

Some categorical remarks on non-associative algebras

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AMS 2025 Fall Virtual Eastern Sectional | October 26, 2025

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Let $\mathbb{k}$ be a field.

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Some examples:

- $\mathbf{Lie}_{\mathbb{k}}$ is the variety of $\mathbb{k}$ -algebras determined by the equations

$$\begin{cases} xx = 0 & \text{multiplication is alternating} \\ x(yz) + y(zx) + z(xy) = 0 & \text{Jacobi identity} \end{cases}$$

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These categories are particularly well behaved: they are *semi-abelian*.

3. Semi-abelian categories

Generalizing *abelian* categories (modules over a ring, sheaves of abelian groups), *semi-abelian* categories were introduced in

[G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*,

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Let us look at the details in the special case of varieties of $\mathbb{k}$ -algebras.

4. Short exact sequences, in general

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In order to define these, we need a **zero object**:

an object $0 := \perp = \top$ which is both **initial**, $\forall Y \exists! (\perp \rightarrow Y)$ and **terminal**, $\forall X \exists! (X \rightarrow \top)$.

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- ▶ any one-element group or non-associative algebra is a zero object;
- ▶ hence **Gp** and any $\mathcal{V} \leq \mathbf{Alg}_{\mathbb{k}}$ are pointed categories; here $0: X \rightarrow Y: x \mapsto 0$;
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A **short exact sequence** is then a pair of arrows (k, q)

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A commutative diagram illustrating a short exact sequence. The sequence of objects is $0 \longrightarrow K \xrightarrow{k} X \xrightarrow{q} Q \longrightarrow 0$. Above the object K is an object L . A vertical dotted arrow labeled $\exists! \bar{\ell}$ points from L to K . A diagonal arrow labeled $\forall \ell$ points from L to X . A curved arrow labeled 0 points from L to Q . The sequence of arrows is $0 \longrightarrow K \xrightarrow{k} X \xrightarrow{q} Q \longrightarrow 0$.

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5. Short exact sequences, in varieties of $\mathbb{k}$ -algebras

In any variety of $\mathbb{k}$ -algebras, kernels exist and are computed in the underlying vector spaces:
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- 1 take the image $f(X) = \{f(x) \in Y \mid x \in X\}$, this is a subalgebra of Y ;
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Any cokernel of f is isomorphic to this one. Up to iso, any short exact sequence is of the form

$$0 \longrightarrow I \xrightarrow{i} X \xrightarrow{q} X/I \longrightarrow 0$$

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- 1 take the image $f(X) = \{f(x) \in Y \mid x \in X\}$, this is a subalgebra of Y ;
- 2 take the smallest ideal I of Y containing $f(X)$;
- 3 let q be the vector space quotient $Y \rightarrow Y/I$ with the induced multiplication.

Any cokernel of f is isomorphic to this one. Up to iso, any short exact sequence is of the form

$$0 \longrightarrow I \xrightarrow{i} X \xrightarrow{q} X/I \longrightarrow 0$$

with i the inclusion of an ideal I of X .

- ▶ A morphism of algebras is a **normal epimorphism** (= a cokernel) iff it is surjective.

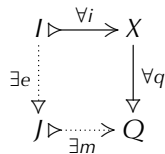
6. Properties of normal epis and normal monos

$$\begin{array}{ccc}
 I & \xrightarrow{\forall i} & X \\
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In a variety of $\mathbb{k}$ -algebras, any $q \circ i$ where i normal mono and q normal epi can be written as $m \circ e$ with e normal epi and m normal mono.

Indeed, $J := qi(I)$ is an ideal of Q : for instance, $JQ \subseteq Q$ because for $x \in X$ and $y \in I$, we have $q(x)qi(y) = q(xi(y))$ and $xi(y) \in I$.

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Results such as the (3×3) -Lemma and the *Short Five Lemma* depend on **protomodularity**:

For $k = \ker(f)$ and $f \circ s = 1_Y$, the images of k and s combined cover X .

$$K \xrightarrow{k} X \begin{array}{c} \xleftarrow{f} \\ \xrightarrow{s} \end{array} Y$$

Here, $x = (x - sf(x)) + sf(x)$ for any $x \in X$; note that $x - sf(x) \in K$ and $sf(x) \in s(Y)$.

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Note that we do not ask for the dual of this condition to hold:
in the present context, that dual is equivalent to abelianness.

7. Limits and colimits

Kernels and cokernels are instances of small limits and colimits, which exist in all varieties of algebras.

