

Part I

Individual Asset Liability
Management.

Chapter 2

The Annuity Puzzle revisited: A deterministic version with lagrangian methods.

This chapter is an adapted version of Hainaut & Devolder (2006a) and revisits the issue of the annuity puzzle. As mentioned in the introduction of the dissertation, our purpose is foremost to determine the optimal consumption strategy of an individual without any bequest motive, and next to value the optimal assets allocation of wealth between life annuities and cash. The budget constraint, which specifies that the individual doesn't borrow to consume, is inserted in the Hamilton Jacobi Bellman equation with a Lagrange multiplier. We arrive at the conclusion that the optimal planning of consumption is divided into two periods and that the transition time can be numerically calculated. We also illustrate the influence of cash return and of age on the optimal allocation of initial wealth.

2.1 Introduction.

Over the last years, life annuitization has become a serious option for retirees. Reasons explaining this trend are the announced reform of the social security, the progressive shift from pure pay as you go systems to funding methods or the increasing volatility of alternative investments. A life annuity protects standards of living and is coupled to an interest rate guarantee which is a long term protection against financial setbacks. However, in practice, few retirees choose this solution. The purpose of this chapter is precisely to provide rational arguments in favour of the integration of life annuities in the individual assets allocation.

Literature counts many articles studying this problem known as the “annuity puzzle”. Yaari (1965) was one of the first to prove under certain assumptions that a consumer, without a bequest motive, should prefer a full annuitization in a market without any risky assets. The underlying intuition is that the annuity rate of return is always higher than the risk free rate because it includes a mortality risk credit. Richard (1975) has solved the problem of optimal consumption/investment for an uncertain lifetime, in a financial markets including risky/non risky assets, and death/life insurances. Kapur and Orszag (1999) have determined the optimal investment/consumption policy for an individual who invests in equities and pure endowments. Milevsky (1998, 2001) and Young (2003) have developed a model in which the

individual defers the purchase of a life annuity until it is not possible to beat the annuity rate of return. Purcal and Piggott (2004) have calibrated the Richard model to the Japanese market in order to explain the demand of life annuity. The Yaari model was discretized by Petrova (2004) and calibrated to the US market.

This work addresses a common problem faced by pensioner without any bequest motive at the date of retirement: how should they consume their savings and allocate their wealth? We consider here that the individual has the opportunity to invest his wealth in a market made up of life annuities and cash, under the budget constraint ensuring that he cannot borrow to finance his consumption. The amount annuitized is determined once for all at the date of retirement. Our contribution is to propose a semi-analytical solution to this issue and to focus on the mechanics of the Lagrange multipliers in deriving the optimal consumption.

In the first part of this chapter, we show that existing approaches are not pertinent and that the consumption pattern may not be found without taking into account the budget constraint. Section 3 proposes a solution to the Hamilton Jacobi Bellman equation wherein the budget constraint is inserted with a Lagrange multiplier. We have found that optimal consumption may be split into two periods. During the first one, the budget constraint is inactive. In the second period, the constraint is active and the optimal consumption is equal to the life annuity. The time of transition between periods is unknown and is computed numerically. In section 4, one calculates the optimal fraction of initial wealth that should be invested in a life annuity to maximize the utility arising from consumption.

2.2 Existing approaches.

We consider an individual retiring at age x who allocates his savings between two securities: life annuities and cash. His initial wealth is denoted W_0 and $P_0 = \alpha.W_0$ is the amount devoted to the purchase of a life annuity, which provides a continuous income B . $F_0 = (1 - \alpha).W_0$ is the part of savings initially deposited in cash and F_t is the amount of cash at time t . As mentioned in the introduction, the decision about the amount annuitized is taken once for all at the date of retirement. The annuity rate B is determined by the formula:

$$B = \frac{\alpha.W_0}{\bar{a}_{x,x+T}} \cdot \frac{1}{1 + \epsilon} \quad (2.2.1)$$

where :

- ϵ : loadings on premium to cover administrative costs.
- $\bar{a}_{x,x+T} = \int_0^T e^{-r \cdot u} \cdot {}_u p_x^{tf} \cdot du$: is a continuous life annuity starting at age x . r is the interest rate granted to the costumer and ${}_u p_x^{tf}$ is the survival probability from age x till age $x + u$. $x + T$ is an upper bound on the age that a human being could reach.
- ${}_u p_x^{tf} = e^{-\int_0^u \mu^{tf}(x+z)dz}$ where $\mu^{tf}(t)$ is the mortality rate used by the insurer to price life annuities. The real survival probability and mortality rate of pensioners are noted

respectively ${}_u p_x$ and $\mu(t)$ and may differ from those used by the insurer to calculate B . ${}_u p_x$ and $\mu(t)$ will be used in the sequel to formulate the problem.

In a first time, we assume that the cash return is equal to r , the guarantee embedded in the life annuity. This assumption will be relaxed in paragraph 2.3.3. The variation, dF_t of the individual's asset at time t , is the sum of the return of cash and of the gap between annuity and consumption rates:

$$dF_t = (r.F_t + B - c_t).dt$$

where c_t is the individual's consumption. In the absence of any bequest motive, the pensioner will try to maximize the discounted utility of consumption till his death. We note $U(c_t)$ the utility drawn from consumption at time t . The value function related to this optimization problem is defined at time t , as follows:

$$v(F_t, t) = \max_{c_t \in \mathbb{R}} \mathbb{E} \left(\int_t^\tau e^{-\rho.(s-t)} . U(c_s) . ds \right)$$

where τ is the random instant of death and ρ is a psychological discount rate. ρ may be interpreted as a measure of the rush to consume. For the sake of simplicity, we choose $\rho = r$, this assumption will be also relaxed in paragraph 2.3.3. Under the additional assumption that the utility function is C.R.R.A. (constant relative risk aversion) with a risk aversion parameter denoted γ ($\gamma < 1$), the value function at time t becomes :

$$v(F_t, t) = \max_{c_t \in \mathbb{R}} \mathbb{E} \left(\int_t^\tau e^{-r.(s-t)} . \frac{c_s^\gamma}{\gamma} . ds | \mathcal{F}_t \right) \quad (2.2.2)$$

where $\mathcal{F}_t$ is the information at time t . According to Yaari (1965) and Richard (1975), the value function (3.2.5) is equivalent to :

$$v(F_t, t) = \max_{c_t \in \mathbb{R}} \int_t^T e^{-r.(s-t)} . {}_{s-t} p_{x+t} . \frac{c_s^\gamma}{\gamma} . ds \quad (2.2.3)$$

