A SIMPLE TWO PERIOD OVERLAPPING GENERATION (OLG) MODEL FOR PUBLIC PENSION SCHEME (PAYG)

Hassana Al-Hassan, Pierre Devolder, Christiana Nayrko, K. Sagary Nokoh

LIDAM Discussion Paper ISBA
2023 / 33

ISBA
Voie du Roman Pays 20 - L1.04.01
B-1348 Louvain-la-Neuve
Email : lidam-library@uclouvain.be
https://uclouvain.be/en/research-institutes/lidam/isba/publication.html

A Simple Two Period Overlapping Generation (OLG) Model For Public Pension Scheme (PAYG)

Hassana Al-Hassan[1], Prof. Pierre Devolder [2], Dr. Christiana Nayrko[3] and Prof. K. Sagary Nokoh[4]

[1,3,4]Faculty of Engineering, Department of Mathematical Sciences, University of Mines and Technology, Ghana
[2] Université Catholique de Louvain, Actuarial Science Department, Belgium

November 7, 2023

[1]hassana@aims.ac.za
[2]pierre.devolder@uclouvain.be
[3]ccnyarko@umat.edu.gh
[4]nokoebiomaths@gmail.com

Abstract

In this paper, we develop in a PAYG public pension system, various ways to share the longevity risk across generations of active affiliates and retirees. We consider a simplified two period Overlapping generation (OLG) model with three major groups: active workers, new retirees and existing retirees. Two levels of risk sharing are proposed; in a first step we develop the sharing between the contributors and the beneficiaries by proposing various designs of the plan, from Defined Benefit to Defined Contribution including hybrid solutions such as Musgrave plans. For this level, the driving force is the dependency ratio. In a second step we consider the sharing between the retirees themselves by considering two important degrees of freedom: the level of the first pension for the new retirees and the revaluation of existing pensions for the older retirees. Different strategies of risk sharing are presented in this framework. We illustrate the concepts by numerical illustrations based on deterministic and stochastic demographic models.

Keywords. PAYG, Dependency Ratio, DB, DC, Musgrave, Risk sharing , Revaluation

1. Introduction

Pensions are crucial in the standard of living of every country. Public Pensions play the role of insurance for retirees and help them live their normal lives after leaving the active work force. Public Pensions serve clearly as a safety net for retirees. Pensions are therefore a very important factor in every economy. Public Pensions differ from country to country. Some countries have generous public Pension schemes whiles others have moderate to zero Pensions. These differences are as a result of the financial status of the countries and also the political decisions surrounding how the Public Pension system is structured. Public Pensions are by definition managed by the State inside a global social security system and generally funded by the contributions of the active working population on a PAYG basis (Pay as you go). Hence the number of active contributors plays a crucial role on the financial sustainability of the Pension system, as well as the ratio between the retired people and the active people called dependency ratio. The ageing process affecting many countries induces an important increase of this dependency ratio. This is as a result of the existing global conditions of low fertility, low mortality and baby boom effect from post-world war II. In a PAYG system, these important changes in the dependency ratio lead to significant challenges on the level of contributions and /or benefits of the scheme. Therefore, some forms of risk sharing between generations seem desirable. In this paper, we consider three ways to adapt the sustainability of the pension system based on automatic balance mechanisms; these three variables are each associated to a particular group of people:

- The contribution rate, affecting the contributors;
- The level of the first pension obtained at retirement, affecting the new retirees;
- The revaluation of old pensions, affecting the people already retired in the past.

A central issue is therefore to model the demography of these three sub groups of the population under study. In this paper we use the approach of a simple stochastic Two Period Overlapping Generation Model to study the Pay as you go (PAYG) Pension design. In the design of the two period Overlapping Generation (OLG) model, we need to understand how the demography behaves by first modelling the evolution of the active generation. Later, we model how an individual retires upon reaching the assigned retirement age. This is the transition from being an active individual who contributes to the Pensions Scheme to eventually being retired. Later, we show how an individual retires and stays retired in the model and receives Pension benefits. Finally, we draw the distinction between the two groups of retirees, the just retired: this is the individual who has just reached the stipulated retirement age whilst the old existing retirees are people retired the previous years. This allows us to model the dependency ratio (ratio of the number retirees (just retired and old existing retirees) and the number of active generation (the contributors)) and to analyse the structure inside the retired population (between new and old retirees).

In the literature, there has been a lot of studies in the area of OLG models on Public Pension design. Mostly we have the works of Rosado Cebrián et al. (2020), Fanti and Gori (2013), Yang (2015) about PAYG using the two period overlapping generation models. In the following papers by Ohtaki (2011), Mauÿner (2009), David (2002) and Meijdam and Verbon (1997) there is the introduction of stochastic overlapping generation models. In Cipriani (2014) we see the impact longevity has on the PAYG system using an OLG approach. In this paper by Buyse et al. (2017), there is the introduction of four period overlapping generation model as well as the addition of some heterogeneous properties. Also in d'Autume (2003), we see a classic introduction of two period OLG models to study the Public Pensions. The paper by Çagaçan Deger (2008), adds some multiple social security systems to the OLG model to make it more realistic with social settings of the demography.

In this paper, we have introduced a simple OLG model with an active population and two groups of retirees: the just retired, and old existing retirees. This gives us some flexibility to calculate the replacement rates of the different generations of retirees separately. Generally, the two period OLG model seems simple but it is the building block of many complicated models. It allows us to propose and compute different mechanisms of automatic adaptation of the pensions to pay to the new retirees as well as the old pensioners.

The main contribution of the paper is to present and illustrate in this OLG structure, different Pension philosophies aiming at answering two basic questions, naturally induced by the ageing effect:

- step 1: how to share the cost of ageing between contributors and retirees by automatic adjustment of the contribution rate and the benefit ratio;

-step 2: once the global benefit level has been designed (step 1), how to share the benefit ratio between the retirees by automatic adjustment of the first pension at retirement age and the revaluation factor on existing pensions.

The rest of this paper is structured as follows: In section 2, we give some basic definitions and discuss the public pension scheme (PAYG). We further derive the equilibrium equation of the PAYG. In particular, we draw the distinction between benefit ratio and replacement rate. In section 3, we discuss the simple two period OLG model as well as the general dynamics of the OLG model. We introduce the risk sharing techniques in the OLG setting as well as the equations for the benefit ratio and the replacement rate. In section 3.3, we develop in particular the different strategies to share the benefits between the new retirees and the existing ones. In section 4, we present various numerical illustrations in a deterministic setting as well as in a stochastic demographic environment. Finally, we conclude in section 5.

