# On the Selection of one Feedback Nash Equilibrium in Discounted Linear-Quadratic Games

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**Abstract.** We study a selection method for a Nash feedback equilibrium of a one-dimensional linear-quadratic nonzero sum game over an infinite horizon : by introducing a change in the time variable, one obtains an associated game over a finite horizon $T > 0$ and with free terminal state. This associated game admits a unique solution which converges to a particular Nash feedback equilibrium of the original problem as the horizon $T$ goes to infinity.

**Key Words.** Linear-quadratic games. Nonzero sum differential games. Nash equilibria. Infinite horizon.

**JEL** : C61, C72.

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# 1 Introduction

In this paper we are interested in nonzero-sum linear-quadratic games with infinite horizon and feedback strategies. For definitions and stantard results, we refer to Basar and Olsder [1].

In general in many economic models with dynamics, the planning period $T$ may not be known and then "$T = \infty$ may very well be the best approximation for the true problem horizon" (Dockner et al., p :61, [2]). It is important, in order to understand our approach, to specify that the infinite horizon games we want to study must be considered as games with a finite horizon $T$, where $T$ is as great as we want. Therefore, in this paper, the finite horizon games we will associate to the given infinite horizon one do not possess any terminal value, i.e. we will consider only problems with free terminal states.

Our goal is then to prove that for scalar 2-player games with a convex quadratic objective, the most natural criterion of selection apply : the unique Nash equilibrium of each finite horizon game, if it exists, converges to a unique Nash equilibrium of the given infinite horizon game.

We know that without convexity assumptions, additional difficulties can arise, such as non-existence of solution for instance or various asymptotic behavior, see Papavassilopoulos and Olsder [3], Mageirou [4]. In infinite horizon, a well-known difficulty arises in optimal control theory : the transversality condition in infinite horizon is not a necessary condition for optimality (see Halkin, [5]). As a consequence, we cannot easily characterize a Nash equilibrium in infinite horizon (for a convex game) by conditions that are necessary and sufficient.

In the seventies and eighties, various papers have characterized the Nash equilibria by a Riccati system of differential equations when the horizon is finite or a Riccati system of algebraic equations when the horizon is infinite ; but only some uniqueness results exist : see Papavassilopoulos et al. [6], Freiling et al. [7]. More recently in the scalar convex case, new results have been established by Engwerda and Weeren.

Weeren et al. [8] propose a study of a general Riccati equation system by the help of differential topology techniques. These authors study the flow of such a quadratic system on the Poincare sphere and, with the Poincare index, they obtain the type and the number of the critical points at infinity of this system. From these results it is possible to conclude (see particularly Remark 2.4 on p :712), that the equilibrium in the finite horizon game converges to one of the equilibria of the infinite horizon game, depending on the terminal value of the game.

In this paper, we propose to establish straightforwardly such results by very simple methods.

On the other hand, Engwerda, in recent work on scalar infinite horizon games [9], gives new conditions on the parameters in order to characterize the existence of a unique Nash equilibrium (see also Weeren et al. [8]). In case of multiplicity, he proves that selection criteria like dynamic stability of equilibria, stability of the closed loop value or maximal total gain are not robust to compare these equilibria.

In this paper, given our wish is to select one particular feedback Nash equilibrium in infinite horizon, we do not need to study the whole set of equilibria (in feedback strategies). For the scalar game we consider, we show that the most natural finite horizon games admit equilibrium strategies that converge to a steady-state (for all fixed data). And this steady-state is precisely the constant linear strategies of one equilibrium in infinite horizon. Again we prove convergence by using only simple methods, characterized in particular by a time inversion.

In section 2, we present the model and spell out the natural convergence criterion in a general multi-dimensional model. In section 3, we prove that these results actually work in the one-dimensional case, leaving their possible extension to the vector case as an open question.