As mentioned earlier, ${}_s p_x$ and $\mu(x + s)$ are respectively the real survival probability and the real mortality rate of the individual. From the theory of stochastic control (see e.g. Fleming and Rishel, 1975), we know that the value function $v(F_t, t)$ is the $\mathbb{C}^{1,1}$ function, solution of the Hamilton Jacobi Bellman (HJB) equation :

$$\begin{aligned} & \frac{\partial v(F_t, t)}{\partial t} - (r + \mu(x + t)) . v(F_t, t) \\ & + \sup_{c_t} \left(\underbrace{L^{c_t} v(F_t, t) + \frac{c_t^\gamma}{\gamma}}_{H, \text{ Hamiltonian}} \right) = 0 \end{aligned} \quad (2.2.4)$$

where $L^{c_t} v(F_t, t)$ is the infinitesimal generator of $v(F_t, t)$:

$$L^{c_t} v(F_t, t) = (r.F_t + B - c_t) . \frac{\partial v(F_t, t)}{\partial F_t}$$

with the terminal condition:

$$v(F_T, T) = 0$$

and subject to the constraint that

$$F_t \geq 0 \quad \forall t \in [0, T]$$

This constraint is known in the literature as the budget constraint. This prevents the individual from borrowing money in order to finance his consumption. Before any further developments, we revisit the solution found by Merton (1969-1971) for the scenario of no annuitization (Note that Merton has considered a market made up of cash and stocks).

2.2.1 No annuitization ($\alpha = 0$).

Let's ignore the budget constraint and solve the HJB equation related to the optimization problem:

$$v_t - (r + \mu(x + t)).v + \sup_{c_t} \left(\underbrace{(r.F_t - c_t).v_F + \frac{c_t^\gamma}{\gamma}}_H \right) = 0 \quad (2.2.5)$$

where v_t and v_F designate respectively the derivative of $v(F_t, t)$ with regards to t and F_t . Differentiating the Hamiltonian leads to the optimal consumption, c_t^* :

$$c_t^* = (v_F)^{\frac{1}{\gamma-1}}$$

If we try a solution of the form:

$$v(F_t, t) = b(t). \frac{F_t^\gamma}{\gamma}$$

We obtain an ODE for $b(t)$:

$$b(t)' + b(t). (r.(\gamma - 1) - \mu(x + t)) + b(t)^{\frac{\gamma}{\gamma-1}}.(1 - \gamma) = 0$$

with the boundary condition $b(T) = 0$. This is a Bernoulli equation and its solution is:

$$b(t)^{\frac{1}{1-\gamma}} = \int_t^T e^{-\int_t^s r + \frac{1}{1-\gamma} \cdot \mu(x+z).dz} ds \quad (2.2.6)$$

Remark that $b(t)^{\frac{1}{1-\gamma}}$ looks like a life annuity and is bigger than one. One can check that the fund remains positive during the individual's life. The optimal consumption is indeed always smaller than the fund.

$$c_t^* = \frac{F_t}{\int_t^T e^{-\int_t^s r + \frac{1}{1-\gamma} \cdot \mu(x+z).dz} ds} < F_t$$

Hence, in case of no annuitization, the unconstrained solution strictly satisfies the budget constraint.

2.2.2 Partial annuitization ($\alpha > 0$).

As in the previous paragraph, let's ignore the budget constraint and solve the HJB equation:

$$v_t - (r + \mu(x+t)).v + \sup_{c_t} \left(\underbrace{(r.F_t + B - c_t).v_F + \frac{c_t^\gamma}{\gamma}}_H \right) = 0 \quad (2.2.7)$$

The optimal consumption is again obtained by differentiating H :

$$c_t^* = (v_F)^{\frac{1}{\gamma-1}}$$

And we try a solution of the form:

$$v(F_t, t) = b(t) \cdot \frac{(F_t + a(t))^\gamma}{\gamma}$$

After calculations, one gets two ODE characterizing $b(t)$ and $a(t)$:

$$b(t)' + b(t) \cdot (r \cdot (\gamma - 1) - \mu(x+t)) + b(t)^{\frac{\gamma}{\gamma-1}} \cdot (1 - \gamma) = 0$$

$$a(t)' - r \cdot a(t) + B = 0$$

With the boundary condition $b(T) = 0$, $b(t)$ is identical to (2.2.6):

$$b(t)^{\frac{1}{1-\gamma}} = \int_t^T e^{-\int_t^s r + \frac{1}{1-\gamma} \cdot \mu(x+z) \cdot dz} ds \quad (2.2.8)$$

The function $a(t)$ is:

$$a(t) = - \int_0^t B \cdot e^{-r \cdot s} \cdot ds + K$$

where K is a constant. The boundary condition on $a(t)$ may not be derived from the constraint $v(T, F_T) = 0$, which is achieved by the terminal condition $b(T) = 0$. If we choose the natural boundary condition $a(T) = 0$, we get that:

$$a(t) = \int_t^T B \cdot e^{-r \cdot (s-t)} \cdot ds$$

The consumption anticipates therefore future incomes from the life annuity:

$$c_t = \frac{F_t + \int_t^T B \cdot e^{-r \cdot (s-t)} \cdot ds}{\int_t^T e^{-\int_t^s r + \frac{1}{1-\gamma} \cdot \mu(x+z) \cdot dz} ds}$$

Figure 2.2.1: Fund evolution.

