

SMOOTH TRANSITION GARCH MODELS: A BAYESIAN
PERSPECTIVE

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December 1998

Abstract

This paper proposes a new kind of asymmetric GARCH where the conditional variance obeys two different regimes with a smooth transition function. In one formulation, the conditional variance reacts differently to negative and positive shocks while in a second formulation, small and big shocks have separate effects. The introduction of a threshold allows for a mixed effect. A Bayesian strategy, based on the comparison between posterior and predictive Bayesian residuals, is built for detecting the presence and the shape of nonlinearities. The method is applied to the Brussels and Tokyo stock indexes. The need for an alternative parameterisation of the GARCH model is emphasised as a solution to numerical problems.

Keywords: Bayesian, asymmetric GARCH, specification tests, nonlinear modelling, stock indexes.

JEL classification: C11, C22, C51, G14.

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The basic idea of this paper (smooth transition GARCH) grew out during the visit of Timo Terasvirta at GREQAM in September 1996. Once a first version of this paper was completed and presented at EC², Florence December 1996, it appeared that a similar effort has been pursued in a classical framework by Gonzales-Riviera (1996) and by Hagerud (1997). Some of the ideas concerning the reparameterisation of GARCH models have been explored in discussions with Rob Engle during his visit in Marseille in June 1992, conversations later pursued with Jean-François Richard. I am grateful to participants of seminars in Marseille, Bordeaux, Louvain, Maastricht, to participants to the conference EC2 “Simulation Methods in Econometrics” held in Florence, December 1996, to the Krakow Workshop on Bayesian Econometrics and Statistics, May 1997. Many discussions and critics with Luc Bauwens helped a lot in the writing of the paper. Usual disclaimers apply.

Support of the European Commission Human Capital and Mobility Program through the network “Econometric inference using simulation methods” is gratefully acknowledged. This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister’s Office, Science Policy Programming. The scientific responsibility is assumed by the author.

1 Introduction

Financial series are characterised by periods of large volatility followed by periods of relative quietness. This type of clustering led to the idea that volatility is predictable, which is of primary importance for option pricing, for portfolio selection and for designing optimal dynamic hedging strategies. The ARCH and GARCH models introduced respectively by Engle (1982) and by Bollerslev (1986) were quite successful in predicting volatility compared to more traditional methods as underlined for instance in Engle, Kane and Noh (1993) or Noh, Engle and Kane (1995). But better predictions are obtained when asymmetries [Engle and Ng (1993)] and non linearities [Pagan and Schwert (1990)] in the response of volatility to news arriving on the market are taken into account. The “leverage effect”¹ that commonly represents the asymmetric impact of good news and bad news on volatility has certainly been the most widely explored with Nelson (1991) who proposed his EGARCH model or Glosten et al (1993) and also Zakoian (1994) who proposed a threshold GARCH or TGARCH. Engle and Ng (1993) provided a summary of asymmetric GARCH models and introduced some new formulations. They tested these models to the daily return series of the Japanese TOPIX index covering 1980-1988. If on average these models perform better than the symmetric GARCH, they found evidence of misspecification in all the models, even if the TGARCH was one of the most successful. This suggests that there is still some room for finding a flexible nonlinear modelling of the skedastic function.

The object of this paper is to study a new class of GARCH models that introduces a smooth transition between two regimes defined by a threshold. I named this model STGARCH for Smooth Transition GARCH. As financial data have very often a high frequency of observation, a smooth transition seems a priori better than an abrupt transition. Engle and Ng (1993) found that the most severe misspecification direction was that the tested models did not take adequately account of the sign asymmetry. The smooth transition model that is proposed in this paper addresses the problem of sign asymmetry. But it is more than a simple generalisation of the TGARCH as it allows for various transition functions that assure a great flexibility to the skedastic function, taking into account sign but also size effects. Finally the specification retained accepts the simple GARCH as a restriction.

The approach of the paper is Bayesian in its spirit. When a nonlinearity is introduced in a model, the likelihood function becomes tricky to maximise as it may be non-differentiable. In such a case, averaging is more secure than maximising, even if the computational burden may be severe. In this paper I shall use a version of the Gibbs sampler that is the Griddy Gibbs sampler developed in Bauwens and

¹The term “leverage effect” comes from the fact that a decrease in the stock price would increase the financial leverage of the firm. This implies a negative correlation between volatility and past returns. In GARCH models, the conditional variance h_t is a function only of the squares of past errors u_{t-j}^2 and not of their signs, which precludes any correlation between h_t and u_{t-j} .

Lubrano (1998). Specification tools appeared rapidly as a necessity because of the computation burden involved in trying to fit different types of nonlinear models. A Bayesian specification search is thus developed in the paper. The strategy is based on the comparison between the posterior and the predictive variance of the Bayesian residuals. It draws on Bauwens and Lubrano (1991). The Bayesian approach provide a nice and convenient framework and the test regressions proposed are compared in the paper to some of the existing tests available in the literature.

The paper is organised as follows. In section 2, I present the general smooth transition GARCH with appropriate transition functions for sign and size asymmetries. In section 3, I compare the impact curves of these different models and stress the impact of a threshold. In section 4, I introduce the methodology for discriminating between different types of asymmetry. In section 5, I study Bayesian inference in GARCH and STGARCH models, insisting on various available parameterisation for the GARCH and on the fact that the posterior density of the transition parameter in STGARCH is not integrable under a flat prior. The next two sections are devoted to empirical applications concerning the Brussels and the Tokyo stock indexes. A last section concludes.

2 A general class of smooth transition GARCH models

The simple GARCH(1,1) is certainly the most widely used model for predicting the volatility of financial series [see Bollerslev, Chou and Kroner (1992) for a good review on the topic]. The regression model with GARCH(1,1) errors can be written as follows:

$$\begin{cases} y_t = x_t' \delta + u_t \\ u_t = \epsilon_t \sqrt{h_t} & \epsilon_t \sim N(0, 1) \\ h_t = \omega + \alpha u_{t-1}^2 + \beta h_{t-1} \end{cases} \quad (1)$$

Most of the time, x_t contains a constant and lagged values of y_t . In (1), the conditional expectation of y_t is $x_t' \delta$ and the unpredicted part of y_t is $y_t - x_t' \delta$. This represents the “news” arriving on the market as defined by Engle and Ng (1993). In the GARCH model, news have a symmetric impact on volatility, whatever their sign or magnitude and whatever the level of y_t . I shall use this model as a starting point to introduce asymmetry and level effects on volatility.

2.1 Some existing models with asymmetry

The possibility of an asymmetric impact has for long been suspected in the literature and confirmed among others by Nelson (1991). The EGARCH model

constitutes the first introduction of an asymmetric effect between negative and positive shocks in an econometric model of volatility. It writes:

$$\log(h_t) = \omega + \alpha g(\epsilon_{t-1}) + \beta \log(h_{t-1}) \quad (2)$$

where the formulation in logarithm relaxes the usual positivity constraints on the parameters. The asymmetric effect is introduced by the nonlinear function $g(\epsilon_t)$:

$$g(\epsilon_t) = \theta \epsilon_t + \gamma[|\epsilon_t| - E(|\epsilon_t|)] \quad (3)$$

which is a function of both the magnitude and the sign of ϵ_t . By construction it has a zero mean. For positive ϵ 's, $g(\epsilon_t)$ is a linear function with slope $\theta + \gamma$. For negative ϵ 's, the slope becomes $\theta - \gamma$. θ allows for possible negative correlation between ϵ_t and future values of the skedastic function h_t (leverage effect). Under the normality assumption, $E(|\epsilon_t|) = \sqrt{2/\pi}$.

Many different models followed in the literature that built on the success of the EGARCH model to represent the asymmetric response of h_t to negative and positive shocks. See Engle and Ng (1993) for a review. The model of Glosten et al (1993) is directly related to the present paper²:

$$h_t = \omega + \alpha_1 u_{t-1}^2 (1 - S_{t-1}) + \alpha_2 u_{t-1}^2 S_{t-1} + \beta h_{t-1} \quad (4)$$

S_t is an indicator function that is zero when u_t is negative and one otherwise. This formulation introduces asymmetry of reaction for the conditional variance. The change of regime occurs when u_t crosses the threshold zero. I call this model a threshold GARCH or TGARCH³. Compared to the EGARCH model, moments are simpler to compute as underlined in Pagan (1996).

2.2 Smooth transition between negative and positive shocks

Threshold GARCH models can be generalised using a smooth transition function $F(u_{t-1}, \gamma)$ taking continuous values between zero and one. The parameter γ governs the smoothness of the transition. Using this smooth transition function, the two regime skedastic function in (4) becomes:

$$\begin{aligned} h_t &= \omega + \alpha_1 u_{t-1}^2 [1 - F(u_{t-1}, \gamma)] + \alpha_2 u_{t-1}^2 F(u_{t-1}, \gamma) + \beta h_{t-1} \\ &= \omega + \alpha_1 u_{t-1}^2 + \lambda u_{t-1}^2 F(u_{t-1}, \gamma) + \beta h_{t-1} \end{aligned} \quad (5)$$

²Zakoian (1994) suggested a model equivalent to:

$$\sqrt{h_t} = \omega + \alpha_1 u_{t-1} (1 - S_{t-1}) + \alpha_2 u_{t-1} S_{t-1} + \beta h_{t-1}$$

The news impact curve of this model is different from that of Glosten et al (1993) due to the squaring of h_t . It is not minimum at $u = 0$.