# 2 A Criterion of Selection for the General Case

In this section, we review the main results in the literature dealing with multi-dimensional linear-quadratic games (Weeren et al. [8], Engwerda [9]). In addition, we introduce the device of time reversal to convert standard Ricatti equations with fixed terminal time into differential equations with fixed initial values. This trick is convenient in the study of the infinite-horizon problem in that it suggests a natural criterion that applies when the limit of the finite-horizon exists and is unique.

For the standard $n$-dimensional linear-quadratic game with two players, we can consider the following evolution equation of the state $Y(t) \in I\!R^n$ :

$$\dot{Y} = \hat{A}Y - B_1V_1 - B_2V_2, \quad Y(0) = Y_0 \tag{1}$$

where $\hat{A}, B_1, B_2$ are constant matrices with appropriate dimensions and where the control of player $i$, $V_i(t) \in I\!R^{m_i}$, corresponds to a linear feedback strategy i.e.

$$V_i(t) = \alpha_i(t)Y(t), \forall t \tag{2}$$

with $\alpha_i(.)$ a continuously differentiable function from $[0,T]$ (respectively $[0,\infty[$) to $I\!R^n$.

The loss function of player $i$ is given by

$$C_i^\delta(V_i(.),V_j(.)) = \int_0^T e^{-\delta t}[Y'(t)Q_iY(t) + V_i'(t)R_iV_i(t)]\, dt \tag{3}$$

with $Q_i$, $R_i$ constant symmetric positive definite matrices, $i \neq j = 1,2$; $\delta \geq 0$; ($Y'$ denotes the transpose of $Y$).

The horizon $T$ of the game is either finite or $+\infty$.

Let us consider the new variables

$$X(t) = e^{-\frac{\delta}{2}t}Y(t) \qquad U_i(t) = e^{-\frac{\delta}{2}t}V_i(t)$$

One easily obtains the new expression of the game

$$\begin{array}{rcl} C_i(U_i(.),U_j(.)) & = & \frac{1}{2}\int_0^T (X'(t)Q_iX(t) + U_i'(t)R_iU_i(t))dt \\ \dot{X}(t) & = & AX(t) - B_1U_1(t) - B_2U_2(t) \\ X(0) & = & X_0 \end{array} \tag{4}$$

with $A = \hat{A} - \frac{\delta}{2}I$ and $U_i = \alpha_i X$.

It is well known from the pioneering paper on nonzero-sum games by Starr and Ho [10], see also Weeren et al.(Theorem 2.2)[8], that a solution $(K_1(.),K_2(.))$ defined on $[0,T]$ of the coupled Riccati differential system

$$\begin{array}{rcl} \dot{K}_1(t) & = & -K_1(t)A - A'K_1(t) \\ & & +K_1(t)S_1K_1(t) + K_1(t)S_2K_2(t) + K_2(t)S_2K_1(t) - Q_1 \\ \dot{K}_2(t) & = & -K_2(t)A - A'K_2(t) \\ & & +K_2(t)S_2K_2(t) + K_2(t)S_1K_2(t) + K_1(t)S_1K_2(t) - Q_2 \end{array} \tag{5}$$

which satisfies the terminal conditions

$$K_1(T) = 0 \text{ and } \mathrm{K_2(T) = 0} \tag{6}$$

and where $S_i = B_iR_i^{-1}B_i'$, defines a feedback Nash equilibrium for the game with finite horizon which is given by

$$(U_1^*(.),U_2^*(.)) = (R_1^{-1}B_1'K_1(.)X(.), R_2^{-1}B_2'K_2(.)X(.))$$

.