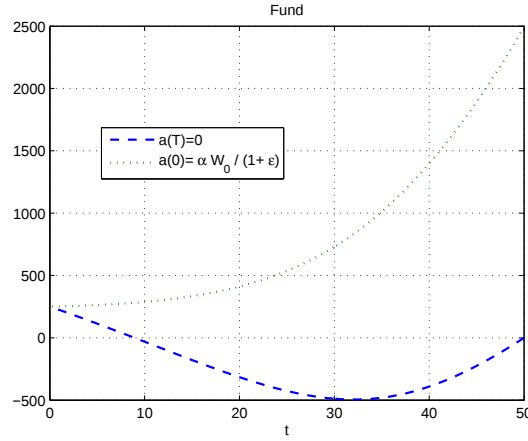
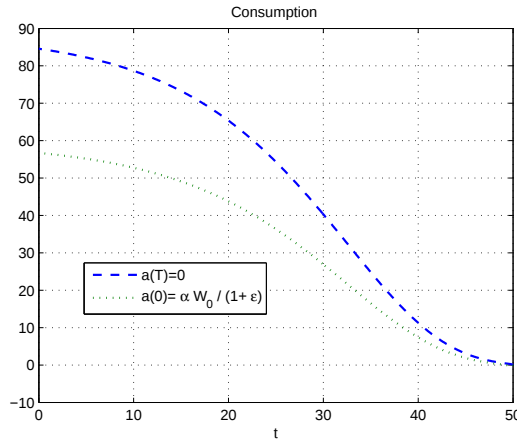


Figure 2.2.2: Consumption evolution.



Figures 2.2.1 and 2.2.2 (curves $a(T) = 0$) illustrate the evolution of fund F_t and of consumption c_t , when a man, 60 years old, invests 75% of his wealth $W_0=1000$, in a life annuity. The risk free rate r is worth 3.25% and the mortality rates (of pensioners and used by the insurer to value B) are given by a Gompertz-Makeham distribution (see appendix for details). There isn't any loading $\epsilon = 0$. The risk aversion parameter γ , is set to -1 . Clearly the consumption is too high and the fund F_t becomes negative after a few years.

As mentioned early, the terminal condition $v(T, F_T) = 0$ is satisfied whatsoever the value of $a(T)$. Hence, we can choose an initial boundary condition on $a(t)$ rather than a terminal one: e.g. $a(0) = \frac{\alpha \cdot W_0}{1+\epsilon}$. In this case (see figures 2.2.1 and 2.2.2), the funds F_t remains non negative during the individual's whole life but the consumption is clearly suboptimal. Solutions found by the Merton's approach either breaks the budget constraint or are suboptimal. It is therefore

absolutely necessary to insert the constraint $F_t \geq 0$, in the HJB equation.

2.2.3 Partial annuitization ($\alpha > 0$) in the Richard model.

This subsection emphasizes that in absence of any bequest motive, the solution developed in the seminal paper of Richard (1975) is not pertinent. Richard has assumed that pensioners can purchase instantaneous life insurances where payments at a premium rate P_t per unit time guarantee a capital $\frac{P_t}{\mu(x+t)}$ in case of death. In this setting, the dynamic of wealth obeys the ODE:

$$dF_t = (r.F_t + B - c_t - P_t).dt$$

Furthermore, the pensioner has a C.R.R.A bequest utility and the value function is now redefined as follows:

$$\begin{aligned} v(F_t, t) = & \max_{c_t, P_t \in \mathbb{R}} \mathbb{E} \left(\int_t^\tau e^{-r.(s-t)} \cdot \frac{c_s^\gamma}{\gamma} . ds \right. \\ & \left. + e^{-r.(\tau-t)} \cdot \frac{u_b}{\gamma} \cdot \left(F_\tau + \frac{P_\tau}{\mu(x+\tau)} \right)^\gamma | \mathcal{F}_t \right) \end{aligned} \quad (2.2.9)$$

where u_b is a constant weight. The HJB equation related to this optimization problem is:

$$\begin{aligned} v_t - (r + \mu(x+t)).v + \\ \sup_{c_t, P_t} \left((r.F_t + B - c_t - P_t).v_F + \frac{c_t^\gamma}{\gamma} \right. \\ \left. + \mu(x+t) \cdot \frac{u_b}{\gamma} \cdot \left(F_t + \frac{P_t}{\mu(x+t)} \right)^\gamma \right) = 0 \end{aligned} \quad (2.2.10)$$

Again we ignore the budget constraint. Optimal consumption and insurance rates are then obtained by differentiating the Hamiltonian:

$$\begin{aligned} c_t^* &= (v_F)^{\frac{1}{\gamma-1}} \\ P_t^* &= \mu(x+t) \cdot \left(\kappa \cdot v_F^{\frac{1}{\gamma-1}} - F_t \right) \end{aligned}$$

where $\kappa = u_b^{\frac{1}{1-\gamma}}$. As in the previous subsection, if we assume that the value function looks like:

$$v(F_t, t) = b(t) \cdot \frac{(F_t + a(t))^\gamma}{\gamma}$$

The HJB equation can be split into two ODE defining $b(t)$ and $a(t)$:

$$\begin{aligned} b(t)' - (r + \mu(x+t)).(1-\gamma).b(t) + \\ (1-\gamma).(1 + \mu(x+t).\kappa).b(t)^{\frac{\gamma}{\gamma-1}} = 0 \\ a(t)' - (r + \mu(x+t)).a(t) + B = 0 \end{aligned}$$

With boundary conditions $b(T) = 0$, $a(T) = 0$, solutions of those ODE are:

$$b(t)^{\frac{1}{1-\gamma}} = \int_t^T (1 + \mu(x+s).\kappa) e^{-\int_t^s r + \mu(x+z).dz} ds \quad (2.2.11)$$

$$\begin{aligned}
a(t) &= B \cdot \int_t^T e^{-\int_t^s r + \mu(x+z).dz} . ds \\
&= B \cdot \bar{a}_{x+t, x+T}
\end{aligned} \tag{2.2.12}$$

