Exact age from two passive tracer concentrations

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Many attempts have been made to estimate tracer or water ages from formulas involving only measured or simulated tracer concentration(s). In most, if not all, cases these age formulations are based on the hypothesis that diffusive processes are negligible. Unfortunately, they rarely are, implying that such formulas may fail to meet their declared objective, leading to an age that is significantly different from the value it was meant to approximate (e.g. Beckers et al. 2001, Deleersnijder et al 2001, Delhez et al. 2003, Delhez and Deleersnijder 2008, Mouchet and Deleersnijder 2008). There are special situations, however, in which the age obtained from tracer concentrations alone is correct whatever the impact of diffusion. An example thereof is discussed in the present working note.

Tracer injection and transport

Let t, $\mathbf{x}$ and Ω denote the time, the position vector and the domain of interest. The latter is limited by the boundary Γ, which is assumed to be impermeable. The velocity field $\mathbf{v}(t,\mathbf{x})$ is divergence-free,

$$\nabla \bullet \mathbf{v} = 0 , \tag{1}$$

and satisfies the impermeability condition

$$\left[\mathbf{v} \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma} = 0 , \tag{2}$$

where the vector $\mathbf{n}$ is the outward unit normal to the boundary of Ω.

Passive tracers are injected into the domain by a point source located at $\mathbf{x}_s$ ($\mathbf{x}_s \in \Omega$). The concentration $C_i(t,\mathbf{x})$ (i=1,2,...) of each tracer is governed by the following equation:

$$\frac{\partial C_i}{\partial t} = q_i\,\delta(\mathbf{x}-\mathbf{x}_s) - \nabla \bullet (\mathbf{v}C_i - \mathbf{K} \bullet \nabla C_i) , \tag{3}$$

where $q_i(t)$ (≥ 0) is the injection rate of the i-th tracer into the domain; $\mathbf{K}(t,\mathbf{x})$ is the diffusivity tensor, which must be symmetric and positive-definite (e.g. Deleersnijder et al. 2001). At the initial instant (t=0), the domain contains no tracer:

$$C_i(0,\mathbf{x}) = 0 . \tag{4}$$

Relation (2) implies that no advective tracer flux crosses the domain boundary. As the latter is impermeable, it is also necessary that the diffusive tracer flux normal to Γ be zero:

$$\left[(-\mathbf{K} \bullet \nabla C_i) \bullet \mathbf{n}\right]_{\mathbf{x}\in\Gamma} = 0 . \tag{5}$$

Clearly, the amount of every tracer present in the domain,

$$m_i(t) = \int_\Omega C_i(t,\mathbf{x})\, d\Omega , \tag{6}$$

increases as follows

$$m_i(t) = \int_0^t q_i(t')\, dt' . \tag{7}$$

This result may be obtained by integrating (3) over the domain of interest and taking into account the impermeability conditions (2) and (5) (Appendix A).

Now assume that for interpreting the transport properties of the flow under consideration it is appropriate to evaluate the age of some of the tracers, *the age of a tracer particle being defined as the time that has elapsed since it was introduced into the domain.* Accordingly, at time t, the age of the tracer parcels injected by the point source during the time interval $[t-t', t-t'+\Delta t']$ tends to t' as $\Delta t' \to 0$. The amount of tracer injected into the domain during the aforementioned time interval obviously is asymptotic to $q_i(t-t')\Delta t'$. Then, using the age-averaging hypothesis introduced by Deleersnijder et al. (2001) as a key ingredient of the Constituent-oriented Age and Residence time Theory[1] (CART, www.climate.be/cart), the mean age at time t of all tracer particles present in the domain of interest is the mass-weighted arithmetic mean of the ages of individual tracer particles. Therefore, the contribution to this mean age of the tracer particles released into the domain by the point source during the time interval $[t-t', t-t'+\Delta t']$ is $t' q_i(t-t')\Delta t' / m_i(t)$, and, in the limit $\Delta t' \to 0$, the sum of all such contributions is the integral

$$\bar{a}_i(t) \ = \ \frac{1}{m_i(t)} \int_0^t t' q(t-t')\, dt' \ . \tag{8}$$

However, this average age is independent of transport processes and, hence, is of no use to diagnose them. Time- and position-dependent ages are needed, $a_i(t,\mathbf{x})$, that, in accordance with the aforementioned age-averaging hypothesis, must satisfy

$$\bar{a}_i(t) \ = \ \frac{1}{m_i(t)} \int_\Omega C_i(t,\mathbf{x})\, a_i(t,\mathbf{x})\, d\Omega \ . \tag{9}$$

That (8) and (9) are consistent with eachother is demontrated in Appendix B.