2. General Pay As You Go (PAYG) Framework

The Public Pension Scheme (PAYG), also known as Social Security is the Pension Scheme used by many states. Simply put, it is the sum of money that an individual is paid upon retirement on condition that they meet the stipulated agreement of contributing to the system during their career life. We note that the contributing age, retirement age, the rate of contribution to the system and replacement rates differ from country to country. These differences and many other factors combined, bring about the different types of Pension plans that exist. The discussion on types of Pension plans is in section (2.1). PAYG runs on the idea of a safety net to individuals in the demography so that they can have a better future after retirement, hence it is non profit. The mechanism is based on the principle of the state collecting contributions at a stipulated age from individuals to the national account and simultaneously paying people who have retired with the money collected. In a pure PAYG system, the income accrued from contributions is transferred to expense directly without any connections to the financial market. The income of the system is simply, the number of people who qualify to contribute to the system multiplied by the contribution rate and the average wage as:

$$\text{Income} = \text{Number of contributors } \times \text{Contribution Rate} \times \text{Mean Wage} \tag{2.0.1}$$

Also, the expenses of the system is simply the number of retires multiplied by the mean pension:

$$\text{Expense} = \text{Number of Retires} \times \text{Mean Pension} \tag{2.0.2}$$

The equilibrium of the system is achieved if income equals the expense: hence we equate equations (2.0.1) and (2.0.2) to get the following:

$$\text{Number of Contributors } \times \text{Contribution Rate} \times \text{Mean Wage} = \text{Number of Retires} \times \text{Mean Pension}$$

$$\text{Contribution Rate} = \frac{\text{No. of Retires}}{\text{No. Contributors}} \times \frac{\text{Mean Pension}}{\text{Mean Wage}}$$

$$\underbrace{\pi(t)}_{\text{Contribution Rate}} = \underbrace{D(t)}_{\text{Dependency Ratio}} \times \underbrace{\delta(t)}_{\text{Benefit Ratio}} \tag{2.0.3}$$

2.1 From DB, DC to Hybrid Pension Plans

Here we consider, the PAYG equilibrium in equation (2.0.3). This is one equation with two unknowns, the contribution rate ($\pi(t)$) and the replacement rate ($\delta(t)$). The number of retires and number of active contributors are known, hence the dependency ratio is known at any time (t). Hence this equation has infinitely many solutions to balance the PAYG equilibrium as shown below. The major questions we ask here are, which solutions are fair? financially sustainable? and socially sound? We explore some of these solutions and discuss some of the consequences to the economy and the different parties involved in the Pension scheme when adopted. Clearly to balance the PAYG equilibrium, we can increase or decrease the contributions and or the replacement rate. Alternatively, we could maintain the contributions and reduce the replacement rate or maintain the replacement rate and increase the contribution rate. More fairly, we could adjust both the contribution rate and the replacement rate to mimic the behavior of the shock as follows, this strategy is what we call hybrid plans.

$$\Uparrow \pi(t) \Downarrow \quad = \underbrace{D(t)}_{\text{Known}} \times \Uparrow \delta(t) \Downarrow \tag{2.1.1}$$

2.1.1 DB. A pure Defined Benefit (DB) is a pension architecture, where the risk sharing is done in such a way that retires are not affected by the demographic shock. Here the replacement rate is kept constant and all the shock of the system is absorbed by the active contributors to the system. Clearly retires are protected by a DB plan whilst workers are treated woefully unfair. Using the PAYG equilibrium the DB system is represented as follows:

$$\underbrace{\pi(t)}_{\text{Adjusted Contributions}} = \underbrace{D(t)}_{\text{Known}} \times \underbrace{\delta(0)}_{\text{Fixed Benefit Ratio}} \tag{2.1.2}$$

2.1.2 DC. In a pure Defined contribution (DC) system, the contributions are fixed and the replacement rates are adjusted to balance the PAYG equilibrium as follows:

$$\underbrace{\pi(0)}_{\text{Fixed Contributions}} = \underbrace{D(t)}_{\text{Known}} \times \underbrace{\delta(t)}_{\text{Adjusted Benefit Ratio}} \tag{2.1.3}$$

Here we see that the risk is borne by retires whilst contributors are not affected by the demographic shock. Clearly, there is no fairness in a DC plan as well.

The imbalance of DB and DC plans has lead to the birth of hybrid plans that seek to achieve some level of fairness and equality in sharing the demographic shock. The two major hybrid plans we discuss in this paper are Musgrave and the convex combination.

2.1.3 Musgrave. Generally for any time t, the Musgrave rule is based on the following invariant :

$$M = \frac{\delta(t)}{(1-\pi(t))} \tag{2.1.4}$$

Here we note that at any given time, the Musgrave constant M will adapt to the contribution rate and replacement rate accordingly. At $t = 0$, the Musgrave constant M, which is the ratio of the initial benefit ratio ($\delta(0)$) and 1 net of the initial contribution rate $\pi(t)$, (1-$\pi(0)$) is given as:

$$M = \frac{\delta(0)}{(1-\pi(0))} \tag{2.1.5}$$

We note that at $t = 0$, $\delta(0)$ the initial benefit ratio is known, and the initial dependency ratio (D(0)) is known. Hence we calculate the initial contribution rate as follows:

$$\pi(0) = D(0)\delta(0) \tag{2.1.6}$$

For example, using the case of Belgium, $\delta(0) = 65\%$, $D(0) = 36\%$ and $\pi(0) = 0.65 * 0.36 = 0.234 = 23.4\%$. Hence at t=0, the Musgrave constant M is given by :

$$M = \frac{0.65}{1 - 0.234} = 0.849$$

The Musgrave framework is an example of what we call **hybrid schemes**.

