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LIQUIDITY CONSTRAINTS AND CYCLES

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Abstract

In this paper, we modify the model of Benhabib and Laroque in which the behaviour of non steady-state equilibria in the neighborhood of the golden rule steady state are studied, to include liquidity constraints. We observe, in an example, the appearance of new cycles.[1]

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Introduction

In this paper, we build a model which has as foundation the paper of Benhabib and Laroque [**1**].

In [**1**], the dynamics of the model of Diamond [**3**] is developed. A two-period overlapping generations economy with production, endogenous labor supply and a constant quantity of outside money is considered. The objective of the paper is to study and characterize the behaviour of non-steady state equilibria near the golden rule equilibrium.

Conclusions to similar questions are presented in Reichlin [**7**], where a non-monetary OLG model with production and endogenous labor supply is studied.

We here introduce a government, which manages the money supply through open market operations and money, endowed with a liquidity constraint. The individuals are required to keep a fraction μ of the value of their purchases until the end of the period in money. This "Clower type" constraint is a particular case of the one imposed in [**5**]. Note that this constraint is also introduced in [**6; Ch 2**]. Hence, money is both a store of value and a medium of exchange.

In the same way as in [**1**], we define what it means for an equilibrium to be close to the golden rule steady state, which is, as we shall see, unrelated to the money constraint parameter μ. We then present the dynamical system that must be satisfied by a sequence of deviations from the golden rule path.

To simplify, we study this dynamical system for an economy in which the government is absent. Our primary concern is to examine the dependence of the economy on the money constraint parameter μ. We shall see in this particular example that cycles, namely flip bifurcations, may appear precisely because of the existence of the money constraint. We do not however examine the behaviour of an economy parametrized both by a given path of fiscal policies and a money constraint on a particular example.

As in most studies on this subject, the elasticity of savings with respect to the interest rate, denoted $\frac{1}{\alpha}$, plays an important role. Here, this parameter depends on the money constraint parameter μ. Cycles appear both for negative and positive values of α.

The case of full depreciation of capital brings about unexpected results. Cycles do appear, but are unrelated to the value of α.

Apart for the case of full depreciation, we observe that for any value of μ smaller than the value for which a cycle appears, the golden rule is indeterminate. For larger values of μ, the golden rule is determinate. In the case of full depreciation, we observe determinacy outside an interval of values of μ containing the precise value for which a cycle appears.

In Section 1, we present the model. We then recall the definition of a critical economy and present the dynamical system on which the analysis relies. In Section 2, we consider a particular example of an economy where the government is absent. We observe the emergence of flip cycles for particular values of the money constraint parameter μ. We make comments on the determinacy of the golden rule.

1 A dynamic economy with a liquidity constraint

1.1 The model

We consider an overlapping generations economy as in [**1**] with money, production, labor and one good. A government manages the money supply through open market operations and a constraint on money is imposed on the individuals. The representative consumer, who lives two periods, must hold a minimal amount of money at the end of the period. A similar transactions constraint is imposed on the government.

The good is the numeraire. The price of money in terms of the good at date t is Q_t and the real wage is W_t. The gross return on capital K_t and the price of a bond B_t are denoted R_t and S_t respectively. In addition, capital and bonds are linked through the arbitrage condition

$$R_{t+1} = \frac{S_{t+1}}{S_t}.$$

Individuals work during the first period of their lives, in which they supply a quantity L_t of labor, $L_t \leq \bar{L}$, for some $\bar{L} > 0$ which represents the individual's labor endowment. In the second period of their lives, the individuals consume C_{t+1}. In a two-period OLG model, to work and consume in each period is equivalent to working only in the first period and consuming only in the second.

Assumption 1.A

> The preferences of the representative individual are described by a utility function separable in labor and consumption,
>
> $$U(C_{t+1}) - V(L_t).$$
>
> The function $U : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, concave, smooth on the interior of $\mathbb{R}_+$ and $\lim_{C\to 0} U'(C) = +\infty$.
>
> The function $V : [0, \bar{L}] \to \mathbb{R}$ is strictly increasing, convex and smooth on $[0, \bar{L})$. In addition, $V'(0) = 0$ and $\lim_{L\to\bar{L}} V'(L) = +\infty$.

The quantity $W_t L_t$ is the individual's wealth. If C_{t+1} is the value of the individual's purchases in period $t + 1$, then $Q_{t+1}M_{t+1} = \mu C_{t+1}$, for some parameter $\mu \in (0, 1)$. In this model, we force the individuals to hold a certain amount of money until the end of the period.

Under perfect foresight, the individual at t chooses a labor supply, L_t, a consumption level, C_{t+1}, savings in the form of capital, K_{t+1}, money, M_{t+1} and bonds, B_{t+1}, to maximize his utility $U(C_{t+1}) - V(L_t)$ subject to the following constraints

$$W_t L_t = K_{t+1} + Q_t M_{t+1} + S_t B_{t+1},$$

$C_{t+1} = R_{t+1}K_{t+1} + Q_{t+1}M_{t+1} + S_{t+1}B_{t+1},$

$Q_{t+1}M_{t+1} = \mu C_{t+1}$, for some $\mu \in (0, 1)$.

Firms act competitively. The *production technology* is described by a production function $F(K_t, L_t)$.

Assumption 1.B

The function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is strictly increasing, strictly concave, homogeneous of degree one and smooth on the interior of $\mathbb{R}_+^2$;

$\lim_{K\to\infty} F(K, L)/K \leq 1 - \delta < 1$, where δ is the depreciation rate;

$\lim_{K\to 0} F_K(K, L) = +\infty, \quad$ for all $L > 0$;

$\lim_{L\to 0} F_L(K, L) = +\infty, \quad$ for all $K > 0$.

The *firm's objective* is to maximize its profit,

$$F(K_t, L_t) - R_tK_t - W_tL_t.$$

At time t, the *government's action* consists in choosing a triple

$$(C_t^g, M_{t+1}^g, B_{t+1}^g),$$

where C_t^g represents its purchase of private goods and M_{t+1}^g and B_{t+1}^g represent its savings in the form of money and bonds respectively, to satisfy, given its present holdings in money and bonds (M_t^g, B_t^g), the following budget and money constraints:

$C_t^g + Q_tM_{t+1}^g + S_tB_{t+1}^g = Q_tM_t^g + S_tB_t^g,$

$Q_tM_t^g = \mu^g C_t^g$, for some $\mu^g \in (0, 1)$.

The term $Q_tM_t^g + S_tB_t^g$ represents the stock of assets that the government holds at t.

