Some categorical remarks on non-associative algebras

Tim Van der Linden

Fonds de la Recherche Scientifique–FNRS
Université catholique de Louvain
Vrije Universiteit Brussel

AMS 2025 Fall Virtual Eastern Sectional | October 26, 2025

1. Introduction

1. Introduction

I am an algebraic topologist turned category theorist.

Nowadays, I mainly study algebraic structures via their categorical properties.

1. Introduction

I am an algebraic topologist turned category theorist.
Nowadays, I mainly study algebraic structures via their categorical properties.

In 2014, I met Xabi García-Martínez, then a PhD student in Santiago de Compostela (Spain), working in the context of non-associative algebras.

1. Introduction

I am an algebraic topologist turned category theorist.
Nowadays, I mainly study algebraic structures via their categorical properties.

In 2014, I met Xabi García-Martínez, then a PhD student in Santiago de Compostela (Spain), working in the context of non-associative algebras.

He is the one who first taught me about non-associative algebras:
how, being formally simple, yet structurally rich,
they might serve as an excellent source of examples and counterexamples.

1. Introduction

I am an algebraic topologist turned category theorist.
Nowadays, I mainly study algebraic structures via their categorical properties.

In 2014, I met Xabi García-Martínez, then a PhD student in Santiago de Compostela (Spain), working in the context of non-associative algebras.

He is the one who first taught me about non-associative algebras:
how, being formally simple, yet structurally rich,
they might serve as an excellent source of examples and counterexamples.

My aim today is to explain some instances of this idea.

1. Introduction

I am an algebraic topologist turned category theorist.
Nowadays, I mainly study algebraic structures via their categorical properties.

In 2014, I met Xabi García-Martínez, then a PhD student in Santiago de Compostela (Spain), working in the context of non-associative algebras.

He is the one who first taught me about non-associative algebras:
how, being formally simple, yet structurally rich,
they might serve as an excellent source of examples and counterexamples.

My aim today is to explain some instances of this idea.

A first result I would like to discuss, is supposed to answer the following question:

> *Can the concept of a derivation be transported from Lie algebras to other types of non-associative algebras, in a way which extends the result that Lie algebra actions are represented by derivations?*

1. Introduction

I am an algebraic topologist turned category theorist.
Nowadays, I mainly study algebraic structures via their categorical properties.

In 2014, I met Xabi García-Martínez, then a PhD student in Santiago de Compostela (Spain), working in the context of non-associative algebras.

He is the one who first taught me about non-associative algebras:
how, being formally simple, yet structurally rich,
they might serve as an excellent source of examples and counterexamples.

My aim today is to explain some instances of this idea.

A first result I would like to discuss, is supposed to answer the following question:

> *Can the concept of a derivation be transported from Lie algebras to other types of non-associative algebras, in a way which extends the result that Lie algebra actions are represented by derivations?*

We claim that the answer is "No".

1. Introduction

I am an algebraic topologist turned category theorist.
Nowadays, I mainly study algebraic structures via their categorical properties.

In 2014, I met Xabi García-Martínez, then a PhD student in Santiago de Compostela (Spain), working in the context of non-associative algebras.

He is the one who first taught me about non-associative algebras:
how, being formally simple, yet structurally rich,
they might serve as an excellent source of examples and counterexamples.

My aim today is to explain some instances of this idea.

A first result I would like to discuss, is supposed to answer the following question:

> *Can the concept of a derivation be transported from Lie algebras to* ***other types of non-associative algebras****, in a way which extends the result that Lie algebra actions are represented by derivations?*

We claim that the answer is "No".

1. Introduction

I am an algebraic topologist turned category theorist.
Nowadays, I mainly study algebraic structures via their categorical properties.

In 2014, I met Xabi García-Martínez, then a PhD student in Santiago de Compostela (Spain), working in the context of non-associative algebras.

He is the one who first taught me about non-associative algebras:
how, being formally simple, yet structurally rich,
they might serve as an excellent source of examples and counterexamples.

My aim today is to explain some instances of this idea.

A first result I would like to discuss, is supposed to answer the following question:

> *Can the concept of a derivation be transported from Lie algebras to other types of non-associative algebras, in a way which extends the result that Lie algebra actions are represented by derivations?*

We claim that the answer is "No".

2. Other types of non-associative algebras?

2. Other types of non-associative algebras?

Let $\mathbb{k}$ be a field.

A $\mathbb{k}$**-algebra** is a $\mathbb{k}$-vector space X with a bilinear multiplication

$$\cdot\colon X \times X \to X\colon (x, y) \mapsto xy = x \cdot y = [x, y].$$

2. Other types of non-associative algebras?

Let $\mathbb{k}$ be a field.

A $\mathbb{k}$**-algebra** is a $\mathbb{k}$-vector space X with a bilinear multiplication

$$\cdot\colon X \times X \to X\colon (x, y) \mapsto xy = x \cdot y = [x, y].$$

Together with the $\mathbb{k}$-algebra morphisms, this defines a category $\boldsymbol{Alg}_{\mathbb{k}}$.

2. Other types of non-associative algebras?

Let $\mathbb{k}$ be a field.

A $\mathbb{k}$**-algebra** is a $\mathbb{k}$-vector space X with a bilinear multiplication

$$\cdot \colon X \times X \to X \colon (x, y) \mapsto xy = x \cdot y = [x, y].$$

Together with the $\mathbb{k}$-algebra morphisms, this defines a category $\boldsymbol{Alg}_{\mathbb{k}}$.

Some examples:

- $\boldsymbol{Lie}_{\mathbb{k}}$ is the variety of $\mathbb{k}$-algebras determined by the equations

$$\begin{cases} xx = 0 & \text{multiplication is alternating} \\ x(yz) + y(zx) + z(xy) = 0 & \text{Jacobi identity} \end{cases}$$

2. Other types of non-associative algebras?

Let $\mathbb{k}$ be a field.
A $\mathbb{k}$**-algebra** is a $\mathbb{k}$-vector space X with a bilinear multiplication

$$\cdot\colon X \times X \to X\colon (x, y) \mapsto xy = x \cdot y = [x, y].$$

Together with the $\mathbb{k}$-algebra morphisms, this defines a category $\boldsymbol{Alg}_{\mathbb{k}}$.

Some examples:

- $\boldsymbol{Lie}_{\mathbb{k}}$ is the variety of $\mathbb{k}$-algebras determined by the equations

$$\begin{cases} xx = 0 & \text{multiplication is alternating} \\ x(yz) + y(zx) + z(xy) = 0 & \text{Jacobi identity} \end{cases}$$

- Associative algebras: $x(yz) = (xy)z$.

2. Other types of non-associative algebras?

Let $\mathbb{k}$ be a field.

A $\mathbb{k}$**-algebra** is a $\mathbb{k}$-vector space X with a bilinear multiplication

$$\cdot\colon X \times X \to X\colon (x, y) \mapsto xy = x \cdot y = [x, y].$$

Together with the $\mathbb{k}$-algebra morphisms, this defines a category $\boldsymbol{Alg}_{\mathbb{k}}$.

Some examples:

- $\boldsymbol{Lie}_{\mathbb{k}}$ is the variety of $\mathbb{k}$-algebras determined by the equations
$$\begin{cases} xx = 0 & \text{multiplication is alternating} \\ x(yz) + y(zx) + z(xy) = 0 & \text{Jacobi identity} \end{cases}$$
- Associative algebras: $x(yz) = (xy)z$.
- Boolean rings: $\mathbb{k} = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$.

2. Other types of non-associative algebras?

Let $\mathbb{k}$ be a field.
A $\mathbb{k}$**-algebra** is a $\mathbb{k}$-vector space X with a bilinear multiplication

$$\cdot \colon X \times X \to X \colon (x, y) \mapsto xy = x \cdot y = [x, y].$$

Together with the $\mathbb{k}$-algebra morphisms, this defines a category $\boldsymbol{Alg}_{\mathbb{k}}$.

Some examples:

- $\boldsymbol{Lie}_{\mathbb{k}}$ is the variety of $\mathbb{k}$-algebras determined by the equations
$$\begin{cases} xx = 0 & \text{multiplication is alternating} \\ x(yz) + y(zx) + z(xy) = 0 & \text{Jacobi identity} \end{cases}$$
- Associative algebras: $x(yz) = (xy)z$.
- Boolean rings: $\mathbb{k} = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$.

The notation $\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means that $\mathscr{V}$ is a **variety of** $\mathbb{k}$**-algebras**:
we fixed a chosen set of additional identities which hold for the algebras in $\mathscr{V}$.

2. Other types of non-associative algebras?

Let $\Bbbk$ be a field. **(In principle, we may consider more general rings.)**
A **$\Bbbk$-algebra** is a $\Bbbk$-vector space X with a bilinear multiplication

$$\cdot \colon X \times X \to X \colon (x, y) \mapsto xy = x \cdot y = [x, y].$$

Together with the $\Bbbk$-algebra morphisms, this defines a category $\boldsymbol{Alg}_{\Bbbk}$.

Some examples:

- $\boldsymbol{Lie}_{\Bbbk}$ is the variety of $\Bbbk$-algebras determined by the equations

$$\begin{cases} xx = 0 & \text{multiplication is alternating} \\ x(yz) + y(zx) + z(xy) = 0 & \text{Jacobi identity} \end{cases}$$

- Associative algebras: $x(yz) = (xy)z$.
- Boolean rings: $\Bbbk = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$.

The notation $\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ means that $\mathscr{V}$ is a **variety of $\Bbbk$-algebras**:
we fixed a chosen set of additional identities which hold for the algebras in $\mathscr{V}$.

2. Other types of non-associative algebras?

Let $\mathbb{k}$ be a field. (In principle, we may consider more general rings.)
A $\mathbb{k}$**-algebra** is a $\mathbb{k}$-vector space X with a bilinear multiplication

$$\cdot \colon X \times X \to X \colon (x, y) \mapsto xy = x \cdot y = [x, y].$$

Together with the $\mathbb{k}$-algebra morphisms, this defines a category $\boldsymbol{Alg}_{\mathbb{k}}$.

Some examples:

- $\boldsymbol{Lie}_{\mathbb{k}}$ is the variety of $\mathbb{k}$-algebras determined by the equations
$$\begin{cases} xx = 0 & \text{multiplication is alternating} \\ x(yz) + y(zx) + z(xy) = 0 & \text{Jacobi identity} \end{cases}$$
- Associative algebras: $x(yz) = (xy)z$.
- Boolean rings: $\mathbb{k} = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$.

The notation $\mathcal{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means that $\mathcal{V}$ is a **variety of** $\mathbb{k}$**-algebras**:
we fixed a chosen set of additional identities which hold for the algebras in $\mathcal{V}$.

2. Other types of non-associative algebras?

Let $\mathbb{k}$ be a field. (In principle, we may consider more general rings.)
A **$\mathbb{k}$-algebra** is a $\mathbb{k}$-vector space X with a bilinear multiplication

$$\cdot \colon X \times X \to X \colon (x, y) \mapsto xy = x \cdot y = [x, y].$$

Together with the $\mathbb{k}$-algebra morphisms, this defines a category $\boldsymbol{Alg}_{\mathbb{k}}$.

Some examples:

- $\boldsymbol{Lie}_{\mathbb{k}}$ is the variety of $\mathbb{k}$-algebras determined by the equations

$$\begin{cases} xx = 0 & \text{multiplication is alternating} \\ x(yz) + y(zx) + z(xy) = 0 & \text{Jacobi identity} \end{cases}$$

- Associative algebras: $x(yz) = (xy)z$.
- Boolean rings: $\mathbb{k} = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$.

The notation $\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means that $\mathscr{V}$ is a **variety of $\mathbb{k}$-algebras**:
we fixed a chosen set of additional identities which hold for the algebras in $\mathscr{V}$.

Such a $\mathscr{V}$ is always a *variety of algebras* the more general sense of Universal Algebra.

2. Other types of non-associative algebras?

Let $\mathbb{k}$ be a field. (In principle, we may consider more general rings.)
A $\mathbb{k}$**-algebra** is a $\mathbb{k}$-vector space X with a bilinear multiplication

$$\cdot\colon X \times X \to X\colon (x, y) \mapsto xy = x \cdot y = [x, y].$$

Together with the $\mathbb{k}$-algebra morphisms, this defines a category $\boldsymbol{Alg}_{\mathbb{k}}$.

Some examples:

- $\boldsymbol{Lie}_{\mathbb{k}}$ is the variety of $\mathbb{k}$-algebras determined by the equations

$$\begin{cases} xx = 0 & \text{multiplication is alternating} \\ x(yz) + y(zx) + z(xy) = 0 & \text{Jacobi identity} \end{cases}$$

- Associative algebras: $x(yz) = (xy)z$.
- Boolean rings: $\mathbb{k} = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$.

The notation $\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means that $\mathscr{V}$ is a **variety of** $\mathbb{k}$**-algebras**:
we fixed a chosen set of additional identities which hold for the algebras in $\mathscr{V}$.