³Zakoian (1994) used TARCH for his model.

where $\lambda = (\alpha_2 - \alpha_1)$. Among the many possible odd smooth transition functions⁴, the logistic function was proved to be very convenient in classical nonlinear modelling by Terasvirta (1994). For a smooth transition GARCH where the objective is to allow for a possible difference of reaction between negative and positive shocks, this function is:

$$F(u_{t-1}, \gamma) = \frac{1}{1 + \exp(-\gamma u_{t-1})} \quad (6)$$

The function $F(\cdot)$ tends to zero when $u \rightarrow -\infty$ and to one for $u \rightarrow +\infty$. So α_1 will characterise negative shocks and α_2 positive ones. For $\gamma \rightarrow \infty$, the transition function becomes equivalent to the Dirac function S_t defined above. This formulation is more flexible than Glosten et al (1993) or Zakoian (1994) as it allows a gradual transition that may be an important feature for high frequency data. For a similar suggestion see Gonzales- Riviera (1996) or Hagerud (1997). I shall call this model LSTGARCH for logistic STGARCH.

2.3 Smooth transition between small and big shocks

A second type of asymmetry is interesting to consider. Periods of important movements are not long in financial series. For instance the great crash of October 1987 gave birth to a great variance in the S&P500, but those movements were dampened very quickly. As underlined by Engle and Mustapha (1992), there seems to be a variable persistence of shocks, small shocks being more persistent than big shocks. Asymmetry between small and big shocks is also considered in Susmel and Engle (1994) for the New-York and London equity market. An even⁵ transition function like the exponential function

$$F(u_{t-1}, \gamma) = 1 - \exp(-\gamma u_{t-1}^2) \quad (7)$$

was made popular by Terasvirta (1994) for modelling size asymmetries in models presenting a non linearity in the mean. Here $F(\pm\infty) = 1$ and $F(0) = 0$. So α_1 will characterise small shocks and α_2 big shocks. For $\gamma \rightarrow \infty$, $F(\cdot)$ becomes an indicator function for the point $u = 0$, which makes our model equivalent to the symmetric GARCH. Hagerud (1997) suggested to use this transition function for size asymmetry in GARCH models. However, this simple exponential function is rather restrictive as it does not give an information on the magnitude of what is really a big shock. An improvement over this function⁶ that says that F goes to zero if u belongs to the interval $[-c, c]$ and goes to one otherwise is given by:

$$F(u_{t-1}, \gamma, c) = \frac{1 - \exp(-\gamma u_{t-1}^2)}{1 + \exp[-\gamma(u_{t-1}^2 - c^2)]} \quad (8)$$

⁴An odd function verifies $f(-x) = -f(x)$. For the logistic function, $F(x) - 1/2$ is odd.

⁵An even transition function verifies $f(-x) = f(x)$.

⁶See the paper of Jansen and Terasvirta (1996) for another type of generalisation of the exponential transition function used for non linear modelling of the mean.

I have now introduced an extra parameter, the threshold c , that determines at which magnitude of past errors the change of regime occurs. The range of c is restricted to positive values for an identification purpose.

The exponential transition function (7) can also be generalised by introducing a threshold parameter c

$$F(u_{t-1}, \gamma, c) = 1 - \exp[-\gamma(u_{t-1} - c)^2] \quad (9)$$

The parameter c translates the exponential curve so that $F(\cdot)$ is now minimum at $u_t = c$. This introduces the possibility of an asymmetry between positive and negative shocks that is now combined with the size effect as will be detailed in the next section.

3 Comparing models through news impact curves

I have proposed a wide range of volatility models that react asymmetrically to the arrival of news using a smooth smooth transition between positive and negative news or between big and small news. A clear idea about the differences between these models is provided by the inspection of the “news impact curve” as defined by Engle and Ng (1993). The news impact curve gives the relation between h_t and u_{t-1} holding constant the other informations. It shows how new information is incorporated to the measure of volatility. We shall see that the different model I have proposed have very different news impact curves. This means that predicted volatility, at least in the short term will be very different according to the model which is used. Consequently valuation of option, hedging may be also rather different.

The news impact curve of the STGARCH is

$$h = \omega + \alpha_1 u^2 + \lambda u^2 F(u, \gamma) + \beta \bar{h} \quad (10)$$

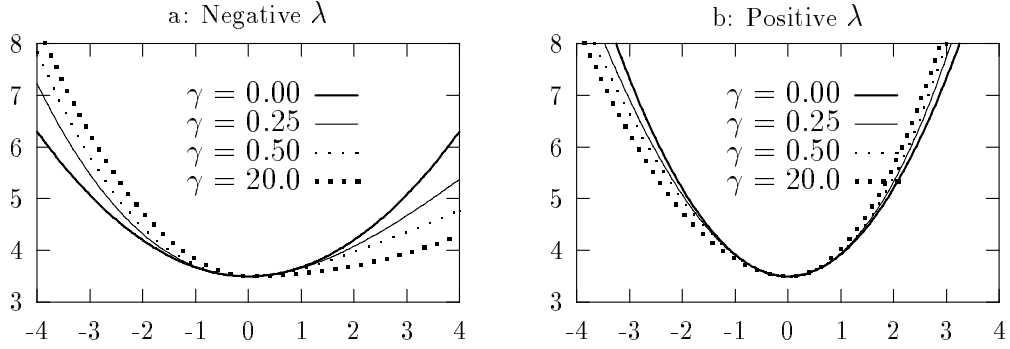
where $\bar{h}$ is set equal to the unconditional mean of the returns. Fixing ω , α_1 , γ and β and giving a value to $\bar{h}$, the function h can be drawn for the whole range of possible values for u . In the next subsection, I shall take a range for u equal to $[-4, 4]$. I shall fix the parameters at the following values: $\omega = 0.25$, $\alpha_1 = 0.3$, $\lambda = -0.25$, $\beta = 0.6$. The parameters γ and c will vary from case to case.

3.1 Logistic and exponential

Let me now draw various news impact curve for specified values of the parameters.

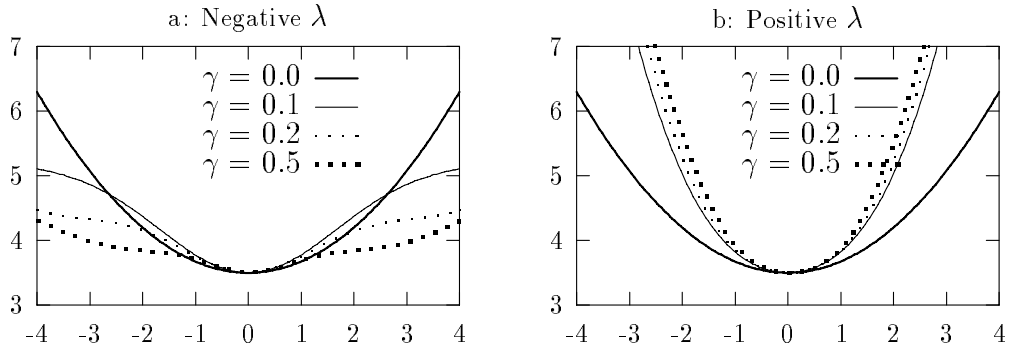
The news impact curve of the simple GARCH is symmetric for negative and positive news. The logistic transition function operates a rotation around $u = 0$ in the LSTGARCH, on the right for negative a λ (see Figure 1.a) or on the left

Figure 1: News impact curves for LSTGARCH



for a positive λ (see Figure 1.b). The importance of the rotation is determined by γ but has a limit given by the TGARCH impact curve. The impact curve of the EGARCH model has a similar general shape, but its tails are of course of the exponential form.

Figure 2: News impact curves for ESTGARCH

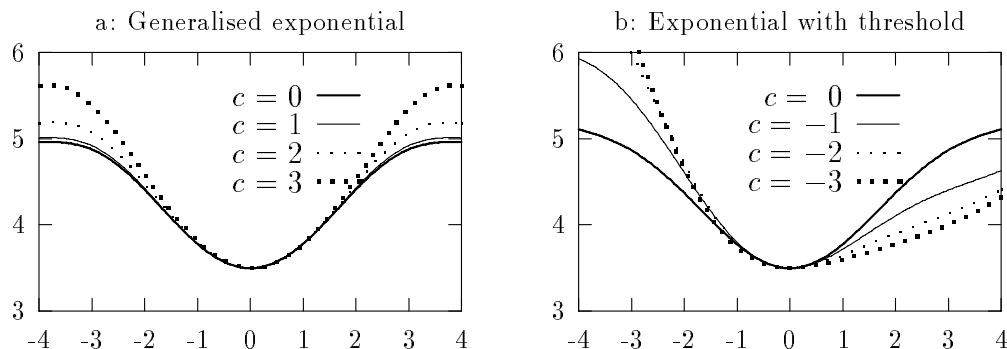


The exponential transition function (with $c = 0$) gives more weight to small news compared to the symmetric GARCH, and softens greatly the impact of big news in the ESTGARCH (see Figure 2.a). The point at which the impact curve crosses the impact curve of the simple GARCH depends on γ . But γ monitors also smoothness of the transition. The model is relatively constrained as for either $\gamma = 0$ or $\gamma \rightarrow \infty$, we recover the symmetric GARCH. A U shape can be given to the news impact curve by changing the sign of λ (see Figure 2.b). In that case, big news receive more weight than small ones.