Definition An *intertemporal competitive equilibrium with perfect foresight* for fixed quantities of money and bonds $(\bar{M}, \bar{B})$ is a sequence

$$\{X_t = (K_t, L_t, C_t, M_t, B_t, C_t^g, M_t^g, B_t^g, R_t, W_t, Q_t, S_t) \in \mathbb{R}^{12} : t \in \mathbb{N}\},$$

satisfying

i. $(L_t, C_{t+1}, K_{t+1}, M_{t+1}, B_{t+1})$ is an optimal action of the consumer born at t, given the price system $(W_t, R_{t+1}, Q_t, Q_{t+1}, S_t, S_{t+1})$ and the money constraint $Q_{t+1}M_{t+1} = \mu C_{t+1}$, for some $\mu \in (0, 1)$;

ii. $(C_t^g, M_{t+1}^g, B_{t+1}^g)$ is a feasible action of the government, given the vector (Q_t, S_t, M_t^g, B_t^g) and the money constraint $Q_tM_t^g = \mu^g C_t^g$, for some parameter $\mu^g \in (0, 1)$;

iii. (K_t, L_t) maximizes the profit of the firm under the technological constraint, given (R_t, W_t);

iv. (good market equilibrium) $F(K_t, L_t) = C_t + C_t^g + K_{t+1}$, for all t;

v. (bond market equilibrium) $\bar{B} = B_t + B_t^g$, for all t;

vi. (money market equilibrium) $\bar{M} = M_t + M_t^g$, for all t.

A path of consumption $\{C_t^g : t \in \mathbb{N}\}$ of the government (a path of fiscal policies) is given.[2] Moreover, the government is endowed with fixed stock of bonds and money holdings B_1^g and M_1^g in period $t = 1$ such that $Q_1 M_1^g = \mu^g C_1^g$.

Proposition 1.1 *A sequence* $\{Y_t = (K_t, L_t, Q_t, S_t) \in \mathbb{R}^4 : t \in \mathbb{N}\}$ *describes an equilibrium of an economy e characterized by* $(U, V, F, \bar{M}, \bar{B}, \mu, \mu^g)$*, given* $\{C_t^g : t \in \mathbb{N}\}$*,* B_1^g *and* M_1^g *such that* $Q_1 M_1^g = \mu^g C_1^g$*, if and only if the following conditions are satisfied:*

$$
(1) \qquad \begin{aligned}
V'(L_t)\left\{1 - \mu + \frac{\mu Q_t}{Q_{t+1}} F_K(K_{t+1}, L_{t+1})\right\} \qquad & \\
-U'(C_{t+1}) F_K(K_{t+1}, L_{t+1}) F_L(K_t, L_t) &= 0, \\
F_L(K_t, L_t) L_t - K_{t+1} - Q_t\left(\bar{M} - \frac{\mu^g C_{t+1}^g}{Q_{t+1}}\right) - S_t(\bar{B} - B_{t+1}^g) &= 0, \\
Q_{t+1}\left(\bar{M} - \frac{\mu^g C_{t+1}^g}{Q_{t+1}}\right)\left(\frac{1-\mu}{\mu}\right) - F_K(K_{t+1}, L_{t+1}) K_{t+1} & \\
-S_{t+1}(\bar{B} - B_{t+1}^g) &= 0, \\
F_K(K_{t+1}, L_{t+1}) - \frac{S_{t+1}}{S_t} &= 0,
\end{aligned}
$$

where

$$
C_{t+1} = \frac{F_K(K_{t+1}, L_{t+1}) F_L(K_t, L_t) L_t - (\bar{B} - B_{t+1}^g)(F_K(K_{t+1}, L_{t+1}) S_t - S_{t+1})}{1 - \mu + \mu F_K(K_{t+1}, L_{t+1}) Q_t / Q_{t+1}}
$$

$$
\text{and} \qquad B_{t+1}^g = B_1^g - \sum_{j=1}^{t} \frac{\left\{C_j^g(1 - \mu^g) + \frac{Q_j}{Q_{j+1}} \mu^g C_{j+1}^g\right\}}{S_j}.
$$

Proof: The Lagrangian associated to the maximization problem of the representative individual is given by:

$$
l = U(C_{t+1}) - V(L_t) - \lambda\left\{C_{t+1} - \frac{R_{t+1} W_t L_t - (\bar{B} - B_{t+1}^g)(R_{t+1} S_t - S_{t+1})}{1 - \mu + \mu R_{t+1} Q_t / Q_{t+1}}\right\}.
$$

[2] C_t^g represents spendings minus taxes in period t.

The first order condition therefore reads:

$$U'(C_{t+1})R_{t+1}W_t = V'(L_t)\left\{1-\mu+\mu R_{t+1}Q_t/Q_{t+1}\right\}.$$

Using the first order conditions $F_K(K_t, L_t) = R_t$ and $F_L(K_t, L_t) = W_t$ obtained from the maximization problem of the firm, one gets, by substitution, the required equations.

Now, let $\{Y_t : t \in \mathbb{N}\}$ be a sequence satisfying conditions (1). Parts i., ii. and iii. of the definition of an equilibrium are met by sufficiency of the first order conditions. Note that from Euler's equation,

$$\begin{aligned}
F(K_t, L_t) &= F_K(K_t, L_t)K_t + F_L(K_t, L_t)L_t \\
&= R_tK_t + W_tL_t \\
&= C_t - Q_tM_t - S_tB_t + K_{t+1} + Q_tM_{t+1} + S_tB_{t+1} \\
&= C_t - Q_t(\bar{M} - M_t^g) - S_t(\bar{B} - B_t^g) + Q_t(\bar{M} - M_{t+1}^g) \\
&\quad + S_t(\bar{B} - B_{t+1}^g) + K_{t+1} \\
&= C_t + Q_tM_t^g + S_tB_t^g - Q_tM_{t+1}^g - S_tB_{t+1}^g + K_{t+1} \\
&= C_t + C_t^g + K_{t+1}.
\end{aligned}$$

Q.E.D.

Let $\tilde{Y}$ denote the sequence $\{Y_t \in \mathbb{R}^4 : t \in \mathbb{N}\}$, and Y be the space of such sequences equipped with the uniform topology[3].

Let e be an economy characterized by $(U, V, F, \bar{M}, \bar{B}, \mu, \mu^g)$ and let us define a mapping $Z(Y_t, Y_{t+1}; e)$ as follows:

$$(Y_t, Y_{t+1}) \longrightarrow (Z_1(Y_t, Y_{t+1}; e), Z_2(Y_t, Y_{t+1}; e), Z_3(Y_t, Y_{t+1}; e), Z_4(Y_t, Y_{t+1}; e))$$

where

$$\begin{aligned}
Z_1(Y_t, Y_{t+1}; e) &\equiv V'(L_t)\left\{1-\mu+\frac{\mu Q_t}{Q_{t+1}}F_K(K_{t+1}, L_{t+1})\right\} \\
&\quad -U'(C_{t+1})F_K(K_{t+1}, L_{t+1})F_L(K_t, L_t); \\
Z_2(Y_t, Y_{t+1}; e) &\equiv F_L(K_t, L_t)L_t - K_{t+1} - Q_t\left(\bar{M} - \frac{\mu^g C_{t+1}^g}{Q_{t+1}}\right) \\
&\quad -S_t(\bar{B} - B_{t+1}^g); \\
Z_3(Y_t, Y_{t+1}; e) &\equiv Q_{t+1}\left(\bar{M} - \frac{\mu^g C_{t+1}^g}{Q_{t+1}}\right)\left(\frac{1-\mu}{\mu}\right) - F_K(K_{t+1}, L_{t+1})K_{t+1} \\
&\quad -S_{t+1}(\bar{B} - B_{t+1}^g); \\
Z_4(Y_t, Y_{t+1}; e) &\equiv F_K(K_{t+1}, L_{t+1}) - \frac{S_{t+1}}{S_t};
\end{aligned}$$

[3] Given an index set J and points $x = (x_\alpha)_{\alpha \in J}$, $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric ρ on $\mathbb{R}^J$ by $\rho(x, y) = \text{lub}\{\min(|x_\alpha - y_\alpha|, 1) : \alpha \in J\}$. The topology induced by ρ is the uniform topology.

where C_{t+1} and B^g_{t+1} are defined as in Proposition 1.1. Note that Z is simply defined by the left-hand side of (1) in Proposition 1.1.

