RISK, INCOME TAXATION, AND PUBLIC INVESTMENT CRITERIA

by

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ABSTRACT

The use of market prices in evaluation of public investments is studied in a one-commodity two-period model with risk and with personal and corporate income taxes. A general measure of riskiness is proposed and employed. Economies with a complete set of insurance markets and with only bond and common stock markets are studied. In general, simple public investment criteria based on market prices exist only under special conditions. Things are simplified if, for instance, all personal tax rates are equal and if the returns to the proposed investment can be related to the average returns to private investments.

This essay is concerned with criteria for the evaluation of "small" risky government investments. Particular stress is placed on the use of market data, along with information on the returns from the contemplated investment and how it is to be financed. The impact of alternative assumptions about the nature of the pre-existing equilibrium is studied, and the effects of personal and corporate income taxes are analyzed under these assumptions.

We follow most of the relevant literature and consider a one-commodity, two-period world. This permits us to ignore the difficult problems of planning long-term capital formation and of the formation and impact of price expectations.[1] It also makes possible simple and unambiguous measurement of the real costs and benefits of public investment, as there are no externalities or public goods in the model. Further, it is assumed that social choice should reflect individual preferences, so that we work with a Bergson social welfare function. Finally, all households in this model are present in both periods, so that no questions of inter-generational equity arise.[2]

The analysis uses the state preference approach throughout. It is kept simple (or old-fashioned) by assuming that all functions are differentiable and that non-negativity constraints are not binding in the solutions to maximization problems.

The following section presents our assumptions about firms, households, and the nature of risk in the economy. An apparently new measure of riskiness is presented there and used in what follows. Section II considers social discounting when the economy possesses complete and perfect insurance markets. The results obtained there are intended to serve as a benchmark to which those of the next two sections may be compared. Section III retains the assumption of complete markets and examines the impact of personal and business income taxation. Section IV considers two models involving incomplete insurance markets. Finally, some conclusions and implications of the analysis are discussed in Section V.

I. Households and Firms

There are H households in the economy, indexed $h=1, \ldots, H$. Let c_o^h denote the consumption of a typical household in the first period. In the second period, one of S possible states of nature will occur; these are indexed $s=1, \ldots, S$. Let c_s^h be the second-period consumption of household h if state s occurs. We assume that π_s, the probability that state s in fact occurs, is objectively known for all s. There is thus risk in the economy, but no uncertainty in the Knightian sense.

Individual welfare is assumed to be measured by a strictly increasing quasi-concave utility function :

$$(1.1)\quad U^h = U^h(c_o^h,\ c_1^h, \ldots,\ c_S^h), \qquad h=1, \ldots, H,$$

and social welfare is assumed to be measured by the following Bergsonian social welfare function :

$$(1.2)\quad W = \sum_{h=1}^{H} w^h U^h,$$

where, without loss of generality, the w^h are taken to be positive constants. It is assumed throughout this study that the w^h cannot be observed directly, so that no public investment criteria that involve these quantities are considered.

For convenience, we define

$$(1.3)\quad MU_i^h = \partial U^h / \partial c_i^h, \qquad h=1, \ldots, H;\ i=0, 1, \ldots, S.$$

The households' marginal rates of substitution between present consumption and uncertain future consumption are then given by

$$(1.4)\quad q_s^h = MU_s^h / MU_o^h, \qquad h=1, \ldots, H;\ s=1, \ldots, S.$$

For any household, q_s^h for any s is in general a function of $(c_o^h,\ c_1^h, \ldots, c_S^h)$;

it gives the per-unit price the household would pay today for a small increase in consumption in state s.

The per-unit price the household would pay today to receive a small amount of certain future consumption must thus be the sum of the q_s^h. Since this price must also be the reciprocal of one plus that household's rate of time preference, ρ^h, that quantity is given by

$$(1.5) \quad \rho^h = \left[1/\sum_{s=1}^{S} q_s^h\right] - 1, \qquad h=1,\ldots,H.$$

To develop a measure of the risk each household attaches to streams of uncertain future consumption, we first define

$$(1.6) \quad y_s^h = 1 - MU_s^h/\left[\pi_s \sum_{j=1}^{S} MU_j^h\right], \qquad h=1,\ldots,H; \; s=1,\ldots,S.$$

These quantities are easiest to interpret in the special case where U^h may be written as

$$(1.7) \quad U^h = \sum_{s=1}^{S} \pi_s u^h(c_o^h, c_s^h), \qquad h=1,\ldots,H.$$

Then if u^h for a given h is strictly concave in its second argument, y_s^h will vary directly with c_s^h. If this function is linear in its second argument, so the household is everywhere risk-neutral, y_s^h is zero for all s. In general, the expected value of y_s^h is zero by construction. Definitions (1.4) - (1.6) and some algebra yield the useful identity.

$$(1.8) \quad q_s^h = \pi_s(1 - y_s^h)/(1 + \rho^h), \qquad h=1,\ldots H; \; s=1,\ldots,S.$$

Suppose now that the household is confronted with a security that promises x_s in state s, s=1,...,S. If these payoffs are small enough, the maximum price the household would pay today for this security is given by

$$(1.9) \quad \sum_{s=1}^{S} q_s^h x_s = \left[\sum_{s=1}^{S} \pi_s x_s - \sum_{s=1}^{S} \pi_s y_s^h x_s \right] / (1 + \rho^h).$$

The second sum in brackets is clearly an allowance for risk; it vanishes if the security is riskless. Accordingly, we define our (marginal) risk measure as[4].

$$(1.10) \quad \phi^h(x) \equiv \sum_{s=1}^{S} \pi_s y_s^h x_s, \qquad h=1,\ldots,H.$$

Since y^h has zero mean, this quantity is equal to the covariance of y^h and x. In the special case where U^h is given by (1.7), and u^h is quadratic in its second argument, ϕ can be shown to be proportional to covariance of y^h and c^h.[5] ϕ is clearly zero if either x_s or y_s^h is the same for all s. It is also zero if x is distributed independently of the risks reflected in the π_s.

To make this last point a bit clearer, suppose that x can take on values $x_1, \ldots, x_I$, with probabilities $\pi_1, \ldots, \pi_I$. Let the probability that state of nature s occurs and $x = x_i$ be π_{is}. The random variable x is then said to be distributed independently of the risks reflected in the π_s if $\pi_s = \pi_{is} = \pi_i \pi_s$, $i=1,\ldots,I$; $s=1,\ldots,S$.
In this case,

$$(1.11) \quad \phi^h(x) = \sum_{s=1}^{S} \sum_{i=1}^{I} \pi_{is} y_s^h x_i = \sum_{i=1}^{I} \pi_i x_i \sum_{s=1}^{S} \pi_s y_s^h = 0, \qquad h=1,\ldots,H.$$

Note that $\phi^h(x)$ need not be positive even if x has a high variance. If (1.7) holds, and u^h is strictly concave in its second argument, this measure gives negative weight to receipts in states in which the consumer is poorly off. Thus it emphasizes the general principle that the riskiness of any asset cannot generally be evaluated in terms of the (marginal) probability distribution of its returns; the consumer's welfare in the various states of nature must be explicitly considered.[6]

Let us now turn to the economy's production possibilities. It is assumed that, in addition to the proposed government investment, there are F other activities carried on in the economy, indexed $f=1,\ldots,F$. Some of these may be government projects, while the others are assumed to be carried on by private firms, who are further assumed to have access to one and only one activity each. For simplicity, we shall refer to all activities as "firms", but it should be kept in mind that some may be operated by the public sector.