Such a $\mathscr{V}$ is always a *variety of algebras* the more general sense of Universal Algebra.
These categories are particularly well behaved: they are *semi-abelian*.

3. Semi-abelian categories

Generalizing *abelian* categories (modules over a ring, sheaves of abelian groups), *semi-abelian* categories were introduced in
[G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra **168** (2002), no. 2–3, 367–386]
as a framework for homological algebra in a non-additive setting.

3. Semi-abelian categories

Generalizing *abelian* categories (modules over a ring, sheaves of abelian groups), *semi-abelian* categories were introduced in
[G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra **168** (2002), no. 2–3, 367–386]
as a framework for homological algebra in a non-additive setting.

The definition unifies older work of Gerstenhaber, Huq, Orzech, and many others; the classical diagram lemmas ((3×3)-*Lemma*, *Snake Lemma, etc.*) are valid here, as are general developments in homological algebra, radical theory and commutator theory.

3. Semi-abelian categories

Generalizing *abelian* categories (modules over a ring, sheaves of abelian groups), *semi-abelian* categories were introduced in
[G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra **168** (2002), no. 2–3, 367–386]
as a framework for homological algebra in a non-additive setting.

The definition unifies older work of Gerstenhaber, Huq, Orzech, and many others; the classical diagram lemmas ((3×3)-*Lemma*, *Snake Lemma, etc.*) are valid here, as are general developments in homological algebra, radical theory and commutator theory.

A category is **semi-abelian** when it is pointed, protomodular, and Barr exact with finite sums.

3. Semi-abelian categories

Generalizing *abelian* categories (modules over a ring, sheaves of abelian groups), *semi-abelian* categories were introduced in
[G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra **168** (2002), no. 2–3, 367–386]
as a framework for homological algebra in a non-additive setting.

The definition unifies older work of Gerstenhaber, Huq, Orzech, and many others; the classical diagram lemmas ((3×3)-*Lemma*, *Snake Lemma, etc.*) are valid here, as are general developments in homological algebra, radical theory and commutator theory.

A category is **semi-abelian** when it is pointed, protomodular, and Barr exact with finite sums.

Examples include groups, (non-unital) rings, near-rings, Lie algebras, associative algebras, crossed modules; loops, Heyting semilattices, cocommutative Hopf algebras.

3. Semi-abelian categories

Generalizing *abelian* categories (modules over a ring, sheaves of abelian groups), *semi-abelian* categories were introduced in
[G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra **168** (2002), no. 2–3, 367–386]
as a framework for homological algebra in a non-additive setting.

The definition unifies older work of Gerstenhaber, Huq, Orzech, and many others; the classical diagram lemmas ((3×3)-*Lemma*, *Snake Lemma, etc.*) are valid here, as are general developments in homological algebra, radical theory and commutator theory.

A category is **semi-abelian** when it is pointed, protomodular, and Barr exact with finite sums.

Examples include groups, (non-unital) rings, near-rings, Lie algebras, associative algebras, crossed modules; loops, Heyting semilattices, cocommutative Hopf algebras.

In fact, any variety of universal algebras whose signature admits a group operation and a unique constant is semi-abelian.

3. Semi-abelian categories

Generalizing *abelian* categories (modules over a ring, sheaves of abelian groups), *semi-abelian* categories were introduced in
[G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra **168** (2002), no. 2–3, 367–386]
as a framework for homological algebra in a non-additive setting.

The definition unifies older work of Gerstenhaber, Huq, Orzech, and many others; the classical diagram lemmas ((3×3)-*Lemma*, *Snake Lemma, etc.*) are valid here, as are general developments in homological algebra, radical theory and commutator theory.

A category is **semi-abelian** when it is pointed, protomodular, and Barr exact with finite sums.

Examples include groups, (non-unital) rings, near-rings, Lie algebras, associative algebras, crossed modules; loops, Heyting semilattices, cocommutative Hopf algebras.

In fact, any variety of universal algebras whose signature
admits a group operation and a unique constant is semi-abelian.

Let us look at the details in the special case of varieties of $\mathbb{k}$-algebras.

4. Short exact sequences, in general

Semi-abelian categories may be described in terms of “good behavior” of their short exact sequences.

4. Short exact sequences, in general

Semi-abelian categories may be described in terms of "good behavior" of their short exact sequences.

In order to define these, we need a **zero object**:
an object $0 \coloneqq \bot = \top$ which is both **initial**, $\forall Y \exists !(\bot \to Y)$ and **terminal**, $\forall X \exists !(X \to \top)$.
A **pointed category** is a category with a zero object. We write $0\colon X \to 0 \to Y$.

4. Short exact sequences, in general

Semi-abelian categories may be described in terms of "good behavior" of their short exact sequences.

In order to define these, we need a **zero object**:
an object $0 \coloneqq \bot = \top$ which is both **initial**, $\forall Y \exists!(\bot \to Y)$ and **terminal**, $\forall X \exists!(X \to \top)$.
A **pointed category** is a category with a zero object. We write $0\colon X \to 0 \to Y$.

Examples:

- any one-element group or non-associative algebra is a zero object;
- hence $\boldsymbol{Gp}$ and any $\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ are pointed categories; here $0\colon X \to Y\colon x \mapsto 0$;
- are not pointed: the category of sets; rings with unit.

4. Short exact sequences, in general

Semi-abelian categories may be described in terms of "good behavior" of their short exact sequences.

In order to define these, we need a **zero object**:
an object $0 \coloneqq \bot = \top$ which is both **initial**, $\forall Y \exists! (\bot \to Y)$ and **terminal**, $\forall X \exists! (X \to \top)$.
A **pointed category** is a category with a zero object. We write $0 \colon X \to 0 \to Y$.

Examples:

- any one-element group or non-associative algebra is a zero object;
- hence $\boldsymbol{Gp}$ and any $\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ are pointed categories; here $0 \colon X \to Y \colon x \mapsto 0$;
- are not pointed: the category of sets; rings with unit.

A **short exact sequence** is then a pair of arrows (k, q)

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "k"'] & X \arrow[r, "q"'] & Q \arrow[r] & 0 \end{tikzcd}$$

where $q \circ k$ factors through zero, k is a kernel of q and q is a cokernel of k.

4. Short exact sequences, in general

Semi-abelian categories may be described in terms of "good behavior" of their short exact sequences.

In order to define these, we need a **zero object**:
an object $0 \coloneqq \bot = \top$ which is both **initial**, $\forall Y \exists! (\bot \to Y)$ and **terminal**, $\forall X \exists! (X \to \top)$.
A **pointed category** is a category with a zero object. We write $0 \colon X \to 0 \to Y$.

Examples:

- any one-element group or non-associative algebra is a zero object;
- hence $\boldsymbol{Gp}$ and any $\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ are pointed categories; here $0 \colon X \to Y \colon x \mapsto 0$;
- are not pointed: the category of sets; rings with unit.

A **short exact sequence** is then a pair of arrows (k, q)

$$
\begin{tikzcd}
 & L \arrow[d, dotted, "\exists!\bar{\ell}"'] \arrow[dr, "\forall\ell"] \arrow[drr, bend left, "0"] & & & \\
0 \arrow[r] & K \arrow[r, "k"'] & X \arrow[r, "q"'] & Q \arrow[r] & 0
\end{tikzcd}
$$

where $q \circ k$ factors through zero, k is a **kernel** of q and q is a cokernel of k.

4. Short exact sequences, in general

Semi-abelian categories may be described in terms of "good behavior"
of their short exact sequences.

In order to define these, we need a **zero object**:
an object $0 \coloneqq \bot = \top$ which is both **initial**, $\forall Y \exists! (\bot \to Y)$ and **terminal**, $\forall X \exists! (X \to \top)$.
A **pointed category** is a category with a zero object. We write $0 \colon X \to 0 \to Y$.

Examples:

- any one-element group or non-associative algebra is a zero object;
- hence $\boldsymbol{Gp}$ and any $\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ are pointed categories; here $0 \colon X \to Y \colon x \mapsto 0$;
- are not pointed: the category of sets; rings with unit.

A **short exact sequence** is then a pair of arrows (k, q)

$$\begin{tikzcd}
 & & & C \\
0 \arrow[r] & K \arrow[r, "k"'] \arrow[urr, bend left, "0"] & X \arrow[r, "q"'] \arrow[ur, "\forall c"] & Q \arrow[r] \arrow[u, dotted, "\exists! \overline{c}"'] & 0
\end{tikzcd}$$

where $q \circ k$ factors through zero, k is a kernel of q and q is a **cokernel** of k.

4. Short exact sequences, in general

Semi-abelian categories may be described in terms of "good behavior" of their short exact sequences.

In order to define these, we need a **zero object**:
an object $0 \coloneqq \bot = \top$ which is both **initial**, $\forall Y \exists!(\bot \to Y)$ and **terminal**, $\forall X \exists!(X \to \top)$.
A **pointed category** is a category with a zero object. We write $0\colon X \to 0 \to Y$.

Examples:

- any one-element group or non-associative algebra is a zero object;
- hence $\boldsymbol{Gp}$ and any $\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ are pointed categories; here $0\colon X \to Y\colon x \mapsto 0$;
- are not pointed: the category of sets; rings with unit.

A **short exact sequence** is then a pair of arrows (k, q)

$$0 \longrightarrow K \xrightarrow[k]{} X \xrightarrow[q]{} Q \longrightarrow 0$$

where $q \circ k$ factors through zero, k is a kernel of q and q is a cokernel of k.

5. Short exact sequences, in varieties of $\mathbb{k}$-algebras

In any variety of $\mathbb{k}$-algebras, kernels exist and are computed in the underlying vector spaces: for $f\colon X \to Y$, we may take $\ker(f)$ to be the inclusion of $\mathrm{Ker}(f) \coloneqq \{x \in X \mid f(x) = 0\}$ into X.

5. Short exact sequences, in varieties of $\mathbb{k}$-algebras

In any variety of $\mathbb{k}$-algebras, kernels exist and are computed in the underlying vector spaces: for $f\colon X \to Y$, we may take $\ker(f)$ to be the inclusion of $\mathrm{Ker}(f) \coloneqq \{x \in X \mid f(x) = 0\}$ into X.

- Any kernel of f will be isomorphic to this one.

5. Short exact sequences, in varieties of $\mathbb{k}$-algebras

In any variety of $\mathbb{k}$-algebras, kernels exist and are computed in the underlying vector spaces: for $f\colon X \to Y$, we may take $\ker(f)$ to be the inclusion of $\mathrm{Ker}(f) \coloneqq \{x \in X \mid f(x) = 0\}$ into X.

- Any kernel of f will be isomorphic to this one.
- A morphism is called a **normal monomorphism** when it is a kernel of some morphism.

5. Short exact sequences, in varieties of $\mathbb{k}$-algebras

In any variety of $\mathbb{k}$-algebras, kernels exist and are computed in the underlying vector spaces: for $f\colon X \to Y$, we may take $\ker(f)$ to be the inclusion of $\mathrm{Ker}(f) \coloneqq \{x \in X \mid f(x) = 0\}$ into X.

- Any kernel of f will be isomorphic to this one.
- A morphism is called a **normal monomorphism** when it is a kernel of some morphism.
- An inclusion of a subalgebra $I \subseteq X$ is a normal mono iff I is an **ideal**: $IX \subseteq I \supseteq XI$.

5. Short exact sequences, in varieties of $\mathbb{k}$-algebras

In any variety of $\mathbb{k}$-algebras, kernels exist and are computed in the underlying vector spaces: for $f\colon X \to Y$, we may take $\ker(f)$ to be the inclusion of $\mathrm{Ker}(f) \coloneqq \{x \in X \mid f(x) = 0\}$ into X.

- Any kernel of f will be isomorphic to this one.
- A morphism is called a **normal monomorphism** when it is a kernel of some morphism.
- An inclusion of a subalgebra $I \subseteq X$ is a normal mono iff I is an **ideal**: $IX \subseteq I \supseteq XI$.

In any variety of non-associative algebras, given a morphism $f\colon X \to Y$, a cokernel $q\colon Y \to Q$ of it exists, and may be obtained as follows:

1. take the image $f(X) = \{f(x) \in Y \mid x \in X\}$, this is a subalgebra of Y;
2. take the smallest ideal I of Y containing $f(X)$;
3. let q be the vector space quotient $Y \to Y/I$ with the induced multiplication.

5. Short exact sequences, in varieties of $\Bbbk$-algebras

In any variety of $\Bbbk$-algebras, kernels exist and are computed in the underlying vector spaces: for $f\colon X \to Y$, we may take $\ker(f)$ to be the inclusion of $\mathrm{Ker}(f) \coloneqq \{x \in X \mid f(x) = 0\}$ into X.