Optimal insurance premium and consumption are finally:

$$\begin{aligned}
P_t^* &= \mu(x+t) \cdot \left(\kappa \cdot \frac{(F_t + B \cdot \bar{a}_{x+t, x+T})}{b(t)^{\frac{1}{1-\gamma}}} - F_t \right) \\
c_t^* &= \frac{(F_t + B \cdot \bar{a}_{x+t, x+T})}{b(t)^{\frac{1}{1-\gamma}}}
\end{aligned}$$

In absence of any bequest motive (when $u_b = \kappa = 0$), optimal premium and consumption rates are respectively worth $-\mu(x+t) \cdot F_t$ and $\left(\frac{F_t}{\bar{a}_{x+t, x+T}} + B \right)$. Contrary to what Purcal and Pigott (2004) claim, the demand for life insurance is in this case not null but well negative. Even if this consumption policy clearly satisfies the budget constraint, it has little sense because an individual cannot sell his mortality risk to the market.

2.3 Optimal Constrained consumption.

As none of the existing solutions are satisfying, we have tried to maximize the utility drawn from consumption, taking explicitly into account the budget constraint. Let $\Lambda(F_t, t)$ be the positive Lagrange multiplier associated to the budget constraint. $\Lambda(F_t, t)$ is strictly positive when the constraint is active ($F_t = 0$) whereas $\Lambda(F_t, t)$ is null when the constraint is not active ($F_t > 0$). The HJB equation of constrained problem is now:

$$\begin{aligned}
&v_t - (r + \mu(x+t)) \cdot v + \Lambda(F_t, t) \cdot F_t \\
&+ sup_{c_t} \left(\underbrace{(r \cdot F_t + B - c_t) \cdot v_F + \frac{c_t^\gamma}{\gamma}}_H \right) = 0
\end{aligned} \tag{2.3.1}$$

Again, we try a solution of the form:

$$v(F_t, t) = b(t) \cdot \frac{(F_t + a(t))^\gamma}{\gamma} \tag{2.3.2}$$

And a Lagrange multiplier that looks like:

$$\Lambda(F_t, t) = \lambda(t) \cdot b(t) \cdot (F_t + a(t))^{\gamma-1} \tag{2.3.3}$$

The choice of this Lagrange multiplier is in fact motivated by the analysis of the full annuitization scenario (see subsection 2.3.1). The optimal consumption c_t^* is still:

$$c_t^* = b(t)^{\frac{1}{\gamma-1}} \cdot (F_t + a(t)) \tag{2.3.4}$$

And if we inject expressions (2.3.2) (2.3.3) (2.3.4) in the HJB equation (2.3.1), we get that:

$$\begin{aligned}
& b(t)' \cdot \frac{(F_t + a(t))^\gamma}{\gamma} + a(t)' \cdot b(t) \cdot (F_t + a(t))^{\gamma-1} \\
& - (r + \mu(x+t)) \cdot b(t) \cdot \frac{(F_t + a(t))^\gamma}{\gamma} \\
& + (r \cdot F_t + r \cdot a(t) - r \cdot a(t) + B) \cdot b(t) \cdot (F_t + a(t))^{\gamma-1} \\
& + (1 - \gamma) \cdot b(t)^{\frac{\gamma}{\gamma-1}} \cdot \frac{(F_t + a(t))^\gamma}{\gamma} \\
& + \lambda(t) \cdot b(t) \cdot (F_t + a(t))^{\gamma-1} \cdot F_t = 0
\end{aligned}$$

Regrouping terms in $\frac{(F_t + a(t))^\gamma}{\gamma}$ and in $(F_t + a(t))^{\gamma-1}$ leads to two ODE :

$$\begin{aligned}
b(t)' - (r \cdot (1 - \gamma) + \mu(x+t) - \lambda(t) \cdot \gamma) \cdot b(t) \\
- b(t)^{\frac{\gamma}{\gamma-1}} \cdot (\gamma - 1) = 0
\end{aligned} \tag{2.3.5}$$

$$a(t)' - r \cdot a(t) + B - \lambda(t) \cdot a(t) = 0 \tag{2.3.6}$$

When the constraint is inactive, the Lagrange multiplier is null, $\lambda(t) = 0$. When the constraint is active, the multiplier is strictly positive, $\lambda(t) > 0$, but the product $\lambda(t) \cdot F_t$ is null. $\lambda(t)$ may be inferred from the study of the full annuitization case ($\alpha = 1$).

2.3.1 Full annuitization.

In case of full annuitization, the optimal consumption is obviously the annuity, $c_t^* = B$, because the utility of consumption is discounted at rate $r + \mu(x+t)$, which is higher than the cash return r . The Lagrangian function $\lambda(t)$ must therefore be the mortality rate, $\mu(x+t)$. Indeed, if $\lambda(t) = \mu(x+t)$ equations (2.3.5) and (2.3.6) become:

$$b(t)' - (r + \mu(x+t)) \cdot (1 - \gamma) \cdot b(t) - b(t)^{\frac{\gamma}{\gamma-1}} \cdot (\gamma - 1) = 0 \tag{2.3.7}$$

$$a(t)' - (r + \mu(x+t)) \cdot a(t) + B = 0 \tag{2.3.8}$$

with boundary conditions $b(T) = 0$ and $a(T) = 0$. The solutions of those ODE are:

$$\begin{aligned}
b(t)^{\frac{1}{1-\gamma}} &= \int_t^T e^{-\int_t^s r + \mu(x+z) \cdot dz} ds \\
&= \bar{a}_{x+t, x+T}
\end{aligned} \tag{2.3.9}$$

$$\begin{aligned}
a(t) &= \int_t^T B \cdot e^{-\int_t^s r + \mu(x+z) \cdot dz} ds \\
&= B \cdot \bar{a}_{x+t, x+T}
\end{aligned} \tag{2.3.10}$$