Let I^f, always assumed positive, be investment in the first period in firm f. It is then assumed that the second-period output of this firm in state s is given by the concave function $R_s^f(I^f)$, $f=1,\ldots,F$; $s=1,\ldots,S$.[7] For each firm and any positive investment level, we define

$$(1.12) \qquad MR_s^f = dR_s^f/dI^f, \qquad\qquad s=1,\ldots,S; \quad f=1,\ldots,F.$$

The marginal rate of return, r^f, on investment in firm f is then given by

$$(1.13) \qquad r^f = \sum_{s=1}^{S} \pi_s MR_s^f - 1, \qquad\qquad f=1,\ldots,F.$$

If we let $AR_s^f = R_s^f/I^f$, $s=1,\ldots,S$; $f=1,\ldots,F$, the firm's average rate of return, $\bar{r}^f$, is simply

$$(1.14) \qquad \bar{r}^f = \sum_{s=1}^{S} \pi_s AR_s^f - 1, \qquad\qquad f=1,\ldots,F.$$

By the concavity of the R_s^f, $\bar{r}^f \geqslant r^f$ for any I^f, $f=1,\ldots,F$.

II. Complete and Perfect Markets

We must now describe the proposed "small" government investment and how it is to be financed. For convenience, let the current cost of the project be unity. If its contingent future returns are then G_s, $s=1,\ldots,S$, its expected rate of return, r^g, is defined by

$$(2.1)\qquad r^g = \sum_{s=1}^{S} \pi_s G_s - 1.$$

We shall derive various tests based on this quantity.

In practice, it may sometimes be possible to express the G_s approximately as follows :

$$(2.2)\quad G_s = K^R + K^I x_s + \sum_{f=1}^{F} K_f^M \; MR_s^f + \sum_{f=1}^{F} K_f^A \; AR_s^f, \quad s=1,\ldots,S,$$

where the K's are constants. The first term is clearly the riskless component of the project's returns. The x_s are assumed to have unit mean and to be distributed independently of the other risks in the economy, as that term was defined above equation (1.11).[8] Finally, the terms involving the K_f^M and the K_f^A give the component of the project's returns that, in a general sense, resembles the returns from existing activities.[9]

For our purposes, it is not necessary to specify in detail what combination of borrowing and taxation would be used to finance the proposed project. It is simply assumed that if the project were adopted, current consumption of household h would fall by α^h $h=1,\ldots,H$, and current investment in firm f would be reduced by β^f, $f=1,\ldots,F$. Let $\bar{\alpha}$ be the sum of the α^h and $\bar{\beta}$ be the sum of the β^f. Then if, for instance, the project were to be financed by borrowing in a perfect capital market, we would have $\bar{\beta} = E^d/(E^d + E^s)$, where E^d and E^s are, respectively, the absolute values of the demand and supply elasticities on that market. Alternatively, if the project were to be financed by a reduction in households' current income, and all households had marginal propensity to consume m^c, we would have $\bar{\alpha} = (1-m^c)$.

Clearly, if the economy is fully-employed and financing is through borrowing or lump-sum taxation, $(\bar{\alpha} + \bar{\beta})$ will equal one. For simplicity, we assume that this is the case.[10]

Let γ_s^h be the fraction of the project's output received by household h in state s, h=1,...,H; s=1,...,S, and let δ_s^{fh} be the fraction of the project-induced fall in firm f's output given up by household h in state s, f=1,...,F; h=1,...,H; s=1,...,S. For any s, the γ_s^h must sum to one across h, while the constraints on the δ_s^{fh} are

$$(2.3)\quad \sum_{f=1}^{F} \beta^f \; MR_s^f = \sum_{f=1}^{F} \sum_{h=1}^{H} \beta^f \delta_s^{fh} \; MR_s^f, \qquad s=1,\ldots,S.$$

Since the proposed project is assumed small, its impact on social welfare can be found by valuing all consumption changes at their social marginal utilities. Under this assumption the project will increase W if and only if

$$(2.4)\quad \sum_{s=1}^{S}\left[\sum_{h=1}^{H} \gamma_s^h w^h MU_s^h\right] G_s > \sum_{h=1}^{H} \alpha^h w^h MU_o^h + \sum_{f=1}^{F}\beta^f \left\{ \sum_{s=1}^{S}\left[\sum_{h=1}^{H} \delta_s^{fh} w^h MU_s^h\right] MR_s^f \right\}.$$

In what follows, we indicate assumptions about the nature of the pre-existing equilibrium that permit this test to be simplified and expressed in terms of (theoretically) observable quantities.

In the simplest case, which provides a useful benchmark, there are no income taxes, and S perfect insurance markets exist, on which households can buy and sell contingent claims. Let $\hat{q}_s$ be the market price in terms of current consumption, established on the s^{th} such market, of basic security s, which entitles its holder to one unit of future consumption if and only if state s occurs, s=1,...,S. The market riskless rate of interest, $\hat{\rho}$, must then be given by

$$(2.5)\quad \hat{\rho} = \left[1/\sum_{s=1}^{S} \hat{q}_s \right] - 1.$$

Similarly, for any random variable x , the market's evaluation of riskiness may be computed as

$$(2.6)\quad \hat{\emptyset}(x) = \sum_{s=1}^{S} \pi_s \hat{y}_s x_s = \sum_{s=1}^{S} \left[\pi_s - (1 + \hat{\rho})\, \hat{q}_s \right] x_s ,$$

where the analog of (1.8) was used to define the $\hat{y}_s$ in terms of the $\hat{q}_s$.

Firms, in general, are allowed to borrow and to issue common stock. Since these securities are composite commodities, their market prices can be expressed in terms of the market prices of the basic securities. Suppose the owners of a typical firm decide to invest I^f, and to finance this by borrowing $b^f I^f$, $0 \leq b^f \leq 1$, and contributing $(1-b^f)I^f$ from their own resources. The firm will then yield its owners,

$$(2.7)\quad Z_s^f = R_s^f(I^f) - (1 + \hat{\rho})b^f I^f, \qquad s=1,\ldots,s;\ f=1,\ldots,F$$

in each state of nature.[11] If we let F' be the set of indices of those firms in which ownership shares are freely tradeable, the market value of a typical fraded firm, $\hat{V}^f$, must then be simply the market value of these contingent claims, minus the owners' original investment :

$$(2.8)\quad \hat{V}^f = \sum_{s=1}^{S} \hat{q}_s Z_s^f - (1-b^f)I^f = \sum_{s=1}^{S} \hat{q}_s R_s^f - I^f, \qquad f \in F'.$$

If a traded firm is operated in its owners' interest, $\hat{V}^f$ must be maximized. Let F'' be the set of indices of those firms for which the right-hand side of (2.8) is maximized. Since government activities, to which one cannot associate market values, may be run efficiently, F'' is not necessarily a subset of F'. Assuming interior solutions, we then have

$$(2.9)\quad \sum_{s=1}^{S} \hat{q}_s MR_s^f = 1 \qquad f \in F''.$$

Since households can buy and sell basic securities, and we ignore corner solutions, we must have $q_s^h = \hat{q}_s$, $h=1,\ldots,H;\ s=1,\ldots,S.$

Finally, if the government can transfer current income freely among households to maximize W, current social marginal utilities will be equalized, along with social marginal utilities in every state of nature.