- Any kernel of f will be isomorphic to this one.
- A morphism is called a **normal monomorphism** when it is a kernel of some morphism.
- An inclusion of a subalgebra $I \subseteq X$ is a normal mono iff I is an **ideal**: $IX \subseteq I \supseteq XI$.

In any variety of non-associative algebras, given a morphism $f\colon X \to Y$,
a cokernel $q\colon Y \to Q$ of it exists, and may be obtained as follows:

1. take the image $f(X) = \{f(x) \in Y \mid x \in X\}$, this is a subalgebra of Y;
2. take the smallest ideal I of Y containing $f(X)$;
3. let q be the vector space quotient $Y \to Y/I$ with the induced multiplication.

Any cokernel of f is isomorphic to this one. Up to iso, any short exact sequence is of the form

$$0 \longrightarrow I \xrightarrow{\;i\;} X \xrightarrow{\;q\;} X/I \longrightarrow 0$$

with i the inclusion of an ideal I of X.

5. Short exact sequences, in varieties of $\Bbbk$-algebras

In any variety of $\Bbbk$-algebras, kernels exist and are computed in the underlying vector spaces: for $f\colon X \to Y$, we may take $\ker(f)$ to be the inclusion of $\mathrm{Ker}(f) \coloneqq \{x \in X \mid f(x) = 0\}$ into X.

- Any kernel of f will be isomorphic to this one.
- A morphism is called a **normal monomorphism** when it is a kernel of some morphism.
- An inclusion of a subalgebra $I \subseteq X$ is a normal mono iff I is an **ideal**: $IX \subseteq I \supseteq XI$.

In any variety of non-associative algebras, given a morphism $f\colon X \to Y$, a cokernel $q\colon Y \to Q$ of it exists, and may be obtained as follows:

1. take the image $f(X) = \{f(x) \in Y \mid x \in X\}$, this is a subalgebra of Y;
2. take the smallest ideal I of Y containing $f(X)$;
3. let q be the vector space quotient $Y \to Y/I$ with the induced multiplication.

Any cokernel of f is isomorphic to this one. Up to iso, any short exact sequence is of the form

$$0 \longrightarrow I \xrightarrow{\;i\;} X \xrightarrow{\;q\;} X/I \longrightarrow 0$$

with i the inclusion of an ideal I of X.

- A morphism of algebras is a **normal epimorphism** (= a cokernel) iff it is surjective.

6. Properties of normal epis and normal monos

$$\begin{tikzcd} I \arrow[r, tail, "\forall i"] \arrow[d, dotted, "\exists e"'] & X \arrow[d, "\forall q"] \\ J \arrow[r, tail, dotted, "\exists m"'] & Q \end{tikzcd}$$

In a variety of $\mathbb{k}$-algebras, any $q \circ i$ where i normal mono and q normal epi can be written as $m \circ e$ with e normal epi and m normal mono.

Indeed, $J \coloneqq qi(I)$ is an ideal of Q: for instance, $JQ \subseteq Q$ because for $x \in X$ and $y \in I$, we have $q(x)qi(y) = q(xi(y))$ and $xi(y) \in I$.

6. Properties of normal epis and normal monos

$$\begin{tikzcd} I \arrow[r, tail, "\forall i"] \arrow[d, dotted, two heads, "\exists e"'] & X \arrow[d, two heads, "\forall q"] \\ J \arrow[r, dotted, tail, "\exists m"'] & Q \end{tikzcd}$$

In a variety of $\mathbb{k}$-algebras, any $q \circ i$ where i normal mono and q normal epi can be written as $m \circ e$ with e normal epi and m normal mono.

Indeed, $J \coloneqq qi(I)$ is an ideal of Q: for instance, $JQ \subseteq Q$ because for $x \in X$ and $y \in I$, we have $q(x)qi(y) = q(xi(y))$ and $xi(y) \in I$.

This already implies that the *Snake Lemma* holds in any variety of $\mathbb{k}$-algebras.

6. Properties of normal epis and normal monos

$$\begin{tikzcd} I \arrow[r, "\forall i", tail] \arrow[d, dotted, "\exists e"'] & X \arrow[d, "\forall q"] \\ J \arrow[r, dotted, tail, "\exists m"'] & Q \end{tikzcd}$$

In a variety of $\Bbbk$-algebras, any $q \circ i$ where i normal mono and q normal epi can be written as $m \circ e$ with e normal epi and m normal mono.

Indeed, $J \coloneqq qi(I)$ is an ideal of Q: for instance, $JQ \subseteq Q$ because for $x \in X$ and $y \in I$, we have $q(x)qi(y) = q(xi(y))$ and $xi(y) \in I$.

This already implies that the *Snake Lemma* holds in any variety of $\Bbbk$-algebras.

Results such as the (3×3)-*Lemma* and the *Short Five Lemma* depend on **protomodularity**:

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "s"] \end{tikzcd}$$

For $k = \ker(f)$ and $f \circ s = 1_Y$, the images of k and s combined cover X.

Here, $x = \big(x - sf(x)\big) + sf(x)$ for any $x \in X$;
note that $x - sf(x) \in K$ and $sf(x) \in s(Y)$.

6. Properties of normal epis and normal monos

$$\begin{tikzcd} I \arrow[r, "\forall i", triangle] \arrow[d, dotted, "\exists e"'] & X \arrow[d, "\forall q"] \\ J \arrow[r, dotted, "\exists m"'] & Q \end{tikzcd}$$

In a variety of $\mathbb{k}$-algebras, any $q \circ i$ where i normal mono and q normal epi can be written as $m \circ e$ with e normal epi and m normal mono.

Indeed, $J := qi(I)$ is an ideal of Q: for instance, $JQ \subseteq Q$ because for $x \in X$ and $y \in I$, we have $q(x)qi(y) = q(xi(y))$ and $xi(y) \in I$.

This already implies that the *Snake Lemma* holds in any variety of $\mathbb{k}$-algebras.

Results such as the (3×3)-*Lemma* and the *Short Five Lemma* depend on **protomodularity**:

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "s"] \end{tikzcd}$$

For $k = \ker(f)$ and $f \circ s = 1_Y$, the images of k and s combined cover X.

Here, $x = \big(x - sf(x)\big) + sf(x)$ for any $x \in X$;
note that $x - sf(x) \in K$ and $sf(x) \in s(Y)$.

Then $f = \operatorname{coker}(k)$, so that we obtain a **split** short exact sequence (a **split extension**).

6. Properties of normal epis and normal monos

$$\begin{tikzcd} I \arrow[r, triangle->, "\forall i"] \arrow[d, dotted, "\exists e"'] & X \arrow[d, "\forall q"] \\ J \arrow[r, dotted, triangle->, "\exists m"'] & Q \end{tikzcd}$$

In a variety of $\mathbb{k}$-algebras, any $q \circ i$ where i normal mono and q normal epi can be written as $m \circ e$ with e normal epi and m normal mono.

Indeed, $J := qi(I)$ is an ideal of Q: for instance, $JQ \subseteq Q$ because for $x \in X$ and $y \in I$, we have $q(x)qi(y) = q(xi(y))$ and $xi(y) \in I$.

This already implies that the *Snake Lemma* holds in any variety of $\mathbb{k}$-algebras.

Results such as the (3×3)-*Lemma* and the *Short Five Lemma* depend on **protomodularity**:

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "s"] \end{tikzcd}$$

For $k = \ker(f)$ and $f \circ s = 1_Y$, the images of k and s combined cover X.

Here, $x = \big(x - sf(x)\big) + sf(x)$ for any $x \in X$;
note that $x - sf(x) \in K$ and $sf(x) \in s(Y)$.

Then $f = \operatorname{coker}(k)$, so that we obtain a **split** short exact sequence (a **split extension**).

Note that we do not ask for the dual of this condition to hold:
in the present context, that dual is equivalent to abelianness.

7. Limits and colimits

Kernels and cokernels are instances of small limits and colimits, which exist in all varieties of algebras.

7. Limits and colimits

Kernels and cokernels are instances of small limits and colimits,
which exist in all varieties of algebras.
(Pullbacks are computed as in sets; hence, a pullback of a normal epi is again a normal epi.)

7. Limits and colimits

Kernels and cokernels are instances of small limits and colimits,
which exist in all varieties of algebras.
(Pullbacks are computed as in sets; hence, a pullback of a normal epi is again a normal epi.)

$$
\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dotted, "\exists!(f{,}g)"] & \\
X & & Y \\
 & X \times Y \arrow[ul, "\pi_X"] \arrow[ur, "\pi_Y"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f"] \arrow[dr, "\iota_X"'] & & Y \arrow[ul, "g"'] \arrow[dl, "\iota_Y"] \\
 & X + Y \arrow[uu, dotted, "\exists!\langle f{,}g\rangle"] &
\end{tikzcd}
$$

Products are cartesian ($X \times Y = X \oplus Y$). Sums (= coproducts) are more complicated.

7. Limits and colimits

Kernels and cokernels are instances of small limits and colimits,
which exist in all varieties of algebras.
(Pullbacks are computed as in sets; hence, a pullback of a normal epi is again a normal epi.)

$$
\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dotted, "\exists!(f{,}g)"] & \\
X & & Y \\
 & X \times Y \arrow[ul, "\pi_X"] \arrow[ur, "\pi_Y"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f"] \arrow[dr, "\iota_X"'] & & Y \arrow[ul, "g"'] \arrow[dl, "\iota_Y"] \\
 & X + Y \arrow[uu, dotted, "\exists!\langle f{,}g\rangle"] &
\end{tikzcd}
$$

Products are cartesian ($X \times Y = X \oplus Y$). Sums (= coproducts) are more complicated.
Elements of $X + Y$ are (equivalence classes of) polynomials with variables in X and Y.

7. Limits and colimits

Kernels and cokernels are instances of small limits and colimits,
which exist in all varieties of algebras.
(Pullbacks are computed as in sets; hence, a pullback of a normal epi is again a normal epi.)

$$
\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dotted, "\exists!(f{,}g)"] & \\
X & & Y \\
& X \times Y \arrow[ul, "\pi_X"] \arrow[ur, "\pi_Y"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& Z & \\
X \arrow[ur, "f"] \arrow[dr, "\iota_X"'] & & Y \arrow[ul, "g"'] \arrow[dl, "\iota_Y"] \\
& X + Y \arrow[uu, dotted, "\exists!\langle f{,}g\rangle"] &
\end{tikzcd}
$$

Products are cartesian ($X \times Y = X \oplus Y$). Sums (= coproducts) are more complicated.
Elements of $X + Y$ are (equivalence classes of) polynomials with variables in X and Y.

If B and Z are free over $\{b\}$ and $\{z\}$, then $B + Z$ is free over $\{b, z\}$.

7. Limits and colimits

Kernels and cokernels are instances of small limits and colimits,
which exist in all varieties of algebras.
(Pullbacks are computed as in sets; hence, a pullback of a normal epi is again a normal epi.)

$$
\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dotted, "\exists!(f,g)" description] & \\
X & & Y \\
 & X \times Y \arrow[ul, "\pi_X"] \arrow[ur, "\pi_Y"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f"] \arrow[dr, "\iota_X"'] & & Y \arrow[ul, "g"'] \arrow[dl, "\iota_Y"] \\
 & X + Y \arrow[uu, dotted, "\exists!\langle f,g\rangle" description] &
\end{tikzcd}
$$

Products are cartesian ($X \times Y = X \oplus Y$). Sums (= coproducts) are more complicated.
Elements of $X + Y$ are (equivalence classes of) polynomials with variables in X and Y.

If B and Z are free over $\{b\}$ and $\{z\}$, then $B + Z$ is free over $\{b, z\}$. The kernel $B\flat Z$ in

$$
\begin{tikzcd}
0 \arrow[r] & B\flat Z \arrow[r] & B + Z \arrow[r, shift right, "\langle 1_B, 0\rangle"'] & B \arrow[l, shift right, "\iota_B"'] \arrow[r] & 0;
\end{tikzcd}
$$

is generated by monomials in b and z involving at least one z.

7. Limits and colimits

Kernels and cokernels are instances of small limits and colimits,
which exist in all varieties of algebras.
(Pullbacks are computed as in sets; hence, a pullback of a normal epi is again a normal epi.)

$$
\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dotted, "\exists!(f{,}g)"] & \\
X & & Y \\
 & X \times Y \arrow[ul, "\pi_X"] \arrow[ur, "\pi_Y"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f"] \arrow[dr, "\iota_X"'] & & Y \arrow[ul, "g"'] \arrow[dl, "\iota_Y"] \\
 & X + Y \arrow[uu, dotted, "\exists!\langle f{,}g\rangle"] &
\end{tikzcd}
$$

Products are cartesian ($X \times Y = X \oplus Y$). Sums (= coproducts) are more complicated.
Elements of $X + Y$ are (equivalence classes of) polynomials with variables in X and Y.