2.1.4 Remark. Benefit Ratio$(\delta(t))$ **Vrs Replacement Rate**$(\gamma_X(t))$ Benefit Ratio is a macroeconomic indicator given by the ratio of the mean pension and the mean wage.

$$\delta(t) = \frac{\text{Mean Pension}}{\text{Mean Wage}}$$

Whereas, replacement rate is a micro-economic indicator given by the ratio of the first pension of an individual say X and the last wage of the individual X.

$$\gamma_X(t) = \frac{\text{First Pension of an individual X}}{\text{Last Wage of an Individual X}}$$

2.1.5 Remark. Alpha $\alpha(t)$, will denote the proportion of new pensioners among the total retires is given as:

$$\alpha(t) = \frac{N_{r1}(t)}{N_r(t)} \tag{2.1.7}$$

2.2 Old and New pensioners

In the last section, we have suggested a mechanism based on an intergenerational fairness between active workers and retirees. Indeed, the Musgrave rule permits us to fix each year on the one hand the contributions rate to be paid by workers and on the other hand the mean benefit ratio. This last coefficient generates a mean pension level for all the retirees a given year. However, the population of retirees is not homogeneous; in particular, each year we can decompose this population into two natural sub groups:

• The "old pensioners" already retired the year before; they had a previous level of pension and it is natural to link each year their new pension with the previous one; the decision to take can then be formalized through a revalorisation coefficient to be applied to the previous pension;

• The "new pensioners" of the year from which a new level pension has to be decided; the decision to take for them can be formalized through a replacement rate.

Starting from a given mean benefit ratio defined for instance by the Musgrave rule (2.1.4) (risk sharing between active and retired populations), we have then, in a second step, to define the revalorisation coefficient to apply on existing pensions and the replacement rate for new retirees in a consistent way (risk sharing among the retirees). There are of course many ways to realize this. Let us imagine for instance that a given year the mean benefit ratio decreases with respect to the previous year. The mean level of pension should then decrease. To achieve this, we could already apply two extremes strategies:

• We pass all the effect on existing retirees by decreasing the revalorisation of existing pensions without decreasing the level of new pensions (protection for new retirees)

• We pass all the effect on new pensioners only by decreasing the replacement rate with respect to older generations and maintain a full revalorisation for old pensions (protection for old retirees).

Of course, intermediate strategies are possible.

The aim of the next sections is precisely to detail this second step of yearly adaptation of pension and to present different methods of sharing between the retirees, in a simple two period OLG model, once the mean benefit ratio has been fixed.

3. The Two Period OLG Model

3.1 The Demographic Assumptions

Here we present the simple two period OLG model which is based on the active population $N_a(t)$, the new retired population $N_{r1}(t)$ and the old existing retired population $N_{r2}(t)$. We model the evolution of how active contributors retire, and are being grouped into two major compartments: the just retired population and the existing retired population. The aim is to use the evolution of retired and the active contributors to model the dependency ratio. Having a fair idea about how the dependency ratio evolves in time, we can model the evolution of the benefit ratio and the contribution rate in time.

First, let Wt be the mean wage at time t; we define the following parameters:

P_t = Mean pension

P_{t1}= New Pension at time t, for the new pensioners

P_{t2} = Old Pension at t, for the old existing Pensioners

π_t = contribution rate at time t

N = number of individuals in the population

γ_t = replacement rate

δ_t = benefit ratio

Here $N_a(t)$ is the active people still contributing, $N_{r1}(t)$ the new group of retirees and $N_{r2}(t)$ is the old group of retirees.

Now we can define the total income at time t as:

$$\text{Income}_t = \pi_t W_t N_a(t) \tag{3.1.1}$$

We choose to model the uncertainty in the arrival of new pensioners by taking the following stochastic process to count the number of individuals just retired:

$$N_{r1}(t) = N_{r1}(0)e^{\eta t} \cdot e^{\sigma W_1(t) - \frac{{\sigma_1}^2 t}{2}} \tag{3.1.2}$$

Consequently, the individuals in the second generation of retires is defined as:

$$N_{r2}(t) = N_{r1}(0)e^{\eta(t-1)} \cdot e^{\sigma W_1(t-1) - \frac{{\sigma_1}^2 (t-1)}{2}} \mathbf{P}_{t-1} = N_{r1}(t-1) \times \mathbf{P}_{t-1} \tag{3.1.3}$$

Where P is the survival probability after retirement. The total number of individuals in the two retired generations is given by:

$$\begin{aligned} N_r(t) &= N_{r1}(t) + N_{r2}(t) \\ &= N_{r1}(0)e^{\eta t} \cdot e^{\sigma W_1(t) - \frac{{\sigma_1}^2 t}{2}} + N_{r1}(0)e^{\eta(t-1)} \cdot e^{\sigma W_1(t-1) - \frac{{\sigma_1}^2 (t-1)}{2}} \mathbf{P}_{t-1} \end{aligned} \tag{3.1.4}$$

In a similar way the number of active individuals is given by :

$$N_a(t) = N_a(0)e^{\rho t} \cdot e^{\sigma_2 W_2(t) - \frac{{\sigma_2}^2 t}{2}} \tag{3.1.5}$$

where W_1 and W_2 are two Brownian motions assumed to be independent.

3.2 The General Dynamic of Pension Benefits

The purpose of this section is to compute in the model previously presented the benefit ratio and to link it with the replacement rates of the different generations of retirees. As seen below, the benefit ratio will be seen as a weighted average of the replacement rates of the retirees. The benefit ratio can be computed following for instance the Musgrave rule (see section 2); then, based on this benefit ratio, we will have to determine the replacement rate of the new retirees and of the old retires in such a way to remain consistent with the global benefit ratio.

In our model the total pension expenses are given by:

$$\begin{aligned}
\text{Outcome}_t &= P_{t1}N_{r1}(t) + P_{t2}N_{r2}(t) \\
\text{Outcome}_t &= P_{t1}N_{r1}(0)e^{\eta t} \cdot e^{\sigma W_1(t) - \frac{{\sigma_1}^2 t}{2}} + P_{t2}N_{r1}(0)e^{\eta(t-1)} \cdot e^{\sigma W_1(t-1) - \frac{{\sigma_1}^2 (t-1)}{2}} \mathbf{P}_{t-1}
\end{aligned} \tag{3.2.1}$$