Dividing (2.4) by the common value of $w^h MU_o^h$, assuming that $\beta^f = 0$ for $f \notin F''$,[12] substituting and simplifying, we obtain the basic acceptance test :

$$(2.10) \qquad \sum_{s=1}^{S} q_s G_s > 1 .$$

This, of course, is no surprise : when markets are complete and perfect, market prices should be used to evaluate government activities. Using (2.5), (2.6), (2.8), and (2.9), we obtain tests in terms of r^g :

$$\begin{aligned}(2.11) \qquad r^g - \hat{\phi}(G) > \hat{\rho} & \\ = [\bar{r}^f - \hat{\phi}(AR^f) - \bar{V}^f] / (1 + \bar{V}^f), & \qquad f \in F' \\ = r^f - \hat{\phi}(MR^f), & \qquad f \in F'',\end{aligned}$$

where $\bar{V}^f = \hat{V}^f / I^f$, $f \in F'$. The project's expected rate of return must be adjusted to reflect its riskiness. The adjusted rate of return may then be compared to the riskless rate of interest or to the adjusted marginal or average rates of return of existing firms. For $f \in F' \cap F''$, $\bar{V}^f$ serves to correct for the difference between average and marginal returns; it would be zero if these were equal in all states.

If we assume that $K_f^A = 0$ for $f \notin F'$, and $K_f^M = 0$ for $f \notin F''$, substitution of (2.2) into (2.10) yields another test :

$$(2.12) \qquad (K^R + K^I) + \sum_{f=1}^{F} K_f^M (1 + \hat{\rho}) + \sum_{f=1}^{F} K_f^A (1 + \bar{V}^f)(1 + \hat{\rho}) > 1 + \hat{\rho} .$$

Note that, as Arrow and Lind [2] have pointed out, independent returns should be treated as if they were riskless.[13]

III. Income Taxation

This section retains the assumption that insurance markets are complete and competitive, but allows for distortions caused by personal and corporate income taxes.[14] Let t^h be the personal tax rate faced by household h, h=1, . . . , H, and let τ^f be the corporate tax rate faced by firm f, f=1, . . . , F. All t^h and τ^f are assumed to be non-negative and less than one. The τ^f corresponding to activities carried on by the government or by unincorporated businesses should be understood to be zero.

Given b^f, τ^f, and I^f, a typical firm will yield its owners

$$(3.1) \quad Z_s^f = R_s^f - (1 + \hat{\rho})b^f I^f - \tau^f [R_s^f - \hat{\rho} b^f I^f - I^f], \quad s=1, \ldots, S; \; f=1, \ldots, F,$$

where, as in the U.S. corporate income tax, interest payments are assumed to be tax-deductible. If, as before, F' is the index set of those firms in which ownership shares are freely traded, the market values of such firms are given by

$$(3.2) \quad \hat{V}^f/(1 - \tau^f) = \sum_{s=1}^{S} \hat{q}_s R_s^f - I^f [(1 + \hat{\rho} B^f)/(1 + \hat{\rho})], \qquad f \in F',$$

where we define

$$(3.3) \quad B^f = (1 - b^f \tau^f)/(1 - \tau^f), \qquad f=1, \ldots, F.$$

For given I^f and $\tau^f > 0$, B^f is a decreasing function of b^f, so that market value is maximized when $b^f = 1$ and all investment is financed by borrowing.[15] It is easy to show that if interest payments were not tax deductable, $\hat{V}^f$ would be independent of b^f, as before, and B^f would be simply $1/(1 - \tau^f)$.

Now it would be clearly unrealistic to assume $b^f = 1$, since in the 1964-71 period, U.S. non-farm non-financial corporations borrowed only about 26.5 % of new funds.[16] On the other hand, this percentage seems too large to neglect; the assumption $b^f = 0$ also seems unwarranted.[17] In the absence of a better alternative, we simply treat the b^f as parameters determined outside the model. They are, in any case, reasonably easily observed.

If, as before, we define F'' as the index set of those firms for which the right-hand side of (3.2) is maximized and ignore corner solutions, then for these efficient firms,

$$(3.4) \quad \sum_{s=1}^{S} \hat{q}_s MR_s^f = (1 + \hat{\rho}\, B^f)/(1 + \hat{\rho}), \qquad f \in F''$$

For $b^f < 1$, the right-hand side of this equation is an increasing function of τ^f. Since, by concavity, the MR_s^f are non-increasing in I^f, the corporate income tax tends to discourage business investment. Further, unless the B^f are equal, total investment will be inefficiently allocated, in the sense that funds could be shifted among firms so as to increase total output in at least one state of nature without lowering it in any state.

Now consider a typical household, with current after-tax income of M_o^h and exogenous income M_s^h in state s, $s=1,\ldots,S$. Let m_s^h be the number of securities it buys that pay off if and only if state s occurs, $s=1,\ldots,S$, and suppose that a fraction d^h, $0 \leq d^h \leq 1$, of this investment is borrowed at the riskless rate of interest.[18] If d^h is taken as a parameter, the household's objective is to choose the m_s^h and total investment, I^h, to maximize U^h, subject to

$$(3.5a) \quad M_o^h - (1-d^h)I^h - c_o^h \geq 0,$$

$$(3.5b) \quad M_s^h + m_s^h - (1+\hat{\rho})d^h I^h - t^h \left[M_s^h + m_s^h - \hat{\rho}\, d^h I^h - I^h\right] - c_s^h \geq 0,$$

$$s=1,\ldots,S,$$

$$(3.5c) \quad I^h - \sum_{s=1}^{S} \hat{q}_s m_s^h \geq 0,$$

where all constraints will hold with equality at the solution. Note that, again following U.S. law, household interest payments are also assumed tax deductible. The first order conditions when $d^h < 1$ yield, after a bit of algebra

$$(3.6) \quad q_s^h = \hat{q}_s \left[(1 + \hat{\rho})/(1 + \hat{\rho} D^h)\right], \qquad s=1,\ldots,S;\ h=1,\ldots,H,$$

where we define

$$(3.7)\ D^h = (1 - t^h), \qquad h=1,\ldots,H.$$

That is, the marginal rates of substitution for any household are all the same multiple of the corresponding market prices. Clearly, unless the D^h are all equal, it will be possible to reallocate consumption so as to make at least one household better off without making any household worse off. Equation (3.6) also implies that $\hat{\phi}(x) = \phi^h(x)$ for all x and h=1,...,H; market prices may be used without adjustment to evaluate riskiness. Further, summation of (3.6) across s establishes that $\rho^h = \hat{\rho} D^h$, h=1,...,H. The impact of the personal income tax on household investment is, like the impact of a change in the market interest rate, ambiguous in general.

Finally, it is of interest to note that D^h does not depend on d^h. In fact, if (3.6) holds, it is easy to show that

$$(3.8) \quad \partial U^h / \partial d^h = I^h MU_o^h \left[1 - (1+\hat{\rho} D^h)\sum_{s=1}^{S} q_s^h\right] = 0, \quad h=1,\ldots,H,$$

so that household are indifferent to changes in their leverage. Thus, while the firms' trades with nature are distorted by the tax-deductibility of interest payments, households' trades with the market are not, and we can ignore the d^h in the remainder of this section.