If B and Z are free over $\{b\}$ and $\{z\}$, then $B + Z$ is free over $\{b, z\}$. The kernel $B\flat Z$ in

$$
\begin{tikzcd}
0 \arrow[r] & B\flat Z \arrow[r] & B + Z \arrow[r, shift right, "\langle 1_B{,}0\rangle"'] & B \arrow[l, shift right, "\iota_B"'] \arrow[r] & 0;
\end{tikzcd}
$$

is generated by monomials in b and z involving at least one z.
So, for instance, $z + (bz)b$ is in $B\flat Z$, but b^2 is not.

8. Groups act by automorphisms, Lie algebras by derivations

[D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. **4** (1998), no. 2, 37–46]

Via a semidirect product construction

$$0 \longrightarrow X \longrightarrow X \rtimes_{\xi} B \underset{\longrightarrow}{\overset{\longleftarrow}{}} B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

8. Groups act by automorphisms, Lie algebras by derivations

[D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. **4** (1998), no. 2, 37–46]

Via a semidirect product construction

$$0 \longrightarrow X \longrightarrow X \rtimes_{\xi} B \underset{\longrightarrow}{\overset{\longleftarrow}{}} B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Next to a general *monadic* description which is always valid,
where the action ξ is codified as a certain morphism $B \flat X \to X$ via $\xi(bx) = {}^{b}x$

8. Groups act by automorphisms, Lie algebras by derivations

[D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. **4** (1998), no. 2, 37–46]

Via a semidirect product construction

$$0 \longrightarrow X \longrightarrow X \rtimes_\xi B \rightleftarrows B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Next to a general *monadic* description which is always valid,
where the action ξ is codified as a certain morphism $B\flat X \to X$ via $\xi(bx) = {}^{b}x$,
these actions may be encoded differently, depending on the context.

8. Groups act by automorphisms, Lie algebras by derivations

[D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. **4** (1998), no. 2, 37–46]

Via a semidirect product construction

$$0 \longrightarrow X \longrightarrow X \rtimes_{\xi} B \rightleftarrows B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Next to a general *monadic* description which is always valid,
where the action ξ is codified as a certain morphism $B\flat X \to X$ via $\xi(bx) = {}^{b}x$,
these actions may be encoded differently, depending on the context. We have:

8. Groups act by automorphisms, Lie algebras by derivations

[D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. **4** (1998), no. 2, 37–46]

Via a semidirect product construction

$$0 \longrightarrow X \longrightarrow X \rtimes_\xi B \underset{\longrightarrow}{\longleftarrow} B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Next to a general *monadic* description which is always valid,
where the action ξ is codified as a certain morphism $B\flat X \to X$ via $\xi(bx) = {}^b x$,
these actions may be encoded differently, depending on the context. We have:

- Groups: homomorphism $\xi\colon B \to Aut(X)$

8. Groups act by automorphisms, Lie algebras by derivations

[D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. **4** (1998), no. 2, 37–46]

Via a semidirect product construction

$$0 \longrightarrow X \longrightarrow X \rtimes_{\xi} B \underset{\longrightarrow}{\longleftarrow} B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Next to a general *monadic* description which is always valid,
where the action ξ is codified as a certain morphism $B\flat X \to X$ via $\xi(bx) = {}^{b}x$,
these actions may be encoded differently, depending on the context. We have:

- Groups: homomorphism $\xi\colon B \to Aut(X)$
- Lie algebras: morphism $\xi\colon B \to Der(X)$

8. Groups act by automorphisms, Lie algebras by derivations

[D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. **4** (1998), no. 2, 37–46]

Via a semidirect product construction

$$0 \longrightarrow X \longrightarrow X \rtimes_{\xi} B \rightleftarrows B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Next to a general *monadic* description which is always valid,
where the action ξ is codified as a certain morphism $B\flat X \to X$ via $\xi(bx) = {}^{b}x$,
these actions may be encoded differently, depending on the context. We have:

- Groups: homomorphism $\xi\colon B \to Aut(X)$
- Lie algebras: morphism $\xi\colon B \to Der(X)$ $\leftsquigarrow$ **details later**

Can we extend the above to $\mathscr{V} \leqslant Alg_{\mathbb{k}}$?

8. Groups act by automorphisms, Lie algebras by derivations

[D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. **4** (1998), no. 2, 37–46]

Via a semidirect product construction

$$0 \longrightarrow X \longrightarrow X \rtimes_{\xi} B \underset{\longrightarrow}{\overset{\longleftarrow}{}} B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Next to a general *monadic* description which is always valid, where the action ξ is codified as a certain morphism $B\flat X \to X$ via $\xi(bx) = {}^{b}x$, these actions may be encoded differently, depending on the context. We have:

- Groups: homomorphism $\xi\colon B \to Aut(X)$
- Lie algebras: morphism $\xi\colon B \to Der(X)$ $\quad\leftsquigarrow$ **details later**

Can we extend the above to $\mathscr{V} \leqslant \mathit{Alg}_{\mathbb{k}}$?

- $\mathscr{V}$-algebras: morphism $\xi\colon B \to$ **???**

9. Actions of Lie algebras

9. Actions of Lie algebras

- Example: if V is a $\Bbbk$–vector space, then $Gl(V)$ is $End(V)$ with bracket
$$f \cdot g \coloneqq f \circ g - g \circ f \quad \text{for} \quad f, g \colon V \to V.$$

9. Actions of Lie algebras

- Example: if V is a $\Bbbk$–vector space, then $Gl(V)$ is $End(V)$ with bracket
$$f \cdot g \coloneqq f \circ g - g \circ f \quad \text{for} \quad f, g \colon V \to V.$$
- Example: if X is a Lie algebra, then $Der(X) \leqslant Gl(X)$ consists of **derivations**,
$$D \colon X \to X \quad \text{such that} \quad D(xy) = D(x)y - xD(y) \text{ for all } x, y \in X.$$

9. Actions of Lie algebras

- Example: if V is a $\Bbbk$–vector space, then $Gl(V)$ is $End(V)$ with bracket
$$f \cdot g \coloneqq f \circ g - g \circ f \quad \text{for} \quad f, g\colon V \to V.$$
- Example: if X is a Lie algebra, then $Der(X) \leqslant Gl(X)$ consists of **derivations**,
$$D\colon X \to X \quad \text{such that} \quad D(xy) = D(x)y - xD(y) \text{ for all } x, y \in X.$$

Via a semidirect product construction
$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$
isomorphism classes of split extensions correspond to so-called *internal actions*.

9. Actions of Lie algebras

- Example: if V is a $\Bbbk$–vector space, then $Gl(V)$ is $End(V)$ with bracket
$$f \cdot g \coloneqq f \circ g - g \circ f \quad \text{for} \quad f, g \colon V \to V.$$
- Example: if X is a Lie algebra, then $Der(X) \leqslant Gl(X)$ consists of **derivations**,
$$D \colon X \to X \quad \text{such that} \quad D(xy) = D(x)y - xD(y) \text{ for all } x, y \in X.$$

Via a semidirect product construction

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Given $\xi \colon B \to Der(X)$, construct $X \rtimes_{\xi} B$ as $X \oplus B$ with
$$(x, b) \cdot (y, c) \coloneqq (xy + \xi(b)(y) - \xi(c)(x), bc)$$

9. Actions of Lie algebras

- Example: if V is a $\Bbbk$–vector space, then $Gl(V)$ is $End(V)$ with bracket
$$f \cdot g \coloneqq f \circ g - g \circ f \quad \text{for} \quad f, g \colon V \to V.$$
- Example: if X is a Lie algebra, then $Der(X) \leqslant Gl(X)$ consists of **derivations**,
$$D \colon X \to X \quad \text{such that} \quad D(xy) = D(x)y - xD(y) \text{ for all } x, y \in X.$$

Via a semidirect product construction

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_\xi B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Given $\xi \colon B \to Der(X)$, construct $X \rtimes_\xi B$ as $X \oplus B$ with
$$(x, b) \cdot (y, c) \coloneqq (xy + \xi(b)(y) - \xi(c)(x), bc),$$
while the sequence determines ξ via $b \mapsto (X \to X \colon x \mapsto s(b)k(x))$.

9. Actions of Lie algebras

- Example: if V is a $\Bbbk$–vector space, then $Gl(V)$ is $End(V)$ with bracket
$$f \cdot g \coloneqq f \circ g - g \circ f \quad \text{for} \quad f, g \colon V \to V.$$
- Example: if X is a Lie algebra, then $Der(X) \leqslant Gl(X)$ consists of **derivations**,
$$D \colon X \to X \quad \text{such that} \quad D(xy) = D(x)y - xD(y) \text{ for all } x, y \in X.$$

Via a semidirect product construction

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_\xi B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Given $\xi \colon B \to Der(X)$, construct $X \rtimes_\xi B$ as $X \oplus B$ with
$$(x, b) \cdot (y, c) \coloneqq (xy + \xi(b)(y) - \xi(c)(x), bc),$$
while the sequence determines ξ via $b \mapsto (X \to X \colon x \mapsto s(b)k(x))$.

How to extend this to other varieties of $\Bbbk$-algebras?

9. Actions of Lie algebras

- Example: if V is a $\mathbb{k}$–vector space, then $Gl(V)$ is $End(V)$ with bracket
$$f \cdot g \coloneqq f \circ g - g \circ f \quad \text{for} \quad f, g \colon V \to V.$$
- Example: if X is a Lie algebra, then $Der(X) \leqslant Gl(X)$ consists of **derivations**,
$$D \colon X \to X \quad \text{such that} \quad D(xy) = D(x)y - xD(y) \text{ for all } x, y \in X.$$

Via a semidirect product construction

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & X \rtimes_\xi B \arrow[r, shift right] & B \arrow[l, shift right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Given $\xi \colon B \to Der(X)$, construct $X \rtimes_\xi B$ as $X \oplus B$ with
$$(x, b) \cdot (y, c) \coloneqq (xy + \xi(b)(y) - \xi(c)(x), bc),$$
while the sequence determines ξ via $b \mapsto (X \to X \colon x \mapsto s(b)k(x))$.

How to extend this to other varieties of $\mathbb{k}$-algebras?
Can the concept of a *derivation* be generalized?

9. Actions of Lie algebras

- Example: if V is a $\Bbbk$–vector space, then $Gl(V)$ is $End(V)$ with bracket
$$f \cdot g \coloneqq f \circ g - g \circ f \quad \text{for} \quad f, g \colon V \to V.$$
- Example: if X is a Lie algebra, then $Der(X) \leqslant Gl(X)$ consists of **derivations**,
$$D \colon X \to X \quad \text{such that} \quad D(xy) = D(x)y - xD(y) \text{ for all } x, y \in X.$$

Via a semidirect product construction

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_\xi B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

isomorphism classes of split extensions correspond to so-called *internal actions*.

Given $\xi \colon B \to Der(X)$, construct $X \rtimes_\xi B$ as $X \oplus B$ with
$$(x, b) \cdot (y, c) \coloneqq (xy + \xi(b)(y) - \xi(c)(x), bc),$$
while the sequence determines ξ via $b \mapsto (X \to X \colon x \mapsto s(b)k(x))$.

How to extend this to other varieties of $\Bbbk$-algebras?
Can the concept of a *derivation* be generalized?
Categorically, what is $Der(X)$?

10. Action representability

10. Action representability

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

We look for a **global** solution: a construction valid for all split extensions in a given variety $\mathscr{V}$.

10. Action representability

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

We look for a **global** solution: a construction valid for all split extensions in a given variety $\mathscr{V}$.

Categorically, this is called **action representability**:
[F. Borceux, G. Janelidze, and G. M. Kelly, *On the representability of actions in a semi-abelian category*, Theory Appl. Categ. **14** (2005), no. 11, 244–286]
Equivalence classes of split extensions in $\mathscr{V}$ by an object X are representable, by an object $[X]$.

$$\mathit{SplitExt}(-, X) \cong \mathit{Hom}(-, [X]) \colon \mathscr{V}^{\mathrm{op}} \to \mathbf{Set}$$

For each X, an object $[X]$ exists such that equivalence classes of split extensions of B by X correspond to morphisms $B \to [X]$, naturally in B.

10. Action representability

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

We look for a **global** solution: a construction valid for all split extensions in a given variety $\mathscr{V}$.

Categorically, this is called **action representability**:
[F. Borceux, G. Janelidze, and G. M. Kelly, *On the representability of actions in a semi-abelian category*, Theory Appl. Categ. **14** (2005), no. 11, 244–286]
Equivalence classes of split extensions in $\mathscr{V}$ by an object X are representable, by an object $[X]$.

$$SplitExt(-, X) \cong Hom(-, [X]) \colon \mathscr{V}^{\mathrm{op}} \to \mathbf{Set}$$

For each X, an object $[X]$ exists such that equivalence classes of split extensions of B by X correspond to morphisms $B \to [X]$, naturally in B.

- Groups: $[X] = Aut(X)$

10. Action representability

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

We look for a **global** solution: a construction valid for all split extensions in a given variety $\mathscr{V}$.