Where $\mathbf{P}$ is the survival probability. Here the benefit ratio δ_t is also defined as:

$$\begin{aligned}
\delta_t &= \frac{\big(P_{t1}N_{r1} + P_{t2}N_{r2}\big)/N_r(t)}{W_t} \\
\delta_t &= \frac{A}{B}
\end{aligned} \tag{3.2.2}$$

where:

$$\begin{aligned}
A &= \frac{P_{t1}N_{r1}(0)e^{\eta t} \cdot e^{\sigma W_1(t) - \frac{{\sigma_1}^2 t}{2}} + P_{t2}N_{r1}(0)e^{\eta(t-1)} \cdot e^{\sigma W_1(t-1) - \frac{{\sigma_1}^2 (t-1)}{2}} \mathbf{P}_{t-1}}{N_{r1}(0)e^{\eta t} \cdot e^{\sigma W_1(t) - \frac{{\sigma_1}^2 t}{2}} + N_{r1}(0)e^{\eta(t-1)} \cdot e^{\sigma W_1(t-1) - \frac{{\sigma_1}^2 (t-1)}{2}} \mathbf{P}_{t-1}} \\
B &= W_t
\end{aligned} \tag{3.2.3}$$

$$\delta_t = \frac{P_{t1}\alpha(t) + P_{t2}(1 - \alpha(t))}{W_t} \tag{3.2.4}$$

where $\alpha(t)$, the proportion of new pensioners among the total retires is:

$$\begin{aligned}
\alpha(t) &= \frac{N_{r1}(t)}{N_r(t)} \\
&= \frac{N_{r1}(0)e^{\eta t} \cdot e^{\sigma W_1(t) - \frac{{\sigma_1}^2 t}{2}}}{N_{r1}(0)e^{\eta t} \cdot e^{\sigma W_1(t) - \frac{{\sigma_1}^2 t}{2}} + N_{r1}(0)e^{\eta(t-1)} \cdot e^{\sigma W_1(t-1) - \frac{{\sigma_1}^2 (t-1)}{2}} \mathbf{P}_{t-1}}
\end{aligned} \tag{3.2.5}$$

We suppose at time $t = 0$, the:

$$P_{01} = P_{02} \qquad \text{and} \qquad N_{r1}(0) = N_{r2}(0) \tag{3.2.6}$$

The two replacement rates are defined as :

$$\gamma_{t1} = \frac{P_{t1}}{W_t}, \qquad \gamma_{t2} = \frac{P_{t2}}{W_t} \tag{3.2.7}$$

Also,

$$\delta_t = \gamma_{t1}\alpha(t) + \gamma_{t2}(1 - \alpha(t)) \tag{3.2.8}$$

Here we try to find a link between the benefit ratio δ_t and the two replacement rates γ_{t1} and γ_{t2}. To do this we consider the following initial conditions :

$\pi(0) = \pi_0, \qquad$ and $\quad \gamma_{01} = \gamma_{02} = \delta_0$

The initial retired population is given by $N_r(0)$ at $t = 0$:

$$N_r(0) = N_{r1}(0) + N_{r2}(0) \tag{3.2.9}$$

Also at $t = 0$ the dependence ratio $D(0)$ is given as:

$$D(0) = \frac{N_r(0)}{N_a(0)} \tag{3.2.10}$$

At any future time , the dependency ratio is random and given by:

$$\begin{aligned} D(t) &= \frac{N_r(t)}{N_a(t)} \\ &= \frac{N_{r1}(0)e^{\eta t} \cdot e^{\sigma W_1(t) - \frac{{\sigma_1}^2 t}{2}} + N_{r1}(0)e^{\eta(t-1)} \cdot e^{\sigma W_1(t-1) - \frac{{\sigma_1}^2(t-1)}{2}} \mathbf{P}_{t-1}}{N_a(0)e^{\rho t} \cdot e^{\sigma_2 W_2(t) - \frac{{\sigma_2}^2 t}{2}}} \end{aligned} \tag{3.2.11}$$

The actuarial equilibrium for the simple two period model is given as:

$$\pi_0 = \delta_0 \cdot D(0) \tag{3.2.12}$$

The contribution rate $\pi(t)$ is given as:

$$\begin{aligned} \pi(t) &= \delta_t \cdot D(t) \\ &= \delta_t \times \frac{N_{r1}(0)e^{\eta t} \cdot e^{\sigma W_1(t) - \frac{{\sigma_1}^2 t}{2}} + N_{r1}(0)e^{\eta(t-1)} \cdot e^{\sigma W_1(t-1) - \frac{{\sigma_1}^2(t-1)}{2}} \mathbf{P}_{t-1}}{N_a(0)e^{\rho t} \cdot e^{\sigma_2 W_2(t) - \frac{{\sigma_2}^2 t}{2}}} \end{aligned} \tag{3.2.13}$$

If we use the Musgrave rule, we get an additional constraint on the two processes δ and π:

$$M = \frac{\delta_t}{1-\pi(t)} = \frac{\delta_0}{1-\pi_0} \tag{3.2.14}$$

Here we introduce β_t the sustainability factor[1] defined by:

$$\gamma_{t2} = \gamma_{t-1 \cdot 1} \cdot \beta_t \tag{3.2.15}$$

Using equation (3.2.8), we have the following :

$$\delta_t = \gamma_{t1} \cdot \alpha(t) + \gamma_{t-1 \cdot 1} \cdot \beta_t \cdot (1-\alpha(t)) \tag{3.2.16}$$

Based on this value for the benefit ratio, we have now to compute the two replacement rates; to do that, we express first the replacement rate of the old pensioners by linking it with their replacement rate the year before, multiplied by a coefficient called sustainability factor β :

Using equation (3.2.14) and (3.2.16), we can get the following :

$$\gamma_{t1} \cdot \alpha(t) + \gamma_{t-1 \cdot 1}\beta_t \cdot (1-\alpha(t)) = M \cdot \left(1-\pi_t\right) \tag{3.2.17}$$

$$\gamma_{t1}\alpha(t) + \gamma_{t-1 \cdot 1}\beta_t \cdot (1-\alpha(t)) = M \cdot \left(1-D(t)\delta_t\right) \tag{3.2.18}$$

We know that

$$\delta_t + MD(t)\delta_t = M \tag{3.2.19}$$

So:

$$\delta_t = \frac{M}{1+MD(t)}, \quad \pi(t) = \frac{MD(t)}{1+MD(t)} \tag{3.2.20}$$

Substituting equation (3.2.20) into equation (3.2.18), results:

$$\gamma_{t1}\alpha(t) + \gamma_{t-1 \cdot 1}\beta_t \cdot (1-\alpha(t)) = \frac{M}{1+MD(t)} \tag{3.2.21}$$

3.3 Particular Risk Sharing Between the Retirees

Starting from relation (3.2.21), we can obtain first a recursive relation between the successive replacement rates for the new Pensioners:

$$\gamma_{t \cdot 1} = \frac{1}{\alpha(t)}\left[\frac{M}{1+M \cdot D(t)} - \beta_t\gamma_{t-1 \cdot 1} \cdot (1-\alpha(t))\right] \tag{3.3.1}$$

In order to generate different strategies of risk sharing between the two generation of Pensioners, let us first develop two extreme solutions:

[1] Sustainability factor shows how the Pension system is sustainable from one generation to the other

- Solution 1: Complete protection of the new Pensions
- Solution 1: Complete protection of the Pension revaluation

3.3.1 Complete protection of the new Pension. In that case, we want to maintain forever the initial replacement rate at retirement age.