Let $\tilde{Z}(\tilde{Y}; e)$ be the sequence of functions $\{Z(Y_t, Y_{t+1}; e) \in \mathbb{R}^4 : t \in \mathbb{N}\}$. The interesting feature of the continuous map $\tilde{Z}(\cdot; e) : Y \to Y$ is that the solutions of the equation

$$\tilde{Z}(\tilde{Y}; e) = 0$$

represent the equilibria of the economy e.

A first approach to the problem of determining the equilibria of the economy e is to focus on stationary equilibria, that is to look at the equilibrium state of the economy where all quantities remain constant.

Let $\{Y_t = Y = (K, L, Q, S) \in \mathbb{R}^4 : t \in \mathbb{N}\}$ be a stationary equilibrium. By Proposition 1.1, it satisfies:

$$\begin{aligned}
V'(L)(1-\mu+\mu F_K(K,L)) - U'(C)F_K(K,L)F_L(K,L) &= 0; \\
F_L(K,L)L - K - Q\left(\bar{M} - \frac{\mu^g C^g}{Q}\right) - S(\bar{B} - B^g) &= 0; \\
Q\left(\bar{M} - \frac{\mu^g C^g}{Q}\right)\left(\frac{1-\mu}{\mu}\right) - F_K(K,L)K - S(\bar{B} - B^g) &= 0; \\
F_K(K,L) - 1 &= 0.
\end{aligned} \tag{2}$$

By introducing money constraints in the model, represented by the parameters μ and μ^g, one might have thought that the golden rule steady state would depend on these parameters. Surprisingly, we get, as in a nonmonetary economy,

$$F_K(K, L) = 1.$$

Combining the second and third equations and using the fact that in equilibrium, $Q\bar{M} = \mu C + \mu^g C^g$, we obtain:

$$C = F_L(K, L)L,$$

which says that consumption and savings are equivalent. Note that under the golden rule, the first equation in (2) reduces to

$$V'(L) = U'(C)F_L(K, L) \qquad \text{or equivalently} \qquad V'(L)L = U'(C)C.$$

This equation corresponds to the first order condition of the following maximization problem:

$$\max U(C) - V(L) \qquad \text{subject to} \qquad C + K = F(K, L).$$

Having gained some information on the golden rule steady state, we shall now study non steady state equilibria which are near the golden rule equilibrium.

1.2 Equilibria near the golden rule

Let e be an economy characterized by $(U, V, F, \bar{M}, \bar{B}, \mu, \mu^g)$ and let $\tilde{Y}^*(e)$ be the corresponding golden rule equilibrium, namely, the sequence $\{Y_t^* : t \in \mathbb{N}\}$ where

$$Y_t^* = (K^*(e), L^*(e), Q^*(e), S^*(e)), \qquad \text{for all } t.$$

By Proposition 1.1, we have $\tilde{Z}(\tilde{Y}^*(e); e) = 0$. Following **[1]**, let us now make precise the notion of closeness underlying the analysis.

Definition An equilibrium for the economy e is said to be close to the golden rule equilibrium $\tilde{Y}^*(e)$ (of the economy e) if there exists a sequence

$$\tilde{y} = \{y_t \in \mathbb{R}^4 : t \in \mathbb{N}\},$$

with $\tilde{y} \neq 0$, where $y_t = (k_t, l_t, q_t, s_t)$, $\|y_t\|$ small for all t and

$$\tilde{Z}(\tilde{Y}^*(e) + \tilde{y}; e) = 0 \qquad (\text{or} \quad Z(Y_t^*(e) + y_t, Y_{t+1}^*(e) + y_{t+1}; e) = 0 \quad \text{for all } t).$$

Let us now recall the definition of a critical point of a differentiable map.

Definition Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined on an open subset U of $\mathbb{R}^n$, we say that $p \in U$ is a *critical point* of F if the differential map $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective. The image $F(p) \in \mathbb{R}^m$ of a critical point p is called a *critical value* of F.

Note that a first order Taylor expansion of $\tilde{Z}$ around $\tilde{Y}^*(e) + \tilde{y}$ evaluated at the golden rule equilibrium $\tilde{Y}^*(e)$ yields

$$\frac{\partial Z}{\partial Y_t}(Y^*(e), Y^*(e))y_t + \frac{\partial Z}{\partial Y_{t+1}}(Y^*(e), Y^*(e))y_{t+1} = 0 \tag{3}$$

for a sequence $\tilde{y} = \{y_t : t \in \mathbb{N}\} \neq 0$ such that $\|y_t\| \leq 1$ for all $t \in \mathbb{N}$.

Our goal will be to characterize the subset of critical economies $\mathbf{C}$ that satisfy (3). Note that 0 is a critical value of $\tilde{Z}$ evaluated at the golden rule equilibrium.

Following **[1]**, let us introduce a few parameters which will appear to be important to describe the set $\mathbf{C}$:

$$\epsilon \; = \; \frac{-K^*(e)F_{KK}(K^*(e), L^*(e))}{F_K(K^*(e), L^*(e))},$$

$$\overline{U} \; = \; \frac{-C^*(e)U''(C^*(e))}{U'(C^*(e))} \qquad \text{and} \qquad \overline{V} \; = \; \frac{L^*(e)V''(L^*(e))}{V'(L^*(e))}.$$

By definition, ϵ is (up to a sign) the inverse of the elasticity of the demand of capital with respect to the real interest rate at the golden rule and $\overline{U}$ and $\overline{V}$ are the coefficients of relative risk aversion at the golden rule.

We shall now define the relationship between savings and the interest rate. Note that the problem of the representative individual can be rewritten as follows:

$$\text{Max } U(C) - V(L) \quad \text{subject to} \quad C = \frac{RWL + (\bar{B} - B^g)(S - RS)}{1 - \mu + \mu R}.$$

The associated first order condition is then

$$U'(C)\frac{RW}{1 - \mu + \mu R} - V'(L) = 0.$$

Let us differentiate this expression with respect to L and R. After some manipulations and using all the information available in equilibrium, we obtain the following relation:

$$\frac{dL}{dR} = \frac{L}{R}\frac{(1-\mu)(1-\overline{U}) + \overline{U}(\bar{B} - B^g)S/C}{(\overline{U} + \overline{V})}.$$

Hence, the elasticity of savings with respect to the interest rate is given by $\frac{1}{\alpha}$, where

$$\alpha = \frac{(\overline{U} + \overline{V})}{(1-\mu)(1-\overline{U}) + \overline{U}(\bar{B} - B^g)S/C}.$$

Thus savings will be increasing or decreasing in the interest rate if and only if α is positive or negative.