Categorically, this is called **action representability**:
[F. Borceux, G. Janelidze, and G. M. Kelly, *On the representability of actions in a semi-abelian category*, Theory Appl. Categ. **14** (2005), no. 11, 244–286]
Equivalence classes of split extensions in $\mathscr{V}$ by an object X are representable, by an object $[X]$.

$$\mathit{SplitExt}(-,X) \cong \mathit{Hom}(-,[X]) \colon \mathscr{V}^{\mathrm{op}} \to \mathbf{Set}$$

For each X, an object $[X]$ exists such that equivalence classes of split extensions of B by X correspond to morphisms $B \to [X]$, naturally in B.

- Groups: $[X] = \mathit{Aut}(X)$
- Lie algebras: $[X] = \mathit{Der}(X)$

10. Action representability

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

We look for a **global** solution: a construction valid for all split extensions in a given variety $\mathscr{V}$.

Categorically, this is called **action representability**:
[F. Borceux, G. Janelidze, and G. M. Kelly, *On the representability of actions in a semi-abelian category*, Theory Appl. Categ. **14** (2005), no. 11, 244–286]
Equivalence classes of split extensions in $\mathscr{V}$ by an object X are representable, by an object $[X]$.

$$SplitExt(-, X) \cong Hom(-, [X]) \colon \mathscr{V}^{\mathrm{op}} \to \boldsymbol{Set}$$

For each X, an object $[X]$ exists such that equivalence classes of split extensions of B by X correspond to morphisms $B \to [X]$, naturally in B.

- Groups: $[X] = Aut(X)$
- Lie algebras: $[X] = Der(X)$
- Vector spaces: $[X] = 0$

10. Action representability

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_\xi B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

We look for a **global** solution: a construction valid for all split extensions in a given variety $\mathscr{V}$.

Categorically, this is called **action representability**:
[F. Borceux, G. Janelidze, and G. M. Kelly, *On the representability of actions in a semi-abelian category*, Theory Appl. Categ. **14** (2005), no. 11, 244–286]

Equivalence classes of split extensions in $\mathscr{V}$ by an object X are representable, by an object $[X]$.

$$SplitExt(-, X) \cong Hom(-, [X]) : \mathscr{V}^{\mathrm{op}} \to \boldsymbol{Set}$$

For each X, an object $[X]$ exists such that equivalence classes of split extensions of B by X correspond to morphisms $B \to [X]$, naturally in B.

- Groups: $[X] = Aut(X)$
- Lie algebras: $[X] = Der(X)$
- Vector spaces: $[X] = 0$

Conditions for $\mathscr{V} \leqslant Alg_{\mathbb{k}}$ to be action representable?

10. Action representability

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_\xi B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

We look for a **global** solution: a construction valid for all split extensions in a given variety $\mathscr{V}$.

Categorically, this is called **action representability**:
[F. Borceux, G. Janelidze, and G. M. Kelly, *On the representability of actions in a semi-abelian category*, Theory Appl. Categ. **14** (2005), no. 11, 244–286]
Equivalence classes of split extensions in $\mathscr{V}$ by an object X are representable, by an object $[X]$.

$$SplitExt(-, X) \cong Hom(-, [X]) \colon \mathscr{V}^{\mathrm{op}} \to \mathbf{Set}$$

For each X, an object $[X]$ exists such that equivalence classes of split extensions of B by X correspond to morphisms $B \to [X]$, naturally in B.

- Groups: $[X] = Aut(X)$
- Lie algebras: $[X] = Der(X)$
- Vector spaces: $[X] = 0$

Conditions for $\mathscr{V} \leqslant Alg_{\mathbb{k}}$ to be action representable?
Then the $[X]$ plays the role of $Der(X)$, so it contains “generalized derivations”.

11. Action representable varieties of $\mathbb{k}$-algebras

$\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means $\mathscr{V}$ consists of $\mathbb{k}$-algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

11. Action representable varieties of $\mathbb{k}$-algebras

$\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means $\mathscr{V}$ consists of $\mathbb{k}$-algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

Are known to be action representable such:

- $\boldsymbol{Vect}_{\mathbb{k}}$: the algebras are **abelian**, $xy = 0$;

11. Action representable varieties of $\mathbb{k}$-algebras

$\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means $\mathscr{V}$ consists of $\mathbb{k}$-algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

Are known to be action representable such:

- $\boldsymbol{Vect}_{\mathbb{k}}$: the algebras are **abelian**, $xy = 0$;
- $\boldsymbol{Lie}_{\mathbb{k}}$;

11. Action representable varieties of $\mathbb{k}$-algebras

$\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means $\mathscr{V}$ consists of $\mathbb{k}$-algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

Are known to be action representable such:

- $\boldsymbol{Vect}_{\mathbb{k}}$: the algebras are **abelian**, $xy = 0$;
- $\boldsymbol{Lie}_{\mathbb{k}}$;
- $\boldsymbol{qLie}_{\mathbb{k}}$ (Jacobi and $xy = -yx$) when $char(\mathbb{k}) = 2$:
  then $\boldsymbol{Lie}_{\mathbb{k}} \lneqq \boldsymbol{qLie}_{\mathbb{k}}$ since $xy = -yx$ does not imply $xx = 0$;

11. Action representable varieties of $\mathbb{k}$-algebras

$\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means $\mathscr{V}$ consists of $\mathbb{k}$-algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

Are known to be action representable such:

- $\boldsymbol{Vect}_{\mathbb{k}}$: the algebras are **abelian**, $xy = 0$;
- $\boldsymbol{Lie}_{\mathbb{k}}$;
- $\boldsymbol{qLie}_{\mathbb{k}}$ (Jacobi and $xy = -yx$) when $char(\mathbb{k}) = 2$:
  then $\boldsymbol{Lie}_{\mathbb{k}} \lneqq \boldsymbol{qLie}_{\mathbb{k}}$ since $xy = -yx$ does not imply $xx = 0$;
- Boolean rings: $\mathbb{k} = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$; here, $[X] = End(X)$.

11. Action representable varieties of $\mathbb{k}$-algebras

$\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means $\mathscr{V}$ consists of $\mathbb{k}$-algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

Are known to be action representable such:

- $\boldsymbol{Vect}_{\mathbb{k}}$: the algebras are **abelian**, $xy = 0$;
- $\boldsymbol{Lie}_{\mathbb{k}}$;
- $\boldsymbol{qLie}_{\mathbb{k}}$ (Jacobi and $xy = -yx$) when $char(\mathbb{k}) = 2$:
  then $\boldsymbol{Lie}_{\mathbb{k}} \lneqq \boldsymbol{qLie}_{\mathbb{k}}$ since $xy = -yx$ does not imply $xx = 0$;
- Boolean rings: $\mathbb{k} = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$; here, $[X] = End(X)$.

Is there anything else?

11. Action representable varieties of $\Bbbk$-algebras

$\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ means $\mathscr{V}$ consists of $\Bbbk$-algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

Are known to be action representable such:

- $\boldsymbol{Vect}_{\Bbbk}$: the algebras are **abelian**, $xy = 0$;
- $\boldsymbol{Lie}_{\Bbbk}$;
- $\boldsymbol{qLie}_{\Bbbk}$ (Jacobi and $xy = -yx$) when $char(\Bbbk) = 2$:
  then $\boldsymbol{Lie}_{\Bbbk} \lneqq \boldsymbol{qLie}_{\Bbbk}$ since $xy = -yx$ does not imply $xx = 0$;
- Boolean rings: $\Bbbk = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$; here, $[X] = End(X)$.

Is there anything else? For infinite $\Bbbk$: **No!**

11. Action representable varieties of $\mathbb{k}$-algebras

$\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ means $\mathscr{V}$ consists of $\mathbb{k}$-algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

Are known to be action representable such:

- $\boldsymbol{Vect}_{\mathbb{k}}$: the algebras are **abelian**, $xy = 0$;
- $\boldsymbol{Lie}_{\mathbb{k}}$;
- $\boldsymbol{qLie}_{\mathbb{k}}$ (Jacobi and $xy = -yx$) when $char(\mathbb{k}) = 2$:
  then $\boldsymbol{Lie}_{\mathbb{k}} \lneqq \boldsymbol{qLie}_{\mathbb{k}}$ since $xy = -yx$ does not imply $xx = 0$;
- Boolean rings: $\mathbb{k} = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$; here, $[X] = End(X)$.

Is there anything else? For infinite $\mathbb{k}$: **No!**

[X. García-Martínez, M. Tsishyn, TVdL, and C. Vienne, *Algebras with representable representations*, Proc. Edinb. Math. Soc. (2) **64** (2021), no. 3, 555–573]

Theorem

When $\mathbb{k}$ is an infinite field, an action representable variety of $\mathbb{k}$-algebras is either $\boldsymbol{Lie}_{\mathbb{k}}$, $\boldsymbol{qLie}_{\mathbb{k}}$, or $\boldsymbol{Vect}_{\mathbb{k}}$.

11. Action representable varieties of $\Bbbk$-algebras

$\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ means $\mathscr{V}$ consists of $\Bbbk$-algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

Are known to be action representable such:

- $\boldsymbol{Vect}_{\Bbbk}$: the algebras are **abelian**, $xy = 0$;
- $\boldsymbol{Lie}_{\Bbbk}$;
- $\boldsymbol{qLie}_{\Bbbk}$ (Jacobi and $xy = -yx$) when $char(\Bbbk) = 2$:
  then $\boldsymbol{Lie}_{\Bbbk} \lneqq \boldsymbol{qLie}_{\Bbbk}$ since $xy = -yx$ does not imply $xx = 0$;
- Boolean rings: $\Bbbk = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$; here, $[X] = End(X)$.

Is there anything else? For infinite $\Bbbk$: **No!**

[X. García-Martínez, M. Tsishyn, TVdL, and C. Vienne, *Algebras with representable representations*, Proc. Edinb. Math. Soc. (2) **64** (2021), no. 3, 555–573]

Theorem

When $\Bbbk$ is an infinite field, an action representable variety of $\Bbbk$-algebras is either $\boldsymbol{Lie}_{\Bbbk}$, $\boldsymbol{qLie}_{\Bbbk}$, or $\boldsymbol{Vect}_{\Bbbk}$.

$Der(X)$ **cannot be generalized in a way which is globally valid.**

12. Representability of representations

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

To prove the theorem, we reduce to **abelian actions**:

B acts on an **abelian** algebra X (where $xy = 0$: X is just a vector space).

12. Representability of representations

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

To prove the theorem, we reduce to **abelian actions**:

B acts on an **abelian** algebra X (where $xy = 0$: X is just a vector space).

(We make representability fail for a simpler subclass of the actions.)

12. Representability of representations

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_\xi B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

To prove the theorem, we reduce to **abelian actions**:
B acts on an **abelian** algebra X (where $xy = 0$: X is just a vector space).

(We make representability fail for a simpler subclass of the actions.)

In the present context, such an action happens to be the same thing as a *representation* or *Beck module*.

12. Representability of representations

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_\xi B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

To prove the theorem, we reduce to **abelian actions**:
B acts on an **abelian** algebra X (where $xy = 0$: X is just a vector space).

(We make representability fail for a simpler subclass of the actions.)

In the present context, such an action happens to be the same thing as a *representation* or *Beck module*.

Representability of representations (RR) means that each functor

$$Rep(-, X) \colon \mathscr{V}^{\mathrm{op}} \to \boldsymbol{Set} \colon B \mapsto \{\text{classes of } B\text{-module structures on } X\}$$

is representable—by an $[\![X]\!]$ for which $Rep(-, X) \cong Hom(-, [\![X]\!])$.

12. Representability of representations

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

To prove the theorem, we reduce to **abelian actions**:
B acts on an **abelian** algebra X (where $xy = 0$: X is just a vector space).

(We make representability fail for a simpler subclass of the actions.)

In the present context, such an action happens to be the same thing as a *representation* or *Beck module*.

Representability of representations (RR) means that each functor

$$Rep(-, X) \colon \mathscr{V}^{\mathrm{op}} \to \boldsymbol{Set} \colon B \mapsto \{\text{classes of } B\text{-module structures on } X\}$$

is representable—by an $[\![X]\!]$ for which $Rep(-, X) \cong Hom(-, [\![X]\!])$.

In $\boldsymbol{Lie}_{\Bbbk}$, we have $[\![X]\!] = Gl(X)$.

12. Representability of representations

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_\xi B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

To prove the theorem, we reduce to **abelian actions**:

B acts on an **abelian** algebra X (where $xy = 0$: X is just a vector space).

(We make representability fail for a simpler subclass of the actions.)

In the present context, such an action happens to be the same thing as a *representation* or *Beck module*.