Therefore, we have the following constraints:

$$\gamma_{t \cdot 1} = \gamma_{t-1 \cdot 1} = \delta_0 \tag{3.3.2}$$

The first Pension, obtained at retirement age is therefore always computed using the original benefit level δ_0.

Of course, the price to pay is that such a strategy can induce a big risk of bad revaluation for older pensioners.

To analyse this effect, let us look at the value of the sustainability factor in that case. Starting from equation (3.2.16) and substituting the values of the replacement rates given by (3.3.2), we get:

$$\delta_t = \delta_0 \cdot \alpha(t) + \delta_0 \cdot \beta(t)(1 - \alpha(t))$$

Therefore the sustainability factor becomes:

$$\beta_t = \frac{\delta_t - \delta_0 \cdot \alpha(t)}{\delta_0 - \delta_0 \cdot \alpha(t)} \tag{3.3.3}$$

For constant benefit levels ($\delta_t = \delta_0$), the sustainability factor is equal to 1 (full revaluation of pension).

However, in case of decrease of the coefficient δ (for instance due to the application of Musgrave rule in an aging environment)

This sustainability factor will be less than one (partial revaluation), and will decrease progressively:

The explicit levels of pension at time t for the two generations are then given by:

$$P_{t \cdot 1} = \delta_0 . W_t \tag{3.3.4}$$

$$P_{t \cdot 2} = \delta_0 \cdot \beta_t \cdot W_t$$

$$P_{t \cdot 2} = \delta_0 \cdot \frac{\delta_t - \delta_0 \alpha(t)}{\delta_0 - \delta_0 \alpha(t)} \cdot W_t$$

$$P_{t \cdot 2} = \left(\delta_0 - \frac{\delta_0 - \delta_t}{1 - \alpha(t)} \right) W_t \tag{3.3.5}$$

From relation (3.3.5), we can observe that the spread between the initial benefit level δ_0 and the constraint benefit level δ_t is exercebated for the second generation $coefficient = \dfrac{1}{1 - \alpha(t)}$

In Particular the Musgrave rule corresponds to the value:

$$\delta_t = \frac{M}{1 + MD(t)}$$

3.3.2 Complete Protection of the Old Pensioners. In this case, the constraint is to maintain at any time a sustainability factor equal to one:

$$\beta_t = 1 \tag{3.3.6}$$

Pensions after retirement are completely reevaluated. The price to pay in this case is a risk of important decrease of the initial pension.

Indeed relation 3.3.1 then becomes:

$$\gamma_{t \cdot 1} = \frac{1}{\alpha(t)} \Big[\delta_t - \gamma_{t-1 \cdot 1} \cdot (1 - \alpha(t)) \Big] \tag{3.3.7}$$

Which can be written as:

$$\gamma_{t1} = \gamma_{t-1 \cdot 1} - \frac{(\gamma_{t-1 \cdot 1} - \delta_t)}{\alpha(t)} \tag{3.3.8}$$

The explicit levels of Pension then become:

$$P_{t \cdot 1} = \gamma_{t \cdot 1} W_t$$

$$P_{t \cdot 2} = \gamma_{t-1 \cdot 1} W_t$$

3.3.3 Same Pension for the Two generation of Retirees. The two previous strategies can be seen as polar situation in the sense that all the effect of a variation of the benefit level δ is affected only to one generation.

Another solution is to affect simultaneously both generations by imposing the following constraints:

$$P_{t1} = P_{t2} = \delta_t \cdot W_t \tag{3.3.9}$$

Then we have :

$$\begin{aligned} \gamma_{t1} = \delta_t &= \frac{M}{1 + MD(t)} \\ \gamma_{t-1 \cdot 1} = \delta_{t-1} &= \frac{M}{1 + MD(t-1)} \end{aligned} \tag{3.3.10}$$

and:

$$\beta_t = \frac{\delta_t}{\delta_{t-1}} = \frac{\gamma_{t \cdot 1}}{\gamma_{t-1 \cdot 1}} \tag{3.3.11}$$

3.3.4 Intermediate Case. We could also consider intermediate cases between the last two choices (Complete protection of the old pensioners versus same pension for all the retirees in (3.3.2) and (3.3.3).

In this case, we could select an intermediate value for the sustainability factor β given by:

$$\beta_t = k\frac{\gamma_{t\cdot 1}}{\gamma_{t-1\cdot 1}} + (1-k), \quad \text{with} \quad 0 \leq k \leq 1 \tag{3.3.12}$$

Introducing this value of the sustainability factor into the general relation (3.3.1), we get the following recursive relation for the replacement rate of the new retirees:

$$\gamma_{t\cdot 1} = \frac{1}{\alpha(t) + k(1-\alpha(t))}\Big[\delta_t - \gamma_{t-1\cdot 1}\cdot(1-\alpha(t))(1-k)\Big] \tag{3.3.13}$$

Subsequently the formula for the second generation using equation (3.2.15) and (3.3.12) results:

$$\gamma_{t\cdot 2} = \gamma_{t-1\cdot 1}\Big[(k\times(\gamma_{t\cdot 1}/\gamma_{t-1\cdot 1})) + (1-k)\Big] \tag{3.3.14}$$

Remark 1

- For $k = 0$, (Complete protection of the old retirees), we obtain the special case (3.3.7).
- For $k = 1$, (Same pension for all retirees), we obtain the special case (3.3.10).

Remark 2

In the special situation of a constant dependency rate, ($D(t) = D, \quad \forall\ t$), all the four cases presented before lead to the same solution:

$$\gamma_{t\cdot 1} = \delta_0$$

This is the same case as a classical DB system, with a constant replacement rate.

