\subset \Omega$ be such that $|\mu|(\Omega \backslash \omega) < \varepsilon$. We then fix $\varphi_0 \in C_0^\infty(\Omega)$ so that $0 \leq \varphi_0 \leq 1$ in Ω and $\varphi_0 = 1$ on ω. Applying (12) with $w = |v_n - v|$ and $\varphi = \varphi_0$, we have

$$\begin{aligned}\int_\Omega |v_n - v| \, d\mu &\leq \int_\omega |v_n - v| \, d\mu + 2C|\mu|(\Omega\backslash\omega) \\ &\leq \int_\Omega |v_n - v|\varphi_0 \, d\mu + 2C\varepsilon = \int_\Omega |v_n - v|\varphi_0 f + \int_\Omega \nabla g \cdot \nabla\big(|v_n - v|\varphi_0\big) + 2C\varepsilon.\end{aligned}$$

By (13), we know that $\int_\Omega \nabla g \cdot \nabla\big(|v_n - v|\varphi_0\big) \to 0$ as $n \to \infty$. Since (v_n) is bounded in L^∞ and $v_n \to v$ in L^2, we have $v_n \rightharpoonup v$ with respect to the weak* topology of L^∞; thus, $\int_\Omega |v_n - v|\varphi_0 f \to 0$. We conclude that $\limsup_{n\to\infty} \int_\Omega |v_n - v| \, d\mu \leq 2C\varepsilon$. Taking $\varepsilon > 0$ arbitrarily small, (11) follows.

Given $k > 0$, we denote by $T_k : \mathbb{R} \to \mathbb{R}$ the truncation operator, i.e. $T_k(s) = s$ if $s \in [-k, k]$ and $T_k(s) = \operatorname{sgn}(s)\, k$ if $|s| > k$. Recall the following standard inequality (see e.g. [4, Lemma 1]):

Lemma 2.2 *Assume $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$. Then, $T_k(u) \in H^1_{\mathrm{loc}}(\Omega)$, $\forall k > 0$; moreover, given $\omega \subset\subset \omega' \subset\subset \Omega$, there exists $C > 0$ such that*

$$\int_\omega \big|\nabla T_k(u)\big|^2 \le k\bigg(\int_{\omega'} |\Delta u| + C\int_{\omega'} |u|\bigg). \tag{14}$$

Another ingredient to prove (5) is our next result, which extends Lemma 2 in [3]:

Proposition 2.1 *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a C^1-convex function such that $0 \le \Phi' \le 1$ on $\mathbb{R}$. If $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$, then*

$$\Delta\Phi(u) \ge \Phi'(u)(\Delta u)_{\mathrm{d}} - (\Delta u)_{\mathrm{c}}^- \quad \text{in } \mathcal{D}'(\Omega). \tag{15}$$

Proof. Without loss of generality, we shall assume that $\Phi \in C^2$ and Φ'' has compact support in $\mathbb{R}$. The general case can be easily deduced by approximation (note that since Φ is convex and Φ' is uniformly bounded, both limits $\Phi'(\pm\infty)$ exist and are finite). We may also assume that $u \in L^1(\Omega)$ and $\int_\Omega |\Delta u| < \infty$.

For every $x \in \Omega$, define $u_n(x) = \rho_n * u(x) = \int_\Omega \rho_n(x-y)u(y)\,dy$, where ρ_n is a family of radial mollifiers such that $\operatorname{supp}\rho_n \subset B_{1/n}$. Since $\Phi'' \ge 0$ in $\mathbb{R}$, we have

$$\Delta\Phi(u_n) = \Phi'(u_n)\Delta u_n + \Phi''(u_n)|\nabla u_n|^2 \ge \Phi'(u_n)\Delta u_n \quad \text{in } \Omega.$$

Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \ge 0$. We multiply both sides of the inequality above by φ and integrate by parts. For every $n \ge 1$ such that $d(\operatorname{supp}\varphi, \partial\Omega) > 1/n$, we have

$$\begin{aligned}\int_\Omega \Phi(u_n)\Delta\varphi &\ge \int_\Omega \Phi'(u_n)\varphi\,\Delta u_n\\ &= \int_\Omega \Big\{\rho_n * \big[\Phi'(u_n)\varphi\big]\Big\}\,\Delta u \ge \int_\Omega \Big\{\rho_n * \big[\Phi'(u_n)\varphi\big]\Big\}\,(\Delta u)_{\mathrm{d}} - \int_\Omega (\rho_n * \varphi)\,(\Delta u)_{\mathrm{c}}^-.\end{aligned}$$

Clearly,

$$\int_\Omega \Phi(u_n)\Delta\varphi \to \int_\Omega \Phi(u)\Delta\varphi \qquad \text{and} \qquad \int_\Omega (\rho_n * \varphi)\,(\Delta u)_{\mathrm{c}}^- \to \int_\Omega \varphi\,(\Delta u)_{\mathrm{c}}^-. \tag{16}$$

We now establish the following:

Claim. $\rho_n * \big[\Phi'(u_n)\varphi\big] \rightharpoonup \Phi'(u)\varphi$ in $H^1(\Omega)$.