3.2 The importance of the threshold c

Introducing a threshold c gives a great flexibility in managing the impact curves of models with exponential and generalised exponential transition functions⁷. Let

Figure 3: News impact curves for STGARCH with a threshold



me consider first the generalised exponential model or GESTGARCH. The introduction of a positive c (or negative as the transition function is symmetric in c) together with a small value of γ modifies roughly speaking the level of the tails, but not their slope (see Figure 3.a). So the constraints present in the initial ESTGARCH are removed. A higher value for γ distorts the impact curve more deeply. Introducing a threshold in the exponential transition function mixes two asymmetry effects: the size and the sign effect. For small values of γ , a negative c rotates the impact curve clockwise and a positive c anticlockwise (see Figure 3.b). A higher value of γ gives rise to humps in the right or the left tail depending on the sign of c . A positive λ gives a U shape to the impact curve and a negative threshold translate and distorts the left tail, while leaving the right tail roughly unchanged.

4 Specification searches

The multiplicity of possible non linear models that can be applied to a given sample renders the econometric analysis difficult. Tools are needed for a preliminary analysis especially here as the computational burden for inference may be important. We must first verify if a GARCH model is necessary and second if the GARCH effect presents a nonlinearity of one of the type described above. Bauwens and Lubrano (1991) have proposed a Bayesian approach to investigate the Bayesian residuals of a linear regression model when heteroskedasticity is suspected. The method compares the posterior expectation of the squared Bayesian residuals computed under the null hypothesis of homoskedasticity H_0 , with the predictive mean

⁷A threshold, when introduced in the logistic transition function, has a very minor influence on the shape of the impact curve. Consequently it is not worth being considered here.

of the squared error term computed under various hypothesis of heteroskedasticity H_i . The posterior distribution of the Bayesian residuals contains information on what the residuals really are, including modelling deficiencies. On the contrary, the predictive density of the error term of a model predict what the residuals should be considering a specific alternative hypothesis. This duality will be at the heart of my specification strategy where I shall compare posterior and predictive expectations by mean of an auxiliary regression using a Bayesian information criterion to select the most likely hypothesis.

4.1 Posterior residuals under H_0

In a linear regression model, a Bayesian residual (as defined for instance in Zellner 1975), is a random variable that is a linear combination of the random variable δ once a given sample (y, X) of dimension T has been observed:

$$u = y - X\delta \quad (11)$$

The variable u also represent the news as defined above. The posterior distribution of u is a linear transformation of the posterior distribution of δ . Under H_0 and a natural conjugate prior density, the latter is a Student density:

$$\varphi(\delta|y) = f_t(\delta|\delta_*, M_*, s_*, \nu_*) \quad (12)$$

where

$$\begin{cases} M_* &= M_0 + X'X \\ \delta_* &= M_*^{-1}(M_0\delta_0 + X'y) \\ s_* &= s_0 + y'y + \delta_0'M_0\delta_0 - \delta_*'M_*\delta_* \\ \nu_* &= \nu_0 + T - d \end{cases} \quad (13)$$

result from the application of standard formulae of natural conjugate analysis⁸. Symbols with a subscript equal to zero represent prior quantities and those with a subscript equal to a star their posterior equivalent. Usual theorems on the distribution of linear transformations of random variables gives the posterior density of Bayesian residuals which is Student:

$$\varphi(u|y) = f_t(u|\hat{u}, P^+, s_*^2, \nu_*) \quad (14)$$

where P^+ is the Moore-Penrose inverse of:

$$P = XM_*^{-1}X' \quad (15)$$

and $\hat{u} = y - X\delta_*$. Under a non informative prior, the posterior mean of the Bayesian residual is given by the classical residual. By marginalisation, I get:

$$\varphi(u_t|y) = f_t(u_t|\hat{u}_t, (P_{tt})^{-1}, s_*^2, \nu_*) \quad (16)$$

⁸See for instance Bauwens et al (1999), chapter 2

and

$$\text{Var}(u_t|y) = \frac{\nu_*}{\nu_* - 2} s_*^2 P_{tt} \quad (17)$$

As $\text{Var}(u_t|y) = E(u_t^2|y) - [E(u_t|y)]^2$, I can deduce that

$$E(u_t^2|y) = E(\sigma^2|y) P_{tt} + \hat{u}_t^2 \quad (18)$$

For $t \rightarrow \infty$, $E(u_t^2|y) \rightarrow \hat{u}_t^2$ as P_{tt} represents a small sample effect that goes to zero. The posterior expectation $E(u_t^2|y)$ will be compared to various predictive expectations by means of the following regression:

$$\hat{u}_t^2 = a_0 + a_1 P_{tt} + a_2 \sum_{j=1}^p \frac{p+1-j}{p(p+1)} g_{H_i}(\hat{u}_{t-j}^2) + \epsilon_t \quad (19)$$

The left hand side plus a_0 and $a_1 P_{tt}$ represent the posterior expectation of u_t^2 under H_0 . The rest of the right hand side represents the predictive expectation of u_t^2 under alternative hypothesis of nonlinearity. To limit the number of regressors which is the plague of this type of regression⁹, I have used the approximation defined by Engle (1982) to enforce positivity constraints for ARCH(p). It is important to keep the number of regressors at a minimum to conduct the search for the type of nonlinearity. Otherwise, the message of the sample may be blurred by too many regressors.

4.2 Predictive errors under H_i

The predictive of the observed sample under H_0 is

$$\begin{aligned} p(u_t) &= \int p(u_t|\sigma^2) \varphi(\sigma^2) d\sigma^2 \\ &= \int f_N(u_t|0, \sigma^2) f_{ig}(\sigma^2|s_0, \nu_0) d\sigma^2 \\ &= f_t(u_t|0, 1, s_0, \nu_0) \end{aligned} \quad (20)$$

where the subscripts N , ig and t stand for the Normal, the inverted gamma² and the Student densities. See Bauwens et al (1999), appendix A for more details. Under H_i , σ^2 becomes time variable so that the predictive density becomes

$$p(u_t) = \int p(u_t|h_t(\theta)) \varphi(\theta) d\theta \quad (21)$$

But this density and its moments have no closed form. It is convenient for the sequel to consider an approximate predictive density with

$$\begin{aligned} p(u_t) &= \int p(u_t|h_t) \varphi(h_t) dh_t \\ &= \int f_N(u_t|0, h_t) f_{ig}(h_t|h_t^0, \nu_0) dh_t \\ &= f_t(u_t|0, 1, h_t^0, \nu_0) \end{aligned} \quad (22)$$

⁹See for instance Granger and Terasvirta (1993), page 73 where a similar type of regression is used for testing linearity in the mean.

where $f_{ig}(h_t|h_t^0, \nu_0)$ is a prior density on h_t with scale parameter h_t^0 and ν_0 degrees of freedom. A part of the model is for the while discarded, but will be reintroduced by an adequate choice of the scale parameter h_t^0 . The approximate predictive density of the squared error term u_t^2 is a Fisher density:

$$u_t^2 \sim F(1, \nu_0) h_t^0 / \nu_0 \quad (23)$$

with expectation:

$$E(u_t^2) = h_t^0 / (\nu_0 - 2) \quad (24)$$

I shall concentrate all my efforts in determining the most reasonable h_t^0 corresponding to each alternative hypothesis.