4. Two Period OLG Pension Simulations For PAYG

4.1 Assumptions

Here we use the following initial parameters to simulate the model:

$T = 10$ years

Initial active people. $= N_a(0) = 150$

Initial retires $= N_{r1}(0) = 50$

Initial retires $= N_{r2}(0) = 50$

Initial replacement rate for first generation $= \gamma_{01}(0) = 50\%$

Initial replacement rate for second generation $= \gamma_{02}(0) = 50\%$

Initial benefit ratio $= \delta_0 = 50\%$

Initial dependence ratio $= D(0) = \frac{Nr(0)}{Na(0)} = \frac{100}{150} = 66.67\%$

Initial contribution rate at time $= \pi_0 = D(0) \times \delta(0) = 66\% * 50\% = 33\%$

Survival Probability($\mathbf{P}$) $= 0.99$

$\eta =$ Increase of young retires $= 5\%$

$\rho =$ Increase of active people $= 2\%$

$\sigma_1 = 0.05$

$\sigma_2 = 0.05$

Musgrave constant $=$ M $= (D[0])/(1 - \pi[0]) = 0.746$

4.2 Deterministic Evolution

In table 4.1 we show we the mean evolution of the number of new retirees, old retires, number of active contributor and the ratio of new retires to the total retires. Based on these deterministic figures, we show in table 4.2 the dependency, benefit ratio and the replacement rate for all the two generations considered.

Table 4.1: Deterministic Calculations

Time	Na	Nr1	Nr2	Alpha
0	150.00	50.00	50.00	50.00 %
1	153.03	52.56	49.50	51.50 %
2	156.12	55.25	52.04	51.50 %
3	159.28	58.09	54.71	51.50 %
4	162.50	61.07	57.61	51.50 %
5	165.78	64.20	60.50	51.00 %
6	169.12	60.49	63.56	51.50 %
7	172.54	70.95	66.82	51.50 %
8	176.03	74.59	70.24	51.50 %
9	179.58	78.42	73.84	51.50 %
10	183.21	82.43	77.63	51.50 %

Table 4.2: Deterministic Calculations

Time	D(t)	δ	π	γ_{t1}: k = 0	γ_{t1}: k = 0.5	γ_{t1}: k=1
0	66.67 %	50.00 %	33.00 %	50.00 %	50.00 %	50.00 %
1	66.69 %	49.83 %	33.23 %	49.66 %	49.77 %	49.83 %
2	68.73 %	49.32 %	33.90 %	49.01 %	49.19 %	49.32 %
3	70.82 %	48.82 %	34.58 %	48.64 %	48.71 %	48.82 %
4	72.98 %	48.31 %	35.25 %	48.00 %	48.19 %	48.31 %
5	75.20 %	47.80 %	35.95 %	47.61 %	47.68 %	47.80 %
6	77.49 %	47.28 %	36.64 %	46.97 %	47.16 %	47.28 %
7	79.85 %	46.76 %	37.33 %	46.56 %	46.63 %	46.76 %
8	82.28 %	46.23 %	38.04 %	45.92 %	46.11 %	46.23 %
9	84.79 %	45.71 %	38.75 %	45.50 %	45.58 %	45.71 %
10	87.37 %	45.17 %	39.47 %	44.87 %	45.04 %	45.17 %

Here, with the active population increasing by 2% and the first retired population increasing by 5%, the increment in the dependency ratio is expected. The decrement in the benefit ratio is also expected in this regard. With an increase in the dependency ratio, definitely the benefits will be reduced for Pensioners. Also, the increments in the contribution rate seen here is expected to create a balance for the PAYG equilibrium. Consequently, all the replacement rates will be reduced as seen here. However, with $k = 1$ the replacement rates are considereably much better compared to the $k = 0, 0.5$.

Table 4.3 gives the mean evolution of the replacement rate for the second generation given values of k (k=0,0.5,1)

Table 4.3: Deterministic Calculations

Time	γ_2 : k = 0	γ_2 : k = 0.5	γ_2 : k = 1
0	50.00 %	50.00 %	50.00 %
1	50.00 %	49.89 %	49.83 %
2	49.66 %	49.48 %	49.32 %
3	48.01 %	48.95 %	48.82 %
4	48.64 %	48.45 %	48.31 %
5	48.00 %	47.93 %	47.80 %
6	47.61 %	47.42 %	47.28 %
7	46.97 %	46.89 %	46.76 %
8	46.56 %	46.37 %	46.23 %
9	45.93%	45.84 %	45.71 %
10	45.50 %	45.31 %	45.17 %

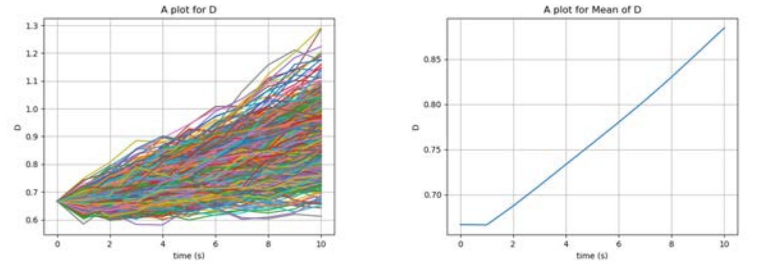
Here, it is observed the replacement rates for γ_2 at $k = 0$, is considerably much better compared to all the different replacement rates. This is consistent with remark (3.3.4), about the complete protection of the older retires. Also the results for γ_1 and γ_2 at $k = 1$ are the same.

These examples represent the evolution of mean values with a deterministic simulation. In the next section we simulate stochastic sample paths of all the processes discussed.

4.3 Stochastic Evolution

Note we consider $N = 1000$, $T = 10$ and show the path simulations for Dependency ratio, contribution rate and benefit ratio.

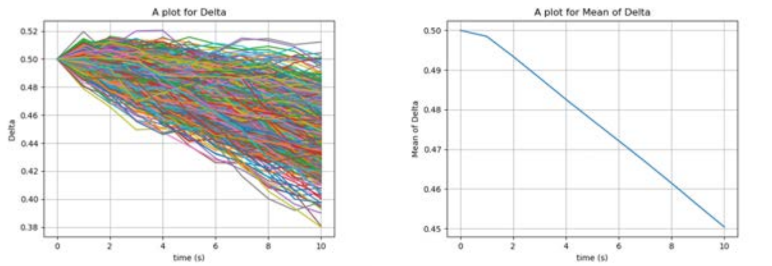
4.3.1 The Stochastic Dependency Ratio. Here, we show the mean dependency ratio in the stochastic environment as follows: Here it is observed, the mean dependency ratio D, increases from 66.67% to 88.60%.