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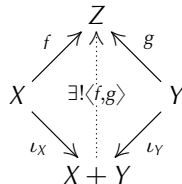
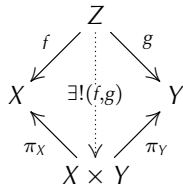
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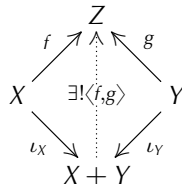
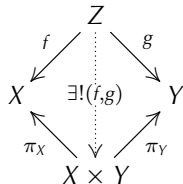


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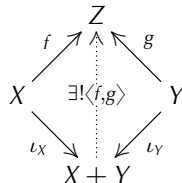
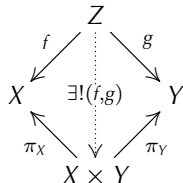


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If B and Z are free over $\{b\}$ and $\{z\}$, then $B + Z$ is free over $\{b, z\}$. The kernel $B \flat Z$ in

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So, for instance, $z + (bz)b$ is in $B \flat Z$, but b^2 is not.

8. Groups act by automorphisms, Lie algebras by derivations

[D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. **4** (1998), no. 2, 37–46]

Via a semidirect product construction

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Categorically, what is $Der(X)$?

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Categorically, this is called **action representability**:

[F. Borceux, G. Janelidze, and G. M. Kelly, *On the representability of actions in a semi-abelian category*, Theory Appl. Categ. **14** (2005), no. 11, 244–286]

Equivalence classes of split extensions in $\mathcal{V}$ by an object X are representable, by an object $[X]$.

$$\text{SplitExt}(-, X) \cong \text{Hom}(-, [X]): \mathcal{V}^{\text{op}} \rightarrow \mathbf{Set}$$

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We look for a **global** solution: a construction valid for all split extensions in a given variety $\mathcal{V}$.

Categorically, this is called **action representability**:

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Equivalence classes of split extensions in $\mathcal{V}$ by an object X are representable, by an object $[X]$.

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Then the $[X]$ plays the role of $\text{Der}(X)$, so it contains “generalized derivations”.

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Der(X) cannot be generalized in a way which is globally valid.

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$$0 \longrightarrow X \xrightarrow{k} X \rtimes_{\xi} B \xrightleftharpoons{s} B \longrightarrow 0$$

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We show that for $\mathcal{V}$ a non-abelian variety of algebras over $\mathbb{k}$ infinite, representability of representations implies that either $\mathcal{V} = \mathbf{Lie}_{\mathbb{k}}$ or $\mathcal{V} = \mathbf{qLie}_{\mathbb{k}}$.

13. *Representability of representations implies the 2-variety property*

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If $\mathcal{V} \leq \mathbf{Alg}_{\mathbb{k}}$ for $\mathbb{k}$ infinite, then for any identity $\phi(x_1, \dots, x_n) = 0$, its homogeneous components are again identities. □

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What we really want is that the variety is determined by a set of homogeneous identities.

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We know that any variety of $\mathbb{k}$ -algebras $\mathcal{V}$ with representable representations over an infinite field $\mathbb{k}$ is a 2-variety: certain degree 3 identities hold.

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- ▶ Anticommutative case: here, from the degree 3 identities we may deduce Jacobi.

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We know that any variety of $\mathbb{k}$ -algebras $\mathcal{V}$ with representable representations over an infinite field $\mathbb{k}$ is a 2-variety: certain degree 3 identities hold.

We want to show that $\mathcal{V} = \mathbf{Lie}_{\mathbb{k}}$ or $\mathcal{V} = \mathbf{qLie}_{\mathbb{k}}$ when $\mathcal{V}$ is non-abelian.

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If a variety of $\mathcal{V} \leq \mathbf{Alg}_{\mathbb{k}}$ satisfies a non-trivial homogeneous identity of degree 2, then $\mathcal{V}$ is either a variety of commutative algebras, or a variety of anticommutative algebras. $\square$

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When this is done, proving that there are no subvarieties besides $\mathbf{Lie}_{\mathbb{k}}$ and $\mathbf{qLie}_{\mathbb{k}}$ is straightforward.