In fact, since $\rho_n * \big[\Phi'(u_n)\varphi\big] \to \Phi'(u)\varphi$ in, say, $L^1(\Omega)$ and since φ has compact support in Ω, it suffices to show that $\big(\Phi'(u_n)\big)$ is bounded in $H^1_{\mathrm{loc}}(\Omega)$. Let $M > 0$ be such that $\operatorname{supp}\Phi'' \subset [-M, M]$. Then,

$$\nabla\Phi'(u_n) = \Phi''(u_n)\nabla u_n = \Phi''(u_n)\nabla T_M(u_n) \quad \text{in } \Omega.$$

Let $\omega \subset\subset \omega' \subset\subset \Omega$. For $n \ge 1$ sufficiently large, it follows from (14) that

$$\int_\omega \big|\nabla\Phi'(u_n)\big|^2 \le \|\Phi''\|_\infty \int_\omega \big|\nabla T_M(u_n)\big|^2 \le CM\bigg(\int_{\omega'} |u_n| + \int_{\omega'} |\Delta u_n|\bigg) \le CM\bigg(\int_\Omega |u| + \int_\Omega |\Delta u|\bigg),$$

for some constant $C > 0$ independent of n.

In view of the previous claim, we can now apply Lemma 2.1 above with $v_n = \rho_n * \big[\Phi'(u_n)\varphi\big]$ and $\mu = (\Delta u)_{\mathrm{d}}$ to conclude that

$$\int\limits_\Omega \Big\{\rho_n * \big[\Phi'(u_n)\varphi\big]\Big\}\,(\Delta u)_{\mathrm{d}} \to \int\limits_\Omega \Phi'(u)\varphi\,(\Delta u)_{\mathrm{d}}. \tag{17}$$

Combining (16) and (17) yields

$$\int\limits_\Omega \Phi(u)\Delta\varphi \geq \int\limits_\Omega \Phi'(u)\varphi\,(\Delta u)_{\mathrm{d}} - \int\limits_\Omega \varphi\,(\Delta u)_{\mathrm{c}}^-, \quad \forall\varphi \in C_0^\infty(\Omega) \text{ with } \varphi \geq 0 \text{ in } \Omega,$$

which is precisely (15).

Proof of (5). Let (Φ_n) be a sequence of smooth convex functions in $\mathbb{R}$ such that $\Phi_n(t) = t$ if $t \geq 0$ and $\big|\Phi_n(t)\big| \leq 1/n$ if $t < 0$. In particular, $0 \leq \Phi' \leq 1$ in $\mathbb{R}$. It follows from the previous proposition that

$$\Delta\Phi_n(u) \geq \Phi_n'(u)(\Delta u)_{\mathrm{d}} - (\Delta u)_{\mathrm{c}}^- \quad \text{in } \mathcal{D}(\Omega).$$

As $n \to \infty$, we get

$$\Delta u^+ \geq \chi_{[u \geq 0]}(\Delta u)_{\mathrm{d}} - (\Delta u)_{\mathrm{c}}^- \quad \text{in } \mathcal{D}(\Omega). \tag{18}$$

In particular, $\Delta u^+ \in \mathcal{M}(\Omega)$. Taking the diffuse part from both sides of (18), we conclude that (5) holds.

3. Proof of (6) in Theorem 1.1

Identity (6) relies on the following:

Theorem 3.1 ("Inverse" maximum principle [6]) *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. If $u \geq 0$ a.e. in Ω, then*

$$(-\Delta u)_{\mathrm{c}} \geq 0 \quad \text{on } \Omega. \tag{19}$$

To complete the proof of Theorem 1.1, we now present:

Proof of (6). From the proof of (5), we already know that Δu^+ is a Radon measure on Ω. Applying the "inverse" maximum principle to u^+, we have $(-\Delta u^+)_{\mathrm{c}} \geq 0$ on Ω. Since $u^+ - u \geq 0$ a.e. in Ω, it also follows from Theorem 3.1 above that $(-\Delta u^+)_{\mathrm{c}} \geq (-\Delta u)_{\mathrm{c}}$ on Ω. Thus,

$$(-\Delta u^+)_{\mathrm{c}} \geq (-\Delta u)_{\mathrm{c}}^+ \quad \text{on } \Omega,$$

which gives the "$\geq$" in (6). The reverse inequality just follows by taking the concentrated part from both sides of (18). In fact,

$$(-\Delta u^+)_{\mathrm{c}} \leq (\Delta u)_{\mathrm{c}}^- = (-\Delta u)_{\mathrm{c}}^+ \quad \text{on } \Omega.$$

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