$\frac{a(t)}{B}$ and $b(t)^{\frac{1}{1-\gamma}}$ are both equal to a life annuity whereas the optimal consumption is well equal to B:

$$c_t^* = b(t)^{\frac{1}{\gamma-1}} \cdot (0 + a(t)) = B$$

The value function is:

$$\begin{aligned} v(F_t, t) &= \frac{B^\gamma}{\gamma} \int_t^T e^{-\int_t^s r + \mu(x+z).dz} . ds \\ &= \frac{B^\gamma}{\gamma} . \bar{a}_{x+t, t+T} \end{aligned} \quad (2.3.11)$$

2.3.2 Partial annuitization.

Subsection 2.2.2 emphasizes that solutions of the unconstrained HJB equation are not optimal or break the budget constraint. Therefore, the optimal solution must activate the budget constraint ($F_t = 0$) at a certain time t^* , which is unknown. And, as showed in paragraph 2.3.1, once that the fund is depleted, the optimal consumption is equal to the annuity B . The value function may then be split into two components: the first one is solution of the unconstrained HJB equation till time t^* and the second one is the solution developed in case of full annuitization:

$$v(F_t, t) = \begin{cases} b_1(t) . \frac{(F_t + a_1(t))^\gamma}{\gamma} & \forall t \leq t^* \\ b_2(t) . \frac{(F_t + a_2(t))^\gamma}{\gamma} & \forall t \geq t^* \end{cases}$$

where $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ are solutions of following ODE:

$$\begin{cases} b_1(t)' - (r.(1-\gamma) + \mu(x+t)).b_1(t) \\ \quad - b_1^{\frac{\gamma}{\gamma-1}} . (\gamma-1) = 0 & \forall t \leq t^* \\ a_1(t)' - r.a_1(t) + B = 0 & \forall t \leq t^* \\ b_2(t)' - (r + \mu(x+t)).(1-\gamma).b_2(t) \\ \quad - b_2^{\frac{\gamma}{\gamma-1}} . (\gamma-1) = 0 & \forall t \geq t^* \\ a_2(t)' - (r + \mu(x+t)).a_2(t) + B = 0 & \forall t \geq t^* \end{cases}$$

with the terminal conditions:

$$\begin{cases} b_1(t^*) = b_2(t^*) & a_1(t^*) = a_2(t^*) \\ b_2(T) = 0 & a_1(T) = 0 \end{cases}$$

As it is not possible to determine t^* analytically, we compute it by a Newton Raphson method, such that the minimum of $F_{t \in [0, T]}$, is equal to $F_{t^*} = 0$.

By definition, the value function is $\mathbb{C}^{1,1}$ everywhere, excepted at (F_{t^*}, t^*) , where it is only continuous. However, we can prove that $v(F_t, t)$ is well differentiable at this point by comparing left and right limits of partial derivatives at this point. By taking into account that $b_1(t^*) = b_2(t^*)$, $a_1(t^*) = a_2(t^*)$ and $F_{t^*} = 0$, we have indeed that

$$\begin{aligned} 0 &= \lim_{t \rightarrow t^*_-} \frac{\partial}{\partial t} v(F_{t^*}, t) - \lim_{t \rightarrow t^*_+} \frac{\partial}{\partial t} v(F_{t^*}, t) \\ &= \lim_{t \rightarrow t^*_-} \mu(x+t).b_1(t).(F_{t^*} + a_1(t))^\gamma - \\ &\quad \lim_{t \rightarrow t^*_+} \mu(x+t).a_2(t).b_2(t).(F_{t^*} + a_2(t))^{\gamma-1} \end{aligned}$$

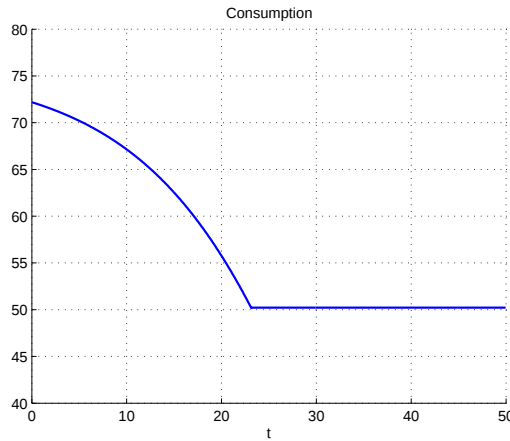
and that

$$\begin{aligned} 0 &= \lim_{F_t \rightarrow F_{t^*}^-} \frac{\partial}{\partial F_t} v(F_t, t^*) - \lim_{F_t \rightarrow F_{t^*}^+} \frac{\partial}{\partial F_t} v(F_t, t^*) \\ &= \lim_{F_t \rightarrow F_{t^*}^-} b_1(t^*) \cdot (F_t + a_1(t^*))^{\gamma-1} - \lim_{F_t \rightarrow F_{t^*}^+} b_2(t^*) \cdot (F_t + a_2(t^*))^{\gamma-1} \end{aligned}$$

Given that $v(F_t, t)$ is a $\mathbb{C}^{1,1}$ solution of the constrained HJB equation, it is well the optimal solution of our optimization problem.

If we apply our results to the example developed in the subsection 2.2.2 (an pensioner of age 60 chooses to invest 75% of his wealth $W_0=1000$, in an annuity with $r = 3.25\%$ and $\gamma = -1$), we obtain that $t^* = 23, 15$. The individual will therefore consume the totality of his fund of cash before the age of 83 years. Figure 2.3.1 presents the optimal pattern of consumption.

Figure 2.3.1: Consumption evolution.