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Assuming there are no identities of degree 2, we may show that there exist $\alpha_i \in \mathbb{Z}[\lambda_1, \dots, \lambda_8, \mu_1, \dots, \mu_8]$ and a non-zero integer m such that $m = \sum_i \alpha_i f_i$, proving the system of equations $(f_i = 0)_{1 \leq i \leq 224}$ to be inconsistent. □

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Using Gröbner bases, the package **Singular** tells us that we may take m equal to the number
145679959084559802430969530553780449546315662406534685329857227054868047
204547211625038600686746894466894057189739717262364992063289026729607565
434350420878447841877721000590295768558884307124148778177377876830064916
666592523291593041749054969370877385813443494870668801865569455851757756
955762099574629327848081243141226057440402445559832044185972204201826090
047396984682860864562787183059873566162910333424228212906065894343670540
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The proof may be extended to infinite fields of prime characteristic.

18. Action representable varieties of $\mathbb{k}$ -algebras

$\mathcal{V} \leq \mathbf{Alg}_{\mathbb{k}}$ means $\mathcal{V}$ consists of $\mathbb{k}$ -algebras $(X, \cdot)$ satisfying a set of (polynomial) equations.

Are known to be action representable such:

- ▶ **$\mathbf{Vect}_{\mathbb{k}}$** : the algebras are **abelian**, $xy = 0$;
- ▶ **$\mathbf{Lie}_{\mathbb{k}}$** ;
- ▶ **$\mathbf{qLie}_{\mathbb{k}}$** (Jacobi and $xy = -yx$) when $\text{char}(\mathbb{k}) = 2$:
then **$\mathbf{Lie}_{\mathbb{k}} \not\leq \mathbf{qLie}_{\mathbb{k}}$** since $xy = -yx$ does not imply $xx = 0$;
- ▶ Boolean rings: $\mathbb{k} = \mathbb{Z}_2$ with $xx = x$, $xy = yx$ and $x(yz) = (xy)z$; here, $[X] = \text{End}(X)$.

Is there anything else? For infinite $\mathbb{k}$: **No!**

[X. García-Martínez, M. Tsishyn, TVdL, and C. Vienne, *Algebras with representable representations*, Proc. Edinb. Math. Soc. (2) **64** (2021), no. 3, 555–573]

Theorem

When $\mathbb{k}$ is an infinite field, an action representable variety of $\mathbb{k}$ -algebras is either **$\mathbf{Lie}_{\mathbb{k}}$** , **$\mathbf{qLie}_{\mathbb{k}}$** , or **$\mathbf{Vect}_{\mathbb{k}}$** .

$\text{Der}(X)$ cannot be generalized in a way which is globally valid.

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Answer: If $\mathbb{k}$ is infinite and $\mathcal{V}$ is non-abelian, then $\mathcal{V} = \mathbf{Lie}_{\mathbb{k}}$ or $\mathcal{V} = \mathbf{qLie}_{\mathbb{k}}$.

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- This implies that the Kaluzhnin–Krasner *Universal Embedding Theorem* cannot be extended from Lie algebras to other varieties of non-associative algebras.

[V. M. Petrogradsky, Y. P. Razmyslov and È. O. Shishkin, *Wreath products and Kaluzhnin-Krasner embedding for Lie algebras*, Proc. Amer. Math. Soc. **135** (2007), no. 3, 625–636]

[B. S. Deval, X. García-Martínez and TVdL, *A universal Kaluzhnin-Krasner embedding theorem*, Proc. Amer. Math. Soc. **152** (2024), no. 12, 5039–5053]

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- ▶ What about *associative* algebras?

Question: Can we capture associativity via a categorical condition?

Answer: Yes, for commutative algebras over an infinite field;

these are associative iff a condition called *cosmash associativity* holds.

[Ü. Reimaa, TVdL and C. Vienne, *Associativity and the cosmash product in operadic varieties of algebras*, Illinois J. Math. **67** (2023), no. 3, 563–598]

21. A question

[M. Culot, F. Renaud, and TVdL, *Non-additive derived functors via chain resolutions*, Glasg. Math. J. **67** (2025), no. 3, 423–466]

I would like to understand the following condition in the context of non-associative algebras:

Given a split extension $(k = \ker(f), f = \operatorname{coker}(k), f \circ s = 1_B)$

$$0 \longrightarrow K \xrightarrow{k} X \xrightleftharpoons[s]{f} Y \longrightarrow 0$$

if the object X is projective, then both K and Y are projective as well.

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When does a variety of non-associative algebras satisfy this condition?

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- ▶ My plan for the near future is to extend the field of study by using these techniques in the context of **algebraic operads**, admitting more than one operation of arities potentially higher than two.

Thank you!