Deriving the age from the Green function

The transport problem under consideration admits the Green function $G(t,t_r,\mathbf{x})$ that is the solution to the following partial differential problem:

$$\frac{\partial G}{\partial t} \ = \ \delta(\mathbf{x}-\mathbf{x}_s)\delta(t-t_r) \ - \ \nabla \bullet (\mathbf{v}G - \mathbf{K} \bullet \nabla G) \ , \tag{10}$$

$$G(t,t_r,\mathbf{x}) = 0 \ , \quad t - t_r \le 0 \ , \tag{11}$$

$$\left[(-\mathbf{K} \bullet \nabla G) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma} \ = \ 0 \ . \tag{12}$$

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[1] Consider, for instance, two particles that are identified by means of subscripts "A" and "B". Their mass and age are denoted m^X and a^X, respectively, with $X=A,B$. The mass of the system "A+B" obvioulsy is $m^{A+B} = m^A + m^B$. This is in agreement with basic physical principles that stipulate that mass is an additive quantity. No such principle exists for the age. Therefore, to obtain the mean age of the system "A+B", an arbitrary decision is to be made. The latter was formulated as the so-called age-averaging hypothesis (Deleersnijder et al., 2001), which is the only arbitrary element of CART. It stipulates that the mean age of a system made up of various particles is the mass-weighted average of the ages of the particles. Accordingly, the mean age of the system "A+B", a^{A+B}, satisfies the follow relation: $m^{A+B}a^{A+B} = m^A a^A + m^B a^B$. Clearly, the age is not an additive quantity, but the age content is — the age content of a particle being defined as the product of its mass and its age. Ages other than CART's, such as the carbon-14 age, implicitly or explicitly rely on an age-averaging hypothesis (Deleersnijder et al., 2001). However, it is believed that CART's age-averaging hypothesis is the simplest one could think of.

Thus, at time t and point $\mathbf{x}$, $G(t,t_r,\mathbf{x})$ is the "concentration" of a tracer of which a unit quantity was suddenly released at time t_r at point $\mathbf{x}_s$.

At time t, the concentration of the i-th tracer injected into the domain by the point source during the time interval $[t-t',t-t'+\Delta t']$ tends to $q_i(t-t')G(t,t-t',\mathbf{x})\Delta t'$ as $\Delta t' \to 0$. Therefore, the total concentration is the sum of all such contributions, which, in the limit $\Delta t' \to 0$, is[2]

$$C_i(t,\mathbf{x}) \;=\; \int_0^t q_i(t-t')G(t,t-t',\mathbf{x})\,dt' \;. \tag{13}$$

By the same token, t' is the age of the particles of the i-th tracer released by the point source during the time interval $[t-t',t-t'+\Delta t']$. Then, using the age-averaging hypothesis, one may derive the mean age[3] of the tracer particles located at point $\mathbf{x}$ at time t:

$$a_i(t,\mathbf{x}) \;=\; \frac{1}{C_i(t,\mathbf{x})}\int_0^t t'q_i(t-t')G(t,t-t',\mathbf{x})\,dt' \;. \tag{14}$$

If the Green function is known, one can derive the concentration and the age of any passive tracer released by a point source located at $\mathbf{x}_s$, whatever the rate of injection $q_i(t)$. However, obtaining the Green function from *in situ* data may not be trivial as a long, high-resolution time series of a measured tracer concentration are needed.

To derive the Green function at location $\mathbf{x}$ from *in situ* data, one could suddenly release a unit amount of a passive tracer at point $\mathbf{x}_s$ and, then, measure at location $\mathbf{x}$ the concentration of this tracer. The latter should be measured for a long period of time with a sufficiently large sampling frequency. This is not always feasible, especially if the flow is time-dependent, which is why other options for estimating tracer ages have been looked for.