Suppose now that the t^h are all equal. Then if the government distributes income so as to equate social marginal utilities of current consumption across households, (3.6) shows that these quantities will also be equated in every future state. Assuming that income is so distributed, we can divide (2.4) by the common value of $w^h MU_o^h$ and, assuming $\beta^f = 0$ for $f \in F''$ as before, use (3.4) and (3.6) to obtain the basic acceptance test :[19]

$$(3.9) \quad (1+\hat{\rho}) \sum_{s=1}^{S} \hat{q}_s G_s > 1 + \hat{\rho} \left[\bar{\alpha} D + \sum_{f=1}^{F} \beta^f B^f \right],$$

where D is the common value of the D^h. Comparing this with (2.10), it is seen that market prices must all be multiplied by the same constant before being used to evaluate the proposed project.

It is straightforword to use (3.2), (3.4), and (3.9) to obtain tests in terms of r^g :

$$(3.10a) \quad r^g - \hat{\emptyset}(G) > \hat{\rho} \left[\bar{\alpha} D + \sum_{f=1}^{F} \beta^f B^f \right],$$

which can also be expressed in terms of firms' returns, since

$$(3.10b) \quad \hat{\rho} = \left[\bar{r}^f - \hat{\emptyset}(AR^f) - \bar{V}^f/(1-\tau^f) \right] / \left[B^f + \bar{V}^f/(1-\tau^f) \right], \quad f \in F', \text{ and}$$

$$(3.10c) \quad \hat{\rho} = \left[r^f - \hat{\emptyset}(MR^f) \right] / B^f \qquad f \in F''.$$

It should be noted that the r's and the $\hat{\emptyset}$'s in the last two tests are computed using returns before tax.

Equation (3.10c) makes it clear that the risk-adjusted marginal rate of return on investment in efficient firms is just $\hat{\rho} B^f$, so that (3.10a) may be interpreted as requiring a comparison between the project's risk-adjusted rate of return and a weighted average of the household rate of time preference and the risk-adjusted marginal rates of return of affected investments, where the weights reflect the project's financing. Since $D \leq 1$ and $B^f \geq 1$, $f=1, \ldots, F$, the distortions due to the personal and corporate taxes tend to be offsetting in this context. For instance, if $\tau^f = .50$ and $b^f = .25$ for all affected firms, $t^h = .15$ for all households, and $\bar{\beta} = .18$,[20] the risk-adjusted rate of return on any proposed project should be compared with 1.012 times the market riskless rate. This multiple would be greater for larger values of $\bar{\beta}$ or τ^f, or smaller values of t^h or b^f.

Finally, if (2.2) holds, and if, as before, $K_f^A = 0$ for $f \notin F'$ and $K_f^M = 0$ for $f \notin F''$, substitution into (3.9) yields a test in terms of the K's :

$$(3.11)\quad (K^R + K^I) + \sum_{f=1}^{F} K_f^M (1+\hat{\rho}B^f) + \sum_{f=1}^{F} K_f^A \left[1+\hat{\rho}B^f + (1+\hat{\rho})\bar{V}^f/(1-\tau^f)\right]$$

$$> 1 + \hat{\rho}\left[\bar{\alpha} D + \sum_{f=1}^{F} \beta^f B^f\right].$$

This may be compared to (2.12), to which it reduces when $D = B^f = (1-\tau^f) = 1, \quad f = 1, \ldots, F.$

When the t^h differ, as they always do at the margin under a progressive income tax, serious complications arise. Since the q_s^h will be different for different households, it will no longer be possible to have both $w^h MU_o^h = w^k MU_o^k$ for all $h, k = 1, \ldots, H$ <u>and</u> $w^h MU_s^h = w^k MU_s^k$ for all $h;k = 1, \ldots, H$ and $s = 1, \ldots, S$. Indeed, suppose a planner were to take the various t^h and households' utility-maximizing trading behaviour as constraints. If he then redistributed income, either current income or (contingent) future income or both, in order to maximize W, none of these equalities would generally be satisfied at the optimum point. On the other hand, differences in the t^h, which make Pareto-efficient allocation of consumption impossible, must have been imposed by the government itself. Since these difference are thus in no sense exogenous to society, it is not clear that the solution to such a planning problem would have any normative significance. In fact, the existence of different marginal personal income tax rates is probably best understood as a second-best response to limitations on governments' abilities to redistribute income, especially contingent future income. Hence, a careful analysis of situations in which the t^h differ would require explicit consideration of the nature and implications of these limitations.

An analysis of this sort would, however, carry us well byond the bounds of this study. In its absence, it is necessary to make more or less arbitrary assumptions in order to obtain further results. Two alternative approaches suggest themselves. First, we might assume that social marginal utilities of consumption have been somehow equated in each state. This would amount to valuing benefits, increases in future consumption, equally, no matter to whom they accrued, and it would free us from any need to identify the project's beneficiaries.[21] Alternatively, we might assume that social marginal utilities of present consumption have been somehow equated. This assumption would have us value costs, foregone present consumption, the same, no matter which households incurred them, and it would enable us to work without information on which households were to make current sacrifices to pay for the project.

Pursuing the first approach, suppose that for state 1,

$$(3.12) \qquad w^h MU_1^h = \lambda \; \hat{q}_1, \qquad\qquad h=1,\ldots,H,$$

where λ is some positive constant. Then, for all other states,

$$(3.13) \quad w^h MU_s^h = w^h MU_o^h q_s^h = w^h MU_1^h q_s^h / q_1^h = w^h MU_1^h \hat{q}_s / \hat{q}_1 = \lambda \; \hat{q}_s,$$

$$s=2,\ldots,S; \; h=1,\ldots,H.$$

using (1.4) and (3.6). Hence, if social marginal utilities are equated in any future state, they will be equated in all future states. Since it is easy to imagine a transfer program that would achieve (3.12), the first approach does not require the assumption of extraordinary government abilities. Given (3.12) and (3.13), we can solve for $w^h MU_o^h$, $h=1,\ldots,H$, in terms of λ, $\hat{\rho}$, and D^h. Substitution of (3.12), (3.13), and this solution into (2.4), cancelation of λ, and use of the assumption that $\beta^f = 0$ for $f \notin F''$ yields the simple basic test.

$$(3.14) \quad (1+\hat{\rho})\sum_{s=1}^{S} \hat{q}_s G_s > 1 + \hat{\rho}\left[\sum_{h=1}^{H} \alpha^h D^h + \sum_{f=1}^{F} \beta^f B^f\right],$$

which looks like the obvious generalization of (3.9). Clearly (3.10) and (3.11) are both changed only by the obvious substitution for $\tilde{\alpha}$ D.