4.2.1 H_1 : ARCH(1) model

In an ARCH(1) model, the skedastic function is $h_t = \omega + \alpha u_{t-1}^2$ with $u_t = y_t - x_t \delta$. I suppose that x_t is univariate for ease of notations. Combining these two expressions gives:

$$h_t = \omega + \alpha y_{t-1}^2 + \alpha \delta^2 x_{t-1} - 2\alpha \delta y_{t-1} x_{t-1} \quad (25)$$

Let me linearise the products $\alpha \delta$ and $\alpha \delta^2$ around $\alpha = 0$ and $\delta = \hat{\delta}$ which is the OLS estimator of δ under H_0 . The resulting candidate for h_t^0 is:

$$h_t^0 = \omega + \alpha \hat{u}_{t-1}^2 \quad (26)$$

4.2.2 H_2 : STARCH(1) model

Combining (5) and (26), the scale parameter of my prior is now of the form:

$$h_t^0 = \omega + \alpha_1 \hat{u}_{t-1}^2 + \lambda \hat{u}_{t-1}^2 F(u_{t-1}, \gamma) \quad (27)$$

This formulation is not very convenient for the while because it still presents the nonlinearity. I shall apply the same recipe as above and linearise $F(u_{t-1}, \gamma)$. This is in accordance with some of the linearity tests reported in the classical literature [see e.g. Luukkonen et al (1988) and Terasvirta (1994)] where a third order Taylor expansion of $F(z)$ around $z = 0$ is recommended. The motivation in the classical literature is to overcome the identification problem present under the null of linearity. Here the question is simply to get a manageable prior scale parameter to compute an approximate conditional predictive density. I shall select the logistic, exponential and generalised exponential transition functions. Their respective third order Taylor expansions are:

$$\begin{aligned} T_{3L}(z) &\propto 24 + 12z - z^3 \\ T_{3E}(z - c) &\propto 3c^2 - 6cz(1 - c^2) + 3z^2(1 - 3c^2) + 2cz^3(3 - 5c^2) \\ T_{3GE}(z^2 - c^2) &\propto z^2(2 - c^2) \end{aligned} \quad (28)$$

Going up to the third order makes very apparent the difference between these three transition functions. The expansion of the logistic is always of the first and third degree, while the expansion of the exponential with $c = 0$ and of the generalised exponential are of the second degree only. The exponential with a threshold has an expansion where all the terms up to the third degree are present. Of course, for $c = 0$, the adequate term vanish.

Let me now replace $F(u_{t-1}, \gamma)$ by its Taylor expansion in the expression of h_t^0 (27). I get three alternative formulations for the prior scale parameter

$$h_t^0 = \omega + \alpha_1 \hat{u}_{t-1}^2 + \lambda \hat{u}_{t-1}^2 \times \begin{cases} (c_0 + c_1 \hat{u}_{t-1} - c_3 \hat{u}_{t-1}^3) \\ (c_0 - c_1 \hat{u}_{t-1} + c_2 \hat{u}_{t-1}^2 + c_3 \hat{u}_{t-1}^3) \\ (c_0 + c_2 \hat{u}_{t-1}^2) \end{cases} \quad (29)$$

which will be the basis of my specification search.

4.3 Comparing hypotheses using an auxiliary model

I can now compare the posterior expectation of u_t^2 to its various possible predictive expectations by mean of the following general regression:

$$\hat{u}_t^2 = a_0 + a_1 P_{tt} + a_2 \sum_{j=1}^p \frac{p+1-j}{p(p+1)} \hat{u}_{t-j}^2 [1 + a_3 \hat{u}_{t-j} + a_4 \hat{u}_{t-j}^2 + a_5 \hat{u}_{t-j}^3] + \epsilon_t \quad (30)$$

I first have to select the optimal p in the full regression (30) using an information criterion. Conditionally on that choice, I shall try to impose the restrictions corresponding to each of the three possible models and choose the model that minimises the Schwarz criterion. This corresponds to selecting the model that has the maximum posterior probability under a diffuse prior information. The set of restrictions can be summarised as follows:

- $a_2 = a_3 = a_4 = a_5 = 0$ mean that there is no ARCH effect.
- $a_3 = a_4 = a_5 = 0$ mean that the ARCH effect is linear.
- $a_4 = 0$ while $a_3 \neq 0$ and $a_5 \neq 0$ means that the nonlinearity is of an odd type corresponding to a logistic transition function.
- $a_3 = a_5 = 0$ and $a_4 \neq 0$ means that the nonlinearity is of an even type that can be modelled either by an exponential with $c = 0$ or a generalised exponential transition function.

- $a_3 \neq 0$, $a_4 \neq 0$ and $a_5 \neq 0$ means that the nonlinearity mixes size and sign effects which can be reasonably modelled by an exponential transition function with a threshold, although this case is not as clear as the others. Many nonlinear transition functions can lead to such a Taylor expansion.

Remark:

Engle and Ng (1993) have proposed in a classical framework sign and size bias test regressions as misspecification tests, but also as exploration tools. They regress $\hat{u}_t^2$ over a constant and S_{t-1} for positive sign bias, $S_{t-1}\hat{u}_{t-1}$ for positive size bias and $(1 - S_{t-1})\hat{u}_{t-1}$ for negative size bias.

5 Bayesian inference in the STARCh model

Bayesian inference in usual GARCH models is made complicated by the fact that the posterior density must be integrated out numerically. Kleibergen and van Dijk (1993) use importance sampling while Geweke (1994) proposes an independent Metropolis algorithm. The usual Gibbs sampler is not applicable here as the posterior density has no property of conjugacy [as defined by Carlin and Gelfand (1991)]. Bauwens and Lubrano (1998) use a Griddy-Gibbs sampler to overcome this difficulty. The algorithm is based on the numerical inversion of each conditional posterior density. As in usual Gibbs samplers, the algorithm encounters numerical difficulties in the case of a strong correlation between the parameters of the model. This may motivate the introduction of an alternative parameterisation of the GARCH model. The presence of a smooth transition function creates some specific problems that concerns the integrability of the posterior density of γ , the smooth transition parameter and a local identification problem at $\gamma = 0$ that corresponds to the linearity of the skedastic function h_t . The threshold c may also cause identification problems.

5.1 Prior and posterior densities

Let me give now the complete notation of the smooth transition GARCH model:

$$\begin{cases} y_t = x_t' \delta + u_t \\ u_t \sim N(0, h_t) \\ h_t = \omega + \alpha_1 u_{t-1}^2 + \lambda u_{t-1}^2 F(u_{t-1}, \gamma, c) + \beta h_{t-1} \end{cases} \quad (31)$$

Let me define the diagonal $(T - 1) \times (T - 1)$ matrix $H(\delta, \theta)$ having h_t as its $[t, t]$ element and where $\theta' = [\omega, \alpha_1, \lambda, \beta, \gamma, c]$. The likelihood function of the $T - 1$ observations of y_t is:

$$l(y; \delta, \theta) \propto |H(\delta, \theta)|^{-1/2} \exp -\frac{1}{2} u' H^{-1}(\delta, \theta) u \quad (32)$$

where $u = y - X\delta$. To evaluate this function, it is necessary to define h_1 . I shall treat this initial condition as fixed. For $u_0 = 0$, $h_1 = \omega$. So h_1 can be taken equal to the empirical variance of the first observations.

I shall be non informative on δ with

$$\varphi(\delta) \propto 1 \quad (33)$$

It is difficult to devise an informative prior for the skedastic parameters. The prior should guaranty or preserve the positivity of h_t . In a usual GARCH, this means that ω , α and β should be positive. Here α_1 and α_2 have to be positive, which means that $\lambda > -\alpha_1$. So the parameterisation in λ may not be the most convenient to impose positivity constraints a priori. In the transition function $F(\cdot)$, γ has to be positive for identification purposes. The same may hold for c in certain transition functions. I shall not try to impose a priori strict stationarity that is necessary for inference. But in general, strict stationarity is verified for most available samples [see e.g. Kleibergen and van Dijk (1993)].

I first propose a non informative prior for the parameters of the skedastic function that cause no problem. I shall discuss below the case of $\varphi(\gamma)$ and of $\varphi(c)$ that raise specific issues such as integrability of the posterior and identification. So:

$$\varphi(\omega, \alpha_1, \alpha_2, \beta) \propto \begin{cases} 1 & \text{if } \omega, \alpha_1, \alpha_2, \beta > 0 \\ 0 & \text{otherwise} \end{cases} \quad (34)$$

The complete posterior density

$$\varphi(\delta, \theta|y) \propto \varphi(\delta) \times \varphi(\omega, \alpha_1, \alpha_2, \beta) \times \varphi(\gamma, c) \times l(y; \delta, \theta) \quad (35)$$

has to be integrated out numerically. No partial analytical integration is possible.

5.2 Integrability for γ

The smooth transition function $F(\cdot)$ becomes a Dirac function for $\gamma \rightarrow \infty$. For instance with the logistic function model (31) becomes in this case observationally equivalent to the model of Glosten et al (1993). Consequently:

Theorem 1 *The posterior density of γ is of the same order of integrability as the prior $\varphi(\gamma)$. In particular posterior moments exist only up to the order of prior moments.*

Proof: For $\gamma \rightarrow \infty$, $F(\gamma, u_t)$ is $O(1)$ in γ . Consequently $\varphi(\gamma|y)$ does not tend to 0 as γ tends to infinity and is not integrable. □

A similar problem arises in regression models with Student errors as underlined in Bauwens and Lubrano (1998). The conclusion is that a prior information is

needed to force the posterior density to tend to zero quickly enough at its right tail in order to be integrable. The prior should at least be $O(\gamma^{1+v})$ with $v > 0$. A convenient minimal prior is the truncated Cauchy density with ¹⁰:

$$\varphi(\gamma) = \begin{cases} (1 + (\gamma - \gamma_0)^2)^{-1} & \text{if } \gamma > 0 \\ 0 & \text{otherwise} \end{cases} \quad (36)$$

Its right tail is $O(\gamma^2)$. A prior that guaranties the existence of all the posterior moments of γ is the exponential prior used for instance by Geweke (1993):

$$\varphi(\gamma) \propto \exp(-\gamma/\gamma_0) \quad \gamma_0, \gamma > 0 \quad (37)$$

The prior mean of γ is equal to γ_0 . In order to be able to specify easily a prior mean for γ , it should be scale free. This is obtained by scaling the observations (dividing the y_t by their empirical standard deviation).