It was shown in Grandmont [**4**] that cycles may appear in an OLG economy which undergoes strong income effects. The argument presented in that paper applies here in a neighborhood of the golden rule steady state. Indeed, we saw that at the golden rule, $LV'(L) = CU'(C)$. Let

$$v(L) = LV'(L) \qquad \text{and} \qquad u(C) = CU'(C).$$

We can then write $L = v^{-1}(u(C))$, as by assumption, v is an increasing function of L. The shape of this "offer curve" depends on the monotonicity of u. As

$$u'(C) = U'(C) + CU''(C) = U'(C)\left(1 + \frac{CU''(C)}{U'(C)}\right) = U'(C)(1 - \overline{U}),$$

the parameter of importance in the determination of the shape of the offer curve is $\overline{U}$. If future return on money, bonds or capital decreases, then future consumption decreases. As leisure today is comparatively less expensive, leisure increases and thus labor supply decreases. This is a substitution effect. Note that lower returns imply a lower income and hence less consumption of the good and of leisure. Labor supply therefore increases. This is an income effect. Consequently, the offer curve is increasing when the substitution effect dominates, which corresponds to $\overline{U} < 1$, and decreasing when the income effect dominates, which corresponds to $\overline{U} > 1$. In the latter case, cycles may appear.

Let us compute the derivatives in (3). Using the identities

$$F_K(k_t, l_t) = r_t = F_{KK}k_t + F_{KL}l_t, \qquad F_{KK}K + F_{KL}L = F_{KL}K + F_{LL}L = 0,$$
$$r_t = -\frac{L}{K}(F_{KL}k_t + F_{LL}l_t),$$

we get, for $\overline{U} \neq 1$, the following dynamical system:

$$\begin{aligned}
\frac{(\overline{U}+\overline{V})}{\alpha}r_{t+1} + \frac{\mu(1-\overline{U})}{Q}q_{t+1} \qquad & \\
-\frac{\overline{U}(\bar{B}-B^g)}{C}s_{t+1} \quad &= \quad \frac{K(1-\overline{U})}{C}r_t + \frac{(\overline{U}+\overline{V})}{L}l_t \\
&\qquad + \frac{\mu(1-\overline{U})}{Q}q_t - \frac{\overline{U}(\bar{B}-B^g)}{C}s_t, \\
\frac{K}{\epsilon}r_{t+1} - \frac{K}{L}l_{t+1} \quad &= \quad Kr_t - \frac{C}{L}l_t + \frac{\mu C}{Q}q_t \\
&\qquad + \left((\bar{B}-B^g)S - C^g\right)\frac{s_t}{S}, \\
K\left(\frac{1}{\epsilon}-1\right)r_{t+1} - \frac{K}{L}l_{t+1} + \left\{\bar{M}\left(\frac{1-\mu}{\mu}\right) + \frac{\mu^g C^g}{Q}\right\}q_{t+1} \qquad & \\
-(\bar{B}-B^g)s_{t+1} \quad &= \quad \frac{-C^g}{S}s_t, \\
r_{t+1} - \frac{1}{S}s_{t+1} \quad &= \quad -\frac{1}{S}s_t,
\end{aligned} \tag{4}$$

where

$$(K, L, Q, S, C, C^g, B^g) := (K^*(e), L^*(e), Q^*(e), S^*(e), C^*(e), C^{g*}(e), B^{g*}(e))$$

and the sequence $\{(r_t, l_t, q_t, s_t) : t \in \mathbb{N}\}$ represents the deviations from the golden rule.

Note that for $\overline{U} = 1$, the first equation in (4) reduces to

$$\frac{(\bar{B}-B^g)S}{C}r_{t+1} - \frac{(\bar{B}-B^g)}{C}s_{t+1} = \frac{(1+\overline{V})}{L}l_t - \frac{(\bar{B}-B^g)}{C}s_t,$$

while the remaining equations are unchanged.

An economy e belongs to $\mathbf{C}$ if there exists a nonzero sequence of deviations $\{y_t = (r_t, l_t, q_t, s_t) : t \in \mathbb{N}\}$ such that $\|y_t\| \leq 1$ for all t and (4) is satisfied. This will be the case if the matrix associated to the system (4) has at least one stable eigenvalue (one eigenvalue of modulus less than or equal to one).

The importance of the set $\mathbf{C}$ results from a proposition of **[1]** that we now recall.

Let E be the set of economies with a monetary golden rule steady state and V be any open neighborhood of the golden rule of an economy e_0. Then every open neighborhood of e_0 in E contains an economy e which possesses an intertemporal equilibrium that differs from the golden rule of e and that lies in V if and only if e_0 lies in $\mathbf{C}$.

We suspect that this proposition remains valid under few modifications in the present setting.

Note that one cannot choose arbitrarily a sequence $\{C_t^g : t \in \mathbb{N}\}$ of fiscal policies. Indeed, the choice of such a sequence determines a path of deficits $\{B_t^g : t \in \mathbb{N}\}$. In equilibrium, B^g, which is nothing but the limit of the B_t^g's, is defined only insofar as the series in

$$\lim_{t\to\infty} B_{t+1}^g = B_1^g - \sum_{j=1}^{\infty} \frac{C_j^g(1-\mu^g)}{S_j} - \sum_{j=1}^{\infty} \frac{\frac{Q_j}{Q_{j+1}}\mu^g C_{j+1}^g}{S_j}$$

converge. This will be the case if, for example, one chooses $C_t^g = C_0 e^{-kt}$ for some $k > 0$. Notice that this choice corresponds to a "zero deficit" equilibrium target.

2 An example with no government

We will now study the dynamical system developed in the preceding section for a particular economy in which the government is absent.

The consumer's problem at t can be stated as follows:

Under perfect foresight, the individual chooses a labor supply, L_t, a consumption level, C_{t+1}, savings in the form of capital, K_{t+1}, money, M_{t+1} and bonds, B_{t+1}, to maximize his utility $U(C_{t+1}) - V(L_t)$ subject to the following constraints

$$W_t L_t = K_{t+1} + Q_t M_{t+1} + S_t B_{t+1},$$

$$C_{t+1} = R_{t+1} K_{t+1} + Q_{t+1} M_{t+1} + S_{t+1} B_{t+1},$$

$$Q_{t+1} M_{t+1} = \mu C_{t+1}, \text{ for some } \mu \in (0,1).$$

We take $\bar{M} = M_t$ and $\bar{B} = B_t$ for all t. Thus the constraints become

$$W_t L_t = K_{t+1} + Q_t \bar{M} + S_t \bar{B},$$

$$C_{t+1} = R_{t+1} K_{t+1} + Q_{t+1} \bar{M} + S_{t+1} \bar{B},$$

$$Q_{t+1} \bar{M} = \mu C_{t+1}, \text{ for some } \mu \in (0,1).$$

With no government, the function Z has three components, (Z_1, Z_2, Z_3), given by

$$\begin{aligned}
Z_1(Y_t, Y_{t+1}; e) &\equiv V'(L_t)\left\{1-\mu+\frac{\mu Q_t}{Q_{t+1}}F_K(K_{t+1}, L_{t+1})\right\} \\
&\quad -U'(C_{t+1})F_K(K_{t+1}, L_{t+1})F_L(K_t, L_t); \\
Z_2(Y_t, Y_{t+1}; e) &\equiv F_L(K_t, L_t)L_t - K_{t+1} - Q_t\bar{M} - S_t\bar{B}; \\
Z_3(Y_t, Y_{t+1}; e) &\equiv F_K(K_{t+1}, L_{t+1}) - \frac{S_{t+1}}{S_t};
\end{aligned}$$

where

$$C_{t+1} = \frac{F_K(K_{t+1}, L_{t+1})K_{t+1} + S_{t+1}\bar{B}}{1-\mu}$$

and

$$Q_{t+1} = \frac{\mu C_{t+1}}{\bar{M}} = \frac{\mu}{\bar{M}}\left\{\frac{F_K(K_{t+1}, L_{t+1})K_{t+1} + S_{t+1}\bar{B}}{1-\mu}\right\}.$$