For infinite horizon, from Papavassilopoulos et al. [6], we know that a constant solution $(K_1, K_2)$ of the algebraic Riccati equation

$$\begin{array}{rcl} -K_1A - A'K_1 + K_1S_1K_1 + K_1S_2K_2 + K_2S_2K_1 - Q_1 & = & 0 \\ -K_2A - A'K_2 + K_2S_2K_2 + K_2S_1K_2 + K_1S_1K_2 - Q_2 & = & 0 \end{array} \quad (7)$$

determines a feedback Nash equilibrium $(U_1^*(.), U_2^*(.))$ where for all $t$

$$U_i^*(t) = R_i^{-1}B_i'K_iX(t)$$

if the corresponding dynamics of $X$ is stable. From Engwerda [9, Theorem 1], if $K_i \geq 0$ solve (7), then $(U_1^*(.), U_2^*(.))$ provides also a feedback Nash equilibrium, and moreover costs are given by $X_0'K_iX_0$ and the closed loop matrix $A_{cl} = A - S_1K_1 - S_2K_2$ is asymptotic stable.

In order to select a particular solution of the infinite horizon game, it seems reasonable to look at the limit of the associated finite horizon game with free terminal state. We particularly emphasize that we don't consider any penalization for the finite horizon games we associate to the infinite game (cf. Introduction. This issue will be discussed again in Section 3.2).

A simple way to study the system (5) with initial conditions (6) is to modify the time variable by reversal. More explicitly, let $(K_1^T(.), K_2^T(.))$ be a solution of (5) and (6) if it exists. Reversal of time means considering, for fixed $T$,

$$H_i^T(t) = K_i^T(T-t), \quad i = 1,2 \quad \forall t \in [0, T]$$

Then the following system

$$\begin{array}{rcl} \dot{H}_1(t) & = & H_1(t)A + A'H_1(t) \\ & & -H_1(t)S_1H_1(t) - H_1(t)S_2H_2(t) - H_2(t)S_2H_1(t) + Q_1 \\ \dot{H}_2(t) & = & H_2(t)A + A'H_2(t) \\ & & -H_2(t)S_2H_2(t) - H_2(t)S_1H_1(t) - H_1(t)S_1H_2(t) + Q_2 \end{array}$$

with initial conditions

$$H_1(0) = 0, \quad H_2(0) = 0$$

admits a unique solution on $[0, \infty[$ if and only if for all $T$ there exists a Nash equilibrium on $[0, T]$ with free terminal state.
This system admits no more that one solution on $[0, \infty[$ with initial condition $H_1(0) = 0, H_2(0) = 0$ because the system is Lipschitz.

As a consequence, when this solution exists and converge to a steady-state $(H_1^*, H_2^*)$, i.e.

$$\forall t \geq 0 \quad \lim_{T\to\infty} (K_1^T(t), K_2^T(t)) = (H_1^*, H_2^*)$$

this limit determines a feedback Nash equilibrium for the infinite horizon game when the corresponding dynamics of $X$ is stable, or when $H_i^* \geq 0$. In this case, one can characterize the unique Nash feedback equilibrium for the infinite horizon game that is the limit of the finite horizon equilibrium.

In the next section, we show that this is indeed the case in the one-dimensional case.

# 3 The one-dimensional case

We denote here the different variables by lower case letters.
Without loss of generality, we may assume $b_1 = b_2 = 1$ and $q_1 = q_2 = 1$ (redefining $u_i$ and $r_i$, $i = 1, 2$).
Then the problem becomes the study of the following system :

$$\begin{array}{rcl} \dot{h}_1(t) & = & 2ah_1(t) - h_1^2(t) - 2h_1(t)h_2(t) + \frac{1}{r_1} \\ \dot{h}_2(t) & = & 2ah_2(t) - h_2^2(t) - 2h_1(t)h_2(t) + \frac{1}{r_2} \end{array}$$

with initial conditions

$$h_1(0) = 0 \quad \textit{and} \quad h_2(0) = 0$$

In the following proposition, we establish that each finite-horizon associated game possesses a Nash Equilibrium.