Under the second assumption, which appears equally defensible, things are not so simple. Division of (2.4) by the common value of $w^h MU_o^h$ yields

$$(3.15) \quad (1+\hat{\rho})\sum_{s=1}^{S}\left[\sum_{h=1}^{H} \gamma_s^h E^h\right]\hat{q}_s G_s > \tilde{\alpha} + (1+\hat{\rho})\sum_{f=1}^{F} \beta^f\left\{\sum_{s=1}^{S}\left[\sum_{h=1}^{H} \delta_s^{fh} E^h\right]\hat{q}_s MR_s^f\right\},$$

where the E^h are defined by

$$(3.16) \qquad E^h = 1/(1+\rho^h) = 1/(1+\hat{\rho}D^h) \qquad\qquad h=1,\ldots,H.$$

This test clearly does not reduce to anything useable without further assumptions. In some cases, for instance, it may be plausible to assume that the δ_s^{fh} are independent of s.[22] Then, under our usual assumption that $\beta^f = 0$ for $f \notin F''$, the right-hand side of (3.15) simplifies, and the test becomes

$$(3.17) \quad (1+\hat{\rho})\sum_{s=1}^{S}\left[\sum_{h=1}^{H} \gamma_s^h E^h\right]\hat{q}_s G_s > \tilde{\alpha} + \sum_{f=1}^{F} \beta^f\left[\sum_{h=1}^{H} \delta^{fh} E^h\right](1+\hat{\rho} B^f).$$

In order to obtain a criterion comparable to (3.14), it is necessary to go even farther, and to assume that the γ_s^h are independent of s. This is a strong assumption, but where it is appropriate, (3.17) becomes

$$(3.18) \quad (1+\hat{\rho})\sum_{s=1}^{S} \hat{q}_s G_s > \left\{\tilde{\alpha} + \sum_{f=1}^{F} \beta^f\left[\sum_{h=1}^{H} \delta^{fh} E^h\right](1+\hat{\rho} B^f)\right\} \Big/ \sum_{h=1}^{H} \gamma^h E^h.$$

A comparison with (3.9) indicates the necessary modifications in (3.10) and (3.11). It is interesting to notice that only (weighted) harmonic means of the D^h appear in (3.18), while only a (weighted) arithmetic mean of these quantities appears in (3.14).[23]

On computational grounds, then, equal valuation of benefits is more convenient than equal valuation of costs, and criteria based on the first assumption clearly require less information. But it seems more than a little backwards to use such standards in assessing the welfare properties of the distribution of income.

IV. Incomplete Markets

It is not immediately apparent how seriously one is led astray by the assumption that capital markets are complete and, except for tax-induced distortions, perfect. Those who work within the state-preference framework tend to emphaize that S is large relative to the number of independent financial markets in any real economy and that there are systematic forces that inhibit the creation of a complete set of such markets.[24] On the other hand, those who work with two-parameter models of risk tend to stress the high quality of existing capital markets.[25] It does not seem clear how important the non-existence of some markets and imperfections in existing ones are for the economy's performance;: it is not even clear how to ask this question precisely.[26] It is even less obvious what impact these factors have on optimal public investment criteria.

A general approach to public investment decisions under incomplete or imperfect insurance markets flows from the work of Bradford [5]. In a world of certainty, Bradford argues that if the government can only make transfers of current income, it should act to equate the $w^h MU_o^h$. If we assume that this has been done, and we divide (2.4) by the common value of this quantity, we obtain

$$(4.1) \quad \sum_{s=1}^{S}\left[\sum_{h=1}^{H} \gamma_s^h q_s^h\right] G_s > \bar{\alpha} + \sum_{f=1}^{F}\beta^f\left\{\sum_{s=1}^{S}\left[\sum_{h=1}^{H}\delta_s^{fh} q_s^h\right] MR_s^f\right\} ,$$

which becomes (3.15) if (3.6) holds. If, as before, we assume the δ_s^{fh} and the γ_s^h do not depend on s, we can obtain a test in terms of r^g :

$$(4.2) \quad (1+r^g)\sum_{h=1}^{H} \gamma^h E^h > \bar{\alpha} + \sum_{f=1}^{F} \beta^f\left\{\sum_{s=1}^{S}\sum_{h=1}^{H}\delta^{fh}E^h\left[1+r^f-\phi^h(MR^f)\right]\right\}+\sum_{h=1}^{H}\gamma^h E^h \phi^h(G),$$

where the E^h are defined by the first equality in (3.16). This reduces to Bradford's harmonic mean criterion when $\bar{\alpha}$ = 1 and the government project is riskless or independent.

On the other hand, the government may be forced to take the present distribution of income as given, and it may be able to make only future transfers. In this case, under certainty, Bradford argues that it should do so in such a way that social marginal utilities of future consumption are equated. The natural analog under risk would be a policy of equating social marginal utilities in each state of nature. We showed above that this is easily done, even when personal tax rates differ, if markets are complete and income taxes are the only distortions. If markets are incomplete, or if other distortions or imperfections are present, this policy may be quite difficult to implement. In general, it will require the government to make many conditional transfers, transfers that are executed if and only if certain states of nature occur. While it is obvious that some transfers of this sort are made in reality, in unemployment insurance programs, for instance, it is also obvious that they are not generally costless. In the absence of a complete set of perfect markets, the government must generally expend resources to verify which state of nature has occurred, as in the administration of unemployment insurance programs.

Still, if this task can be considered accomplished, simple tests result. If we assume

$$(4.3) \quad w^h MU_s^h = \lambda_s, \qquad h=1,\ldots,H;\ s=1,\ldots,S,$$

it follows that $\phi^h(x) = \phi(x)$ for all random variables x, $h=1,\ldots,H$, and that the $w^h MU_o^h$ are proportional to $(1+\rho^h)$. Making the appropriate substitutions in (2.4) and simplifying yields

$$(4.4) \quad r^g - \phi(G) \quad \sum_{h=1}^{H} \alpha^h \rho^h + \sum_{f=1}^{F} \beta^f [r^f - \phi(MR^f)] ,$$

which reduces to Bradford's arithmetic mean criterion if $\bar{\alpha} = 1$ and the project is riskless or independant.

However, there is a basic problem with this approach. In order to employ (4.1) or (4.4), the decision-maker would need to have all the ρ^h, y_s^h, and MR_s^f, along with detailed information on the incidence of costs or benefits. If S, F, and H are large, it is simply unreasonable to expect him to obtain all these data from the households and firms involved. But, unless we can specify the nature of the economy's market failures, there is no way to relate these quentities to observable market prices.

As a second approach, then let us explicitly consider equilibrium conditions for an economy with an incomplete set of markets. We retain the personal and corporate income taxes of the last section, but we now assume that the only marketable assets are riskless bonds and ownership shares in firms, common stock. Households are assumed to take market prices and firms' investment decisions as given, while firms take households' preferences and shareholdings as given. This, of course, is exactly the stock market economy proposed by Diamond [7] and studied by Sandmo [22] and Drèze [8, 10], with the addition of income taxes.[27] It is true that stocks and bonds are not the only assets available to households in real economies, but it is also true that many possible assets do not exist in these economies. Hence, if other imperfections are not too important, one might expect the results derived below and those of the last section to bracket, in some sense, those that would hold for an actual economy.

It is simplest to begin by considering a typical household. Let θ^{fh} be the fraction of the common stock of firm f held by household h, $f \in F'$; $h=1,\ldots,H$. These quantities sum to unity accross h, and they may be positive or negative if short selling is allowed. A typical household must pay $\theta^{fh}\hat{V}^f$ for its shares in firm f, and it must contribute $(1-b^f)\,\theta^{fh}I^f$ towards the firm's investment. In turn, it will receive $\theta^{fh}Z^f_s$ in each of the states of nature, where the Z^f_s are defined by (3.1).[28]

A typical household will then seek to maximize U^h subject to

$$(4.5a) \quad M^h_o - (1-d^h)I^h - c^h_o \geq 0,$$

$$(4.5b) \quad M^h_s + (1-t^h)\left[x_o(1+\hat{\rho}) + \sum_{f \in F'} \theta^{fh} Z^f_s\right] + t^h(1+\hat{\rho}\, d^h)I^h - d^h(1+\hat{\rho})I^h - c^h_s \geq 0,$$

$$s=1,\ldots,S,$$

$$(4.5c) \quad I^h - x_o - \sum_{f \in F'} \theta^{fh}\left[\hat{V}^f + (1-b^f)I^f\right] \geq 0,$$

where x_o is the amount invested in riskless bonds. (These may be compared to (3.5).). These constraints will hold with equality at a solution.