It is also possible to derive an analytical expression of the consumption c_t . We know that:

$$\begin{cases} c_t^* = b_1(t)^{\frac{1}{\gamma-1}} \cdot (F_t + a_1(t)) & \forall t \leq t^* \\ c_t^* = B & \forall t \geq t^* \end{cases}$$

If we differentiate c_t^* , for $t < t^*$, we obtain that:

$$dc_t^* = -\frac{1}{1-\gamma} \cdot \mu(x+t) \cdot c_t^* \cdot dt$$

The consumption is then ruled by the following dynamic:

$$\begin{aligned} c_t^* &= c_0 \cdot e^{-\int_0^t \frac{\mu(x+z)}{1-\gamma} \cdot dz} \\ c_0 &= b_1(0)^{\frac{1}{\gamma-1}} \cdot (F_{t=0} + a_1(0)) \end{aligned}$$

The optimal consumption is therefore a decreasing function of the initial consumption $c_{t=0}$. The consumption is reduced by a factor which is similar to a probability of decease corrected by $\frac{1}{1-\gamma}$:

$$c_t^* = c_0 \cdot \left(e^{-\int_0^t \mu(x+z).dz} \right)^{\frac{1}{1-\gamma}} = c_0 \cdot ({}_t p_x)^{\frac{1}{1-\gamma}}$$

The evolution of the fund, F_t is also easily calculated:

$$\begin{cases} F_t = F_{t=0}.e^{r.t} + B.\int_0^t e^{r.(t-s)} ds \\ \quad - c_0.\int_0^t ({}_s p_x)^{\frac{1}{1-\gamma}} .e^{r.(t-s)} ds & \forall t \leq t^* \\ F_t = 0 & \forall t \geq t^* \end{cases}$$

2.3.3 Influence of the cash return and of the discount rate on the consumption policy.

We consider now that the cash return and the discount rate are different from the guarantee of the life annuity. Let r_F be the cash return. The difference between the guarantee, r , and the cash return r_F , is noted $\theta = r_F - r$. The dynamic of the individual's fund is then described by the ODE:

$$dF_t = ((r + \theta).F_t + B - c_t).dt$$

Furthermore, the psychological discount rate is now equal to $\rho = r + \phi$. The HJB equation becomes:

$$\begin{aligned} & v_t - (r + \phi + \mu(x + t)).v + \Lambda(F_t, t).F \\ & + \sup_{c_t} \left(\underbrace{((r + \theta).F_t + B - c_t).v_F + \frac{c_t^\gamma}{\gamma}}_H \right) = 0 \end{aligned} \quad (2.3.12)$$

Again, we try a value function and a Lagrange multiplier given by (2.3.2) and by (2.3.3). After calculations, components $a(t)$ and $b(t)$ of the value function are solutions of:

$$\begin{aligned} & b(t)' - (r.(1 - \gamma) + \phi + \mu(x + t) - \theta.\gamma - \lambda(t).\gamma).b(t) \\ & - b^{\frac{\gamma}{\gamma-1}}.(\gamma - 1) = 0 \end{aligned} \quad (2.3.13)$$

$$a(t)' - (r + \theta).a(t) + B - \lambda(t).a(t) = 0 \quad (2.3.14)$$

with the terminal conditions $a(T) = 0$ and $b(T) = 0$. As in the previous section, we assume that there exists a instant t^* such that the optimal consumption c_t^* for $t \geq t^*$, is equal to the life annuity B . The budget constraint imposes then that:

$$\lambda(t) = \mu(x + t) - \theta + \phi \quad \forall t \in [t^*, T]$$

Note that the Lagrange multiplier must remain positive. So we must have $t^* \geq \bar{t}$ with $\bar{t}$ such that $\lambda(\bar{t}) = \mu(x + \bar{t}) - \theta + \phi = 0$. There is no easy way to prove that $t^* \geq \bar{t}$ but it may be checked after computations. Before t^* , the funds F_t is positive and therefore the Lagrangian

function $\lambda(t)$ is null (the constraint $F_t \geq 0$ is inactive). The solution is again split into two components:

$$v(F_t, t) = \begin{cases} b_1(t) \cdot \frac{(F_t + a_1(t))^\gamma}{\gamma} & \forall t \leq t^* \\ b_2(t) \cdot \frac{(F_t + a_2(t))^\gamma}{\gamma} & \forall t \geq t^* \end{cases}$$

where $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ are solutions of following ODE:

$$\begin{cases} b_1(t)' - (r \cdot (1 - \gamma) + \mu(x + t) + \phi - \theta \cdot \gamma) \cdot b_1(t) \\ \quad - b_1^{\frac{\gamma}{\gamma-1}} \cdot (\gamma - 1) = 0 & \forall t \leq t^* \\ a_1(t)' - (r + \theta) \cdot a_1(t) + B = 0 & \forall t \leq t^* \\ b_2(t)' - (r + \mu(x + t) + \phi) \cdot (1 - \gamma) \cdot b_2(t) \\ \quad - b_2^{\frac{\gamma}{\gamma-1}} \cdot (\gamma - 1) = 0 & \forall t \geq t^* \\ a_2(t)' - (r + \mu(x + t) + \phi) \cdot a_2(t) + B = 0 & \forall t \geq t^* \end{cases}$$

with the following terminal conditions:

$$\begin{cases} b_1(t^*) = b_2(t^*) & a_1(t^*) = a_2(t^*) \\ b_2(T) = 0 & a_1(T) = 0 \end{cases}$$

We can prove in the same way as subsection 2.3.2, that the value function is well $\mathbb{C}^{1,1}$ everywhere. We solve numerically the ODE, applied to the example developed in the subsection 2.2.2 (a pensioner of age 60 chooses to invest 75% of his wealth in an annuity with $r = 3.25\%$ and $\gamma = -1$, $W_0 = 1000$), for different cash returns. For the sake of simplicity, we let the discount rate ρ equal to r ($\phi = 0$).

Figure 2.3.2: Consumption in function of θ .