Representability of representations (RR) means that each functor

$$\mathit{Rep}(-, X) \colon \mathscr{V}^{\mathrm{op}} \to \boldsymbol{Set} \colon B \mapsto \{\text{classes of } B\text{-module structures on } X\}$$

is representable—by an $[\![X]\!]$ for which $\mathit{Rep}(-, X) \cong \mathit{Hom}(-, [\![X]\!])$.

In $\boldsymbol{Lie}_{\Bbbk}$, we have $[\![X]\!] = Gl(X)$. $\quad Gl(X)$ is typical for Lie algebras:

12. Representability of representations

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \underset{}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

To prove the theorem, we reduce to **abelian actions**:
B acts on an **abelian** algebra X (where $xy = 0$: X is just a vector space).

(We make representability fail for a simpler subclass of the actions.)

In the present context, such an action happens to be the same thing as a *representation* or *Beck module*.

Representability of representations (RR) means that each functor

$$Rep(-, X) \colon \mathscr{V}^{\mathrm{op}} \to \textbf{\textit{Set}} \colon B \mapsto \{\text{classes of } B\text{-module structures on } X\}$$

is representable—by an $[\![X]\!]$ for which $Rep(-, X) \cong Hom(-, [\![X]\!])$.

In $\textbf{\textit{Lie}}_{\Bbbk}$, we have $[\![X]\!] = Gl(X)$. $\quad Gl(X)$ is typical for Lie algebras:

We show that for $\mathscr{V}$ a non-abelian variety of algebras over $\Bbbk$ infinite, representability of representations implies that either $\mathscr{V} = \textbf{\textit{Lie}}_{\Bbbk}$ or $\mathscr{V} = \textbf{\textit{qLie}}_{\Bbbk}$.

13. *Representability of representations* implies the *2-variety property*

Theorem

For $\mathbb{k}$ infinite and $\mathscr{V}$ a variety of $\mathbb{k}$-algebras, TFAE and hold under (RR):

13. *Representability of representations* implies the *2-variety property*

Theorem

For $\mathbb{k}$ infinite and $\mathscr{V}$ a variety of $\mathbb{k}$-algebras, TFAE and hold under (RR): $b(xy)$

(i) $\mathscr{V}$ is **algebraically coherent**: $B\flat X + B\flat Y \to B\flat(X+Y)$ is a surjection, where

$$0 \longrightarrow B\flat Z \longrightarrow B+Z \underset{\langle 1_B, 0\rangle}{\overset{\longleftarrow}{\longrightarrow}} B \longrightarrow 0;$$

[A. S. Cigoli, J. R. A. Gray, and TVdL, *Algebraically coherent categories*, Theory Appl. Categ. **30** (2015), no. 54, 1864–1905]

13. *Representability of representations* implies the *2-variety property*

Theorem

For $\Bbbk$ infinite and $\mathscr{V}$ a variety of $\Bbbk$-algebras, TFAE and hold under (RR):

(i) $\mathscr{V}$ is **algebraically coherent**: $B\flat X + B\flat Y \to B\flat(X+Y)$ is a surjection, where

$$0 \longrightarrow B\flat Z \longrightarrow B+Z \underset{\langle 1_B,0\rangle}{\overset{\longleftarrow}{\longrightarrow}} B \longrightarrow 0;$$

[A. S. Cigoli, J. R. A. Gray, and TVdL, *Algebraically coherent categories*, Theory Appl. Categ. **30** (2015), no. 54, 1864–1905]

(ii) for $\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8$ in $\Bbbk$, the following identities hold in $\mathscr{V}$:

$z(xy) = \lambda_1 y(zx) + \lambda_2 x(yz) + \lambda_3 y(xz) + \lambda_4 x(zy) + \lambda_5 (zx)y + \lambda_6 (yz)x + \lambda_7 (xz)y + \lambda_8 (zy)x,$

$(xy)z = \mu_1 y(zx) + \mu_2 x(yz) + \mu_3 y(xz) + \mu_4 x(zy) + \mu_5 (zx)y + \mu_6 (yz)x + \mu_7 (xz)y + \mu_8 (zy)x;$

Sketch of proof.

(i) $\Leftrightarrow$ (ii) [X. García-Martínez and TVdL, *A characterisation of Lie algebras amongst anti-commutative algebras*, J. Pure Appl. Algebra **223** (2019), 4857–4870]

13. *Representability of representations* implies the *2-variety property*

Theorem

For $\mathbb{k}$ infinite and $\mathscr{V}$ a variety of $\mathbb{k}$-algebras, TFAE and hold under (RR):

(i) $\mathscr{V}$ is **algebraically coherent**: $B\flat X + B\flat Y \to B\flat(X+Y)$ is a surjection, where

$$0 \longrightarrow B\flat Z \longrightarrow B+Z \underset{\langle 1_B,0\rangle}{\leftrightarrows} B \longrightarrow 0;$$

[A. S. Cigoli, J. R. A. Gray, and TVdL, *Algebraically coherent categories*, Theory Appl. Categ. **30** (2015), no. 54, 1864–1905]

(ii) for $\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8$ in $\mathbb{k}$, the following identities hold in $\mathscr{V}$:

$$z(xy) = \lambda_1 y(zx) + \lambda_2 x(yz) + \lambda_3 y(xz) + \lambda_4 x(zy) + \lambda_5 (zx)y + \lambda_6 (yz)x + \lambda_7 (xz)y + \lambda_8 (zy)x,$$
$$(xy)z = \mu_1 y(zx) + \mu_2 x(yz) + \mu_3 y(xz) + \mu_4 x(zy) + \mu_5 (zx)y + \mu_6 (yz)x + \mu_7 (xz)y + \mu_8 (zy)x;$$

(iii) $\mathscr{V}$ is a **2-variety**: for every ideal I in every algebra, I^2 is also an ideal.

Sketch of proof.

(i) $\Leftrightarrow$ (ii) [X. García-Martínez and TVdL, *A characterisation of Lie algebras amongst anti-commutative algebras*, J. Pure Appl. Algebra **223** (2019), 4857–4870]

(ii) $\Leftrightarrow$ (iii) [T. Anderson, *The Levitzki radical in varieties of algebras*, Math. Ann. **194** (1971), 27–34]

13. *Representability of representations* implies the *2-variety property*

Theorem

For $\Bbbk$ infinite and $\mathscr{V}$ a variety of $\Bbbk$-algebras, TFAE and **hold under (RR)**:

(i) $\mathscr{V}$ is **algebraically coherent**: $B\flat X + B\flat Y \to B\flat(X+Y)$ is a surjection, where

$$0 \longrightarrow B\flat Z \longrightarrow B + Z \underset{\langle 1_B, 0\rangle}{\overset{\longleftarrow}{\longrightarrow}} B \longrightarrow 0;$$

[A. S. Cigoli, J. R. A. Gray, and TVdL, *Algebraically coherent categories*, Theory Appl. Categ. **30** (2015), no. 54, 1864–1905]

(ii) for $\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8$ in $\Bbbk$, the following identities hold in $\mathscr{V}$:

$$z(xy) = \lambda_1 y(zx) + \lambda_2 x(yz) + \lambda_3 y(xz) + \lambda_4 x(zy) + \lambda_5 (zx)y + \lambda_6 (yz)x + \lambda_7 (xz)y + \lambda_8 (zy)x,$$
$$(xy)z = \mu_1 y(zx) + \mu_2 x(yz) + \mu_3 y(xz) + \mu_4 x(zy) + \mu_5 (zx)y + \mu_6 (yz)x + \mu_7 (xz)y + \mu_8 (zy)x;$$

(iii) $\mathscr{V}$ is a **2-variety**: for every ideal I in every algebra, I^2 is also an ideal.

Sketch of proof.

(i) $\Leftrightarrow$ (ii) [X. García-Martínez and TVdL, *A characterisation of Lie algebras amongst anti-commutative algebras*, J. Pure Appl. Algebra **223** (2019), 4857–4870]

(ii) $\Leftrightarrow$ (iii) [T. Anderson, *The Levitzki radical in varieties of algebras*, Math. Ann. **194** (1971), 27–34]

Finally, $Rep(-,X) \cong Hom(-,[\![X]\!])\colon \mathscr{V}^{\mathrm{op}} \to \boldsymbol{Set}$ sends sums in $\mathscr{V}$ to products in $\boldsymbol{Set}$. $\square$

13. *Representability of representations* implies the *2-variety property*

Theorem

For $\Bbbk$ infinite and $\mathscr{V}$ a variety of $\Bbbk$-algebras, TFAE and hold under (RR):

(i) $\mathscr{V}$ is **algebraically coherent**: $B\flat X + B\flat Y \to B\flat(X+Y)$ is a surjection, where

$$0 \longrightarrow B\flat Z \longrightarrow B+Z \underset{\langle 1_B, 0\rangle}{\overset{\longleftarrow}{\longrightarrow}} B \longrightarrow 0;$$

[A. S. Cigoli, J. R. A. Gray, and TVdL, *Algebraically coherent categories*, Theory Appl. Categ. **30** (2015), no. 54, 1864–1905]

(ii) for $\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8$ in $\Bbbk$, the following identities hold in $\mathscr{V}$:

$$z(xy) = \lambda_1 y(zx) + \lambda_2 x(yz) + \lambda_3 y(xz) + \lambda_4 x(zy) + \lambda_5 (zx)y + \lambda_6 (yz)x + \lambda_7 (xz)y + \lambda_8 (zy)x,$$
$$(xy)z = \mu_1 y(zx) + \mu_2 x(yz) + \mu_3 y(xz) + \mu_4 x(zy) + \mu_5 (zx)y + \mu_6 (yz)x + \mu_7 (xz)y + \mu_8 (zy)x;$$

(iii) $\mathscr{V}$ is a **2-variety**: for every ideal I in every algebra, I^2 is also an ideal.

Sketch of proof.

(i) $\Leftrightarrow$ (ii) [X. García-Martínez and TVdL, *A characterisation of Lie algebras amongst anti-commutative algebras*, J. Pure Appl. Algebra **223** (2019), 4857–4870]

(ii) $\Leftrightarrow$ (iii) [T. Anderson, *The Levitzki radical in varieties of algebras*, Math. Ann. **194** (1971), 27–34]

Finally, $Rep(-,X) \cong Hom(-,[\![X]\!])\colon \mathscr{V}^{\mathrm{op}} \to \mathbf{Set}$ sends sums in $\mathscr{V}$ to products in $\mathbf{Set}$. $\square$

14. Why an infinite field?

The assumption that $\Bbbk$ is an *infinite* field is relevant for the following reason.

14. Why an infinite field?

The assumption that $\Bbbk$ is an *infinite* field is relevant for the following reason.

Theorem

If $\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ for $\Bbbk$ infinite, then for any identity $\phi(x_1, \ldots, x_n) = 0$, its homogeneous components are again identities. $\square$

[K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, *Rings that are nearly associative*, Academic Press, 1982]

14. Why an infinite field?

The assumption that $\Bbbk$ is an *infinite* field is relevant for the following reason.

Theorem

If $\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ for $\Bbbk$ infinite, then for any identity $\phi(x_1, \ldots, x_n) = 0$, its homogeneous components are again identities. $\square$

[K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, *Rings that are nearly associative*, Academic Press, 1982]

For instance:

- $xy + yx + x(yz) + (xy)z = 0$ entails $xy = -yx$ and $x(yz) = -(xy)z$;
- $xx = x$ entails $x = 0$, so the multiplication cannot be idempotent (no "Boolean rings over an infinite field").

14. Why an infinite field?

The assumption that $\Bbbk$ is an *infinite* field is relevant for the following reason.

Theorem

If $\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ for $\Bbbk$ infinite, then for any identity $\phi(x_1, \ldots, x_n) = 0$, its homogeneous components are again identities. $\square$

[K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, *Rings that are nearly associative*, Academic Press, 1982]

For instance:

- $xy + yx + x(yz) + (xy)z = 0$ entails $xy = -yx$ and $x(yz) = -(xy)z$;
- $xx = x$ entails $x = 0$, so the multiplication cannot be idempotent (no "Boolean rings over an infinite field").

What we really want it that the variety is determined by a set of homogeneous identities.

15. Structure of the proof

We know that any variety of $\Bbbk$-algebras $\mathscr{V}$ with representable representations over an infinite field $\Bbbk$ is a 2-variety: certain degree 3 identities hold.

15. Structure of the proof

We know that any variety of $\mathbb{k}$-algebras $\mathscr{V}$ with representable representations over an infinite field $\mathbb{k}$ is a 2-variety: certain degree 3 identities hold.
We want to show that $\mathscr{V} = \boldsymbol{Lie}_{\mathbb{k}}$ or $\mathscr{V} = \boldsymbol{qLie}_{\mathbb{k}}$ when $\mathscr{V}$ is non-abelian.

15. Structure of the proof

We know that any variety of $\mathbb{k}$-algebras $\mathscr{V}$ with representable representations over an infinite field $\mathbb{k}$ is a 2-variety: certain degree 3 identities hold.
We want to show that $\mathscr{V} = \boldsymbol{Lie}_{\mathbb{k}}$ or $\mathscr{V} = \boldsymbol{qLie}_{\mathbb{k}}$ when $\mathscr{V}$ is non-abelian.