A two-tracer case study

Consider two tracers, the first one being released at a constant rate,

$$q_1(t) = Q \;, \tag{15}$$

while the rate of injection of the second increases linearly in time,

$$q_2(t) \;=\; \frac{t}{\tau}Q \;, \tag{16}$$

where Q is a constant; the constant τ is an appropriate timescale. Now assume that *the age of the first tracer is to be determined solely from concentration measurements.*

A few decades ago, i.e. before the development of general theories of tracer transport and the associated timescales, one would have relied on an *ad hoc* approach for reaching the goal set above. It is likely that one would have considered a one-dimensional flow without diffusion in an infinite pipe ($x \in]-\infty,\infty[$). The corresponding velocity is the constant U,

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[2] For the present problem, CART's concentration distribution function (Delhez et al. 1999), $c_i(t,\mathbf{x},t')$, is equal to $q_i(t-t')G(t,t-t',\mathbf{x})\,\upsilon(t-t')$, where $\upsilon(t-t')$ is the Heaviside step function, which is equal to 1 or 0 according to whether $t-t'$ is positive or negative.

[3] The integral $\int_0^t t'\,q_i(t-t')G(t,t-t',\mathbf{x})\,dt'$ actually is CART's age concentration, which is denoted $\alpha_i(t,\mathbf{x})$.

which can be assumed to be positive without any loss of generality. Setting the point source at x=0, for $x \in [0,Ut]$, the tracer concentrations and the age of the first tracer are readily seen to be

$$C_1(t,x) = \frac{Q}{U}, \tag{17}$$

$$C_2(t,x) = \frac{Ut-x}{U\tau}\frac{Q}{U} \tag{18}$$

and

$$a_1(t,x) = \frac{x}{U}. \tag{19}$$

For $x \notin [0,Ut]$ no tracer concentration and age are defined, as there is no tracer particle outside the subdomain defined by $x \in [0,Ut]$. Combining (17)-(18) easily yields

$$a_1(t,x) = t - \frac{C_2(t,x)}{C_1(t,x)}\tau, \tag{20}$$

an expression that holds true for $x \in [0,Ut]$ and is irrelevant in the rest of the domain. Though formula (20) is valid only for a one-dimensional flow with no-diffusion, one would have sought inspiration in it, leading to the assumption that the age of the first tracer could be estimated at any time and location in realistic flow domain from a similar formula, i.e.

$$\boxed{a_1(t,\mathbf{x}) = t - \frac{C_2(t,\mathbf{x})}{C_1(t,\mathbf{x})}\tau} \tag{21}$$

Surprisingly, (21) provides the exact age of the first tracer, i.e. the age that is in accordance with modern age theories. This can be seen by manipulating the formulas that were established above by means of the Green function.

Substituting (15)-(16) into (13)-(14) leads to

$$C_1(t,\mathbf{x}) = Q\int_0^t G(t,t-t',\mathbf{x})\,dt', \tag{22}$$

$$C_2(t,\mathbf{x}) = \frac{Q}{\tau}\int_0^t (t-t')G(t,t-t',\mathbf{x})\,dt' \tag{23}$$

and

$$a_1(t,\mathbf{x}) = \frac{\int_0^t t'G(t,t-t',\mathbf{x})\,dt'}{\int_0^t G(t,t-t',\mathbf{x})\,dt'}. \tag{24}$$

Then, using (22)-(24), the ratio of the tracer concentrations may be transformed as follows:

$$\frac{C_2(t,\mathbf{x})}{C_1(t,\mathbf{x})} = \frac{1}{\tau}\frac{\int_0^t (t-t')G(t,t-t',\mathbf{x})\,dt'}{\int_0^t G(t,t-t',\mathbf{x})\,dt'} = \frac{t}{\tau} - \frac{1}{\tau}\underbrace{\frac{\int_0^t t'G(t,t-t',\mathbf{x})\,dt'}{\int_0^t G(t,t-t',\mathbf{x})\,dt'}}_{a_1(t,\mathbf{x})} = \frac{t}{\tau} - \frac{a_1(t,\mathbf{x})}{\tau}, \tag{25}$$

which is readily seen to be equivalent (21). QED.

Combining (7), (8) and (15) leads to $m_1(t) = Qt$ and $\overline{a}_1(t) = t/2$. Similarly, the amount of the second tracer present in the domain and the mean age thereof are $m_2(t) = Qt^2/(2\tau)$ and $\overline{a}_2(t) = t/3$, respectively.