**Proposition 1** *For any finite $T > 0$, the solution $(k_1^T(.), k_2^T(.))$ of (5) and (6) exists, is positive and is uniformly bounded (independently of $T$). This solution is characterized by reversal of time for the solution of*

$$\begin{array}{rcl} \dot{h}_1(t) & = & 2ah_1(t) - h_1^2(t) - 2h_1(t)h_2(t) + \frac{1}{r_1} \\ \dot{h}_2(t) & = & 2ah_2(t) - h_2^2(t) - 2h_1(t)h_2(t) + \frac{1}{r_2} \end{array} \tag{8}$$

*with initial conditions*

$$h_1(0) = 0 \quad \textit{and} \quad h_2(0) = 0 \tag{9}$$

*This means that $(k_1^T(T-t), k_2^T(T-t))$ is the restriction to $[0, T]$ of the solution $(h_1(t), h_2(t))$ of (8) and (9).*

**Proof** Clearly $(k_1^T(t), k_2^T(t))$ exists if and only if $h_i^T(t) = k_i^T(T-t)$, $i = 1, 2$, is a solution of (8) and (9), or equivalently if and only if the solution $h_i(t)$, $i = 1, 2$ of (8) and (9) exists on $[0, T]$.

Let us study this solution. Initially , at $t = 0$ , $h_1(t)$ and $h_2(t)$ are increasing (the derivatives $\dot{h}_i(0) = \frac{1}{r_i}$ are positive). Moreover, as long as the solution exists, $h_i(t)$ cannot become non-positive for $i = 1, 2$. Indeed, at the first date $t_0 > 0$ such that one $h_i(t)$ becomes non-positive, we have $h_i(t_0) = 0$ and $\dot{h}_i(t_0) = \frac{1}{r_i} > 0$; but this would imply $h_i(t) < 0$ for $t < t_0$, $t$ near $t_0$. As a consequence, as long as the solution is defined, we have for $i = 1, 2$ :

$$\dot{h}_i(t) < 2ah_i(t) - h_i^2(t) + \frac{1}{r_i}$$

This implies that $h_i(t)$ cannot becomes larger than the positive root $\bar{h}_i$ of

$$2ah_i - h_i^2 + \frac{1}{r_i} = 0$$

i.e. $\bar{h}_i = a + \sqrt{a^2 + \frac{1}{r_i}}$. The reason for that is : before reaching $\bar{h}_i$, the derivative $\dot{h}_i(t)$ becomes negative. We have shown that the solution of (8) and (9) verifies $0 < h_i(t) < \bar{h}_i$ , $i = 1, 2$ for all $t > 0$ such that the solution exists on $[0, t]$. This implies that the solution exists on the whole interval $[0, \infty[$, is positive and uniformly bounded. QED

In order to describe now the behaviour at infinity, we consider two cases. In the symmetric case, i.e. when $r_1 = r_2 = r$, the two equations (8) are identical and $h_1(.) = h_2(.) = h(.)$ is the solution on $[0, \infty[$ of the unique differential equation

$$\dot{h}(t) = 2ah(t) - 3h^2(t) + \frac{1}{r}, \quad with \quad h(0) = 0$$

whose solution is

$$h(t) = \hat{h} + \frac{1}{\lambda e^{bt} - \frac{3}{b}}$$

where $\hat{h} = \frac{1}{3}(a + \sqrt{a^2 + \frac{3}{r}})$, $b = 2\sqrt{a^2 + \frac{3}{r}} > 0$ and $\lambda = -\frac{1}{b\hat{h}}(\sqrt{a^2 + \frac{3}{r}} - a) < 0$.

The limit of $h(t)$, as $t$ goes to $+\infty$, is equal to $\hat{h}$, $(\hat{h}, \hat{h})$ is a steady state of (8) and therefore a feedback Nash equilibrium for the infinite game because $\hat{h} > 0$.

In the non symmetric case, say with $r_1 < r_2$, we consider the difference $y(t) = h_2(t) - h_1(t)$ and the sum $z(t) = h_2(t) + h_1(t)$.