Lemma

If a variety of $\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ satisfies a non-trivial homogeneous identity of degree 2, then $\mathscr{V}$ is either a variety of commutative algebras, or a variety of anticommutative algebras. $\square$

15. Structure of the proof

We know that any variety of $\mathbb{k}$-algebras $\mathscr{V}$ with representable representations over an infinite field $\mathbb{k}$ is a 2-variety: certain degree 3 identities hold.
We want to show that $\mathscr{V} = \boldsymbol{Lie}_{\mathbb{k}}$ or $\mathscr{V} = \boldsymbol{qLie}_{\mathbb{k}}$ when $\mathscr{V}$ is non-abelian.

Lemma

If a variety of $\mathscr{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ satisfies a non-trivial homogeneous identity of degree 2, then $\mathscr{V}$ is either a variety of commutative algebras, or a variety of anticommutative algebras. $\square$

- Commutative case: we may show that representations are not representable in $\mathscr{V}$;

15. Structure of the proof

We know that any variety of $\mathbb{k}$-algebras $\mathcal{V}$ with representable representations over an infinite field $\mathbb{k}$ is a 2-variety: certain degree 3 identities hold.
We want to show that $\mathcal{V} = \boldsymbol{Lie}_{\mathbb{k}}$ or $\mathcal{V} = \boldsymbol{qLie}_{\mathbb{k}}$ when $\mathcal{V}$ is non-abelian.

Lemma

If a variety of $\mathcal{V} \leqslant \boldsymbol{Alg}_{\mathbb{k}}$ satisfies a non-trivial homogeneous identity of degree 2, then $\mathcal{V}$ is either a variety of commutative algebras, or a variety of anticommutative algebras. $\square$

- Commutative case: we may show that representations are not representable in $\mathcal{V}$;
- Anticommutative case: here, from the degree 3 identities we may deduce Jacobi.

15. Structure of the proof

We know that any variety of $\Bbbk$-algebras $\mathscr{V}$ with representable representations over an infinite field $\Bbbk$ is a 2-variety: certain degree 3 identities hold.
We want to show that $\mathscr{V} = \boldsymbol{Lie}_{\Bbbk}$ or $\mathscr{V} = \boldsymbol{qLie}_{\Bbbk}$ when $\mathscr{V}$ is non-abelian.

Lemma

If a variety of $\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ satisfies a non-trivial homogeneous identity of degree 2, then $\mathscr{V}$ is either a variety of commutative algebras, or a variety of anticommutative algebras. □

- Commutative case: we may show that representations are not representable in $\mathscr{V}$;
- Anticommutative case: here, from the degree 3 identities we may deduce Jacobi.

Can we prove that an identity of degree 2 must hold in $\mathscr{V}$?

15. Structure of the proof

We know that any variety of $\Bbbk$-algebras $\mathscr{V}$ with representable representations over an infinite field $\Bbbk$ is a 2-variety: certain degree 3 identities hold.
We want to show that $\mathscr{V} = \boldsymbol{Lie}_{\Bbbk}$ or $\mathscr{V} = \boldsymbol{qLie}_{\Bbbk}$ when $\mathscr{V}$ is non-abelian.

Lemma

If a variety of $\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ satisfies a non-trivial homogeneous identity of degree 2, then $\mathscr{V}$ is either a variety of commutative algebras, or a variety of anticommutative algebras. $\square$

- Commutative case: we may show that representations are not representable in $\mathscr{V}$;
- Anticommutative case: here, from the degree 3 identities we may deduce Jacobi.

Can we prove that an identity of degree 2 must hold in $\mathscr{V}$? **Yes!**

15. Structure of the proof

We know that any variety of $\Bbbk$-algebras $\mathcal{V}$ with representable representations over an infinite field $\Bbbk$ is a 2-variety: certain degree 3 identities hold.
We want to show that $\mathcal{V} = \boldsymbol{Lie}_{\Bbbk}$ or $\mathcal{V} = \boldsymbol{qLie}_{\Bbbk}$ when $\mathcal{V}$ is non-abelian.

Lemma

If a variety of $\mathcal{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ satisfies a non-trivial homogeneous identity of degree 2, then $\mathcal{V}$ is either a variety of commutative algebras, or a variety of anticommutative algebras. $\square$

- Commutative case: we may show that representations are not representable in $\mathcal{V}$;
- Anticommutative case: here, from the degree 3 identities we may deduce Jacobi.

Can we prove that an identity of degree 2 must hold in $\mathcal{V}$? **Yes!**

When this is done, proving that there are no subvarieties besides $\boldsymbol{Lie}_{\Bbbk}$ and $\boldsymbol{qLie}_{\Bbbk}$ is straightforward.

16. Finding non-trivial identities of degree 2

Proposition

Let $\mathbb{k}$ be a field of characteristic 0 and $\mathscr{V}$ a variety of $\mathbb{k}$-algebras.

If $\mathscr{V}$ has representable representations, then $\mathscr{V}$ satisfies a non-trivial identity of degree 2.

16. Finding non-trivial identities of degree 2

Proposition

Let $\mathbb{k}$ be a field of characteristic 0 and $\mathscr{V}$ a variety of $\mathbb{k}$-algebras.
If $\mathscr{V}$ has representable representations, then $\mathscr{V}$ satisfies a non-trivial identity of degree 2.

Sketch of proof.

In $\mathscr{V}$ with (RR), from the 2-variety property we deduce a system of polynomial equations, which is inconsistent when there are no degree 2 identities.

16. Finding non-trivial identities of degree 2

Proposition

Let $\mathbb{k}$ be a field of characteristic 0 and $\mathscr{V}$ a variety of $\mathbb{k}$-algebras.
If $\mathscr{V}$ has representable representations, then $\mathscr{V}$ satisfies a non-trivial identity of degree 2.

Sketch of proof.

In $\mathscr{V}$ with (RR), from the 2-variety property we deduce a system of polynomial equations, which is inconsistent when there are no degree 2 identities.

We let X be a certain 79-dimensional vector space and consider three copies B^1, B^2 and B^3 of the free $\mathscr{V}$-algebra on a single generator, each with a chosen (abelian) action on X.

16. Finding non-trivial identities of degree 2

Proposition

Let $\mathbb{k}$ be a field of characteristic 0 and $\mathscr{V}$ a variety of $\mathbb{k}$-algebras.
If $\mathscr{V}$ has representable representations, then $\mathscr{V}$ satisfies a non-trivial identity of degree 2.

Sketch of proof.

In $\mathscr{V}$ with (RR), from the 2-variety property we deduce a system of polynomial equations, which is inconsistent when there are no degree 2 identities.

We let X be a certain 79-dimensional vector space and consider three copies B^1, B^2 and B^3 of the free $\mathscr{V}$-algebra on a single generator, each with a chosen (abelian) action on X.
The isomorphism

$$Rep(B^1 + B^2 + B^3, X) \cong Rep(B^1, X) \times Rep(B^2, X) \times Rep(B^3, X)$$

gives us an action ξ of $B^1 + B^2 + B^3$ on X

16. Finding non-trivial identities of degree 2

Proposition

Let $\mathbb{k}$ be a field of characteristic 0 and $\mathscr{V}$ a variety of $\mathbb{k}$-algebras.
If $\mathscr{V}$ has representable representations, then $\mathscr{V}$ satisfies a non-trivial identity of degree 2.

Sketch of proof.

In $\mathscr{V}$ with (RR), from the 2-variety property we deduce a system of polynomial equations, which is inconsistent when there are no degree 2 identities.

We let X be a certain 79-dimensional vector space and consider three copies B^1, B^2 and B^3 of the free $\mathscr{V}$-algebra on a single generator, each with a chosen (abelian) action on X.
The isomorphism

$$Rep(B^1 + B^2 + B^3, X) \cong Rep(B^1, X) \times Rep(B^2, X) \times Rep(B^3, X)$$

gives us an action ξ of $B^1 + B^2 + B^3$ on X, and the 2-variety property in $X \rtimes_\xi (B^1 + B^2 + B^3)$ gives rise to a system $(f_i = 0)_{1 \leqslant i \leqslant 224}$ of 224 polynomial equations (in λ_j and μ_j, $1 \leqslant j \leqslant 8$).

16. Finding non-trivial identities of degree 2

Proposition

Let $\mathbb{k}$ be a field of characteristic 0 and $\mathscr{V}$ a variety of $\mathbb{k}$-algebras.
If $\mathscr{V}$ has representable representations, then $\mathscr{V}$ satisfies a non-trivial identity of degree 2.

Sketch of proof.

In $\mathscr{V}$ with (RR), from the 2-variety property we deduce a system of polynomial equations, which is inconsistent when there are no degree 2 identities.

We let X be a certain 79-dimensional vector space and consider three copies B^1, B^2 and B^3 of the free $\mathscr{V}$-algebra on a single generator, each with a chosen (abelian) action on X.
The isomorphism

$$Rep(B^1 + B^2 + B^3, X) \cong Rep(B^1, X) \times Rep(B^2, X) \times Rep(B^3, X)$$

gives us an action ξ of $B^1 + B^2 + B^3$ on X, and the 2-variety property in $X \rtimes_{\xi} (B^1 + B^2 + B^3)$ gives rise to a system $(f_i = 0)_{1 \leqslant i \leqslant 224}$ of 224 polynomial equations (in λ_j and μ_j, $1 \leqslant j \leqslant 8$).

Assuming there are no identities of degree 2, we may show that there exist $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$ and a non-zero integer m such that $m = \sum_i \alpha_i f_i$, proving the system of equations $(f_i = 0)_{1 \leqslant i \leqslant 224}$ to be inconsistent. $\square$

17. Finding $m = \sum_i \alpha_i f_i$ for some $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$

17. Finding $m = \sum_i \alpha_i f_i$ for some $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$

To find an integer m and polynomials $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$ for which $m = \sum_i \alpha_i f_i$, we had to use a computer.

17. Finding $m = \sum_i \alpha_i f_i$ for some $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$

To find an integer m and polynomials $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$ for which $m = \sum_i \alpha_i f_i$, we had to use a computer.

Using Gröbner bases, the package **Singular** tells us that we may take m equal to the number
145679959084559802430969530553780449546315662406534685329857227054868047
204547211625038600686746894466894057189739717262364992063289026729607565
434350420878447841877721000590295768558884307124148778177377876830064916
666592523291593041749054969370877385813443494870668801865569455851757756
955762099574629327848081243141226057440402445559832044185972204201826090
047396984682860864562787183059873566162910333424228212906065894343670540
539725147842861513488173278220217745776941965011882278131563632704266203
6615068825734248984068309403420950318 of 541 digits.

17. Finding $m = \sum_i \alpha_i f_i$ for some $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$

To find an integer m and polynomials $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$ for which $m = \sum_i \alpha_i f_i$, we had to use a computer.

Using Gröbner bases, the package **Singular** tells us that we may take m equal to the number
145679959084559802430969530553780449546315662406534685329857227054868047
204547211625038600686746894466894057189739717262364992063289026729607565
434350420878447841877721000590295768558884307124148778177377876830064916
666592523291593041749054969370877385813443494870668801865569455851757756
955762099574629327848081243141226057440402445559832044185972204201826090
047396984682860864562787183059873566162910333424228212906065894343670540
539725147842861513488173278220217745776941965011882278131563632704266203
6615068825734248984068309403420950318 of 541 digits.
The file that contains the corresponding α_i is over 8 megabyte large.

17. Finding $m = \sum_i \alpha_i f_i$ for some $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$

To find an integer m and polynomials $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$ for which $m = \sum_i \alpha_i f_i$, we had to use a computer.

Using Gröbner bases, the package **Singular** tells us that we may take m equal to the number
145679959084559802430969530553780449546315662406534685329857227054868047
204547211625038600686746894466894057189739717262364992063289026729607565
434350420878447841877721000590295768558884307124148778177377876830064916
666592523291593041749054969370877385813443494870668801865569455851757756
955762099574629327848081243141226057440402445559832044185972204201826090
047396984682860864562787183059873566162910333424228212906065894343670540
539725147842861513488173278220217745776941965011882278131563632704266203
6615068825734248984068309403420950318 of 541 digits.
The file that contains the corresponding α_i is over 8 megabyte large.

Checking that the equality does indeed hold is relatively simple.

17. Finding $m = \sum_i \alpha_i f_i$ for some $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$

To find an integer m and polynomials $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$ for which $m = \sum_i \alpha_i f_i$, we had to use a computer.