The main result obtained in this working note, i.e. formula (21), might hold true under hypotheses less that are restrictive than those invoked herein. Determinig the hypotheses to be relaxed and establishing the related theoretical results depends on the tracing experiments that one actually wishes to conduct in the field.

Appendix A

It is to be seen that (7) holds valid. To do so, one must first integrate the tracer budget equation (3) over the domain of interest, leading to

$$\underbrace{\frac{d}{dt}\int_{\Omega} C_i(t,\mathbf{x})\, d\Omega}_{\substack{=m_i(t) \\ \text{by virtue of (6)}}} \;=\; q_i(t) \underbrace{\int_{\Omega} \delta(\mathbf{x}-\mathbf{x}_s)\, d\Omega}_{=1} \;-\; \int_{\Omega} \nabla \bullet (\mathbf{v}C_i - \mathbf{K} \bullet \nabla C_i)\, d\Omega \;. \tag{A.1}$$

Then, using the divergence theorem, the last term in (A.1) may be seen to be zero:

$$\begin{aligned} \int_{\Omega} \nabla \bullet (\mathbf{v}C_i - \mathbf{K} \bullet \nabla C_i)\, d\Omega \;&=\; \int_{\Gamma} (\mathbf{v}C_i - \mathbf{K} \bullet \nabla C_i) \bullet \mathbf{n}\;\, d\Gamma \\ &=\; \int_{\Gamma} \underbrace{(\mathbf{v}C_i) \bullet \mathbf{n}}_{\substack{=0,\text{ according} \\ \text{to (2)}}}\;\, d\Gamma \;+\; \int_{\Gamma} \underbrace{(-\mathbf{K} \bullet \nabla C_i) \bullet \mathbf{n}}_{=0,\text{ according to (5)}}\;\, d\Gamma \;=\; 0 \end{aligned} \tag{A.2}$$

Thus, (A.1) reduces to $dm_i/dt = q_i$, which leads the global tracer mass budget (7). QED.

Appendix B

To demonstrate that relation (9) is consistent with (8), one must first recall a key property of the Green function. The mass budget (7) applies to the Green function, as the latter is the solution to a problem that is similar to (2)-(5). In this case, the amount of tracer present at any time $t > 0$ is equal to unity, i.e.

$$\int_{\Omega} G(t, t-t', \mathbf{x})\, d\Omega \;=\; 1 \;. \tag{B.1}$$

Next, one substitutes (22) and (24) into (9), leading to

$$\begin{aligned} \overline{a}_i(t) \;&=\; \frac{1}{m_i(t)} \int_{\Omega} C_i(t,\mathbf{x})\, a_i(t,\mathbf{x})\, d\Omega \\ &=\; \frac{1}{m_i(t)} \int_{\Omega} \int_0^t t' q(t-t') G(t, t-t', \mathbf{x})\, dt'\, d\Omega \end{aligned}$$

$$= \frac{1}{m_i(t)} \int_0^t t' q(t-t') \underbrace{\int_\Omega G(t,t-t',\mathbf{x})\, d\Omega}_{\substack{=1,\ \text{according} \\ \text{to (B.1)}}} \; dt' \qquad \text{(B.2)}$$

$$= \frac{1}{m_i(t)} \int_0^t t' q(t-t') \; dt'$$

QED.

References

Beckers J.-M., E.J.M. Delhez and E. Deleersnijder, 2001, Some properties of generalised age-distribution equations in fluid dynamics, *SIAM Journal on Applied Mathematics*, 61, 1526-1544

Deleersnijder E., J.-M. Campin and E.J.M. Delhez, 2001, The concept of age in marine modelling: I. Theory and preliminary model results, *Journal of Marine Systems*, 28, 229-267

Delhez E.J.M. and E. Deleersnijder, 2008, Age and the time lag method, *Continental Shelf Research*, 28, 1057-1067

Delhez E.J.M., J.-M. Campin, A.C. Hirst and E. Deleersnijder, 1999, Toward a general theory of the age in ocean modelling, *Ocean Modelling*, 1, 17-27

Delhez E.J.M., E. Deleersnijder, A. Mouchet and J-M. Beckers, 2003, A note on the age of radioactive tracers, *Journal of Marine Systems,* 38, 277-286

Mouchet A. and E. Deleersnijder, 2008, The leaky funnel model, a metaphor of the ventilation of the World Ocean as simulated in an OGCM, *Tellus,* 60A, 761-774