Treating d^h, and the Z^f_s, b^f, and I^f as parameters, the first-order conditions yield

$$(4.6) \quad \rho^h = \hat{\rho}\, D^h, \qquad h=1,\ldots,H, \text{ and}$$

$$(4.7) \quad \hat{V}^f = \frac{1+\hat{\rho}\, D^h}{1+\hat{\rho}} \sum_{s=1}^{S} q^h_s Z^f_s - (1-b^f)I^f, \qquad f \in F';\ h=1,\ldots,H,$$

where the D^h are defined by (3.7).

Again, utility is independent of d^h, and this parameter can be ignored. The first term on the right-hand side of (4.7) may be interpreted as the household's valuation of the Z_s^f, corrected for the difference between its rate of time preference and the market rate of interest.

To consider firm behaviour, we first use (3.1) to re-write (4.7) as

$$(4.8)\quad (1+\hat{\rho})\hat{V}^f/(1-\tau^f) = (1+\hat{\rho}D^h)\sum_{s=1}^{S} q_s^h R_s^f - I^f(1+\hat{\rho}B^f), \quad f \in F', \ h=1,\ldots,H.$$

This would reduce to (3.2) if (3.6) held, which will generally not be the case in this model.[29] If firm f took the θ^{fh} as given, and if it were compelled to give households corresponding weights in its decision making, it might be led to maximize

$$(4.9)\quad A^f = \sum_{h=1}^{H} \theta^{fh}(1+\hat{\rho}D^h)\left[\sum_{s=1}^{S} q_s^h R_s^f\right] - I^f(1+\hat{\rho}B^f), \qquad f=1,\ldots,F.$$

A^f is simply a constant times a weighted average of owners' valuations of the firm, using ownership shares as weights. If this quantity is maximized, we must have[30]

$$(4.10)\quad \sum_{s=1}^{S}\left[\sum_{h=1}^{H} \theta^{fh}(1+\hat{\rho}D^h)q_s^h\right] MR_s^f = (1+\hat{\rho}B^f), \qquad f \in F'',$$

which may be compared to (3.4).

In the Appendix it is shown that if $t^h = \tau^f = 0$, $h=1,\ldots,H$; $f=1,\ldots,F$, and if the government can transfer only present consumption, bonds, and ownership shares among households, then if an allocation is a welfare optimum subject to this constraint, (4.6), (4.8), and (4.10) must hold and, in addition, social marginal utilities of current consumption must be equated. Unless we allow the government to costlessly make contingent transfers, something the private market cannot do by assumption, social marginal utilities of future consumption will not generally be equated in any state, even in the absence of taxes.

This suggests that the only plausible approach is to assume that costs, not benefits, are valued equally for all households.

If we consider a world with more than two periods, however, things no longer appear so simple. Today's reality was just one of a large set of possible states of nature yesterday. If, yesterday, the government had been unable to equate social marginal utilities in each future state, it would only be by wildest chance that social marginal utilities of current consumption would be equal today. Thus, if we assume that transfers are made, instantly and costlessly, each period in order to equate social marginal utilities of current concumption, we are, in effect, assuming that the government can costlessly make contingent transfers in each preceding period.

A little more thought points up the importance of household expectations. If households are continually surprised by the transfers they receive, then, even with different t^h, the $w^h MU_o^h$ can be equated in each period _ex post_, even though _ex ante_ private marginal utilities of future consumption will not vary proportionally across states of nature. On the other hand, if households are told at the start of each period exactly what transfers they would receive in each state of nature in the next period, and if their _ex ante_ social marginal utilities are thereby equated in every state, then, with different t^h, it will be impossible to equate _ex post_ marginal utilities of current consumption. It is not immediately obvious whether a planner, in either situation, should consider _ex ante_ or _ex post_ social marginal utilities.

All of this suggests the need for an explicit multi-period analysis of public investment under risk. Even in the absence of such an analysis, the foregoing arguments indicate that if constraints on governments' ability to make contingent transfers are recognized, there is no presumption that (constrained) welfare maximization implies equalization of either current of future (contingent) social marginal utilities. Still, unless one of these equalizations is assumed, it appears impossible to derive public investment criteria that do not depend on the unobservable w^h. Hence, simply because no better approach seems available, we shall, as before, investigate the implications of these two alternative arbitrary assumptions.

As before, the assumption that conditions (4.3) are satisfied yields simpler results. Solving these for the $w^h MU_o^h$, assuming once again that $\beta^f = 0$ for $f \notin F''$, and substituting into (2.4) yields the basic criterion.

$$(4.11) \quad \sum_{s=1}^{S} \lambda'_s \; G_s > 1 + \rho \left[\sum_{h=1}^{H} \alpha^h D^h + \sum_{f=1}^{F} \beta^f B^f \right],$$

where the λ'_s are defined by

$$(4.12) \qquad \lambda'_s = \lambda_s / \sum_{s=1}^{S} \lambda_s = (1 + \hat{\rho} D^h) q_s^h, \qquad s=1,\ldots,S;\ h=1,\ldots,H.$$

Since, as before, $\phi^h(x) = \phi(x)$ for all x, $h=1,\ldots,H$, (4.11) becomes immediately

$$(4.13) \quad r^g - \phi(G) > \hat{\rho}\left[\sum_{h=1}^{H} \alpha^h D^h + \sum_{f=1}^{F} \beta^f B^f\right].$$

In this economy, however, it will not be possible, in general, to compute ϕ (G) using only market prices. Tests involving the K's in (2.2) are thus of more interest. Substituting, under the usual assumption that $K_f^A = 0$ for $f \notin F'$ and $K_f^M = 0$ for $f \notin F''$, we obtain a test that is simply (3.11) with the obvious substitution for $\bar{\alpha}$ D.

If, alternatively, we assume the $w^h MU_o^h$ are equalized, division of (2.4) by their common value yields

$$(4.14) \quad \sum_{s=1}^{S} \left[\sum_{h=1}^{H} \gamma_s^h q_s^h \right] G_s > \bar{\alpha} + \sum_{f=1}^{F} \beta^f \left\{ \sum_{s=1}^{S} \left[\sum_{h=1}^{H} \delta_s^{fh} q_s^h \right] MR_s^f \right\}.$$

Let us first consider the right-hand side. The only way to simplify the summation is to assume $\beta^f = 0$ for $f \notin F''$, as before, and to employ (4.10). This would seem to require the assumption that $\delta_s^{fh} = \theta^{fh}$, $f=1,\ldots,F$; $h=1,\ldots,H$; $s=1,\ldots,S$.[31] Unfortunately, even this assumption does not permit use of (4.10) unless the D^h are all equal. If (and, in general, only if) both these strong assumptions are made, (4.14) becomes

$$(4.15)\quad (1+\hat{\rho}\,D)\sum_{s=1}^{S}\left[\sum_{h=1}^{H}\gamma_s^h q_s^h\right]G_s > 1+\rho\left[\bar{\alpha}\,D+\sum_{f=1}^{F}\beta^f B^f\right].$$