Using (8) and (9), we obtain the following system

$$\begin{array}{rcl} \dot{y}(t) & = & 2ay(t) - y(t)z(t) + \delta \\ \dot{z}(t) & = & 2az(t) - \frac{3}{2}z^2(t) + \frac{1}{2}y^2(t) + \sigma \end{array} \tag{10}$$

where $\tau = \frac{1}{r_2} - \frac{1}{r_1} > 0$, $\sigma = \frac{1}{r_2} + \frac{1}{r_1} > \delta$ and with :

$$y(0) = 0 \quad and \quad z(0) = 0 \tag{11}$$

Since the solution of (8) and (9) exists on $[0, \infty[$, is positive and bounded by $\bar{h}_i = a + \sqrt{a^2 + \frac{1}{r_i}}$, $i = 1, 2$, the solution of (10) and (11) exists on $[0, \infty[$, $z(t) = h_2(t) + h_1(t)$ is positive and bounded by $\bar{z} = \bar{h}_2 + \bar{h}_1$ , and $y(t) = h_2(t) - h_1(t)$ is bounded by $\bar{y} = \bar{h}_2$.

Moreover, near $t = 0$, $y(t)$ is positive for $t > 0$ because $\dot{y}(0) = \tau > 0$ ; $y(t)$ remains positive for all $t > 0$ because $y(t) = 0$ implies $\dot{y}(t) = \tau > 0$. Thus the solution of (10) and (11) is uniquely defined on $[0, \infty[$ and it satisfies, for all $t > 0$,

$$0 < y(t) < \bar{y} \quad and \quad 0 < z(t) < \bar{z}$$

**Proposition 2** *There exists a unique steady state $(y^*, z^*)$ of the dynamics (10) such that $y^*$ is non-negative. This steady state is locally stable, and the solution of (10) with initial condition (11) converges to $(y^*, z^*)$.*

**Proof** A steady state of (10) satisfies

$$(z^* - 2a)y^* = \tau > 0$$

Thus, if $y^*$ is non-negative, $z^*$ satisfies

$$z^* > 2a$$

By substitution in the second steady state condition, we obtain that $\phi(z^*) = 0$, where

$$\phi(z) = 2az - \frac{3}{2}z^2 + \frac{\tau^2}{2(z - 2a)^2} + \sigma \tag{12}$$

For $z > 2a$, we have $\phi'(z) < 0$ and $\phi(z)$ decreases from $+\infty$ to $-\infty$ when $z$ increases from $2a$ to $+\infty$. Moreover, $\phi(0) > 0$.Thus there exists a unique solution $z^*$ of $\phi(z^*) = 0$ satisfying :

$$z^* > max\{0, 2a\}$$

One can verify that $z^* < \bar{z}$ and that $y^* = \frac{\tau}{z^*-2a} < \bar{y}$.
We have shown that there exists a unique steady state $(y^*, z^*)$ of (10) on $[0,\bar{y}] \times [0,\bar{z}]$.

The partial derivatives of (10) at this steady-state are

$$\frac{\partial \dot{y}}{\partial y} = 2a - z^*, \quad \frac{\partial \dot{y}}{\partial z} = -y^*, \quad \frac{\partial \dot{z}}{\partial y} = y^*, \quad \frac{\partial \dot{z}}{\partial z} = 2a - 3z^*$$

The sum $4a - 4z^*$ of the two eigenvalues is negative and their product $(z^* - 2a)(3z^* - 2a) + y^{*2}$ is positive. Thus the steady state is locally stable.
We now show that there exists no closed orbit. Note first that as long as $z(t) \leq \frac{4a}{3}$ in the case $a > 0$, or $z \geq 0$ in the case $a \leq 0$, we have

$$\dot{z}(t) > \frac{1}{2}y^2(t) + \sigma \geq \sigma > 0$$

Thus there exists some $t_0$ such that $z(t) > \frac{4a}{3}$ (or $z(t) > 0$ if $a \leq 0$) for all $t > t_0$. This implies that

$$\frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = 4a - 2z(t) < 0$$

for all $t > t_0$. From Bendixson's criterion (see Guckenheimer and Holmes [11]), there exists no closed orbit in the set

$$\{0 \leq y \leq \bar{y} \quad and \quad max\{0, \frac{4a}{3}\} \leq z \leq \bar{z}\}$$