5.3 Local identification for λ

The smooth transition function becomes constant at the point $\gamma = 0$. Consequently the skedastic function presents a perfect collinearity and the parameter λ becomes not identified. If for some samples, the point $\gamma = 0$ is not in the useful integration range and so the problem has no practical importance, this is not the general situation, especially for high frequency data. Consequently it is recommended to exclude a priori the point $\gamma = 0$ from the integration range. It may also be wise to use a positive γ_0 in the Cauchy prior, so as to eventually translate the mode of the posterior away from zero. Supposing γ strictly positive is not a bias in favour of asymmetry as there is still the possibility that $\lambda = 0$.

5.4 The special case of the threshold parameter c

In usual nonlinear regression models, the posterior density of the threshold c may be very badly behaved [see for instance Lubrano (1998) or Osiewalski and Welfe (1998)]. Here, the threshold parameter determines what is the most likely value for a big shock when the generalised exponential transition function is chosen. When a threshold is used in the exponential transition function, it determines where, on the scale of the u is the asymmetry between the positive and negative shocks. Of course the domain of definition of c is determined by the observed sample. But there will be an identification problem every time there is not enough observations left between the end points of the integration range and the minimum and maximum of the sample. So c has to be integrated out on a restricted range.

¹⁰A flat prior on $1/\gamma$ yields $\varphi(\gamma) \propto 1/\gamma^2$. But this prior creates a singularity at $\gamma = 0$ which is a point of interest. The Cauchy prior simply translates the singularity outside of the region $\gamma > 0$.

5.5 Another possible parameterisation

The general notation for the GARCH model assumes that in (1) the variance of ϵ_t is not constrained to be one, but is equal to σ^2 . Adopting this formulation for the STGARCH model gives:

$$\begin{cases} y_t = x_t' \delta + u_t \\ u_t = \epsilon_t \sqrt{h_t} & \epsilon_t \sim N(0, \sigma^2) \\ \sigma^2 h_t = \omega + \alpha_1 u_{t-1}^2 + \lambda u_{t-1}^2 F(u_{t-1}, \gamma, c) + \beta \sigma^2 h_{t-1} \end{cases} \quad (38)$$

I can divide by σ^2 both members of the definition of h_t :

$$h_t = \frac{\omega}{\sigma^2} + \frac{\alpha_1}{\sigma^2} u_{t-1}^2 + \frac{\lambda}{\sigma^2} u_{t-1}^2 F(u_{t-1}, \gamma, c) + \beta h_{t-1} \quad (39)$$

This shows that in this parameterisation only $\tilde{\omega} = \omega/\sigma^2$, $\tilde{\alpha}_1 = \alpha_1/\sigma^2$ and $\tilde{\lambda} = (\alpha_2 - \alpha_1)/\sigma^2$ are identified. On the contrary, β is always identified. The traditional parameterisation insures identification by setting $\sigma^2 = 1$. Another possible simple identification rule consists in setting $\tilde{\omega} = 1$ which means that $\omega = \sigma^2$. The drawback of this parameterisation is that $\tilde{\alpha}_1$ and $\tilde{\lambda}$ now depend on the scale of the y_t . Its advantage is that σ^2 can be integrated out analytically. The size of the numerical integration is thus reduced by one. The skedastic function becomes:

$$h_t = 1 + \tilde{\alpha}_1 u_{t-1}^2 + \tilde{\lambda} u_{t-1}^2 F(u_{t-1}, \gamma, c) + \beta h_{t-1} \quad (40)$$

As σ^2 will be integrated out analytically, there will be no possible numerical trade off between the constant and the variable part of the skedastic function. This certainly will improve numerical stability [see e.g. Robert and Mengersen (1995) for an analysis of reparameterisation issues on the performance of the Gibbs sampler].

The likelihood function (32) of the $T - 1$ observations of y_t is transformed into:

$$l(y; \delta, \sigma^2, \tilde{\theta}) \propto \sigma^{-(T-1)} |H(\delta, \tilde{\theta})|^{-1/2} \exp - \frac{1}{2\sigma^2} u' H^{-1}(\delta, \tilde{\theta}) u \quad (41)$$

where $\tilde{\theta}' = [\tilde{\alpha}_1, \tilde{\lambda}, \beta, \gamma, c]$. Note that in order to evaluate this likelihood function, I have now $h_1 = 1$ as a starting value.

The prior densities on σ^2 and $\tilde{\theta}$ have to be made compatible with those on θ . A usual prior density on σ^2 which is a scale parameter is

$$\varphi(\sigma^2) = f_{ig}(\sigma^2 | s_0, \nu_0) \quad (42)$$

or simply $\varphi(\sigma^2) \propto \sigma^{-2}$ if we are non informative. As the new parameterisation implies that $\omega = \sigma^2$, the prior on ω must have the same form. The prior density

(34) on α and λ was uniform. Due to the Jacobian of the transformation from α_i to $\tilde{\alpha}_i$, the resulting prior is

$$\varphi(\alpha_1, \lambda) \propto \sigma^{-4} \quad (43)$$

The priors on β , γ and c remain the same as before. With these priors, I can get nice expressions for the posterior densities

Theorem 2 *The marginal posterior density of δ and $\tilde{\theta}$ is given by*

$$\begin{aligned} \varphi(\delta, \tilde{\theta}|y) &\propto \int_{\sigma^2 > 0} l(y; \delta, \sigma^2, \tilde{\theta}) \varphi(\sigma^2) \varphi(\delta) \varphi(\tilde{\theta}) d\sigma^2 \\ &\propto |H(\delta, \tilde{\theta})|^{-1/2} \times [s_0 + u'H^{-1}(\delta, \tilde{\theta})u]^{-(\nu_0+T+3)/2} \end{aligned}$$

The conditional posterior density of σ^2 is an inverted gamma2 with:

$$\varphi(\sigma^2|\delta, \tilde{\theta}, y) = f_{ig}(\sigma^2|s_*(\delta, \tilde{\theta}), \nu_0 + T + 3) \quad (44)$$

where

$$s_*(\delta, \tilde{\theta}) = s_0 + u'H^{-1}(\delta, \tilde{\theta})u \quad (45)$$

The marginal posterior density of σ^2 can be simulated by:

$$\sigma_j^2 \sim s_*(\delta_j, \tilde{\theta}_j)/\chi^2(\nu_0 + T - 1) \quad (46)$$

where δ_j and $\tilde{\theta}_j$ are draws of the corresponding parameters made from their marginal posterior density.

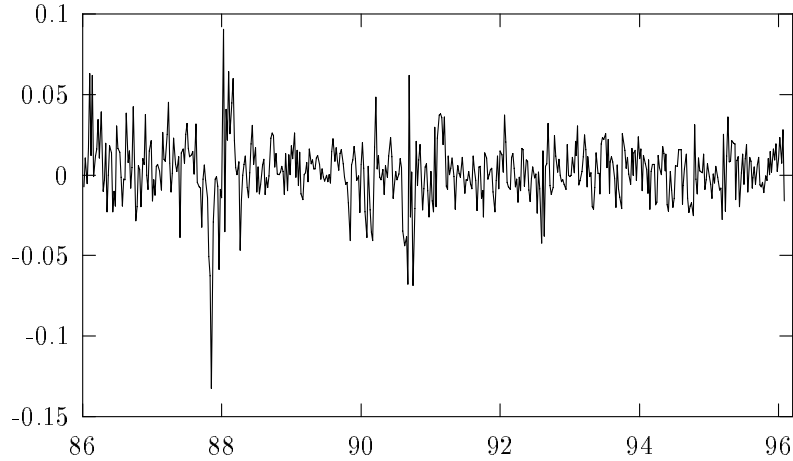
Proof: The proof follows from the integrating constants of the inverted gamma2 density. The simulation procedure is simply based on the property of usual algorithms for simulating an inverted gamma2. □

Of course, we are interested in the posterior density of the original parameterisation which is the most common one and not of the transformed parameters. The backward transformation can be done easily. Posterior draws for α_1 and λ are obtained using the non linear transformation: $\alpha_{1j} = \tilde{\alpha}_{1j} \times \sigma_j^2$ and $\lambda_j = \tilde{\lambda}_j \times \sigma_j^2$.

6 Specification and inference for the Brussels index

As a first illustration of the method, I shall use data of the Brussels Spot Market Index. I have weekly data collected at the closure of the market each Friday covering the period 03/01/86 to 26/01/96. The data have been corrected for bank holidays. This make 509 observations that I will take in differences of logs. The transformed observations are displayed in Figure 4. These data present very large fluctuations after the melt down of October 1987, some large fluctuations around 1991 and are relatively quiet after 1993.