Using Gröbner bases, the package **Singular** tells us that we may take m equal to the number
$1456799590845598024309695305537804495463156624065346853298572270548680472045472116250386006867468944668940571897397172623649920632890267296075654343504208784478418777210005902957685588843071241487781773778768300649166665925232915930417490549693708773858134434948706688018655694558517577569557620995746293278480812431412260574404024455598320441859722042018260900473969846828608645627871830598735661629103334242282129060658943436705405397251478428615134881732782202177457769419650118822781315636327042662036615068825734248984068309403420950318$ of 541 digits.
The file that contains the corresponding α_i is over 8 megabyte large.

Checking that the equality does indeed hold is relatively simple.
The proof may be extended to infinite fields of prime characteristic.

18. Action representable varieties of $\Bbbk$-algebras

$\mathscr{V} \leqslant \boldsymbol{Alg}_{\Bbbk}$ means $\mathscr{V}$ consists of $\Bbbk$-algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

Are known to be action representable such:

- $\boldsymbol{Vect}_{\Bbbk}$: the algebras are **abelian**, $xy = 0$;
- $\boldsymbol{Lie}_{\Bbbk}$;
- $\boldsymbol{qLie}_{\Bbbk}$ (Jacobi and $xy = -yx$) when $char(\Bbbk) = 2$:
  then $\boldsymbol{Lie}_{\Bbbk} \lneqq \boldsymbol{qLie}_{\Bbbk}$ since $xy = -yx$ does not imply $xx = 0$;
- Boolean rings: $\Bbbk = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$; here, $[X] = End(X)$.

Is there anything else? For infinite $\Bbbk$: **No!**

[X. García-Martínez, M. Tsishyn, TVdL, and C. Vienne, *Algebras with representable representations*, Proc. Edinb. Math. Soc. (2) **64** (2021), no. 3, 555–573]

Theorem

When $\Bbbk$ is an infinite field, an action representable variety of $\Bbbk$-algebras is either $\boldsymbol{Lie}_{\Bbbk}$, $\boldsymbol{qLie}_{\Bbbk}$, or $\boldsymbol{Vect}_{\Bbbk}$.

$Der(X)$ **cannot be generalized in a way which is globally valid.**

19. Similar results, I

19. Similar results, I

- We saw that a variety $\mathscr{V}$ is a 2-variety iff $B\flat X + B\flat Y \to B\flat(X+Y)$ is always a surjection.

19. Similar results, I

- We saw that a variety $\mathscr{V}$ is a 2-variety iff $B\flat X + B\flat Y \to B\flat(X+Y)$ is always a surjection. Question: What if it is always an isomorphism?

19. Similar results, I

- We saw that a variety $\mathscr{V}$ is a 2-variety iff $B\flat X + B\flat Y \to B\flat(X+Y)$ is always a surjection.
  Question: What if it is always an isomorphism?
  Answer: If $\mathbb{k}$ is infinite and $\mathscr{V}$ is non-abelian, then $\mathscr{V} = \boldsymbol{Lie}_{\mathbb{k}}$ or $\mathscr{V} = \boldsymbol{qLie}_{\mathbb{k}}$.
  [X. García-Martínez and TVdL, *A characterisation of Lie algebras via algebraic exponentiation*, Adv. Math. **341** (2019), 92–117]

19. Similar results, I

- We saw that a variety $\mathscr{V}$ is a 2-variety iff $B\flat X + B\flat Y \to B\flat(X+Y)$ is always a surjection.
  Question: What if it is always an isomorphism?
  Answer: If $\mathbb{k}$ is infinite and $\mathscr{V}$ is non-abelian, then $\mathscr{V} = \boldsymbol{Lie}_{\mathbb{k}}$ or $\mathscr{V} = \boldsymbol{qLie}_{\mathbb{k}}$.

  [X. García-Martínez and TVdL, *A characterisation of Lie algebras via algebraic exponentiation*, Adv. Math. **341** (2019), 92–117]

- This implies that the Kaluzhnin–Krasner *Universal Embedding Theorem* cannot be extended from Lie algebras to other varieties of non-associative algebras.

  [V. M. Petrogradsky, Y. P. Razmyslov and È. O. Shishkin, *Wreath products and Kaluzhnin-Krasner embedding for Lie algebras*, Proc. Amer. Math. Soc. **135** (2007), no. 3, 625–636]

  [B. S. Deval, X. García-Martínez and TVdL, *A universal Kaluzhnin-Krasner embedding theorem*, Proc. Amer. Math. Soc. **152** (2024), no. 12, 5039–5053]

20. Similar results, II

20. Similar results, II

- A non-trivial 2-variety over an infinite field is $\boldsymbol{Lie}_{\mathbb{k}}$ iff it is a **Schreier variety**: every subalgebra of a free algebra is again free.

  [V. V. Dotsenko and X. García-Martínez, *A characterisation of Lie algebras using ideals and subalgebras*, Bull. Lond. Math. Soc. **56** (2024), no. 7, 2408–2423]

20. Similar results, II

- A non-trivial 2-variety over an infinite field is $\boldsymbol{Lie}_{\mathbb{k}}$ iff it is a **Schreier variety**: every subalgebra of a free algebra is again free.

  [V. V. Dotsenko and X. García-Martínez, *A characterisation of Lie algebras using ideals and subalgebras*, Bull. Lond. Math. Soc. **56** (2024), no. 7, 2408–2423]

- What about *associative* algebras?

20. Similar results, II

- A non-trivial 2-variety over an infinite field is $\textbf{\textit{Lie}}_{\mathbb{k}}$ iff
  it is a **Schreier variety**: every subalgebra of a free algebra is again free.

  [V. V. Dotsenko and X. García-Martínez, *A characterisation of Lie algebras using ideals and subalgebras*, Bull. Lond. Math. Soc. **56** (2024), no. 7, 2408–2423]
- What about *associative* algebras?
  Question: Can we capture associativity via a categorical condition?

20. Similar results, II

- A non-trivial 2-variety over an infinite field is $\boldsymbol{Lie}_{\mathbb{k}}$ iff
  it is a **Schreier variety**: every subalgebra of a free algebra is again free.

  [V. V. Dotsenko and X. García-Martínez, *A characterisation of Lie algebras using ideals and subalgebras*, Bull. Lond. Math. Soc. **56** (2024), no. 7, 2408–2423]

- What about *associative* algebras?
  Question: Can we capture associativity via a categorical condition?
  Answer: Yes, for commutative algebras over an infinite field;
  these are associative iff a condition called *cosmash associativity* holds.

  [Ü. Reimaa, TVdL and C. Vienne, *Associativity and the cosmash product in operadic varieties of algebras*, Illinois J. Math. **67** (2023), no. 3, 563–598]

21. A question

[M. Culot, F. Renaud, and TVdL, *Non-additive derived functors via chain resolutions*, Glasg. Math. J. **67** (2025), no. 3, 423–466]

I would like to understand the following condition in the context of non-associative algebras:

Given a split extension $(k = \ker(f),\, f = \mathrm{coker}(k),\, f \circ s = 1_B)$

$$0 \longrightarrow K \overset{k}{\rightarrowtail} X \underset{s}{\overset{f}{\rightleftarrows}} Y \longrightarrow 0$$

if the object X is projective, then both K and Y are projective as well.

21. A question

[M. Culot, F. Renaud, and TVdL, *Non-additive derived functors via chain resolutions*, Glasg. Math. J. **67** (2025), no. 3, 423–466]

I would like to understand the following condition in the context of non-associative algebras:

> *Given a split extension* $(k = \mathrm{ker}(f),\, f = \mathrm{coker}(k),\, f \circ s = 1_B)$
>
> $$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, tail, "k"] & X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$
>
> *if the object X is projective, then both K and Y are projective as well.*

Note that Y is *always* projective when so is X, because f is a retraction.

21. A question

[M. Culot, F. Renaud, and TVdL, *Non-additive derived functors via chain resolutions*, Glasg. Math. J. **67** (2025), no. 3, 423–466]

I would like to understand the following condition in the context of non-associative algebras:

Given a split extension $(k = \ker(f),\ f = \operatorname{coker}(k),\ f \circ s = 1_B)$

$$0 \longrightarrow K \xrightarrow{k} X \underset{s}{\overset{f}{\rightleftarrows}} Y \longrightarrow 0$$

if the object X is projective, then both K and Y are projective as well.

Note that Y is *always* projective when so is X, because f is a retraction.

- Known to be true in all abelian categories: then K is a retract of X, because $X \cong K \oplus Y$.

21. A question

[M. Culot, F. Renaud, and TVdL, *Non-additive derived functors via chain resolutions*, Glasg. Math. J. **67** (2025), no. 3, 423–466]

I would like to understand the following condition in the context of non-associative algebras:

Given a split extension $(k = \ker(f),\ f = \operatorname{coker}(k),\ f \circ s = 1_B)$

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k"] & X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}
$$

if the object X is projective, then both K and Y are projective as well.

Note that Y is *always* projective when so is X, because f is a retraction.

- Known to be true in all abelian categories: then K is a retract of X, because $X \cong K \oplus Y$.
- True also for groups and Lie algebras over a field:
  These are *Schreier varieties*, any subalgebra of a free algebra is again free.

21. A question

[M. Culot, F. Renaud, and TVdL, *Non-additive derived functors via chain resolutions*, Glasg. Math. J. **67** (2025), no. 3, 423–466]

I would like to understand the following condition in the context of non-associative algebras:

Given a split extension $(k = \ker(f),\ f = \operatorname{coker}(k),\ f \circ s = 1_B)$

$$0 \longrightarrow K \overset{k}{\rightarrowtail} X \underset{s}{\overset{f}{\rightleftarrows}} Y \longrightarrow 0$$

if the object X is projective, then both K and Y are projective as well.

Note that Y is *always* projective when so is X, because f is a retraction.

- Known to be true in all abelian categories: then K is a retract of X, because $X \cong K \oplus Y$.
- True also for groups and Lie algebras over a field:
  These are *Schreier varieties*, any subalgebra of a free algebra is again free.
- True for Lie algebras over any commutative ring with unit, *despite the fact that these need not be Schreier*. The proof is based on Lazard's Elimination Theorem.

21. A question

[M. Culot, F. Renaud, and TVdL, *Non-additive derived functors via chain resolutions*, Glasg. Math. J. **67** (2025), no. 3, 423–466]

I would like to understand the following condition in the context of non-associative algebras:

Given a split extension $(k = \ker(f),\ f = \operatorname{coker}(k),\ f \circ s = 1_B)$

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, tail, "k"] & X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

if the object X is projective, then both K and Y are projective as well.

Note that Y is *always* projective when so is X, because f is a retraction.

- Known to be true in all abelian categories: then K is a retract of X, because $X \cong K \oplus Y$.
- True also for groups and Lie algebras over a field:
  These are *Schreier varieties*, any subalgebra of a free algebra is again free.
- True for Lie algebras over any commutative ring with unit, *despite the fact that these need not be Schreier*. The proof is based on Lazard's Elimination Theorem.

When does a variety of non-associative algebras satisfy this condition?

22. Outlook

22. Outlook

- The study of non-associative algebras has helped me to better understand constructions and conditions occurring naturally in categorical algebra.

22. Outlook

- The study of non-associative algebras has helped me to better understand constructions and conditions occurring naturally in categorical algebra.

  This context turned out to be surprisingly rich and highly appropriate for this aim.

22. Outlook

- The study of non-associative algebras has helped me to better understand constructions and conditions occurring naturally in categorical algebra.

  This context turned out to be surprisingly rich and highly appropriate for this aim.
- Reciprocally, categorical algebra offers

22. Outlook

- The study of non-associative algebras has helped me to better understand constructions and conditions occurring naturally in categorical algebra.

  This context turned out to be surprisingly rich and highly appropriate for this aim.
- Reciprocally, categorical algebra offers
  - a set of standard techniques which may be useful in the study of non-associative algebras,

22. Outlook

- The study of non-associative algebras has helped me to better understand constructions and conditions occurring naturally in categorical algebra.

  This context turned out to be surprisingly rich and highly appropriate for this aim.
- Reciprocally, categorical algebra offers
  - a set of standard techniques which may be useful in the study of non-associative algebras,
  - an alternative perspective on things which are likely known,

22. Outlook

- The study of non-associative algebras has helped me to better understand constructions and conditions occurring naturally in categorical algebra.

  This context turned out to be surprisingly rich and highly appropriate for this aim.
- Reciprocally, categorical algebra offers
  - a set of standard techniques which may be useful in the study of non-associative algebras,
  - an alternative perspective on things which are likely known,
  - questions which may eventually lead to new insights.

22. Outlook

- The study of non-associative algebras has helped me to better understand constructions and conditions occurring naturally in categorical algebra.

  This context turned out to be surprisingly rich and highly appropriate for this aim.
- Reciprocally, categorical algebra offers
  - a set of standard techniques which may be useful in the study of non-associative algebras,
  - an alternative perspective on things which are likely known,
  - questions which may eventually lead to new insights.
- My plan for the near future is to extend the field of study
  by using these techniques in the context of **algebraic operads**,
  admitting more than one operation of arities potentially higher than two.

Thank you!