It is clear that even if we further assume that the γ_s^h are independent of s, the test involving r^g will involve a weighted average of the (generally) unobservable $\phi^h(G)$, as in (4.2). A criterion of any operational interest seems obtainable only by making this assumption and assuming further that (2.2) holds with $K_f^A = 0$ for $f \notin F'$, and $K_f^M = 0$ for $f \notin F''$. Substitution then yields

$$(4.16)\quad (K^R + K^I) + (1+\hat{\rho}\,D)\sum_{f=1}^{F}K_f^M\left[\sum_{h=1}^{H}\gamma^h q_s^h\right]MR_s^f$$

$$+\sum_{f=1}^{F}K_f^A\left[1+\hat{\rho}\,B^f+(1+\hat{\rho})\bar{V}^f/(1-\tau^f)\right] > 1+\hat{\rho}\left[\bar{\alpha}\,D+\sum_{f=1}^{F}\beta^f B^f\right],$$

which, again, may be compared to (3.11). Since average, not marginal returns are traded on the stock market, the γ^h vanish in the second summation but not in the first. The summation in the K_f^M would simplify, via (4.10), if (and, in general, only if) only one of these constants, say K_1^M, were non-zero and $\gamma^h = \theta^{1h}$, $h=1,\ldots,H$.[32]

V. Conclusions and Implications

Section III showed that in a simple one-commodity two-period world, if personal income tax rates are equal, if a complete set of insurance markets exists, and if the distribution of income may be assumed optimal, simple public investment criteria based on market prices may be derived. Unfortunately, the rest of that Section and Section IV then indicated that if either of the first two assumptions are violated, things become difficult rapidly. One of two equally plausible arbitrary assumptions about the normative properties of the income distribution must be made to obtain any results, and the results are sensitive to the assumption chosen. Hence, the tests given above should be interpreted as illustrations of the implications of various assumptions, rather than as guides for social policy. To emphasize this point, a few additional words about our basic model are in order.

Since everything in this paper has been in real terms, it should be clear that the riskless market rate of interest, $\hat{\rho}$, must be a real rate of interest. Further, it should be clear that $\hat{\rho}$ is at best approximately observable in any real economy. As Arrow and Kurz [1], p. xiv, put it, "There is, stictly speaking, no single risk-free rate of interest, so long as there is uncertainty about price movements and about the future of the rate of interest itself". Further, one can imagine many states of nature (involving, for instance, nuclear war), in which all financial assets would be worthless. Hence, all market rates embody some degree of risk, so that it might be reasonable to suppose that they all over-estimate the "true" riskless rate.

If, as this argument seems to imply, the rate on long-term government bonds is in fact above households' (tax-adjusted) marginal rates of time preference, one might wonder why households are observed to borrow large amounts at rates well above the government bond rate. While there are a variety of explanations for this, they all point to imperfections in existing markets. At the very least, most households face different borrowing and lending rates, so that neither, in general, can give a good indication of rates of time preference. One must certainly question the relevance of any policy conclusions derived from a model that assumes away such obvious and important market imperfections. On the other hand, it is not yet clear how these facts of life might be built into a tractable formal model.

Further, we have limited ourselves to a world with two periods and one commodity. Under certainly, however, results derived in two-period models do not always carry over to the multi-period case.[33] As the same phenomenon seems certain to arise under risk, there is a need to examine multi-period generalizations of these models. These will not be easy to construct, especially since the assumption of perfect and complete markets certainly must be dropped. Price expectations and the timing of firms' investment decisions will then become important, and these are difficult topics.[34] Further, as we indicated in the last section, assessing the normative significance of the income distribution will become much more difficult.[35] Finally, a multi-period model would seem likely to both permit and require formal treatment of the differential taxation of dividends and capital gains.

In a world of complete and perfect markets, a one-commodity does not seem too restrictive, although it prohibits consideration of externalities and public goods. On the other hand, interesting problems would arise in a stock market model if additional commodities were introduced, especially if leisure were added to the list of goods.[36] Further, this addition must be made if the effects of the personal income tax on households' labor supply decisions are to be considered.

Still, if the very simple model of this paper yields simple formulae only under restrictive assumptions, can one really hope that anything useable will flow from more complicated models ? While the answer is not obvious, it seems at least possible that neoclassical searches for rigorous, simple, general, and useful public investment criteria, which do not depend on the w^h, may be doomed to failure. We may have to give up one or more of the other attributes if useful criteria are to be obtained.

APPENDIX

If M_o is the society's initial stock of the single good, and T^h is the transfer of riskless future income to household h (equivalent to a current transfer of riskless bonds), $h=1,\ldots,H$, the planner's problem in a stock market economy is to maximize W subject to

$$\text{(A.1)} \quad [\alpha] \; M_o - \sum_{h=1}^{H} c_o^h - \sum_{f=1}^{F} I^f \geq 0.$$

$$\text{(A.2)} \quad [\beta_s^h] \; T^h + \sum_{f=1}^{F} \theta^{fh} R_s^f(I^f) - c_s^h \geq 0, \qquad h=1,\ldots,H; \; s=1,\ldots,S.$$

$$\text{(A.3)} \quad [\gamma^f] \qquad 1 - \sum_{h=1}^{H} \theta^{fh} \geq 0, \qquad f=1,\ldots,F.$$

$$\text{(A.4)} \quad [\delta] \qquad - \sum_{h=1}^{H} T^h \geq 0.$$

Two points should be noted here. First, (A. 2) expresses the constraints on households' consumption bundles that flow from the incompleteness of markets; Diamond [7] calls these "linearity constraints". The presence of these restrictions generally implies that the solution to the planner's problem will not be fully Pareto-efficient; see Mossin [21] , ch. 6, for a detailed discussion. Second, Drèze [8, 10] has pointed out that the set defined by (A. 2) is not convex, so the feasible set for the planner's problem is not convex. Even though W is quasi-concave, this in turn, implies that the Kuhn-Tucker conditions are necessary (under differentiability) but not sufficient for a maximum. (For a detailed discussion, see Drèze [8] , Section 3).

Since all social marginal utilities are positive, (A. 1) - (A. 4) must hold with equality at a solution to the planner's problem.

Assuming that non-negativity constraints on the I^f (and, possibly, on the θ^{fh}) are not binding, we can form the relevant Lagrangian, using the quantities in brackets above as multipliers, differentiate, and substitute to obtain the necessary conditions

$$\text{(A. 5)} \quad w^h MU_o^h = \alpha, \qquad h=1, \ldots, H.$$

$$\text{(A. 6)} \quad \sum_{s=1}^{S} q_s^h = \delta/\alpha, \qquad h=1, \ldots, H.$$

$$\text{(A. 7)} \quad \sum_{s=1}^{S} q_s^h R_s^f = \gamma^f/\alpha, \qquad h=1, \ldots, H; \; f=1, \ldots, F.$$

$$\text{(A. 8)} \quad \sum_{s=1}^{S} \left[\sum_{h=1}^{H} \theta^{fh} q_s^h \right] MR_s^f = 1, \qquad f=1, \ldots, F.$$

The last three of these are equivalent to (4. 6), (4. 8), and (4. 10) in the text, with appropriate interpretation of the multipliers, when all income tax rates are zero . Hence it follows that any constrained welfare optimum, at which social marginal utilities of current consumption must be equal by (A. 5), can be supported as an equilibrium in the stock market economy. But, since (A. 5) - (A. 8) are only necessary conditions for a welfare optimum, it does not follow that every stock market equilibrium at which (A. 5) also holds is a welfare optimum. In fact, Dreze [10] , Section 4, has provided counterexamples.