Figure 4: Brussels Spot Index returns



6.1 Specification search

The returns are first filtered by an AR(2) model to remove autocorrelations:

$$\Delta y_t = \underset{[1.74]}{0.08} + \underset{[4.41]}{0.19}\Delta y_{t-1} + \underset{[3.68]}{0.16}\Delta y_{t-2} + \hat{u}_t$$

(t statistics are given between square brackets). From this regression, I get 506 residuals $\hat{u}_t$ that are analysed by mean of the auxiliary regression (30). A BIC determined an optimal p of 4. The resulting regression is:

$$\hat{u}_t^2 = \underset{[1.10]}{0.15} + \underset{[3.10]}{0.00025} P_{tt} + \underset{[2.56]}{0.50} \left[\sum_{j=1}^4 \frac{p+1-j}{p(p+1)} \hat{u}_{t-j}^2 \right] \times \\ \left[1 - \underset{[-8.42]}{0.51} \hat{u}_{t-j} - \underset{[-1.25]}{0.011} \hat{u}_{t-j}^2 + \underset{[8.09]}{0.019} \hat{u}_{t-j}^3 \right] + e_t$$

Let me now investigate various types of restrictions in this regression and select

Table 1: Which GARCH for SPOTW data

	linear	non linear	odd transition	even trans.
BIC	1.729	1.567	1.555	1.672

the most probable model by mean of its Schwarz criterion. From Table 1, I can conclude that there is definitely an ARCH effect, that this effect is nonlinear and that the nonlinearity distinguishes between negative and positive shocks. The

nonlinearity is generated by the large fluctuations of the first half of the sample. If I discard the first 250 observations, the ARCH effect becomes linear. But if I discard only the first 100 observations nonlinearity is still present, despite the fact that the melt down of 1987 is excluded. For each sub sample analysis, I refiltered the data with an updated AR(2).

6.2 Inference results

A LSTGARCH(1,1) model was fitted to the filtered returns that were standardised (division by the empirical standard deviation). I used a Cauchy prior for γ with

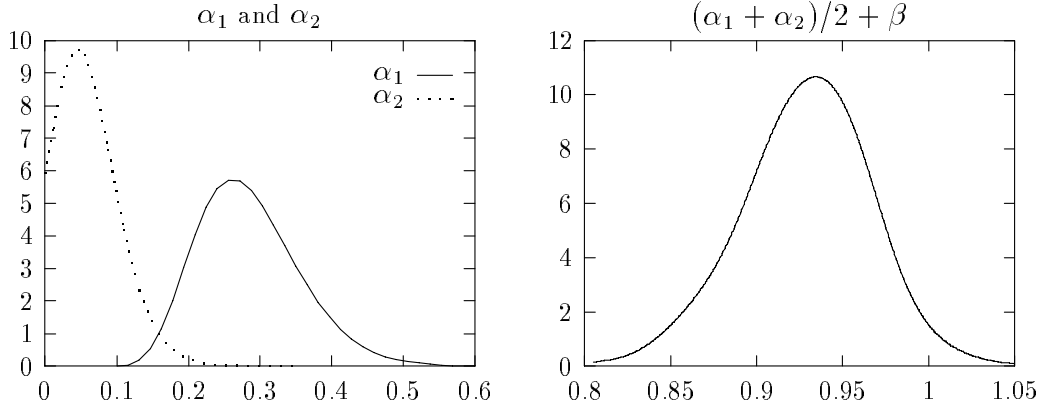
Table 2: MLE and posterior results
for SPOTW data

Method	ω	α_1	α_2	β	γ
MLE	0.055	0.24	0.051	0.79	1.84
St dev	0.020	0.061	0.042	0.042	2.60
Griddy Gibbs	0.069	0.29	0.064	0.75	1.91
St dev	0.026	0.072	0.042	0.051	1.60

$\gamma_0 = 0$. The prior on all the other parameters is uniform. The posterior density was integrated using the Griddy Gibbs sampler of Bauwens and Lubrano (1998). The draws displayed a certain amount of negative correlation between β and (ω, α_1) of respectively -0.79 and -0.63. Other correlations are negligible. For 1000+150 draws (the first 150 draws were discarded for warming up the chain), I get the following results reported in table 2. It took 19 minutes on a Pentium 130 for the 1150 draws. Convergence was checked using CUMSUM graphs. There is not much difference between classical and Bayesian estimates. On average, Bayesian standard deviations are slightly greater than their classical homologues, but the ratio of posterior means over posterior standard deviations is in general lower than the classical counterpart.

The joint graph of the posterior densities of α_1 and α_2 shows in Figure 5 that negative (α_1) and positive (α_2) shocks have a quite different impact. Negative shocks have a greater contribution to the conditional variance than positive ones. If the graphs of the posteriors slightly overlap, the posterior probability that $\lambda > 0$ is equal to zero when estimated directly from the draws. The posterior graphs are obtained, not from the draws, but using a technique of variance reduction explained in Bauwens and Lubrano (1998).

Figure 5: Posterior densities for BRUSSELS

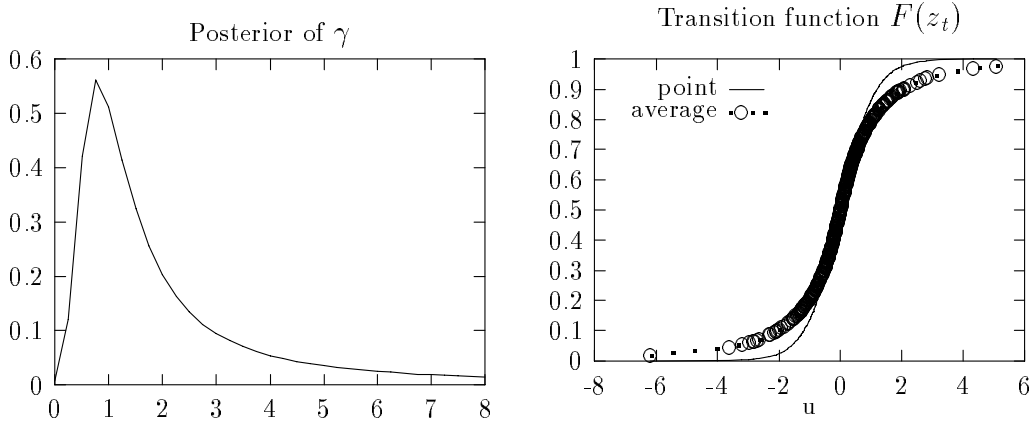


The shocks are not persistent as the weak stationarity condition (see the appendix for a derivation) is verified with a probability of 0.98. The posterior density of $(\alpha_1 + \alpha_2)/2 + \beta$, given in Figure 5, was estimated with a kernel method, using the draws of the Gibbs sampler.

6.3 Analysis of nonlinearity

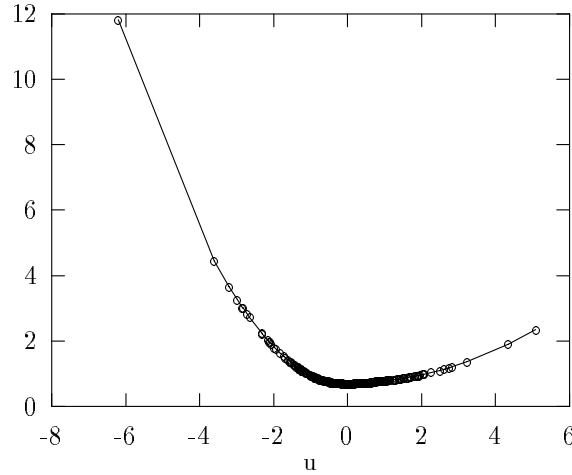
The posterior density of γ , as given in Figure 6, is fairly concentrated on low values of γ which put credit to a smooth transition between the regimes. The moments

Figure 6: The smooth transition function for SPOTW data



of γ were computed on the truncated range $[0, 8]$, because untruncated posterior moments do not exist with a Cauchy prior. The graph of the smooth transition function $F(u_{t-1})$, given in Figure 6, is obtained as a transformation of the posterior draws. On the same graph, I give the same transition function, but evaluated at

Figure 7: News impact curve for SPOTW data



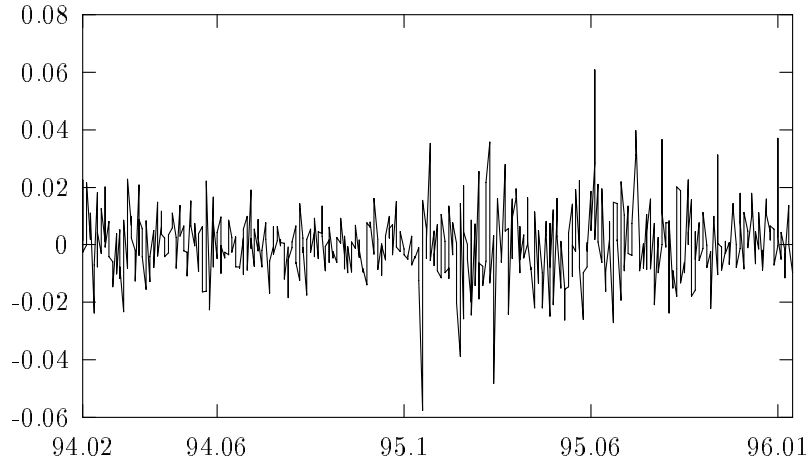
the posterior expectation. This combination shows that the Bayesian transition function is much softer than the transition function obtained by a point estimate. As the filtered returns were standardised, the scale is expressed in term of standard deviations. Most of the observation are situated within ± 2 standard deviations. They correspond to the quasi linear part of the smooth transition function. The transition function is lower than 0.053 (respectively 0.10) or greater than 0.94 (respectively 0.90) only for four observations (respectively 10 and 8 observations which corresponds to the melt down of October 1987 and a period in July 1990. This demonstrates the fact that the smooth transition is important and that the change of regime is not an abrupt function of the signs of the shocks, but evolves gradually with their magnitude. This is a soft transition. An abrupt transition concerns only very few observations in the sample.