(a) Dependence Ratio (D) (b) Mean of Dependency Ratio

Figure 4.1: Dependency Ratio (D)

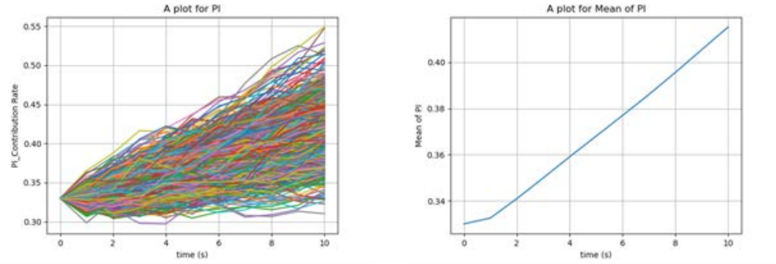
Now we model the benefit ratio using the Musgrave framework. From figure (4.2) we see that the benefit ratio will be adjusted per the increment in the dependence ratio.The mean benefit ratio decreases gradually from the initial 50% to 45% over the course of the period considered.



(a) δ (b) Mean of δ

Figure 4.2: Benefit Ratio (δ) in Musgrave Framework

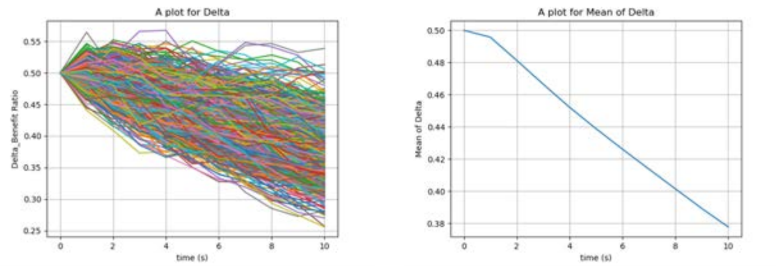
Also, in the Musgrave framework the mean contribution rate increases from the initial 33% to 41.8% with the 10 year period projected.



(a) π (b) Mean of π

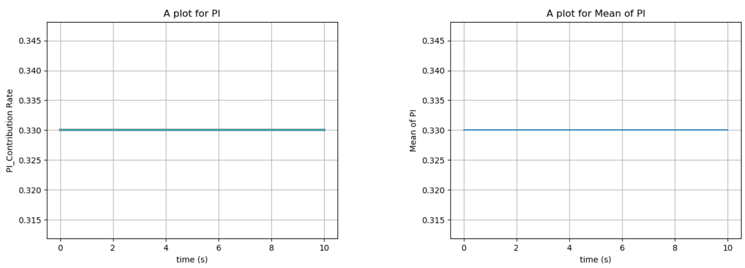
Figure 4.3: Contribution Rate (π) in Musgrave Framework

In the pure DC case, the contribution rate remains a constant through out the period. However, the benefit ratio bears all the risk. Here the mean benefit ratio will decrease monotonically to 38% for the period projected.



(a) δ (b) Mean of δ

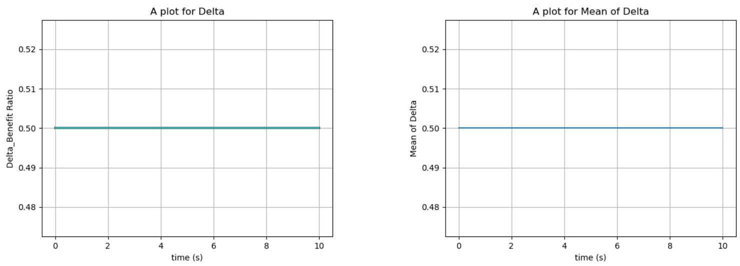
Figure 4.4: Benefit Ratio (δ) in DC



(a) π (b) Mean of π

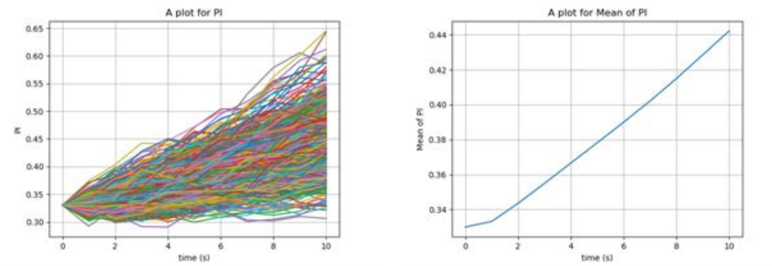
Figure 4.5: Contribution Rate (π) in DC

In the pure DB case, the benefit ratio remains a constant. Here the contribution rate increases to adjust to all the risks of the increase in the dependence ratio. Therefor for the period considered the mean contribution rate will increase to 44%.



(a) δ (b) Mean of δ

Figure 4.6: Benefit Ratio (δ) in DB



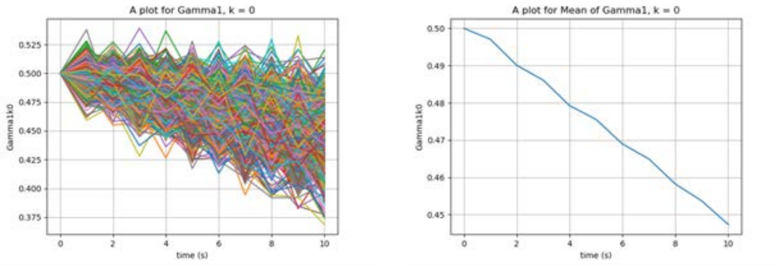
(a) π (b) Mean of π

Figure 4.7: Contribution Rate ($\pi(t)$) in DB

We conclude here that with any increments in the dependency ratio which is expected, the corresponding parameters will increase or decrease depending on the Pension model used, to adjust the pay as you equilibrium. In the case of Musgrave it is a shared risk among benefit ratio and contribution rate.

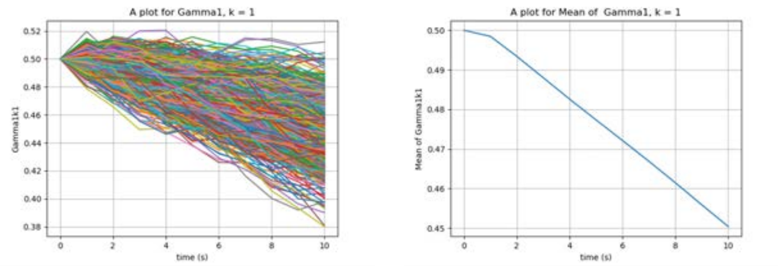
4.4 Simulation Results for Replacement Rates

4.4.1 First Replacement Rate. Here we show the results for the first replacement rate at $k = 0$: Here we show the results for the first replacement rate at $k = 1$:



(a) γ_1 (b) Mean of γ_1

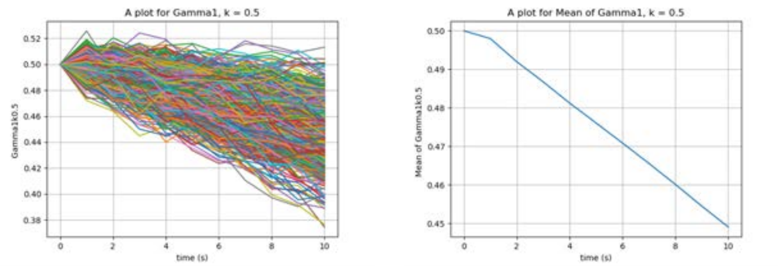
Figure 4.8: Replacement Rate (γ_1) in Musgrave Framework



(a) γ_1 (b) Mean of γ_1

Figure 4.9: Replacement Rate (γ_1) in Musgrave Framework

Here we show the results for the first replacement rate at $k = 0.5$:

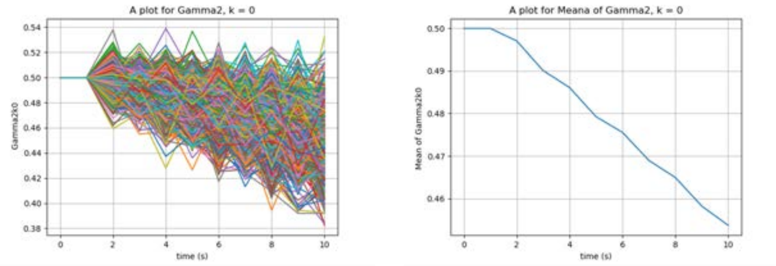


(a) γ_1 (b) Mean of γ_1

Figure 4.10: Replacement Rate (γ_1) in Musgrave Framework

4.4.2 The second replacement Rate. Here we show the results for the second replacement rate at $k = 0.5, 0, 1$:

Here the mean of the second replacement rate for $k = 0$ decreases from the initial 50% to 45.37% for the period projected.

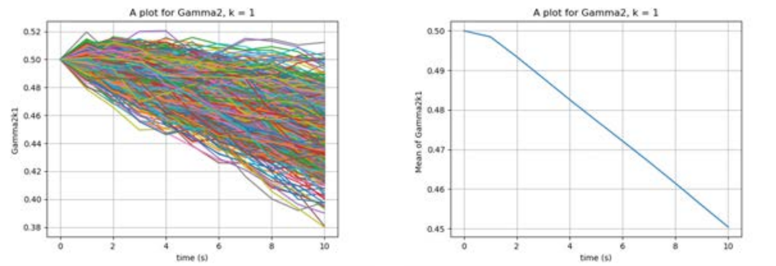


(a) γ_2 (b) Mean of γ_2

Figure 4.11: Replacement Rate (γ_2) in Musgrave Framework

Here we show the results for the second replacement rate at $k = 1$:

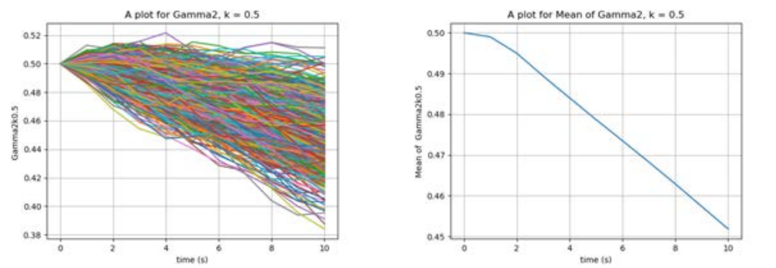
Here the mean of the second replacement rate for $k = 1$ decreases from the initial 50% to 45.4% for the period projected.



(a) γ_2 (b) Mean of γ_2

Figure 4.12: Replacement Rate (γ_2) in Musgrave Framework

The second replacement rate at $k = 0.5$:



(a) γ_2 (b) Mean of γ_2

Figure 4.13: Replacement Rate (γ_2) in Musgrave Framework

Here the mean of the second replacement rate for $k = 0.5$ decreases from the initial 50% to 45% for the period projected.

5. Conclusions

In this paper we introduced two successive levels of adaptations of a public PAYG pension scheme in order to face the effects of ageing: at first we control the sharing of cost between contributors and retirees and a second one leading to various sharing of the global pension budget between the new retirees and the people already retired in the past. The formulas and illustrations have been implemented in a simple stochastic OLG demographic model. In the first level, classical DB and DC architectures appear as extremes while intermediate solutions can be easily developed such as the Musgrave rule. In the second level (between retirees), some extreme strategies can be also used such as a complete revaluation of existing pensions on wages and intermediate solutions are also possible. The simple structure of the demographic model enables us to simulate easily all these mechanisms. For further research, the demographic model could be generalized, for instance by introducing more age classes.

References

Buyse, T., Heylen, F., and Van De Kerckhove, R. (2017). Pension reform in an olg model with heterogeneous abilities. *Journal of Pension Economics and Finance*, 16(2):144–172.

Cipriani, G. P. (2014). Population aging and payg pensions in the olg model. *Journal of Population Economics*, 27(1):251–256.

d'Autume, A. (2003). Ageing and retirement age. what can we learn from the overlapping generations model ? Université paris1 panthéon-sorbonne (post-print and working papers), HAL.

David, D. L. C. (2002). *A theory of economic growth : dynamics and policy in overlapping generations*. Cambridge University Press, Cambridge.

Fanti, L. and Gori, L. (2013). Complex dynamics in an olg model of neoclassical growth with endogenous retirement age and public pensions. *Nonlinear Analysis Real World Applications*, 14.

Ihori, T. (2017). *The Public Pension*, pages 169–201.

Maußner, B. H. (2009). *Stochastic Overlapping Generations Models*, pages 507–552. Springer Berlin Heidelberg, Berlin, Heidelberg.

Meijdam, L. and Verbon, H. A. A. (1997). Aging and public pensions in an overlapping-generations model. *Oxford Economic Papers*, 49(1):29–42.

Ohtaki, E. (2011). A note on the existence of monetary equilibrium in a stochastic olg model with a finite state space. *Economics Bulletin*, 31:485–492.

Rosado Cebrián, B., Peris-Ortiz, M., and Rueda-Armengot, C. (2020). *Adequacy of Public Pension Systems*, pages 173–191.

Yang, Z. (2015). *Chinese public pensions analyzed by OLG models*, pages 1–213.

Çagaçan Deger (2008). Pension Reform in an OLG Model with Multiple Social Security Systems. ERC Working Papers 0805, ERC - Economic Research Center, Middle East Technical University.