By Proposition 1.1, at a stationary equilibrium

$$\{Y_t = Y = (K, L, S) : t \in \mathbb{N}\},$$

we have

$$\begin{aligned}
V'(L)(1-\mu+\mu F_K(K,L)) - U'(C)F_K(K,L)F_L(K,L) &= 0; \\
F_L(K,L)L - K - Q\bar{M} - S\bar{B} &= 0; \\
F_K(K,L) - 1 &= 0.
\end{aligned} \tag{5}$$

We thus get:

$$F_K = 1, \qquad F_L L = C, \qquad Q\bar{M} = \mu C.$$

Let us now compute the derivatives in (3). Using the identities

$$F_K(k_t, l_t) = r_t = F_{KK}k_t + F_{KL}l_t, \qquad F_{KK}K + F_{KL}L = F_{KL}K + F_{LL}L = 0,$$

$$r_t = -\frac{L}{K}(F_{KL}k_t + F_{LL}l_t),$$

we get, for $\overline{U} \neq 1$, the following dynamical system (denoted (6)):

$$\begin{aligned}
&\frac{K}{C}\frac{(\overline{U}-\mu)}{(1-\mu)}\left\{r_{t+1}\left(1-\frac{1}{\epsilon}\right)+\frac{1}{L}l_{t+1}\right\} - r_{t+1}(1-\mu) + s_{t+1}\frac{\bar{B}}{C}\frac{(\overline{U}-\mu)}{(1-\mu)} \quad = \\
&\qquad -\frac{K}{C}\frac{\mu}{(1-\mu)}\left\{r_t\left(1-\frac{1}{\epsilon}\right)+\frac{1}{L}l_t\right\} - \frac{K}{C}r_t - \frac{\overline{V}}{L}l_t - s_t\frac{\bar{B}}{C}\frac{\mu}{(1-\mu)}
\end{aligned}$$

$$
\text{(6)} \qquad \begin{aligned}
K\left\{\frac{r_{t+1}}{\epsilon}-\frac{1}{L}l_{t+1}\right\} &= r_t\frac{K}{(1-\mu)}\left(1-\frac{\mu}{\epsilon}\right) \\
&\quad -\frac{C}{L}l_t\left\{1-\frac{K}{C}\frac{\mu}{(1-\mu)}\right\}+s_t\frac{\bar{B}}{(1-\mu)} \\
r_{t+1}-\frac{1}{S}s_{t+1} &= -\frac{1}{S}s_t
\end{aligned}
$$

where $(K, L, C, S) := (K^*(e), L^*(e), C^*(e), S^*(e))$ and $\{(r_t, l_t, s_t) : t \in \mathbb{N}\}$ is a sequence of deviations from the golden rule.

Note that for $\overline{U} = 1$, the first equation in (6) reduces to

$$
\begin{aligned}
&\frac{K}{C}\left\{r_{t+1}\left(1-\frac{1}{\epsilon}\right)+\frac{1}{L}l_{t+1}\right\}-r_{t+1}(1-\mu)+s_{t+1}\frac{\bar{B}}{C} \quad = \\
&-\frac{K}{C}\frac{\mu}{(1-\mu)}\left\{r_t\left(1-\frac{1}{\epsilon}\right)+\frac{1}{L}l_t\right\}-\frac{K}{C}r_t-\frac{\overline{V}}{L}l_t-s_t\frac{\bar{B}}{C}\frac{\mu}{(1-\mu)}
\end{aligned}
$$

while the remaining equations are unchanged. For $\overline{U} \neq 1$, (6) can also be written as

$$
\begin{pmatrix} r_{t+1} \\ \frac{l_{t+1}}{L} \\ \frac{s_{t+1}}{S} \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{pmatrix} \begin{pmatrix} r_t \\ \frac{l_t}{L} \\ \frac{s_t}{S} \end{pmatrix}
$$

where

$$
\begin{aligned}
\tilde{A}_{11} &= \frac{K}{C(1-\mu)}\left(1+\tilde{\gamma}\left(1-\frac{1}{\epsilon}\right)\right) \\
\tilde{A}_{12} &= \frac{\tilde{\alpha}}{(1-\mu)}+\mu(\overline{V}+1)\tilde{\beta}+\frac{K}{C}\frac{\tilde{\gamma}}{(1-\mu)} \\
\tilde{A}_{13} &= \frac{\bar{B}S\tilde{\beta}}{C}\left(1-\mu+\frac{\overline{U}-1}{1-\mu}\right)-1+\frac{K}{C}(\overline{U}-\mu)\tilde{\beta}-\frac{1-\mu}{\overline{U}-1} \\
\tilde{A}_{21} &= \left(\frac{K}{\epsilon C(1-\mu)}-1\right)\left(1+\tilde{\gamma}\left(1-\frac{1}{\epsilon}\right)\right) \\
\tilde{A}_{22} &= \frac{C}{K}+\frac{\tilde{\alpha}}{\epsilon(1-\mu)}-\tilde{\gamma}\left(1-\frac{K}{\epsilon C(1-\mu)}\right)+\frac{\mu(\overline{V}+1)\tilde{\beta}}{\epsilon} \\
\tilde{A}_{23} &= \frac{\bar{B}S\tilde{\beta}}{\epsilon C}\left(1-\mu+\frac{\overline{U}-1}{1-\mu}\right)-\frac{\bar{B}S}{K(1-\mu)}-\frac{1}{\epsilon} \\
&\quad +\frac{1}{\epsilon(\overline{U}-1)}\left(\frac{K}{C}\frac{\overline{U}-\mu}{(1-\mu)}-(1-\mu)\right) \\
\tilde{A}_{31} &= \frac{K}{C(1-\mu)}\left(1+\tilde{\gamma}\left(1-\frac{1}{\epsilon}\right)\right) \\
\tilde{A}_{32} &= \frac{K}{C}\frac{\tilde{\gamma}}{(1-\mu)}+\frac{\tilde{\alpha}}{(1-\mu)}+\mu\tilde{\beta}(\overline{V}+1) \\
\tilde{A}_{33} &= \frac{\bar{B}S\tilde{\beta}}{C}\left(1-\mu+\frac{\overline{U}-1}{1-\mu}\right)+\frac{K\tilde{\beta}(\overline{U}-\mu)}{C}-\frac{1-\mu}{\overline{U}-1}
\end{aligned}
$$

and

$$\tilde{\alpha} = \frac{\overline{U} + \overline{V}}{1 - \overline{U}}, \qquad \tilde{\beta} = \frac{1}{(\overline{U} - 1)(1 - \mu)}, \qquad \tilde{\gamma} = \frac{\mu}{(1 - \mu)}.$$