But the solution $(y(t), z(t))$ of (10) and (11) belongs to this set for $t > t_0$. From Andronov et al. [12], we conclude that this solution converges to the unique steady state $(y^*, z^*)$ in this set. QED

The corresponding steady-state $(h_1^*, h_2^*)$ of the system (8) is then well determined : $h_1^* + h_2^* = z^*$ is the solution of $\phi(z^*) = 0$ (equation (12)) such that $z^* > 2a$, and we have

$$h_2^* - h_1^* = y^* = \frac{\tau}{(z^* - 2a)}$$

and

$$h_1^* = \frac{1}{2}(z^* - y^*) \text{ and } h_2^* = \frac{1}{2}(z^* + y^*)$$

## 3.1 Main Result

**Theorem 1** *The steady-state* $(h_1^*, h_2^*)$ *(resp.* $(\hat{h}_1, \hat{h}_2)$ *in the symmetric case) defines a feedback Nash equilibrium for infinite horizon of the standard one-dimensional linear-quadratic game with the strategies* $u_i(t) = h_i^* x(t)$*, (resp.* $u_i(t) = \hat{h}_i x(t)$*),* $i = 1, 2$*.*
*These strategies are the limits of the strategies of the finite horizon game : for all* $t \geq 0$ *we have*

$$\lim_{T \to +\infty} (k_1^T(t), k_2^T(t)) = (h_1^*, h_2^*) \ \ (resp. (\hat{h}_1, \hat{h}_2)) \tag{13}$$

**Proof** By definition, $(h_1^*, h_2^*)$ satisfy the system (7) and $h_i^* > 0$. We conclude that $(h_1^*, h_2^*)$ is a feedback Nash equilibrium for infinite horizon. By definition, we also have for all $t \geq 0$ and all $T > t$

$$k_i^T(t) = h_i(T - t) \quad i = 1, 2$$

The solutions $(h_1(\tau), h_2(\tau))$ of (8) and (9) converge to $(h_1^*, h_2^*)$ when $\tau$ goes to $+\infty$. This implies the limit property (13). QED

We also recall that for this game, we can obtain from Weeren et al.[8] the existence of more than one Nash Equilibrium. In the symmetric case, $r_1 = r_2 = r$, if $a > 0$ and $a^2 - r^{-1} > 0$, three Nash equilibria exist. In the non-symmetric case, Engwerda [9] gives conditions for existence of one or three Nash equilibrium.
**Interpretation :**
The unique Nash equilibrium $(h_1^*, h_2^*)$ which is selected is easy to interpret. The condition $h_i^* > 0$ means that the player $i's$ strategy stabilizes the state variable : $u_i(t) = r_i^{-1} h_i^* x(t)$ has the same sign as $x(t)$ and the speed of convergence to 0 of $x$, $-a + h_1^* + h_2^*$, is increased. The condition $h_2^* > h_1^*$ means a larger effort for stabilization is exerted by player 2, who has the lower cost of control (as $r_2 < r_1$).

## 3.2 Discussion on this selection principle

There is an important dynamic difference between the symmetric and the non symmetric case : the selected steady state is locally stable in the symmetric case but unstable (in the sense of saddle point) in the non symmetric case. We elaborate on this point below.

If one wants to study the robustness of this principle of selection with perturbations on the initial value : $h_1(0) = \epsilon_1$ and $h_2(0) = \epsilon_2$, things go wrong in the symmetrical case with asymmetric perturbations. More precisely in the asymmetric case, i.e. with $r_2 \neq r_1$, the stability of the steady state implies that for all sufficiently small perturbations, the same limit is obtained.