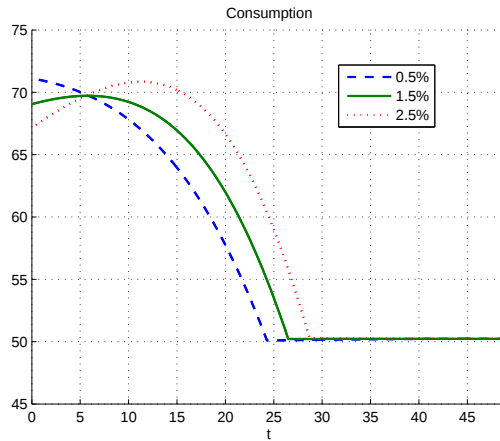


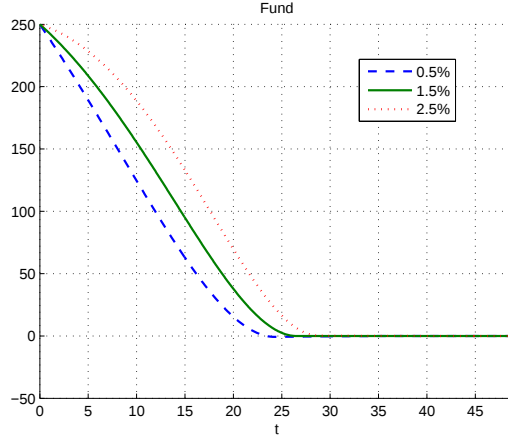
Figure 2.3.3: Fund in function of θ .

Figure 2.3.2 clearly reveals that the higher is the cash return, the lower is the consumption during the first years. It is indeed in the individual's advantage to delay his consumption in order to benefit from cash return which is higher than the rate $(r + \mu(x + t))$, used to discount utility. This policy of consumption also influences the evolution of the fund (see figure 2.3.3), and delays the time t^* at which the fund will be depleted. Table 2.3.1 shows the instants t^* for different θ . Clearly, we have $t^* \geq \bar{t}$ with $\bar{t}$ solution to the equation: $\lambda(\bar{t}) = \mu(x + \bar{t}) - \theta + \phi = 0$. It confirms that Lagrange multipliers are positive for each tests.

Table 2.3.1: Time t^* and $\mu(x + t^*)$

$\theta - \phi$	t^*	$\mu(60 + t^*)$	$\bar{t}$
0.5%	24.30	8.66%	-6.45
1.0%	25.35	9.57%	1.30
1.5%	26.50	10.69%	5.75
2.0%	27.60	11.88%	8.85
2.5%	28.70	13.20%	11.25
3.0%	29.90	14.81%	13.15

As in section 2.3, it is possible to obtain an expression of optimal consumption c_t^* in function of c_0 , for time $t \leq t^*$. The dynamic of c_t^* is:

$$dc_t^* = -\frac{1}{1-\gamma} \cdot (\mu(x+t) + \phi - \theta \cdot \gamma) \cdot c_t^* \cdot dt$$

And by integration we get that :

$$c_t^* = c_0 \cdot \left(e^{-\int_0^t \mu(x+z) + \phi - \theta \cdot \gamma dz} \right)^{\frac{1}{1-\gamma}}$$

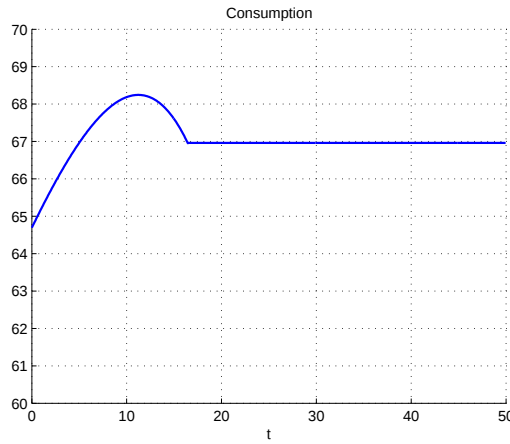
As we have observed on examples, the consumption planning can count two phases before t^* : one of increase and one of decrease. The instant of transition between those two periods is

noted $\tilde{t}$ and is solution of the following equation:

$$\int_0^{\tilde{t}} (\mu(x+z) + \phi - \theta.\gamma) dz = 0$$

It's interesting to observe on an example that optimal consumption may be lower than the annuity. Figure 2.3.4 depicts this for a man who invest 100% of his wealth in a life annuity, at the age of 60 years ($r = 3.25\%$, $\gamma = -1$, $W_0 = 1000$ and $\theta = 2.5\%$). From 60 years until 65.1 years, a part of the life annuity is reinvested in cash. The consumption increases till $\tilde{t} = 11.65$ and next falls to B at $t^* = 16.96$. This observation motives developments done in the next section.

Figure 2.3.4: Consumption, purchase of an annuity when 50 years old.

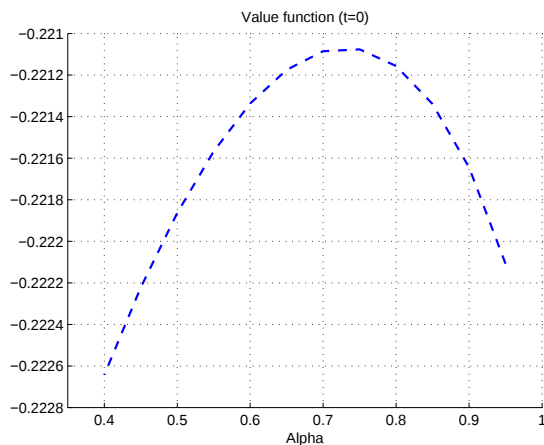
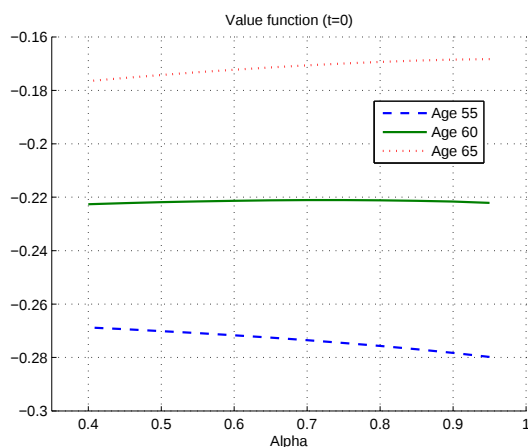


2.4 Optimal level of annuitization.

In the previous section, the retiree invests a fixed fraction α of his initial wealth in an annuity. If cash provides a higher return than the insurance contract ($r_F > r$), there must exist an optimal ratio α^* maximizing the discounted value of the utility drawn from consumption.

$$\alpha^* = \arg \sup_{\alpha} v ((1 - \alpha).W_0 , 0)$$

We will see on an example that this optimal ratio α^* depends on the individual's age, on the spread between the cash return and the guarantee. Again, we consider the case of a pensioner, 60 years old, who invests $\alpha.W_0$ in an annuity ($r = 3.25\%$, $\gamma = -1$, $W_0 = 1000$, $\rho = r$) and $(1 - \alpha).W_0$ in cash providing a return $r_F = r + 2.5\% = 5.75\%$ ($\theta = 2.5\%$) .