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FOOTNOTES

1. Public investment criteria in multi-period models under certainty have been examined by Kenneth J. Arrow and Mordecai Kurz [1] , Jacques H. Drèze [9], and Daniel Mc Fadden [18], among others.

2. These questions have been discussed by Stephen A. Marglin [19], Gordon Tullock [27], and Robert Lind [17], among others.

3. This assumption both simplifies our notation and removes the interesting but difficult problems treated by Ross M. Starr [25].

4. As far as I know, this measure has not appeared previously, though equation (3') of Richard Kihlstrom and Mark Pauly [16] is very closely related to (1. 8) above. This measure is consistent with the discussion of risk-neutrality in Richard Schmalensee [24]. The "risk margins" of Agnar Sandmo [22] provide a different but not unrelated measure of riskiness.

5. William Brainard and F. Trenery Dolbear [6] prove a closely related proposition.

6. See Brainard and Dolbear [6] on this principle. In mean-variance analysis, the analogous point is that covariances must be considered along with, and are often more important than, variances; see, for instance, Eugene F. Fama and Merton H. Miller [11], chs. 6 and 7.

7. This assumption was introduced by Peter Diamond [7]. It does not allow substitution of output in one state for output in another when investment is constant and is thus rather restrictive. Recently, Jacques Drèze [8, 10] has presented and employed a much more general specification. Adopting his approach would, however, greatly complicate the analysis that follows without, I think, altering its basic conclusions.

8. Projects for which (2. 2) holds but only K^1 is non-zero have been analyzed by Kenneth J. Arrow and Robert Lind [2].

9. Jack Hirschleifer and David L. Shapiro [15], among others, have argued that such resemblances are common.

10. If the economy were under-employed, one might expect $(\bar{\alpha} + \beta)$ to be less than unity; see Robert Haveman and John V. Krutilla [14]. On the other hand, if non-lump-sum taxes were used to finance the project, this sum might exceed unity because of induced efficiency losses.

11. We are implicitly assuming either that all Z_s^f are non-negative or that owners do not enjoy limited liability. If these assumptions are dropped, firms' bonds will no longer be riskless in general, and they will not be able to borrow at the riskless rate. If markets are complete, allowing for this possibility complicates the analysis but does not alter its conclusions; see Fama and Miller [11], ch. 4. On the other hand, Joe E. Stiglitz [26] has shown that if these assumptions are dropped in a model with incomplete markets, like the stock market model of Section IV, this approach is no longer valid.

12. It is thus only necessary to assume that all affected activities are carried on efficiently. But if, for instance, the project is to be financed in part by reducing other government investments, no general test can be derived unless it is assumed that those investments were initially at their optimal levels.

13. This is the main result of Section I of Arrow and Lind [2]. In their Section II, a stronger result of this sort is proven, without assuming complete insurance markets. As this is a limiting proposition, derived by holding project size fixed and allowing H to increase without bound, the structure of our model does not lend itself to its presentation of interpretation.

14. For treatments of the implications of distortions induced by income taxation under certainty, see William J. Baumol [4], Arnold C. Harberger [12], and Agnar Sandmo and Jacques H. Drèze [23]. Harberger's analysis has been extended in the mean-variance model of risk by Martin J. Bailey and Michael C. Jensen [3]. It should be clear that, following all these authors, we deal only with proportional taxes with full loss offset.

15. This point was first made formally by Franco Modigliani and Merton H. Miller [20].

16. See the 1973 Economic Report of the President [28], Tables C-54 and C-76.

17. This assumption seems to be made by Bailey and Jensen [3], Harberger [12], and Baumol [4]. Sandmo and Drèze [23] assume interest payments are not tax-deductable.

18. The M_s^h are probably best thought of as lump-sum government transfer payments. I see no obvious reason to restrict the m_s^h to be non-negative by prohibiting short sales. If these contraints are imposed, however, (3.6) holds only at an interior solution. Note that the household may purchase stocks and bonds, as well as basic securities, but these may clearly be expressed and priced in terms of the m_s^h they yield. Finally, if we assume that MU_i^h increases without bound as c_i^h goes to zero, $i = 0, 1, \ldots, S$, it is clear that at a maximum of U^h the household will always be able to meet its interest payments.

19. Since this test translates immediately into a first-order condition for determining the optimal level of investment in government activities, because the G_s may be interpreted as MR_s^f, it is clear that (3.4), with $\tau^f = 0$, should not hold for such activities. The correct value of B^f in these cases is, in fact, given by the quantity in brackets on the right-hand side of (3.9).

20. These numbers are meant to be representative of recent U.S. experience; see Tables C-2 and C-17 of the 1973 Economic Report of the President [28].

21. This approach to applied welfare economics in the presence of market imperfection or market failure has been defended at length by Arnold C. Harberger [13]; it has been used in the present context by Harberger [12] and by Bailey and Jensen [3].

22. When tax rates are non-zero, adoption of the project would, in general, change the government's tax receipts in every state of nature. In a two-period model like this one, assuming a balanced budget, these changes must be reflected in changes in the government's transfer payments to households. Thus, in general, the δ_s^{fh} will reflect both households' ownership of firms and the government's (marginal) transfer policies.

23. This same difference appears in the alternative criteria of David F. Bradford [5], which are discussed in Section IV, below.

24. See, for instance, the discussions in Arrow and Lind [2] and Schmalensee [24].

25. See, for instance, Bailey and Jensen [3] and Fama and Miller [11], ch. 8.

26. See Brainard and Dolbear [6] for a discussion ofandevidence on some aspects of this question.

27. It should be noted that Drèze [8, 10] does not assume the existence of a market for riskless bonds. An excellent exposition of the stock market model is provides by Jan Mossin [21].

An interesting alternative to this model is developed in Section V of Sandmo [22], where it is assumed that firms are owned by individuals who cannot trade their ownership shares. The operating policies of these firms, along with information on the R_s^f functions, can be used to derive some information about the q_s^h. Unfortunately, no operational public investment criteria emerge from Sandmo's model except in very special cases. Further, his model seems too restrictive; some risky assets are, in fact, tradeable.

28. See footnote 11.

29. If the matrix of the R_s^f had rank S, it would be possible to solve (4.8) for the q_s^h, which then would satisfy (3.6). In this case, of course, the economy has what amounts to a complete system of markets.

30. As Diamond [7] and Mossin [21], ch. 8, have emphasized, maximization of A^f is not a simple task. Firms must have a great deal of information about shareholders' preferences. The basic problem is that a firm's investment decision, because it affects the welfare of all its stockholders, assumes the character of a (semi-) public good. Drèze [8,10] has emphasized this aspect of the situation and has constructed algorithms, which resemble those that have been devised for the classical public goods problem, that can guide a stock market economy to an equilibrium at which, with no taxes, (4.6), (4.8), and (4.10) hold.

31. See footnote 22. Without taxes, this assumption would be an identity. Here, we are assuming that the government's (marginal) transfer policy involves lump-sum tax rebates. It should be noted that these are necessarily contingent transfers.

32. Sansmo [22] considers only investments of this type.

33. See Arrow and Kurz [1], Drèze [9], and Mc Fadden [18], for instance.

34. This last point is discussed by Drèze [10], sect. 7.

35. These difficulties would increase still more if the π_s were not assumed objectively known; see Starr [25].

36. Again, see Drèze [10], sect. 7.