In the present model, bonds and capital are substitutes while in [**1**], money and capital are substitutes. By setting $\mu = 0$ in the previous formulation, we get back the system presented in [**1**], where money in [**1**] corresponds to bonds in our model. More precisely,

$$\begin{pmatrix} r_{t+1} \\ \frac{l_{t+1}}{L} \\ \frac{s_{t+1}}{S} \end{pmatrix} = \begin{pmatrix} \frac{K}{C} & \tilde{\alpha} & 0 \\ \frac{K}{\epsilon C} - 1 & \frac{C}{K} + \frac{\tilde{\alpha}}{\epsilon} & 1 - \frac{C}{K} \\ \frac{K}{C} & \tilde{\alpha} & 1 \end{pmatrix} \begin{pmatrix} r_t \\ \frac{l_t}{L} \\ \frac{s_t}{S} \end{pmatrix}.$$

Let the utility function of the representative consumer be

$$U(C_{t+1}) - V(L_t) = \frac{C_{t+1}^{\gamma}}{\gamma} - \frac{L_t^{\beta}}{\beta},$$

where $\gamma \neq 0$, $\gamma < 1$ and $\beta \geq 1$. Then $\overline{U} = 1 - \gamma$ and $\overline{V} = \beta - 1$. Assume the production function F is defined as follows:

$$F(K_t, L_t) = K_t^{\rho} L_t^{1-\rho} + (1 - \delta) K_t,$$

where $0 < \rho < 1 - \rho < 1$. At the golden rule equilibrium, the following relationships hold:

$$\frac{L}{K} = \left(\frac{\delta}{\rho}\right)^{\frac{1}{1-\rho}}, \qquad \frac{C}{K} = (1 - \rho)\frac{\delta}{\rho}, \qquad \epsilon = (1 - \rho)\delta,$$

$$Q\bar{M} = \mu C, \qquad \frac{\bar{B}S}{C} = 1 - \frac{K}{C} - \mu, \qquad \text{and} \qquad L^{\beta} = C^{\gamma},$$

where the last equation comes from the identity $LV' = CU'$. Moreover,

$$\alpha = \frac{\overline{U} + \overline{V}}{(1 - \mu)(1 - \overline{U}) + \overline{U}\bar{B}S/C} = \frac{\overline{U} + \overline{V}}{1 - \mu - \overline{U}K/C},$$

where $\frac{1}{\alpha}$ is the elasticity of savings with respect to the interest rate.

We shall now study, by fixing the preference and technology parameters and a given set of golden rule values (K, L, C, S), the sensitivity of the economy to μ. The behaviour of the economy will be described through the analysis of the eigenvalues of the Jacobian matrix of the system (6).

Let us first state a proposition, adapted from [**2**], which makes precise the behaviour of a linear system, once the associated eigenvalues are known.

Proposition 2.1 *Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear diagonalisable transformation. Suppose the eigenvalues are λ_1, λ_2, and λ_3 (not necessarily distinct) and*

i. $|\lambda_1| < 1$,

ii. $|\lambda_2| > 1$,

iii. $\lambda_3 = -1$.

Then there are 3 lines, W^s, W^u, and W^c on which

i. *if $x \in W^s$, then $L(x) \in W^s$ and $L^n(x) \to 0$ as $n \to \infty$.*

ii. *if $x \in W^u$, then $L(x) \in W^u$ and $L^{-n}(x) \to 0$ as $n \to \infty$.*

iii. *if $x \in W^c$, then $L(x) \in W^c$ and $L^2(x) = x$.*

Case 1 Let $\gamma = 0.9$, $\beta = 2$, $\rho = 0.25$, $\delta = 0.1$. Then

$$\overline{U} = 0.1, \quad \overline{V} = 1, \quad \epsilon = 0.075, \quad \frac{C}{K} = \frac{3}{10}, \quad \frac{L}{K} = \left(\frac{2}{5}\right)^{\frac{4}{3}}, \quad C^{0.9} = L^2.$$

The graph of the eigenvalues parametrized by the money constraint parameter μ is presented in Figure 2.1. A closer look to the eigenvalues in a neighborhood of 0 is given in Figure 2.2.

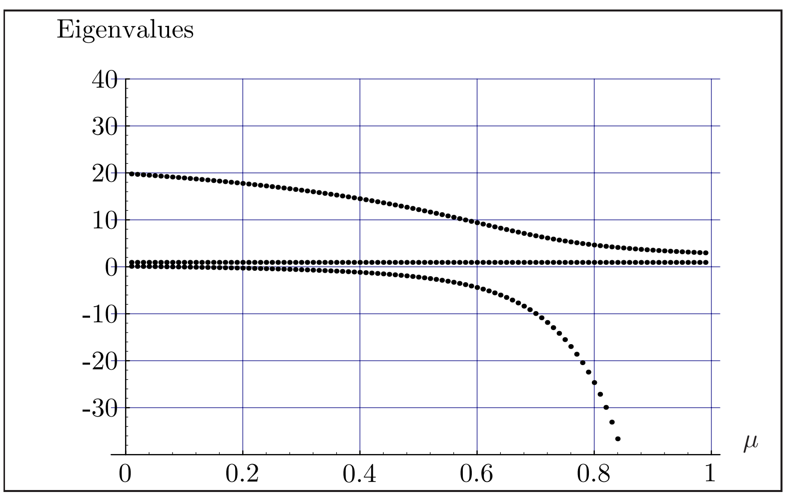

Figure 2.1

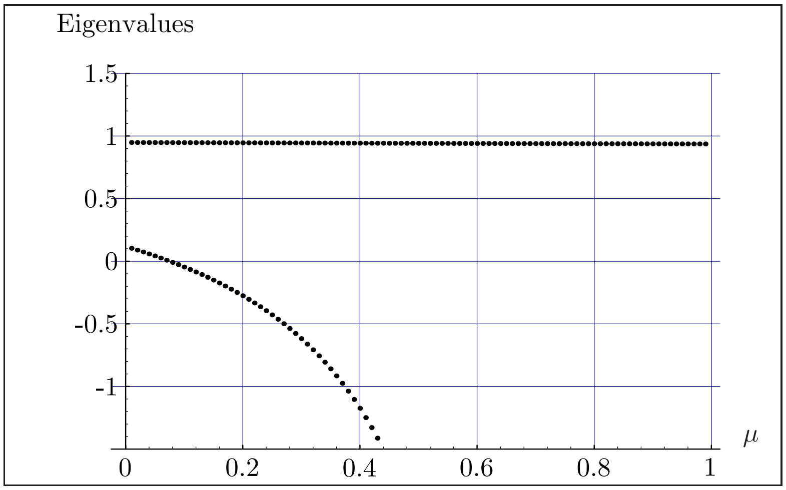

Figure 2.2

We observe the following results:

1. For $\mu \approx 0.3742$, one of the eigenvalues is -1, one eigenvalue is stable (it has a modulus less than one) and one eigenvalue is unstable (it has a modulus greater than one).