Figure 2.4.1: $v((1 - \alpha).W_0, 0)$ in function of α , age : 60 years.Figure 2.4.2: $V((1 - \alpha).W_0, 0)$ in function of the age of annuity purchase.

The graph 2.4.1 depicts the value function at time $t = 0$, for different levels of annuitization. Even if the influence of the ratio α on the value function is relatively small (due to the choice of $\gamma = -1$), a maximum is reached for $\alpha = 75\%$. This optimum also depends on the age of annuity purchase (see figure 2.4.2). At 55 years old, the pensioner has not any advantage to buy a life annuity if he can get a cash return of 5.75%. Whereas at 65 years old, it is interesting to invest the whole capital in a life annuity because the average return of the annuity ($r + \mu(x + t)$) is always higher than 5.75%.

The parameter α influences widely the consumption planning (see figure 2.4.3) : the higher is the amount devoted to purchase life annuities, the higher is the consumption at the individual's end of life.

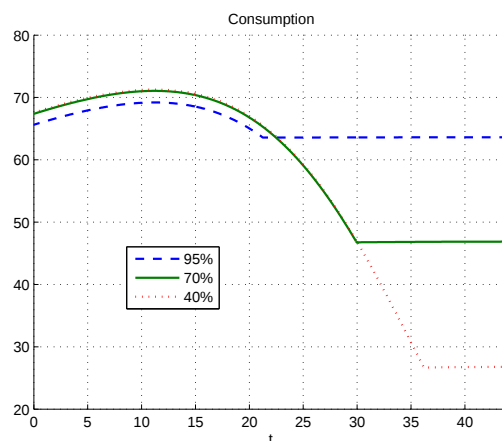
Figure 2.4.3: Consumption in function of α , 60 years old.

Table 2.4.1 confirms that solutions found are optimal : all Lagrange multipliers are positive since $\mu(x + t)$ are always bigger than the cash account premium $\theta=2.5\%$.

Table 2.4.1: Time t^* and $\mu(x + t^*)$

	55 y		65 y	
α	t^*	$\mu(55 + t^*)$	t^*	$\mu(65 + t^*)$
95%	27.15	7.05%	15.95	6.29%
90%	29.50	8.83%	18.65	8.14%
85%	31.35	10.54%	20.60	9.81%
80%	32.85	12.17%	22.15	11.38%
75%	34.20	13.85%	23.50	12.95%
70%	35.40	15.54%	24.75	14.60%
65%	36.55	17.35%	25.85	16.22%
60%	37.65	19.28%	26.95	18.03%
55%	38.65	21.22%	28.40	20.72%
50%	39.65	23.36%	28.95	21.84%
45%	40.65	25.72%	29.90	23.93%
40%	41.60	28.18%	30.90	26.34%

2.5 Conclusions.

This chapter analyses the optimal consumption and assets allocation of an individual, in absence of any bequest motive. Contrary to the classical consumption/investment problem solved by Merton, solutions of the unconstrained HJB equation either breaks the budget constraint or are suboptimal. Whereas, Richard's solution is not realistic.

We have then tried to insert the budget constraint in the HJB equation with a Lagrange multiplier and have found that the optimal consumption planning is brokedown into two periods. During the first one, the budget constraint is inactive and the consumption is proportional to the initial consumption. In the second period, the constraint is active and the optimal consumption is equal to the annuity. The instant of transition between those periods is calculated numerically.

Next, we observe that the higher is the cash return, the lower is the consumption during the first years. Moreover, the consumption may be lower than the annuity when the cash return is high and when the individual dedicates his whole wealth to purchase a life annuity. We also show that consumption is explicitly dependent on the gap between cash return and guarantee of the annuity.

In the last part of this work, we show the influence of the cash return on the optimal allocation of initial wealth between cash and a life annuity. An individual old enough and without any bequest motive should dedicate a part of his wealth to purchase a life annuity. Before a certain age, which is function of the individual's mortality rate and of the cash return, it is in the individual's advantage to invest partly his wealth in a cash account.

The next chapter addresses numerically an identical issue, in a market made up of cash, stocks and a life annuity. In this setting, the time of activation of the budget constraint becomes random and there doesn't exist any semi-analytical solution.

Appendix.

In examples presented in this chapter, we assume that mortality rates of pensioners and mortality rates used for pricing, $\mu(x + t)$ are given by a Gompertz-Makeham distribution. The parameters are those defined by the Belgian regulator for the pricing of a life annuity purchased by males. For an individual of age x , the mortality rate is :

$$\mu(x) = \mu_1(x) = a_\mu + b_\mu \cdot c^x$$

$$a_\mu = -\ln(s_\mu)$$

$$b_\mu = \ln(g_\mu) \cdot \ln(c_\mu)$$

where the parameters s_μ , g_μ , c_μ take the values showed in the table 6.8.1. Table 3.6.2 presents the evolution of mortality rates.

Table 2.5.1: Belgian legal mortality, for life insurance products, and for a male population.

s_μ :	0.999441703848
g_μ :	0.999733441115
c_μ :	1.116792453830

Table 2.5.2: Mortality rates.

Age x	$\mu(x)$
30	0.10%
40	0.18%
50	0.37%
60	0.88%
70	2.23%
80	5.74%