The news impact curve, as given in Figure 7 is very much tilted, revealing a large asymmetry between negative and positive shocks, even for small shocks. It does not matter in this case if the curve is evaluated at the posterior mean or computed directly.

7 Specification and inference for the Tokyo index

Here I try to illustrate the even transition function with the daily NIKKEI 225 index covering the period 17 Feb 1994 to 18 Jan 1996. This makes 475 observations corresponding to the closure of the market. The data, displayed in Figure 8, are taken in differences of logarithms. A first look at this graph reveals that we have another type of sample configuration. The first year of observations is characterised by a relative quietness of the market. The second year corresponds to a greater variance with many peaks.

Figure 8: Tokyo NIKKEI 225 returns



7.1 Specification search

It appeared that it was not necessary to filter the data. I only standardised $u_t = \Delta \log(y_t)$ where y_t is the level of the index. The test regression was estimated with an optimal lag of $p = 1$:

$$\hat{u}_t^2 = 0.72 + 0.38 \hat{u}_{t-1}^2 \times [1 + 0.0025 \hat{u}_{t-1} - 0.016 \hat{u}_{t-1}^2 - 0.00017 \hat{u}_{t-1}^3] + e_t$$

$\begin{matrix} [5.78] & [3.84] \\ [0.071] & [-3.02] & [-0.10] \end{matrix}$

(t statistics between square brackets). A sequence of BIC, as reported in table 3 clearly indicates the even (exponential) transition model (ESTGARCH) as the preferred model.

Table 3: Which GARCH for the daily NIKKEI data

	linear	nonlinear	odd	even
BIC	1.612	1.635	1.640	1.605

7.2 Inference results

The Tokyo data set is tricky to analyse as there is a very high negative correlation between ω and β (around -0.95) that impeded the convergence of the Monte Carlo Chain algorithm. However using the alternative parameterisation of the GARCH

solves in this case many problems as correlation in the draws completely disappears. As few as 100 draws were enough to have a clear idea of the shape of the posterior and 1000+150 produced a very reasonable convergence of the chain as checked on CUMSUM graphs.

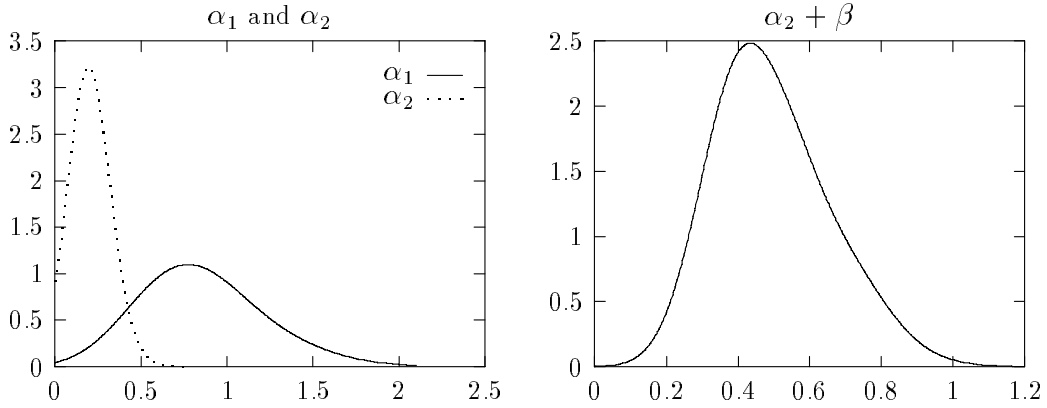
Table 4: Posterior results for the NIKKEI 250 index

	ω	α_1	α_2	β	γ	c
MLE	0.0020	0.59	0.069	0.87	6.28	-0.82
S.D.	0.023	0.17	0.027	0.045	4.97	0.17
Mean	0.015	0.86	0.20	0.30	2.22	-1.05
St dev	0.0051	0.33	0.075	0.17	1.72	0.32

In preliminary tries not reported here, I have adjusted to the data a STGARCH with three different even transition functions. The model with an exponential transition and a threshold seemed to give the better fit and the best account of non linearity. Final results are given in table 4. All the estimates, classical and Bayesian, are given in the original parameterisation. The maximum likelihood estimates were computed with the normalisation $\sigma^2 = 1$. The Bayesian estimates were computed with $\omega = 1$. The reported Bayesian estimates were obtained by applying the necessary transformations. There are some substantial differences between the Bayesian and the classical estimates. But in general classical and Bayesian confidence intervals are overlapping, except for β . The posterior density of β is in fact bimodal. A Dickey-Savage ratio would favour the restriction $\beta = 0$, restriction which is in accordance with the specification search that retained an optimal lag $p = 1$ for the test regression. The probability that $\lambda = \alpha_2 - \alpha_1 > 0$ is equal to 0.006. It goes up to 0.24 when c is imposed to be zero. The presence of a threshold is essential here for modelling the non linear effect that totally disappears otherwise.

It is very difficult to say that big shocks have a permanent effect. The posterior expectation of $ws = \alpha_2 + \beta$, which gives indications on stationarity and persistence (see the appendix for a derivation), is 0.50 with a standard deviation of 0.15. The probability that $ws > 1$ is equal to 0.001. This probability does not change much when $c = 0$. The impact of bigger shocks given by α_2 is much smaller than that of smaller shocks (α_1). However, the posterior density of α_1 is flatter, indicating a larger uncertainty about the effective impact of small shocks on volatility.

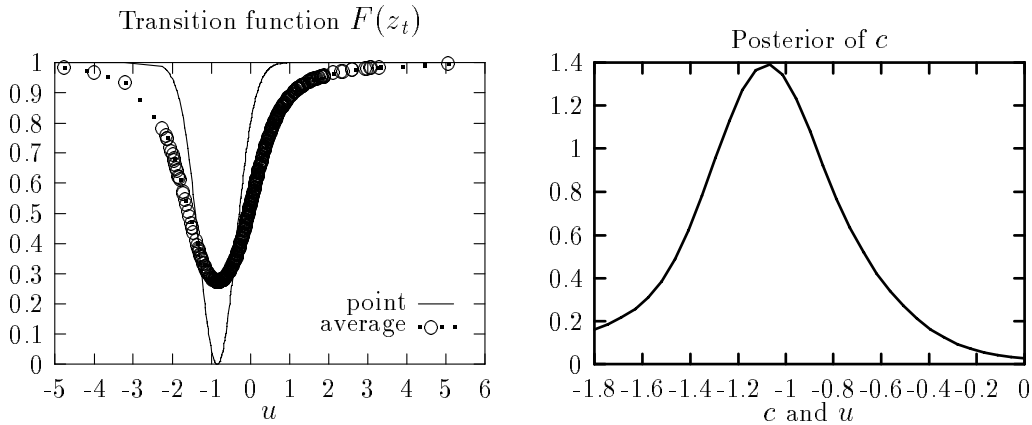
Figure 9: Posterior densities for Tokyo



7.3 Analysis of nonlinearity

The even transition function is plotted against the standardised shocks. Most of the shocks are within ± 2 standard deviations. The transition function sharper

Figure 10: Smooth transition for Tokyo

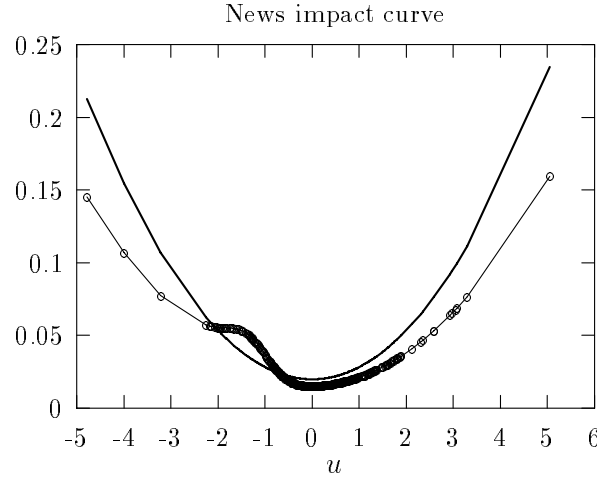


when evaluated at posterior expectation than when computed directly. Consequently the "Bayesian" actual transition is smoother than its "classical" counterpart. Averaging has also the consequence that the transition function is not zero at its minimum, except when $c = 0$. The presence of a threshold makes that there are much more observations on the right hand side of the transition function than on the left hand side even if there are as many positive than negative shocks in the data.