2. For $\mu \in (0, 0.3742)$, two eigenvalues are stable and one is unstable.

3. For $\mu \in (0.3742, 1)$, one eigenvalue is stable and two are unstable.

4. $\alpha = \frac{11/10}{2/3-\mu}$, which is positive for $\mu = 0.3742$.

5. $\tilde{\alpha} > 0$.

Case 2 Let $\gamma = 0.5$, $\beta = 2$, $\rho = 0.25$, $\delta = 0.1$. For these values,

$$\overline{U} = 0.5, \quad \overline{V} = 1, \quad \epsilon = 0.075, \quad \frac{C}{K} = \frac{3}{10}, \quad \frac{L}{K} = \left(\frac{2}{5}\right)^{\frac{4}{3}}, \quad C^{\frac{1}{2}} = L^2.$$

The graph of the eigenvalues parametrized by the money constraint parameter μ is presented in Figure 2.3. A closer look to the eigenvalues in a neighborhood of 0 is given in Figure 2.4.

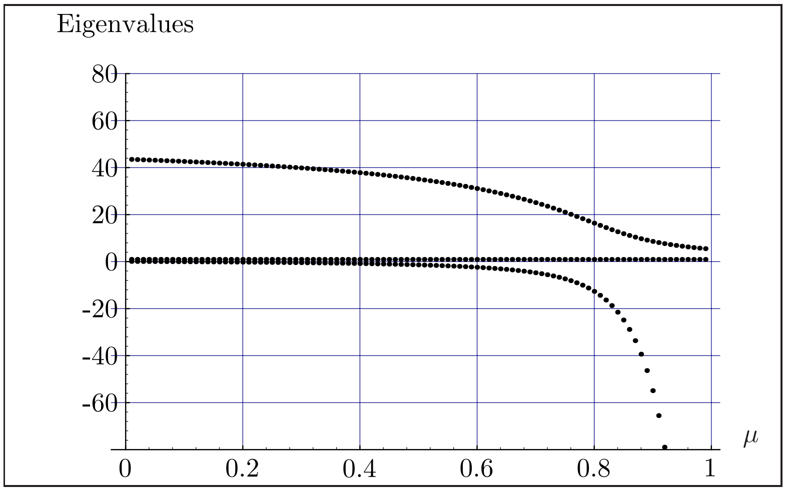

Figure 2.3

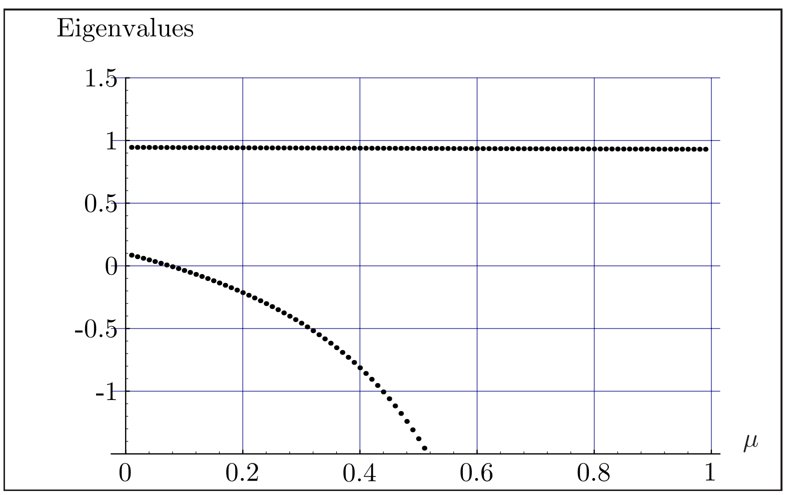

Figure 2.4

In this particular case, we obtain the following results:

1. For $\mu \approx 0.438899$, one of the eigenvalues is -1, one eigenvalue is stable (it has a modulus less than one) and one eigenvalue is unstable (it has a modulus greater than one).
2. For $\mu \in (0, 0.438899)$, two eigenvalues are stable and one is unstable.
3. For $\mu \in (0.438899, 1)$, one eigenvalue is stable and two are unstable.
4. For any $\mu \in (0,1)$, $\alpha = \frac{-9}{4+6\mu} < 0$.
5. $\tilde{\alpha} > 0$.

Case 3 Let $\gamma = -1$, $\beta = 2$, $\rho = 0.25$, $\delta = 0.1$. Then

$$\overline{U} = 2, \quad \overline{V} = 1, \quad \epsilon = 0.075, \quad \frac{C}{K} = \frac{3}{10}, \quad \frac{L}{K} = \left(\frac{2}{5}\right)^{\frac{4}{3}}, \quad C^{-1} = L^2.$$

Again, the graph of the eigenvalues parametrized by the money constraint parameter μ is presented in Figure 2.5. A closer look to the eigenvalues in a neighborhood of 0 is given in Figure 2.6.

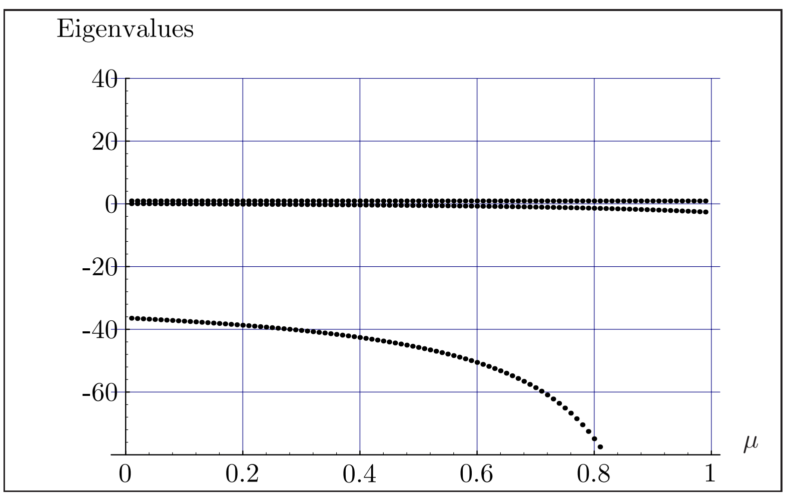

Figure 2.5

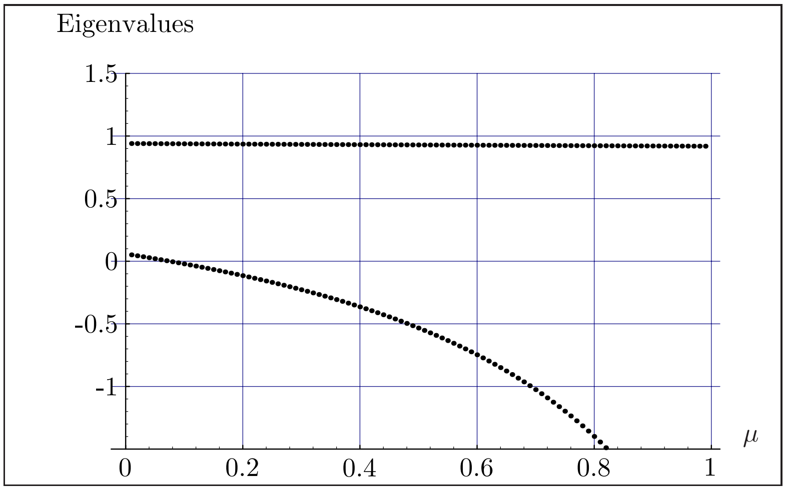

Figure 2.6

We get the following results:

1. For $\mu \approx 0.69197$, one of the eigenvalues is -1, one eigenvalue is stable (it has a modulus less than one) and one eigenvalue is unstable (it has a modulus greater than one).
2. For $\mu \in (0, 0.69197)$, two eigenvalues are stable and one is unstable.
3. For $\mu \in (0.69197, 1)$, one eigenvalue is stable and two are unstable.
4. For any $\mu \in (0,1)$, $\alpha = \frac{-9}{17+3\mu} < 0$.
5. $\tilde{\alpha} < 0$.

Modifying the value of the parameter β implies no considerable modification to the above results.

For $\beta = 2$, $\rho = 0.25$ and $\delta = 0.1$, as γ decreases, or equivalently, as $\overline{U}$ increases, the value of μ for which an eigenvalue is -1 increases.

Although no uncertainty is modelled here, one possible explanation for this relation between γ and μ comes from analyzing the attitude toward risk of the individual. A low γ (or equivalently, a large $\overline{U}$) indicates that an individual is quite risk averse. Such an individual may feel that money is not the safest asset to hold. For this reason, we might require this individual to keep a larger fraction of his purchases in money.

Alternatively, if we think of γ as giving information about the wealth of an individual (in the sense that the larger γ is, the wealthier an individual is), one may thing that the larger is γ, the easier it is for an individual to obtain credit. Therefore, the individual should not be constrained to keep a large fraction of his purchases in money.

Note that changing ρ in cases 2 and 3 does not have any significant effect on the results. In fact, with the specifications of these cases, α is negative for any value of μ and ρ. By changing ρ in case 1, we observe that α remains positive for some interval of values of μ.

It is interesting to look at the graph of α as a function of μ, where $\frac{1}{\alpha}$ is the elasticity of savings with respect to the interest rate. By doing so, we notice that for the technology parameters set as above (i.e. $\rho = 0.25$ and $\delta = 0.1$), there is an interval of values of μ, $(0, x)$, such that α is positive. It corresponds to values of $\overline{U}$ in the interval $(0, 0.3)$, or equivalently, to values of γ in the interval $(0.7, 1)$.

The parameter $\frac{1}{\tilde{\alpha}}$ is the elasticity of savings with respect to the interest rate in **[1]**. As noted earlier, system (6) can be written in a form in which the parameter $\tilde{\alpha}$ plays, apparently, an important role. We observe that cycles appear both for negative and positive values of $\tilde{\alpha}$.

It is well-known that the comparison of the number of stable eigenvalues to the number of predetermined variables in an economy gives information on the determinacy or indeterminacy of the steady states of the economy.[4]

In this model, the labor supply and the price of money are free variables. By fixing labor, capital becomes a predetermined variable.

In all the above cases and for any value of μ in the interval $(0, x)$, where x is the value of μ for which an eigenvalue is -1, there are two stable eigenvalues and one unstable eigenvalue. Therefore, for those values of μ, there is one degree of indeterminacy in our model. On the other hand, for any $\mu \in (x, 1)$, the golden rule steady state is determinate.

Case 4 Assume there is full depreciation of capital (i.e. $\delta = 1$). Keeping ρ and β fixed to 0.25 and 2 respectively and letting $\overline{U}$ take, in turn, the values 2, 0.5 and 0.1, we observe that for $\mu = 0.8$, one eigenvalue is equal to -1, independently of the value taken by $\overline{U}$, or equivalently, independently of the sign of α. Moreover, independently of the value of $\overline{U}$, for $\mu \in (2/3, 0.8)$, the golden rule steady state is indeterminate (two stable eigenvalues, one unstable eigenvalue).

Recall that an economy is in **C** if there is a nonzero sequence

$$\{y_t = (r_t, l_t, s_t) : t \in \mathbb{N}\}$$

with $\|y_t\| \leq 1$ for all t such that (3) or equivalently (6) is satisfied. In the above four cases, for any initial condition $y_0 = (r_0, l_0, s_0) \in W^s$, where W^s is a stable

[4] We shall say that the golden rule steady state is indeterminate if there exists many equilibrium paths along which the corresponding equilibria converge to the golden rule.

eigenspace, the vector

$$\begin{pmatrix} r_t \\ l_t \\ s_t \end{pmatrix} = A^t \begin{pmatrix} r_0 \\ l_0 \\ s_0 \end{pmatrix}$$

is of norm less than or equal to one for all t. Thus each economy specified in these cases belongs to **C**.

The elasticity of substitution between capital and labor at the golden rule, $\sigma(K/L)$, is

$$\sigma(K/L) = \frac{1}{\delta(1-\rho)+\rho} = \frac{1}{\epsilon+\rho}.$$

For our specification, $\sigma(K/L) \geq 1$. In **[7]**, cycles appear precisely when this parameter is positive and close to 0.

Note that for all values of μ for which an eigenvalue λ is negative and less than one in absolute value, the sequence of vectors $\{v_\lambda, Av_\lambda, A^2v_\lambda, ..., A^tv_\lambda, ...\}$, where v_λ is an eigenvector associated to the eigenvalue λ, converges to the golden rule steady state in a cyclical fashion. Hence, it may take a long time before the golden rule steady state is effectively reached.

In our search for cycles, we were unable to find Hopf bifurcations (pairs of complex conjugate eigenvalues of modulus equal to one) by varying the parameters. Although we have no proof that with a money constraint, such cycles do not appear for a Cobb-Douglas production function, it was noted in **[1]** that the set of Hopf cycles is empty in the Cobb-Douglas case.

Conclusion

In this work, we were interested in analyzing how the equilibria near the golden rule steady state of a productive economy with money and a government would respond to the addition of a money constraint. After describing the dynamical system that allows such an analysis to be performed, an example of an economy with no government was treated.

Among the parameters usually of interest in the search for cycles in OLG economies with production are the elasticity of savings with respect to the interest rate, $\frac{1}{\alpha}$, and the elasticity of substitution between capital and labor, $\sigma(K/L)$. For our specification, flip bifurcations appear both for positive and negative values of α, which depends on the money constraint parameter μ; the parameter $\sigma(K/L)$ appears to be uninteresting.

It would be of interest to find general conditions for the existence of cycles in the neighborhood of the golden rule steady state of an economy parametrized both by a money constraint and a sequence of fiscal policies.

In our example, we observed indeterminacy of the golden rule steady state for some values of the money constraint parameter μ. It would be interesting to investigate whether, by choosing appropriately a sequence of fiscal policies, the indeterminacy disappears or not.

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