In the symmetric case, if we consider correlated perturbations, i.e. $\epsilon_1 = \epsilon_2$, the problem remains symmetric and we obtain the selected Nash equilibrium, which is a saddle point for the two-dimensional dynamics, but is stable for the one-dimensional dynamics resulting from symmetry.

Now for the symmetric case with independent perturbations, say for $\epsilon_1 < \epsilon_2$, the resulting dynamics is truly two-dimensional and we obtain convergence to an equilibrium that is not the selected one. But it is important to recall that the finite horizon games we associate to the infinite horizon game have no reason to reflect an end-state penalty of the sort implied by our perturbations here.

All together then, it is only in the symmetric case with independent perturbations that difficulties arise when one attempts to obtain the selected equilibrium numerically.

An open question is suggested by our results. Consider a criterion leading to an unstable path for the original system, but at the same time allowing for the definition of an equivalent subsystem that is stable. Should this criterion be considered valid and applicable ?

### 3.3 A final remark on the discounted case

We recall that in this one-dimensional case the game is

$$\begin{array}{rcl} C_i^\delta(v_i(.), v_j(.)) & = & \frac{1}{2}\int_0^\infty e^{-\delta t}(y^2(t) + r_i v_i^2(t))dt \\ \dot{y}(t) & = & \hat{a}y(t) - v_1(t) - v_2(t) \\ y(0) & = & y_0 \end{array} \tag{14}$$

and the undiscounted associated game has a state evolution given by

$$\dot{x}(t) = (\hat{a} - \frac{\delta}{2})x(t) - u_1(t) - u_2(t)$$

As in the selection process studied before $a := \hat{a} - \frac{\delta}{2}$ does not matter, and all the preceding results apply. We have therefore for this infinite horizon game selected a unique Nash Equilibrium $(v_1^*(.), v_2^*(.))$ which corresponds to the game obtained by our main result.

As $C_1^\delta(v_1^*(.), v_2^*(.)) = C_1(u_1^*(.), u_2^*(.))$, the objective functions $C_1^\delta(v_1^*(.), v_2^*(.))$ converges. Now from the stability of the state's equation we have

$$\hat{a} - \frac{\delta}{2} - (h_1^* + h_2^*) < 0$$

and therefore in terms of the coordinnates $y, v_1, v_2$ the selected process requires that

$$\hat{a} - (h_1^* + h_2^*) < \frac{\delta}{2}$$

then as $\hat{a} - (h_1^* + h_2^*) > 0$ is feasible, the dynamic associated to this Nash equilibrium given by (14) is no longer asymptotically stable, the state grows infinite with time. For instance, in the symmetric case ($r_1 = r_2$), this is the case if $a^2 + \frac{4}{r} < 2a\delta$.
Finally, from the maximum principle we also derive that the costate variable $p_i(t)$ goes to infinity with time ($p_i(t) = h_i^* x(t)$). Therefore in this case, for all finite horizon associated games, we have the transversality condition $p_i^T(T) = 0$ but $\lim_{t\to\infty} p_i(t) = +\infty$.

# 4 Conclusion

We have used a simple method, that reversing the dynamics allows one to characterize the feedback Nash equilibrium for finite horizon $T$ of the standard one-dimensional linear-quadratic game. Assuming a free terminal state, time being reversed, this equilibrium is the restriction to the interval $[0, T]$ of the solution of the simple system (8) and (9). When time goes to infinity, this solution converges to a well determined steady state, and this steady state defines the linear strategies of a feedback Nash equilibrium for infinite horizon. In the standard one-dimensional linear-quadratic game, this limit property gives an interesting method for selecting one Nash feedback equilibrium for infinite horizon which can be easily interpreted. To be robust by numerical computations in the symmetric case, this criterium assume that perturbations are symmetric in such a way that the dynamics is defined by a one-dimensional equation. This selection criterion is both simple and meaningful, because the infinite horizon game must be interpreted as the limit of finite horizon games.

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