The graph of the posterior density of c shows that the negative threshold $c = -1.05$ is relatively precisely estimated. The range of integration for c was truncated for identification reasons (greater than -1.8). The positive segment receives none

of the probability (as verified on experiments not reported here). As a matter of fact, the maximum likelihood routine did not converge when c was left free, when c was constrained to be positive, but did converge when c was constrained to be negative.

Figure 11: News impact curves for Tokyo



The (Bayesian) news impact curve displays a fairly large asymmetry when compared to the symmetric GARCH. Figure 11 indicates that if most of the points concern small shocks between ± 2 standard deviations, the deformation of the impact curve is situated inside that interval. Inside that interval, positive and negative shocks have a different impact. The volatility is higher for small negative shocks (between 0 and 1.5 standard deviations) than for positive ones. Over 2 standard deviations, positive and negative shocks have the same influence on volatility. The impact curve for the linear GARCH gives a greater weight to big shocks.

8 Conclusion

In this paper, I have considered a new type of asymmetric GARCH(1,1) with different choices for the asymmetry. It appears clearly that the Bayesian approach was successful. First of all, the advantage of the Bayesian approach is that problems are clearly pointed out and in particular the difficulties coming from inference on the smooth transition parameter that are frequently reported in classical analysis of nonlinear models [see e.g. Granger and Terasvirta (1993), p.123] are given here a sound theoretical basis. Secondly those inference difficulties receive a correct numerical treatment as the posterior density of the STGARCH model reveal easier to integrate than to maximise. Point estimates are not the same with the two

methods because posterior densities are skewed.

The asymmetric GARCH models proposed by Glosten et al (1993) or by Zakoian (1994) had an abrupt transition. With financial data, and especially at high frequency, the smooth transition proved to be relevant in the two empirical illustrations given in this paper. And the Bayesian approach produces a transition that is even smoother than the transition obtained with point estimates.

The presence or absence of non linearity may depend on the frequency of the observations. In empirical investigations not reported in paper, I analysed the CAC40 Paris index. Weekly data did not present non linearity according to the specification tests, but daily data did. This exercise demonstrates the usefulness of the specification tools introduced in this paper to detect the presence or absence of non linearity in the data and their ability to give some information on the nature of non linearity. The question still remains to know if the models I have proposed and implemented capture all the asymmetry present in the data.

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APPENDIX

A Stationarity and persistence

Nelson (1990) discusses stationarity and persistence in the GARCH(1,1) model. I shall extend some of Nelson's results to the STARCH model. The general skedastic function (5) can be factorised as:

$$h_t = \omega + h_{t-1}(\alpha_1 \epsilon_{t-1}^2 + \lambda \epsilon_{t-1}^2 F_{t-1} + \beta) \quad (47)$$

Repeatedly substituting for h_{t-i} in the above formula, we have for $t \geq 2$:

$$\begin{aligned} h_t = h_0 \prod_{i=1}^t [\alpha_1 \epsilon_{t-i}^2 + \lambda \epsilon_{t-i}^2 F_{t-i} + \beta] \\ + \omega [1 + \sum_{k=1}^{t-1} \prod_{i=1}^k (\alpha_1 \epsilon_{t-i}^2 + \lambda \epsilon_{t-i}^2 F_{t-i} + \beta)] \end{aligned} \quad (48)$$

This equation defines the conditional process of the h_t as starting from h_0 . I shall suppose h_0 finite and strictly positive with probability one. Two questions will be addressed:

- When is the process of the h_t strictly stationary
- When does a shock given to the conditional process h_t decay as $t \rightarrow \infty$?

A first theorem, largely inspired from Nelson (1990), gives conditions for strict stationarity:

Theorem 3 When $\omega = 0$, the conditional process of the h_t goes to zero almost surely if

$$E(\log[\alpha_1 \epsilon_t^2 + \lambda \epsilon_t^2 F_t + \beta]) < 0 \quad (49)$$

and goes to infinity for the reverse case.

When $\omega > 0$, the conditional process of the h_t is strictly stationary if (49) holds.

Proof: The proof follows directly from the appendix of Nelson (1990). □

I shall now concentrate on a particular definition of persistence, persistence in L^1 which corresponds to Nelson's formula (36) and is related the integrated GARCH of Engle and Bollerslev (1986). This type of persistence implies that the process is not weak stationary. I shall study persistence analysing the impact of an initial shock ϵ_0 on the conditional variance h_t .

Theorem 4 A shock will be said to be persistent in L^1 unless

$$\lim_{t \rightarrow \infty} [\alpha_1 E(\epsilon_t^2) + \lambda E(\epsilon_t^2 F_t) + \beta]^{t-1} = 0 \quad (50)$$

Integration in the sense of Engle and Bollerslev corresponds to

$$\alpha_1 E(\epsilon_t^2) + \lambda E(\epsilon_t^2 F_t) + \beta = 1 \quad (51)$$

Proof: The impact of a normalised shock ϵ_0^2 on h_t is defined by

$$\begin{aligned} \frac{1}{E(h_0)} E \frac{\partial h_t}{\partial \epsilon_0^2} &= E \left(\frac{\partial \prod_{i=1}^t [\alpha_1 \epsilon_{t-i}^2 + \lambda \epsilon_{t-i}^2 F_{t-i} + \beta]}{\partial \epsilon_0^2} \right) \\ &= E \left(\frac{\partial [\alpha_1 \epsilon_0^2 + \lambda \epsilon_0^2 F_0 + \beta]}{\partial \epsilon_0^2} \right) E \left(\prod_{i=1}^{t-1} [\alpha_1 \epsilon_{t-i}^2 + \lambda \epsilon_{t-i}^2 F_{t-i} + \beta] \right) \\ &= (\alpha_1 + \lambda [E(F_0) + E(\epsilon_0^2 \frac{\partial F_0}{\partial \epsilon_0^2})]) \times [\alpha_1 E(\epsilon_t^2) + \lambda E(\epsilon_t^2 F_t) + \beta]^{t-1} \end{aligned} \quad (52)$$

using the factorisation (48). When $t \rightarrow \infty$

$$[\alpha_1 E(\epsilon_t^2) + \lambda E(\epsilon_t^2 F_t) + \beta]^{t-1}$$

goes to zero or to infinity depending if the inside bracket is lower or greater than one. □

There is presumably no analytical result available to compute $E(\epsilon_t^2 F_t)$, except when F is an indicator function. In this case, I can compute truncated moments for the two above cases: negative versus positive shocks and small versus big shocks. I shall assume that ϵ has a Normal distribution with zero mean and unit variance.

A.1 Negative and positive shocks

F is now zero when ϵ is negative and one otherwise. So as the distribution of ϵ is symmetric, negative and positive values are equally probable and:

$$E(\epsilon_t^2 F_t) = \int_{-\infty}^0 \epsilon^2 f(\epsilon) d\epsilon = 1/2 \quad (53)$$

The weak stationarity (non-persistence) condition becomes

$$(\alpha_1 + \alpha_2)/2 + \beta < 1 \quad (54)$$

and persistence is measured by α_1 for negative shocks and by α_2 for positive ones. For $\lambda = 0$, we recover the results of Engle and Bollerslev (1986).

A.2 Small and big shocks

F now is zero when $\epsilon \in [-c, c]$ and one otherwise. Because of the symmetry of the distribution of ϵ :

$$E(\epsilon_t^2 F_t) = 1 - \int_{-c}^c \epsilon^2 f(\epsilon) d\epsilon$$

Assuming normality

$$\begin{aligned} E(\epsilon_t^2 F_t) &= [1 - \text{ERF}(c \frac{\sqrt{2}}{2}) + c \frac{\sqrt{2}}{\sqrt{\pi}} \exp(-\frac{c^2}{2})] \\ &= (1 - g(c)) \end{aligned} \quad (55)$$

where $\text{ERF}(z)$ is the error function defined as the integral of the Gaussian function from zero to z . $g(c)$ is an increasing function of $c > 0$ taking values in $[0, 1]$. The weak stationarity condition becomes a not trivial function of c with:

$$(\alpha_1 + \lambda[1 - g(c)]) + \beta < 1 \quad (56)$$

For $c \rightarrow \infty$, we recover the usual condition for weak stationarity in linear GARCH with $\alpha_1 + \beta < 1$. For $c = 0$, we have the stationarity condition for the simple exponential transition function which is $\alpha_2 + \beta < 1$.